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q-Rung Neutrosophic Sets and Topological Spaces

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Abstract: The concept of q-rung orthopair neutrosophic set is introduced in this paper and fundamental properties of it are studied. Also the ordinary notion of topological space is extended to q-rung orthopair neutrosophic environment, as well as the fundamental concepts of convergence, continuity, compactness and Hausdorff topological space. All these generalizations are illustrated by suitable examples.

Keywords: Fuzzy Set; Intuitionistic Fuzzy Set; Neutrosophic Set; q-Rung Orthopair Fuzzy Set; q-Rung Orthopair Neutrosophic Set; q-Rung Orthopair Neutrosophic Topological Space.

1. Introduction

Zadeh, in 1965, extended the concept of a crisp set to that of a *fuzzy set* [1], on the purpose of tackling mathematically the existing in everyday life partial truths, as well as the definitions having no clear boundaries, like "high mountains", "clever people", "good players", etc. Zadeh's idea was to replace the objective function of crisp sets with the membership function in fuzzy sets taking values in the interval [0, 1]. In this way a membership degree between 0 and 1 is assigned to each element of the universal set within the corresponding fuzzy set.

Before the introduction of fuzzy sets, probability used to be the unique mathematical tool in hands of the experts for managing the existing in real world uncertainty, caused by the shortage of knowledge about an observed phenomenon. Several types of uncertainty appear in everyday life, including *randomness, imprecision, vagueness, ambiguity, inconsistency*, etc. [2]**.** The uncertainty due to randomness is related to well-defined events whose outcomes cannot be predicted in advance, like the turn of a coin, the throwing of a die, etc. Imprecision occurs when the corresponding events are well defined, but the possible outcomes cannot be expressed with an exact numerical value; e.g. "The temperature tomorrow will be over 30^{o} C". Vagueness is created when one is unable to clearly differentiate between two properties, like a good and a mediocre student. In case of ambiguity the existing information leads to several interpretations by different observers. For example, the phrase "Boy no girl" written as "Boy, no girl" means boy, but written as "Boy no, girl" means girl. Inconsistency appears when two or more pieces of information cannot be true at the same time. As a result the obtainable in this case information is conflicted or undetermined. For example, "The chance of raining tomorrow is 80%, but this does not mean that the chance of not raining is 20%, because they might be hidden weather factors".

Probability, however, was proved to be effective only for tackling the uncertainty due to randomness, in contrast to fuzzy sets which were proved to be suitable for tackling other forms of uncertainty as well, and in particular the uncertainty due to vagueness [3]. Following the introduction of fuzzy sets, several generalizations of them and theories related to them have been proposed on the purpose of tackling more effectively all the forms of the existing uncertainty, e.g. see [4].None of these generalizations or theories, however, was proved to be suitable for tackling all the forms of the uncertainty alone, but the synthesis of all of them forms an effective framework for this purpose.

In 1986, Atanassov expanded the definition of fuzzy set to *intuitionistic fuzzy set* by adding the degree of non-membership to Zadeh's degree of membership [5]. Intuitionistic fuzzy sets can be used everywhere the ordinary fuzzy sets can be applied, but this is not always necessary. An example where the use of intuitionistic fuzzy sets is necessary is the case of an election, where a candidate can have voted for (membership), or voted against (non-membership) by the electoral vote. The intuitionistic fuzzy sets are suitable for tackling the uncertainty due to imprecision, which appears frequently in human reasoning [6].

Yager introduced later the concept of the *q-rung orthopair fuzzy set* with q a positive integer, in which the sum of the q-th powers of the membership and non-membership degrees of the elements of the universal set is bounded by 1 [7]. When q=1 we have an intuitionistic fuzzy set, when q=2 a *Pythagorean fuzzy set* [8] and when q=3 a *Fermatean fuzzy set* [9]. It has been established that Pythagorean and Fermatean fuzzy sets have stronger ability than intuitionistic fuzzy sets for tackling the uncertainty in decision making problems [10].

In 1995, Smarandache proposed the concept of the degree of indeterminacy or neutrality and further expanded the idea of intuitionistic fuzzy set to that of *neutrosophic set* [11], drawing inspiration from the neutralities that frequently surface in everyday life, like <friend, neutral, enemy>, <win, draw, defeat>, <high, medium, short>, etc. Neutrosophic sets are effective for tackling the uncertainty due to inconsistency and ambiguity.

