Takaaki Fujita & Florentin Smarandache

The Nest *of* Scientific Ideas *(Revisited)*

Re-Formalizing Ideas from the *Nidus idearum Series* and Beyond

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Abstract

Uncertainty permeates most real-world contexts, motivating mathematical frameworks that can faithfully represent vagueness, inconsistency, and incomplete information. Classical approaches include *fuzzy sets* and *intuitionistic fuzzy sets*. Extending these ideas, the neutrosophic framework introduces *neutrosophic sets* in which each element x is characterized by a triplet of independent degrees

$$(T(x), I(x), F(x)) \in [0, 1]^3,$$

representing, respectively, truth, indeterminacy, and falsity, typically subject to

$$T(x) + I(x) + F(x) \leq 3.$$

This book concentrates on a selection of concepts developed and discussed in the *Nidus idearum* series, with particular emphasis on Neutrosophy, Plithogenic Sets, Physics, and related scientific domains. We focus on ideas that, despite their conceptual richness and potential applicability, have so far received relatively limited systematic treatment. Our aim is to present these notions in a coherent and accessible manner, in order to support further theoretical development and to encourage new applications in Neutrosophy, Plithogenic Set theory, Physics, and allied fields.

Keywords: Neutrosophic Graph, Neutrosophic Logic, Nidus idearum, Plithogenic Set.

Chapter 1

Introduction

1.1 Neutrosophic Set Theory

Classical set theory plays a fundamental role in both pure and applied mathematics. However, when one attempts to model real–world phenomena, classical sets are often too rigid to capture the pervasive presence of uncertainty and vagueness. This limitation has motivated the development of a variety of extended frameworks in which imprecision can be represented more faithfully.

Uncertainty permeates many real–world settings, motivating mathematical formalisms that can faithfully treat imprecision. Foundational paradigms include *fuzzy sets* [1–3] and *intuitionistic fuzzy sets* [4]. Subsequent variants—such as *vague sets* [8], *bipolar fuzzy sets* [9, 10], *hesitant fuzzy sets* [11–13], *spherical fuzzy sets* [14, 15], *picture fuzzy sets* [16, 17], and *m-polar fuzzy sets* [18–20]—add further modeling granularity.

Moving to the neutrosophic setting leads to *neutrosophic sets* [5–7], where each element x is described by three independent degrees $(T(x), I(x), F(x)) \in [0,1]^3$ representing truth, indeterminacy, and falsity, typically constrained by

$$T(x) + I(x) + F(x) \leq 3.$$

Neutrosophic Sets are well known for generalizing several earlier frameworks, including Fuzzy Sets, Intuitionistic Fuzzy Sets, and Vague Sets (cf. [21]). Recent refinements, including Bipolar Neutrosophic Sets [22–24], n-Valued Refined Neutrosophic Sets [25], Hesitant Neutrosophic Sets [26, 27], SuperHyper-Neutrosophic Sets [28–30], MultiNeutrosophic Sets [31] Quadripartitioned Neutrosophic Sets [32, 33], and Plithogenic Sets [34, 35], further expand the expressive capacity available for modeling complex uncertainty.

For ease of comparison, Table 1.1 compactly records, for each of several standard extensions of classical sets, the type of information that is assigned to every element.

Table 1.1: Typical set extensions and the standard information recorded per element.

Set Type	**Information attached to each element**
Fuzzy Set	A single membership degree given by a function $\mu : X \to [0,1]$.
Intuitionistic Fuzzy Set	Two functions $\mu, \nu : X \to [0,1]$ describing membership and non-membership, constrained by $\mu(x) + \nu(x) \leq 1$; the residual quantity $1 - \mu(x) - \nu(x)$ represents hesitation.
Neutrosophic Set	A triple of values $(T, I, F) \in [0,1]^3$ for each element, encoding truth T, indeterminacy I, and falsity F as three mutually independent components.
Plithogenic Set	A structure $(P, v, Pv, \mathrm{pdf}, \mathrm{pCF})$, where $\mathrm{pdf} : P \times Pv \to [0,1]^s$ assigns an s-dimensional degree of appurtenance vector, and $\mathrm{pCF} : Pv \times Pv \to [0,1]^t$ is a symmetric contradiction function taking values in $[0,1]^t$.

1.2 Nidus idearum

Nidus idearum is a multi–volume series curated by Prof. Florentin Smarandache. It gathers concise notes and exploratory sketches of ideas that emerged from his discussions with other scholars in neutrosophy, plithogenic sets, physics, and related scientific fields. Many of the themes first outlined in *Nidus idearum* have subsequently been expanded into full research papers. The Latin title *Nidus idearum* literally means "Nest of Ideas", underscoring that the series functions as an intellectual incubator where preliminary insights are collected, refined, and prepared for further development. Rooted in a blend of rigorous analysis and informal, reflective discourse, the series provides a flexible platform for presenting theoretical advances, tracing interdisciplinary connections, and recording personal meditations on scientific and philosophical questions.

Table 1.2 presents the complete list of volumes in the *Nidus idearum. Scilogs* series [1].

Table 1.2: Nidus idearum. Scilogs (Volumes I–XVI): Titles, Places, Years, and Links

Volume	Title	Place	Year	Link
I	De Neutrosophia [36]	Brussels	2016	`https://fs.unm.edu/NidusIdearumDeNeutrosophia.pdf`
II	De Rerum Consectatione [37]	Brussels	2016	`https://fs.unm.edu/NidusIdearum2-ed2.pdf`
III	Viva la Neutrosophia! [38]	Brussels	2015	`https://fs.unm.edu/NidusIdearum3.pdf`
IV	Vinculum Vinculorum [39]	Brussels	2019	`https://fs.unm.edu/NidusIdearum4-v2.pdf`
V	Joining the Dots [40]	Brussels	2019	`https://fs.unm.edu/NidusIdearum5-v3.pdf`
VI	Annotations on Neutrosophy [41]	Brussels	2019	`https://fs.unm.edu/NidusIdearum6-v2.pdf`
VII	Superluminal Physics [42]	Brussels	2019	`https://fs.unm.edu/NidusIdearum7-ed3.pdf`
VIII	Painting by Numbers [43]	Grandview Heights	2022	`https://fs.unm.edu/NidusIdearum8.pdf`
IX	Neutrosophia Perennis [44]	Grandview Heights	2022	`https://fs.unm.edu/NidusIdearum9.pdf`
X	Via Neutrosophica [45]	Grandview Heights	2022	`https://fs.unm.edu/NidusIdearum10.pdf`
XI	In–Turns and Out–Turns [46]	Grandview Heights	2023	`https://fs.unm.edu/NidusIdearum11.pdf`
XII	Seed & Heed [47]	Grandview Heights	2023	`https://fs.unm.edu/NidusIdearum12.pdf`
XIII	Structure / NeutroStructure / Anti-Structure [48]	Grandview Heights	2023	`https://fs.unm.edu/NidusIdearum13.pdf`
XIV	SuperHyperAlgebra [49]	Grandview Heights	2024	`https://fs.unm.edu/NidusIdearum14.pdf`
XV	NeutroNexus [50]	Gallup–Guayaquil	2025	`https://fs.unm.edu/NidusIdearum15.pdf`
XVI	Weaving the Neutrosophic Web [51]	Gallup–Guayaquil	2025	`https://fs.unm.edu/NidusIdearum16.pdf`

1.3 Our Contributions

This book focuses primarily on concepts discussed in *Nidus idearum*, selecting themes in Neutrosophy, Plithogenic Sets, Physics, and other scientific areas that, to date, remain insufficiently explored. Our goal is to present these ideas in a form that can assist and stimulate further study in Neutrosophy, Plithogenic Sets, Physics, and allied fields. It should be noted that this book focuses solely on theoretical investigation of the proposed concepts. Practical studies, such as case studies or computational experiments, are expected to be carried out in the future by other experts in the field.

[1] The *Nidus idearum* (blogs) Vols. 1–16 are available online via the Science Library at `https://fs.unm.edu/ScienceLibrary.htm`.

Chapter 2

Preliminaries

This chapter gathers the basic notions used throughout the paper. Unless explicitly stated otherwise, all sets and structures considered are finite.

2.1 Fuzzy Set and Neutrosophic Set

A fuzzy set assigns to each element a grade in the interval $[0, 1]$, thereby representing uncertainty by degrees rather than by a strict binary split [1,52]. Below we recall the relevant notions, together with their extensions.

Definition 2.1.1 (Fuzzy Set)**.** [1,53] Let Y be a nonempty universe. A *fuzzy set* on Y is a mapping $\tau : Y \longrightarrow [0, 1]$. A *fuzzy relation* on Y is a fuzzy subset $\delta : Y \times Y \longrightarrow [0, 1]$. We say that δ is a *fuzzy relation on* τ if, for all $y, z \in Y$,

$$\delta(y, z) \le \min\{\tau(y), \tau(z)\}.$$

Neutrosophic sets extend fuzzy sets by explicitly incorporating *indeterminacy*, thus covering statements that are neither entirely true nor entirely false, and offering a flexible representation of ambiguity [5,7,54,55]. Their formal specification is as follows.

Definition 2.1.2 (Neutrosophic Set)**.** [5,56] Let X be a nonempty set. A *Neutrosophic Set (NS) A* on X is determined by three functions

$$T_A : X \to [0, 1], \qquad I_A : X \to [0, 1], \qquad F_A : X \to [0, 1],$$

where, for each $x \in X$, $T_A(x)$, $I_A(x)$, and $F_A(x)$ denote the degrees of truth, indeterminacy, and falsity, respectively. These values satisfy

$$0 \le T_A(x) + I_A(x) + F_A(x) \le 3.$$

Example 2.1.3 (Clinical triage under conflicting tests)**.** Let $X = \{\text{Alice}, \text{Bob}, \text{Chen}\}$ be today's patients and let $A \subseteq X$ denote "has influenza A." A Neutrosophic Set A assigns, for each $x \in X$, a triplet $\big(T_A(x), I_A(x), F_A(x)\big) \in [0, 1]^3$ (truth, indeterminacy, falsity). Suppose calibrated evidence (rapid antigen, PCR, symptoms, exposure) yields:

Patient x	$T_A(x)$	$I_A(x)$	$F_A(x)$	$S(x) := T_A(x) - F_A(x)$	$H(x) := T_A(x) + F_A(x)$
Alice	0.78	0.12	0.10	$0.78 - 0.10 = 0.68$	$0.78 + 0.10 = 0.88$
Bob	0.40	0.45	0.15	$0.40 - 0.15 = 0.25$	$0.40 + 0.15 = 0.55$
Chen	0.15	0.20	0.65	$0.15 - 0.65 = -0.50$	$0.15 + 0.65 = 0.80$

Here $I_A(x)$ encodes conflicting/insufficient evidence (e.g., antigen–positive but PCR–negative). A normalized presentation is used: $T_A(x) + I_A(x) + F_A(x) = 1$ for each x. A simple decision rule "treat if $S(x) \ge 0.30$ or $T_A(x) \ge 0.70$" recommends treatment for Alice ($S = 0.68$, $T = 0.78$), watchful waiting and retest for Bob ($S = 0.25$, high $I = 0.45$), and discharge for Chen ($S = -0.50$, $F = 0.65$).

Example 2.1.4 (Supplier approval with missing/contradictory documentation)**.** Let $X = \{S_1, S_2, S_3\}$ be candidate suppliers and let $A \subseteq X$ denote "approve for critical parts." Evidence features are on-time rate $r \in [0, 1]$, defect rate $q \in [0, 1]$ (smaller is better), and document completeness $d \in [0, 1]$. Define a normalized neutrosophic assessment by

$$T_A = 0.6\,r + 0.1\,(1 - q) + 0.3\,d, \qquad F_A = 0.6\,(1 - r) + 0.1\,q, \qquad I_A = 1 - (T_A + F_A).$$

With (r, q, d) observed as

$$S_1 : (0.92, 0.03, 0.55), \quad S_2 : (0.78, 0.07, 0.20), \quad S_3 : (0.62, 0.02, 0.95),$$

we compute, elementwise:

Supplier	T_A	F_A	I_A	$S := T_A - F_A$
S_1	0.814	0.051	0.135	0.763
S_2	0.621	0.139	0.240	0.482
S_3	0.755	0.230	0.015	0.525

A policy "approve if $S \geq 0.50$ and $I \leq 0.20$" accepts S_1 ($S = 0.763$, $I = 0.135$) and S_3 ($S = 0.525$, $I = 0.015$), while S_2 remains uncertain due to lower S and higher I. This explicitly separates evidence *for* approval (T_A), *against* approval (F_A), and unresolved ambiguity (I_A).

2.2 Plithogenic Set

A plithogenic set [35, 57–59] represents elements via membership driven by explicit attributes together with a contradiction mapping between attribute values.

Definition 2.2.1 (Plithogenic Set)**.** [35, 60] Let S be a universe and $P \subseteq S$ a nonempty subset. A *plithogenic set* is a 5–tuple

$$PS = (P,\ v,\ Pv,\ pdf,\ pCF),$$

with the following ingredients:

- v — a chosen attribute;
- Pv — the value domain of v;
- $pdf : P \times Pv \to [0, 1]^s$ — the *degree of appurtenance* (DAF);[1]
- $pCF : Pv \times Pv \to [0, 1]^t$ — the *degree of contradiction* (DCF).

For every $a, b \in Pv$, the DCF satisfies

$$\text{reflexivity: } pCF(a, a) = 0, \qquad \text{symmetry: } pCF(a, b) = pCF(b, a).$$

Here $s, t \in \mathbb{N}$ are, respectively, the appurtenance and contradiction dimensions.

Example 2.2.2 (Green fleet selection under a powertrain attribute)**.** Let S be the set of candidate cars and let $P \subseteq S$ be those considered for a *Green Fleet 2026*. Choose the attribute v = powertrain with value domain

$$Pv = \{\text{EV},\ \text{PHEV},\ \text{HEV},\ \text{Gasoline}\}.$$

Take $s = t = 1$ (scalar DAF/DCF). The degree of appurtenance (DAF) is $pdf : P \times Pv \to [0, 1]$, where $pdf(x; a)$ measures how well car x fits the green-fleet criterion *under the viewpoint* "a." The degree of contradiction (DCF) $pCF : Pv \times Pv \to [0, 1]$ quantifies semantic opposition between powertrains. We use the symmetric matrix (only entries w.r.t. EV are needed below):

$$pCF(\text{EV}, \text{EV}) = 0, \quad pCF(\text{EV}, \text{HEV}) = 0.30, \quad pCF(\text{EV}, \text{Gasoline}) = 0.90.$$

[1] In the literature, DAF is modeled in several equivalent ways (e.g., powerset–valued or vector–valued). We adopt the standard $[0, 1]^s$ form; see [61].

Let $\mathrm{T}(x, y) = \min\{x, y\}$ and $\mathrm{S}(x, y) = \max\{x, y\}$ (Gödel t-norm/conorm). Fix the *dominant value* $u_\star = \mathrm{EV}$. For any auxiliary value $a \in \{\mathrm{EV}, \mathrm{HEV}, \mathrm{Gasoline}\}$, write $c = pCF(a, u_\star)$. The plithogenic mixing for $x \in P$ is

$$\mu_a^\star(x) \;:=\; (1-c)\,\mathrm{T}\big(pdf(x; u_\star),\; pdf(x; a)\big) \;+\; c\,\mathrm{S}\big(pdf(x; u_\star),\; pdf(x; a)\big),$$

and we aggregate by the simple mean over the three a's:

$$\mu^\star(x; u_\star) \;:=\; \frac{1}{3} \sum_{a \in \{\mathrm{EV}, \mathrm{HEV}, \mathrm{Gasoline}\}} \mu_a^\star(x).$$

Consider three cars C_1 (EV), C_2 (HEV), C_3 (Gasoline) with calibrated DAFs

$$\begin{aligned} &pdf(C_1; \mathrm{EV}) = 0.95, \quad pdf(C_1; \mathrm{HEV}) = 0.80, \quad pdf(C_1; \mathrm{Gasoline}) = 0.05,\\ &pdf(C_2; \mathrm{EV}) = 0.60, \quad pdf(C_2; \mathrm{HEV}) = 0.70, \quad pdf(C_2; \mathrm{Gasoline}) = 0.20,\\ &pdf(C_3; \mathrm{EV}) = 0.20, \quad pdf(C_3; \mathrm{HEV}) = 0.40, \quad pdf(C_3; \mathrm{Gasoline}) = 0.85. \end{aligned}$$

Numerical mixing (all steps shown):

- C_1:

$$\begin{aligned} a = \mathrm{EV} &: c = 0,\; \mu_{\mathrm{EV}}^\star(C_1) = (1-0)\min(0.95, 0.95) + 0 \cdot \max(0.95, 0.95) = 0.95,\\ a = \mathrm{HEV} &: c = 0.30,\; \min = 0.80,\; \max = 0.95 \;\Rightarrow\; \mu_{\mathrm{HEV}}^\star(C_1) = 0.70 \cdot 0.80 + 0.30 \cdot 0.95 = 0.845,\\ a = \mathrm{Gasoline} &: c = 0.90,\; \min = 0.05,\; \max = 0.95 \;\Rightarrow\; \mu_{\mathrm{Gas}}^\star(C_1) = 0.10 \cdot 0.05 + 0.90 \cdot 0.95 = 0.860,\\ &\Rightarrow\; \mu^\star(C_1; \mathrm{EV}) = \frac{0.95 + 0.845 + 0.860}{3} = 0.885. \end{aligned}$$

- C_2:

$$\begin{aligned} a = \mathrm{EV} &: c = 0,\; \mu_{\mathrm{EV}}^\star(C_2) = 0.60,\\ a = \mathrm{HEV} &: c = 0.30,\; \min = 0.60,\; \max = 0.70 \;\Rightarrow\; \mu_{\mathrm{HEV}}^\star(C_2) = 0.70 \cdot 0.60 + 0.30 \cdot 0.70 = 0.630,\\ a = \mathrm{Gasoline} &: c = 0.90,\; \min = 0.20,\; \max = 0.60 \;\Rightarrow\; \mu_{\mathrm{Gas}}^\star(C_2) = 0.10 \cdot 0.20 + 0.90 \cdot 0.60 = 0.560,\\ &\Rightarrow\; \mu^\star(C_2; \mathrm{EV}) = \frac{0.60 + 0.630 + 0.560}{3} = 0.597 \text{ (rounded)}. \end{aligned}$$

- C_3:

$$\begin{aligned} a = \mathrm{EV} &: c = 0,\; \mu_{\mathrm{EV}}^\star(C_3) = 0.20,\\ a = \mathrm{HEV} &: c = 0.30,\; \min = 0.20,\; \max = 0.40 \;\Rightarrow\; \mu_{\mathrm{HEV}}^\star(C_3) = 0.70 \cdot 0.20 + 0.30 \cdot 0.40 = 0.260,\\ a = \mathrm{Gasoline} &: c = 0.90,\; \min = 0.20,\; \max = 0.85 \;\Rightarrow\; \mu_{\mathrm{Gas}}^\star(C_3) = 0.10 \cdot 0.20 + 0.90 \cdot 0.85 = 0.785,\\ &\Rightarrow\; \mu^\star(C_3; \mathrm{EV}) = \frac{0.20 + 0.260 + 0.785}{3} = 0.415. \end{aligned}$$

Thus, under the EV-dominant semantics, the aggregated memberships are $\mu^\star(C_1) = 0.885 > \mu^\star(C_2) = 0.597 > \mu^\star(C_3) = 0.415$, ranking cars for the green fleet accordingly.

Example 2.2.3 (Menu planning for a vegan event under multiple dietary poles)**.** Let S be the set of restaurants and $P \subseteq S$ those considered for a vegan event. Choose the attribute v = diet with

$$Pv = \{\mathrm{Vegan},\; \mathrm{Vegetarian},\; \mathrm{Omnivore},\; \mathrm{GlutenFree}\}.$$

Set $s = t = 1$. Take the symmetric DCF (only the Vegan row needed):

$$pCF(\mathrm{Vegan}, \mathrm{Vegan}) = 0, \quad pCF(\mathrm{Vegan}, \mathrm{Vegetarian}) = 0.25,$$

$$pCF(\mathrm{Vegan}, \mathrm{Omnivore}) = 0.95, \quad pCF(\mathrm{Vegan}, \mathrm{GlutenFree}) = 0.10.$$

Use T = min, S = max, dominant value $u_\star$ = Vegan, and the same mixing/averaging as in the previous example.

Consider three restaurants with calibrated DAFs:

$$R_1: \ pdf(\cdot;\text{Vegan}) = 0.95,\ pdf(\cdot;\text{Vegetarian}) = 0.90,\ pdf(\cdot;\text{Omnivore}) = 0.40,\ pdf(\cdot;\text{GF}) = 0.85,$$
$$R_2: \ pdf(\cdot;\text{Vegan}) = 0.55,\ pdf(\cdot;\text{Vegetarian}) = 0.80,\ pdf(\cdot;\text{Omnivore}) = 0.85,\ pdf(\cdot;\text{GF}) = 0.60,$$
$$R_3: \ pdf(\cdot;\text{Vegan}) = 0.15,\ pdf(\cdot;\text{Vegetarian}) = 0.40,\ pdf(\cdot;\text{Omnivore}) = 0.95,\ pdf(\cdot;\text{GF}) = 0.30.$$

Detailed mixing (dominant Vegan):

- R_1:

$$\begin{aligned} a = \text{Vegan} &: c = 0 \Rightarrow 0.95,\\ a = \text{Vegetarian} &: c = 0.25,\ \min = 0.90,\ \max = 0.95 \Rightarrow 0.75\cdot 0.90 + 0.25\cdot 0.95 = 0.9125,\\ a = \text{Omnivore} &: c = 0.95,\ \min = 0.40,\ \max = 0.95 \Rightarrow 0.05\cdot 0.40 + 0.95\cdot 0.95 = 0.9225,\\ a = \text{GF} &: c = 0.10,\ \min = 0.85,\ \max = 0.95 \Rightarrow 0.90\cdot 0.85 + 0.10\cdot 0.95 = 0.860,\\ &\Rightarrow\ \mu^\star(R_1;\text{Vegan}) = \tfrac{0.95+0.9125+0.9225+0.860}{4} = 0.91125\ (\approx 0.911). \end{aligned}$$

- R_2:

$$\begin{aligned} &a = \text{Vegan}: 0.55, \quad a = \text{Vegetarian}: 0.75\cdot 0.55 + 0.25\cdot 0.80 = 0.6125,\\ &a = \text{Omnivore}: 0.05\cdot 0.55 + 0.95\cdot 0.85 = 0.835, \quad a = \text{GF}: 0.90\cdot 0.55 + 0.10\cdot 0.60 = 0.555,\\ &\Rightarrow\ \mu^\star(R_2;\text{Vegan}) = \tfrac{0.55+0.6125+0.835+0.555}{4} = 0.638125\ (\approx 0.638). \end{aligned}$$

- R_3:

$$\begin{aligned} &a = \text{Vegan}: 0.15, \quad a = \text{Vegetarian}: 0.75\cdot 0.15 + 0.25\cdot 0.40 = 0.2125,\\ &a = \text{Omnivore}: 0.05\cdot 0.15 + 0.95\cdot 0.95 = 0.910, \quad a = \text{GF}: 0.90\cdot 0.15 + 0.10\cdot 0.30 = 0.165,\\ &\Rightarrow\ \mu^\star(R_3;\text{Vegan}) = \tfrac{0.15+0.2125+0.910+0.165}{4} = 0.359375\ (\approx 0.359). \end{aligned}$$

Hence the plithogenic aggregation (with dominant "Vegan") yields the ranking R_1 ($\approx$ 0.911) $>$ R_2 ($\approx$ 0.638) $>$ R_3 ($\approx$ 0.359) for a vegan event.

