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# Foundation of Revolutionary Topologies: An Overview, Examples, Trend Analysis, Research Issues, Challenges, and Future Directions

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**Abstract:** We now found nine new topologies, such as: NonStandard Topology, Largest Extended NonStandard Real Topology, Neutrosophic Triplet Weak/Strong Topologies, Neutrosophic Extended Triplet Weak/Strong Topologies, Neutrosophic Duplet Topology, Neutrosophic Extended Duplet Topology, Neutrosophic MultiSet Topology, and recall and improve the seven previously founded topologies in the years (2019-2023), namely: NonStandard Neutrosophic Topology, NeutroTopology, AntiTopology, Refined Neutrosophic Topology, Refined Neutrosophic Crisp Topology, SuperHyperTopology, and Neutrosophic SuperHyperTopology. They are called avantgarde topologies because of their innovative forms.

**Keywords:** Classical Topology; Topological Space; NeutroSophication; AntiSophication; NeutroTopology; AntiTopology; Refined Neutrosophic Topology; Refined Neutrosophic Crisp Topology; SuperHyperTopology; Neutrosophic SuperHyperTopology; Extended NonStandard Real Set; NonStandard Topology; NonStandard Neutrosophic Topology; Largest Extended NonStandard Real Topology; left monad; Right Monad; Pierced Binad; Left Monad Closed to the Right; Right Monad Closed to the Left, Unpierced Binad; Neutrosophic OverTopology; Neutrosophic UnderTopology; Neutrosophic OffTopology; (Fuzzy & Fuzzy-Extensions) Over/Under/Off-Topologies; Neutrosophic MultiSet Topology.

## 1. Introduction

The foundation of new topologies raised from development of other fields such as NeutroAlgebra and AntiAlgebra (that gave birth to NeutroTopology and AntiTopology), SuperHyperAlgebra and Neutrosophic SuperHyperAlgebra (that gave birth to SuperHyperTopology and Neutrosophic SuperHyperTopology), Refined Crisp Set (that gave birth to the Refined Crisp Topology), and Refined Neutrosophic Set (that gave birth to refined Neutrosophic Topology), and NonStandard Set (that gives birth to NonStandard Topology and NonStandard Neutrosophic Topology), Neutrosophic Triplet Set, Neutrosophic Extended Triplet Set, Neutrosophic Dual Set, Neutrosophic Extended Dual Set, and Neutrosophic MultiSet.

This is almost a virgin territory of research since little research has been done, mostly about the AntiTopology [8]. Nevertheless, it is a promising field to study in the future, since it better reflects our real world, where the laws (axioms) do not apply in the same degree to all people (powerful people are above the law, others immune to the law, and many feel the full hardship of the law); since the world as a dynamic system is formed by sub-systems, and each sub-system by sub-sub-systems and so on (whence the necessity to introduce the SuperHyperStructure based on the n-th PowerSet of a Set, whose particular cases are the SuperHyperAlgebra and SuperHyperTopology), etc.

We recall the classical definition of Topology, then the procedures of NeutroSophication and respectively AntiSophication of it, that result in adding in two new types of topologies: NeutroTopology and respectively AntiTopology.

Then we define topology on Refined Neutrosophic Set (2013), Refined Neutrosophic Crisp Set [3]. Afterwards, we extend the topology on the framework of SuperHyperAlgebra [6], then the NonStandard Neutrosophic Set to NonStandard Topology and NonStandard Neutrosophic Topology (never defined before).
The corresponding neutrosophic topological spaces are presented.
This research is an improvement of paper [7] and book [12, sections 4.8 and 4.9].

## 2. Classical Topology

Let $\mathcal{U}$ be a non-empty set, and $P(\mathcal{U})$ the power set of $\mathcal{U}$.
Let $\tau \subseteq P(\mathcal{U})$ be a family of subsets of $\mathcal{U}$.
Then $\tau$ is called a Classical Topology on $\mathcal{U}$ if it satisfies the following axioms: (CT-1) $\phi$ and $\mathcal{U}$ belong to $\tau$.
(CT-2) The intersection of any finite number of elements in $\tau$ is in $\tau$.
(CT-3) The union of any finite or infinite number of elements in $\tau$ is in $\tau$.
All three axioms are totally (100%) true (or T = 1, I = 0, F = 0). We simply call them (classical) *Axioms*.
Then $(\mathcal{U}, \tau)$ is called a *Classical Topological Space* on $\mathcal{U}$.

## 3. NeutroSophication of the Topological Axioms

NeutroSophication of the topological axioms means that the axioms become partially true, partially indeterminate, and partially false. They are called NeutroAxioms.
(NCT-1) Either $\{\phi \notin \tau$ and $\mathcal{U} \in \tau\}$, or $\{\phi \in \tau$ and $\mathcal{U} \notin \tau\}$.

(NCT-2) There exist a finite number of elements in $\tau$ whose intersection belong to $\tau$ (degree of truth *T*); and a finite number of elements in $\tau$ whose intersection is indeterminate (degree of indeterminacy *I*); and a finite number of elements in $\tau$ whose intersection does not belong to $\tau$ (degree of falsehood *F*); where $(T, I, F) \notin \{(1,0,0),(0,0,1)\}$ since (1, 0, 0) represents the above Classical Topology, while (0, 0, 1) the below AntiTopology.

(NCT-3) There exist a finite or infinite number of elements in $\tau$ whose union belongs to $\tau$ (degree of truth *T*); and a finite or infinite number of elements in $\tau$ whose union is indeterminate (degree of indeterminacy *I*); and a finite or infinite number of elements in $\tau$ whose union does not belong to $\tau$ (degree of falsehood *F*); where of course $(T, I, F) \notin \{(1,0,0),(0,0,1)\}$.

## 4. AntiSophication of the Classical Topological Axioms

AntiSophication of the topological axioms means to negate (anti) the axioms, the axioms become totally (100%) false (or T = 0, I = 0, F = 1). They are called *AntiAxioms*.
(ACT-1) $\phi \notin \tau$ and $\mathcal{U} \notin \tau$.
(ACT-2) The intersection of any finite number ($n \geq 2$) of elements in $\tau$ is not in $\tau$.
(ACT-3) The union of any finite or infinite number ($n \geq 2$) of elements in $\tau$ is not in $\tau$.

## 5. <Topology, NeutroTopology, AntiTopology>

As such, we have a neutrosophic triplet of the form:

<Axiom(1, 0, 0), NeutroAxiom(*T, I, F*), AntiAxiom(0, 0, 1)>,

where $(T, I, F) \neq (1, 0, 0)$ and $(T, I, F) \neq (0, 0, 1)$.

Correspondingly, one has:

<Topology, NeutroTopology, AntiTopology>.

Therefore, in general:

*(Classical) Topology* is a topology that has all axioms totally true. We simply call them *Axioms*.

*NeutroTopology* is a topology that has at least one *NeutroAxiom* and the others are all *classical Axioms* [therefore, no AntiAxiom].

*AntiTopology* is a topology that has one or more *AntiAxioms*, no matter what the others are (*classical Axioms*, or *NeutroAxioms*).

## 6. Theorem on the number of Structures/NeutroStructures/AntiStructures

If a Structure has $m$ axioms, with $m \geq 1$, then after NeutroSophication and AntiSophication one obtains $3^m$ types of structures, categorized as follows:

1*Classical Structure* + $(2^m - 1)$*NeutroStructures* + $(3^m - 2^m)$*AntiStructures* = $3^m$ *Structures.*

## 7. Consequence on the number of Topologies/NeutroTopologies/AntiTopologies

As a particular case of the previous theorem, from a Topology which has $m = 3$ axioms, one makes, after NeutrosSophication and AntiSophication, $3^3 = 27$ types of structures, as follows: 1 classical Topology, $2^3 - 1 = 7$ NeutroTopologies, and $3^3 - 2^2 = 19$ AntiTopologies.

1*Classical Topology* + 7*NeutroTopologies* + 19*AntiTopologies* = $3^3$ Topologies are presented below:

There is 1 (one) type of Classical Topology, whose axioms are listed below:

1 *Classical Topology*

$$\begin{pmatrix} CT-1 \\ CT-2 \\ CT-3 \end{pmatrix}$$

## 8. Definition of NeutroTopology [4, 5]

It is a topology that has at least one topological axiom which is partially true, partially indeterminate, and partially false, or *(T, I, F)*, where T = True, *I* = Indeterminacy, *F* = False, and no topological axiom is totally false, in other words: $(T, I, F) \notin \{(1,0,0),(0,0,1)\}$, where *(1, 0, 0)* represents the classical Topology, while *(0, 0, 1)* represents the below AntiTopology.

