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On The Special Gamma Function Over The Complex Two-Fold Algebras

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Abstract:

The concept of special functions plays an important role in mathematical analysis and physics as well. In this paper, we study some different types of the special Gamma function defined on the two-fold fuzzy complex field, where we combine the classical Gamma function with the two-fold fuzzy algebra defined on complex numbers. On the other hand, many elementary properties of this new special function will be determined in terms of theorems and proofs.

Keywords: Gamma function, two-fold fuzzy algebra, complex field, special function.

Introduction

The theory of special functions is considered one of the most comprehensive and important theories in mathematics due to its wide applications in various fields of knowledge and physics [1-3]. The gamma function is one of the most famous functions in mathematics that plays a central role in number theory, probability, and the calculation of random processes [6-7].

Neutrosophic logic as a good generalization of fuzzy logic plays a central role in modern studies that are related to algebra and analysis [4-5], with very wide applications in decision-making and geometry [8-9].

In [10], Smarandache proposed the concept of two-fold algebras, and then these idea was used in the study of fuzzy number theoretical relations [11], and in module theory [12].

In this work, we are motivated to use the two-fold fuzzy complex algebra with Gamma functions to generate a new analytical structure and to study its properties. This study may be very helpful in the future because it opens a wide door to use of two-fold algebra in defining and presenting some new types of special functions that can be applied in other fields of knowledge.

Main Discussion

Definition:

Let $\mathbb{C}$ be the complex field, $\mu: \mathbb{R} \times \mathbb{R} \to [0.1]$, we define the complex twofold fuzzy algebra as follows.

$\mathbb{C}_f = \{(a + bi)_{\mu(x)} \quad ; a.b.x \quad \in \mathbb{R} \ . \quad i^2 = -1\}$,

Binary operation:

$(*): C_f \times C_f \to C_f$ such that:

$(a + bi)_{\mu(x)} * (c + di)_{\mu(y)} = [(a + c) + (b + d)i]_{\max(\mu(x).\mu(y))}$

Theorem1:

Let $(C_f.*)$ be the two fold complex fuzzy algebra defined previously, then:

1] $(*)$ is well defined.

2] $(*)$ is commutative.

3] $(*)$ is associative.

4] For each $(a + bi)_{\mu(x)} \in C_f$, there exists $o_{\mu(x)} \in C_f$ such that:

$$(a + bi)_{\mu(x)} * o_{\mu(x)} = (a + bi)_{\mu(x)}.$$

Example:

For $: \mathbb{R} \times \mathbb{R} \to [0.1]$; $\mu(x) = \begin{cases} e^x & ; x \leq 0 \\ e^{\frac{1}{x}} & ; x \geq 0 \end{cases}$, we have:

$A = (3 + 2i)_{\mu(3)} \quad .B = (2 - 5i)_{\mu(-2)}$, then:

$A * B = \ (5-3i)_{\max\left(\frac{1}{e^3},\frac{1}{e^2}\right)} = \ (5-3i)_{\frac{1}{e^2}}.$

Definition:

Let $A = (a+bi)_{\mu(x)} \in C_f$, we define:

1] $\bar{A} = (a-bi)_{\mu(x)}$

2] $|A| = (\sqrt{a^2+b^2})_{\mu(x)}$

Theorem 2:

For $A = (a+bi)_{\mu(x)} \in C_f$. $B = (c+di)_{\mu(y)} \in C_f$, we have:

1] $A * \bar{A} = (2a)_{\mu(x)}$

2] $\overline{A * B} = \bar{A} * \bar{B}$

Example:

Take $A = \ (1+2i)_{\mu(5)}$ $.B = \ (3-i)_{\mu(-10)}$; $\mu(x) = \begin{cases} e^x & ; x \le 0 \\ e^{\frac{1}{x}} & ; x > 0 \end{cases}$