In 2019 the concept of Pythagorean fuzzy set was extended to neutrosophic environments [12] and in 2021 the same happened with the Fermatean fuzzy set [13].

In this work we introduce the concept of the *q-rung orthopair neutrosophic set* focusing on fundamental properties of this kind of set and on *q-rung orthopair neutrosophic topological spaces.* The rest of the paper is formulated as follows: Section 2 contains the necessary mathematical background for the understanding of the paper. The concept of the q-rung orthopair neutrosophic set is presented in Section 3 together with basic properties of these sets. In Section 4 the classical notion of topological space is extended to *q-rung orthopair neutrosophic topological spaces* together with fundamental properties and concepts like convergence, continuity, compactness and Hausdorff topological spaces. The paper closes with the final conclusions and some hints for further research included in its last Section 5.

2. Mathematical Background

The exact definition of a fuzzy set [1] is the following:

Definition 1: A fuzzy set A in the universal set of the discourse U is a set of ordered pairs of the form:

$$A = \{(x, m(x)): x \in U, m(x) \in [0, 1]\} \quad (1)$$

In Eq. (1) m: U → [0, 1] is the membership function of A, and the real value y = m(x) is the membership degree of x in A, for all x in U. The greater m(x), the better x satisfies the characteristic property of A.

The definition of the membership function of A is not unique depending on the personal goals of each observer. For example, if A is the fuzzy set of "tall men", one may consider all the men with heights greater than 1.90 m tall and another one all those with heights greater than 2 m. As a result, the first observer will assign membership degree 1 to all men with heights between 1.90 m and 2 m, whereas the second one will assign membership degrees <1 to them. The only restriction for the definition of the membership function is to be compatible with common sense; otherwise it does not give a creditable description of the real situation represented by the corresponding fuzzy set. This could happen for instance in the previous example, if men with heights less than 1.60 m had membership degrees > 0.5. Most of the properties and operations of crisp sets can be extended in a natural way to fuzzy sets; e.g. see [2].

Atanassov extended the concept of fuzzy set to that of intuitionistic fuzzy set [5] as follows:

Definition 2: An intuitionistic fuzzy set A in the universal set U is of the form

$$A = \{(x, m(x), n(x)): x \in U, m(x), n(x) \in [0, 1], \quad 0 \leq m(x) + n(x) \leq 1\} \quad (2)$$

In Eq. (2) m: U → [0, 1] is the membership function and n: U → [0, 1] is the non-membership function of A and m(x), n(x) are the degrees of membership and non-membership respectively for each x in A. For simplicity we write A = <m, n>. Further, h(x) = 1 - m(x) - n(x), is said to be the degree of hesitation of x in A. If h(x) = 0, then A is a fuzzy set.

The term intuitionistic fuzzy set is due to Atanassov's collaborator Gargov [14] in analogy to the idea of intuitionism introduced by Brouwer at the beginning of the last century [15]. It is recalled that intuitionism rejects Aristotle's law of the excluded middle by stating that a proposition is either true, or not true, or we do not know if it is true or not true. The first part of this statement corresponds to Zadeh's membership degree, the second to Atanassov's non-membership and the third to the degree of hesitation.

Example 1: Let U be the set of the students of a class, let A be the intuitionistic fuzzy set of the good students of the class. Then each student x of U is characterized by an *intuitionistic fuzzy pair* (m, n) with respect to A, with m, n in [0, 1].For example, if (x, 0.5, 0.3) is in A, then, there is a 50% belief that x is a good student, but also a 30% belief that he is not a good student and a 20% hesitation to characterize him as a good student or not.

The properties and operations of fuzzy sets can be extended to intuitionistic fuzzy sets; e.g. see [6].

When defining the membership and non-membership degrees of the elements of the universal set U, it could happen that m(x) + n(x) >1. In such cases the corresponding structure cannot be treated as an intuitionistic fuzzy set. This motivated Yager to define the wider class of q-rung orthopair fuzzy sets [7] as follows:

Definition 3: A q-rung orthopair fuzzy set A in the universal set U, where q is a positive integer, is of the fo

$$A = \{(x, m(x), n(x)): x \in U, m(x), n(x) \in [0, 1], \quad 0 \leq [m(x)]^q + [n(x)]^q \leq 1\} \quad (3)$$

In Eq. (3) m(x) is the membership and n(x) is the non-membership degree of x in A respectively.