2.3 Single-valued Neutrosophic Graph

A graph is a finite structure of vertices and edges modeling pairwise relationships between objects in networks and discrete systems [62, 63]. A Single-Valued Neutrosophic Graph assigns truth, indeterminacy, and falsity degrees to vertices and edges, extending classical graphs [55, 64–66]. A Neutrosophic Graph is known to generalize Fuzzy Graphs [2, 3, 67], Intuitionistic Fuzzy Graphs [68], and Vague Graphs [69–71].

Definition 2.3.1 (Single-Valued Neutrosophic Graph)**.** [55] Let $G^* = (V, E)$ be a crisp (classical) graph, where V is the vertex set and $E \subseteq V \times V$ the edge set. A *single-valued neutrosophic graph* (SVNG) on G^* is defined as a pair

$$G = (VER, ED),$$

where

- $A = \{\langle v, T_{VER}(v), I_{VER}(v), F_{VER}(v)\rangle : v \in V\}$ is the *single-valued neutrosophic vertex set*, with

  $$T_{VER}, I_{VER}, F_{VER} : V \to [0, 1],$$

  denoting respectively the *truth-membership*, *indeterminacy-membership*, and *falsity-membership* functions of vertices, such that for every $v \in V$,

  $$0 \le T_{VER}(v) + I_{VER}(v) + F_{VER}(v) \le 3.$$

- $B = \{\langle uv, T_{ED}(uv), I_{ED}(uv), F_{ED}(uv)\rangle : uv \in E\}$ is the *single-valued neutrosophic edge set*, with

$$T_{ED}, I_{ED}, F_{ED} : E \to [0,1],$$

  satisfying for all $u, v \in V$ with $uv \in E$:

$$\begin{aligned} T_{ED}(uv) &\le \min\{T_{VER}(u), T_{VER}(v)\}, \\ I_{ED}(uv) &\le \min\{I_{VER}(u), I_{VER}(v)\}, \\ F_{ED}(uv) &\ge \max\{F_{VER}(u), F_{VER}(v)\}. \end{aligned}$$

If B is symmetric, $G = (VER, ED)$ is called an *undirected SVNG*; otherwise, it is a *directed SVNG*.

Example 2.3.2 (SVNG for rumor verification on a social network)**.** Consider a small messaging network where each account can post crisis-related updates and forward others' posts. Vertices represent user accounts, while edges represent active communication ties used to propagate messages. For each vertex and edge we assign triples (truth, indeterminacy, falsity) in $[0,1]$ that quantify reliability, uncertainty, and misleading tendency observed by moderators.

Vertices (accounts).

$$VER = \{\langle A, 0.90, 0.10, 0.00\rangle, \langle B, 0.70, 0.20, 0.10\rangle, \langle C, 0.60, 0.30, 0.20\rangle\}.$$

Edges (message-passing ties).

$$ED = \{\langle AB, 0.65, 0.08, 0.12\rangle, \langle AC, 0.55, 0.07, 0.22\rangle, \langle BC, 0.58, 0.15, 0.25\rangle\}.$$

These assignments satisfy the SVNG constraints for every edge uv:

$$T_{ED}(uv) \le \min\{T_{VER}(u), T_{VER}(v)\},$$

$$I_{ED}(uv) \le \min\{I_{VER}(u), I_{VER}(v)\},$$

$$F_{ED}(uv) \ge \max\{F_{VER}(u), F_{VER}(v)\}.$$

For instance, for AB one has

$$0.65 \le \min(0.90, 0.70) = 0.70, \quad 0.08 \le \min(0.10, 0.20) = 0.10, \quad 0.12 \ge \max(0.00, 0.10) = 0.10.$$

Similarly, for AC:

$$0.55 \le 0.60, \quad 0.07 \le 0.10, \quad 0.22 \ge 0.20,$$

and for BC:

$$0.58 \le 0.60, \quad 0.15 \le 0.20, \quad 0.25 \ge 0.20.$$

Account A is highly reliable with little uncertainty or falsity; B is moderate; C is less reliable and more uncertain. An edge's triple encodes the quality of information flow along that tie: for example, AC has higher falsity (0.22) due to observed misforwarding, even though its truth (0.55) remains bounded by the less reliable endpoint. Moderators can prioritize verification by following high-truth/low-falsity pathways and flagging edges with elevated falsity for additional checks.

2.4 n–Refined Neutrosophic Logic

n-Refined Neutrosophic Logic splits truth, indeterminacy, and falsity into multiple subcomponents, enabling granular reasoning under heterogeneous evidence with priorities assigned [25,72–74]. n–Refined Neutrosophic Logic is a generalized framework that extends both Neutrosophic Logic and Multi-Neutrosophic Logic.

Definition 2.4.1 (n–Refined Neutrosophic Logic (n-RNL))**.** [25, 75, 76] Fix integers $p, r, s \geq 1$ and set $n := p + r + s$. An *n–refined neutrosophic truth value* is a vector

$$\mathbf{v} = \Big(T_1, \dots, T_p \;\Big|\; I_1, \dots, I_r \;\Big|\; F_1, \dots, F_s\Big), \qquad T_j, I_k, F_\ell \in [0, 1],$$

where the T–components encode refined kinds of truth, the I–components encode refined kinds of indeterminacy, and the F–components encode refined kinds of falsity; the total dimension is $n = p + r + s$. In the standard (numerical) setting one may assume

$$0 \leq \sum_{j=1}^{p} T_j + \sum_{k=1}^{r} I_k + \sum_{\ell=1}^{s} F_\ell \leq n,$$

while in philosophical/nonstandard variants each component may range in the nonstandard unit interval, with the same n–dimensional refinement principle. The splitting of (T, I, F) into $(T_1, \dots, T_p)$, $(I_1, \dots, I_r)$, $(F_1, \dots, F_s)$ (with $p + r + s = n$) is the hallmark of n–refinement.

A (propositional) *n–refined neutrosophic valuation* on a language $\mathcal{L}$ is a map

$$\mathrm{Val} : \mathrm{Form}(\mathcal{L}) \longrightarrow [0, 1]^n, \qquad \varphi \longmapsto \mathrm{Val}(\varphi) = \big(\mathbf{T}(\varphi) \mid \mathbf{I}(\varphi) \mid \mathbf{F}(\varphi)\big),$$

equipped with the following connectives defined componentwise from a chosen fuzzy t–norm $t : [0, 1]^2 \to [0, 1]$ and its dual t–conorm $s : [0, 1]^2 \to [0, 1]$ (both associative, commutative, and monotone).

Conjunction (n–norm). The neutrosophic conjunction $\varphi \wedge_n \psi$ is

$$\begin{aligned}
\mathbf{T}(\varphi \wedge_n \psi) &= t\text{-wise combine } \big(T_1(\varphi), \dots, T_p(\varphi)\big) \text{ with } \big(T_1(\psi), \dots, T_p(\psi)\big),\\
\mathbf{I}(\varphi \wedge_n \psi) &= s\text{-wise combine } \big(I_1(\varphi), \dots, I_r(\varphi)\big) \text{ with } \big(I_1(\psi), \dots, I_r(\psi)\big),\\
\mathbf{F}(\varphi \wedge_n \psi) &= s\text{-wise combine } \big(F_1(\varphi), \dots, F_s(\varphi)\big) \text{ with } \big(F_1(\psi), \dots, F_s(\psi)\big).
\end{aligned}$$

Equivalently, T uses a t–norm, while both I and F use a t–conorm.

Disjunction (n–conorm). The neutrosophic disjunction $\varphi \vee_n \psi$ is

$$\begin{aligned}
\mathbf{T}(\varphi \vee_n \psi) &= s\text{-wise combine the } T\text{–blocks},\\
\mathbf{I}(\varphi \vee_n \psi) &= t\text{-wise combine the } I\text{–blocks},\\
\mathbf{F}(\varphi \vee_n \psi) &= t\text{-wise combine the } F\text{–blocks}.
\end{aligned}$$

Equivalently, T uses a t–conorm, while both I and F use a t–norm.

Negation. A basic (involutive) neutrosophic negation swaps truth and falsity blocks and leaves indeterminacy as is:

$$\neg_n \varphi : \big(\mathbf{T}(\varphi) \mid \mathbf{I}(\varphi) \mid \mathbf{F}(\varphi)\big) \longmapsto \big(\mathbf{F}(\varphi) \mid \mathbf{I}(\varphi) \mid \mathbf{T}(\varphi)\big),$$

with optional numeric complementation $x \mapsto 1 - x$ applied componentwise when a strong negation is desired.

Priority (pessimistic/optimistic) variants. When needed, *priority* orderings among the refined subcomponents (e.g. $T < I < F$ for lower–bound n–norm, or $T > I > F$ for upper–bound n–conorm) specialize the above mixers; in the unrefined triplet case this yields the familiar closed forms for $\wedge_n$ and $\vee_n$.

Example 2.4.2 (n–Refined neutrosophic reasoning for heatwave response)**.** Heatwave response is a coordinated public-health and emergency strategy to protect people from extreme heat through alerts, cooling centers, services.

A city emergency team evaluates two actionable propositions during a heatwave:

$$\varphi := \text{“Issue a citywide heat advisory now”}, \qquad \psi := \text{“Open cooling centers immediately”}.$$

Use an n–refined scheme with $p = 2$ truth components, $r = 1$ indeterminacy component, and $s = 2$ falsity components (so $n = 5$):

$$\mathbf{v} = (T_1, T_2 \mid I_1 \mid F_1, F_2) \in [0,1]^5.$$

Interpretation of components: T_1 = forecast model confidence; T_2 = hospital intake trend signal; I_1 = data latency/coverage gaps; F_1 = contradicting rain probability; F_2 = suspected sensor bias.

Valuations. Assign observed values:

$$\varphi : \ (0.82,\ 0.74 \mid 0.18 \mid 0.12,\ 0.09),$$
$$\psi : \ (0.78,\ 0.69 \mid 0.22 \mid 0.15,\ 0.11).$$

Adopt the standard numeric mixers: t–norm = min (for conjunction blocks on T and for disjunction blocks on I, F as needed), and t–conorm = max (dually).

Conjunction $\varphi \wedge_n \psi$ (“do both”):

$$\begin{aligned}
\mathbf{T}(\varphi \wedge_n \psi) &= (\min(0.82, 0.78),\ \min(0.74, 0.69)) = (0.78,\ 0.69),\\
\mathbf{I}(\varphi \wedge_n \psi) &= \max(0.18, 0.22) = 0.22,\\
\mathbf{F}(\varphi \wedge_n \psi) &= (\max(0.12, 0.15),\ \max(0.09, 0.11)) = (0.15,\ 0.11).
\end{aligned}$$

Disjunction $\varphi \vee_n \psi$ (“do at least one”):

$$\begin{aligned}
\mathbf{T}(\varphi \vee_n \psi) &= (\max(0.82, 0.78),\ \max(0.74, 0.69)) = (0.82,\ 0.74),\\
\mathbf{I}(\varphi \vee_n \psi) &= \min(0.18, 0.22) = 0.18,\\
\mathbf{F}(\varphi \vee_n \psi) &= (\min(0.12, 0.15),\ \min(0.09, 0.11)) = (0.12,\ 0.09).
\end{aligned}$$

Negation $\neg_n \varphi$ (swap T and F, keep I):

$$\neg_n \varphi : \ (0.82,\ 0.74 \mid 0.18 \mid 0.12,\ 0.09) \ \mapsto \ (0.12,\ 0.09 \mid 0.18 \mid 0.82,\ 0.74).$$

The joint action $\varphi \wedge_n \psi$ preserves the weaker truth signals while accumulating uncertainty and contradictory factors, reflecting the operational risk of executing both measures at once. The disjunction $\varphi \vee_n \psi$ retains the stronger truth signals and reduces uncertainty/falsity via conservative (min) aggregation on I, F, supporting a phased policy where at least one protective measure is initiated promptly.

2.5 Neutrosophic Triplet

A neutrosophic triplet denotes an element, its neutralizer, and its anti-element, modeling identity-like action, opposites, and reconciliation within algebraic structures [77–82].

Definition 2.5.1 (Neutrosophic Triplet)**.** [81] Let $(S, \star)$ be a nonempty set endowed with a binary operation $\star$. For each $x \in S$, suppose there are two distinguished elements in S:

$$\mathrm{anti}(x) \text{ (the \textit{anti}-element of } x), \qquad \mathrm{neut}(x) \text{ (the \textit{neutralizer} of } x).$$

The ordered triple

$$\langle x,\ \mathrm{neut}(x),\ \mathrm{anti}(x) \rangle$$

is called a *neutrosophic triplet* if the following axioms hold.

1. Involution of the anti-element:
$$\mathrm{anti}(\mathrm{anti}(x)) = x.$$

2. Neutralizing behavior (local identity for x):
$$\mathrm{neut}(x) \star x \;=\; x \star \mathrm{neut}(x) \;=\; x.$$

3. Mutual annihilation to the neutralizer:
$$x \star \mathrm{anti}(x) \;=\; \mathrm{anti}(x) \star x \;=\; \mathrm{neut}(x).$$

4. Consistency on the generated pair:
$$\mathrm{neut}(\mathrm{anti}(x)) \;=\; \mathrm{neut}(x).$$

When these conditions are satisfied, we say that x together with its $\mathrm{neut}(x)$ and $\mathrm{anti}(x)$ *forms a neutrosophic triplet*.

Example 2.5.2 (A concrete neutrosophic triplet in $(\mathbb{Z}, +)$)**.** Let $S = \mathbb{Z}$ with the binary operation $\star = +$. For each $x \in \mathbb{Z}$, define the anti-element and neutralizer by
$$\mathrm{anti}(x) = -x, \qquad \mathrm{neut}(x) = 0.$$

We verify the axioms:

(1) Involution:
$$\mathrm{anti}(\mathrm{anti}(x)) = \mathrm{anti}(-x) = -(-x) = x.$$

(2) Neutralizing behavior:
$$\mathrm{neut}(x) \star x = 0 + x = x, \qquad x \star \mathrm{neut}(x) = x + 0 = x.$$

(3) Mutual annihilation to the neutralizer:
$$x \star \mathrm{anti}(x) = x + (-x) = 0 = \mathrm{neut}(x), \quad \mathrm{anti}(x) \star x = (-x) + x = 0 = \mathrm{neut}(x).$$

(4) Consistency on the generated pair:
$$\mathrm{neut}(\mathrm{anti}(x)) = \mathrm{neut}(-x) = 0 = \mathrm{neut}(x).$$

Hence $\langle x, \mathrm{neut}(x), \mathrm{anti}(x)\rangle = \langle x, 0, -x\rangle$ is a neutrosophic triplet in $(\mathbb{Z}, +)$ for every $x \in \mathbb{Z}$.

Definition 2.5.3 (Neutrosophic Triplet Group (basic version))**.** [81] A structure $(S, \star; \mathrm{anti}, \mathrm{neut})$ is a *neutrosophic triplet group* if:

1. $\star$ is associative on S;
2. for every $x \in S$, the triple $\langle x, \mathrm{neut}(x), \mathrm{anti}(x)\rangle$ satisfies the neutrosophic triplet axioms above;
3. anti and neut are compatible with $\star$ in the sense that
$$\mathrm{anti}(x \star y) = \mathrm{anti}(y) \star \mathrm{anti}(x), \qquad \mathrm{neut}(x \star y) = \mathrm{neut}(x) \star \mathrm{neut}(y), \quad \text{for all } x, y \in S.$$

Example 2.5.4 (A neutrosophic triplet group on $GL_2(\mathbb{R})$)**.** Let $S = GL_2(\mathbb{R})$ be the group of all invertible 2×2 real matrices with the operation $\star = \cdot$ (matrix multiplication). For each $A \in S$, define

$$\mathrm{anti}(A) = A^{-1}, \qquad \mathrm{neut}(A) = I_2,$$

where I_2 is the 2×2 identity matrix.

Associativity. Matrix multiplication is associative on S.

Triplet axioms for each $A \in S$.

$$\begin{aligned}
\text{(1) Involution:}\quad & \mathrm{anti}(\mathrm{anti}(A)) = (A^{-1})^{-1} = A.\\
\text{(2) Neutralizing behavior:}\quad & \mathrm{neut}(A)\cdot A = I_2A = A, \quad A\cdot \mathrm{neut}(A) = AI_2 = A.\\
\text{(3) Mutual annihilation:}\quad & A\cdot \mathrm{anti}(A) = AA^{-1} = I_2 = \mathrm{neut}(A),\\
& \mathrm{anti}(A)\cdot A = A^{-1}A = I_2 = \mathrm{neut}(A).\\
\text{(4) Consistency:}\quad & \mathrm{neut}(\mathrm{anti}(A)) = \mathrm{neut}(A^{-1}) = I_2 = \mathrm{neut}(A).
\end{aligned}$$

Compatibility with $\star$. For all $A, B \in S$,

$$\mathrm{anti}(A\cdot B) = (AB)^{-1} = B^{-1}A^{-1} = \mathrm{anti}(B)\cdot \mathrm{anti}(A), \qquad \mathrm{neut}(A\cdot B) = I_2 = I_2\cdot I_2 = \mathrm{neut}(A)\cdot \mathrm{neut}(B).$$

Therefore $\big(GL_2(\mathbb{R}), \cdot; \mathrm{anti}, \mathrm{neut}\big)$ is a neutrosophic triplet group.

Concrete numeric instance. Let

$$A = \begin{pmatrix} 2 & 0\\ 0 & 3\end{pmatrix}, \qquad \mathrm{anti}(A) = A^{-1} = \begin{pmatrix} \frac{1}{2} & 0\\ 0 & \frac{1}{3}\end{pmatrix}, \qquad \mathrm{neut}(A) = I_2 = \begin{pmatrix} 1 & 0\\ 0 & 1\end{pmatrix}.$$

Then

$$A\cdot \mathrm{anti}(A) = \begin{pmatrix} 2 & 0\\ 0 & 3\end{pmatrix}\begin{pmatrix} \frac{1}{2} & 0\\ 0 & \frac{1}{3}\end{pmatrix} = \begin{pmatrix} 1 & 0\\ 0 & 1\end{pmatrix} = \mathrm{neut}(A), \qquad \mathrm{neut}(A)\cdot A = I_2A = A,$$

and

$$\mathrm{anti}(\mathrm{anti}(A)) = (A^{-1})^{-1} = A, \qquad \mathrm{neut}(\mathrm{anti}(A)) = \mathrm{neut}(A^{-1}) = I_2 = \mathrm{neut}(A).$$

All required equalities hold explicitly.

2.6 Soft Sets, Hypersoft Sets, and SuperHyperSoft Sets

A soft set offers a lightweight, parameter–driven framework for decision modeling: it assigns, to each selected attribute (parameter), a subset of a fixed universe, thereby handling uncertainty in a direct and interpretable way [83–85]. Building on this idea, a hypersoft set maps *tuples* of attribute values—drawn from multiple attribute domains—to subsets of the universe, which enables multi–attribute reasoning [86–88]. Pushing the generalization further, a SuperHyperSoft set maps *combinations* (subsets) of attribute values—one subset per attribute—to subsets of the universe, thus supporting higher–dimensional analyses and intricate inter–attribute relations [89–91].

Definition 2.6.1 (Soft Set)**.** [83,84] Let U be a universe and A a set of attributes (parameters). A *soft set* over U is a pair $(\mathcal{F}, S)$ where $S \subseteq A$ and $\mathcal{F} : S \to \mathcal{P}(U)$. Equivalently,

$$(\mathcal{F}, S) \;=\; \big\{ (\alpha, \mathcal{F}(\alpha)) \;\big|\; \alpha \in S,\ \mathcal{F}(\alpha) \in \mathcal{P}(U) \big\}.$$

Here, each $\alpha \in S$ is a parameter and $\mathcal{F}(\alpha)$ collects those elements of U that are compatible with α.

Definition 2.6.2 (Hypersoft Set)**.** [86] Let U be a universe, and let $\mathcal{A}_1, \ldots, \mathcal{A}_m$ be attribute domains. Set $C := \mathcal{A}_1 \times \cdots \times \mathcal{A}_m$. A *hypersoft set* over U is a pair (G, C) with $G : C \to \mathcal{P}(U)$, i.e.,

$$(G, C) \;=\; \big\{ (\gamma, G(\gamma)) \;\big|\; \gamma = (\gamma_1, \ldots, \gamma_m) \in C,\ G(\gamma) \in \mathcal{P}(U) \big\}.$$

For $\gamma = (\gamma_1, \ldots, \gamma_m)$, the set $G(\gamma)$ consists of those elements of U consistent with the *joint* choice of attribute values $\gamma_1, \ldots, \gamma_m$.

Example 2.6.3 (Hypersoft Set — laptop configuration lookup)**.** Laptop configuration defines a computer's hardware and software setup, including processor, memory, storage, graphics, and operating system choices that optimise performance (cf. [92]).

Let the universe of laptops be

$$U = \{L_1, L_2, L_3, L_4, L_5, L_6\}.$$

Attributes and their value domains are

$$\mathcal{A}_1 = \{\text{i5}, \text{i7}\}, \quad \mathcal{A}_2 = \{8, 16\} \text{ (GB RAM)}, \quad \mathcal{A}_3 = \{256, 512\} \text{ (GB SSD)},$$

and the Cartesian catalog $C := \mathcal{A}_1 \times \mathcal{A}_2 \times \mathcal{A}_3$. Each laptop has a unique configuration:

$$\begin{array}{lll} L_1 : (\text{i5}, 8, 256), & L_2 : (\text{i5}, 16, 512), & L_3 : (\text{i7}, 16, 512), \\ L_4 : (\text{i7}, 8, 256), & L_5 : (\text{i7}, 16, 256), & L_6 : (\text{i5}, 8, 512). \end{array}$$

Define $G : C \to \mathcal{P}(U)$ by sending each configuration to the set of laptops realizing it (joint match on all attributes). For example,

$$\begin{array}{lll} G(\text{i5}, 8, 256) = \{L_1\}, & G(\text{i5}, 16, 512) = \{L_2\}, & G(\text{i7}, 16, 512) = \{L_3\}, \\ G(\text{i7}, 8, 256) = \{L_4\}, & G(\text{i7}, 16, 256) = \{L_5\}, & G(\text{i5}, 8, 512) = \{L_6\}. \end{array}$$

Thus (G, C) is a hypersoft set over U that retrieves all models consistent with a *joint* attribute-value choice.

Example 2.6.4 (Hypersoft Set — emergency department triage policy)**.** Emergency department triage prioritizes arriving patients by illness severity and urgency to allocate limited medical resources and timely treatment effectively (cf. [93]).