$\bar{A} = \ (1-2i)_{\mu(5)} = \ (1-2i)_{e^{-5}}$ $.\bar{B} = \ (3+i)_{\mu(-10)} = \ (3+i)_{e^{-10}}$

$A * B = \ (4+i)_{e^{-5}}$ $.\ \overline{A} * \bar{B} = \ (4-i)_{e^{-5}}$ $.\overline{A * B} = \ (4-i)_{e^{-5}}.$

$|A| = \sqrt{5}_{e^{-5}}$. $|A| = \sqrt{10}_{e^{-10}}$

Definition:

We define the following binary operation on C_f:

$\circ: C_f \times C_f \to C_f$: $(a+bi)_{\mu(x)} \circ \ (c+di)_{\mu(y)}$

$= [ac - bd + (ad+bc)i]_{\min(\mu(x).\mu(y))}$

Theorem 3:

1] $(\circ)$ is well defined.

2] $(\circ)$ is commutative.

3] $(\circ)$ is associative.

4] $(\circ)$ is distributive on $(*)$.

5] For each $(a+bi)_{\mu(x)} \in C_f$, there exists $1_{\mu(x)} \in C_f$ such that:

$(a+bi)_{\mu(x)} \circ 1_{\mu(x)} = \ (a+bi)_{\mu(x)}.$

Theorem 4:

Let $A = (a + bi)_{\mu(x)}$. $B = (c + di)_{\mu(y)} \in C_f$, then:

1] $|A \circ B| = |A| \circ |B|$

2] $|\bar{A}| = |A|$

3]$A \circ \bar{A} = |A^2|$

Definition:

Let $A = (a + bi)_{\mu(x)} \in C_f$, we define the first special function Gamma on C_f as follows:

$\Gamma_1(A_{\mu(x)}) = (\Gamma(A))_{\mu(\Gamma(x))}$; $a.x > 0$.

The second Gamma function on C_f is defined as follows:

$\Gamma_2(A_{\mu(x)}) = (\Gamma(A))_{\mu(\Gamma(x))}$; $a > 0$. $x \in \mathbb{R}$.

The third Gamma function on C_f is defined as follows:

$\Gamma_3(A_{\mu(x)}) = (A)_{\mu(\Gamma(x))}$; $x > 0$. $A \in C_f$

Theorem5:

Consider $\Gamma_1 . \Gamma_2 . \Gamma_3$ the three types of special Gamma functions defined over C_f , then:

1] $A_{\mu(x)} \circ \Gamma_2(A_{\mu(x)}) = \Gamma_2[(A + 1)_{\mu(x)}]$

2] $A_{\mu(x)} \circ \Gamma_1(A_{\mu(x)}) = \Gamma_1[(A + 1)_{\mu(x)}]$

3] for $A = (a)_{\mu(x)} \in C_f$; $a \in \mathbb{R}^+$, we have:

$$\begin{cases} \lim_{a \to 0^+} A_{\mu(x)} \circ \Gamma_2(A_{\mu(x)}) = 1_{\mu(x)} \\ \lim_{a \to 0^+} A_{\mu(x)} \circ \Gamma_1(A_{\mu(x)}) = 1_{\mu(x)} \end{cases}$$

4] for $A = a_{\mu(x)} \in C_f$; $a.x \in \mathbb{R}^+$, we have:

$$\begin{cases} \Gamma_1\left(A_{\mu(x)}\right) = (2\int_0^\infty e^{-t^2} \cdot t^{2a-1}\, dt)_{\mu\left(2\int_0^\infty e^{-t^2} \cdot t^{2x-1}\, dt\right)} \\ \Gamma_2\left(A_{\mu(x)}\right) = (2\int_0^\infty e^{-t^2} \cdot t^{2a-1}\, dt)_{\mu(x)} \\ \Gamma_3\left(A_{\mu(x)}\right) = (a)_{\mu\left(2\int_0^\infty e^{-t^2} \cdot t^{2x-1}\, dt\right)} \end{cases}$$