For q = 1, a 1-rung orthopair fuzzy set is an ordinary intuitionistic fuzzy set. Further, a 2-rung orthopair fuzzy set is referred to as a Pythagorean fuzzy set [8] and a 3-rung orthopair fuzzy set is referred to as a Fermatean fuzzy set [9].

The following Proposition helps to clarify the Yager's motivation for introducing the notion of the q-rung orthopair fuzzy set:

Proposition 1: Let q_1, q_2 be positive integers, with $q_2 > q_1$. Then the set $\{(m, n): m, n \in [0, 1], 0 \leq m^{q_2} + n^{q_2} \leq 1\}$ is larger than the set $\{(m, n): m, n \in [0, 1], 0 \leq m^{q_1} + n^{q_1} \leq 1\}$.

Proof: Since m, n ∈ [0, 1], is $m^{q_2} + n^{q_2} < m^{q_1} + n^{q_1}$. Consequently, if $m^{q_1} + n^{q_1} \leq 1$, it is also $m^{q_2} + n^{q_2} \leq 1$ and the result follows.

Example 2: Let (x, 0.8, 9.7) be an element of the q-rung orthopair fuzzy set A. Then, since 0.8 + 0.7 >1, A is not an intuitionistic fuzzy set, Also, since $(0.8)^2 + (0.7)^2 = 0.64 + 0.49 > 1$, A is not a Pythagorean fuzzy set too. But $(0.8)^3 + (0.7)^3 = 0.512 + 0.343 < 1$. Thus A could be a Fermatean fuzzy set, this depending on the form of its other elements.

Proposition 2: Let A be q_1-rung and B be q_2-rung orthopair fuzzy sets respectively, with $q_2 > q_1$. Then A is also a q_2-rung orthopair fuzzy set.

Proof: Let x(m, n) be an element of A. Then $0 \leq m^{q_1} + n^{q_1} \leq 1$, with m, n ∈ [0, 1]. But $q_2 > q_1$, therefore, $0 \leq m^{q_2} + + n^{q_2} \leq m^{q_1} + n^{q_1} \leq 1$ and the result follows.

In particular, an intuitionistic fuzzy set is a Pythagorean fuzzy set, which is a Fermatean fuzzy set.

The simplest form of a neutrosophic set is the *single valued neutrosophic set*, which is defined as follows [16]:

Definition 4: A single valued neutrosophic set A in the universe U is of the form

$$A = \{(x, m(x), i(x), n(x)): x \in U, m(x), i(x), n(x) \in [0,1], 0 \leq m(x)+i(x)+n(x) \leq 3\} \quad (4)$$

In Eq. (4) m(x), i(x), n(x) are the degrees of membership (or truth), indeterminacy (or neutrality) and non-membership (or falsity) with respect to A, for all x in U, referred to as the neutrosophic components of x. For simplicity, we write A = <m, i, n>.

The etymology of the term "neutrosophy" comes from the adjective "neutral" and the Greek word "sophia" (wisdom) and, according to Smarandanche, who introduced it, means the "knowledge of the neutral thought".

Example 3: Let U be the set of the players of a football team and let A be the single valued neutrosophic set of the good players of U. Then each player x of U is characterized by a *neutrosophic triplet* (m, i, n) with respect to A, with m, i, n in [0, 1]. For example, x(0.6, 0.2, 0.4) ∈ A, means that there is a 60% belief that x is a good player, but simultaneously a 20% doubt about and a 40% belief that x may not be a good player. In particular, x(0,1,0) ∈ A means that we do not know absolutely nothing about x's affiliation with A.

The concepts and operations defined on intuitionistic fuzzy sets can be extended in a natural way to neutrosophic sets [11].

Remark 1:

(i). Indeterminacy is understood to be everything which is between the opposites of truth and falsity [17]. In an intuitionistic fuzzy set the indeterminacy is equal by default with the hesitancy, i.e. we have i(x)=1- m(x) – n(x). Also, in a fuzzy set is i(x) = 0 and n(x) = 1 – m(x), whereas in a crisp set it is m(x) =1 (or 0) and n(x)= 0 (or 1). In other words, crisp sets, fuzzy sets and intuitionistic fuzzy sets are special cases of single valued neutrosophic sets.

