Hypersoft Sets in a Game Theory-Based Decision Making Model

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Abstract

In this study, we offer a hypersoft set theory-based game model for handling uncertainties. The term "hypersoft game" refers to this newly suggested game. Four techniques of game solution are identified: hypersoft saddle points, hypersoft upper and lower values, hypersoft dominated strategy and hypersoft nash equilibrium. We build a two-person hypersoft game first. Additionally, we present real-world problems that the hypersoft saddle point approaches and hypersoft dominating strategy are used to tackle. In conclusion, we expand the hypersoft games from two players to n players.

Keywords: hypersoft set, two person hypersoft games, hypersoft payoff functions, hypersoft dominated strategies, hypersoft lower and hypersoft upper values, hypersoft nash equilibrium.

1 Introduction

Soft set theory was developed by Molodtsov[18] as a generic Mathematical instrument for handling uncertainties. Soft sets have been widely employed to solve decision-making issues. A soft set application for decision-making problems was proposed by Maji et al.,[17] Cagman and Enginoglu.[10]

By replacing the function with a multi argument function specified in the cartesian product with a new set of parameters, Smarandache[21] extended the idea of a soft set to a hypersoft set. Compared to soft sets, this idea is more flexible and more suited to problems requiring decision-making. The hypersoft sets were effectively employed in the COVID-19 decision-making approach by Inthumathi et al.[11] In addition, they presented hypersoft matrices[12] and their uses.[13] They have introduced the concept of hypersoft semi-open sets.[14]

The study of strategic decision-making is known as game theory. The two-person zero-sum game and its proof served as the foundation for contemporary game theory, which was first presented by Neumann and Morgenstern[19] in 1944. Aliprantis and Chakrabarti[2] provided decision-making games. The concepts of fuzzy sets have been embedded into several intriguing game theory applications. Numerous writers have examined the two-person zero sum games with fuzzy objectives and payoffs in game theory.[4,5,7–9,16,22] Additionally, the max-min solution for the degree of attainment of a fuzzy objective has been examined. Soft games, as described by Irfan Deli and Naim Cagman,[15] are applicable to challenges that involve ambiguity and uncertainty.

In this research, we develop a novel uncertainty-handling game model for games dubbed hypersoft games. This game is highly practical and easy to apply in practise since the hypersoft payoff functions of the hypersoft game are set valued functions and the solution of the hypersoft games acquired by applying the operations of sets.

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The organisation of our article is as follows: The next part includes some fundamental definitions of hypersoft sets that are necessary for our work. The two-person hypersoft games that we construct in section 3 have four different ways to solve them: hypersoft saddle points, hypersoft upper and lower values, hypersoft dominating strategy, and hypersoft nash equilibrium. We distribute applications for two-person hypersoft games in area 4. We provide n-person hypersoft games in section 5, which are an expansion of the two-person hypersoft games. The last portion contains the conclusion.

2 Hypersoft sets

Here, we review some fundamental terms and findings related to hypersoft sets.

Definition 2.1. [21] "Let $\mathcal{U}$ be a universe of discourse, $\mathcal{P}(\mathcal{U})$ the power set $\mathcal{U}$ and $E_1, E_2, ..., E_n$ the pairwise disjoint sets of parameters. Let A_i be the non-empty subset of E_i for each $i = 1, 2, ..., n.$ A hypersoft set can be identified by the pair$(\mathcal{F}, A_1 \times A_2 \times ... \times A_n)$, $\mathcal{F} : A_1 \times A_2 \times ... \times A_n \longrightarrow \mathcal{P}(\mathcal{U})$.
For sake of simplicity, we write the symbols $\boldsymbol{\mathcal{E}}$ for $E_1 \times E_2 \times ... \times E_n$, $\boldsymbol{\mathcal{A}}$ for $A_1 \times A_2 \times ... \times A_n$ and $\tilde{\boldsymbol{a}}$ for an element of the set $\boldsymbol{\mathcal{A}}$. We also suppose that none of the set A_i is empty.

Example 2.2. Let $\mathcal{U} = \{x_1, x_2, x_3, x_4\}$.
Let the attributes be $a_1 = size, a_2 = colour, a_3 = gender, a_4 = nationality$, and their attributes values respectively:

size = $A_1 = \{small, medium, tall\}$
colour =$A_2 = \{white, yellow, red, black\}$
gender =$A_3 = \{male, female\}$
nationality = $A_4 = \{American, French, Italian\}$

Let the function be $\mathcal{F} : A_1 \times A_2 \times A_3 \times A_4 \longrightarrow \mathcal{P}(\mathcal{U})$.
Then the hypersoft set is
$\mathcal{F}(\{tall, white, female, italian\}) = \{x_1, x_3\}$.

Definition 2.3. [1] Let $\mathcal{U}$ be a universe of discourse and $\boldsymbol{\mathcal{A}}$ a subset of $\boldsymbol{\mathcal{E}}$. Then $(\mathcal{F}, \boldsymbol{\mathcal{A}})$ is called

1. a null hypersoft set if for each parameter $\tilde{\boldsymbol{a}} \in \boldsymbol{\mathcal{A}}$, $\mathcal{F}(\tilde{\boldsymbol{a}})$ is an 0. We will denote it by $\Phi_{\boldsymbol{\mathcal{A}}}$.
2. an absolute hypersoft set if for each parameter $\tilde{\boldsymbol{a}} \in \boldsymbol{\mathcal{A}}$, $\mathcal{F}(\tilde{\boldsymbol{a}}) = \mathcal{U}$. We will denote it by $\mathcal{U}_{\boldsymbol{\mathcal{A}}}$.

Definition 2.4. [1] Let $(\mathcal{F}, \boldsymbol{\mathcal{A}})$ and $(\mathcal{G}, \boldsymbol{\mathcal{B}})$ be two hypersoft sets over $\mathcal{U}$. Then $(\mathcal{F}, \boldsymbol{\mathcal{A}})$ is called a hypersoft subset of $(\mathcal{G}, \boldsymbol{\mathcal{B}})$ if $\boldsymbol{\mathcal{A}} \subset \boldsymbol{\mathcal{B}}$ and $\mathcal{F}(\tilde{\boldsymbol{a}}) \subseteq \mathcal{G}(\tilde{\boldsymbol{a}})$ for all $\tilde{\boldsymbol{a}} \in \boldsymbol{\mathcal{A}}$. We denote this by $(\mathcal{F}, \boldsymbol{\mathcal{A}}) \subseteq (\mathcal{G}, \boldsymbol{\mathcal{B}})$.

Definition 2.5. [1] The complement of a hypersoft set $(\mathcal{F}, \boldsymbol{\mathcal{A}})$ is denoted by $(\mathcal{F}, \boldsymbol{\mathcal{A}})^c$ and is defined as $(\mathcal{F}, \boldsymbol{\mathcal{A}})^c = (\mathcal{F}^c, \boldsymbol{\mathcal{A}})$ where $\mathcal{F}^c(\tilde{\boldsymbol{a}})$ is the complement of $\mathcal{F}(\tilde{\boldsymbol{a}})$ for each $\tilde{\boldsymbol{a}} \in \boldsymbol{\mathcal{A}}$.