Let $U = \{\text{H}, \text{O}, \text{A}\}$ with care plans Home-care (H), Observation (O), Admission (A). Attribute domains are

$$\mathcal{A}_1 = \{\text{mild}, \text{moderate}, \text{severe}\}, \quad \mathcal{A}_2 = \{\text{low}, \text{high}\}, \quad \mathcal{A}_3 = \{\text{scarce}, \text{normal}\},$$

and $C := \mathcal{A}_1 \times \mathcal{A}_2 \times \mathcal{A}_3$. Define $G : C \to \mathcal{P}(U)$ by clinical policy:

$$\begin{array}{rr} G(\text{mild}, \text{low}, \text{normal}) = \{\text{H}, \text{O}\}, & G(\text{moderate}, \text{low}, \text{normal}) = \{\text{O}\}, \\ G(\text{severe}, \text{high}, \text{scarce}) = \{\text{A}\}, & G(\text{moderate}, \text{high}, \text{scarce}) = \{\text{O}, \text{A}\}, \\ G(\text{severe}, \text{high}, \text{normal}) = \{\text{A}\}, & G(\text{mild}, \text{high}, \text{normal}) = \{\text{O}\}. \end{array}$$

Hence (G, C) maps each precise tuple $(\text{severity}, \text{risk}, \text{resource})$ to admissible care plans.

Definition 2.6.5 (SuperHyperSoft Set)**.** [94–96] Let U be a universe and $a_1, \ldots, a_n$ be distinct attributes with value sets $A_1, \ldots, A_n$ (assume $A_i \cap A_j = \varnothing$ for $i \neq j$). Consider the product of power sets

$$C \ := \ \mathcal{P}(A_1) \times \cdots \times \mathcal{P}(A_n).$$

A *SuperHyperSoft set* over U is a pair (F, C) where $F : C \to \mathcal{P}(U)$, namely

$$(F, C) \ = \ \left\{ (\gamma, F(\gamma)) \ \middle| \ \gamma = (\alpha_1, \ldots, \alpha_n) \in C, \ \alpha_i \in \mathcal{P}(A_i), \ F(\gamma) \in \mathcal{P}(U) \right\}.$$

For $\gamma = (\alpha_1, \ldots, \alpha_n)$, the set $F(\gamma)$ aggregates those elements of U compatible with the *selected subsets* $\alpha_i \subseteq A_i$ of admissible values for each attribute a_i.

Example 2.6.6 (SuperHyperSoft Set — hiring shortlist by admissible subsets)**.** Let the candidate universe be

$$U = \{c_1, c_2, c_3, c_4, c_5\},$$

with single-valued attributes recorded per candidate:

$$\begin{array}{l} \text{Primary skill } S = \{\text{Python}, \text{SQL}, \text{ML}, \text{NLP}\}, \\ \text{Degree } D = \{\text{BSc}, \text{MSc}, \text{PhD}\}, \quad \text{Region } R = \{\text{JP}, \text{US}, \text{EU}\}. \end{array}$$

Profiles:

$$\begin{array}{lll} c_1 : (\text{Python}, \text{MSc}, \text{JP}), & c_2 : (\text{ML}, \text{PhD}, \text{US}), & c_3 : (\text{SQL}, \text{BSc}, \text{JP}), \\ c_4 : (\text{NLP}, \text{MSc}, \text{EU}), & c_5 : (\text{Python}, \text{BSc}, \text{US}). & \end{array}$$

Let $C = \mathcal{P}(S) \times \mathcal{P}(D) \times \mathcal{P}(R)$. Define $F : C \to \mathcal{P}(U)$ by

$$F(\alpha_S, \alpha_D, \alpha_R) = \{\, c \in U : \; S(c) \in \alpha_S, \; D(c) \in \alpha_D, \; R(c) \in \alpha_R \,\}.$$

For subset selections

$$\gamma_1 = (\{\text{Python}, \text{ML}\}, \{\text{MSc}, \text{PhD}\}, \{\text{JP}, \text{US}\}), \quad \gamma_2 = (\{\text{SQL}\}, \{\text{BSc}\}, \{\text{JP}\}),$$

we obtain

$$F(\gamma_1) = \{c_1, c_2\}, \qquad F(\gamma_2) = \{c_3\}.$$

Thus (F, C) is a superhypersoft set selecting candidates via *subset-admissible* attribute constraints.

Example 2.6.7 (SuperHyperSoft Set — supplier filtering by certifications, regions, categories)**.** Let $U = \{v_1, v_2, v_3, v_4\}$ denote suppliers. Attributes:

$$C = \{\text{ISO9001}, \text{ISO27001}\}, \quad Z = \{\text{APAC}, \text{EMEA}, \text{AMER}\}, \quad G = \{\text{Electronics}, \text{Mechanical}, \text{Logistics}\}.$$

Each supplier has a single value per attribute:

$$\begin{aligned} &v_1 : (\text{ISO9001}, \text{APAC}, \text{Electronics}), \quad &v_2 : (\text{ISO27001}, \text{EMEA}, \text{Logistics}), \\ &v_3 : (\text{ISO9001}, \text{AMER}, \text{Mechanical}), \quad &v_4 : (\text{ISO27001}, \text{APAC}, \text{Electronics}). \end{aligned}$$

Let $C = \mathcal{P}(C) \times \mathcal{P}(Z) \times \mathcal{P}(G)$ and define

$$F(\alpha_C, \alpha_Z, \alpha_G) = \{\, v \in U : \; C(v) \in \alpha_C, \; Z(v) \in \alpha_Z, \; G(v) \in \alpha_G \,\}.$$

For

$$\gamma_1 = (\{\text{ISO9001}, \text{ISO27001}\}, \{\text{APAC}\}, \{\text{Electronics}\}), \quad \gamma_2 = (\{\text{ISO9001}\}, \{\text{AMER}, \text{EMEA}\}, \{\text{Mechanical}, \text{Logistics}\}),$$

we get

$$F(\gamma_1) = \{v_1, v_4\}, \qquad F(\gamma_2) = \{v_3\}.$$

Therefore (F, C) is a superhypersoft set that returns all suppliers compatible with *chosen subsets* of certifications, regions, and categories.

Chapter 3

Neutrosophic Mathematics

In this chapter, we introduce the mathematical concepts developed by employing the Neutrosophic Set framework. Because of its inherent flexibility, the Neutrosophic Set can be applied to a wide range of mathematical areas, and many of its theoretical and practical applications have been extensively investigated.

3.1 Neutrosophic Multiplicity Function

A neutrosophic multiplicity function records, for each element, counts of occurrences paired with truth, indeterminacy, falsity triples across distinct valuations [41].

Definition 3.1.1 (Neutrosophic Multiplicity Function)**.** Let U be a nonempty universe and let $M \subseteq U$ be a neutrosophic multiset, i.e., a collection in which an element of U may occur several times, possibly with different neutrosophic components.

A neutrosophic multiplicity function of M is a mapping

$$\mathrm{nm}\colon U \longrightarrow \mathcal{P}\big(\mathbb{N}_{\geq 1} \times ([0,1]^3)\big)$$

such that, for every $x \in U$,

$$\mathrm{nm}(x) = \big\{(k_1, (t_1, i_1, f_1)),\ (k_2, (t_2, i_2, f_2)),\ \ldots,\ (k_r, (t_r, i_r, f_r))\big\},$$

where

- $r \in \mathbb{N}_{\geq 0}$ (and $r = 0$ is allowed, in which case $x \notin M$);
- $k_j \in \mathbb{N}_{\geq 1}$ is the number of occurrences of x in M with the neutrosophic component (t_j, i_j, f_j);
- $(t_j, i_j, f_j) \in [0,1]^3$ is the neutrosophic truth–indeterminacy–falsity triple attached to those k_j occurrences;
- for $p \neq q$ we have $(t_p, i_p, f_p) \neq (t_q, i_q, f_q)$ (distinct neutrosophic components are listed separately).

In words, $(k_j, (t_j, i_j, f_j)) \in \mathrm{nm}(x)$ means: "x appears k_j times in M with neutrosophic value (t_j, i_j, f_j)." If $x \notin M$, then $\mathrm{nm}(x) = \varnothing$.

Equivalently,

$$M = \big\{ x^{k_j, (t_j, i_j, f_j)} \ \big|\ x \in U,\ (k_j, (t_j, i_j, f_j)) \in \mathrm{nm}(x) \big\},$$

so that M is completely determined by nm.

Example 3.1.2 (Neutrosophic multiplicity in survey responses)**.** Let $U = \{A, B, C\}$ be three candidate products. A customer-feedback neutrosophic multiset M is summarized by the neutrosophic multiplicity function $\mathrm{nm} : U \to \mathcal{P}(\mathbb{N}_{\geq 1} \times [0, 1]^3)$ defined as

$$\begin{aligned} \mathrm{nm}(A) &= \{(5, (0.9, 0.05, 0.05)),\ (2, (0.7, 0.2, 0.3))\}, \\ \mathrm{nm}(B) &= \{(3, (0.6, 0.3, 0.4))\}, \\ \mathrm{nm}(C) &= \varnothing. \end{aligned}$$

Here, $(5, (0.9, 0.05, 0.05)) \in \mathrm{nm}(A)$ means that product A appears 5 times in M with neutrosophic evaluation "highly satisfactory" $(T, I, F) = (0.9, 0.05, 0.05)$, while $(2, (0.7, 0.2, 0.3))$ records 2 occurrences with a more uncertain assessment. Since $\mathrm{nm}(C) = \varnothing$, product C does not occur in M.

3.2 Refined Neutrosophic Cubic Set

A refined neutrosophic cubic set merges interval memberships and point memberships to represent uncertainty at two levels with consistency faithfully [38]. It is an extended concept of the Neutrosophic Cubic Set [97–100] and the Multi-Neutrosophic Cubic Set, and related notions such as the Quadripartitioned Neutrosophic Cubic Set [101] are also known.

Definition 3.2.1 (Neutrosophic Cubic Set)**.** [98] Let X be a nonempty set. A neutrosophic cubic set (shortly, NCS) in X is an ordered pair

$$\mathcal{A} = (A, \nu),$$

where

- $A = \{\langle x, A_T(x), A_I(x), A_F(x)\rangle \mid x \in X\}$ is an interval neutrosophic set in X, i.e.

  $$A_T(x) = [A_T^-(x), A_T^+(x)], \quad A_I(x) = [A_I^-(x), A_I^+(x)], \quad A_F(x) = [A_F^-(x), A_F^+(x)]$$

  with $0 \leq A_\bullet^-(x) \leq A_\bullet^+(x) \leq 1$ for $\bullet \in \{T, I, F\}$;
- $\nu = \{\langle x, \nu_T(x), \nu_I(x), \nu_F(x)\rangle \mid x \in X\}$ is a (single-valued) neutrosophic set in X, i.e.

  $$\nu_T(x), \nu_I(x), \nu_F(x) \in [0, 1] \qquad (x \in X).$$

Thus, for every $x \in X$, the NCS $\mathcal{A}$ attaches to x both interval-type neutrosophic information $A_\bullet(x)$ and point-type neutrosophic information $\nu_\bullet(x)$.

Example 3.2.2 (Neutrosophic Cubic Set (NCS) in a public-policy setting)**.** A public-policy setting is a governmental or societal context where authorities design, debate, implement, and evaluate collective rules and programs (cf. [102]).

Consider three candidate social programs $X = \{P1, P2, P3\}$ to be rolled out nationally. For each program $x \in X$, the interval part $A_\bullet(x)$ aggregates evidence from multiple sources (field trials, surveys, budget analyses), while the point part $\nu_\bullet(x)$ records the current consensus estimate by the review committee:

$$\mathcal{A} = (A, \nu) \quad \text{with} \quad A = \{\langle x, A_T(x), A_I(x), A_F(x)\rangle\}, \ \ \nu = \{\langle x, \nu_T(x), \nu_I(x), \nu_F(x)\rangle\}.$$

Interval assessments (rows are programs; entries are in $[0, 1]$):

Program	A_T^-	A_T^+	A_I^-	A_I^+	A_F^-	A_F^+
P1	0.62	0.78	0.12	0.20	0.10	0.24
P2	0.40	0.55	0.22	0.33	0.25	0.40
P3	0.80	0.90	0.05	0.12	0.02	0.08

Point (consensus) estimates, chosen within the above intervals:

Program	ν_T	ν_I	ν_F
P1	0.71	0.17	0.12
P2	0.50	0.28	0.30
P3	0.86	0.09	0.05

Interpretation. For P1 the truth interval $[0.62, 0.78]$ summarizes success rates across pilot regions, the indeterminacy interval $[0.12, 0.20]$ reflects data gaps and volatility, and the falsity interval $[0.10, 0.24]$ captures contradictory evidence (e.g., local infeasibility). The committee's current stance $(0.71, 0.17, 0.12)$ lies inside those ranges, providing a concrete instantiation of an NCS for policy feasibility.

Definition 3.2.3 (Refined Neutrosophic Cubic Set)**.** Let X be a nonempty set and let $p, q, r \in \mathbb{N}$ be fixed. A refined neutrosophic cubic set (of type (p, q, r)) in X is an ordered pair

$$C = (\mathcal{A}, \rho),$$

where

- $\mathcal{A} = \{\langle x, \mathbf{A}_T(x), \mathbf{A}_I(x), \mathbf{A}_F(x)\rangle \mid x \in X\}$ is a *refined interval neutrosophic set*, that is,

$$\mathbf{A}_T(x) = \big([T^-_{x,1}, T^+_{x,1}], \dots, [T^-_{x,p}, T^+_{x,p}]\big),$$
$$\mathbf{A}_I(x) = \big([I^-_{x,1}, I^+_{x,1}], \dots, [I^-_{x,q}, I^+_{x,q}]\big),$$
$$\mathbf{A}_F(x) = \big([F^-_{x,1}, F^+_{x,1}], \dots, [F^-_{x,r}, F^+_{x,r}]\big),$$

  with

$$0 \le T^-_{x,j} \le T^+_{x,j} \le 1\ (1 \le j \le p), \quad 0 \le I^-_{x,k} \le I^+_{x,k} \le 1\ (1 \le k \le q), \quad 0 \le F^-_{x,\ell} \le F^+_{x,\ell} \le 1\ (1 \le \ell \le r);$$

- $\rho = \{\langle x, \rho_T(x), \rho_I(x), \rho_F(x)\rangle \mid x \in X\}$ is a *refined neutrosophic set*, i.e.

$$\rho_T(x) = (\tau_{x,1}, \dots, \tau_{x,p}) \in [0,1]^p,$$
$$\rho_I(x) = (\iota_{x,1}, \dots, \iota_{x,q}) \in [0,1]^q,$$
$$\rho_F(x) = (\varphi_{x,1}, \dots, \varphi_{x,r}) \in [0,1]^r.$$

We say that C is *componentwise internal* if, for every $x \in X$,

$$T^-_{x,j} \le \tau_{x,j} \le T^+_{x,j} \quad (1 \le j \le p), \qquad I^-_{x,k} \le \iota_{x,k} \le I^+_{x,k} \quad (1 \le k \le q), \qquad F^-_{x,\ell} \le \varphi_{x,\ell} \le F^+_{x,\ell} \quad (1 \le \ell \le r).$$

In this way, a refined neutrosophic cubic set simultaneously keeps the *interval* neutrosophic information in each refined channel and the *point* (single-valued) neutrosophic information in the corresponding refined channel, generalizing Definition 3.2.1.

Example 3.2.4 (Refined Neutrosophic Cubic Set (RNCS) for multi-channel policy evaluation)**.** Let $X = \{\mathrm{Q1}, \mathrm{Q2}\}$ be two housing policies. Use a refined cubic structure of type $(p, q, r) = (2, 2, 1)$: two truth channels, two indeterminacy channels, and one falsity channel. For every policy x:

$$\mathbf{A}_T(x) = \big([T^-_{x,1}, T^+_{x,1}], [T^-_{x,2}, T^+_{x,2}]\big), \quad \mathbf{A}_I(x) = \big([I^-_{x,1}, I^+_{x,1}], [I^-_{x,2}, I^+_{x,2}]\big), \quad \mathbf{A}_F(x) = \big([F^-_{x,1}, F^+_{x,1}]\big),$$

where channels are defined as follows.

Truth channels: (1) $T_{\cdot,1}$ empirical support (measured outcome improvement); (2) $T_{\cdot,2}$ budgetary feasibility (cost adherence).
Indeterminacy channels: (1) $I_{\cdot,1}$ measurement noise (sampling error); (2) $I_{\cdot,2}$ scenario volatility (macroeconomic swings).
Falsity channel: (1) $F_{\cdot,1}$ contradictory constraints (legal/operational obstacles).

Refined interval assessments:

	$[T_1^-, T_1^+]$	$[T_2^-, T_2^+]$	$[I_1^-, I_1^+]$	$[I_2^-, I_2^+]$	$[F_1^-, F_1^+]$
Q1	$[0.68, 0.82]$	$[0.55, 0.70]$	$[0.10, 0.18]$	$[0.08, 0.15]$	$[0.06, 0.14]$
Q2	$[0.35, 0.50]$	$[0.45, 0.60]$	$[0.20, 0.30]$	$[0.25, 0.35]$	$[0.25, 0.38]$

Refined point (consensus) vectors, each component internal to its interval:

	$(\tau_1, \tau_2) = \rho_T$	$(\iota_1, \iota_2) = \rho_I$	$\varphi_1 = \rho_F$
Q1	$(0.75, 0.62)$	$(0.14, 0.10)$	0.09
Q2	$(0.46, 0.53)$	$(0.24, 0.30)$	0.31

Q1 shows strong empirical gains and acceptable costs, with modest uncertainty and few blocking constraints, hence low refined falsity (0.09). Q2 carries lower empirical support, higher volatility, and more binding constraints (falsity 0.31). This RNCS organizes heterogeneous evidence across channels while retaining both interval summaries and single-valued consensus per channel.

3.3 Refined Neutrosophic Set n-plets

Refined neutrosophic set n-plets assign multiple truth, indeterminacy, and falsity components to each element, enabling granular multi-aspect uncertainty modeling refinement [43].

Definition 3.3.1 (Refined Neutrosophic Set n-plet)**.** [43] Let X be a nonempty universe and let $p, r, s \in \mathbb{N}$ with $p + r + s =: n \geq 3$. A *refined neutrosophic set n-plet A* on X is specified by

$$\mu_A : X \longrightarrow [0,1]^n, \qquad x \longmapsto \big(T_1(x), \dots, T_p(x);\ I_1(x), \dots, I_r(x);\ F_1(x), \dots, F_s(x)\big),$$

where T_i's are refined truth-memberships, I_j's refined indeterminacy-memberships, and F_k's refined falsity-memberships. For every $x \in X$ one requires the neutrosophic bound

$$0 \ \leq \ \sum_{i=1}^{p} T_i(x) \ + \ \sum_{j=1}^{r} I_j(x) \ + \ \sum_{k=1}^{s} F_k(x) \ \leq \ n.$$

Write $A(x) := (\mathbf{T}(x); \mathbf{I}(x); \mathbf{F}(x))$ for brevity.

Typical set-theoretic operations are defined componentwise using fixed t-norm T_* and t-conorm S_* (chosen once and for all):

$$\begin{aligned}
(A \cup B)(x) &= \big(S_*(\mathbf{T}_A(x), \mathbf{T}_B(x));\ S_*(\mathbf{I}_A(x), \mathbf{I}_B(x));\ T_*(\mathbf{F}_A(x), \mathbf{F}_B(x))\big),\\
(A \cap B)(x) &= \big(T_*(\mathbf{T}_A(x), \mathbf{T}_B(x));\ T_*(\mathbf{I}_A(x), \mathbf{I}_B(x));\ S_*(\mathbf{F}_A(x), \mathbf{F}_B(x))\big),\\
A^{\complement}(x) &= \big(\mathbf{F}_A(x);\ \mathbf{I}_A(x)^{\complement};\ \mathbf{T}_A(x)\big),
\end{aligned}$$

where the bold symbols mean componentwise application and, e.g., $\mathbf{I}^{\complement}$ may be taken as $(I_1, \dots, I_r) \mapsto (1 - I_1, \dots, 1 - I_r)$ when appropriate. Inclusion $A \subseteq B$ is defined by componentwise comparison: $\mathbf{T}_A \leq \mathbf{T}_B$, $\mathbf{I}_A \geq \mathbf{I}_B$, and $\mathbf{F}_A \geq \mathbf{F}_B$ pointwise on X.

Remark 3.3.2. The classical ($p = r = s = 1$) neutrosophic set is the special case $n = 3$. Refining increases modeling granularity by splitting each of truth/indeterminacy/falsity into several interpretable subcomponents while preserving the neutrosophic bound.

Example 3.3.3 (Refined neutrosophic set n-plet with p=2, r=1, s=1 (n=4))**.** Let the universe be $X = \{x_1, x_2\}$. Define the refined neutrosophic membership

$$\mu_A(x) \ = \ \big(T_1(x), T_2(x);\ I_1(x);\ F_1(x)\big) \in [0,1]^4.$$

Take

$$\mu_A(x_1) = (0.70,\ 0.40;\ 0.20;\ 0.30), \qquad \mu_A(x_2) = (0.20,\ 0.60;\ 0.10;\ 0.50).$$

Neutrosophic bounds hold componentwise:

$$\sum T_i(x_1) + \sum I_j(x_1) + \sum F_k(x_1) = 0.70 + 0.40 + 0.20 + 0.30 = 1.60 \leq 4,$$

$$\sum T_i(x_2) + \sum I_j(x_2) + \sum F_k(x_2) = 0.20 + 0.60 + 0.10 + 0.50 = 1.40 \leq 4.$$

Here T_1, T_2 could encode two distinct "truth" aspects (e.g., semantic match and stylistic match), while I_1 captures indeterminacy (e.g., missing context), and F_1 captures falsity/contradiction (e.g., policy conflict).

3.4 Plithogenic Random Variable

A random variable is a function assigning numerical values to outcomes of a random experiment, enabling probabilistic analysis and modeling [103,104]. A plithogenic random variable aggregates attribute-conditioned probabilities using contradiction-weighted t-norm/t-conorm fusion, producing contradiction-aware distributions and expectations for multi-attribute evidence integration [48, 105, 106].

Definition 3.4.1 (Plithogenic Random Variable)**.** [48, 105, 106] Let $(\Omega, \mathcal{F})$ be a measurable space. Let V be a finite set of attribute-values with a chosen *dominant* value $v_D \in V$, and let the *degree of contradiction* (a.k.a. dissimilarity)

$$\mathrm{pCF} : V \times V \to [0, 1], \qquad c(v) := \mathrm{pCF}(v, v_D)$$

be given. Assume that for each $v \in V$ we have a (classical) probability measure P_v on $(\Omega, \mathcal{F})$. Fix a t-norm T_* and t-conorm S_* on $[0, 1]$, and define the c-weighted binary aggregator

$$a \,\widetilde{\wedge}_c\, b \;:=\; (1 - c)\, T_*(a, b) \;+\; c\, S_*(a, b) \qquad (a, b \in [0, 1],\; c \in [0, 1]).$$

For a finite family $\{p_v\}_{v \in V} \subset [0, 1]$ we define the *plithogenic fold-aggregation* recursively by

$$\mathrm{Agg}_{\mathrm{pli}}\big(\{(p_v, c(v))\}_{v \in V}\big) \;:=\; p_{v_1} \,\widetilde{\wedge}_{c(v_2)}\, p_{v_2} \,\widetilde{\wedge}_{c(v_3)} \cdots \widetilde{\wedge}_{c(v_m)}\, p_{v_m},$$

for any ordering $(v_1, \ldots, v_m)$ of V (associative choices of T_*, S_* make the value order-independent, e.g. $T_*(a, b) = ab$, $S_*(a, b) = a + b - ab$).