(ii). When the sum m(x) + i(x) + n(x) of the neutrosophic components of x ∈ U in a neutrosophic set in U is < 1, then it leaves room for incomplete information about x, when it is equal to 1 for complete information and when is greater than 1 for inconsistent (i.e. contradiction tolerant) information about x. A single valued neutrosophic set may contain simultaneously elements leaving room for all the previous types of information.

(iii). When m(x) + i(x) + n(x) < 1, ∀ x ∈ U, then the corresponding single valued neutrosophic set is usually referred to as *picture fuzzy set* [18]. In this case 1- m(x) -i(x) -n(x) is called the degree of refusal membership of x in A. The picture fuzzy sets based models are suitable for describing situations where we face human opinions involving answers of types yes, abstain, no and refusal to express an opinion. Voting is a representative example of such a situation.

(iv). The difference between the general definition of a neutrosophic set and of the previously given definition of a single valued neutrosophic set is that in the general definition m(x), i(x) and n(x) may take values in the non-standard unit interval]−0, 1+[including values < 0 or > 1. This is something that can happen in everyday life situations; e.g. see an example, in [11].

3. Extending the Concept of Orthopair Fuzzy Set to Neutrosophic Environment

The concept of Pythagorean fuzzy set was extended to neutrosophic environment [12] as follows:

Definition 5: A Pythagorean neutrosophic set A in the universe U is of the form:

$$A = \{(x, m(x), i(x), n(x)): x \in U, m(x), i(x), n(x) \in [0,1], 0 \leq m^2(x) + n^2(x) \leq 1\} \quad (5)$$

In Eq. (5) m(x), i(x), n(x) are the degrees of membership, indeterminacy and non-membership with respect to A, for all x in U. Since $0 \leq m^2(x)+n^2(x) \leq 1$, the neutrosophic components m(x) and n(x) are dependent and the component i(x) is independent.

Proposition 3: Let A be a Pythagorean neutrosophic set, then $0 \leq m^2(x) + i^2(x) + n^2(x) \leq 2$, for all x in U.

Proof: Since $i(x) \in [0, 1]$, is $0 \leq i^2(x) \leq 1$ and the result follows by $0 \leq m^2(x) + n^2(x) \leq 1$.

Remark 2: By Corollary 1 an intuitionistic fuzzy set is a Pythagorean fuzzy set too. A neutrosophic set, however, need not be a Pythagorean neutrosophic set. For example, let A be a neutrosophic set and let x(0.8, 0.3. 0.7) be an element of A. Then $(0.8)^2 + (0.7)^2 = 0.64 + 0.49 > 1$, therefore A is not a Pythagorean neutrosophic set.

The concept of Fermatean neutrosophic set has also been defined [13] as follows:

Definition 6: A Fermatean neutrosophic set A in the universe U is of the form

$$A = \{(x, m(x), i(x), n(x)): x \in U, m(x), i(x), n(x) \in [0,1], 0 \leq m^3(x) + n^3(x) \leq 1\} \quad (6)$$

In Eq. (6) m(x), i(x), n(x) are the degrees of membership, indeterminacy and non-membership with respect to A, for all x in U. The components m(x) and n(x) are dependent and the component i(x) is independent. Similarly with Proposition 3 it can be shown that $0 \leq m^3(x) + i^3(x) + n^3(x) \leq 2$, for all x in U.

Pythagorean and Fermatean neutrosophic sets have found some interesting applications; e.g. see [12, 19, 20-23], etc.

Here we extend the concept of a Pythagorean (Fermatean) neutrosophic set as follows:

Definition 7: A *q-rung orthopair neutrosophic set* A, with q a positive integer, is of the form

$$A = \{(x, m(x), i(x), n(x)): x \in U, m(x), i(x), n(x) \in [0,1], 0 \leq m^q(x) + n^q(x) \leq 1\} \quad (7)$$

In Eq. (7) m(x), i(x), n(x) are the degrees of membership, indeterminacy and non-membership with respect to A, for all x in U. The components m(x) and n(x) are dependent and the component i(x) is independent. Similarly with Proposition 3 it can be shown that $0 \leq m^q(x) + i^q(x) + n^q(x) \leq 2$, for all x in U. For simplicity we write A = <m, i, n>.