Definition 2.6. [1] Let $(\mathcal{F}, \boldsymbol{\mathcal{A}})$ and $(\mathcal{G}, \boldsymbol{\mathcal{B}})$ be two hypersoft sets over $\mathcal{U}$.
Then

1. Union of $(\mathcal{F}, \boldsymbol{\mathcal{A}})$ and $(\mathcal{G}, \boldsymbol{\mathcal{B}})$ is denoted by $(\mathcal{H}, \boldsymbol{\mathcal{C}}) = (\mathcal{F}, \boldsymbol{\mathcal{A}}) \cup (\mathcal{G}, \boldsymbol{\mathcal{B}})$ with $\boldsymbol{\mathcal{C}} = C_1 \times C_2 \times ... \times C_n$, where $C_i = A_i \cup B_i$ for $i = 1, 2, ..., n$, and $\mathcal{H}$ is defined by
$$\mathcal{H}(\tilde{\boldsymbol{a}}) = \begin{pmatrix} \mathcal{F}(\tilde{\boldsymbol{a}}), & if\ \tilde{\boldsymbol{a}} \in \boldsymbol{\mathcal{A}} - \boldsymbol{\mathcal{B}} \\ \mathcal{G}(\tilde{\boldsymbol{a}}), & if\ \tilde{\boldsymbol{a}} \in \boldsymbol{\mathcal{B}} - \boldsymbol{\mathcal{A}} \\ \mathcal{F}(\tilde{\boldsymbol{a}}) \cup \mathcal{G}(\tilde{\boldsymbol{a}}), & if\ \tilde{\boldsymbol{a}} \in \boldsymbol{\mathcal{A}} \cap \boldsymbol{\mathcal{B}} \\ 0, & else \end{pmatrix}$$
where $\tilde{\boldsymbol{a}} = (c_1, c_2, ...c_n) \in \boldsymbol{\mathcal{C}}$.
2. Intersection of $(\mathcal{F}, \boldsymbol{\mathcal{A}})$ and $(\mathcal{G}, \boldsymbol{\mathcal{B}})$ is denoted by $(\mathcal{H}, \boldsymbol{\mathcal{C}}) = (\mathcal{F}, \boldsymbol{\mathcal{A}}) \cap (\mathcal{G}, \boldsymbol{\mathcal{B}})$, where $\boldsymbol{\mathcal{C}} = C_1 \times C_2 \times ... \times C_n$, where $C_i = A_i \cap B_i$ for $i = 1, 2, ..., n$, and $\mathcal{H}$ is defined as
$\mathcal{H}(\tilde{\boldsymbol{a}}) = \mathcal{F}(\tilde{\boldsymbol{a}}) \cap \mathcal{G}(\tilde{\boldsymbol{a}})$. where $\tilde{\boldsymbol{a}} = (c_1, c_2, ...c_n) \in \boldsymbol{\mathcal{C}}$.

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3 Two Person Hypersoft Games

We create two-person hypersoft games with hypersoft payoffs in this section. We then provide four ways to solve the games. For further information and fundamental terminology, see.[3,6,20]

Throughout the work, we write the symbols $\mathfrak{P}$ for $P_1 \times P_2 \times ... \times P_n$, $\mathfrak{Q}$ for $Q_1 \times Q_2 \times .... \times Q_n$, $\mathfrak{p}$ for an element of $\mathfrak{P}$ and $\mathfrak{q}$ for an element of $\mathfrak{Q}$

Definition 3.1. Let $\mathfrak{P}, \mathfrak{Q}$ are a sets of strategies. A choice of behavior is called an action. The elements of $\mathfrak{P} \times \mathfrak{Q}$ are called action pairs. That is, $\mathfrak{P} \times \mathfrak{Q}$ is the set of available actions.

Definition 3.2. Let $\mathcal{U}$ be a set of alternatives, $\mathcal{P}(\mathcal{U})$ be the power set of $\mathcal{U}$ and $\mathfrak{P}, \mathfrak{Q}$ are set of strategies. Then a set valued function $\Im_{\mathcal{HS}} : \mathfrak{P} \times \mathfrak{Q} \longrightarrow \mathcal{P}(\mathcal{U})$ is called a hypersoft payoff function. For each $(\mathfrak{p}, \mathfrak{q}) \in \mathfrak{P} \times \mathfrak{Q}$, the value $\Im_{\mathcal{HS}}(\mathfrak{p}, \mathfrak{q})$ is called a hypersoft payoff.

Definition 3.3. Let $\mathfrak{P}$ and $\mathfrak{Q}$ be a set of strategies of player 1 and 2 respectively, $\mathcal{U}$ be a set of alternatives and $\Im_{(\mathcal{HS})_k} : \mathfrak{P} \times \mathfrak{Q} \longrightarrow \mathcal{P}(\mathcal{U})$ be a hypersoft payoff function for player k, $(k = 1, 2)$. Then for each player k, a two person hypersoft game is defined by a hypersoft set over $\mathcal{U}$ as

$$(\mathcal{HS})_k = \{((\mathfrak{p}, \mathfrak{q}), \Im_{(\mathcal{HS})_k}(\mathfrak{p}, \mathfrak{q})) : (\mathfrak{p}, \mathfrak{q}) \in \mathfrak{P} \times \mathfrak{Q}\}$$

The two person hypersoft game is played as follows:
At a certain time player 1 chooses a stratagy $\mathfrak{p}_i \in \mathfrak{P}$, simultaneously player 2 chooses a strategy $\mathfrak{q}_i \in \mathfrak{Q}$ and once this is done each player $k(k = 1, 2)$ receives the hypersoft payoff $\Im_{(\mathcal{HS})_k}(\mathfrak{p}_i, \mathfrak{q}_j)$

If $\mathfrak{P} = \{\mathfrak{p}_1, \mathfrak{p}_2, ..., \mathfrak{p}_m\}$ and $\mathfrak{Q} = \{\mathfrak{q}_1, \mathfrak{q}_2, ..., \mathfrak{q}_n\}$ then the hypersoft payoffs of $(\mathcal{HS})_k$ can be organized in the form of $m \times n$ matrix shown in table 1.

Table 1: The two person hypersoft game

$(\mathcal{HS})_k$	$\mathfrak{q}_1$	$\mathfrak{q}_2$	$\cdots$	$\mathfrak{q}_n$
$\mathfrak{p}_1$	$\Im_{(\mathcal{HS})_k}(\mathfrak{p}_1, \mathfrak{q}_1)$	$\Im_{(\mathcal{HS})_k}(\mathfrak{p}_1, \mathfrak{q}_2)$	$\cdots$	$\Im_{(\mathcal{HS})_k}(\mathfrak{p}_1, \mathfrak{q}_n)$
$\mathfrak{p}_2$	$\Im_{(\mathcal{HS})_k}(\mathfrak{p}_2, \mathfrak{q}_1)$	$\Im_{(\mathcal{HS})_k}(\mathfrak{p}_2, \mathfrak{q}_2)$	$\cdots$	$\Im_{(\mathcal{HS})_k}(\mathfrak{p}_2, \mathfrak{q}_n)$
$\vdots$	$\cdots$	$\cdots$	$\cdots$	$\cdots$
$\mathfrak{p}_m$	$\Im_{(\mathcal{HS})_k}(\mathfrak{p}_m, \mathfrak{q}_1)$	$\Im_{(\mathcal{HS})_k}(\mathfrak{p}_m, \mathfrak{q}_2)$	$\cdots$	$\Im_{(\mathcal{HS})_k}(\mathfrak{p}_m, \mathfrak{q}_n)$

We can now provide an example of a two-player hypersoft game.