Definition:

The two fold neutrosophic complex algebra is defined as follows:

$$C_N = \{ \ (a+bi)_{(t.j.f)} \quad ; \quad a.b \in \mathbb{R} \quad . \quad t.j.f \in [0.1] \ . \ i^2 = -1 \ \}$$

We define the following binary operations:

$$*: C_N \times C_N \to C_N \quad ; \quad (a+bi)_{(t_1.j_1.f_1)} * \ (c+di)_{(t_2.j_2.f_2)}$$

$$= [a + c + (d+b)i]_{(t.j.f)}$$

Where: $\begin{cases} t = \max(t_1.t_2) \\ f = \min(f_1.f_2) \\ j = \min(j_1.j_2) \end{cases}$

$$\circ: C_N \times C_N \to C_N \quad ; \quad (a+bi)_{(t_1.j_1.f_1)} \circ \ (c+di)_{(t_2.j_2.f_2)}$$

$$= [ac - bd + (ad+bc)i]_{(t.j.f)}$$

Where: $\begin{cases} t = \min(t_1.t_2) \\ j = \max(j_1.j_2) \\ f = \max(f_1.f_2) \end{cases}$

Example:

Let $A = (2+5i)_{\left(\frac{1}{2}.\frac{1}{3}.\frac{1}{5}\right)}$ $.B = (1-i)_{\left(0.\frac{1}{2}.\frac{1}{3}\right)}$, we have:

$A * B = (3+4i)_{\left(\frac{1}{2}.\frac{1}{3}.\frac{1}{5}\right)}$ $.A \circ B = (7+3i)_{\left(0.\frac{1}{2}.\frac{1}{3}\right)}$

Definition:

Let $A = (a+bi)_{(t.j.f)} \in C_N$, we define:

1] $\bar{A} = (a-bi)_{(t.j.f)}$

2] $|A| = (\sqrt{a^2+b^2})_{(t.j.f)}$

Theorem 6:

Let $(C_N.\circ \ .*)$ be the two fold neutrosophic complex algebra, then:

1] $(*)$ is well defined.

2] $(\circ)$ is well defined.

3] $(*).(\circ)$ are commutative.

4] $(*).(\circ)$ are associative.

5] $(\circ)$ is distributive on $(*)$.

6] for each $A = (a+bi)_{(t.j.f)} \in C_N$, there exists: $0_{(t.j.f)}.1_{(t.j.f)}$

Such that $\begin{cases} A * 0 = A \\ A \circ 1 = A \end{cases}$

Theorem 7:

Let $(C_N.\circ \ .*)$ be the neutrosophic two fold algebra, then:

For $A \ .B \in C_N$, we have:

1] $\overline{A * B} = \bar{A} * \bar{B}$

2] $\overline{A \circ B} = \bar{A} \circ \bar{B}$

3] $A \circ \bar{A} = |A^2|$

4] $|A \circ B| = |A| \circ |B|$

Definition:

Let $(C_N.\circ \ .*)$ be the neutrosophic two fold algebra, we define the following types of Gamma special function:

$$\Gamma_1(A_{(t.j.f)}) = (\Gamma(A))_{(\Gamma(t).\Gamma(j).\Gamma(f))} \qquad ;a > 0.$$

$$\Gamma_2(A_{(t.j.f)}) = (\Gamma(A))_{(t.j.f)} \qquad ;a > 0.$$

$$\Gamma_3(A_{(t.j.f)}) = (A)_{(\Gamma(t).\Gamma(j).\Gamma(f))}$$

Theorem 8:

Consider $\Gamma_1.\Gamma_2.\Gamma_3$ the three types of Gamma functions over $(C_N.\circ \ .*)$, we have:

1] $A_{(t.j.f)} \ \Gamma_2(A_{(t.j.f)}) = \Gamma_2(A+1)_{(t.j.f)}$

2] $$\Gamma_1(A_{(t.j.f)}) = (L)_{(a_1.b_1.c_1)} \quad ;where: \begin{cases} L = 2\int_0^\infty e^{-t^2} \cdot t^{2a-1}\,dt \\ a_1 = 2\int_0^\infty e^{-t^2} \cdot t^{2(t)-1}\,dt \\ b_1 = 2\int_0^\infty e^{-t^2} \cdot t^{2j-1}\,dt \\ c_1 = 2\int_0^\infty e^{-t^2} \cdot t^{2f-1}\,dt \end{cases}$$

3] For $A = a \in \mathbb{R}^+$; $\Gamma_2(A_{(t.j.f)}) = (L)_{(t.j.f)}$ $.where \ \ L = 2\int_0^\infty e^{-t^2} \cdot t^{2a-1}\,dt.$

4] For any $A \in C_N$. $\Gamma_3(A_{(t.j.f)}) = (A)_{(a_1.b_1.c_1)}$; where $\begin{cases} a_1 = 2\int_0^\infty e^{-t^2} \cdot t^{2(t)-1}\,dt \\ b_1 = 2\int_0^\infty e^{-t^2} \cdot t^{2j-1}\,dt \\ c_1 = 2\int_0^\infty e^{-t^2} \cdot t^{2f-1}\,dt \end{cases}$

Proof of theorem (1):

1] assume that: $\begin{cases} (a+bi)_{\mu(x)} = (a'+b'i)_{\mu(x')} \\ (c+di)_{\mu(y)} = (c'+d'i)_{\mu(y')} \end{cases}$

Then: $\begin{cases} a = a' \ .b = b' \\ c = c' \ .d = d' \\ \mu(y) = \mu(y') \ .\mu(x) = \mu(x') \end{cases}$

Hence: $(a+bi)_{\mu(x)} * (c+di)_{\mu(y)} = [a+c+(d+b)i]_{\max(\mu(x).\mu(y))}$

$= [a'+c'+(d'+b')i]_{\max(\mu(x').\mu(y'))} = (a'+b'i)_{\mu(x')} * (c'+d'i)_{\mu(y')}$.

2] $(a+bi)_{\mu(x)} * (c+di)_{\mu(y)} = [a+c+(b+d)i]_{\max(\mu(x).\mu(y))} = (c+di)_{\mu(y)} * (a+bi)_{\mu(x)}$.

3] Let $X = (a+bi)_{\mu(x)}$ $.Y = (c+di)_{\mu(y)}$ $.Z = (m+ni)_{\mu(z)}$, then:

$X * (Y * Z) = X * [c+m+i(d+n)]_{\max(\mu(y).\mu(z))}$

$= [a+c+m+i(b+d+n)]_{\max(\mu(x).\mu(y).\mu(z))}$

$= (a+c+i(b+d))_{\max(\mu(x).\mu(y))} * (m+ni)_{\mu(z)} = (X*Y)*Z$.

4] It is clear that:

$(a+bi)_{\mu(x)} * o_{\mu(x)} = (a+0+ib)_{\max(\mu(x).\mu(x))} = (a+bi)_{\mu(x)}$.

Proof of theorem (2):

1] $A * \bar{A} = (a+bi)_{\mu(x)} * (a-bi)_{\mu(x)} = (2a)_{\mu(x)}$.

2] $\overline{A * B} = \overline{(a+c+\imath(d+b))}_{\max(\mu(x).\mu(y))} = (a+c-i(d+b))_{\max(\mu(x).\mu(y))} = (a-ib)_{\mu(x)} * (c-id)_{\mu(y)} = \bar{A} * \bar{B}$.