A 1-rung orthopair neutrosophic set is referred to as *intuitionistic neutrosophic set*; e.g. see Table 1of [19]. Also, for q = 2 we have a Pythagorean neutrosophic set and for q = 3 a Fermatean neutrosophic set. Following the proof of Proposition 2 one can show:

Proposition 4: Let A be q_1-rung and B be q_2-rung orthopair neutrosophic sets respectively, with $q_2 > q_1$, then A is also a q_2-rung orthopair neutrosophic set.

Proof: The same with the proof of Proposition 2.

In particular, an intuitionistic neutrosophic set is a Pythagorean neutrosophic set, which is a Fermatean neutrosophic set.

The classical operations on crisp sets can be generalized for q-rung orthopair neutrosophic sets. Here we define the subset and the complement of a q-rung orthopair neutrosophic set, as well as the union and intersection of two such sets.

Definition 8: Let $A = <m_A, i_A, n_A>$ and $B = <m_B, i_B, n_B>$ be two q-rung orthopair neutrosophic sets in the universe U. Then:

i. A is called a *subset* of B ($A \subseteq B$), if, and only if, $m_A(x) \leq m_B(x)$, $i_A(x) \leq i_B(x)$ and $n_A(x) \geq n_B(x)$, $\forall x \in U$. If we have simultaneously $A \subseteq B$ and $B \subseteq A$, then A and B are called *equal* q-rung orthopair neutrosophic sets (A=B).
ii. The *complement* of $A = <m_A, i_A, n_A>$ is the q-rung orthopair neutrosophic set $A^C = <n_A, 1-i_A, m_A>$ in U.
iii. The *union* A∪B is the q-rung orthopair neutrosophic set $C = <m_C, i_C, n_C>$ in U with $m_C = \max\{m_A, m_B\}$, $i_C = \max\{i_A, i_B\}$ and $n_C = \min\{n_A, n_B\}$.
iv. The *intersection* A∩B is the q-rung orthopair neutrosophic set $D = <m_D, i_D, n_D>$ in U with $m_D = \min\{m_A, m_B\}$, $i_C = \min\{i_A, i_B\}$ and $n_C = \max\{n_A, n_B\}$.

Remark 3:

(i). It is easy to check that all the above relations are well defined. For the union, for example, set $m = \max\{m_A, m_B\}$ and $n = \min\{n_A, n_B\}$. If $m = m_A$ and $n = n_B$, then $0 \leq m^q + n^q = (m_A)^q + (n_B)^q \leq (m_A)^q + (n_A)^q \leq 1$. In an analogous way one can show that we always have $m^q + n^q \leq 1$ for all the other possible combinations, which means that A∪B is a q-rung orthopair neutrosophic set.

(ii). When A and B are crisp sets, it is straightforward to check that the previous definitions are reduced to the corresponding ordinary definitions for crisp sets.

(iii). With the help of the previous definitions it is straightforward to check that most of the laws and properties of crisp sets are also true for q-rung orthopair neutrosophic sets, like the commutative and associative laws for the union and intersection, the distributive law of the union with respect to intersection and vice versa, the double complement property $(A^C)^C = A$, etc.

Example 4: Let U= {x_1, x_2, x_3} be the universal set and let A={(0.3,0.3,0.6,x_1),(0.5,0.3,0.4,x_2), (0.7,0.2,0.5,x_3)}and B={(0.6,0.1,0.2,x_1), (0.3,0.2,0.5, x_2), (0.3,0.1,0.6,x_3)}be two q-rung orthopair neutrosophic set in U, q>1. Then:

(i). Neither $A \subseteq B$, nor $B \subseteq A$

(ii). c(A)={(0.6,0.7,0.3,x_1),(0.4,0.7,0.5,x_2), (0.5,0.8, 0.7,x_3)} and c(B)={(0.2,0.9,0.6,x_1),0.5,0.8,0.3,x_2) (0.6,0.9,0.3..,x_3)}.

(iii). A∪B={(0.6,0.3,0.2,x_1),(0.5,0.3,0.4,x_2), (0.7,0.2,0.5,x_3)}

(iv). A∩B={(0.3,0.1,0.6,x_1),(0.3,0.2,0.5,x_2), (0.3,0.1,0.6,x_3)}

Definition 9:

(i). The *empty* q-rung orthopair neutrosophic set in the universe U is defined to be $\emptyset_U$ = {(x, 0, 0, 1): x ∈ U}.