Example 3.4. Let $\mathcal{R} = \{r_1, r_2, r_3, ..., r_8\}$ be a set of alternatives, $\mathcal{P}(\mathcal{R})$ be the power set of $\mathcal{R}$, $\mathfrak{P} = \{\mathfrak{p}_1, \mathfrak{p}_3, \mathfrak{p}_5\}$ and $\mathfrak{Q} = \{\mathfrak{q}_1, \mathfrak{q}_2, \mathfrak{q}_4\}$ be a set of the strategies of player 1 and player 2 respectively. If player 1 constructs a two person hypersoft game as follows,

$$(\mathcal{HS})_1 = \left\{ \begin{array}{c} ((\mathfrak{p}_1, \mathfrak{q}_1), \{r_1, r_2, r_5, r_8\}), ((\mathfrak{p}_1, \mathfrak{q}_2), \{r_1, r_2, r_3, r_4, r_5, r_8\}), \\ ((\mathfrak{p}_1, \mathfrak{q}_4), \{r_3, r_8\}), ((\mathfrak{p}_3, \mathfrak{q}_1), \{r_1, r_3, r_7\}), \\ ((\mathfrak{p}_3, \mathfrak{q}_2), \{r_1, r_2, r_3, r_5, r_6, r_7\}), ((\mathfrak{p}_3, \mathfrak{q}_4), \{r_1, r_2, r_3\}), \\ ((\mathfrak{p}_5, \mathfrak{q}_1), \{r_3, r_4, r_5, r_8\}), \\ ((\mathfrak{p}_5, \mathfrak{q}_2), \{r_1, r_2, r_3, r_4, r_5, r_6, r_8\}), ((\mathfrak{p}_5, \mathfrak{q}_4), \{r_1, r_2, r_3, r_8\}). \end{array} \right\}$$

The game's hypersoft payoffs may then be set up as shown in table 2.

Table 2

$(\mathcal{HS})_1$	$\mathfrak{q}_1$	$\mathfrak{q}_2$	$\mathfrak{q}_4$
$\mathfrak{p}_1$	$\{r_1, r_2, r_5, r_8\}$	$\{r_1, r_2, r_3, r_4, r_5, r_8\}$	$\{r_3, r_8\}$
$\mathfrak{p}_3$	$\{r_1, r_3, r_7\}$	$\{r_1, r_2, r_3, r_5, r_6, r_7\}$	$\{r_1, r_2, r_3\}$
$\mathfrak{p}_5$	$\{r_3, r_4, r_5, r_8\}$	$\{r_1, r_2, r_3, r_4, r_5, r_6, r_8\}$	$\{r_1, r_2, r_3, r_8\}$

Let's go over a few of this game's components:

The value of the game will be a set $\{r_1, r_2, r_3, r_8\}$ if player 1 selects $\mathfrak{p}_5$ and player 2 selects $\mathfrak{q}_4$. The set of choices $\{r_1, r_2, r_3, r_8\}$ is won by player 1 in this scenario, whereas player 2 loses the identical set of alternatives. Likewise, if player 2 builds the following two-player hypersoft game,

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$$(\mathcal{HS})_2 = \left\{ \begin{array}{c} ((\mathfrak{p_1}, \mathfrak{q_1}), \{r_3, r_4, r_6, r_7\}), ((\mathfrak{p_1}, \mathfrak{q_2}), \{r_6, r_7\}), \\ ((\mathfrak{p_1}, \mathfrak{q_4}), \{r_1, r_2, r_4, r_5, r_6, r_7\}), ((\mathfrak{p_3}, \mathfrak{q_1}), \{r_2, r_4, r_5, r_6, r_8\}), \\ ((\mathfrak{p_3}, \mathfrak{q_2}), \{r_4, r_8\}), ((\mathfrak{p_3}, \mathfrak{q_4}), \{r_4, r_5, r_6, r_7, r_8\}), \\ ((\mathfrak{p_5}, \mathfrak{q_1}), \{r_1, r_2, r_6, r_7\}), ((\mathfrak{p_5}, \mathfrak{q_2}), \{r_7\}), \\ ((\mathfrak{p_5}, \mathfrak{q_4}), \{r_4, r_5, r_6, r_7\}). \end{array} \right\}$$

Then the hypersoft payoffs of the game can be arranged as in table 3.

Table 3

$(\mathcal{HS})_2$	$\mathfrak{q_1}$	$\mathfrak{q_2}$	$\mathfrak{q_4}$
$\mathfrak{p_1}$	$\{r_3, r_4, r_6, r_7\}$	$\{r_6, r_7, \}$	$\{r_1, r_2, r_4, r_5, r_6, r_7\}$
$\mathfrak{p_3}$	$\{r_2, r_4, r_5, r_6, r_8\}$	$\{r_4, r_8\}$	$\{r_4, r_5, r_6, r_7, r_8\}$
$\mathfrak{p_5}$	$\{r_1, r_2, r_6, r_7\}$	$\{r_7, \}$	$\{r_4, r_5, r_6, r_7\}$

Let us explain some element of this two person hypersoft game. If player 1 select $\mathfrak{p_1}$ and player 2 select $\mathfrak{q_2}$, then the value of game will be a set $\{r_6, r_7\}$.

$$\Im_{(\mathcal{HS})_2}(\mathfrak{p_1}, \mathfrak{q_2}) = \{r_6, r_7\}.$$

In this case, player 1 wins the set of alternatives $\{r_6, r_7\}$ and player 2 lost $\{r_6, r_7\}$.

Definition 3.5. Let $(\mathcal{HS})_k = \{(\mathfrak{p}, \mathfrak{q}), \Im_{(\mathcal{HS})_k}((\mathfrak{p}, \mathfrak{q})) : ((\mathfrak{p}, \mathfrak{q}))\in\mathfrak{P} \times \mathfrak{Q}\}$ be a two person hypersoft game and $(\mathfrak{p_i}, \mathfrak{q_j}), (\mathfrak{p_r}, \mathfrak{q_s})\in\mathfrak{P}\times\mathfrak{Q}$. Then player k is rational if the player's hypersoft payoff satisfies the following conditions.

1. Either $\Im_{(\mathcal{HS})_k}(\mathfrak{p_i}, \mathfrak{q_j}) \supseteq \Im^k_{\mathfrak{P}\times\mathfrak{Q}}(\mathfrak{p_r}, \mathfrak{q_s})$ or $\Im_{(\mathcal{HS})_k}(\mathfrak{p_r}, \mathfrak{q_s}) \supseteq \Im^k_{\mathfrak{P}\times\mathfrak{Q}}(\mathfrak{p_i}, \mathfrak{q_j})$

2. When $\Im_{(\mathcal{HS})_k}(\mathfrak{p_i}, \mathfrak{q_j}) \supseteq \Im^k_{\mathfrak{P}\times\mathfrak{Q}}(\mathfrak{p_r}, \mathfrak{q_s})$ and $\Im_{(\mathcal{HS})_k}(\mathfrak{p_r}, \mathfrak{q_s}) \supseteq \Im^k_{\mathfrak{P}\times\mathfrak{Q}}(\mathfrak{p_i}, \mathfrak{q_j})$, then $\Im_{(\mathcal{HS})_k}(\mathfrak{p_i}, \mathfrak{q_j}) = \Im^k_{\mathfrak{P}\times\mathfrak{Q}}(\mathfrak{p_r}, \mathfrak{q_s})$.