Proof of theorem (3):

1] assume that: $\begin{cases} (a+bi)_{\mu(x)} = & (a'+b'i)_{\mu(x')} \\ (c+di)_{\mu(y)} = & (c'+d'i)_{\mu(y')} \end{cases}$

Then: $\begin{cases} a=a' \quad .b=b' \\ c=c' \quad .d=d' \\ \mu(x)=\mu(x') \quad .\mu(y)=\mu(y') \end{cases}$

So that: $(a+bi)_{\mu(x)} \circ (c+di)_{\mu(y)} = [ac-bd+i(ad+bc)]_{\min(\mu(x).\mu(y))}$

$= [a'c' - b'd' + i(a'd'+b'c')]_{\min(\mu(x').\mu(y'))} = (a'+b'i)_{\mu(x')} \circ (c'+d'i)_{\mu(y')}$.

2] $(a+bi)_{\mu(x)} \circ (c+di)_{\mu(y)} = [ac-bd+i(ad+bc)]_{\min(\mu(x).\mu(y))} = [ca-db+i(da+cb)]_{\min(\mu(x).\mu(y))} = (c+di)_{\mu(y)} \circ (a+bi)_{\mu(x)}$.

3] Let $X=(a+bi)_{\mu(x)}$. $Y=(c+di)_{\mu(y)}$ $.Z = (m+ni)_{\mu(z)}$, then:

$X \circ (Y \circ Z) = X \circ [(c+di)(m+ni)]_{\min(\mu(y).\mu(z))} = [(a+bi)(c+di)(m+ni)]_{\min(\mu(x).\mu(y).\mu(z))}$

$= [(a+bi)(c+di)]_{\min(\mu(x).\mu(y))} \circ (m+ni)_{\mu(z)} = (X \circ Y) \circ Z$.

4] $X \circ (Y * Z) = (a+bi)_{\mu(x)} \circ [(c+m)+i(d+n)]_{\max(\mu(y).\mu(z))} = [(a+bi)(c+di)+(a+bi)(m+ni)]_{\min(\mu(x).\max(\mu(y).\mu(z)))}$

Also $(X \circ Y) * (X \circ Z) = [(a+bi)(c+di)]_{\min(\mu(x).\mu(y))} * [(a+bi)(m+ni)]_{\min(\mu(x).\mu(z))}$

$= [(a+bi)(c+di)+(a+bi)(m+ni)]_{\max(\min(\mu(x).\mu(y)).\min(\mu(x).\mu(z)))} = X \circ (Y * Z)$

5] it holds directly from the definition.

Proof of theorem (4):

1] $|A \circ B| = [|(a+bi)(c+di)|]_{\min(\mu(x).\mu(y))} = [\sqrt{a^2+b^2} \cdot \sqrt{c^2+d^2}]_{\min(\mu(x).\mu(y))} = [\sqrt{a^2+b^2}]_{\mu(x)} \circ [\sqrt{c^2+d^2}]_{\mu(y)} = |A| \circ |B|$

2] $|\bar{A}| = (\sqrt{a^2+b^2})_{\mu(x)} = |A|$.

3]$A \circ \bar{A} = [(a+bi)(a-bi)]_{\min(\mu(x).\mu(x))} = (a^2+b^2)_{\mu(x)} = |A^2|$.

Proof of theorem (5):

Before the proof get started, it will be useful to write the formulas of Gamma function:

1] $\Gamma(Z) = 2\int_0^\infty e^{-t} \cdot t^{z-1}\, dt \quad ; \quad \mathbb{R}_e(Z) > 0 \;\; .z \in \mathbb{C}$

2] $\Gamma(Z+1) = Z \;\; \Gamma(Z)$

3] $\lim_{x\to 0^+} x\Gamma(x) = 1 \quad ; \quad x \in \mathbb{R}^+$

4] $\Gamma(x) = 2\int_0^\infty e^{-t^2} \cdot t^{2x-1}\, dx \quad ; 0 < x < \infty$

Now we prove the first part:

1 $A_{\mu(x)} \circ \Gamma_2\left(A_{\mu(x)}\right) = (A\Gamma(A))_{\mu(x)} = (\Gamma(A+1))_{\mu(x)} = \Gamma_2[(A+1)_{\mu(x)}]$

2 $A_{\mu(x)} \circ \Gamma_1\left(A_{\mu(x)}\right) = (A\Gamma(A))_{\mu(\Gamma(x))} = (\Gamma(A+1))_{\mu(\Gamma(x))} = \Gamma_1[(A+1)_{\mu(x)}]$

3 $\lim_{a\to 0^+} A_{\mu(x)} \circ \Gamma_2\left(A_{\mu(x)}\right) = \lim_{a\to 0^+} [a\Gamma(a)]_{\mu(x)} = 1_{\mu(x)}$

$\lim_{a\to 0^+} A_{\mu(x)} \circ \Gamma_1\left(A_{\mu(x)}\right) = \lim_{a\to 0^+} [A\Gamma(A)]_{\mu(\Gamma(x))} = \lim_{a\to 0^+} [a\Gamma(a)]_{\mu(\Gamma(x))} = 1_{\mu(x)}$

4 $since \begin{cases} \Gamma(A) = 2\int_0^\infty e^{-t} \cdot t^{2a-1}\, dt \\ \Gamma(x) = 2\int_0^\infty e^{-t^2} \cdot t^{2x-1}\, dt \end{cases}$

We get: $\begin{cases} \Gamma_1\left(A_{\mu(x)}\right) = (2\int_0^\infty e^{-t^2} \cdot t^{2a-1}\, dt)_{\mu\left(2\int_0^\infty e^{-t^2} \cdot t^{2x-1}\, dt\right)} \\ \Gamma_2\left(A_{\mu(x)}\right) = (2\int_0^\infty e^{-t^2} \cdot t^{2a-1}\, dt)_{\mu(x)} \\ \Gamma_3\left(A_{\mu(x)}\right) = (a)_{\mu\left(2\int_0^\infty e^{-t^2} \cdot t^{2x-1}\, dt\right)} \end{cases}$

Proof of theorem (6):

1] Assume that: $(a+bi)_{(t_1.j_1.f_1)} = (a'+b'i)_{(t_1'.j_1'.f_1')}$. $(c+di)_{(t_2.j_2.f_2)} = (c'+d'i)_{(t_2'.j_2'.f_2')}$.

Then: $\begin{cases} a = a' \quad .b = b' \quad .c = c' \quad .d = d' \\ t_1 = t_1' \;.t_2 = t_2' \\ j_1 = j_1' \;.j_2 = j_2' \\ f_1 = f_1' \;. \quad f_2 = f_2' \end{cases}$

$$(a+bi)_{(t_1.j_1.f_1)} * \ (c+di)_{(t_2.j_2.f_2)} = [a+c+(b+d)i]_{(t_3.j_3.f_3)}$$

$$= [a'+c'+(b'+d')i]_{(t_4.j_4.f_4)}$$

Where $\begin{cases} t_3 = \max(t_1.t_2) = t_4 = \max(t_1' \;.t_2') \\ f_3 = \min(f_1.f_2) = f_4 = \min(f_1' \;.f_2') \\ j_3 = \min(j_1.j_2) = j_4 = \min(j_1' \;.j_2') \end{cases}$

2] Assume that: $\begin{cases} (a+bi)_{(t_1.j_1.f_1)} = (a'+b'i)_{(t_1'.j_1'.f_1')} \\ (c+di)_{(t_2.j_2.f_2)} = (c'+d'i)_{(t_2'.j_2'.f_2')} \end{cases}$

We get: $\begin{cases} a = a' \quad .b = b' \quad .c = c' \quad .d = d' \\ t_1 = t_1' \;.t_2 = t_2' \\ j_1 = j_1' \;.j_2 = j_2' \\ f_1 = f_1' \;. \quad f_2 = f_2' \end{cases}$