Therefore, the NeutroTopology is a topology in between the classical Topology and the AntiTopology.

There are 7 types of different NeutroTopologies, whose axioms, for each type, are listed below:

7 *NeutroTopologies*

$$\begin{pmatrix} NCT-1 \\ CT-2 \\ CT-3 \end{pmatrix}, \begin{pmatrix} CT-1 \\ NCT-2 \\ CT-3 \end{pmatrix}, \begin{pmatrix} CT-1 \\ CT-2 \\ NCT-3 \end{pmatrix},$$

$$\begin{pmatrix} NCT-1 \\ NCT-2 \\ CT-3 \end{pmatrix}, \begin{pmatrix} CT-1 \\ NCT-2 \\ NCT-3 \end{pmatrix}, \begin{pmatrix} NCT-1 \\ CT-2 \\ NCT-3 \end{pmatrix},$$

$$\begin{pmatrix} NCT-1 \\ NCT-2 \\ NCT-3 \end{pmatrix}.$$

## 9. Definition of AntiTopology [4, 5]

It is a topology that has at least one topological axiom that is 100% false (T, I, F) = (0, 0, 1). The NeutroTopology and AntiTopology are particular cases of NeutroAlgebra and AntiAlgebra [4] and, in general, they all are particular cases of the NeutroStructure and AntiStructure respectively, since we consider "Structure" in any field of knowledge [5].

There are 19 types of different AntiTopologies, whose axioms, for each type, are listed below:

$$19\ AntiTopologies$$

$$\begin{pmatrix} ACT-1 \\ CT-2 \\ CT-3 \end{pmatrix}, \begin{pmatrix} CT-1 \\ ACT-2 \\ CT-3 \end{pmatrix}, \begin{pmatrix} CT-1 \\ CT-2 \\ ACT-3 \end{pmatrix},$$

$$\begin{pmatrix} ACT-1 \\ ACT-2 \\ CT-3 \end{pmatrix}, \begin{pmatrix} CT-1 \\ ACT-2 \\ ACT-3 \end{pmatrix}, \begin{pmatrix} ACT-1 \\ CT-2 \\ ACT-3 \end{pmatrix},$$

$$\begin{pmatrix} ACT-1 \\ NCT-2 \\ NCT-3 \end{pmatrix}, \begin{pmatrix} NCT-1 \\ ACT-2 \\ NCT-3 \end{pmatrix}, \begin{pmatrix} NCT-1 \\ NCT-2 \\ ACT-3 \end{pmatrix},$$

$$\begin{pmatrix} ACT-1 \\ ACT-2 \\ NCT-3 \end{pmatrix}, \begin{pmatrix} NCT-1 \\ ACT-2 \\ ACT-3 \end{pmatrix}, \begin{pmatrix} ACT-1 \\ NCT-2 \\ ACT-3 \end{pmatrix},$$

$$\begin{pmatrix} ACT-1 \\ NCT-2 \\ CT-3 \end{pmatrix}, \begin{pmatrix} CT-1 \\ ACT-2 \\ NCT-3 \end{pmatrix}, \begin{pmatrix} NCT-1 \\ CT-2 \\ ACT-3 \end{pmatrix},$$

$$\begin{pmatrix} ACT-1 \\ CT-2 \\ NCT-3 \end{pmatrix}, \begin{pmatrix} CT-1 \\ NCT-2 \\ ACT-3 \end{pmatrix}, \begin{pmatrix} NCT-1 \\ ACT-2 \\ CT-3 \end{pmatrix},$$

$$\begin{pmatrix} ACT-1 \\ ACT-2 \\ ACT-3 \end{pmatrix}.$$

## 10. Refined Neutrosophic Set

Let $U$ be a universe of discourse, and a non-empty subset $R$ of it,

$$R = \begin{Bmatrix} x\left(T_1(x), T_2(x), \dots, T_p(x)\right); \\ \left(I_1(x), I_2(x), \dots, I_r(x)\right); \\ \left(F_1(x), F_2(x), \dots, F_s(x)\right); \end{Bmatrix}$$

with all $T_j, I_k, F_l \in [0,1], 1 \le j \le p, 1 \le k \le r, 1 \le l \le s,$ and no restriction on their sums $0 \le T_m + I_m + F_m \le 3$, with $1 \le m \le \max\{p, r, s\}$, where $p, r, s \ge 0$ are fixed integers, and at least one of them is $\ge 2$, in order to ensure the refinement (sub-parts) or multiplicity (multi-parts) – depending on the application, of at least one neutrosophic component amongst $T$ (truth), $I$ (indeterminacy), $F$ (falsehood); and of course $x \in \mathcal{U}$.

By notation we consider that index zero means the empty-set, i.e. $T_0 = I_0 = F_0 = \phi$ (or zero), and the same for the missing sub-parts (or multi-parts).

For example, the below (2,3,1)-Refined Neutrosophic Set is identical to a (3,3,3)-Refined Neutrosophic Set: $(T_1, T_2; I_1, I_2, I_3; F_1) \equiv (T_1, T_2, 0; I_1, I_2, I_3; F_1, 0, 0)$, where the missing components $T_3$, and $F_2$, $F_3$ were replaced each of them by 0 (zero) $R$ is called a *(p, r, s)-refined neutrosophic set* { or *(p, r, s)-RNT* }.

The neutrosophic set has been extended to the Refined Neutrosophic Set (Logic, and Probability) by Smarandache [1] in 2013, where there are multiple parts of the neutrosophic components, as such T was split into subcomponents $T_1, T_2, \dots, T_p$, and I into $I_1, I_2, \dots, I_r$, and F into $F_1, F_2, \dots, F_s$, with p + r + s

= n ≥ 2 and integers p, r, s ≥ 0 and at least one of them is ≥ 2 in order to ensure the refinement (or multiplicity) of at least one neutrosophic component amongst T, I, and F.

Even more: the subcomponents $T_j$, $I_k$, and/or $F_l$ can be countable or uncountable infinite subsets of [0, 1].

This definition also includes the *Refined Fuzzy Set*, when $r = s = 0$ and $p \geq 2$;

and the definition of the *Refined Intuitionistic Fuzzy Set*, when $r = 0$, and either $p \geq 2$ and $s \geq 1$, or $p \geq 1$ and $s \geq 2$.

All other fuzzy extension sets (Pythagorean Fuzzy Set, Spherical Fuzzy Set, Fermatean Fuzzy Set, q-Rung Orthopair Fuzzy Set, etc.) can be refined/multiplicated in a similar way.

## 11. Definition of Refined Neutrosophic Topology

Let $\mathcal{U}$ be a universe of discourse, and $\mathcal{P}(\mathcal{U})$ be the family of all (*p, r, s*)-refined neutrosophic subsets of $\mathcal{U}$.
Let $\tau_{RNT} \subseteq \mathcal{P}(\mathcal{U})$ be a family of (*p, r, s*)-refined neutrosophic subsets of $\mathcal{U}$.
Then $\tau_{RNT}$ is called a *Refined Neutrosophic Topology* (*RNT*) if it satisfies the axioms:
(RNT-1) $\phi$ and $\mathcal{U}$ belong to $\tau_{RNT}$;
(RNT-2) The intersection of any finite number of elements in $\tau_{RNT}$ is in $\tau_{RNT}$;
(RNT-3) The union of any finite or infinite number of elements in $\tau_{RNT}$ is in $\tau_{RNT}$;
Then $(\mathcal{U}, \tau_{RNT})$ is called a Refined Neutrosophic Topological Space on $\mathcal{U}$.

The *Refined Neutrosophic Topology* is a topology defined on a Refined Neutrosophic Set.
{Similarly, the Refined Fuzzy Topology is defined on a Refined Fuzzy Set, while the Refined Intuitionistic Fuzzy Topology is defined on a Refined Intuitionistic Fuzzy Set, etc.
And, as a generalization, on any type of fuzzy extension set [such as: Pythagorean Fuzzy Set, Spherical Fuzzy Set, Fermatean Fuzzy Set, q-Rung Orthopair Fuzzy Set, etc.] one can define a corresponding fuzzy extension topology}.

## 12. Neutrosophic Crisp Set

The *Neutrosophic Crisp Set* was defined by Salama and Smarandache in 2014 and 2015.

Let X be a non-empty fixed space. And let *D* be a Neutrosophic Crisp Set [2],

where $D = \langle A, B, C \rangle$, with *A, B, C* as subsets of *X*.