(ii). The *universal* q-rung orthopair neutrosophic set in U is defined to be I_U = {(x, 1, 1, 0): x ∈ U}. It is straightforward to check that for each q-rung orthopair neutrosophic set A in U is $A \cup I_U = I_U$, $A \cap I_U = A$, $A \cup \emptyset_U = A$ and $A \cap \emptyset_U = \emptyset_U$.

4. q-Rung Orthopair Neutrosophic Topological Spaces

Topological spaces are the most general category of mathematical spaces, on which fundamental properties like convergence, continuity, compactness, etc. are defined [24]. The ordinary notion of topological space has been extended to fuzzy [25], to intuitionistic fuzzy [26], to soft [27], to neutrosophic topological space [26], etc. Here we generalize the notion of topological space to the notion to *q-rung orthopair neutrosophic topological space* and we study the previously mentioned properties on such kind of spaces.

Definition 10: A *q-rung orthopair neutrosophic topology* T on a non-empty set U is defined as a collection of q-rung orthopair neutrosophic sets in U such that:

1 I_U and $\emptyset_U$ belong to T.

2 The intersection of any two elements of T belongs to T.

3 The union of any number (finite or infinite) of elements of T belongs also to T.

Trivial examples are the *discrete q-rung neutrosophic topology* of all q-rung orthopair neutrosophic sets in U and the *non-discrete q-rung neutrosophic topology* T= {I_U, $\emptyset_U$}.

The elements of a q-rung neutrosophic topology Ton U are called *open* q-rung orthopair neutrosophic sets of U and their complements are called *closed* q-rung orthopair neutrosophic sets of U. The pair (U, T) is referred to as a *q-rung neutrosophic topological space* on U.

Example 5: Let U = {x} and let A = {(x ,0.5,0.5,0.4)}, B = {(x ,0.4,0.6,0.8)}, C = {(x ,0.5,0.6,0.4)}, D ={(x ,0.4,0.5,0.8)} be q-rung orthopair neutrosophic sets in U, q > 1. Then it is straightforward to check that the collection T = {$\emptyset_U$, I_U, A, B, C, D} is a q-rung orthopair neutrosophic topology on U.

We close this work by extending the concepts of *convergence*, *continuity*, *compactness* and Hausdorff topological space to q-rung orthopair neutrosophic topological spaces.

Definition 11: Given two q-rung orthopair neutrosophic sets A and B of the q-rung neutrosophic topological space (U, T), B is said to be a *neighborhood* of A, if there exists an open q-rung orthopair neutrosophic set Q such that $A \subseteq Q \subset B$. Further, we say that a sequence {A_n} of q-rung orthopair neutrosophic sets of (U, T) *converges* to the q-rung orthopair neutrosophic set A of (U, T), if there exists a positive integer m such that for each integer n≥m and each neighborhood B of A we have that $A_n \subset$ B.

The following Proposition generalizes Zadeh's extension principle for fuzzy sets (see [2], pp. 20-21) to q-rung orthopair neutrosophic sets.

Proposition 5: Let U and V be two non-empty crisp sets and let g: U $\to$ V be a function. Then g can be extended to a function G mapping q-rung orthopair neutrosophic sets in U to q-rung orthopair neutrosophic sets in V.

Proof: Let A<m_A, i_A, n_A> be a q-rung orthopair neutrosophic set in U. Then its image G(A) is a q-rung orthopair neutrosophic set B in V, whose neutrosophic components are defined as follows: Given y in V, consider the set $g^{-1}(y)=\{x \in U: g(x)=y\}$. If $g^{-1}(y) = \emptyset$, then $m_B(y)=0$, and if $g^{-1}(v)\neq\emptyset$, then $m_B(y)$ is equal to the maximal value of all $m_A(x)$ such that $x \in g^{-1}(y)$. Conversely, the inverse image $G^{-1}(B)$ is the q-rung orthopair neutrosophic set A in U with membership function $m_A(x)=m_B(g(ux)$, for each x $\in$ U. In an analogous way one can determine the neutrosophic components i_B and n_B of B.