Hence: $(a+bi)_{(t_1.j_1.f_1)} \circ \ (c+di)_{(t_2.j_2.f_2)} = [ac-bd+i(ad+bc)]_{(t_3.j_3.f_3)}$

$$= [a'c' \ - b'd' + i(a'd'+b'c')]_{(t_3.j_3.f_3)} = (a'+b'i)_{(t_1'.j_1'.f_1')} \circ (c'+d'i)_{(t_2'.j_2'.f_2')}$$

Where $\begin{cases} t_3 = \min(t_1.t_2) = \min(t_1' \;.t_2') \\ j_3 = \max(j_1.j_2) = \max(j_1' \;.j_2') \\ f_3 = \max(f_1.f_2) = \max(f_1' \;.f_2') \end{cases}$

3]$(a+bi)_{(t_1.j_1.f_1)} \circ \ (c+di)_{(t_2.j_2.f_2)} = [ac-bd+i(ad+bc)]_{(t_3.j_3.f_3)}$

$$= [ca-db+i(da+cb)]_{(t_3.j_3.f_3)} = \ (c+di)_{(t_2.j_2.f_2)} \circ (a+bi)_{(t_1.j_1.f_1)}$$

, Where $\begin{cases} t_3 = \min(t_1.t_2) = \min(t_2.t_1) \\ j_3 = \max(j_1.j_2) = \max(j_2.j_1) \\ f_3 = \max(f_1.f_2) = \max(f_2.f_1) \end{cases}$

$(a+bi)_{(t_1.j_1.f_1)} * \ (c+di)_{(t_2.j_2.f_2)} = \ (c+di)_{(t_2.j_2.f_2)} * \ (a+bi)_{(t_1.j_1.f_1)}$ by a similar argument.

4] $(a+bi)_{(t_1.j_1.f_1)} \circ [\ (c+di)_{(t_2.j_2.f_2)} \circ \ (m+ni)_{(t_3.j_3.f_3)}] = [(a+bi)(c+di)(m+ni)]_{(t.j.f)} = L$

Where $\begin{cases} t = \min(t_1.t_2.t_3) \\ j = \max(j_1.j_2.j_3) \\ f = \max(f_1.f_2.f_3) \end{cases}$

So that: $L = [(a+bi)_{(t_1.j_1.f_1)} \circ (c+di)_{(t_2.j_2.f_2)}] \circ (m+ni)_{(t_3.j_3.f_3)}$, hence $(\circ)$ is associative.

The associativity of (*) can be proved by the same.

5] $(a+bi)_{(t_1.j_1.f_1)} \circ [\ (c+di)_{(t_2.j_2.f_2)} * (m+ni)_{(t_3.j_3.f_3)}] = [(a+bi)[(c+di)+(m+ni)]]_{(t.j.f)} = L.$

Where $\begin{cases} t = \min(t_1.\max(t_2.t_3)) \\ j = \max(j_1.\min(j_2.j_3)) \\ f = \max(f_1.\min(f_2.f_3)) \end{cases}$

Thus: $L = [(a+bi)_{(t_1.j_1.f_1)} \circ (c+di)_{(t_2.j_2.f_2)}] * [(a+bi)_{(t_1.j_1.f_1)} \circ (m+ni)_{(t_3.j_3.f_3)}]$.

6] $\begin{cases} A * 0 = (A+0)_{(t.j.f)} = A_{(t.j.f)} \\ A \circ 1 = (A \cdot 1)_{(t.j.f)} = A_{(t.j.f)} \end{cases}$

Proof of theorem (7):

1] $\overline{A * B} = \overline{(A+B)}_{(t.j.f)} = (\bar{A}+\bar{B})_{(t.j.f)} = \bar{A}_{(t.j.f)} * \bar{B}_{(t.j.f)}$.