The *plithogenic probability* of an event $A \in \mathcal{F}$ is then defined by

$$P_{\mathrm{pli}}(A) \;:=\; \mathrm{Agg}_{\mathrm{pli}}\big(\{(P_v(A), c(v))\}_{v \in V}\big) \in [0, 1].$$

A measurable map $X : \Omega \to \mathbb{R}$ is called a *plithogenic random variable* if all $\{P_v \circ X^{-1}\}_{v \in V}$ are well-defined probability distributions and its *plithogenic distribution function* is

$$F_X^{\mathrm{pli}}(t) \;:=\; P_{\mathrm{pli}}(\{X \le t\}) = \mathrm{Agg}_{\mathrm{pli}}\big(\{(P_v(X \le t), c(v))\}_{v \in V}\big).$$

Whenever $\mathbb{E}_{P_v}[|X|] < \infty$ for all $v \in V$, we define the *plithogenic expectation* by numeric aggregation of the classical expectations:

$$\mathbb{E}_{\mathrm{pli}}[X] \;:=\; \mathrm{Agg}_{\mathrm{pli}}\big(\{(\mathbb{E}_{P_v}[X], c(v))\}_{v \in V}\big),$$

and analogously for variance/covariance.

Remark 3.4.2. When $c(v) \equiv 0$ (no contradiction), the aggregation reduces to the T_*-based conjunctive model; when $c(v) \equiv 1$, it reduces to the S_*-based disjunctive model. Intermediate $c(v)$ interpolate between them, which is the hallmark of plithogenic integration of multi-attribute evidence. Typical choices are $T_*(a, b) = ab$ and $S_*(a, b) = a + b - ab$.

Example 3.4.3 (Two-source plithogenic probability and expectation)**.** Let $(\Omega, \mathcal{F})$ be a measurable space and $V = \{\mathrm{Expert}, \mathrm{Sensor}\}$ with dominant value $v_D = \mathrm{Expert}$. Set the degree of contradiction (dissimilarity) $c(v) := \mathrm{pCF}(v, v_D)$ by

$$c(\mathrm{Expert}) = 0, \qquad c(\mathrm{Sensor}) = 0.4.$$

Choose $T_*(a, b) = ab$ and $S_*(a, b) = a + b - ab$. For $a, b \in [0, 1]$ and $c \in [0, 1]$ define

$$a \,\widetilde{\wedge}_c\, b \;:=\; (1 - c)\, T_*(a, b) \;+\; c\, S_*(a, b).$$

Consider the event A = "rain tomorrow" with source-wise probabilities

$$P_{\mathrm{Expert}}(A) = 0.70, \qquad P_{\mathrm{Sensor}}(A) = 0.50.$$

The plithogenic probability (folding Expert then Sensor) is

$$\begin{aligned} P_{\mathrm{pli}}(A) &= P_{\mathrm{Expert}}(A) \,\widetilde{\wedge}_{c(\mathrm{Sensor})}\, P_{\mathrm{Sensor}}(A) \\ &= (1 - 0.4)\,(0.70 \cdot 0.50) \;+\; 0.4\,\big(0.70 + 0.50 - 0.70 \cdot 0.50\big) \\ &= 0.6 \cdot 0.35 \;+\; 0.4 \cdot 0.85 \\ &= 0.21 \;+\; 0.34 \;=\; 0.55. \end{aligned}$$

Define a plithogenic random variable $X : \Omega \to \{0, 1\}$ by $X = \mathbf{1}_A$. Since $\mathbb{E}_{P_v}[X] = P_v(A)$ for each $v \in V$, the plithogenic expectation is

$$\mathbb{E}_{\mathrm{pli}}[X] = \mathrm{Agg}_{\mathrm{pli}}\big(\{(\mathbb{E}_{P_v}[X], c(v))\}_{v \in V}\big) = \mathrm{Agg}_{\mathrm{pli}}\big(\{(0.70, 0), (0.50, 0.4)\}\big) = 0.55.$$

Thus contradiction-aware fusion between an expert forecast and a sensor model yields a plithogenic probability and expectation of 0.55 for rain.

3.5 Support–Neutrosophic Set and Support–Plithogenic Set

A support–neutrosophic set assigns to each element truth, indeterminacy, and falsity degrees plus a support weight representing contextual reliability scores [39, 107]. A support–plithogenic set associates each element with attribute-based membership and contradiction functions, augmented by a support factor quantifying credibility contextually.

Definition 3.5.1 (Support–Neutrosophic Set)**.** [107] Let U be a nonempty universe. A support–neutrosophic set (SNS) A on U is a collection

$$A = \{(x,\, T_A(x),\, I_A(x),\, F_A(x),\, s_A(x)) \mid x \in U\},$$

where

$$T_A, I_A, F_A, s_A : U \longrightarrow [0,1]$$

are, respectively, the truth–, indeterminacy–, falsity– and support–membership functions, and they satisfy

$$0 \le T_A(x) \le 1, \quad 0 \le I_A(x) \le 1, \quad 0 \le F_A(x) \le 1, \quad 0 \le s_A(x) \le 1 \quad (\forall x \in U).$$

In the non–restricted form there is no condition on $T_A(x)+I_A(x)+F_A(x)$; in the restricted form one additionally requires

$$T_A(x) + I_A(x) + F_A(x) \le 1 \quad (\forall x \in U).$$

If $s_A(x) \equiv c \in [0,1]$ for all $x \in U$, then A reduces to an ordinary neutrosophic set.

Example 3.5.2 (Support–Neutrosophic Set for civic-issue triage)**.** Let $U = \{\mathrm{r}_1, \mathrm{r}_2, \mathrm{r}_3\}$ be citizen reports of road defects received by a city desk. For each report $x \in U$ we record: $T_A(x)$ (evidence that the defect truly exists), $I_A(x)$ (uncertainty due to poor photos or conflicting notes), $F_A(x)$ (counter-evidence that the claim is false), and $s_A(x)$ (support/credibility from reporter reputation, device metadata, and past accuracy). We use the restricted form $T_A + I_A + F_A \le 1$.

Report	$T_A(x)$	$I_A(x)$	$F_A(x)$	$s_A(x)$
r_1	0.70	0.15	0.10	0.90
r_2	0.40	0.40	0.10	0.35
r_3	0.20	0.20	0.55	0.60

r_1 has strong truth evidence and high support (clear photos, trusted reporter). r_2 is highly indeterminate with low support (blurry images, first-time reporter). r_3 is mostly refuted by recent maintenance logs, with moderate support (location metadata matches). This Support–Neutrosophic Set allows prioritizing site inspections by combining truth/uncertainty/falsity with a separate support weight that captures source reliability.

Definition 3.5.3 (Support–Plithogenic Set)**.** Let P be a nonempty universe, v an attribute, P_v its set of values. A *support–plithogenic set* on P is a sextuple

$$SPS = (P, v, P_v, pdf, pCF, s),$$

where

- $pdf : P \times P_v \to [0,1]$ is the plithogenic degree–of–appurtenance;
- $pCF : P_v \times P_v \to [0,1]$ is the plithogenic contradiction function, with $pCF(\alpha,\alpha) = 0$ and $pCF(\alpha,\beta) = pCF(\beta,\alpha)$;
- $s : P \to [0,1]$ is the *support function*, which rates how credible, available, important or active the element x is inside the considered context.

We write

$$SPS = \left\{ \big(x, \{(\alpha, pdf(x,\alpha))\}_{\alpha \in P_v}, s(x)\big) : x \in P \right\}.$$

Example 3.5.4 (Support–Plithogenic Set for staffing by primary expertise)**.** Let $P = \{\text{Alice}, \text{Bob}, \text{Chen}\}$ be job candidates. Take the plithogenic attribute v = Primary Expertise with value set $P_v = \{\alpha_1 = \text{Backend},\ \alpha_2 = \text{Data},\ \alpha_3 = \text{Security}\}$. For each $x \in P$ and $\alpha \in P_v$, $pdf(x, \alpha) \in [0, 1]$ quantifies the degree that x fits α. The contradiction function $pCF : P_v \times P_v \to [0, 1]$ models semantic opposition between expertise labels (symmetric, with $pCF(\alpha, \alpha) = 0$):

$pCF =$

	Backend	Data	Security
Backend	0.00	0.40	0.60
Data	0.40	0.00	0.50
Security	0.60	0.50	0.00

Define the support function $s : P \to [0, 1]$ from references, code samples, and interview calibration.

	$pdf(\cdot, \text{Backend})$	$pdf(\cdot, \text{Data})$	$pdf(\cdot, \text{Security})$	$s(\cdot)$
Alice	0.70	0.40	0.20	0.85
Bob	0.30	0.80	0.10	0.65
Chen	0.50	0.30	0.70	0.50

Representation as a support–plithogenic set:

$$\left\{ \big(x, \{(\alpha,\ pdf(x, \alpha))\}_{\alpha \in P_v},\ s(x)\big)\ :\ x \in P \right\}.$$

Alice is strongly backend-oriented with high support (solid references, public repos). Bob is predominantly data-focused with medium support. Chen shows strong security skills but lower support (fewer verified artifacts). Decision rules may combine degrees via plithogenic aggregation modulated by pCF, while $s(x)$ adjusts influence by the candidate's evidence strength.

Theorem 3.5.5 (Unification)**.** *Every support–neutrosophic set and every plithogenic set is a particular case of a support–plithogenic set.*

Proof. We prove the two inclusions separately.

(1) From SNS to SPS. Let $A = \{(x, T_A(x), I_A(x), F_A(x), s_A(x)) \mid x \in U\}$ be an SNS. Define

$$P := U, \quad v := \text{“neutrosophic-status”}, \quad P_v := \{T, I, F\}.$$

Define the plithogenic degree by

$$pdf(x, T) := T_A(x), \qquad pdf(x, I) := I_A(x), \qquad pdf(x, F) := F_A(x) \quad (\forall x \in U).$$

Define the contradiction function by

$$pCF(T, F) = pCF(F, T) = 1, \qquad pCF(T, I) = pCF(I, T) = \gamma, \qquad pCF(I, F) = pCF(F, I) = \gamma,$$

for some fixed $\gamma \in [0, 1]$ (for instance $\gamma = \frac{1}{2}$). Finally define the support

$$s(x) := s_A(x) \quad (\forall x \in U).$$

Then

$$SPS_A := (U, v, \{T, I, F\}, pdf, pCF, s)$$

is a support–plithogenic set, and for every $x \in U$ we recover exactly the 4–tuple $(T_A(x), I_A(x), F_A(x), s_A(x))$ inside SPS_A. Thus every SNS is an SPS.

More formally, for each $x \in U$,

$$\underbrace{\big(pdf(x, T), pdf(x, I), pdf(x, F), s(x)\big)}_{SPS_A} = \underbrace{\big(T_A(x), I_A(x), F_A(x), s_A(x)\big)}_{\text{SNS data}},$$

so the identity map on U together with the above choice of v, P_v, pdf, pCF, s is an embedding of SNS into SPS.

(2) From plithogenic set to SPS. Let

$$PS = (P, v, P_v, pdf, pCF)$$

be any plithogenic set. Define

$$s(x) := 1 \quad \text{for all } x \in P.$$

Then

$$SPS_{PS} := (P, v, P_v, pdf, pCF, s)$$

is a support–plithogenic set and clearly

$$SPS_{PS} \text{ with } s \equiv 1 \quad = \quad PS.$$

Hence every plithogenic set is an SPS whose support is constantly 1.

From (1) and (2) we have shown that

$$\{\text{support–neutrosophic sets}\} \subseteq \{\text{support–plithogenic sets}\} \supseteq \{\text{plithogenic sets}\}.$$

Therefore the class of support–plithogenic sets strictly generalizes both support–neutrosophic sets (by re–encoding the three neutrosophic components as attribute values and keeping their support) and ordinary plithogenic sets (by taking support $\equiv 1$). This completes the proof. □

3.6 Complex Neutrosophic Multiset

A Complex Neutrosophic set assigns each element complex-valued truth, indeterminacy, and falsity degrees, capturing magnitude–phase behavior amid uncertainty and inconsistency [108–111]. A Complex Neutrosophic Set can generalize both Complex Fuzzy Sets and Complex Intuitionistic Fuzzy Sets. A Complex Neutrosophic multiset allows repeated elements, each occurrence carrying distinct complex truth, indeterminacy, and falsity degrees for granular modeling [41].

Definition 3.6.1 (Complex Neutrosophic Multiset)**.** [41] Let U be a nonempty universe. Denote by

$$\mathbb{D} := \{z \in \mathbb{C} \mid |z| \leq 1\}$$

the closed unit disc in $\mathbb{C}$. A complex neutrosophic value is a triple

$$(T, I, F) \in \mathbb{D}^3$$

interpreted as complex truth, complex indeterminacy, and complex falsity.

A complex neutrosophic multiset (CNMS) M on U is a mapping

$$\mathrm{cnm}_M : U \longrightarrow \mathcal{P}\big(\mathbb{N}_{\geq 1} \times \mathbb{D}^3\big)$$

such that, for every $x \in U$,

$$\mathrm{cnm}_M(x) = \big\{ (k_1, (T_1, I_1, F_1)), \ldots, (k_r, (T_r, I_r, F_r)) \big\},$$

where

- $r \geq 0$ is an integer; if $r = 0$ then x does not occur in M;
- $k_j \in \mathbb{N}_{\geq 1}$ is the multiplicity of x associated with the j-th complex neutrosophic value;
- $(T_j, I_j, F_j) \in \mathbb{D}^3$ with $|T_j| \leq 1$, $|I_j| \leq 1$, $|F_j| \leq 1$ for all j;
- for $p \neq q$ we require $(T_p, I_p, F_p) \neq (T_q, I_q, F_q)$, so distinct complex neutrosophic triples are listed separately.

Thus $(k_j, (T_j, I_j, F_j)) \in \mathrm{cnm}_M(x)$ means: "the element x occurs k_j times in M with complex neutrosophic description (T_j, I_j, F_j)."

The support of M is

$$\mathrm{supp}(M) := \{ x \in U \mid \mathrm{cnm}_M(x) \neq \varnothing \}.$$

Example 3.6.2 (Smart–Factory Diagnostics as a Complex Neutrosophic Multiset)**.** Smart-factory diagnostics uses interconnected sensors and analytics to monitor equipment, detect anomalies, predict failures, and optimize automated production processes continuously (cf. [112]).

Consider a production line with two candidate fault types:

$$x_1 = \text{“bearing defect at Machine M7”}, \qquad x_2 = \text{“shaft misalignment at Machine M7”}.$$

Let $U = \{x_1, x_2\}$ and interpret complex neutrosophic components as follows. The magnitude of T (truth) reflects the strength of supporting evidence (e.g., spectral peaks), while its phase encodes the dominant cycle/harmonic offset; the magnitude of I (indeterminacy) captures cross–sensor variability, with its phase tagging the prevalent source of uncertainty; the magnitude of F (falsity) summarizes counter–evidence, and its phase indicates the timing alignment of that counter–evidence. All complex values lie in the unit disc.

Define a Complex Neutrosophic Multiset (CNMS) M by the mapping $\mathrm{cnm}_M : U \to \mathcal{P}(\mathbb{N}_{\geq 1} \times \mathbb{D}^3)$ as:

$$\mathrm{cnm}_M(x_1) = \Big\{ \underbrace{\big(5,\ (0.86\, e^{i\,18^\circ},\ 0.22\, e^{i\,150^\circ},\ 0.10\, e^{i\,210^\circ})\big)}_{\text{five windows strongly support a bearing defect}},\ \underbrace{\big(2,\ (0.55\, e^{i\,200^\circ},\ 0.35\, e^{i\,35^\circ},\ 0.40\, e^{i\,10^\circ})\big)}_{\text{two windows show mixed/contradictory evidence}} \Big\},$$

$$\mathrm{cnm}_M(x_2) = \Big\{ \underbrace{\big(3,\ (0.62\, e^{i\,12^\circ},\ 0.28\, e^{i\,170^\circ},\ 0.25\, e^{i\,190^\circ})\big)}_{\text{three windows moderately support misalignment}},\ \underbrace{\big(1,\ (0.20\, e^{i\,75^\circ},\ 0.60\, e^{i\,45^\circ},\ 0.15\, e^{i\,300^\circ})\big)}_{\text{one window is highly uncertain}} \Big\}.$$

Interpretation.

- For x_1, five consecutive analysis windows present strong, phase–coherent signatures ($|T| = 0.86$ at 18°), with low counter–evidence ($|F| = 0.10$). Two additional windows exhibit mixed support ($|T| = 0.55$) and notable contradiction ($|F| = 0.40$), reflecting transient operating changes.
- For x_2, three windows moderately support misalignment ($|T| = 0.62$ near 12°), while one window carries high indeterminacy ($|I| = 0.60$), e.g., due to tool change or sensor drift.
- Multiplicities $(5, 2, 3, 1)$ encode how many windows (or inspectors/tests) delivered each distinct complex neutrosophic assessment.

The multiset support is $\mathrm{supp}(M) = \{x_1, x_2\}$, and all complex components respect $|T|, |I|, |F| \leq 1$. This CNMS compactly aggregates repeated, phase–aware evidence and counter–evidence about competing fault hypotheses in a real manufacturing setting.

3.7 Imaginary Indeterminacy

Imaginary indeterminacy quantifies uncertainty arising from subconscious, dream, or latent cognition, yet behaves identically to ordinary indeterminacy within neutrosophic operations [38, 42].

Definition 3.7.1 (Imaginary Indeterminacy)**.** [38, 42] Let X be a neutrosophic universe and let a neutrosophic value be written as

$$(T, I, F) \in [0, 1]^3.$$

We say that I is of *imaginary indeterminacy type*, and write $I = I_{\mathrm{im}}$, if

1. I_{im} does not arise from consciously available or awake-time information sources, but

2. I_{im} quantifies uncertainty coming from sub–conscious, latent, dream–state, sleep–time, or otherwise non–manifest cognitive processes, and

3. $I_{\text{im}} \in [0, 1]$ is handled in the neutrosophic operations exactly as an ordinary indeterminacy degree.

Formally, an *imaginary–indeterminate* neutrosophic value is

$$(T, I_{\text{im}}, F) \in [0, 1]^3,$$

and any neutrosophic structure that allows such I_{im} is said to support imaginary indeterminacy.

Example 3.7.2 (Dream–influenced therapy selection with imaginary indeterminacy)**.** Dream-influenced therapy uses patients' dream reports and sleep patterns to guide treatment choices, integrating subconscious cues with clinical evidence assessment.

A clinician must choose between two therapies for a patient:

$$X = \{\text{A}, \text{B}\}.$$

Clinical evidence yields (T, F), while a sleep–diary and dream–report analysis contribute an *imaginary indeterminacy* score I_{im}, which does not come from awake–time information but is processed exactly like ordinary indeterminacy in computations.

Assign neutrosophic triples

$$(T, I_{\text{im}}, F)(\text{A}) = (0.78,\ 0.25,\ 0.12), \qquad (T, I_{\text{im}}, F)(\text{B}) = (0.42,\ 0.10,\ 0.51).$$

To aggregate, define a decision index that downweights evidence by the indeterminacy factor:

$$\text{DI}(H) \;:=\; \big(T(H) - F(H)\big)\,\big(1 - I_{\text{im}}(H)\big), \qquad H \in \{\text{A}, \text{B}\}.$$

Compute for A:

$$T(\text{A}) - F(\text{A}) = 0.78 - 0.12 = 0.66, \quad 1 - I_{\text{im}}(\text{A}) = 1 - 0.25 = 0.75,$$

$$\text{DI}(\text{A}) = 0.66 \times 0.75 = 0.495.$$

Compute for B:

$$T(\text{B}) - F(\text{B}) = 0.42 - 0.51 = -0.09, \quad 1 - I_{\text{im}}(\text{B}) = 1 - 0.10 = 0.90,$$

$$\text{DI}(\text{B}) = -0.09 \times 0.90 = -0.081.$$

Since $\text{DI}(\text{A}) = 0.495 > \text{DI}(\text{B}) = -0.081$, therapy A is preferred. Here I_{im} reflects dream–state, subconscious uncertainty (e.g., recurring nightmares about adverse effects), yet it is treated identically to ordinary indeterminacy in the neutrosophic calculus, as required by the definition.

3.8 Neutrosophic Soft OffSet and Neutrosophic HyperSoft OffSet

A neutrosophic OffSet permits truth, indeterminacy, or falsity degrees beyond $[0, 1]$, modeling over/under estimation and paradoxical evidence in extreme scenarios [113–117]. A further extended concept known as Plithogenic Offsets has also been introduced [118–122]. A neutrosophic soft OffSet maps each parameter to an OffSet on a universe, enabling attribute-conditioned overset/underset uncertainty modeling and analysis [46]. A neutrosophic hypersoft OffSet indexes OffSets by multi-attribute tuples, supporting higher-dimensional parameter interactions with controlled overset/underset membership excursions and dependencies [46].

Definition 3.8.1 (Neutrosophic OffSet (single set))**.** [46] On a universe U, a *single-valued neutrosophic set* A is given by three functions $T_A, I_A, F_A : U \to \mathbb{R}$. It is a *neutrosophic OffSet* if some component is *off* the unit interval, i.e., there exists $x \in U$ such that $T_A(x) > 1$ or $I_A(x) > 1$ or $F_A(x) > 1$ (overset), or $T_A(x) < 0$ or $I_A(x) < 0$ or $F_A(x) < 0$ (underset).

Example 3.8.2 (Neutrosophic OffSet: urban road safety)**.** Urban road safety focuses on reducing crashes, protecting vulnerable users, and improving traffic environments through design, enforcement, and education measures (cf. [123]).

Let $U = \{s_1, s_2, s_3\}$ be road segments, and consider the neutrosophic predicate "safe today" with $T, I, F : U \to \mathbb{R}$.

x	$T(x)$	$I(x)$	$F(x)$
s_1	0.80	0.10	0.10
s_2	0.30	0.20	0.60
s_3	**1.05**	0.10	**−0.05**

Here s_3 is an *OffSet* instance (overset $T > 1$ from redundant sensors; underset $F < 0$ after bias correction), capturing overconfident yet conflicting evidence.

Definition 3.8.3 (Neutrosophic Soft OffSet)**.** [46] Let U be a universe and E a set of parameters. A *neutrosophic soft offset* over (U, E) is a pair (F, E) where $F : E \to \mathsf{NS}_{\text{off}}(U)$ assigns to each $e \in E$ a neutrosophic OffSet $F(e)$ on U. Equivalently, for each e there are $T_e, I_e, F_e : U \to \mathbb{R}$ and at least one $x \in U$ with a component outside $[0, 1]$.

Example 3.8.4 (Neutrosophic Soft OffSet: diagnosis by stage)**.** Let the universe $U = \{p_1, p_2, p_3\}$ be patients and parameters $E = \{\text{early}, \text{late}\}$ (disease stage). A neutrosophic soft OffSet (F, E) assigns to each $e \in E$ an OffSet $F(e) = (T_e, I_e, F_e)$ on U for the statement "requires aggressive therapy". For e = early:

x	$T_{\text{early}}(x)$	$I_{\text{early}}(x)$	$F_{\text{early}}(x)$
p_1	0.20	0.15	0.70
p_2	**1.10**	0.05	−0.05
p_3	0.40	0.20	0.50

For e = late:

x	$T_{\text{late}}(x)$	$I_{\text{late}}(x)$	$F_{\text{late}}(x)$
p_1	0.60	0.10	0.35
p_2	0.80	0.05	0.20
p_3	0.50	0.15	**−0.05**

Each parameter induces its own OffSet (e.g., early–p_2 overset truth 1.10 from duplicated positives; late–p_3 underset falsity).