Depending on the intersections and unions between these three sets *A, B, C* one gets several:

Types of Neutrosophic Crisp Sets [2, 3].

The object having the form $D = \langle A, B, C \rangle$ is called:

(a) A neutrosophic crisp set of Type 1 (NCS-Type1) if it satisfies:
$A \cap B = B \cap C = C \cap A = \phi$ (empty set).

(b) A neutrosophic crisp set of Type 2 (NCS-Type2) if it satisfies:
$A \cap B = B \cap C = C \cap A = \phi$ and $A \cup B \cup C = X$.

(c) A neutrosophic crisp set of Type 3 (NCS-Type3) if it satisfies:
$A \cap B \cap C = \phi$ and $A \cup B \cup C = X$.

Of course, more types of Neutrosophic Crisp Sets may be defined by modifying the intersections and unions of the subsets *A, B,* and *C*.

## 13. Refined Neutrosophic Crisp Set

The *Refined Neutrosophic Crisp Set* [3] was introduced by Smarandache in 2019, by refining/multiplication of *D* (and denoting it by *RD* = Refined D) by refining/multiplication of its sets *A, B, C* into sub-subsets/multi-sets as follows:
$RD = (A_1, \ldots, A_p; B_1, \ldots, B_r; C_1, \ldots, C_s)$, with $p, r, s \geq 1$ be positive integers and at least one of them be $\geq 2$ in order to ensure the refinement/multiplication of at least one component amongs A, B, C, where

$$A = \bigcup_{i=1}^{p} A_i, B = \bigcup_{j=1}^{r} B_j, C = \bigcup_{k=1}^{s} C_k$$

and many types of Refined Neutrosophic Crisp Sets may be defined by modifying the intersections or unions of the subsets/multisets $A_i, B_j, C_k, 1 \leq i \leq p, 1 \leq j \leq r, 1 \leq k \leq s$, depending on each application.

## 14. Definition of Refined Neutrosophic Crisp Topology

Let $\mathcal{U}$ be a universe of discourse, and $\mathcal{P}(\mathcal{U})$ be the family of all (*p, r, s*)-refined neutrosophic crisp subsets of $\mathcal{U}$.
Let $\tau_{RNCT} \subseteq \mathcal{P}(\mathcal{U})$ be a family of (*p, r, s*)-refined neutrosophic crisp subsets of $\mathcal{U}$.
Then $\tau_{RNCT}$ is called a *Refined Neutrosophic Crisp Topology* (*RNCT*) if it satisfies the axioms:
(RNCT-1) $\phi$ and $\mathcal{U}$ belong to $\tau_{RNCT}$;
(RNCT-2) The intersection of any finite number of elements in $\tau_{RNCT}$ is in $\tau_{RNCT}$;
(RNCT-3) The union of any finite or infinite number of elements in $\tau_{RNCT}$ is in $\tau_{RNCT}$.
Then $(\mathcal{U}, \tau_{RNCT})$ is called a Refined Neutrosophic Crisp Topological Space on $\mathcal{U}$.
Therefore, the *Refined Neutrosophic Crisp Topology* is a topology defined on the Refined Neutrosophic Crisp Set.

## 15. Definition of the n^th^-PowerSets $P^n(H)$ and $P_*^n(H)$.

The n^th^-PowerSets $P^n(H)$ and $P_*^n(H)$ of the set *H,* that the SuperHyperTopology and respectively Neutrosophic SuperHyperTopology are based on, better describe our real world, since a system *H* (that may be a set, company, institution, country, region, etc.) is organized in sub-systems, which in their turn are organized each of them in sub-sub-systems, and so on.
The n^th^-PowerSet $P^n(H)$ is defined recursively:

$$P^0(H) \stackrel{def}{=} H$$
$$P^1(H) = P(H)$$
$$P^2(H) = P(P(H))$$
$$P^3(H) = P(P^2(H)) = P(P(P(H)))$$
$$\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots$$
$$P^n(H) = P(P^{n-1}(H)) = \underbrace{P(P(\ldots P(H)\ldots))}_{n}$$

where *P* is repeated *n* times into the last formula, and the empty-set $\phi$ (that represents indeterminacy, uncertainty) is allowed in all sequence terms:

$$H, P(H), P^2(H), P^3(H), \ldots, P^n(H).$$

Similarly,

The n[th]-PowerSet $P_*^n(H)$ is defined recursively:

$$P_*^0(H) \overset{def}{=} H$$
$$P_*^1(H) = P_*(H)$$
$$P_*^2(H) = P_*(P_*(H))$$
$$P_*^3(H) = P_*(P_*^2(H)) = P_*(P_*(P_*(H)))$$
$$\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots$$
$$P_*^n(H) = P_*(P_*^{n-1}(H)) = \underbrace{P_*(P_*(\ldots P_*(H)\ldots))}_{n}$$

where $P$ is repeated $n$ times into the last formula, and the empty-set $\phi$ (that represents indeterminacy, uncertainty) is not allowed in none of the sequence terms: $H, P_*(H), P_*^2(H), P_*^3(H), \ldots, P_*^n(H)$.

## 16. SuperHyperOperation

We recall our 2016 concepts of SuperHyperOperation, SuperHyperAxiom, SuperHyperAlgebra, and their corresponding Neutrosophic SuperHyperOperation Neutrosophic SuperHyperAxiom and Neutrosophic SuperHyperAlgebra [6].

Let $P_*^n(H)$ be the n[th]-powerset of the set $H$ such that none of $P(H)$, $P^2(H)$, …, $P^n(H)$ contain the empty set $\phi$.

Also, let $P_n(H)$ be the n[th]-powerset of the set $H$ such that at least one of the $P(H)$, $P^2(H)$, …, $P^n(H)$ contain the empty set $\phi$. For any subset A, we identify {A} with A.

The SuperHyperOperations are operations whose codomain is either $P_*^n(H)$ and in this case one has classical-type SuperHyperOperations, or $P^n(H)$ and in this case one has Neutrosophic SuperHyperOperations, for integer $n \geq 2$.

## 17. The n[th]-PowerSet better describe our real world

The n[th]-PowerSets $P^n(H)$ and $P_*^n(H)$, that the SuperHyperTopology and respectively Neutrosophic SuperHyperTopology are based on, better describe our real world, since a system $H$ (that may be a set, company, institution, country, region, etc.) is organized in sub-systems, which in their turn are organized each in sub-sub-systems, and so on.

## 18. SuperHyperAxiom

A classical-type SuperHyperAxiom or more accurately a *(m, n)*-SuperHyperAxiom is an axiom based on classical-type SuperHyperOperations.

Similarly, a Neutrosophic SuperHyperAxiom {or Neutrosphic (m, n)-SuperHyperAxiom} is an axiom based on Neutrosophic SuperHyperOperations.

There are:

- Strong SuperHyperAxioms, when the left-hand side is equal to the right-hand side as in non-hyper axioms.
- And Week SuperHyperAxioms, when the intersection between the left-hand side and the right-hand side is non-empty.

## 19. SuperHyperAlgebra and SuperHyperStructure

A SuperHyperAlgebra or more accurately *(m-n)*-SuperHyperAlgebra is an algebra dealing with SuperHyperOperations and SuperHyperAxioms.

Again, a Neutrosophic SuperHyperAlgebra {or Neutrosphic (m, n)-SuperHyperAlgebra} is an algebra dealing with Neutrosophic SuperHyperOperations and Neutrosophic SuperHyperOperations.

In general, we have SuperHyperStructures {or *(m-n)*-SuperHyperStructures}, and corresponding Neutrosophic SuperHyperStructures.

For example, there are SuperHyperGrupoid, SuperHyperSemigroup, SuperHyperGroup, SuperHyperRing, SuperHyperVectorSpace, etc.

## 20. Distinction between SuperHyperAlgebra vs. Neutrosophic SuperHyperAlgebra

- If none of the power sets $P^k(H)$, $1 \le k \le n$, do not include the empty set $\phi$, then one has a classical-type SuperHyperAlgebra;
- If at least one power set, $P^k(H)$, $1 \le k \le n$, includes the empty set $\phi$, then one has a Neutrosophic SuperHyperAlgebra.