Definition 12: Let (U, T) and (V, S) be two q-rung neutrosophic topological spaces on the non-empty crisp sets U and V respectively and let g be a function g: U $\to$ V. Then, according to Proposition 5, g can be extended to a function G which maps q-rung orthopair neutrosophic sets of U to q-rung orthopair neutrosophic sets of V. We say then that g is a *q-rung orthopair neutrosophical continuous* function, if, and only if, the inverse image of each open q-rung orthopair neutrosophic set of V through G is an open q-rung orthopair neutrosophic set of U.

Definition 13: A family A = $\{A_i, i\in I\}$ of q-rung orthopair neutrosophic sets of the q-rung orthopair neutrosophic topological space (U, T) is called a *cover* of U, if U =.$\bigcup_{i\in I} A_i$. If the elements of A are open q-rung orthopair neutrosophic sets, then A is called an *open cover* of U. Also, each q-rung orthopair neutrosophic subset of A which is also a cover of U is called a sub-cover of A. The q-rung orthopair neutrosophic topological space (U, T) is said to be *compact*, if every open cover of U contains a sub-cover with finitely many elements.

Definition 14: A q-rung orthopair neutrosophic topological space (U, T) is called a *T_1- q-rung orthopair neutrosophic topological space* if, and only if, for each pair of elements x_1, x_2 of U with $x_1 \neq x_2$, there exist at least two open q-rung orthopair neutrosophic sets Q_1 and Q_2 such that $x_1 \in Q_1$, $x_2 \notin O_1$ and $x_2 \in Q_2$, $x_1 \notin O_2$.

Definition 15: A q-rung orthopair neutrosophic topological space (U, T) is called a *T_2- q-rung orthopair neutrosophic topological space* if, and only if, for each pair of elements x_1, x_2 of U with $x_1 \neq x_2$, there exist at least two open q-rung orthopair neutrosophic sets Q_1 and Q_2 such that $x_1\in O_1$, $x_2\in O_2$ and $O_1\cap O_2 = \emptyset_U$.

A T_2-q-rung orthopair neutrosophic topological space is also called a *Hausdorff* or a *separable* q-rung orthopair neutrosophic topological space. Obviously a T_2-rung orthopair neutrosophic topological space is always a T_1- q-rung orthopair neutrosophic topological space.

5. Conclusions

In this work we introduced the concept of q-rung orthopair neutrosophic set and we extended the classical notion of topological space and the fundamental properties of convergence, continuity, compactness and Hausdorff space defined in it to q-rung orthopair neutrosophic topological spaces. Examples were also given to illustrate our results.

It looks that proper combinations of the theories developed for tackling the existing in real life uncertainty is a promising tool for obtaining better results in a variety of human activities characterized by uncertainty (see also [29, 30]). This is, therefore, a fruitful area for future research

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Data availability

The datasets generated during and/or analyzed during the current study are not publicly available due to the privacy-preserving nature of the data but are available from the corresponding author upon reasonable request.

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Conflict of interest

The authors declare that there is no conflict of interest in the research.

Ethical approval

This article does not contain any studies with human participants or animals performed by any of the authors.