Definition 3.8.5 (Neutrosophic HyperSoft OffSet)**.** [46] Let attributes $\mathcal{A} = \{A_1, \ldots, A_k\}$ have value domains $V_1, \ldots, V_k$. A (hyper)parameter is a tuple $v = (v_1, \ldots, v_k) \in V_1 \times \cdots \times V_k$. A *neutrosophic hypersoft offset* on $(U, \mathcal{A})$ is a mapping

$$F : V_1 \times \cdots \times V_k \longrightarrow \mathsf{NS}_{\text{off}}(U),$$

so each attribute–value tuple v selects a neutrosophic OffSet $F(v) = (T_v, I_v, F_v)$ on U with some component off $[0, 1]$.

Example 3.8.6 (Neutrosophic HyperSoft OffSet: drone flyability by weather–time)**.** Drone flyability by weather–time evaluates if drones can safely operate under specific combinations of weather conditions and time periods given (cf. [124]).

Let attributes be $\mathcal{A} = \{\text{Weather}, \text{Time}\}$ with $V_{\text{Weather}} = \{\text{sunny}, \text{rainy}\}$ and $V_{\text{Time}} = \{\text{day}, \text{night}\}$. Universe $U = \{\text{Z1}, \text{Z2}\}$ are flight zones. Define $F : V_{\text{Weather}} \times V_{\text{Time}} \to \mathsf{NS}_{\text{off}}(U)$ for the predicate "flyable now".

For $(\text{sunny}, \text{day})$:

x	$T(x)$	$I(x)$	$F(x)$
Z1	0.95	0.03	0.05
Z2	0.90	0.05	0.08

For (rainy, night) (OffSet due to contradictory feeds):

x	$T(x)$	$I(x)$	$F(x)$
Z1	**1.12**	0.08	**−0.20**
Z2	0.30	0.40	0.70

The tuple (rainy, night) produces overset/underset values for Z1 (overconfident clearance from redundant radars; negative falsity after anomaly filtering), exemplifying a *neutrosophic hypersoft OffSet*.

3.9 n-Refined Neutrosophic Set Ranking

An n-refined neutrosophic ranking aggregates weighted truth, indeterminacy-penalized, and falsity components across alternatives, producing comparable scalar scores for decisions selection [49]. Related concepts include fuzzy ranking [125–127] and neutrosophic ranking [128, 129], which are widely studied in the literature.

Definition 3.9.1 (n-Refined Neutrosophic Set Ranking)**.** [49] Let $\mathcal{N}_n(X)$ denote the class of all n-refined neutrosophic sets on X. A ranking on $\mathcal{N}_n(X)$ is a functional

$$\mathsf{Rank} : \mathcal{N}_n(X) \longrightarrow \mathbb{R}$$

defined for $A \in \mathcal{N}_n(X)$ by

$$\mathsf{Rank}(A) \ := \ \frac{1}{|X|} \sum_{x \in X} \sum_{j=1}^{n} w_j \big(T_j(x) - \alpha I_j(x) - F_j(x)\big),$$

where

$$w_j \geq 0, \quad \sum_{j=1}^{n} w_j = 1, \qquad \alpha \in [0, 1]$$

are user–chosen importance weights and an indeterminacy–penalty. For two n-refined neutrosophic sets $A, B \in \mathcal{N}_n(X)$ we say

$$A \succ B \quad \Longleftrightarrow \quad \mathsf{Rank}(A) > \mathsf{Rank}(B),$$
$$A \sim B \quad \Longleftrightarrow \quad \mathsf{Rank}(A) = \mathsf{Rank}(B).$$

This gives a total (or preorder) ranking of n-refined neutrosophic sets: higher truth and lower falsity/indeterminacy over the n refined channels produce a better rank.

Example 3.9.2 (Real-world: Selecting a Marketing Plan with n-Refined Neutrosophic Set Ranking)**.** A marketing plan outlines goals, target customers, messages, channels, timing, and budget for promoting products or services over a period (cf. [130]).

We compare two marketing plans A and B across three target cities $X = \{C_1, C_2, C_3\}$. Each plan provides, for every city $x \in X$ and each refined channel $j = 1, \ldots, n$ with $n = 4$, a triplet $\big(T_j(x), I_j(x), F_j(x)\big) \in [0, 1]^3$. We fix importance weights

$$(w_1, w_2, w_3, w_4) = (0.40,\ 0.30,\ 0.20,\ 0.10),$$

and an indeterminacy penalty $\alpha = 0.50$. The ranking score of a plan on a specific city is

$$\text{Score}_x(\text{Plan}) = \sum_{j=1}^{4} w_j \Big(T_j(x) - \alpha\, I_j(x) - F_j(x)\Big),$$

and the plan's overall rank is the mean over cities:

$$\text{Rank}(\text{Plan}) = \frac{1}{|X|} \sum_{x \in X} \text{Score}_x(\text{Plan}).$$

Data (Plan A).

	(T_1, I_1, F_1)	(T_2, I_2, F_2)	(T_3, I_3, F_3)	(T_4, I_4, F_4)
C_1	(0.85, 0.10, 0.05)	(0.80, 0.15, 0.10)	(0.75, 0.20, 0.15)	(0.70, 0.10, 0.20)
C_2	(0.60, 0.20, 0.25)	(0.65, 0.15, 0.20)	(0.55, 0.25, 0.30)	(0.58, 0.18, 0.25)
C_3	(0.72, 0.12, 0.18)	(0.70, 0.20, 0.15)	(0.68, 0.22, 0.20)	(0.66, 0.15, 0.22)

Data (Plan B).

	(T_1, I_1, F_1)	(T_2, I_2, F_2)	(T_3, I_3, F_3)	(T_4, I_4, F_4)
C_1	$(0.82, 0.12, 0.08)$	$(0.77, 0.18, 0.12)$	$(0.70, 0.24, 0.18)$	$(0.68, 0.15, 0.22)$
C_2	$(0.62, 0.22, 0.23)$	$(0.60, 0.20, 0.22)$	$(0.57, 0.28, 0.29)$	$(0.55, 0.20, 0.30)$
C_3	$(0.70, 0.15, 0.17)$	$(0.69, 0.21, 0.16)$	$(0.66, 0.24, 0.22)$	$(0.65, 0.18, 0.23)$

Step-by-step computation for A at city C_1.

$$\begin{aligned}
\text{Score}_{C_1}(A) &= \sum_{j=1}^{4} w_j \big(T_j - \alpha I_j - F_j\big) \\
&= 0.40\,(0.85 - 0.50 \cdot 0.10 - 0.05) + 0.30\,(0.80 - 0.50 \cdot 0.15 - 0.10) \\
&\quad + 0.20\,(0.75 - 0.50 \cdot 0.20 - 0.15) + 0.10\,(0.70 - 0.50 \cdot 0.10 - 0.20) \\
&= 0.40\,(0.85 - 0.05 - 0.05) + 0.30\,(0.80 - 0.075 - 0.10) \\
&\quad + 0.20\,(0.75 - 0.10 - 0.15) + 0.10\,(0.70 - 0.05 - 0.20) \\
&= 0.40 \cdot 0.75 + 0.30 \cdot 0.625 + 0.20 \cdot 0.50 + 0.10 \cdot 0.45 \\
&= 0.3000 + 0.1875 + 0.1000 + 0.0450 \\
&= 0.6325.
\end{aligned}$$

Scores for A at all cities.

$$\text{Score}_{C_1}(A) = 0.6325, \quad \text{Score}_{C_2}(A) = 0.2615, \quad \text{Score}_{C_3}(A) = 0.4375.$$

Hence

$$\text{Rank}(A) = \frac{0.6325 + 0.2615 + 0.4375}{3} = \frac{1.3315}{3} = 0.4438333333.$$

Step-by-step computation for B at city C_1 (one line per channel).

$$\begin{aligned}
j = 1 &: 0.40\,(0.82 - 0.50 \cdot 0.12 - 0.08) = 0.40\,(0.82 - 0.06 - 0.08) = 0.40 \cdot 0.68 = 0.2720, \\
j = 2 &: 0.30\,(0.77 - 0.50 \cdot 0.18 - 0.12) = 0.30\,(0.77 - 0.09 - 0.12) = 0.30 \cdot 0.56 = 0.1680, \\
j = 3 &: 0.20\,(0.70 - 0.50 \cdot 0.24 - 0.18) = 0.20\,(0.70 - 0.12 - 0.18) = 0.20 \cdot 0.40 = 0.0800, \\
j = 4 &: 0.10\,(0.68 - 0.50 \cdot 0.15 - 0.22) = 0.10\,(0.68 - 0.075 - 0.22) = 0.10 \cdot 0.385 = 0.0385.
\end{aligned}$$

Thus

$$\text{Score}_{C_1}(B) = 0.2720 + 0.1680 + 0.0800 + 0.0385 = 0.5585.$$

Scores for B at all cities.

$$\text{Score}_{C_1}(B) = 0.5585, \quad \text{Score}_{C_2}(B) = 0.2390, \quad \text{Score}_{C_3}(B) = 0.4065.$$

Hence

$$\text{Rank}(B) = \frac{0.5585 + 0.2390 + 0.4065}{3} = \frac{1.2040}{3} = 0.4013333333.$$

Decision.

$$\text{Rank}(A) - \text{Rank}(B) = 0.4438333333 - 0.4013333333 = 0.0425 > 0,$$

so $A \succ B$ under the specified w_j and α.

3.10 Neutrosophic Infinity

A neutrosophic infinity is a formal inverse of a subindeterminacy unit, canceling its effect within an extended neutrosophic algebra framework [131, 132].

Definition 3.10.1 (Neutrosophic Infinity)**.** With notation as in Definition 8.4.1, a *neutrosophic infinity* associated to a subindeterminacy I_s (with $s \in Z(R) \cup \{0\}$) is a formal symbol I_s^{-1} such that

$$I_s \cdot I_s^{-1} = 1$$

in a stipulated multiplicative extension of R_I (when such a product is defined). In particular, in $\mathbb{Z}_n$ one adjoins the symbols $\text{NeutrosophicInfinity}_s := I_s^{-1}$ corresponding to $s \in Z(\mathbb{Z}_n) \cup \{0\}$.

Example 3.10.2 (Real-world normalization of ambiguous process-phase tags via neutrosophic infinity)**.** Consider a six-stage production line with phases encoded modulo 6, i.e. $R = \mathbb{Z}_6$. Ambiguous causes tied to specific periodicities are recorded as *subindeterminacy tags* I_s for $s \in Z(\mathbb{Z}_6) \cup \{0\} = \{0, 2, 3, 4\}$. To allow "tag removal," we work in a stipulated commutative multiplicative extension $\mathcal{M}$ of R that contains all I_s and new symbols I_s^{-1} (the *neutrosophic infinities*) subject only to the relations

$$I_s \cdot I_s^{-1} = I_s^{-1} \cdot I_s = 1, \qquad \text{and each } I_s^{\pm 1} \text{ commutes with } R \text{ and with } I_t^{\pm 1}.$$

(We emphasize $\mathcal{M}$ is a *multiplicative* extension; no ring inverses of zero divisors are assumed.)

Suppose an event occurs at phase $e = 5 \in \mathbb{Z}_6$, but the technician cannot distinguish whether the anomaly stems from an "even-phase" confounder; they therefore record the datum with the tag I_2:

$$D := I_2 \cdot e \in \mathcal{M}.$$

Later, during automated analysis, the system applies the corresponding neutrosophic infinity I_2^{-1} to normalize (remove) this subindeterminacy:

$$\text{Normalize}(D) = I_2^{-1} \cdot D = I_2^{-1} \cdot (I_2 \cdot e) = (I_2^{-1} \cdot I_2) \cdot e = 1 \cdot e = e.$$

Hence the base phase $e = 5$ is recovered exactly once the appropriate neutrosophic infinity is applied.

If two independent ambiguous causes were present, say an even-phase and a triple-phase confounder, the recorded datum would be

$$D' := I_2 \cdot I_3 \cdot e,$$

and the combined normalization uses the product of the corresponding neutrosophic infinities in reverse order:

$$I_3^{-1} \cdot I_2^{-1} \cdot D' = I_3^{-1} \cdot I_2^{-1} \cdot (I_2 \cdot I_3 \cdot e) = (I_3^{-1} I_3) \cdot (I_2^{-1} I_2) \cdot e = 1 \cdot 1 \cdot e = e.$$

The subindeterminacy tags I_s annotate measurements with unresolved periodic causes; the neutrosophic infinities I_s^{-1} serve as formal "cancellers" that remove exactly those annotations when the cause is later identified. The algebraic cancellations above are the precise mathematical counterpart of that data-cleaning step.

3.11 Subset–Valued Complex Neutrosophic Set

A subset-valued neutrosophic set assigns each element subsets of truth, indeterminacy, and falsity degrees, modeling heterogeneous or interval-like uncertainty information [133]. A Subset–Valued Complex Neutrosophic Set assigns each element subsets of complex neutrosophic truth, indeterminacy, falsity values capturing heterogeneous evidence sources [134].

Definition 3.11.1 (Subset–Valued Complex Neutrosophic Set)**.** [134] Let X be a nonempty universe and let

$$\mathbb{CN} := \{ m + n\mathbf{I} : m, n \in \mathbb{C}, \mathbf{I}^2 = \mathbf{I} \}$$

be the set of *complex neutrosophic numbers*. A *subset–valued complex neutrosophic set* A on X assigns to each $x \in X$ a triple

$$A(x) = \big(T_A(x), I_A(x), F_A(x)\big),$$

where $T_A(x), I_A(x), F_A(x)$ are nonempty subsets of $\mathbb{CN}$. When each of $T_A(x), I_A(x), F_A(x)$ is a singleton contained in $\{m + n\mathbf{I} \in \mathbb{CN} : \Re(m), \mathrm{Im}(m) \in [0, 1]\}$, one gets a single–valued complex neutrosophic set; interval–like cases are modeled by complex–neutrosophic intervals.

Example 3.11.2 (Multi-source clinical risk label as a subset–valued complex neutrosophic set)**.** Clinical risk quantifies the probability and severity of adverse health outcomes for a patient, guiding diagnosis, monitoring, and treatment decisions (cf. [135]).

Let $\mathbf{I}$ be an idempotent unit with $\mathbf{I}^2 = \mathbf{I}$, and let complex neutrosophic numbers have the form $m + n\mathbf{I}$ with $m, n \in \mathbb{C}$. Consider a hospital triage universe $X = \{\mathrm{p_A}\}$ consisting of a single patient $\mathrm{p_A}$. Define a subset–valued complex neutrosophic set A on X by specifying three nonempty subsets:

$$T_A(\mathrm{p_A}) = \{\, 0.72 + 0.10\mathbf{I},\ 0.64 + 0.15\mathbf{I}\,\},$$
$$I_A(\mathrm{p_A}) = \{\, 0.25 + 0.30\mathbf{I},\ 0.18 + 0.20\mathbf{I}\,\},$$
$$F_A(\mathrm{p_A}) = \{\, 0.12 + 0.05\mathbf{I}\,\}.$$

Interpretation of sources. The two truth entries come from (i) a lab-based classifier and (ii) a physician's consensus score. The two indeterminacy entries come from (i) a missing-data model and (ii) an overnight nurse note with partially conflicting observations. The falsity entry is contributed by an imaging model that contradicts a suspected diagnosis. Keeping *subsets* for each component preserves these heterogeneous, simultaneous assessments without forcing premature aggregation.

A simple decision aid (using real parts) can rank urgency:

$$\mathrm{score}(\mathrm{p_A}) := \max_{t \in T_A(\mathrm{p_A})} \Re(t) \;-\; \min_{f \in F_A(\mathrm{p_A})} \Re(f) \;-\; \frac{1}{|I_A(\mathrm{p_A})|} \sum_{i \in I_A(\mathrm{p_A})} \Re(i).$$

Numerically,

$$\max \Re(T_A) = 0.72, \qquad \min \Re(F_A) = 0.12, \qquad \frac{1}{2}\big(\Re(0.25+0.30\mathbf{I}) + \Re(0.18+0.20\mathbf{I})\big) = \tfrac{0.25+0.18}{2} = 0.215,$$

hence

$$\mathrm{score}(\mathrm{p_A}) = 0.72 - 0.12 - 0.215 = 0.385.$$

If the triage threshold is 0.30, the patient is escalated for specialist review. The $\mathbf{I}$-components remain stored for advanced fusion rules that weight latent ambiguity.

3.12 Subset–Valued Fuzzy / Neutrosophic / Plithogenic Sets

A subset–valued fuzzy set assigns each element a nonempty subset of membership degrees, modeling heterogeneous evidence, intervals, and uncertainty information. A subset–valued plithogenic set maps each item and attribute value to subsets of membership vectors, aggregating contradictions via degree-of-contradiction weights.

Definition 3.12.1 (Subset–Valued Fuzzy Set (SVFS))**.** Let X be a nonempty universe and $\mathcal{P}([0,1])$ the powerset of $[0,1]$. A *subset–valued fuzzy set* on X is a map

$$A : X \to \mathcal{P}([0,1]) \setminus \{\varnothing\}, \qquad x \longmapsto A(x),$$

where $A(x) \subseteq [0,1]$ is the (possibly non-singleton) set of admissible membership degrees of x. When each $A(x)$ is a singleton, one recovers an ordinary fuzzy set.

Example 3.12.2 (Subset-Valued Fuzzy Set: consumer satisfaction)**.** Consumer satisfaction measures how well products or services meet customer expectations, influencing loyalty, repeat purchases, and brand perception over time (cf. [136]).

Let $X = \{\mathrm{A}, \mathrm{B}, \mathrm{C}\}$ be three smartwatches, and consider the linguistic label "highly satisfactory" based on three expert scores for each product.

Define a subset–valued fuzzy set $A : X \to \mathcal{P}([0,1]) \setminus \{\varnothing\}$ by

$$A(\mathrm{A}) = \{0.75,\ 0.80,\ 0.85\},$$
$$A(\mathrm{B}) = \{0.50,\ 0.55\},$$
$$A(\mathrm{C}) = \{0.30,\ 0.40,\ 0.45\}.$$

Here $A(\mathrm{A})$ is the set of admissible membership degrees assigned to product A by the three experts, reflecting slight disagreement but consistently high satisfaction. Similarly, $A(\mathrm{B})$ and $A(\mathrm{C})$ collect all plausible degrees for B and C. If one later chooses to aggregate each subset (e.g. by sup or average), an ordinary fuzzy set is obtained.

Definition 3.12.3 ((Recall)Subset–Valued Neutrosophic Set (SVNS))**.** [133] Let X be a nonempty universe. A *subset–valued neutrosophic set* on X assigns to each $x \in X$ a triple

$$\mathcal{A}(x) = \big(T_{\mathcal{A}}(x),\ I_{\mathcal{A}}(x),\ F_{\mathcal{A}}(x)\big),$$

with $T_{\mathcal{A}}(x), I_{\mathcal{A}}(x), F_{\mathcal{A}}(x) \in \mathcal{P}([0,1]) \setminus \{\emptyset\}$. Writing inf and sup for the usual bounds (with the convention on closedness as needed), one requires

$$0 \le \inf T_{\mathcal{A}}(x) + \inf I_{\mathcal{A}}(x) + \inf F_{\mathcal{A}}(x) \le \sup T_{\mathcal{A}}(x) + \sup I_{\mathcal{A}}(x) + \sup F_{\mathcal{A}}(x) \le 3 \qquad (\forall x \in X).$$

When each of $T_{\mathcal{A}}(x), I_{\mathcal{A}}(x), F_{\mathcal{A}}(x)$ is a singleton, this reduces to a single–valued neutrosophic set.

Definition 3.12.4 (Subset–Valued Plithogenic Set (SVPS))**.** A *plithogenic context* is a tuple

$$PS = \big(P,\ v,\ Pv,\ pdf,\ pCF\big),$$

where P is a universe of items, v is a fixed attribute with value domain Pv, $pdf : P \times Pv \to \mathcal{P}([0,1]^s) \setminus \{\emptyset\}$ (for some fixed $s \in \mathbb{N}$) is the degree-of-appurtenance mapping that returns a *subset* of $[0,1]^s$, and $pCF : Pv \times Pv \to [0,1]$ is the degree of contradiction, with $pCF(a,a) = 0$ and $pCF(a,b) = pCF(b,a)$. A *subset–valued plithogenic set* over PS is the data (P, v, Pv, pdf, pCF) itself; for each $x \in P$ and $u \in Pv$, the set $pdf(x;u) \subseteq [0,1]^s$ collects admissible membership vectors under attribute value u.[1]

Example 3.12.5 (Subset-Valued Plithogenic Set: eco-rating of office buildings)**.** Let $P = \{B_1, B_2\}$ be two office buildings and consider the attribute v = "energy standard" with value domain

$$Pv = \{\text{"national code"},\ \text{"green certification"}\}.$$

For each building $x \in P$ and $u \in Pv$, we attach a subset of neutrosophic membership triples in $[0,1]^3$: truth (compliance), indeterminacy (data gaps), and falsity (noncompliance evidence). Thus

$$pdf : P \times Pv \to \mathcal{P}([0,1]^3) \setminus \{\emptyset\}.$$

Define

$$pdf(B_1; \text{national code}) = \big\{(0.82, 0.10, 0.12),\ (0.78, 0.15, 0.14)\big\},$$

$$pdf(B_1; \text{green certification}) = \big\{(0.60, 0.20, 0.25),\ (0.65, 0.18, 0.22)\big\},$$

$$pdf(B_2; \text{national code}) = \big\{(0.55, 0.25, 0.30)\big\},$$

$$pdf(B_2; \text{green certification}) = \big\{(0.35, 0.30, 0.50),\ (0.40, 0.28, 0.45)\big\}.$$

The degree of contradiction function $pCF : Pv \times Pv \to [0,1]$ encodes how dissimilar the standards are, e.g.

$$pCF(\text{national code, green certification}) = pCF(\text{green certification, national code}) = 0.7, \quad pCF(u,u) = 0.$$

For B_1, different energy auditors under the "national code" and "green certification" criteria deliver several candidate triples, stored as subsets in pdf. The plithogenic structure uses pCF to aggregate or compare these subset–valued assessments while accounting for the high contradiction between the two standards. Thus (P, v, Pv, pdf, pCF) forms a subset–valued plithogenic set describing eco–ratings of buildings under multiple, partially conflicting standards.

Theorem 3.12.6 (SVPS generalizes SVFS and SVNS)**.** *Every subset–valued fuzzy set and every subset–valued neutrosophic set can be represented as a subset–valued plithogenic set by suitable choices of* (s, Pv, pdf, pCF).

[1] Operations (conjunction/aggregation) are typically defined via a pCF–weighted blend of a fixed t–norm/t–conorm acting componentwise on $[0,1]^s$. This is not needed for the reduction theorem below; we use only the representational layer.

Proof. We prove the two reductions by explicit embeddings.

(1) SVFS $\hookrightarrow$ SVPS. Let $A : X \to \mathcal{P}([0,1]) \setminus \{\emptyset\}$ be a subset–valued fuzzy set. Choose $P := X$, $s := 1$, a *singleton* attribute domain $Pv := \{u_*\}$, and let pCF be arbitrary (it is never used when Pv is a singleton). Define

$$pdf(x; u_*) \;:=\; A(x) \;\subseteq [0,1] \quad (\subseteq [0,1]^s).$$

Then for each $x \in X$, the plithogenic membership subset under u_* *is exactly* the fuzzy membership subset $A(x)$. Hence (P, v, Pv, pdf, pCF) is a SVPS that represents A.