2] $\overline{A \circ B} = \overline{(A \cdot B)}_{(t.j.f)} = (\bar{A} \cdot \bar{B})_{(t.j.f)} = \bar{A}_{(t.j.f)} \circ \bar{B}_{(t.j.f)}$.

3 . 4 hold directly from the definition.

Proof of theorem (8):

It can be proved by a similar argument to that of theorem 5.

Conclusion

In this paper, we defined some different types of the special Gamma function on the two-fold fuzzy complex field, where we combined the classical Gamma function with the two-fold fuzzy algebra defined on complex numbers. On the other hand, many elementary properties of this new special function are determined and presented.

References

[1] ABRAMOWITZ, M. and STEGUN, I.A., (EDS.), "Gamma (Factorial) Function" and "Incomplete Gamma Function." §6.1 and §6.5 in Handbook of Mathematical Functions and

Formulas, Graphs and Mathematical Tables, 9th printing, Dover, New York, 1992, pp. 255-258 and pp 260-263.

[2] ANDREWS, G.E., ASKEY, R. and ROY, R., Special Functions, Cambridge University Press, Cambridge, 2001.

[3] Josef Al Jumayel. (2023). On Some Applications of Gamma Function. Journal of Galoitica: Journal of Mathematical Structures and Applications, 4 (1), 21-25
(Doi : https://doi.org/10.54216/GJMSA.040103)

[4] Mohammad Abobala. (2020). n-Cyclic Refined Neutrosophic Algebraic Systems of Sub-Indeterminacies, An Application to Rings and Modules. International Journal of Neutrosophic Science, 12 (2), 81-95
(**Doi :** https://doi.org/10.54216/IJNS.120203)

[5] Noor Edin Rabeh, Othman Al-Basheer, Sara Sawalmeh, Rozina Ali. (2023). An Algebraic Approach to n-Plithogenic Square Matrices For $\mathbf{18} \le \boldsymbol{n} \le \mathbf{19}$. Journal of Neutrosophic and Fuzzy Systems, 7 (2), 08-23

[6]ARTIN, E. The Gamma Function, HOLT, RINEHART, and WINSTO, New York,2002.

[7] BAMEW, E.W., "The Theory of the Gamma Function," Messenger Math., (2),Vol. 29, 2004, pp.64-128.

[8] Hasan Dadaş. (2024). Advanced Decision-Making Techniques with Generalized Close Sets in Neutrosophic Soft Bitopological Spaces. Galoitica: Journal of Mathematical Structures and Applications, 10 (1), 08-16
(**Doi :** https://doi.org/10.54216/GJMSA.0100101).

[9] Hasan DADAS , Necati Olgun. (2023). Neutrosophic Soft n-Topological Spaces: A Framework for Decision-Making Problems. Galoitica: Journal of Mathematical Structures and Applications, 9 (2), 15-21
(**Doi:** https://doi.org/10.54216/GJMSA.090202).

[10] Florentine Smarandache. "Neutrosophic Two-Fold Algebra", Plithogenic Logic and Computation, Vol.1, No.1 2024. PP.11-15.

[11] Mohammad Abobala. (2023). On a Two-Fold Algebra Based on the Standard Fuzzy Number Theoretical System. Journal of Neutrosophic and Fuzzy Systems, 7 (2), 24-29 (**Doi:** https://doi.org/10.54216/JNFS.070202).

[12] Ahmed Hatip, Necati Olgun. (2023). On the Concepts of Two- Fold Fuzzy Vector Spaces and Algebraic Modules. Journal of Neutrosophic and Fuzzy Systems, 7 (2), 46-52 (**Doi :** https://doi.org/10.54216/JNFS.070205).

[13] Djamal Lhiani. (2024). On Some Different Types of Algebraic Linear Equations In Several Different Symbolic 2-Plithogenic Rings. Journal of Neutrosophic and Fuzzy Systems, 8 (1), 39-45 (**Doi :** https://doi.org/10.54216/JNFS.080105)

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