## 21. Definition of SuperHyperTopology (SHT) [6]

It is a topology designed on the n[th]-PowerSet of a given non-empty set $H$, that excludes the empty-set, denoted as $P_*^n(H)$, built as follows:

$P_*(H)$ is the first powerset of the set *H*, and the index $*$ means without the empty-set (Ø);

$P_*^2(H) = P_*(P_*(H))$ is the second powerset of *H* (or the powerset of the powerset of *H*), without the empty-sets; and so on, the *n*-th powerset of *H*,

$P_*^n(H) = P_*(P_*^{n-1}(H)) = \underbrace{P_*(P_*(...P_*(H)...))}_{n}$, where $P_*$ is repeated *n* time ( $n \ge 2$ ), and without the empty-sets.

Let consider $\tau_{SHT}$ be a family of subsets of $P_*^n(H)$.

Then $\tau_{SHT}$ is called a Neutrosophic SuperHyperTopology on $P_*^n(H)$, if it satisfies the following axioms:

(SHT-1) $\phi$ and $P_*^n(H)$ belong to $\tau_{SHT}$.

(SHT-2) The intersection of any finite number of elements in $\tau_{SHT}$ is in $\tau_{SHT}$.

(SHT-3) The union of any finite or infinite number of elements in $\tau_{SHT}$ is in $\tau_{SHT}$.

Then $(P_*^n(H), \tau_{SHT})$ is called a SuperHyperTopological Space on $P_*^n(H)$.

## 22. Definition of Neutrosophic SuperHyperTopology (NSHT) [6]

It is, similarly, a topology designed on the *n*-th PowerSet of a given non-empty set *H*, but includes the empty-sets [that represent indeterminacies] too.

As such, in the above formulas, $P_*(H)$ that excludes the empty-set, is replaced by $P(H)$ that includes the empty-set.

$P(H)$ is the first powerset of the set *H*, including the empty-set (Ø);

$P^2(H) = P(P(H))$ is the second powerset of *H* (or the powerset of the powerset of *H*), that includes the empty-sets; and so on, the *n*-th powerset of *H*,

$$P^n(H) = P(P^{n-1}(H)) = \underbrace{P(P(...P(H)...))}_{n}$$

where $P$ is repeated $n$ times ( $n \geq 2$ ), and includes the empty-sets (∅).
Let consider $\tau_{NSHT}$ be a family of subsets of $P^n(H)$.
Then $\tau_{NSHT}$ is called a Neutrosophic SuperHyperTopology on $P^n(H)$, if it satisfies the following axioms:
(NSHT-1) $\phi$ and $P^n(H)$ belong to $\tau_{NSHT}$.
(NSHT-2) The intersection of any finite number of elements in $\tau_{NSHT}$ is in $\tau_{NSHT}$.
(NSHT-3) The union of any finite or infinite number of elements in $\tau_{NSHT}$ is in $\tau_{NSHT}$.
Then $(P^n(H), \tau_{NSHT})$ is called a Neutrosophic SuperHyperTopological Space on $P^n(H)$.

## 23. Introduction to NonStandard Analysis [9-12]

An ***infinitesimal*** [or infinitesimal number] ( $\varepsilon$ ) is a number $\varepsilon$ such that $|\varepsilon| < 1/n$, for any non-null positive integer *n*. An infinitesimal is close to zero, and so small that it cannot be measured.

The infinitesimal is a number smaller, in absolute value, than anything positive nonzero. Infinitesimals are used in calculus.
An ***infinite*** [or infinite number] *( ω )* is a number greater than anything:
*1 + 1 + 1 + … + 1* (for any finite number terms)
The infinites are reciprocals of infinitesimals.
The set of ***hyperreals*** (or *non-standard reals*), denoted as $R^*$, is the extension of set of the real numbers, denoted as *R*, and it comprises the infinitesimals and the infinites, that may be represented on the *hyperreal number line*
$1/\varepsilon = \omega/1$.

The set of hyperreals satisfies the *transfer principle*, which states that the statements of first order in *R* are valid in $R^*$ as well.

A *monad* (*halo*) of an element $a \in R^*$, denoted by $\mu(a)$, is a subset of numbers infinitesimally close to *a*.

## 24. First Extension of NonStandard Analysis [13]

Let's denote by $R_+^*$ the set of positive nonzero hyperreal numbers.

We consider the left monad and right monad, and the (*pierced*) *binad* that we have introduced as extension in 1998 [5]:
**Left Monad** { that we denote, for simplicity, by ($^-a$) or only $^-a$ } is defined as:

$\mu(^-a) = (^-a) = {}^-a = \overset{-}{a} = \{a - x, x \in R_+^* \mid x \text{ is infinitesimal}\}$.

**Right Monad** { that we denote, for simplicity, by ($a^+$) or only by $a^+$ } is defined as:

$\mu(a^+) = (a^+) = a^+ = \overset{+}{a} = \{a + x, x \in R_+^* \mid x \text{ is infinitesimal}\}$.

**Pierced Binad** { that we denote, for simplicity, by ($^-a^+$) or only $^-a^+$ } is defined as:

$\mu(^-a^+) = (^-a^+) = {}^-a^+ = \overset{-+}{a} =$
$= \{a - x, x \in R_+^* \mid x \text{ is infinitesimal}\} \cup \{a + x, x \in R_+^* \mid x \text{ is infinitesimal}\}$
$= \{a \pm x, x \in R_+^* \mid x \text{ is infinitesimal}\}$.

The left monad, right monad, and the pierced binad are subsets of $R^*$.

## 25. Second Extension of NonStandard Analysis

For necessity of doing calculations that will be used in NonStandard neutrosophic logic in order to calculate the NonStandard neutrosophic logic operators (conjunction, disjunction, negation, implication, equivalence) and in order to have the NonStandard Real MoBiNad Set closed under arithmetic operations, Smarandache extended in 2019: the left monad to the Left Monad Closed to the Right, the right monad to the Right Monad Closed to the Left; and the Pierced Binad to the Unpierced Binad, defined as follows:

**Left Monad Closed to the Right**

$\mu\left(\overset{-0}{a}\right) = \left(\overset{-0}{a}\right) = \overset{-0}{a}$ = {a – x | x = 0, or x ∈ $R_+^*$ and x is infinitesimal} = μ($^-$a) ∪ {a} = ($^-$a) ∪ {a} = $^-$a ∪ {a}.

**Right Monad Closed to the Left**

$\mu\left(\overset{0+}{a}\right) = \left(\overset{0+}{a}\right) = \overset{0+}{a}$ = {a + x | x = 0, or x ∈ $R_+^*$ and x is infinitesimal} = μ($a^+$) ∪ {a} = ($a^+$) ∪ {a} = $a^+$ ∪ {a}.

**Unpierced Binad**

$\mu\left(\overset{-0+}{a}\right) = \left(\overset{-0+}{a}\right) = \overset{-0+}{a}$ = {a – x | x ∈ $R_+^*$ and x is infinitesimal} ∪ {a + x | x ∈ $R_+^*$ and x is infinitesimal} ∪ {a} =
= {$a \pm x$ | x = 0, or x ∈ $R_+^*$ and x is infinitesimal}
= μ($^-a^+$) ∪ {a} = ($^-a^+$) ∪ {a} = $^-a^+$ ∪ {a}

The element *{a}* has been included into the left monad, right monad, and pierced binad respectively.

### 26. NonStandard Neutrosophic Topology

The previous two extensions of NonStandard Analysis, used in the construction of NonStandard Neutrosophic Logic, NonStandard Neutrosophic Set, and NonStandard Neutrosophic Probability, were defined on the NonStandard Unit Interval

$I_{nontandard} = ]^-0,1^+[$ ,

we have founded [13] since 1998, and we have previously [13-15] proposed it, where:

$I_{nontandard} = ]^-0,1^+[ = \{\overset{0}{x}; \overset{-}{x}, \overset{+}{x}, \overset{-0}{x}, \overset{0+}{x}, \overset{-+}{x}, \overset{-0+}{x}; 0 \le x \le, x \in R\}$, where *R* is the set of real numbers.

Let $P(]^-0,1^+[)$ be the powerset of $]^-0,1^+[$ .

Let τ = $P(]^-0,1^+[)$, which means that τ is the family of all subsets of $P(]^-0,1^+[)$. Of course:

(i). $\phi$ and $]^-0,1^+[$ belong to τ.
(ii). The intersection of any finite number of elements in τ is in τ.
(iii). The union of any number of finite or infinite number of elements in τ is in τ.

Therefore, τ is a NonStandard Neutrosophic Topology.
Then ( $]^-0,1^+[$ , τ) is called a NonStandard Neutrosophic Topological Space.

### 27. NonStandard Topology

As a generalization of NonStandard Neutrosophic Topology one propose now the NonStandard Topology.