(2) SVNS $\hookrightarrow$ SVPS. Let $\mathcal{A} : X \to \big(\mathcal{P}([0,1]) \setminus \{\emptyset\}\big)^3$, $x \mapsto \big(T_{\mathcal{A}}(x), I_{\mathcal{A}}(x), F_{\mathcal{A}}(x)\big)$ be a subset–valued neutrosophic set. Choose $P := X$, $s := 3$, $Pv := \{u_*\}$, and again any pCF. Define the subset of *membership triples* by the Cartesian product

$$pdf(x; u_*) \;:=\; T_{\mathcal{A}}(x) \;\times\; I_{\mathcal{A}}(x) \;\times\; F_{\mathcal{A}}(x) \;\subseteq [0,1]^3.$$

By construction, every admissible triple $(t, i, f) \in pdf(x; u_*)$ satisfies the neutrosophic bounds of $\mathcal{A}(x)$, and conversely each choice of $(t, i, f) \in T_{\mathcal{A}}(x) \times I_{\mathcal{A}}(x) \times F_{\mathcal{A}}(x)$ appears as a plithogenic membership vector for x at u_*. Thus $\mathcal{A}$ is represented as a SVPS.

In both cases the attribute domain is collapsed to a singleton, so the contradiction function pCF is immaterial for representation; taking $s = 1$ recovers SVFS and taking $s = 3$ with product sets recovers SVNS. Therefore, the class of SVPS strictly contains the classes of SVFS and SVNS as special cases. □

3.13 Nonstandard Fuzzy / Neutrosophic / Plithogenic Sets

A nonstandard fuzzy set assigns each element a hyperreal membership near $[0,1]$, allowing infinitesimal underset/overset deviations and analysis via monads. A nonstandard neutrosophic set assigns hyperreal truth, indeterminacy, falsity degrees near $[0,1]$, permitting infinitesimal inconsistencies and refined uncertainty modeling variability [133]. A nonstandard plithogenic set maps items and attribute values to hyperreal membership vectors, aggregating evidence using contradiction-weighted t-norm/t-conorm operators internally.

Definition 3.13.1 (Nonstandard primitives)**.** [133] Let ${}^*\mathbb{R}$ be a hyperreal field extending $\mathbb{R}$. An element $\varepsilon \in {}^*\mathbb{R}$ is *infinitesimal* if $|\varepsilon| < \frac{1}{n}$ for all $n \in \mathbb{N}$. Write the *halo* of $[0,1]$ as

$$[0,1]_{\mathrm{ns}} \;:=\; \{\, x \in {}^*\mathbb{R} \mid \exists y \in [0,1] \text{ with } x \approx y \,\},$$

and, more liberally, for a fixed positive infinitesimal δ, set the *nonstandard band*

$$[0,1]^{(\delta)} \;:=\; [-\delta,\; 1+\delta] \;\subset {}^*\mathbb{R}.$$

The *standard part* map st sends near–standard $x \in {}^*\mathbb{R}$ to the unique $y \in \mathbb{R}$ with $x \approx y$.

Definition 3.13.2 (Nonstandard Fuzzy Set (NSF))**.** Let U be a universe. A *nonstandard fuzzy set* on U is a map

$$\mu_A : U \longrightarrow [0,1]^{(\delta)} \subset {}^*\mathbb{R},$$

for some fixed infinitesimal $\delta > 0$. Set operations are defined by internal t-norm/t-conorms extended to ${}^*\mathbb{R}$:

$$\mu_{A\cup B}(x) = S_*\big(\mu_A(x), \mu_B(x)\big), \quad \mu_{A\cap B}(x) = T_*\big(\mu_A(x), \mu_B(x)\big), \quad \mu_{A^{\complement}}(x) = 1 - \mu_A(x).$$

The *standard shadow* is $\mathrm{st} \circ \mu_A : U \to [0,1]$ when $\mu_A(x) \in [0,1]_{\mathrm{ns}}$.

Example 3.13.3 (Nonstandard Fuzzy Set: micro–sensor drift in air–quality monitoring)**.** Air-quality monitoring continuously measures pollutants in air to assess health risks, regulatory compliance, trends, and environmental management for urban communities (cf. [137]).

Let $U = \{\mathrm{S}_1, \mathrm{S}_2, \mathrm{S}_3\}$ be three $\mathrm{PM}_{2.5}$ sensors on the same urban roof. Consider the fuzzy predicate

$$A = \text{“currently well–calibrated sensor”}.$$

Because of micro–drift and rounding, engineers use a nonstandard fuzzy set $\mu_A : U \to [0,1]^{(\delta)}$ with infinitesimals $\varepsilon_1, \varepsilon_2, \varepsilon_3$ satisfying $|\varepsilon_j| < \delta$:

$$\mu_A(\mathrm{S}_1) = 0.95 + \varepsilon_1, \quad \mu_A(\mathrm{S}_2) = 1 + \varepsilon_2, \quad \mu_A(\mathrm{S}_3) = 0.82 - \varepsilon_3.$$

Here $\mu_A(\mathrm{S}_2)$ may lie slightly above 1 (overset) due to combining two nearly identical lab calibrations, while $\mu_A(\mathrm{S}_3)$ is slightly below its nominal value after a correction. The standard shadows satisfy

$$\mathrm{st}\big(\mu_A(\mathrm{S}_1)\big) = 0.95, \quad \mathrm{st}\big(\mu_A(\mathrm{S}_2)\big) = 1, \quad \mathrm{st}\big(\mu_A(\mathrm{S}_3)\big) = 0.82,$$

so decision rules can be designed in the classical $[0,1]$ scale, while the infinitesimals retain fine–grained information about overconfidence and drift.

Definition 3.13.4 (Nonstandard Neutrosophic Set (NSN))**.** [133] A *nonstandard neutrosophic set* on U is a triple of maps

$$T_A, I_A, F_A : U \longrightarrow [0,1]^{(\delta)} \subset {}^*\mathbb{R},$$

optionally subject to the near–standard neutrosophic bound

$$T_A(x) + I_A(x) + F_A(x) \;\le\; 3 + \delta \qquad (x \in U).$$

Set operations act componentwise via an internal t-norm T_* and t-conorm S_*:

$$\begin{aligned}
(T,I,F)_{A\cup B}(x) &= \big(S_*(T_A,T_B),\ S_*(I_A,I_B),\ T_*(F_A,F_B)\big),\\
(T,I,F)_{A\cap B}(x) &= \big(T_*(T_A,T_B),\ T_*(I_A,I_B),\ S_*(F_A,F_B)\big),\\
(T,I,F)_{A^{\complement}}(x) &= \big(F_A(x),\ 1 - I_A(x),\ T_A(x)\big).
\end{aligned}$$

Example 3.13.5 (Nonstandard Neutrosophic Set: telemedicine connection stability)**.** Telemedicine delivers remote healthcare using telecommunications technologies, enabling consultations, diagnosis, monitoring, and follow-up care without in-person clinic visits for patients (cf. [138]).

Let $U = \{\mathrm{h_d}, \mathrm{h_r}\}$ be two home–internet profiles for telemedicine: $\mathrm{h_d}$ = downtown apartment, $\mathrm{h_r}$ = rural house. Consider the neutrosophic predicate

$$A = \text{“sufficiently stable for high–resolution video today”}.$$

A nonstandard neutrosophic set attaches

$$(T_A, I_A, F_A) : U \to [0,1]^{(\delta)} \times [0,1]^{(\delta)} \times [0,1]^{(\delta)}$$

with infinitesimals $\varepsilon_T, \varepsilon_I, \varepsilon_F$:

$$\begin{aligned}
(T_A, I_A, F_A)(\mathrm{h_d}) &= \big(0.88 + \varepsilon_T,\ 0.07 - \varepsilon_I,\ 0.10 + \varepsilon_F\big),\\
(T_A, I_A, F_A)(\mathrm{h_r}) &= \big(0.62 - \varepsilon_T,\ 0.25 + \varepsilon_I,\ 0.30 + \varepsilon_F\big).
\end{aligned}$$

For $\mathrm{h_d}$, repeated speed–tests and call logs give very strong truth ($T \approx 0.88$) and tiny indeterminacy; overset/underset perturbations $0.88 + \varepsilon_T$, $0.07 - \varepsilon_I$ record almost negligible measurement noise. For $\mathrm{h_r}$, higher $I_A(\mathrm{h_r})$ captures inconsistent latency readings. In both cases, the standard parts $\mathrm{st}(T_A), \mathrm{st}(I_A), \mathrm{st}(F_A)$ lie in $[0,1]$ and obey

$$\mathrm{st}\big(T_A(x) + I_A(x) + F_A(x)\big) \le 3,$$

while the infinitesimals preserve subtle uncertainty about line stability.

Definition 3.13.6 (Nonstandard Plithogenic Set (NSP))**.** Fix a plithogenic context

$$PS = \big(P,\ v,\ Pv,\ pdf,\ pCF\big),$$

where P is a universe, v is an attribute with value set Pv, the *degree of appurtenance* is

$$pdf:\ P \times Pv \ \longrightarrow\ \big([0,1]^{(\delta)}\big)^s \subset ({}^*\mathbb{R})^s,$$

and the *degree of contradiction* is

$$pCF:\ Pv \times Pv \ \longrightarrow\ [0,1]^{(\delta)} \ \subset {}^*\mathbb{R} \quad \text{with} \quad pCF(a,a) = 0,\ \ pCF(a,b) = pCF(b,a).$$

A *nonstandard plithogenic set* is a selection of (P, v, Pv, pdf, pCF) together with fixed internal t-norm/t-conorm (T_*, S_*) used to aggregate across contradictory values. For $a, b \in [0,1]^{(\delta)}$ and $c \in [0,1]^{(\delta)}$, define the DCF–weighted binary aggregator

$$a \mathbin{\widetilde{\wedge}_c} b \;:=\; (1-c)\,T_*(a,b) \;+\; c\,S_*(a,b),$$

and extend componentwise to vectors in $\big([0,1]^{(\delta)}\big)^s$. Given a finite multiset of attribute values $\{u_1, \dots, u_m\} \subset Pv$, the *aggregated membership* of $x \in P$ is

$$\mu_{\mathrm{NSP}}(x;\{u_j\}) \;:=\; pdf(x;u_1) \mathbin{\widetilde{\wedge}_{c_{12}}} pdf(x;u_2) \mathbin{\widetilde{\wedge}_{c_{13}}} \cdots \mathbin{\widetilde{\wedge}_{c_{1m}}} pdf(x;u_m), \quad c_{1j} := pCF(u_1, u_j).$$

When only one value u is considered, $\mu_{\mathrm{NSP}}(x;\{u\}) = pdf(x;u)$.

Example 3.13.7 (Nonstandard Plithogenic Set: creditworthiness across almost–consistent sources)**.** Creditworthiness measures a borrower's ability and willingness to repay debt reliably, based on income, history, assets, and obligations to lenders (cf. [139]).

Let $P = \{\mathrm{c}_1\}$ be a single loan applicant and let v be the attribute "evidence source" with value set

$$Pv = \{\text{bureau},\ \text{bank},\ \text{fintech}\}.$$

We study the plithogenic predicate

"creditworthy for requested limit",

using scalar memberships ($s = 1$). The nonstandard degree of appurtenance

$$pdf : P \times Pv \to [0,1]^{(\delta)}$$

is given (for a fixed infinitesimal $\delta > 0$) by

$$\begin{aligned} pdf(\mathrm{c}_1;\text{bureau}) &= 0.81 + \varepsilon_b, \\ pdf(\mathrm{c}_1;\text{bank}) &= 0.78 - \varepsilon_k, \\ pdf(\mathrm{c}_1;\text{fintech}) &= 0.74 + \varepsilon_f, \end{aligned}$$

where each $\varepsilon_\bullet$ is infinitesimal with $|\varepsilon_\bullet| < \delta$. The contradiction degrees

$$pCF(\text{bureau}, \text{bank}) = 0.1 + \eta_1, \quad pCF(\text{bureau}, \text{fintech}) = 0.2 + \eta_2, \quad pCF(\text{bank}, \text{fintech}) = 0.15 + \eta_3$$

are also nonstandard, reflecting tiny disagreements between scoring models. Using an internal t-norm T_* and t-conorm S_*, the aggregated membership for c_1 can be computed via the DCF–weighted operator

$$a \mathbin{\widetilde{\wedge}_c} b = (1-c)\,T_*(a,b) + c\,S_*(a,b) \quad (c \in [0,1]^{(\delta)}).$$

Even if the internal aggregation yields a value slightly outside $[0,1]$, its standard part

$$\mathrm{st}\big(\mu_{\mathrm{NSP}}(\mathrm{c}_1)\big) \in [0,1]$$

is used by the bank's decision rule, while the infinitesimals encode how close the sources were to complete agreement about the applicant's creditworthiness.

Theorem 3.13.8 (NSP generalizes NSF and NSN)**.** *There exist injective embeddings*

$$\iota_{\mathrm{F}} :\ \mathrm{NSF}(U) \hookrightarrow \mathrm{NSP}(P{=}U), \qquad \iota_{\mathrm{N}} :\ \mathrm{NSN}(U) \hookrightarrow \mathrm{NSP}(P{=}U)$$

such that $\pi_{\mathrm{F}} \circ \iota_{\mathrm{F}} = \mathrm{id}$ *and* $\pi_{\mathrm{N}} \circ \iota_{\mathrm{N}} = \mathrm{id}$ *for suitable projections* $\pi_{\mathrm{F}}, \pi_{\mathrm{N}}$*. Hence every nonstandard fuzzy set and every nonstandard neutrosophic set is a special case of a nonstandard plithogenic set.*

Proof. For NSF, fix $s = 1$ and a singleton attribute value set $Pv = \{u_0\}$. Given $\mu_A : U \to [0,1]^{(\delta)}$, define $pdf(x;u_0) := \mu_A(x)$ and choose arbitrary pCF (irrelevant for a singleton). Set $P := U$, v arbitrary. Then for each x,

$$\mu_{\mathrm{NSP}}(x;\{u_0\}) = pdf(x;u_0) = \mu_A(x),$$

so $\iota_F(\mu_A)$ is recovered by the projection $\pi_F(pdf)(x) := pdf(x; u_0)$. Injectivity is immediate.

For NSN, fix $s = 3$ with coordinates interpreted as (T, I, F), and again take $Pv = \{u_0\}$. Given $(T_A, I_A, F_A) : U \to \big([0,1]^{(\delta)}\big)^3$, define

$$pdf(x; u_0) := \big(T_A(x), I_A(x), F_A(x)\big), \qquad pCF \text{ arbitrary.}$$

Then for each x,

$$\mu_{NSP}(x; \{u_0\}) = pdf(x; u_0) = (T_A(x), I_A(x), F_A(x)),$$

so the projection $\pi_N(pdf)(x) := pdf(x; u_0)$ returns the original NSN. Again injectivity is obvious. Therefore NSP strictly contains NSF and NSN as submodels, completing the proof. □

3.14 Hybrid Complex Neutrosophic Set

A Hybrid Complex Neutrosophic Set assigns each element mixed single/interval complex truth, indeterminacy, falsity values, preserving heterogeneous evidence from sources [38].

Definition 3.14.1 (Hybrid Complex Neutrosophic Set)**.** [38] Let X be a nonempty universe. A *hybrid complex neutrosophic set* A on X is a mapping $x \mapsto \big(T_A(x), I_A(x), F_A(x)\big)$ such that at least one component among $T_A(x), I_A(x), F_A(x)$ is subset–valued in $\mathbb{CN}$ (as in Definition 3.11.1) while the remaining components are single–valued in $\mathbb{CN}$. Hybridization thus mixes subset–valued and single–valued complex neutrosophic information coordinatewise.

Example 3.14.2 (Hybrid Complex Neutrosophic Set for misinformation triage)**.** Misinformation is false or inaccurate information shared without intent to deceive, yet it misleads people and distorts understanding, public perception (cf. [140]).

Let $\mathbb{D} = \{z \in \mathbb{C} : |z| \le 1\}$ be the closed unit disc and denote rectangular complex sets by

$$[a, b] + i[c, d] \ := \ \{\, x + iy \in \mathbb{C} \mid x \in [a, b],\ y \in [c, d] \,\}.$$

Consider a universe $X = \{\text{post}_1\}$ of social-media items about a breaking event. Define a Hybrid Complex Neutrosophic Set H on X by assigning to post_1 the triple

$$\big(T_H(\text{post}_1),\ I_H(\text{post}_1),\ F_H(\text{post}_1)\big),$$

where the components mix interval- and single-valued complex entries:

$$T_H(\text{post}_1) = [0.70, 0.85] + i[0.10, 0.20], \qquad I_H(\text{post}_1) = 0.25 + 0.30\,i,$$
$$F_H(\text{post}_1) = \{\, 0.10 + 0.05\,i,\ 0.12 + 0.02\,i \,\}.$$

Interpretation. T_H aggregates heterogeneous *truth* evidence (fact-checkers, agency feeds) as a set of plausible complex values; I_H is a single complex *indeterminacy* summarizing unresolved contradictions; F_H lists discrete *falsity* indicators (bot detector, prior debunk).

All entries lie in $\mathbb{D}$:

$$|0.85 + 0.20i|^2 = 0.85^2 + 0.20^2 = 0.7225 + 0.0400 = 0.7625 \le 1,$$
$$|0.25 + 0.30i|^2 = 0.1525 \le 1, \quad |0.12 + 0.05i|^2 = 0.0169 \le 1.$$

A simple decision functional (real-part prototype) picks representatives and scores:

$$t^\star = \frac{0.70 + 0.85}{2} + i\,\frac{0.10 + 0.20}{2} = 0.775 + 0.15i,$$
$$i^\star = I_H(\text{post}_1) = 0.25 + 0.30i,$$
$$f^\star = \arg \max_{z \in F_H(\text{post}_1)} \Re(z) = 0.12 + 0.02i.$$

Score (higher is better):

$$\text{Score} = \Re(t^\star) - \Re(i^\star) - \Re(f^\star) = 0.775 - 0.25 - 0.12 = 0.405.$$

With an operational threshold 0.35, we have $0.405 \ge 0.35$, thus post_1 is provisionally classified as *likely true*. The imaginary parts encode phase/volatility signals (timing, source drift) usable by more advanced fusion rules.

3.15 General Neutrosophic Complex Set

A General Neutrosophic Complex Set assigns each element complex-valued truth, indeterminacy, falsity components, possibly as singletons, intervals, or subsets collections [38].

Definition 3.15.1 (General Neutrosophic Complex Set)**.** [38] Let X be a nonempty universe. A *general neutrosophic complex set* A on X is any assignment

$$A(x) \;=\; \big(T_A(x),\, I_A(x),\, F_A(x)\big), \qquad x \in X,$$

where each component is either a single value in $\mathbb{C}\mathcal{N}$ or a nonempty subset of $\mathbb{C}\mathcal{N}$. The subunitary/real cases are obtained by restricting to $\mathbb{C}\mathcal{N}$ elements whose real parts lie in $[0,1]$ and whose imaginary/indeterminate parts are constrained as desired; classical single–valued neutrosophic sets are recovered when all components are real singletons in $[0,1]$.

Example 3.15.2 (General Neutrosophic Complex Set for credit risk under asynchronous signals)**.** Credit risk is the possibility that a borrower fails to meet debt obligations, causing financial loss to lenders or investors (cf. [141]).

Let $X = \{\mathrm{app}_1\}$ be loan applications. A General Neutrosophic Complex Set G assigns to app_1 a triple of complex sets $(T_G, I_G, F_G) \subseteq \mathbb{C}\times\mathbb{C}\times\mathbb{C}$:

$$T_G(\mathrm{app}_1) = [0.65, 0.80] + i[0.10, 0.18],$$

$$I_G(\mathrm{app}_1) = \{\, 0.30 + 0.40i,\ 0.20 + 0.25i \,\},$$

$$F_G(\mathrm{app}_1) = \{\, 0.15 + 0.10i \,\} \;\cup\; \big([0.05, 0.12] + i[0.02, 0.06]\big).$$

All entries are bounded (unit-disc check for corner values):

$$|0.80 + 0.18i|^2 = 0.80^2 + 0.18^2 = 0.6400 + 0.0324 = 0.6724 \le 1,$$

$$|0.30 + 0.40i|^2 = 0.09 + 0.16 = 0.25 \le 1,$$

$$|0.15 + 0.10i|^2 = 0.0225 + 0.0100 = 0.0325 \le 1.$$

Decision aggregation (explicit, reproducible):

$$t^* = \mathrm{mid}\big(T_G(\mathrm{app}_1)\big) = \frac{0.65 + 0.80}{2} + i\,\frac{0.10 + 0.18}{2} = 0.725 + 0.14i,$$

$$i^* = \mathrm{mean}\big(I_G(\mathrm{app}_1)\big) = \frac{(0.30 + 0.40i) + (0.20 + 0.25i)}{2} = 0.25 + 0.325i,$$

$$f^* = \arg\max_{z \in F_G(\mathrm{app}_1)} \Re(z) = 0.15 + 0.10i \quad (\text{since } 0.15 > \mathrm{mid}[0.05, 0.12] = 0.085).$$

Score with indeterminacy penalty $\beta = 0.5$:

$$\mathrm{Score} = \Re(t^*) - \beta\,\Re(i^*) - \Re(f^*) = 0.725 - 0.5 \cdot 0.25 - 0.15 = 0.725 - 0.125 - 0.15 = 0.450.$$

With an approval threshold $\theta = 0.40$, we obtain $0.450 \ge \theta$, hence app_1 is *approved*. The imaginary parts encode phase/timing of signals (e.g., reporting delay, market drift) and can be exploited by more advanced fusion rules. This example is "general" because T_G is an interval set, I_G is a finite set, and F_G is a hybrid union of a singleton and a rectangle.

3.16 Neutrosophic Triplet HyperStructures and Hypertopologies

Hyperstructures generalize algebraic operations to set-valued outcomes, supporting multi-result composition, generalized associativity, and closure across complex combination systems and models [142–145]. Hypertopologies extend topological frameworks to hyperoperations, requiring openness to be preserved under left and right hyper-translations and compatible neutral–opposite constraints throughout the space [146, 147]. Related concepts such as SuperHyperStructures [148–150] and SuperHypertopologies [147, 151] have also been developed. A Neutrosophic Triplet HyperStructure equips a hyperoperation with elementwise neutral and opposite mappings, forming constrained triples that link operation, neutrality, and opposition [51]. A Neutrosophic Triplet Hypertopology is a topology stable under hyper-translations, respecting the corresponding neutral and opposite elements associated with each point in the structure [51].

Definition 3.16.1 (Hyperoperation and Semihypergroup)**.** [152–154] Let H be a nonempty set. A *hyperoperation* on H is a map

$$\circ : H \times H \longrightarrow \mathcal{P}^*(H),$$

where $\mathcal{P}^*(H) = \mathcal{P}(H) \setminus \{\varnothing\}$, and for $x, y \in H$ we write $x \circ y := \circ(x, y) \subseteq H$, $x \circ Y := \bigcup_{y \in Y}(x \circ y)$, $X \circ y := \bigcup_{x \in X}(x \circ y)$. A pair $(H, \circ)$ is a *semihypergroup* if $\circ$ is associative:

$$(x \circ y) \circ z \;=\; x \circ (y \circ z) \qquad (\forall x, y, z \in H).$$

If, in addition, $x \circ H = H \circ x = H$ holds for all $x \in H$ (reproductive axiom), then $(H, \circ)$ is a *hypergroup*.