Definition 3.6. Let $(\mathcal{HS})_k = \{((\mathfrak{p}, \mathfrak{q}), \Im_{(\mathcal{HS})_k}(\mathfrak{p}, \mathfrak{q})) : (\mathfrak{p}, \mathfrak{q})\in\mathfrak{P} \times \mathfrak{Q}\}$ be a two person hypersoft game.Then an action $(\mathfrak{p}^*, \mathfrak{q}^*)\in\mathfrak{P} \times \mathfrak{Q}$ is called an optimal action if

$$\Im_{(\mathcal{HS})_k}(\mathfrak{p}^*, \mathfrak{q}^*) \supseteq \Im_{(\mathcal{HS})_k}(\mathfrak{p}, \mathfrak{q}) \text{ for all } (\mathfrak{p}, \mathfrak{q})\in\mathfrak{P} \times \mathfrak{Q}.$$

Definition 3.7. Let $(\mathcal{HS})_k = \{((\mathfrak{p}, \mathfrak{q}), \Im_{(\mathcal{HS})_k}(\mathfrak{p}, \mathfrak{q})) : (\mathfrak{p}, \mathfrak{q})\in\mathfrak{P} \times \mathfrak{Q}\}$ be a two person hypersoft game.Then,

1. If $\Im_{(\mathcal{HS})_k}(\mathfrak{p_i}, \mathfrak{q_j} \supset \Im_{(\mathcal{HS})_k}(\mathfrak{p_r}, \mathfrak{q_s})$, we says that a player strictly prefers action pair $(\mathfrak{p_i}, \mathfrak{q_j})$ over action $(\mathfrak{p_r}, \mathfrak{q_s})$,
2. If $\Im_{(\mathcal{HS})_k}(\mathfrak{p_i}, \mathfrak{q_j}) = \Im_{(\mathcal{HS})_k}(\mathfrak{p_r}, \mathfrak{q_s})$, we says that a player is indifferent between the two actions,
3. $\Im_{(\mathcal{HS})_k}(\mathfrak{p_i}, \mathfrak{q_j}) \supseteq \Im_{(\mathcal{HS})_k}(\mathfrak{p_r}, \mathfrak{q_s})$, we says that a player either prefers $(\mathfrak{p_i}, \mathfrak{q_j})$ to $(\mathfrak{p_r}, \mathfrak{q_s})$ or is indifferent between the two actions.

Definition 3.8. Let $(\mathcal{HS})_k = \{((\mathfrak{p}, \mathfrak{q}), \Im_{(\mathcal{HS})_k}(\mathfrak{p}, \mathfrak{q})) : (\mathfrak{p}, \mathfrak{q})\in\mathfrak{P} \times \mathfrak{Q}\}$ be a two person hypersoft game for $k = 1, 2$. Then,

1. If $\Im_{(\mathcal{HS})_k}(\mathfrak{p}, \mathfrak{q}) = \Phi$ for all $(\mathfrak{p}, \mathfrak{q})\in\mathfrak{P} \times \mathfrak{Q}$, then $(\mathcal{HS})_k$ is called a empty hypersoft game denoted by $(\mathcal{HS})_\Phi$.
2. $\Im_{(\mathcal{HS})_k}(\mathfrak{p}, \mathfrak{q}) = \mathcal{U}$ for all $(\mathfrak{p}, \mathfrak{q})\in\mathfrak{P} \times \mathfrak{Q}$, then $(\mathcal{HS})_k$ is called a full hypersoft game, denoted by $(\mathcal{HS})_\mathcal{U}$

In hypersoft game theory, the two-person zero sum game from classical game theory will now be a two-person disjoint game. The definition that follows provides it.

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Definition 3.9. If there is an empty set at the intersection of the players' hypersoft payoffs for every action pair, the game is referred to as a two-person disjoint hypersoft game.
For instance, example 3.4 is a two person disjoint hypersoft game.

Proposition 3.10. *Let* $(\mathcal{HS})_k = \{((\mathfrak{p},\mathfrak{q}), \Im_{(\mathcal{HS})_k}(\mathfrak{p},\mathfrak{q})) : (\mathfrak{p},\mathfrak{q}) \in \mathfrak{P} \times \mathfrak{Q}\}$ *be a two person disjoint hypersoft game for* $k = 1, 2$. *Then,*

1. $((\mathcal{HS})_k^c)^c = (\mathcal{HS})_k, k = 1, 2$.
2. $(\mathcal{HS})_1 \setminus (\mathcal{HS})_2 = (\mathcal{HS})_1$
3. $(\mathcal{HS})_2 \setminus (\mathcal{HS})_1 = (\mathcal{HS})_2$
4. $(\mathcal{HS})_1 \cap (\mathcal{HS})_2 = (\mathcal{HS})_\Phi$

Proof. Proof is trivial. □

Definition 3.11. If each action pair's union of the players' hypersoft payoffs is universally established, the game is referred to as a two-person universal hypersoft game.
As an illustration, example 3.4 is a universal hypersoft game for two players.

Proposition 3.12. *Let* $(\mathcal{HS})_k = \{((\mathfrak{p},\mathfrak{q}), \Im_{(\mathcal{HS})_k}(\mathfrak{p},\mathfrak{q})) : (\mathfrak{p},\mathfrak{q}) \in \mathfrak{P} \times \mathfrak{Q}\}$ *be a two person universal hypersoft game for* $k = 1, 2$. *Then,*

1. $((\mathcal{HS})_k^c)^c = (\mathcal{HS})_k, k = 1, 2$
2. $(\mathcal{HS})_1^c = (\mathcal{HS})_2$
3. $(\mathcal{HS})_2^c = (\mathcal{HS})_1$
4. $(\mathcal{HS})_1 \cup (\mathcal{HS})_2 = (\mathcal{HS})_\mathcal{U}$.

Proof. Proof is straightforward. □

Proposition 3.13. *Let* $(\mathcal{HS})_k = \{((\mathfrak{p},\mathfrak{q}), \Im_{(\mathcal{HS})_k}(\mathfrak{p},\mathfrak{q})) : (\mathfrak{p},\mathfrak{q}) \in \mathfrak{P} \times \mathfrak{Q}\}$ *be a two person both universal and disjoint hypersoft game for* $k = 1, 2$. *Then,*

1. $(\mathcal{HS})_1 \setminus (\mathcal{HS})_2 = (\mathcal{HS})_1$
2. $(\mathcal{HS})_2 \setminus (\mathcal{HS})_1 = (\mathcal{HS})_2$
3. $(\mathcal{HS})_1 \cap (\mathcal{HS})_2 = (\mathcal{HS})_\Phi$
4. $(\mathcal{HS})_1 \cup (\mathcal{HS})_2 = (\mathcal{HS})_\mathcal{U}$

Proof. Proof is straightforward. □

3.1 Method 1 : Hypersoft Saddle Point Method

Definition 3.14. Let $\Im_{(\mathcal{HS})_k}$ be a hypersoft payoff funtion of a two person hypersoft game $(\mathcal{HS})_k$. If the following propertices hold.

1. $\bigcup_{i=1}^{m} \Im_{(\mathcal{HS})_k}(\mathfrak{p}_\mathfrak{i}, \mathfrak{q}_\mathfrak{j}) = \Im_{(\mathcal{HS})_k}(\mathfrak{p},\mathfrak{q})$

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2. $\bigcap_{j=1}^{n} \Im_{(\mathcal{HS})_k}(\mathfrak{p_i}, \mathfrak{q_j}) = \Im_{(\mathcal{HS})_k}(\mathfrak{p}, \mathfrak{q})$

Then $\Im_{(\mathcal{HS})_k}(\mathfrak{p}, \mathfrak{q})$ is called a hypersoft saddle point value and $(\mathfrak{p}, \mathfrak{q})$ is called a hypersoft saddle point of player k's in the two person hypersoft game.
Note that if $\Im_{(\mathcal{HS})_1}(\mathfrak{p}, \mathfrak{q})$ is a hypersoft saddle point of a two person hypersoft game $(\mathcal{HS})_1$, then player 1 can win atleast by choosing the strategy $\mathfrak{p} \in \mathfrak{P}$ and player 2 can keep loss to atmost $\Im_{(\mathcal{HS})_1}(\mathfrak{p}, \mathfrak{q})$ by choosing the strategy $\mathfrak{q} \in \mathfrak{Q}$. Hence the hypersoft saddle point is a value of the two person hypersoft game.