Let's consider the real numbers $a, b \in R$ and the real interval $[a, b]$. Let's extend it to a non-standard interval $]^{-}a, b^{+}[$ is the same way as for the NonStandard Neutrosophic Logic and Set.

Let's have by convention the same meaning of the following notations:

$x \overset{def}{=} \overset{0}{x}$, and $^{-}x = \overset{-}{x}$, also $x^{+} = \overset{+}{x}$ for any real number *x*.

Then:

$]^{-}a, b^{+}[ = \{x; \overset{0}{x}, \overset{-}{x}, \overset{+}{x}, \overset{-0}{x}, \overset{0+}{x}, \overset{-+}{x}, \overset{-0+}{x}; a \le x \le b, x \in R\}$, where *R* is the set of real numbers.

Let $U_{NonStandard}$ = $]^{-}a, b^{+}[$ be a NonStandard interval, for $a < b$, where *a* and *b* are real numbers, and $P(U_{NonStandard})$ be the power set of $U_{NonStandard}$.

Then $P(U_{NonStandard})$ is formed by the empty set ($\phi$) and itself $U_{NonStandard}$, together with all standard and NonStandard subsets of $]^{-}a, b^{+}[$.

The finite intersections, and finite or infinite unions of any standard and NonStandard subsets are still (standard or NonStandard) subsets of $U_{NonStandard}$.

Let $\tau_{NonStandard} \subseteq P(U_{NonStandard})$ be a family of standard or NonStandard subsets of $P(U_{NonStandard})$.

Then $\tau_{NonStandard}$ is called a NonStandard Topology on $U_{NonStandard}$ if it satisfies the following axioms:

(i). The empty set ($\phi$) and $U_{NonStandard}$ belong to $\tau_{NonStandard}$.

(ii). The intersection of finite number of elements in $\tau_{NonStandard}$ is stil in $\tau_{NonStandard}$.

(iii). The union of any finite or infinite number of elements in $\tau_{NonStandard}$ is stil in $\tau_{NonStandard}$.

Then ($U_{NonStandard}$, $\tau_{NonStandard}$) is called a NonStandard Topological Space.

## 28. Extended NonStandard Real Set ( $\overset{-0+}{ER}$ )

We introduce it now for the first time:

$\overset{-0+}{ER} = \{x; \overset{0}{x}, \overset{-}{x}, \overset{+}{x}, \overset{-0}{x}, \overset{0+}{x}, \overset{-+}{x}, \overset{-0+}{x}; x \in R\}$, actually:

$\overset{-0+}{ER} = \overset{0}{R} \cup \overset{-}{R} \cup \overset{+}{R} \cup \overset{-0}{R} \cup \overset{0+}{R} \cup \overset{-+}{R} \cup \overset{-0+}{R}$,

where one uses the notations:

$$\overset{0}{R} \overset{def}{\equiv} R$$

$$\overset{-}{R} = \{\overset{-}{x}, x \in R\}$$

$$\overset{+}{R} = \{\overset{+}{x}, x \in R\}$$

$$\overset{-0}{R} = \{\overset{-0}{x}, x \in R\}$$

$$\overset{0+}{R} = \{\overset{0+}{x}, x \in R\}$$

$$\overset{-+}{R} = \{\overset{-+}{x}, x \in R\}$$

$$\overset{-0+}{R} = \{\overset{-0+}{x}, x \in R\}$$

## 29. Largest Extended NonStandard Real Topology

$P\left(\overset{-0+}{ER}\right)$, which is the powerset of $\overset{-0+}{ER}$, generates the Largest Extended NonStandard Real Topology on the whole Extended NonStandard Real Set $\overset{-0+}{ER}$.

## 30. Over/Under/Off-Sets and Logics and Probabilities

The Neutrosophic Set was extended [Smarandache, 2007] to Neutrosophic Overset (when some Neutrosophic component is > 1), since we observed that, for example, an employee working overtime deserves a degree of membership > 1, with respect to an employee that only works regular full-time and whose degree of membership = 1;

and to Neutrosophic Underset (when some Neutrosophic component is < 0), since, for example, an employee making more damage than benefit to his company deserves a degree of membership < 0, with respect to an employee that produces benefit to the company and has the degree of membership > 0;

and to and to Neutrosophic Offset (when some Neutrosophic components are off the interval [0, 1], i.e. some Neutrosophic component > 1 and some Neutrosophic component < 0).

Similarly for Over/Under/Off-Logic and respectively Over/Under/Off-Topology [16 - 19].

Since these ideas look counter-intuitive and totally different from the mainstream framework, we present below elementary examples from our real world of such degrees that are outside the box {we mean outside the interval [0, 1]}.

## 31. Real Example of OverMembership and UnderMembership

In a company a full-time employer works 40 hours per week. Let's consider the last week period.

Helen worked part-time, only 30 hours, and the other 10 hours she was absent without payment; hence, her membership degree was 30/40 = 0.75 < 1.

John worked full-time, 40 hours, so he had the membership degree 40/40 = 1, with respect to this company.

But George worked overtime 5 hours, so his membership degree was (40+5)/40 = 45/40 = 1.125 > 1.

Thus, we need to make distinction between employees who work overtime, and those who work full-time or part-time. That's why we need to associate a degree of membership strictly greater than 1 to the overtime workers.

Now, another employee, Jane, was absent without pay for the whole week, so her degree of membership was 0/40 = 0.

Yet, Richard, who was also hired as a full-time, not only didn't come to work last week at all (0 worked hours), but he produced, by accidentally starting a devastating fire, much damage to the company, which was estimated at a value half of his salary (i.e. as he would have gotten for working 20 hours that week). Therefore, his membership degree has to be less that Jane's (since Jane produced no damage). Whence, Richard's degree of membership, with respect to this company, was - 20/40 = - 0.50 < 0.

Consequently, we need to make distinction between employees who produce damage, and those who produce profit, or produce neither damage no profit to the company.

Therefore, the membership degrees > 1 and < 0 are real in our world, so we have to take them into consideration.

Then, similarly, the Neutrosophic Logic/Measure/Probability/Statistics etc. were extended to respectively *Neutrosophic Over-/Unde-r/Off-Logic, -Measure, -Probability, - Statistics* etc. (Smarandache, 2007).

**32. Definition of the Single-Valued Neutrosophic OverSet**

Let $U_{over}$ be an OverUniverse of Discourse {i.e. there exist some elements in $U_{over}$ whose degrees of membership are > 1 }, and the Neutrosophic OverSet $A_{over} \subseteq U_{over}$.

Let *T(x), I(x), F(x)* be the functions that describe the degrees of membership, indeterminate-membership, and nonmembership respectively, of a generic element $x \in U_{over}$, with respect to the Neutrosophic OverSet $A_{over}$:

$T(x), I(x), F(x) : U_{over} \to [0, \Omega]$ where $0 < 1 < \Omega$, and $\Omega$ is called OverLimit,

$T(x), I(x), F(x) \in [0, \Omega]$, for all $x \in U_{over}$.

A Single-Valued Neutrosophic OverSet $A_{over}$ is defined as:

$A_{over}$ = *{(x, <T(x), I(x), F(x)>), x* $\in U_{over}$*}*, such that there exist some elements in $A_{over}$ that have at least one neutrosophic component that is > 1.

**33. Definition of the Single-Valued Neutrosophic OverTopology**

Let $U_{over}$ be an OverUniverse of Discourse, and $P(U_{over})$ the powerset of $U_{over}$.

Let $\tau_{over} \subseteq P(U_{over})$ be a family of Single-Valued Neutrosophic OverSets of $U_{over}$.

Then $\tau_{over}$ is called a Single-Valued Neutrosophic OverTopology on $U_{over}$ if it satisfies the following axioms:

(i). $\phi$ and $U_{over}$ belong to $\tau_{over}$.

(ii). The intersection of any finite number of single-valued Neutrosophic OverSets in $\tau_{over}$ is in $\tau_{over}$.

(iii). The union of any finite or infinite number of single-valued Neutrosophic OverSets in $\tau_{over}$ is in $\tau_{over}$.

Then ($U_{over}$, $\tau_{over}$) is called a Neutrosophic OverTopological Space.

**34. Definition of the Single-Valued Neutrosophic UnderSet**

The previous two extensions of NonStandard Analysis, used in the construction of NonStandard Neutrosophic Logic, NonStandard Neutrosophic Set, and NonStandard Neutrosophic Probability, were defined on the NonStandard Unit Interval.

Let $U_{under}$ be an UnderUniverse of Discourse { i.e. there exist some elements in $U_{under}$ whose degrees of membership are < 0 }, and the Neutrosophic UnderSet $A_{under} \subseteq U_{under}$.