Definition 3.16.2 (Neutrosophic Triplet HyperStructure (single–valued form))**.** [51] A *Neutrosophic Triplet HyperStructure* is a quadruple

$$\mathbf{H} = (H, \circ, \mathrm{neut}, \mathrm{anti})$$

where $(H, \circ)$ is a hypergroupoid (i.e. equipped with a hyperoperation), and neut, anti : $H \to H$ are maps such that for every $x \in H$,

$$x \in \big(\mathrm{neut}(x) \circ x\big) \cap \big(x \circ \mathrm{neut}(x)\big), \qquad \mathrm{neut}(x) \in \big(\mathrm{anti}(x) \circ x\big) \cap \big(x \circ \mathrm{anti}(x)\big).$$

For each $x \in H$, the triple $(x, \mathrm{neut}(x), \mathrm{anti}(x))$ is called a (hyper-)neutrosophic triplet. If $(H, \circ)$ is a semihypergroup (respectively, a hypergroup), we call **H** a *Neutrosophic Triplet Semihypergroup* (respectively, a *Neutrosophic Triplet Hypergroup*).

Definition 3.16.3 (Neutrosophic Triplet HyperStructure (set–valued form))**.** [51] Equivalently, one may allow multiple neutrals and opposites by taking

$$\mathrm{NEUT}, \mathrm{ANTI} : H \to \mathcal{P}^*(H),$$

and requiring that for each $x \in H$ there exist $n \in \mathrm{NEUT}(x)$ and $a \in \mathrm{ANTI}(x)$ with

$$x \in (n \circ x) \cap (x \circ n), \qquad n \in (a \circ x) \cap (x \circ a).$$

Any choice of such n, a yields a (hyper-)neutrosophic triplet (x, n, a).

Remark 3.16.4 (Strong/weak forms)**.** If the maps neut, anti (single–valued case) exist for all $x \in H$, the structure is *strong*. If only the set–valued version is assumed and the conditions hold with some $n \in \mathrm{NEUT}(x)$, $a \in \mathrm{ANTI}(x)$ for each x, the structure is *weak*.

Example 3.16.5 (Neutrosophic Triplet HyperStructure on a three–element hypergroupoid)**.** Let $H = \{-1, 0, 1\}$ and define the symmetric hyperoperation

$$x \circ y \;:=\; \{x, y, 0\} \qquad (\forall x, y \in H).$$

Set $\mathrm{neut}(x) := 0$ and $\mathrm{anti}(x) := -x$ for every $x \in H$. Then, for all $x \in H$,

$$(\mathrm{neut}(x) \circ x) = 0 \circ x = \{0, x\} \ni x, \qquad (x \circ \mathrm{neut}(x)) = x \circ 0 = \{x, 0\} \ni x,$$

and

$$(\mathrm{anti}(x) \circ x) = (-x) \circ x = \{-x, x, 0\} \ni \mathrm{neut}(x) = 0, \quad (x \circ \mathrm{anti}(x)) = x \circ (-x) = \{x, -x, 0\} \ni 0.$$

Hence $(H, \circ, \mathrm{neut}, \mathrm{anti})$ is a (single–valued) Neutrosophic Triplet HyperStructure. (Associativity is not required here; we only exhibit a hypergroupoid satisfying the triplet conditions.)

Definition 3.16.6 (Neutrosophic Triplet Hypertopology)**.** [51] Let $\mathbf{H} = (H, \circ, \text{neut}, \text{anti})$ be a Neutrosophic Triplet HyperStructure (single– or set–valued as above). A *Neutrosophic Triplet Hypertopology* on $\mathbf{H}$ is a family $\tau \subseteq \mathcal{P}(H)$ such that:

1. $\varnothing, H \in \tau$; if $\{U_i\}_{i\in I} \subseteq \tau$ then $\bigcup_{i\in I} U_i \in \tau$; if $U, V \in \tau$ then $U \cap V \in \tau$ (topology axioms).
2. (Translation–invariance under $\circ$) For all $U \in \tau$ and all $a \in H$, the *left* and *right* hyper–translates are open:
$$a \circ U := \bigcup_{u\in U}(a \circ u) \in \tau, \qquad U \circ a := \bigcup_{u\in U}(u \circ a) \in \tau.$$
3. (Neutrosophic compatibility) For every $x \in H$ there exist $U_x, V_x \in \tau$ such that
$$x \in U_x \subseteq \big(\text{neut}(x) \circ \{x\}\big) \cap \big(\{x\} \circ \text{neut}(x)\big),$$
and, writing $n_x \in \text{NEUT}(x)$ for the single–valued case $n_x = \text{neut}(x)$, there exists $a_x \in \text{ANTI}(x)$ (single–valued: $a_x = \text{anti}(x)$) with
$$n_x \in V_x \subseteq \big(a_x \circ \{x\}\big) \cap \big(\{x\} \circ a_x\big).$$

Then $(H, \circ, \text{neut}, \text{anti}; \tau)$ is called a *Neutrosophic Triplet Hypertopological Space*. If $(H, \circ)$ is a semihypergroup (resp. hypergroup), we may say *neutrosophic triplet semihypertopology* (resp. *hypertopology*).

Remark 3.16.7 (Bases generated by translates)**.** If $\mathcal{B} \subseteq \tau$ is a base for τ, then the collection

$$\mathcal{B}_\circ := \{\, A \circ B \circ C \mid A, C \subseteq H \text{ finite},\ B \in \mathcal{B} \,\}$$

is also a base and is closed under left/right hyper–translation. In particular, requiring item (2) is equivalent to demanding that τ admit such a translation–invariant base.

Example 3.16.8 (Neutrosophic Triplet Hypertopology compatible with the above hyperoperation)**.** Work with the same H and $\circ$ as above. Consider the topology

$$\tau := \big\{\, \varnothing,\ \{0\},\ \{0,1\},\ \{0,-1\},\ H \,\big\}.$$

Topological axioms: τ contains $\varnothing, H$, is closed under unions and finite intersections (e.g. $\{0,1\}\cap\{0,-1\} = \{0\}$, $\{0,1\} \cup \{0,-1\} = H$).

Hyper–translation invariance: For any $a \in H$ and $U \in \tau$,

$$a \circ U = \bigcup_{u\in U}\{a, u, 0\} \in \{\varnothing, \{0\}, \{0,1\}, \{0,-1\}, H\} = \tau,$$

and similarly $U \circ a \in \tau$ by symmetry of $\circ$.

Neutrosophic compatibility: For each $x \in H$ choose

$$U_x := \begin{cases} \{0\}, & x = 0, \\ \{0, x\}, & x \in \{-1, 1\}, \end{cases} \qquad V_x := \{0\}.$$

Then $x \in U_x \subseteq (\text{neut}(x) \circ \{x\}) \cap (\{x\} \circ \text{neut}(x)) = \{x, 0\}$, and with $a_x := \text{anti}(x) = -x$ we have

$$\text{neut}(x) = 0 \in V_x \subseteq (a_x \circ \{x\}) \cap (\{x\} \circ a_x) = \{x, -x, 0\}.$$

Thus $(H, \circ, \text{neut}, \text{anti}; \tau)$ is a Neutrosophic Triplet Hypertopological Space.

3.17 n-ary Neutrosophic Triplet

An n-ary neutrosophic triplet couples an element with its neutral and opposite under an n-ary operation, enabling generalized symmetry properties. Let (M, f) be an n-ary magma, i.e., $f : M^n \to M$ is an n-ary operation ($n \geq 2$) [51].

Definition 3.17.1 (n-ary neutral and inverse for an element)**.** [51] Fix $x \in M$. An element $e \in M$ is called an *n-ary neutral for x* if

$$f(x, \underbrace{e, \ldots, e}_{n-1}) = x \quad \text{and} \quad f(\underbrace{e, \ldots, e}_{n-1}, x) = x.$$

An element $y \in M$ is called an *n-ary inverse of x (w.r.t. e)* if

$$f(x, y, \underbrace{e, \ldots, e}_{n-2}) = e \quad \text{and} \quad f(y, x, \underbrace{e, \ldots, e}_{n-2}) = e.$$

Definition 3.17.2 (n-ary neutrosophic triplet (weak/strong))**.** [51] A *weak n-ary neutrosophic triplet* is a triple $(x, \mathrm{neut}_n(x), \mathrm{anti}_n(x)) \in M^3$ where $\mathrm{neut}_n(x)$ is an n-ary neutral for x and $\mathrm{anti}_n(x)$ is an n-ary inverse of x w.r.t. that neutral. If, in addition, there exists $e \in M$ such that for every position $k \in \{1, \ldots, n\}$ and every $x \in M$,

$$f(e, \ldots, e, \overset{k}{x}, e, \ldots, e) = x,$$

then e is a *global n-ary neutral* and $(x, e, \mathrm{anti}_n(x))$ is called a *strong* n-ary neutrosophic triplet.

Definition 3.17.3 (n-ary neutrosophic triplet set and group)**.** A *weak (resp. strong) n-ary neutrosophic triplet set* is a nonempty $S \subseteq M$ such that every $x \in S$ admits a weak (resp. strong) n-ary neutrosophic triplet within S. If, moreover, f satisfies the n-ary associativity law on S, then (S, f) is called a (weak/strong) *n-ary neutrosophic triplet group*.

Example 3.17.4 (Concrete n=3 case: ternary addition on integers)**.** Let $M = \mathbb{Z}$ and define the ternary operation

$$f : \mathbb{Z}^3 \longrightarrow \mathbb{Z}, \qquad f(a, b, c) := a + b + c.$$

Step 1: Global 3*-ary neutral.* Take $e := 0$. For any $x \in \mathbb{Z}$ and any position $k \in \{1, 2, 3\}$ we have

$$f(x, 0, 0) = x + 0 + 0 = x,$$
$$f(0, x, 0) = 0 + x + 0 = x,$$
$$f(0, 0, x) = 0 + 0 + x = x.$$

Hence $e = 0$ is a global 3-ary neutral element for f in the sense of

$$f(e, e, x) = f(e, x, e) = f(x, e, e) = x \qquad (\forall x \in \mathbb{Z}).$$

Step 2: 3*-ary inverse with respect to* $e = 0$*.* Fix $x \in \mathbb{Z}$ and set $y := -x$. Then

$$f(x, y, 0) = x + (-x) + 0 = 0 = e,$$
$$f(y, x, 0) = (-x) + x + 0 = 0 = e,$$

so $y = -x$ is a 3-ary inverse of x with respect to the neutral element $e = 0$.

Thus, for every $x \in \mathbb{Z}$ the triple

$$\big(x,\ \mathrm{neut}_3(x),\ \mathrm{anti}_3(x)\big) := \big(x,\ 0,\ -x\big)$$

is a *strong* ternary neutrosophic triplet: the neutral component does not depend on x (it is the global neutral $e = 0$), and the inverse component is $-x$.

Step 3: 3*-ary associativity and triplet group.* The operation f is 3-ary associative, because for all $a_1, a_2, a_3, a_4, a_5 \in \mathbb{Z}$,

$$f\big(f(a_1, a_2, a_3), a_4, a_5\big) = (a_1 + a_2 + a_3) + a_4 + a_5 = a_1 + a_2 + a_3 + a_4 + a_5,$$
$$f\big(a_1, f(a_2, a_3, a_4), a_5\big) = a_1 + (a_2 + a_3 + a_4) + a_5 = a_1 + a_2 + a_3 + a_4 + a_5,$$
$$f\big(a_1, a_2, f(a_3, a_4, a_5)\big) = a_1 + a_2 + (a_3 + a_4 + a_5) = a_1 + a_2 + a_3 + a_4 + a_5.$$

Hence all standard 3-ary associativity identities hold. Therefore the pair

$$(S, f) = (\mathbb{Z}, +\_3), \quad \text{with } +\_3(a, b, c) := a + b + c,$$

is a *strong 3-ary neutrosophic triplet group*, since every $x \in S$ admits a strong ternary neutrosophic triplet $(x, 0, -x)$ and f is 3-ary associative on S.

3.18 Refined Neutrosophic Metric Space

A neutrosophic metric space assigns three-component distances (truth, indeterminacy, falsity) between points, obeying nonnegativity, symmetry, and triangle inequalities componentwise [155, 156]. A refined neutrosophic metric space assigns multi-component distances splitting truth, indeterminacy, falsity into subparts, satisfying nonnegativity, symmetry, and triangle inequalities [157].

Definition 3.18.1 (Refined Neutrosophic Numbers)**.** [76, 158, 159] Fix nonnegative integers p, r, s with $n = p + r + s \geq 1$. A *refined neutrosophic number* is a vector

$$\mathbf{z} = (T_1, \dots, T_p;\ I_1, \dots, I_r;\ F_1, \dots, F_s) \in [0, \infty)^n,$$

whose subcomponents represent refined truth-, indeterminacy-, and falsity–grades. Write $\mathbf{0} = (0, \dots, 0)$ and define the componentwise order and addition on $[0, \infty)^n$.

Example 3.18.2 (Refined neutrosophic number)**.** Let $p = 2$, $r = 1$, $s = 2$, so $n = p + r + s = 5$. Consider the evaluation of a medical treatment option. A refined neutrosophic number is

$$\mathbf{z} = (T_1, T_2;\ I_1;\ F_1, F_2) = (0.82, 0.75;\ 0.10;\ 0.05, 0.08) \in [0, \infty)^5.$$

Here T_1 and T_2 represent two distinct truth evidences (e.g. clinical trial and real–world study), I_1 measures residual uncertainty in the data, while F_1 and F_2 encode two sources of falsity (e.g. side–effect reports and conflicting expert opinion). Componentwise order and addition act in the usual way on $\mathbb{R}^5$.

Definition 3.18.3 (Refined Neutrosophic Metric Space)**.** [157] Let X be a nonempty set and $n = p + r + s \geq 1$. A map $D : X \times X \to [0, \infty)^n$, $D(x, y) = (\mathbf{T}(x, y);\ \mathbf{I}(x, y);\ \mathbf{F}(x, y))$, is a *refined neutrosophic metric* if, for all $x, y, z \in X$,

1. $D(x, y) \succeq \mathbf{0}$, and $D(x, y) = \mathbf{0}$ iff $x = y$;
2. $D(x, y) = D(y, x)$ (symmetry, componentwise);
3. $D(x, z) \preceq D(x, y) + D(y, z)$ (triangle inequality, componentwise).

Then (X, D) is called a *refined neutrosophic metric space*. An optional scalarization is given by a weight vector $w \in (0, \infty)^n$ to define $\|D(x, y)\|_w = \sum_{j=1}^{n} w_j D_j(x, y)$.

Example 3.18.4 (Refined neutrosophic metric space)**.** Let $X = \mathbb{R}$, and choose $p = r = s = 1$, so $n = 3$. Define, for $x, y \in \mathbb{R}$,

$$D(x, y) := (\mathbf{T}(x, y);\ \mathbf{I}(x, y);\ \mathbf{F}(x, y)) = (|x - y|;\ 0.3|x - y|;\ 0.1|x - y|) \in [0, \infty)^3.$$

Then:

$$D(x, y) \succeq \mathbf{0} \text{ and } D(x, y) = \mathbf{0} \Leftrightarrow |x - y| = 0 \Leftrightarrow x = y;$$

$$D(x, y) = D(y, x) \quad \text{since } |x - y| = |y - x|;$$

and by the usual triangle inequality on $\mathbb{R}$,

$$\mathbf{T}(x, z) = |x - z| \leq |x - y| + |y - z| = \mathbf{T}(x, y) + \mathbf{T}(y, z),$$
$$\mathbf{I}(x, z) = 0.3|x - z| \leq 0.3|x - y| + 0.3|y - z| = \mathbf{I}(x, y) + \mathbf{I}(y, z),$$
$$\mathbf{F}(x, z) = 0.1|x - z| \leq 0.1|x - y| + 0.1|y - z| = \mathbf{F}(x, y) + \mathbf{F}(y, z),$$

so $D(x, z) \preceq D(x, y) + D(y, z)$ componentwise. Hence $(\mathbb{R}, D)$ is a refined neutrosophic metric space.

3.19 Real Refined Neutrosophic Numbers

Refined neutrosophic numbers represent quantities as a base value plus multiple sub-indeterminacy components, enabling granular truth, indeterminacy, falsity modeling capabilities [76, 158–160]. Real refined neutrosophic numbers use real coefficients for base and sub-indeterminacy parts, forming an $(n + 1)$-dimensional real vector space with basis. Complex refined neutrosophic numbers allow complex coefficients for base and sub-indeterminacy components, yielding an $(n + 1)$-dimensional complex vector space with basis [51].

Definition 3.19.1 ((Recall) Refined Neutrosophic Numbers)**.** [76, 158] Fix an integer $n \geq 2$ and introduce n distinct *literal sub-indeterminacies* $I_1, \dots, I_n$. Let S be a commutative ring (e.g., $\mathbb{R}$ or $\mathbb{C}$). The set of *refined neutrosophic numbers over S with n sub-indeterminacies* is

$$\mathrm{RN}_n(S) := \Big\{ a + \sum_{j=1}^{n} b_j I_j \;\Big|\; a, b_1, \dots, b_n \in S \Big\}.$$

Two elements are equal iff their coefficients match:

$$a + \sum_{j=1}^{n} b_j I_j = a' + \sum_{j=1}^{n} b'_j I_j \quad \Longleftrightarrow \quad a = a' \text{ and } b_j = b'_j \ (1 \leq j \leq n).$$

We endow $\mathrm{RN}_n(S)$ with the *S–module* structure given by coefficientwise operations:

$$\Big(a + \sum\nolimits_j b_j I_j\Big) + \Big(c + \sum\nolimits_j d_j I_j\Big) := (a + c) + \sum\nolimits_j (b_j + d_j) I_j,$$

$$\lambda \cdot \Big(a + \sum\nolimits_j b_j I_j\Big) := (\lambda a) + \sum\nolimits_j (\lambda b_j) I_j \qquad (\lambda \in S).$$

Thus $\mathrm{RN}_n(S)$ is canonically the free S–module of rank $n + 1$ with basis $\{1, I_1, \dots, I_n\}$. (No multiplicative relations among the I_j are imposed here.)

Definition 3.19.2 (Real Refined Neutrosophic Numbers (RRN))**.** The *real* refined neutrosophic numbers with n sub-indeterminacies are

$$\mathrm{RRN}_n := \mathrm{RN}_n(\mathbb{R}) = \Big\{ a + \sum_{j=1}^{n} b_j I_j \;\Big|\; a, b_j \in \mathbb{R} \Big\}.$$

With the coefficientwise addition and real scalar multiplication above, RRN_n is a real vector space of dimension $n + 1$ spanned by $\{1, I_1, \dots, I_n\}$.

Example 3.19.3 (Real refined neutrosophic numbers (RRN), $n = 3$)**.** Work in $\mathrm{RRN}_3 = \{ a + b_1 I_1 + b_2 I_2 + b_3 I_3 \mid a, b_j \in \mathbb{R} \}$. Let

$$x = 2.5 + 0.4 I_1 - 1.2 I_2 + 0\, I_3, \qquad y = -1 + 2\, I_1 + 0\, I_2 + 3\, I_3.$$

Coefficientwise addition (in the basis $\{1, I_1, I_2, I_3\}$) gives

$$x + y = (2.5 - 1) + (0.4 + 2) I_1 + (-1.2 + 0) I_2 + (0 + 3) I_3 = 1.5 + 2.4 I_1 - 1.2 I_2 + 3 I_3.$$

Real scalar operations act componentwise; for instance

$$3x - 2y = (7.5 + 1.2 I_1 - 3.6 I_2 + 0\, I_3) - (-2 + 4 I_1 + 0\, I_2 + 6 I_3) = 9.5 - 2.8 I_1 - 3.6 I_2 - 6 I_3.$$

Thus $\mathrm{RRN}_3 \cong \mathbb{R}^4$ as a real vector space via $a + \sum_{j=1}^{3} b_j I_j \longleftrightarrow (a, b_1, b_2, b_3)$.

Definition 3.19.4 (Complex Refined Neutrosophic Numbers (CRN))**.** The *complex* refined neutrosophic numbers with n sub-indeterminacies are

$$\mathrm{CRN}_n := \mathrm{RN}_n(\mathbb{C}) = \Big\{ a + \sum_{j=1}^{n} b_j I_j \;\Big|\; a, b_j \in \mathbb{C} \Big\}.$$

With coefficientwise addition and complex scalar multiplication, CRN_n is a complex vector space of dimension $n + 1$ with basis $\{1, I_1, \dots, I_n\}$.

Example 3.19.5 (Complex refined neutrosophic numbers (CRN), $n = 2$)**.** Work in $\mathrm{CRN}_2 = \{ a + b_1 I_1 + b_2 I_2 \mid a, b_j \in \mathbb{C} \}$. Let

$$z = (1 + i) + (2 - 3i)I_1 + (-i)I_2, \qquad w = (-i) + (1 + i)I_1 + 3\, I_2.$$

Coefficientwise addition yields

$$z + w = \underbrace{(1 + i - i)}_{=1} + \underbrace{(2 - 3i + 1 + i)}_{=3-2i} I_1 + \underbrace{(-i + 3)}_{=3-i} I_2 = 1 + (3 - 2i)I_1 + (3 - i)I_2.$$

Complex scalar multiplication is also componentwise; for example

$$i\, z = i(1 + i) + i(2 - 3i)I_1 + i(-i)I_2 = (-1 + i) + (3 + 2i)I_1 + 1 \cdot I_2.$$

Hence $\mathrm{CRN}_2 \cong \mathbb{C}^3$ as a complex vector space via $a + \sum_{j=1}^{2} b_j I_j \longleftrightarrow (a, b_1, b_2)$.

3.20 Hypersoft Vague Set and SuperHypersoft Vague Set

A Hypersoft Vague Set maps attribute tuples to vague subsets of a universe, assigning each element truth and falsity bounds [50]. A SuperHypersoft Vague Set maps powerset attribute combinations to vague subsets, representing multivalued criteria with per element truth and falsity [50].

Definition 3.20.1 ((Recall) Hypersoft Set)**.** [88] Let U be a universe and let $A_1, \dots, A_m$ be attribute domains. Set $C := A_1 \times \cdots \times A_m$. A *Hypersoft Set* on U is a pair (G, C) with a mapping

$$G : C \longrightarrow \mathcal{P}(U), \qquad (G, C) \;=\; \{ (\gamma, G(\gamma)) : \gamma \in C,\; G(\gamma) \subseteq U \},$$

so that each attribute tuple $\gamma = (\gamma_1, \dots, \gamma_m)$ indexes the subset $G(\gamma) \subseteq U$.

Definition 3.20.2 ((Recall) SuperHyperSoft Set)**.** [94] Let U be a universe and let $a_1, \dots, a_n$ be distinct attributes with value-sets A_i (pairwise disjoint). Define

$$C \;:=\; \mathcal{P}(A_1) \times \cdots \times \mathcal{P}(A_n).$$

A *SuperHyperSoft Set* on U is a pair (F, C) with a mapping

$$F : C \longrightarrow \mathcal{P}(U),$$

so that each $\gamma = (\alpha_1, \dots, \alpha_n) \in C$ (with $\alpha_i \subseteq A_i$) is assigned a subset $F(\gamma) \subseteq U$.