Example 3.15. Let $\mathcal{R} = \{r_1, r_2, r_3, ..., r_{10}\}$ be a set of alternatives, $\mathfrak{P} = \{\mathfrak{p_1}, \mathfrak{p_2}, \mathfrak{p_3}, \mathfrak{p_4}\}$ and $\mathfrak{Q} = \{\mathfrak{q_1}, \mathfrak{q_2}, \mathfrak{q_3}\}$ be the strategies for player 1 and 2 respectively. Then two person hypersoft game of player 1 is given as in Table 4.

Table 4

$(\mathcal{HS})_1$	$\mathfrak{q_1}$	$\mathfrak{q_2}$	$\mathfrak{q_3}$
$\mathfrak{p_1}$	$\{r_1, r_4, r_7\}$	$\{r_4, \}$	$\{r_4\}$
$\mathfrak{p_2}$	$\{r_5\}$	$\{r_7\}$	$\{r_4, r_7\}$
$\mathfrak{p_3}$	$\{r_2, r_4, r_5, r_7, r_8, r_{10}\}$	$\{r_4, r_8\}$	$\{r_7, r_8\}$
$\mathfrak{p_4}$	$\{r_2, r_4, r_5, r_7, r_8\}$	$\{r_1, r_4, r_7, r_8\}$	$\{r_4, r_7, r_8\}$

Clearly,

$$\bigcup_{i=1}^{4} \Im_{(\mathcal{HS})_1}(\mathfrak{p_i}, \mathfrak{q_1})=\{r_1, r_2, r_4, r_5, r_7, r_8, r_{10}\},$$
$$\bigcup_{i=1}^{4} \Im_{(\mathcal{HS})_1}(\mathfrak{p_i}, \mathfrak{q_2})=\{r_1, r_4, r_7, r_8\},$$
$$\bigcup_{i=1}^{4} \Im_{(\mathcal{HS})_1}(\mathfrak{p_i}, \mathfrak{q_3})=\{r_4, r_7, r_8\},$$
and
$$\bigcap_{j=1}^{3} \Im_{(\mathcal{HS})_1}(\mathfrak{p_1}, \mathfrak{q_j})=\{r_4\},$$
$$\bigcap_{j=1}^{3} \Im_{(\mathcal{HS})_1}(\mathfrak{p_2}, \mathfrak{q_j})=\Phi,$$
$$\bigcap_{j=1}^{3} \Im_{(\mathcal{HS})_1}(\mathfrak{p_3}, \mathfrak{q_j})=\{r_8\},$$
$$\bigcap_{j=1}^{3} \Im_{(\mathcal{HS})_1}(\mathfrak{p_4}, \mathfrak{q_j})=\{r_4, r_7, r_8\}.$$

The intersection of the fourth row and the union of the third column make $\{r_4, r_7, r_8\}$ a hypersoft saddle point in the two-person hypersoft game. The two-player hypersoft game value is therefore $\{r_4, r_7, r_8\}$.

Note:
Every two person hypersoft game (tphs-game) has not a hypersoft saddle point. In the above example, if $\{r_4, r_7, r_8\}$ is replaced with $\{r_4, r_7, r_8, r_9\}$ in hypersoft payoff $\Im_{(\mathcal{HS})_1}(\mathfrak{p_4}, \mathfrak{q_3})$, then a hypersoft saddle point of the game can not be found.

3.2 Method 2 : Hypersoft Upper and Hypersoft Lower Value Method

A tphs-game cannot employ a saddle point; instead, the hypersoft higher and hypersoft lower values of the tphs-game may be used, as stated in the description below.

Definition 3.16. Let $\Im_{(\mathcal{HS})_k}$ be a hypersoft payoff function of a tphs-game $(\mathcal{HS})_k$. Then,

1. Hypersoft upper value of the tphs-game, denoted $\overline{v}$ is defined by
$$\overline{v} = \bigcap_{\mathfrak{q}\in\mathfrak{Q}} \left(\bigcup_{\mathfrak{p}\in\mathfrak{P}} (\Im_{(\mathcal{HS})_k}(\mathfrak{p},\mathfrak{q})) \right)$$
2. Hypersoft lower value of the tphs-game, denoted $\underline{v}$ is defined by
$$\underline{v} = \bigcup_{\mathfrak{p}\in\mathfrak{P}} \left(\bigcap_{\mathfrak{q}\in\mathfrak{Q}} (\Im_{(\mathcal{HS})_k}(\mathfrak{p},\mathfrak{q})) \right)$$
3. If hypersoft upper and hypersoft lower value of the tphs-game are equal, they are called value of the tphs-game, denoted by v. (ie) $v = \underline{v} = \overline{v}$.

Example 3.17. Let us consider Table 4 in Example 3.15 . It is clear that hypersoft upper value $\overline{v} = \{r_4, r_7, r_8\}$ and hypersoft lower value $\underline{v} = \{r_4, r_7, r_8\}$, hence $\underline{v} = \overline{v}$. It means that value of the tphs-game is $\{r_4, r_7, r_8\}$.

Theorem 3.18. *If $\underline{v}$ and $\overline{v}$ be a hypersoft lower and hypersoft upper value of a tphs-game, respectively. Then the hypersoft lower value is subset or equal to the hypersoft upper value, that is $\underline{v} \subseteq \overline{v}$.*

Proof. Assume that $\underline{v}$ be a hypersoft lower value, $\overline{v}$ be a hypersoft upper value of a tphs-game and $\mathfrak{P} = (\mathfrak{p}_1, \mathfrak{p}_2, ..., \mathfrak{p}_m)$ and $\mathfrak{Q} = (\mathfrak{q}_1, \mathfrak{q}_2, ..., \mathfrak{q}_n)$ are sets of the strategies for player 1 and 2 respectively, we choose $\mathfrak{p}_i^* \in \mathfrak{P}$ and $\mathfrak{q}_j^* \in \mathfrak{Q}$. Then

$$\begin{aligned}
\underline{v} &= \bigcup_{\mathfrak{p}\in\mathfrak{P}} \left(\bigcap_{\mathfrak{q}\in\mathfrak{Q}} (\Im_{\mathfrak{P}\times\mathfrak{Q}}(\mathfrak{p},\mathfrak{q})) \right) \\
&\subseteq \bigcap_{\mathfrak{q}\in\mathfrak{Q}} (\Im_{\mathfrak{P}\times\mathfrak{Q}}(\mathfrak{p}^*,\mathfrak{q})) \\
&\subseteq \Im_{\mathfrak{P}\times\mathfrak{Q}}(\mathfrak{p}^*,\mathfrak{q}^*) \\
&\subseteq \bigcup_{\mathfrak{q}\in\mathfrak{P}} (\Im_{\mathfrak{P}\times\mathfrak{Q}}(\mathfrak{p},\mathfrak{q}^*)) \\
&\subseteq \bigcap_{\mathfrak{q}\in\mathfrak{Q}} \left(\bigcup_{\mathfrak{p}\in\mathfrak{P}} (\Im_{\mathfrak{P}\times\mathfrak{Q}}(\mathfrak{p},\mathfrak{q})) \right) \\
&= \overline{v}
\end{aligned}$$

hence $\underline{v} \subseteq \overline{v}$

□

Example 3.19. Let us consider hypersoft upper value $\overline{v}$ and hypersoft lower $\underline{v}$ in example 3.17. It is clear that $\underline{v} = \{r_4, r_7, r_8\} \subseteq \overline{v} = \{r_4, r_7, r_8\}$, hence $\underline{v} \subseteq \overline{v}$.