Let *T(x), I(x), F(x)* be the functions that describe the degrees of membership, indeterminate-membership, and nonmembership respectively, of a generic element $x \in U_{under}$, with respect to the Neutrosophic UnderSet $A_{under}$:

$T(x), I(x), F(x) : U_{under} \to [\Psi, 1]$

where $\Psi < 0 < 1$, and $\Psi$ is called UnderLimit,

$T(x), I(x), F(x) \in [\Psi, 1]$, for all $x \in U_{under}$.

A Single-Valued Neutrosophic UnderSet $A_{under}$ is defined as:

$A_{under}$ = {(x, <T(x), I(x), F(x)>), x ∈ $U_{under}$}, such that there exist some elements in $A_{under}$ that have at least one neutrosophic component that is < 0.

**35. Definition of the Single-Valued Neutrosophic UnderTopology**

Let $U_{under}$ be an UnderUniverse of Discourse, and $P(U_{under})$ the powerset of $U_{under}$.

Let $\tau_{under} \subseteq P(U_{under})$ be a family of Single-Valued Neutrosophic UnderSets of $U_{under}$.

Then $\tau_{under}$ is called a Single-Valued Neutrosophic UnderTopology on $U_{under}$ if it satisfies the following axioms:

(i). $\phi$ and $U_{under}$ belong to $\tau_{under}$.

(ii). The intersection of any finite number of single-valued neutrosophic undersets in $\tau_{under}$ is in $\tau_{under}$.

(iii). The union of any finite or infinite number of single-valued neutrosophic undersets in $\tau_{under}$ is in $\tau_{under}$.

Then ($U_{under}$, $\tau_{under}$) is called a Neutrosophic UnderTopological Space.

**36. Definition of the Single-Valued Neutrosophic OffSet**

Let $U_{off}$ be an OffUniverse of Discourse {i.e. there exist elements of $U_{off}$ whose degrees of membership are outside the interval [0, 1], some < 0 and others > 1}, and the Neutrosophic OffSet $A_{off} \subseteq U$.

Let *T(x), I(x), F(x)* be the functions that describe the degrees of membership, indeterminate-membership, and nonmembership respectively, of a generic element *x* ∈ $U_{off}$, with respect to the neutrosophic offset $A_{off}$:

$T(x), I(x), F(x) : U_{off} \to [\Psi, \Omega]$

where $\Psi < 0 < 1 < \Omega$, and $\Psi$ is called UnderLimit, while $\Omega$ is called OverLimit, $T(x), I(x), F(x) \in [\Psi, \Omega]$, for all $x \in U_{off}$.

A Single-Valued Neutrosophic Offset $A_{off}$ is defined as:

$A_{off}$ = {(x, <T(x), I(x), F(x)>), x ∈ $U_{off}$}, such that there exist some elements in $A_{off}$ that have at least one neutrosophic component that is > 1, and at least one neutrosophic component that is < 0.

**37. Definition of the Single-Valued Neutrosophic OffTopology**

The previous two extensions of NonStandard Analysis, used in the construction of NonStandard Neutrosophic Logic, NonStandard Neutrosophic Set, and NonStandard Neutrosophic Probability, were defined on the NonStandard Unit Interval.

Let $U_{off}$ be an OffUniverse of Discourse, and $P(U_{off})$ the powerset of $U_{off}$.

Let $\tau_{off} \subseteq P(U_{off})$ be a family of Single-Valued Neutrosophic OffSets of $U_{off}$.

Then $\tau_{off}$ is called a Single-Valued Neutrosophic OffTopology on $U_{off}$ if it satisfies the following axioms:

(i). $\phi$ and $U_{off}$ belong to $\tau_{off}$.

(ii). The intersection of any finite number of single-valued neutrosophic offsets in $\tau_{off}$ is in $\tau_{off}$.

(iii). The union of any finite or infinite number of single-valued neutrosophic offsets in $\tau_{off}$ is in $\tau_{off}$.

Then ($U_{off}$, $\tau_{off}$) is called a Neutrosophic OffTopological Space.

**38. Neutrosophic Triplet Weak/Strong Set (N)**

Let *(N, *)* be a groupoid, or non-empty set endowed with a well-defined binary operation *.
A **Neutrosophic Triplet** is an object of the form *<x, neut(x), anti(x)>*, for $x \in N$,
where *neut(x)* $\in N$ is the neutral of *x*, different from the classical algebraic unitary element if any, such that:

$$x * neut(x) = neut(x) * x = x$$

and *anti(x)* $\in N$ is the opposite of x such that:

$$x * anti(x) = anti(x) * x = neut(x).$$

In general, an element x may have more neutrals (*neut*'s) and more opposites (*anti*'s).
The neutrosophic triplets and their neutrosophic triplet algebraic structures were first introduced by Florentin Smarandache and Mumtaz Ali [20 - 23] in 2014 - 2016.

**39. Definition of the Neutrosophic Triplet Weak Set (NTS, *)** is a set such that each element $a \in NTS$ is part of a neutrosophic triplet *<b, neut(b), anti(b)>*, i.e. *a = b*, or *a = neut(b)*, or *a = anti(b)*.

**40. Definition of the Single-Valued Neutrosophic Triplet Weak Topology**

Let $U_{Triplet-Weak}$ be a Universe of Discourse which has the structure of a Neutrosophic Triplet Weak Set, and $P(U_{Triplet-Weak})$ the powerset of $U_{Triplet-Weak}$.
Let $\tau_{Triplet-Weak} \subseteq P(U_{Triplet-Weak})$ be a family of Single-Valued Neutrosophic Triplet Weak Sets of $U_{Triplet-Weak}$.
Then $\tau_{Triplet-Weak}$ is called a Single-Valued Neutrosophic Triplet Weak Topology on $U_{Triplet-Weak}$ if it satisfies the following axioms:

(i). $\phi$ and $U_{Triplet-Weak}$ belong to $\tau_{Triplet-Weak}$.

(i). The intersection of any finite number of single-valued neutrosophic triplet weak sets in $\tau_{Triplet-Weak}$ is in $\tau_{Triplet-Weak}$.

(ii). The union of any finite or infinite number of single-valued neutrosophic triplet weak sets in $\tau_{Triplet-Weak}$ is in $\tau_{Triplet-Weak}$.

Then ($U_{Triplet-Weak}$, $\tau_{Triplet-Weak}$) is called a Neutrosophic Triplet Weak Topological Space.

**41. Definition of Neutrosophic Triplet Strong Set (or Neutrosophic Triplet Set)**

The groupoid *(N, *)* is called a neutrosophic triplet strong set if for any $a \in N$ there exist some neutral of *a*, denoted *neut(a)* $\in N$, different from the classical algebraic unitary element (if any), and some opposite of *a*, called *anti(a)* $\in N$.

**Table 1**. Example of Neutrosophic Triplet Strong Set.

| * | 1 | 2 |
|---|---|---|
| **1** | 2 | 1 |
| **2** | 1 | 1 |

The set ({1,2}, * ) is a groupoid, without classical unit element.
Then <1, 2, 1> and <2, 1, 2> and are neutrosophic triplets.
The neutrosophic triplet strong set is N = {1, 2}.

## 42. Theorem on the Neutrosophic Triplet Strong and Weak Sets

Any neutrosophic triplet strong set is a neutrosophic triplet weak set, but not conversely.
*Proof.*

Let *(N, *)* be a neutrosophic triplet strong set. If $a \in N$, then is also included in N, therefore there exists a neutrosophic triplet in *N* that includes a, whence N is a neutrosophic triplet weak set.

Conversely, we prove by using a counterexample.

Let $Z_3 = \{0, 1, 2\}$, embedded with the multiplication × modulo *3*, which is a well-defined law. The classical unitary element in $Z_3$ is *1*.

$(Z_3, \times)$ is a neutrosophic triplet weak set, since the neutrosophic triplets formed in $Z_3$ with respect to the law × contain all elements *0, 1, 2,*

i.e. *<0, 0, 0>*, *<0, 0, 1>*, and *<0, 0, 2>*.

But ($Z_3$, × ) is not a neutrosophic triplet strong set, since, for example, for $2 \in Z_3$ there is no $neut(2) \neq 1$ and no *anti(2).*

## 43. Definition of the Single-Valued Neutrosophic Triplet Strong Topology

Let $U_{Triplet-Strong}$ be a Universe of Discourse which has the structure of a Neutrosophic Triplet Strong Set, and $P(U_{Triplet-Strong})$ the powerset of $U_{Triplet-Strong}$.