References

1. Zadeh, L.A. (1965). Fuzzy Sets. Inf. Control, 8, 338–353.
2. Klir, G.J. & Folger, T.A. (1988). Fuzzy sets, Uncertainty and Information. Prentice-Hall, London, UK.
3. Kosko, B. (1990). Fuzziness Vs Probability. Int. J. of General Systems, 17(2-3), 211-240.
4. Voskoglou, M.Gr. (2019). Generalizations of Fuzzy Sets and Related Theories. In M. Voskoglou (Ed.), An Essential Guide to Fuzzy Systems, pp. 345-352, Nova Science Publishers, NY.
5. Atanassov, K.T. (1986). Intuitionistic Fuzzy Sets. Fuzzy Sets and Systems, 20(1), 87-96.
6. Atanassov, K.T. (1999). Intuitionistic Fuzzy Sets. Physica-Verlag, Heidelberg, N.Y.
7. Yager, R.R. (2017). Generalized orthopair fuzzy sets. IEEE Transactions on Fuzzy Systems, 25(5), 1222–1230.
8. Yager, R.R. (2013). Pythagorean fuzzy subsets. Proceedings of Joint IFSA World Congress and NAFIPS Annual Meeting, 57–61, Edmonton, Canada.
9. Senapati, T., Yager, R.R. (2020). Fermatean fuzzy sets. J. Ambient Intell. Human Comput., 11, 663–674.
10. Zhang, Z., Hu, Z. (2014). Extension of TOPSIS to Multiple Criteria Decision Making with Pythagorean Fuzzy Sets. Int. J. of Information Systems, 29(12), 1061-1078.
11. Smarandache, F. (1998). Neutrosophy / Neutrosophic Probability, Set, and Logic. Proquest, Michigan, USA.
12. Jansi, R., Mohana, K., Smarandache, F. (2019). Correlation Measure for Pythagorean Neutrosophic Sets with T and F as Dependent Neutrosophic Components. Neutrosophic Sets and Systems, 30, 202-212.
13. Sweety, C.A.C. Jansi, R. (2021). Fermatean Neutrosophic Sets. International Journal of Advanced Research in Computer and Communication Engineering, 10(6), 23-27.
14. Atanassov, K.T. (2008). 25 years of Intuitionistic Fuzzy Sets or: The most important mistakes and results of mine. 7th International Workshop on Intuitionistic Fuzzy Sets and Generalizations, Warsaw, Poland, retrieved from https://ifigenia.org/images/4/49 /7IWIFSGN-Atanassov.pdf.
15. Brouwer, L.E.J. (1983). Intuitionism and Formalism. In Philosophy of Mathematics, Benacerraf, P., Putnam, H., Eds., pp. 77-89, Cambridge University Press: Cambridge, UK, Prentice Hall, Englewood Cliffs, N.J., USA.
16. Wang, H., Smarandanche, F., Zhang, Y. & Sunderraman, R. (2010). Single valued neutrosophic sets. Review of the Air Force Academy (Brasov), 1(16), 10-14.
17. Smarandache, F. (2021). Indeterminacy in neutrosophic theories and their applications. International Journal of Neutrosophic Science, 15(2), 89-97.
18. Cuong, B.C. (2014). Picture Fuzzy Sets. Journal of Computer Science and Cybernetics, 30(4), 409-420.
19. Saeed, M., Safique, I. (2024). Relation of Fermatean Neutrosophic Soft Sets with Application to Sustainable Agriculture, HyperSoft Set Methods in Engineering, 1, 21-33.
20. Broumi, S., Sundareswaran, R., Bakali, Q.A. & Talea, M. (2022). Theory and Applications of Fermatean Neutrosophic Graphs. Neutrosophic Sets and Systems, 50, 248-256.
21. Raut, P.K., Behera, S.V., Broumi, S. & Mishra, D. (2023). Calculation of Short Path on Fermatean Neutrosophic Networks. Neutrosophic Sets and Systems, 57, 328-341.
22. Broumi, S., Pradha, S.K. (2023). Fermatean Neutrosophic Graphs and their Basic Operations. Neutrosophic Sets and Systems, 58, 572-595.

23. Saraswathi , Y.K., Broumi, S. (2024), An Efficient Approach for Solving Time-Dependent Shortest Path Problem under Fermatean Neutrosophic Environment. Neutrosophic Sets and Systems, 63, 82-94.
24. Willard, S. (2004). General Topology. Dover Publ. Inc., N.Y.
25. Chang, S.L. (1968). Fuzzy topological spaces. Journal of Mathematical Analysis and Applications, 24(1), 182-190.
26. Luplanlez, F.G. (2006). On intuitionistic fuzzy topological spaces, Kybernetes, 35(5), 743-747.
27. Shabir, M., Naz, M. (2011). On soft topological spaces. Computers and Mathematics with Applications. 61, 1786-1799.
28. Salama, A.A., Alblowi, S.A. (2013). Neutrosophic sets and neutrosophic topological spaces. IOSR Journal of Mathematics, 3(4), 31-35.
29. Voskoglou, M.Gr., Broumi, S., Smarandache, F. (2022). A Combined Use of Soft and Neutrosophic Sets for Student Assessment with Qualitative Grades. Journal of Neutrosophic and Fuzzy Systems, 4(1), 15-20.
30. Voskoglou, M.Gr. (2023). An Application of Neutrosophic Sets to Decision-Making. Neutrosophic Sets and Systems, 53, 1-9.

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