Remark 3.20. Let $\Im_{(\mathcal{HS})_k}(\mathfrak{p},\mathfrak{q})$ be a hypersoft saddle point, $\underline{v}$ be a hypersoft lower value, $\overline{v}$ be a hypersoft upper value of a tphs-game. Then, $\underline{v} \subseteq \Im_{(\mathcal{HS})_k}(\mathfrak{p}^*,\mathfrak{q}^*) \subseteq \overline{v}$.

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Proof. Assume that $\Im_{(\mathcal{HS})_k}(\mathfrak{p}^*, \mathfrak{q}^*)$ be a hypersoft saddle point, $\underline{v}$ be a hypersoft lower value, $\overline{v}$ be a hypersoft upper value of a tphs-game and $\mathfrak{P} = \{\mathfrak{p}_1, \mathfrak{p}_2, ..., \mathfrak{p}_m\}$ and $\mathfrak{Q} = \{\mathfrak{q}_1, \mathfrak{q}_2, ..., \mathfrak{q}_n\}$ are sets of the strategies for player 1 and 2 respectively.
We choose $\mathfrak{p}_i^* \in \mathfrak{P}$ and $\mathfrak{q}_j^* \in \mathfrak{Q}$.
Since $\Im_{(\mathcal{HS})_k}(\mathfrak{p}^*, \mathfrak{q}^*)$ is a hypersoft saddle point, we have

$$\bigcup_{i=1}^{m}\Im_{(\mathcal{HS})_k}(\mathfrak{p}_i, \mathfrak{q}_j) = \bigcap_{j=1}^{n}\Im_{(\mathcal{HS})_k}(\mathfrak{p}_i, \mathfrak{q}_j) = \Im_{(\mathcal{HS})_k}(\mathfrak{p}^*, \mathfrak{q}^*)$$

$$\begin{aligned}
\underline{v} &= \bigcup_{\mathfrak{p}\in\mathfrak{P}}\left(\bigcap_{\mathfrak{q}\in\mathfrak{Q}}(\Im_{\mathfrak{P}\times\mathfrak{Q}}(\mathfrak{p}, \mathfrak{q}))\right)\\
&\subseteq \bigcup_{i=1}^{m}\Im_{(\mathcal{HS})_k}(\mathfrak{p}_i, \mathfrak{q}_j)\\
&= \Im_{(\mathcal{HS})_k}(\mathfrak{p}^*, \mathfrak{q}^*)\\
&= \bigcap_{j=1}^{n}\Im_{(\mathcal{HS})_k}(\mathfrak{p}_i, \mathfrak{q}_j)\\
&\subseteq \bigcap_{\mathfrak{q}\in\mathfrak{Q}}\left(\bigcup_{\mathfrak{p}\in\mathfrak{P}}(\Im_{\mathfrak{P}\times\mathfrak{Q}}(\mathfrak{p}, \mathfrak{q}))\right)\\
&= \overline{v}
\end{aligned}$$

hence $\underline{v} \subseteq \Im_{(\mathcal{HS})_k}(\mathfrak{p}^*, \mathfrak{q}^*) \subseteq \overline{v}$.

□

Corollary 3.21. *Let $\Im_{(\mathcal{HS})_k}(\mathfrak{p}, \mathfrak{q})$ be a hypersoft saddle point, $\underline{v}$ be a hypersoft lower value and $\overline{v}$ be a hypersoft upper value of a tphs-game. If $v = \underline{v} = \overline{v}$, then $\Im_{(\mathcal{HS})_k}(\mathfrak{p}, \mathfrak{q})$ is exactly v.*

Example 3.22. Let us consider hypersoft upper value $\overline{v}$ and hypersoft lower value $\underline{v}$ in example 3.17. It is clear that hypersoft saddle point $\Im_{(\mathcal{HS})_k}(\mathfrak{p}, \mathfrak{q})$ is exactly $v = \underline{v} = \overline{v} = \{r_4, r_7, r_8\}$.

Note: In every tphs-game, the hypersoft lower value $\underline{v}$ can not be equals to the hypersoft upper value $\overline{v}$.
In the above example if $\{r_4, r_7, r_8\}$ is replaced with $\{r_4, r_7, r_8, r_9\}$ in hypersoft payoff $\Im_{(\mathcal{HS})_k}(\mathfrak{p}_4, \mathfrak{q}_3)$, then the hypersoft lower value $\underline{v}$ can not be equals to the hypersoft upper value $\overline{v}$.

3.3 Method 3 : Hypersoft Dominated Strategy Method

If in a tphs-game $\underline{v} \neq \overline{v}$, then to get the solution of the game hypersoft dominated strategy may be used. We define hypersoft dominated strategy for tphs-game as follows.

Definition 3.23. Let $(\mathcal{HS})_1$ be a tphs- game with its hypersoft payoff function $\Im_{(\mathcal{HS})_1}$. Then

1. a strategy $\mathfrak{p}_i \in \mathfrak{P}$ is called a hypersoft dominated to another strategy $\mathfrak{p}_r \in \mathfrak{P}$, if $\Im_{(\mathcal{HS})_1}(\mathfrak{p}_i, \mathfrak{q}) \supseteq \Im_{(\mathcal{HS})_1}(\mathfrak{p}_r, \mathfrak{q}))$ for all $\mathfrak{q} \in \mathfrak{Q}$.
2. a strategy $\mathfrak{q}_j \in \mathfrak{Q}$ is called a hypersoft dominated to another strategy $\mathfrak{q}_s \in \mathfrak{Q}$, if $\Im_{(\mathcal{HS})_1}(\mathfrak{p}, \mathfrak{q}_j) \subseteq \Im_{(\mathcal{HS})_1}(\mathfrak{p}, \mathfrak{q}_s))$ for all $\mathfrak{p} \in \mathfrak{P}$.

Rows and columns can be removed from tphs-games in order to shrink them utilising the hypersoft dominating method. Sometimes, this process of getting rid of hypersoft-dominated tactics finds us the answer to a tphs-game. The hypersoft elimination method is the name given to this type of tphs-game solving technique. This approach may be used to solve the following tphs-game.

Example 3.24. Let $\mathcal{R} = \{r_1, r_2, r_3, ..., r_{10}\}$ be a set of alternatives, $\mathfrak{P} = \{\mathfrak{p}_1, \mathfrak{p}_2, \mathfrak{p}_3\}$ and $\mathfrak{Q} = \{\mathfrak{q}_1, \mathfrak{q}_2, \mathfrak{q}_3\}$ be the strategies for player 1 and player 2 respectively.
Then tphs-game of player 1 is given as in Table 5."