Let $\tau_{Triplet-Strong} \subseteq P(U_{Triplet-Strong})$ be a family of Single-Valued Neutrosophic Triplet Strong Sets of $U_{Triplet-Strong}$.

Then $\tau_{Triplet-Strong}$ is called a Single-Valued Neutrosophic Triplet Strong Topology on $U_{Triplet-Strong}$ if it satisfies the following axioms:

(i). $\phi$ and $U_{Triplet-Strong}$ belong to $\tau_{Triplet-Strong}$.

(ii). The intersection of any finite number of single-valued neutrosophic triplet strong sets in $\tau_{Triplet-Strong}$ is in $\tau_{Triplet-Strong}$.

(iii). The union of any finite or infinite number of single-valued neutrosophic triplet strong sets in $\tau_{Triplet-Strong}$ is in $\tau_{Triplet-Strong}$.

Then ($U_{Triplet-Strong}$, $\tau_{Triplet-Strong}$) is called a Neutrosophic Triplet Strong Topological Space.

## 44. Neutrosophic Extended Triplet

A neutrosophic extended triplet is a neutrosophic triplet, defined as above, but where the *neutral* of *x* {denoted by $_e neut(x)$ and called "extended neutral", where "e" in front stands for 'extended'} is allowed to be equal to the classical algebraic unitary element (if any) of the law * defined on the set. Therefore, the restriction "different from the classical algebraic unitary element if any" is released.

Thus, a neutrosophic extended triplet is an object of the form <*x*, $_e$*neut(x)*, $_e$*anti(x)*>, for *x*∈*N*, where $_e$*neut(x)*∈*N* is the *extended neutral* of *x*, which can be equal or different from the classical algebraic unitary element if any, such that:

*X* * $_e$*neut(x)* = $_e$*neut(x)* * *x* = *x*

and *anti(x)*∈*N* is the *extended opposite* of *x* such that:

*x**$_e$*anti(x)* = $_e$*anti(x)* * *x* = $_e$*neut(x)*.

In general, for each *x* ∈ N there are exist many $_e$*neut*'s (extended neutrals) and $_e$*anti*'s (extended opposites). The neutrosophic extended triplets were introduced by Smarandache in *2016*.

**45. Definition of Neutrosophic Extended Triplet Weak Set**

The set *N* is called a neutrosophic extended triplet weak set if for any *x*∈*N* there exist a neutrosophic extended triplet <*y*, $_e$*neut(y)*, $_e$*anti(y)*> included in *N*, such that *x* = *y* or *x* = $_e$*neut(y)* or *x* = $_e$*anti(y)*.

**46. Definition of the Single-Valued Neutrosophic Extended Triplet Weak Topology**

Let $U_{Extended-Triplet-Weak}$ be a Universe of Discourse which has the structure of a Neutrosophic Extended Triplet Weak Set, and $P(U_{Extended-Triplet-Weak})$ the powerset of $U_{Extended-Triplet-Weak}$.

Let $\tau_{Extended-Triplet-Weak} \subseteq P(U_{Extended-Triplet-Weak})$ be a family of Single-Valued Neutrosophic Extended Triplet Weak Sets of $U_{Extended-Triplet-Weak}$.

Then $\tau_{Extended-Triplet-Weak}$ is called a Single-Valued Neutrosophic Extended Triplet Weak Topology on $U_{Extended-Triplet-Weak}$ if it satisfies the following axioms:

(i). $\phi$ and $U_{Extended-Triplet-Weak}$ belong to $\tau_{Extended-Triplet-Weak}$.

(ii). The intersection of any finite number of single-valued neutrosophic extended triplet weak sets in $\tau_{Extended-Triplet-Weak}$ is in $\tau_{Extended-Triplet-Weak}$.

(iii). The union of any finite or infinite number of single-valued neutrosophic extended triplet weak sets in $\tau_{Extended-Triplet-Weak}$ is in $\tau_{Extended-Triplet-Weak}$.

Then ( $U_{Extended-Triplet-Weak}$, $\tau_{Extended-Triplet-Weak}$ ) is called a Neutrosophic Extended Triplet Weak Topological Space.

**47. Definition of Neutrosophic Extended Triplet Strong Set**

The set *N* is called a neutrosophic extended triplet strong set if for any *x* ∈ *N* there exist $_e$*neut(x)* ∈ *N* and $_e$*anti(x)* ∈ *N*.

**48. Definition of the Single-Valued Neutrosophic Extended Triplet Strong Topology**

Let $U_{Extended-Triplet-Strong}$ be a Universe of Discourse which has the structure of a Neutrosophic Extended Triplet Strong Set, and $P(U_{Extended-Triplet-Strong})$ the powerset of $U_{Extended-Triplet-Strong}$.

Let $\tau_{Extended-Triplet-Strong} \subseteq P(U_{Extended-Triplet-Strong})$ be a family of Single-Valued Neutrosophic Extended Triplet Strong Sets of $U_{Extended-Triplet-Strong}$.

Then $\tau_{Extended-Triplet-Strong}$ is called a Single-Valued Neutrosophic Extended Triplet Strong Topology on $U_{Extended-Triplet-Strong}$ if it satisfies the following axioms:

(i). $\phi$ and $U_{Extended-Triplet-Strong}$ belong to $\tau_{Extended-Triplet-Strong}$.

(ii). The intersection of any finite number of single-valued neutrosophic extended triplet strong sets in $\tau_{Extended-Triplet-Strong}$ is in $\tau_{Extended-Triplet-Strong}$.

(iii). The union of any finite or infinite number of single-valued neutrosophic extended triplet strong sets in $\tau_{Extended-Triplet-Strong}$ is in $\tau_{Extended-Triplet-Strong}$.

Then ($U_{Extended-Triplet-Strong}$, $\tau_{Extended-Triplet-Strong}$) is called a Neutrosophic Extended Triplet Strong Topological Space.

## 49. Neutrosophic Duplets

The Neutrosophic Duplets and the Neutrosophic Duplet Algebraic Structures were introduced by Florentin Smarandache in 2016.

Let *U* be a universe of discourse, and a set *D* included in *U*, endowed with a well-defined law #.

## 50. Definition of the Neutrosophic Duplet

We say that <*a, neut(a)*>, where *a*, and its neutral *neut(a)* belong to D, is a neutrosophic duplet if:

(i). *neut(a)* is different from the unitary element of *D* with respect to the law # (if any);

(ii). *a # neut(a) = neut(a) # a = a;*

(iii). there is no opposite *anti(a)* belonging to *D* for which *a # anti(a) = anti(a) # a = neut(a)*.

## 51. Example of Neutrosophic Duplets

In *(*$Z_8$*, #)*, the set of integers with respect to the regular multiplication modulo 8, one has the following neutrosophic duplets:

<2, 5 >, <4, 3>, <4, 5>, <4, 7>, and <6, 5>.

Proof:

Let $Z_8 = \{0, 1, 2, 3, 4, 5, 6, 7\}$, having the unitary element *1* with respect to the multiplication *# modulo 8*.

*2 # 5 = 5 # 2 = 10 = 2 (mod 8)*, so *neut(2) = 5* $\neq$ *1*.

There is no *anti(2)* ∈ $Z_8$, because:

2 # *anti(2) = 5 (mod 8)*, or *2y = 5 (mod 8)* by denoting *anti(2) = y*, is equivalent to:

*2y - 5 =* $M_8$ {multiple of *8*}, or *2y - 5 = 8k*, where *k* is an integer, or *2(y - 4k) = 5*, where both *y* and *k* are integers, or: e*ven number = odd number*, which is impossible.

Therefore, we proved that <*2, 5*> is a neutrosophic duplet.

Similarly for <*4, 5*>, <*4, 3*>, <*4, 7*>, and <*6, 5*>.

A counter-example: <*0, 0*> is not a neutrosophic duplet, because it is a neutrosophic triplet: <*0, 0, 0*>, where there exists an *anti(0) = 0*.

## 52. Definition of the Single-Valued Neutrosophic Duplet Topology

Let $U_{Duplet}$ be a Universe of Discourse which has the structure of a Neutrosophic Duplet Set, and $P(U_{Duplet})$ the powerset of $U_{Duplet}$.

Let $\tau_{Duplet} \subseteq P(U_{Duplet})$ be a family of Single-Valued Neutrosophic Duplet Sets of $U_{Duplet}$.

Then $\tau_{Duplet}$ is called a Single-Valued Neutrosophic Duplet Topology on $U_{Duplet}$ if it satisfies the following axioms:

(i). $\phi$ and $U_{Duplet}$ belong to $\tau_{Duplet}$.