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Table 5

$(\mathcal{HS})_1$	$\mathbf{q_1}$	$\mathbf{q_2}$	$\mathbf{q_3}$
$\mathfrak{p}_1$	$\{r_2, r_4, r_7\}$	$\{r_4\}$	$\{r_4\}$
$\mathfrak{p}_2$	$\{r_5\}$	$\{r_7\}$	$\{r_4, r_7\}$
$\mathfrak{p}_3$	$\{r_2, r_4, r_5, r_7, r_8, r_{10}\}$	$\{r_4, r_7, r_8\}$	$\{r_4, r_7, r_8\}$

The last column is dominated by the middle column. After removing the final column, Table 6 looks like this:

Table 6

$(\mathcal{HS})_1$	$\mathbf{q_1}$	$\mathbf{q_2}$
$\mathfrak{p}_1$	$\{r_2, r_4, r_7\}$	$\{r_4\}$
$\mathfrak{p}_2$	$\{r_5\}$	$\{r_7\}$
$\mathfrak{p}_3$	$\{r_2, r_4, r_5, r_7, r_8, r_{10}\}$	$\{r_4, r_7, r_8\}$

Now, as you can see from Table 6 (Note that this is not the case in Table 5), the bottom row dominates the top row. After removing the top row, Table 7 looks like this:

Table 7

$(\mathcal{HS})_1$	$\mathbf{q_1}$	$\mathbf{q_2}$
$\mathfrak{p}_2$	$\{r_5\}$	$\{r_7\}$
$\mathfrak{p}_3$	$\{r_2, r_4, r_5, r_7, r_8, r_{10}\}$	$\{r_4, r_7, r_8\}$

In Table 7, player 1 has a hypersoft dominate strategy $\mathfrak{p}_3$ so that $\mathfrak{p}_2$ is now eliminated. player 2 can now choose between $\mathbf{q_1}$ and $\mathbf{q_2}$. Player 2 will clearly choose $\mathbf{q_2}$.
The solution using the method is $(\mathfrak{p}_3, \mathbf{q_2})$, value of the tphs-game is $\{r_4, r_7, r_8\}$.

3.4 Method 4 : Hypersoft Nash Equilibrium Method

Certain tphs-games without hypersoft dominating tactics are ineligible for using the hypersoft elimination procedure. The hypersoft nash equilibrium, which is described as follows, may be applied in this situation.

Definition 3.25. Let $(\mathcal{HS})_k$ be a tphs-game with its hypersoft payoff function $\Im_{(\mathcal{HS})_k}$ for $k = 1, 2$. If the following properties hold:

1. $\Im_{(\mathcal{HS})_1}(\mathfrak{p}^*, \mathbf{q}^*) \supseteq \Im_{(\mathcal{HS})_1}(\mathfrak{p}, \mathbf{q}^*)$ for each $\mathfrak{p} \in \mathfrak{P}$.
2. $\Im_{(\mathcal{HS})_2}(\mathfrak{p}^*, \mathbf{q}^*) \supseteq \Im_{(\mathcal{HS})_2}(\mathfrak{p}^*, \mathbf{q})$ for each $\mathbf{q} \in \mathfrak{Q}$.

Then $(\mathfrak{p}^*, \mathbf{q}^*) \in \mathfrak{P} \times \mathfrak{Q}$ is a hypersoft nash equilibrium of a tphs-game.

Note that if $(\mathfrak{p}^*, \mathbf{q}^*) \in \mathfrak{P} \times \mathfrak{Q}$ is a hypersoft nash equilibrium of a tphs-game, then player 1 can win atleast $\Im_{(\mathcal{HS})_1}(\mathfrak{p}^*, \mathbf{q}^*)$ by choosing strategy $\mathfrak{p}^* \in \mathfrak{Q}$ and player 2 can win atleast $\Im_{(\mathcal{HS})_2}(\mathfrak{p}^*, \mathbf{q}^*)$ by choosing strategy $\mathbf{q}^* \in \mathfrak{Q}$. Hence the hypersoft nash equilibrium is an optimal action for tphs-game, therefore $\Im_{(\mathcal{HS})_k}(\mathfrak{p}^*, \mathbf{q}^*)$ is the solution of the tphs-game for player k, $k = 1, 2$.
The following tphs-game can be solved by using this method.

Example 3.26. Let $\mathcal{R} = \{r_1, r_2, r_3, ..., r_{10}\}$ be a set of alternatives, $\mathfrak{P} = \{\mathfrak{p}_1, \mathfrak{p}_2, \mathfrak{p}_3\}$ and $\mathfrak{Q} = \{\mathbf{q}_1, \mathbf{q}_2, \mathbf{q}_3\}$ be the strategies for player 1 and player 2 respectively.
Then tphs-game of player 1 is given as in Table 8.

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Table 8

$(\mathcal{HS})_1$	$\mathfrak{q}_1$	$\mathfrak{q}_2$	$\mathfrak{q}_3$
$\mathfrak{p}_1$	$\{r_1, r_2, r_4, r_7, r_8, r_9\}$	$\{r_1, r_2, r_4, r_7, r_8\}$	$\{r_1, r_2, r_3, r_4, r_7, r_8\}$
$\mathfrak{p}_2$	$\{r_1, r_2, r_3, r_5\}$	$\{r_1, r_4, r_7, r_8\}$	$\{r_1, r_2, r_3, r_4, r_5, r_9\}$
$\mathfrak{p}_3$	$\{r_2, r_5, r_7, r_8, r_{1o}\}$	$\{r_2, r_4, r_7, r_8\}$	$\{r_4, r_5, r_7, r_8, r_{10}\}$

and tphs-game of player 2 is given as in Table 9.

Table 9

$(\mathcal{HS})_2$	$\mathfrak{q}_1$	$\mathfrak{q}_2$	$\mathfrak{q}_3$
$\mathfrak{p}_1$	$\{r_3, r_5, r_6, r_{10}\}$	$\{r_3, r_5, r_6, r_9, r_{10}\}$	$\{r_5, r_6, r_9, r_{10}\}$
$\mathfrak{p}_2$	$\{r_4, r_6, r_7, r_8, r_9, r_{10}\}$	$\{r_2, r_3, r_5, r_6, r_9, r_{10}\}$	$\{r_6, r_8, r_9, r_{10}\}$
$\mathfrak{p}_3$	$\{r_1, r_3, r_4, r_6, r_9\}$	$\{r_1, r_3, r_5, r_6, r_9, r_{10}\}$	$\{r_1, r_2, r_3, r_6, r_9\}$

From the tables we have

1. $\Im_{(\mathcal{HS})_1}(\mathfrak{p}_1, \mathfrak{q}_2) \supseteq \Im_{(\mathcal{HS})_1}(\mathfrak{p}, \mathfrak{q}_2)$ for each $\mathfrak{p} \in \mathfrak{P}$.
2. $\Im_{(\mathcal{HS})_2}(\mathfrak{p}_1, \mathfrak{q}_2) \supseteq \Im_{(\mathcal{HS})_2}(\mathfrak{p}_1, \mathfrak{q})$ for each $\mathfrak{q} \in \mathfrak{Q}$.

Thus, $(\mathfrak{p}_1, \mathfrak{q}_2) \in \mathfrak{P} \times \mathfrak{Q}$ is a hypersoft nash equilibrium.
so, $\Im_{(\mathcal{HS})_1}(\mathfrak{p}_1, \mathfrak{q}_2) = \{r_1, r_2, r_4, r_7, r_8\}$ and
$\Im_{(\mathcal{HS})_2}(\mathfrak{p}_1, \mathfrak{q}_2) = \{r_3, r_5, r_6, r_9, r_{10}\}$
is the solution of the tphs-game for player 1 and player 2 respectively.