(ii). The intersection of any finite number of single-valued neutrosophic duplet sets in $\tau_{Duplet}$ is in $\tau_{Duplet}$.

(iii). The union of any finite or infinite number of single-valued neutrosophic duplet sets in $\tau_{Duplet}$ is in $\tau_{Duplet}$.

Then ($U_{Duplet}$, $\tau_{Duplet}$) is called a Neutrosophic Duplet Topological Space.

### 53. Definition of the Neutrosophic Extended Duplet

Let *U* be a universe of discourse, and a set *D* included in *U*, endowed with a well-defined law #. We say that <*a*, *${}_e$neut(a)*>, where *a*, and its extended neutral *${}_e$neut(a)* belong to *D*, such that:

(i). *${}_e$neut(a)* may be equal or different from the unitary element of *D* with respect to the law # (if any);

(ii). *a # ${}_e$neut(a) = ${}_e$neut(a) # a = a;*

(iii). There is no extended opposite *${}_e$anti(a)* belonging to *D* for which *a # ${}_e$anti(a) = ${}_e$anti(a) # a = ${}_e$neut(a).*

### 54. Definition of the Single-Valued Neutrosophic Extended Duplet Topology

Let $U_{Extended-Duplet}$ be a Universe of Discourse which has the structure of a Neutrosophic Extended Duplet Set, and $P(U_{Extended-Duplet})$ the powerset of $U_{Extended-Duplet}$.

Let $\tau_{Extended-Duplet} \subseteq P(U_{Extended-Duplet})$ be a family of Single-Valued Neutrosophic Duplet Sets of $U_{Extended-Duplet}$.

Then $\tau_{Extended-Duplet}$ is called a Single-Valued Neutrosophic Duplet Topology on $U_{Extended-D}$ if it satisfies the following axioms:

(i). $\phi$ and $U_{Extended-Duplet}$ belong to $\tau_{Extended-Duplet}$.

(ii). The intersection of any finite number of single-valued neutrosophic extended duplet sets in $\tau_{Extended-Duplet}$ is in $\tau_{Extended-Duplet}$.

(iii). The union of any finite or infinite number of single-valued neutrosophic extended duplet sets in $\tau_{Extended-Duplet}$ is in $\tau_{Extended-Duplet}$.

Then ($U_{Extended-Duplet}$, $\tau_{Extended-Duplet}$) is called a Neutrosophic Extended Duplet Topological Space.

### 55. Definition of Neutrosophic MultiSet

The Neutrosophic MultiSet and the Neutrosophic Multiset Algebraic Structures were introduced by Florentin Smarandache [23] in 2016.

Let $\mathcal{U}$ be a universe of discourse, and a set $M \subseteq U$.

A *Neutrosophic Multiset M* is a neutrosophic set where one or more elements are repeated with the same neutrosophic components, or with different neutrosophic components.

It is an extension of the classical multiset, fuzzy multiset, intuitionistic fuzzy multiset, etc.

### 56. Examples of Neutrosophic MultiSets

$A$ = {(0.6, 0.3, 0.1), (0.8, 0.4, 0.2), $c$(0.5, 0.1, 0.3)} is a neutrosophic set (not multiset).
But $B$ = {(0.6, 0.3, 0.1),(0.6, 0.3, 0.1),$b$(0.8, 0.4, 0.2)} is a neutrosophic multiset, since the element $a$ is repeated; we say that the element $a$ has the *neutrosophic multiplicity* 2 with the same neutrosophic components.
While $C$ = {(0.6, 0.3, 0.1), (0.7, 0.1, 0.2), $a$(0.5, 0.4, 0.3), $c$(0.5, 0.1, 0.3)} is also a neutrosophic multiset, because the element $a$ is repeated (it has the *neutrosophic multiplicity* 3), but with different neutrosophic components, since, for example, during the time, the neutrosophic membership of an element may change.

If the element $a$ is repeated $k$ times, keeping the same neutrosophic components ($ta,,fa$), we say that $a$ has *multiplicity* $k$.
But if there is some change in the neutrosophic components of $a$, we say that $a$ has the *neutrosophic multiplicity* $k$.

Therefore, we define in general the *Neutrosophic Multiplicity Function (nm)*:

$nm$: $\mathcal{U} \to \mathbb{N} = \{1, 2, 3, \ldots, \infty\}$, and for any $a \in A$ one has

$(a) = \{(k_1, \langle t_1, i_1, f_1\rangle), (k_2, \langle t_2, i_2, f_2\rangle), \ldots, (k_j, \langle t_j, i_j, f_j\rangle), \ldots\}$ which means that

$a$ is repeated $k_1$ times with the neutrosophic components $\langle t_1, i_1, f_1\rangle$;

$a$ is repeated $k_2$ times with the neutrosophic components $\langle t_2, i_2, f_2\rangle$, ...,

$a$ is repeated $k_j$ times with the neutrosophic components $\langle t_j, i_j, f_j\rangle$, ..., and so on.

Then, a neutrosophic multiset $A$ can be written as:

$$A = \{(a, (a)), \text{ for } a \in A)\}.$$

### 57. Examples of operations with neutrosophic multisets

Let's have:

$A = \{5\langle 0.6, 0.3, 0.2\rangle, 5\langle 0.6, 0.3, 0.2\rangle, 5\langle 0.4, 0.1, 0.3\rangle, 6\langle 0.2, 0.7, 0.0\rangle\}$;
$B = \{5\langle 0.6, 0.3, 0.2\rangle, 5\langle 0.8, 0.1, 0.1\rangle, 6\langle 0.9, 0.0, 0.0\rangle\}$;
$C = \{5\langle 0.6, 0.3, 0.2\rangle, 5\langle 0.6, 0.3, 0.2\rangle\}$.

Then:

*Intersection of Neutrosophic Multisets.*

$A \cap B = \{5\langle 0.6, 0.3, 0.2\rangle\}$.

*Union of Neutrosophic Multisets*

$A \cup B = \{5\langle 0.6, 0.3, 0.2\rangle, 5\langle 0.6, 0.3, 0.2\rangle, 5\langle 0.4, 0.1, 0.3\rangle, 5\langle 0.8, 0.1, 0.1\rangle, 6\langle 0.2, 0.7, 0.0\rangle, 6\langle 0.9, 0.0, 0.0\rangle\}$.

*Inclusion of Neutrosophic Multisets*

$C \subset A$, but $C \not\subset B$.

### 58. Definition of the Single-Valued Neutrosophic MultiSet Topology

Let $U_{MultiSet}$ be a Universe of Discourse which has the structure of a Neutrosophic MultiSet, and $P(U_{MultiSet})$ the powerset of $U_{MultiSet}$.
Let $\tau_{MultiSet} \subseteq P(U_{MultiSet})$ be a family of Single-Valued Neutrosophic MultiSets of $U_{MultiSet}$.
Then $\tau_{MultiSet}$ is called a Single-Valued Neutrosophic MultiSet Topology on $U_{MultiSet}$ if it satisfies the following axioms:

(i). $\phi$ and $U_{MultiSet}$ belong to $\tau_{MultiSet}$.

(ii). The intersection of any finite number of single-valued neutrosophic multisets in $\tau_{MultiSet}$ is in $\tau_{MultiSet}$.

(iii). The union of any finite or infinite number of single-valued neutrosophic multisets in $\tau_{MultiSet}$ is in $\tau_{MultiSet}$.

Then $(U_{MultiSet}, \tau_{MultiSet})$ is called a Neutrosophic MultiSet Topological Space.

## 59. Conclusion

These eight new avantgarde topologies, together with the previous six new topologies and their corresponding topological space, were introduced by Smarandache in 2019-2023, but they have not yet been much studied and applied, except the NeutroTopologies and AntiTopologies [8] which got some attention from researchers. While NonStandard Neutrosophic Topology, Neutrosophic Triplet Weak/Strong Topologies, Neutrosophic Extended Triplet Weak/Strong Topologies, Neutrosophic Duplet topology, Neutrosophic Extended Duplet Topology, Neutrosophic MultiSet Topology are proposed now for the first time. As future research would be to study their large applications in our real world.

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**Data availability**
The datasets generated during and/or analyzed during the current study are not publicly available due to the privacy-preserving nature of the data but are available from the corresponding author upon reasonable request.

**Conflict of interest**
The authors declare that there is no conflict of interest in the research.

**Ethical approval**
This article does not contain any studies with human participants or animals performed by any of the authors.

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