4 An Application

This section presents a real-world problem that is handled by combining the hypersoft saddle point approach and hypersoft dominating strategy. There are two businesses, let's call player 1 and player 2, that are in competition with one another to boost product sales in the nation. Therefore, they give advertisements. Assume that two companies have a set of different products $\mathcal{R} = \{r_1, r_2, r_3, ..., r_8\}$. For $i = 1, 2, 3, ...8$, the product r_i stands for Scanner, Printer, Monitor, Mouse, Keyboard, Hard Disc, Computer and Laptop respectively.
The products can be characterized by a set of strategy $\mathfrak{P} = \mathfrak{Q} = \{\mathfrak{p}_i, i = 1, 2, 3\}$ which contains styles of advertisement. For $i = 1, 2, 3$, the strategies $\mathfrak{p}_i$ stands for Social Media, Newspaper and Television respectively. Suppose that $\mathfrak{P} = \{\mathfrak{p}_1, \mathfrak{p}_2, \mathfrak{p}_3\}$ and $\mathfrak{Q} = \{\mathfrak{q}_1 = \mathfrak{p}_1, \mathfrak{q}_2 = \mathfrak{p}_2, \mathfrak{q}_3 = \mathfrak{p}_3\}$ are strategies of players 1 and 2 respectively. Then tphs-game of player 1 is given as in Table 10.

Table 10

$(\mathcal{HS})_1$	$\mathfrak{q}_1$	$\mathfrak{q}_2$	$\mathfrak{q}_3$
$\mathfrak{p}_1$	$\{r_1, r_2, r_3, r_5, r_8\}$	$\{r_1, r_2, r_3, r_4, r_5, r_8\}$	$\{r_3\}$
$\mathfrak{p}_2$	$\{r_1, r_3, r_7\}$	$\{r_1, r_2, r_3, r_5, r_6, r_7\}$	$\{r_2, r_3\}$
$\mathfrak{p}_3$	$\{r_1, r_2, r_3, r_4, r_5\}$	$\{r_1, r_2, r_3, r_4, r_5, r_6, r_8\}$	$\{r_1, r_2, r_3\}$

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In Table 10, let us explain action pair $(\mathfrak{p}_2, \mathfrak{q}_3)$:

If player 1 select $\mathfrak{p}_2$ = "Newspaper" and
player 2 select $\mathfrak{q}_3$ = "Television"

then the hypersoft payoff of player 1 is a set $\{r_2, r_3\}$. ie. $\Im_{(\mathcal{HS})_1}(\mathfrak{p}_2, \mathfrak{q}_3) = \{r_2, r_3\}$.
In this case , player 1 increase the sale of $\{r_2, r_3\}$ and player 2 decrease sale of $\{r_2, r_3\}$.
Now we solve the game. It is seen in Table 10.

$$\{r_1, r_2, r_3, r_5, r_8\} \subseteq \{r_1, r_2, r_3, r_4, r_5, r_8\}$$
$$\{r_1, r_3, r_7\} \subseteq \{r_1, r_2, r_3, r_5, r_6, r_7\}$$
$$\{r_1, r_2, r_3, r_4, r_5\} \subseteq \{r_1, r_2, r_3, r_4, r_5, r_6, r_8\}$$

the middle column is dominated by the right column. We then deleting the middle column we obtain Table 11.

Table 11

$(\mathcal{HS})_1$	$\mathfrak{q}_1$	$\mathfrak{q}_3$
$\mathfrak{p}_1$	$\{r_1, r_2, r_3, r_5, r_8\}$	$\{r_3\}$
$\mathfrak{p}_2$	$\{r_1, r_3, r_7\}$	$\{r_2, r_3\}$
$\mathfrak{p}_3$	$\{r_1, r_2, r_3, r_4, r_5\}$	$\{r_1, r_2, r_3\}$

In Table 11, there is no another hypersoft dominated strategy, we can use hypersoft saddle point method.

$$\bigcup_{i=1}^{3} \Im_{(\mathcal{HS})_1}(\mathfrak{p}_i, \mathfrak{q}_1) = \{r_1, r_2, r_3, r_4, r_5, r_7, r_8\}$$
$$\bigcup_{i=1}^{3} \Im_{(\mathcal{HS})_1}(\mathfrak{p}_i, \mathfrak{q}_3) = \{r_1, r_2, r_3\}$$
$$\bigcap_{j=1,3} \Im_{(\mathcal{HS})_1}(\mathfrak{p}_1, \mathfrak{q}_j) = \{r_3\}$$
$$\bigcap_{j=1,3} \Im_{(\mathcal{HS})_1}(\mathfrak{p}_2, \mathfrak{q}_j) = \{r_3\}$$
$$\bigcap_{j=1,3} \Im_{(\mathcal{HS})_1}(\mathfrak{p}_3, \mathfrak{q}_j) = \{r_1, r_2, r_3\}.$$

Here optimal strategy for the game is $(\mathfrak{p}_3, \mathfrak{q}_3)$.
Since $\bigcup_{i=1}^{3} \Im_{(\mathcal{HS})_1}(\mathfrak{p}_i, \mathfrak{q}_3) = \bigcap_{j=1,3} \Im_{(\mathcal{HS})_1}(\mathfrak{p}_3, \mathfrak{q}_j)$.
Therefore, value of the tphs-game is $\{r_1, r_2, r_3\}$.

5 n-Person Hypersoft Games

The hypersoft games may frequently be played by more than two players in numerous applications. Consequently, it is possible to expand two-person hypersoft games (tphs-game) to n-person hypersoft games (nphs-game).

Definition 5.1. Let $\mathcal{U}$ be a set of alternatives, $\mathcal{P}(\mathcal{U})$ be the power set of $\mathcal{U}$ and $\mathfrak{P}_k$ is the set of strategies of player $k, (k = 1, 2, ..., n)$. Then for each player k, an n-person hypersoft game (nphs-game) is defined by a hypersoft set over $\mathcal{U}$ as:
$(\mathcal{HS})_k^n = \{((\mathfrak{p}_1, \mathfrak{p}_2, , .., \mathfrak{p}_n), \Im_{(\mathcal{HS})_k^n}(\mathfrak{p}_1, \mathfrak{p}_2, , .., \mathfrak{p}_n)) : (\mathfrak{p}_1, \mathfrak{p}_2, , .., \mathfrak{p}_n) \in (\mathfrak{P}_1 \times \mathfrak{P}_2 \times ... \times \mathfrak{P}_n)\}$
where $\Im_{(\mathcal{HS})_k^n}$ is a hypersoft payoff function of player k.
The nphs-game is played as follows:
At a certain time player 1 choose a strategy $\mathfrak{p}_1 \in \mathfrak{P}_1$ and simultaneously each player $k, (k = 2, 3...n)$ choose a strategy $\mathfrak{p}_k \in \mathfrak{P}_k$ and once this is done each player k receives the hypersoft payoff $\Im_{(\mathcal{HS})_k^n}(\mathfrak{p}_1, \mathfrak{p}_2, , .., \mathfrak{p}_n)$.

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6 Conclusion

We define two-person hypersoft games (tphs-game) with hypersoft payoffs in this work. In addition, we provided four approaches to solving the tphs-games and an example demonstrating how the approaches may be effectively used to solve real-world issues. Lastly, we expanded the hypersoft games from two players (tphs-games) to n players (nphs-games). In the future, we may use nphs-games to solve decision-making issues related to daily life.

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