Definition 3.20.3 (Hypersoft Vague Set)**.** [50] Let U be a universe and write the *vague universe*

$$U(t, 1 - f) := \big\{ x\big(t_x,\; 1 - f_x\big) : x \in U,\; t_x, f_x \in [0, 1] \big\},$$

where t_x (truth-membership) and f_x (false-nonmembership) parameterize the vague membership interval $[t_x,\; 1 - f_x]$. Given attribute domains $A_1, \dots, A_n$ and $C := A_1 \times \cdots \times A_n$, a *Hypersoft Vague Set* is a mapping

$$F : C \longrightarrow \mathcal{P}\big(U(t, 1 - f)\big),$$

assigning to each attribute tuple a vague subset of U.

Example 3.20.4 (Hypersoft Vague Set: Laptop suitability by attributes)**.** Laptop suitability describes how well a laptop matches user needs, including performance, portability, battery life, reliability, price, ergonomics, durability (cf. [161]).

Let the universe be the set of laptops $U = \{\mathrm{L}_1, \mathrm{L}_2, \mathrm{L}_3\}$, where an item $x \in U$ is labeled as $x(t_x, 1 - f_x)$ with $t_x, f_x \in [0, 1]$ and $t_x + f_x \le 1$. Take two attribute domains:

$$A_1 = \{\text{Budget}, \text{Flagship}\}, \qquad A_2 = \{\text{Compact}, \text{Large}\},$$

and $C = A_1 \times A_2$. Define a hypersoft vague set $F : C \to \mathcal{P}\big(U(t, 1 - f)\big)$ that records “suitability for a student buyer”:

$$F(\text{Budget}, \text{Compact}) = \{\, L_1(0.70,\ 0.80),\ L_2(0.50,\ 0.70)\,\},$$
$$F(\text{Budget}, \text{Large}) = \{\, L_2(0.60,\ 0.85)\,\},$$
$$F(\text{Flagship}, \text{Compact}) = \{\, L_3(0.65,\ 0.90)\,\},$$
$$F(\text{Flagship}, \text{Large}) = \{\, L_3(0.80,\ 0.90)\,\}.$$

Here, e.g., $L_1(0.70, 0.80)$ encodes the vague membership interval $[t,\ 1-f] = [0.70, 0.80]$, meaning "student suitability" for L_1 under (Budget, Compact) lies between 0.70 and 0.80. All entries satisfy $t_x \le 1 - f_x$ (equivalently $t_x + f_x \le 1$).

Definition 3.20.5 (SuperHyperSoft Vague Set)**.** With the same vague universe $U(t, 1-f)$ as above, and with attributes $a_1, \ldots, a_n$ having value-sets A_i, set

$$C \ := \ \mathcal{P}(A_1) \times \cdots \times \mathcal{P}(A_n).$$

A *SuperHyperSoft Vague Set* is a mapping

$$F : C \longrightarrow \mathcal{P}\big(U(t, 1-f)\big),$$

so each superhyper combination $\gamma = (\alpha_1, \ldots, \alpha_n)$ (with $\alpha_i \subseteq A_i$) is sent to a vague subset of U, i.e., a set of labeled elements $x\big(t_x, 1-f_x\big)$.

Example 3.20.6 (SuperHyperSoft Vague Set: Vaccination site suitability by grouped attributes)**.** Let the universe of candidate sites be $U = \{C_1, C_2, C_3\}$ (clinics). A site x is labeled $x(t_x, 1-f_x)$ with $t_x, f_x \in [0, 1]$ and $t_x + f_x \le 1$. Consider attributes and value-sets:

$$A_1 = \{\text{Child}, \text{Adult}, \text{Senior}\}, \qquad A_2 = \{\text{LowRisk}, \text{MediumRisk}, \text{HighRisk}\}.$$

A superhyper parameter is a pair of subsets $\gamma = (\alpha_1, \alpha_2) \in \mathcal{P}(A_1) \times \mathcal{P}(A_2)$. Define $F : \mathcal{P}(A_1) \times \mathcal{P}(A_2) \to \mathcal{P}\big(U(t, 1-f)\big)$ to encode "suitability as a vaccination hub" for grouped populations:

$$\gamma^\star = \big(\{\text{Adult}, \text{Senior}\},\ \{\text{MediumRisk}, \text{HighRisk}\}\big), \qquad F(\gamma^\star) = \{\, C_1(0.80,\ 0.90),\ C_3(0.60,\ 0.80)\,\}.$$

Interpretation: for the combined target cohort "Adult/Senior at Medium/High risk," clinic C_1 has a vague suitability interval $[0.80, 0.90]$, while C_3 has $[0.60, 0.80]$. Different superhyper combinations (other γ) map analogously to vague subsets of U.

3.21 Neutrosophic Hypercubic Set

A Neutrosophic Hypercubic Set assigns each element neutrosophic triples across multidimensional indices, enabling slices, marginals, and parameterized uncertainty analysis tasks [40].

Definition 3.21.1 (Neutrosophic Hypercubic Set)**.** [40] Fix a dimension $d \in \mathbb{N}$ and nonempty index sets $I_1, \ldots, I_d$ (e.g. $I_j = \{0, 1, \ldots, n_j\}$ or $I_j = [0, 1]$). Write the hypercubic index space

$$\mathbb{I} \ := \ I_1 \times \cdots \times I_d.$$

Let U be a universe of discourse. A *neutrosophic hypercubic set of dimension d on U indexed by* $\mathbb{I}$ is a mapping

$$\mu :\ U \times \mathbb{I} \longrightarrow [0, 1]^3, \qquad (x, \alpha) \longmapsto \mu(x; \alpha) = \big(T(x; \alpha),\ I(x; \alpha),\ F(x; \alpha)\big),$$

assigning to each element $x \in U$ and each hypercubic index $\alpha = (\alpha_1, \ldots, \alpha_d) \in \mathbb{I}$ a neutrosophic triple, with $T, I, F \in [0, 1]$ and (optionally) $0 \le T + I + F \le 3$.

Equivalently, a neutrosophic hypercubic set is a family $\{\mu_\alpha : U \to [0, 1]^3\}_{\alpha \in \mathbb{I}}$ of neutrosophic sets parametrized by the d-dimensional hypercube $\mathbb{I}$. For $S \subseteq \{1, \ldots, d\}$ and $\beta \in \prod_{j \in S} I_j$ the *slice at β* is the neutrosophic set

$$\mu^S_\beta(x) \ := \ \mu\big(x;\ (\alpha_1, \ldots, \alpha_d)\big) \quad \text{with } \alpha_j = \beta_j \ (j \in S) \text{ and } (\alpha_k)_{k \notin S} \text{ free.}$$

Aggregations across the complementary indices (e.g. componentwise sup/inf or t-norm/t-conorm with renormalization) yield *marginals* along coordinate subcubes when needed.

Remark 3.21.2. When $d = 0$ (i.e. $\mathbb{I}$ is a singleton), a neutrosophic hypercubic set reduces to an ordinary neutrosophic set on U. When U is a singleton, μ becomes a neutrosophic labeling of the vertices of the d-dimensional hypercube $\mathbb{I}$.

Example 3.21.3 (Public-health campaign effectiveness across district, week, and channel)**.** Let $U = \{\text{mask\_mandate}\}$. Choose dimension $d = 3$ with index sets $I_1 = \{\text{North}, \text{South}\}$ (district), $I_2 = \{\text{W1}, \text{W2}\}$ (week), $I_3 = \{\text{SMS}, \text{TV}\}$ (communication channel). Thus $\mathbb{I} = I_1 \times I_2 \times I_3$.

Define a neutrosophic hypercubic set $\mu : U \times \mathbb{I} \to [0, 1]^3$ by the table below for $x = \text{mask\_mandate}$, where $\mu(x; \alpha) = (T, I, F)$ encodes estimated public acceptance (truth), uncertainty (indeterminacy), and rejection (falsity).

North district

	SMS	TV
W1	(0.78, 0.12, 0.10)	(0.70, 0.15, 0.15)
W2	(0.82, 0.10, 0.08)	(0.74, 0.14, 0.12)

South district

	SMS	TV
W1	(0.66, 0.20, 0.14)	(0.60, 0.22, 0.18)
W2	(0.69, 0.18, 0.13)	(0.62, 0.21, 0.17)

Slice (fix channel = SMS). For $\alpha_3 = \text{SMS}$ the slice consists of:

$$\mu(x; \text{North}, \text{W1}, \text{SMS}) = (0.78, 0.12, 0.10), \quad \mu(x; \text{South}, \text{W1}, \text{SMS}) = (0.66, 0.20, 0.14),$$
$$\mu(x; \text{North}, \text{W2}, \text{SMS}) = (0.82, 0.10, 0.08), \quad \mu(x; \text{South}, \text{W2}, \text{SMS}) = (0.69, 0.18, 0.13).$$

Marginal over districts (pessimistic falsity, optimistic indeterminacy). For each week with channel SMS, aggregate by

$$T^{\text{marg}} = \max_{\text{district}} T, \qquad I^{\text{marg}} = \min_{\text{district}} I, \qquad F^{\text{marg}} = \max_{\text{district}} F.$$

Week W1:

$$T^{\text{marg}} = \max(0.78, 0.66) = 0.78, \quad I^{\text{marg}} = \min(0.12, 0.20) = 0.12, \quad F^{\text{marg}} = \max(0.10, 0.14) = 0.14.$$

Week W2:

$$T^{\text{marg}} = \max(0.82, 0.69) = 0.82, \quad I^{\text{marg}} = \min(0.10, 0.18) = 0.10, \quad F^{\text{marg}} = \max(0.08, 0.13) = 0.13.$$

Overall marginal across all indices. Using the same rule over all 8 combinations:

$$T^* = \max(0.78, 0.70, 0.82, 0.74, 0.66, 0.60, 0.69, 0.62) = 0.82,$$
$$I^* = \min(0.12, 0.15, 0.10, 0.14, 0.20, 0.22, 0.18, 0.21) = 0.10,$$
$$F^* = \max(0.10, 0.15, 0.08, 0.12, 0.14, 0.18, 0.13, 0.17) = 0.18.$$

Decision score (with indeterminacy penalty $\alpha = 0.5$).

$$\text{Score} = T^* - \alpha I^* - F^* = 0.82 - 0.5 \cdot 0.10 - 0.18 = 0.82 - 0.05 - 0.18 = 0.59.$$

If the acceptance threshold is $\theta = 0.50$, then $0.59 \geq \theta$ and the campaign is recommended, with SMS in week W2 offering the strongest projected acceptance.

Theorem 3.21.4 (Neutrosophic Hypercubic Sets generalize Neutrosophic Cubic Sets)**.** *Let U be a nonempty universe. Denote by* $\mathrm{NCS}(U)$ *the class of neutrosophic cubic sets on U as in Definition 3.2.1, and by* $\mathrm{NHS}(U; d, \mathbb{I})$ *the class of neutrosophic hypercubic sets on U of dimension d with index space $\mathbb{I}$. There exists an injective map*

$$\Phi : \mathrm{NCS}(U) \hookrightarrow \mathrm{NHS}\big(U;\ 2,\ \{\mathrm{int}, \mathrm{pt}\} \times [0,1]\big),$$

together with a cubic reduction *operator*

$$\mathcal{R} : \mathrm{NHS}\big(U;\ 2,\ \{\mathrm{int}, \mathrm{pt}\} \times [0,1]\big) \longrightarrow \mathrm{NCS}(U),$$

such that $\mathcal{R} \circ \Phi = \mathrm{id}_{\mathrm{NCS}(U)}$. Consequently, every neutrosophic cubic set is (canonically) realized as a neutrosophic hypercubic set of dimension 2, hence neutrosophic hypercubic sets strictly generalize neutrosophic cubic sets.

Proof. Fix $\mathbb{I} := \{\mathrm{int}, \mathrm{pt}\} \times [0,1]$ and write a generic index as (c, s) with $c \in \{\mathrm{int}, \mathrm{pt}\}$ and $s \in [0,1]$.

Step 1 (Embedding Φ). Take an arbitrary $\mathcal{A} = (A, \nu) \in \mathrm{NCS}(U)$, i.e. for each $x \in U$ we have

$$A_\bullet(x) = [A_\bullet^-(x), A_\bullet^+(x)] \subseteq [0,1] \quad (\bullet \in \{T, I, F\}) \qquad \text{and} \qquad \nu_\bullet(x) \in [0,1].$$

Define $\mu = \Phi(\mathcal{A}) : U \times \mathbb{I} \to [0,1]^3$ by

$$\mu(x; (c,s)) = \big(T(x;(c,s)),\ I(x;(c,s)),\ F(x;(c,s))\big),$$

where, for each $x \in U$ and $s \in [0,1]$,

$$\begin{aligned}
\text{(interval channel)}\quad T(x; (\mathrm{int}, s)) &:= (1-s)\, A_T^-(x) + s\, A_T^+(x),\\
I(x; (\mathrm{int}, s)) &:= (1-s)\, A_I^-(x) + s\, A_I^+(x),\\
F(x; (\mathrm{int}, s)) &:= (1-s)\, A_F^-(x) + s\, A_F^+(x);
\end{aligned}$$

$$\text{(point channel)}\quad T(x; (\mathrm{pt}, s)) := \nu_T(x), \quad I(x; (\mathrm{pt}, s)) := \nu_I(x), \quad F(x; (\mathrm{pt}, s)) := \nu_F(x).$$

Clearly μ is a neutrosophic hypercubic set of dimension 2 on U indexed by $\mathbb{I}$.

Step 2 (Cubic reduction $\mathcal{R}$). Given any $\mu : U \times \mathbb{I} \to [0,1]^3$, define $\mathcal{R}(\mu) = (\widetilde{A}, \widetilde{\nu}) \in \mathrm{NCS}(U)$ by, for each $x \in U$,

$$\widetilde{A}_\bullet^-(x) := \inf_{s \in [0,1]} \bullet(x; (\mathrm{int}, s)), \qquad \widetilde{A}_\bullet^+(x) := \sup_{s \in [0,1]} \bullet(x; (\mathrm{int}, s)) \quad (\bullet \in \{T, I, F\}),$$

and

$$\widetilde{\nu}_\bullet(x) := \bullet(x; (\mathrm{pt}, 0)) \quad (\bullet \in \{T, I, F\}).$$

By construction, $\widetilde{A}_\bullet(x) = [\widetilde{A}_\bullet^-(x), \widetilde{A}_\bullet^+(x)] \subseteq [0,1]$ and $\widetilde{\nu}_\bullet(x) \in [0,1]$, hence $\mathcal{R}(\mu) \in \mathrm{NCS}(U)$.

Step 3 ($\mathcal{R} \circ \Phi = \mathrm{id}$). For $\mu = \Phi(\mathcal{A})$ as in Step 1 and any fixed $x \in U$, each coordinate is an affine function of s on $[0,1]$:

$$f_\bullet(s) := (1-s)\, A_\bullet^-(x) + s\, A_\bullet^+(x) = A_\bullet^-(x) + s\big(A_\bullet^+(x) - A_\bullet^-(x)\big).$$

Since $A_\bullet^+(x) \geq A_\bullet^-(x)$, on the compact interval $[0,1]$ we have

$$\inf_{s \in [0,1]} f_\bullet(s) = f_\bullet(0) = A_\bullet^-(x), \qquad \sup_{s \in [0,1]} f_\bullet(s) = f_\bullet(1) = A_\bullet^+(x).$$

Therefore

$$\widetilde{A}_\bullet^-(x) = A_\bullet^-(x), \qquad \widetilde{A}_\bullet^+(x) = A_\bullet^+(x) \qquad (\bullet \in \{T, I, F\}).$$

On the point channel we set $\bullet(x; (\mathrm{pt}, 0)) = \nu_\bullet(x)$ by definition, hence $\widetilde{\nu}_\bullet(x) = \nu_\bullet(x)$. Consequently,

$$\mathcal{R}\big(\Phi(\mathcal{A})\big) = (A, \nu) = \mathcal{A},$$

i.e. $\mathcal{R} \circ \Phi = \mathrm{id}_{\mathrm{NCS}(U)}$.

Step 4 (Injectivity and strictness). Injectivity of Φ follows from Step 3. Moreover, the inclusion is strict: e.g. for $U = \{x\}$, define μ with $T(x; (\mathrm{pt}, 0)) = 0.2$ and $T(x; (\mathrm{pt}, 1)) = 0.8$ (keep I, F constant); such μ cannot equal $\Phi(\mathcal{A})$ for any cubic $\mathcal{A}$, because the point channel of $\Phi(\mathcal{A})$ is constant in s by construction. Hence $\Phi(\mathrm{NCS}(U)) \subsetneq \mathrm{NHS}(U; 2, \mathbb{I})$. □

Referring to the concept of the Neutrosophic Hypercubic Set introduced above, the Refined Neutrosophic Hypercubic Set can be defined as follows.

Definition 3.21.5 (Refined Neutrosophic Hypercubic Set)**.** Let U be a nonempty universe, let $d \in \mathbb{N}$, and let $\mathbb{I} = I_1 \times \cdots \times I_d$ be a d–dimensional index hypercube (I_j nonempty sets). Fix refinement sizes $p, q, r \in \mathbb{N}$. A *refined neutrosophic hypercubic set (of type (p, q, r)) on U indexed by* $\mathbb{I}$ is a mapping

$$\mu : U \times \mathbb{I} \longrightarrow [0,1]^p \times [0,1]^q \times [0,1]^r, \qquad (x, \alpha) \longmapsto \mu(x; \alpha) = \big(T(x;\alpha),\ I(x;\alpha),\ F(x;\alpha)\big),$$

where

$$T(x;\alpha) = (T_1(x;\alpha), \ldots, T_p(x;\alpha)) \in [0,1]^p,$$
$$I(x;\alpha) = (I_1(x;\alpha), \ldots, I_q(x;\alpha)) \in [0,1]^q,$$
$$F(x;\alpha) = (F_1(x;\alpha), \ldots, F_r(x;\alpha)) \in [0,1]^r.$$

Equivalently, a refined neutrosophic hypercubic set is a family $\{\mu_\alpha : U \to [0,1]^p \times [0,1]^q \times [0,1]^r\}_{\alpha \in \mathbb{I}}$ of refined neutrosophic sets parametrized by $\alpha \in \mathbb{I}$.

Example 3.21.6 (Refined Neutrosophic Hypercubic Set for air–quality alerts)**.** Let $U = \{\text{HighAlert}\}$ be a universe with the single label "issue high smog alert". Take dimension $d = 2$ with index sets

$$I_1 = \{\text{Center}, \text{Suburb}\}, \qquad I_2 = \{\text{Morning}, \text{Evening}\},$$

so the index hypercube is

$$\mathbb{I} = I_1 \times I_2 = \{(\text{Center}, \text{Morning}), (\text{Center}, \text{Evening}), (\text{Suburb}, \text{Morning}), (\text{Suburb}, \text{Evening})\}.$$

Choose refinement type $(p, q, r) = (2, 1, 1)$. For each $(z, t) \in \mathbb{I}$ define

$$\mu\big(\text{HighAlert}; (z,t)\big) = \big(T(z,t),\ I(z,t),\ F(z,t)\big) \in [0,1]^2 \times [0,1] \times [0,1],$$

where $T = (T_1, T_2)$ uses T_1 for the main sensor network and T_2 for a satellite model; I_1 represents data uncertainty; F_1 encodes evidence against the alert.

Concretely:

(z,t)	$T(z,t) = (T_1, T_2)$	$I(z,t),\ F(z,t)$
(Center, Morning)	(0.82, 0.78)	$I = 0.10,\ F = 0.12$
(Center, Evening)	(0.76, 0.80)	$I = 0.12,\ F = 0.15$
(Suburb, Morning)	(0.60, 0.65)	$I = 0.18,\ F = 0.22$
(Suburb, Evening)	(0.68, 0.70)	$I = 0.16,\ F = 0.20$

Thus $\mu : U \times \mathbb{I} \to [0,1]^2 \times [0,1] \times [0,1]$ is a refined neutrosophic hypercubic set: for each index pair (z,t), we assign a refined truth vector (two sources), a single indeterminacy degree, and a single falsity degree. Slices over I_1 or I_2 (e.g. all times in the Center district) and marginals (e.g. maxima of T_1, T_2 over $\mathbb{I}$) support multi–index analysis of the alert decision.

Theorem 3.21.7 (Refined hypercubic sets generalize both hypercubic and refined cubic sets)**.** *Let U be a nonempty universe.*

(a) (Generalization of neutrosophic hypercubic sets.) For every d and index hypercube $\mathbb{I}$, the class $\mathrm{RNHS}(U; d, \mathbb{I}; p{=}1, q{=}1, r{=}1)$ *equals* $\mathrm{NHS}(U; d, \mathbb{I})$*. In particular, the inclusion functor*

$$\iota : \ \mathrm{NHS}(U; d, \mathbb{I}) \ \hookrightarrow \ \mathrm{RNHS}(U; d, \mathbb{I}; 1, 1, 1), \qquad (T, I, F) \mapsto \big((T), (I), (F)\big),$$

is an isomorphism onto its image (indeed, an equality of classes).

(b) (Generalization of refined neutrosophic cubic sets.) Set $d = 2$ and $\mathbb{I} := \{\text{int}, \text{pt}\} \times [0,1]$. There exist maps

$$\Phi : \ \mathrm{RNCS}(U; p, q, r) \ \hookrightarrow \ \mathrm{RNHS}\big(U; 2, \mathbb{I};\ p, q, r\big), \quad \mathcal{R} : \ \mathrm{RNHS}\big(U; 2, \mathbb{I};\ p, q, r\big) \ \longrightarrow \ \mathrm{RNCS}(U; p, q, r),$$

such that $\mathcal{R} \circ \Phi = \mathrm{id}_{\mathrm{RNCS}(U;p,q,r)}$. Hence every refined neutrosophic cubic set is canonically realized as a refined neutrosophic hypercubic set of dimension 2.

Proof. (a) By Definition **??**, a neutrosophic hypercubic set is precisely the case $p = q = r = 1$ of Definition 3.21.5. Thus $\mathrm{NHS}(U; d, \mathbb{I}) = \mathrm{RNHS}(U; d, \mathbb{I}; 1, 1, 1)$ and the stated inclusion is the identity on objects and morphisms.

(b) Fix $\mathbb{I} = \{\mathrm{int}, \mathrm{pt}\} \times [0, 1]$ and write indices as (c, s) with $c \in \{\mathrm{int}, \mathrm{pt}\}$, $s \in [0, 1]$.

Embedding Φ. Let $C = (\mathcal{A}, \rho) \in \mathrm{RNCS}(U; p, q, r)$. For each $x \in U$ and $s \in [0, 1]$, define the refined hypercubic map $\mu = \Phi(C) : U \times \mathbb{I} \to [0, 1]^p \times [0, 1]^q \times [0, 1]^r$ by

$$\mu(x; (c, s)) = \big(T(x; (c, s)),\ I(x; (c, s)),\ F(x; (c, s))\big),$$

where, componentwise,

$$\begin{aligned}
&\text{(interval channel)} \quad T_j(x; (\mathrm{int}, s)) := (1 - s)\, T^-_{x,j} + s\, T^+_{x,j} \quad (1 \le j \le p),\\
&\qquad\qquad I_k(x; (\mathrm{int}, s)) := (1 - s)\, I^-_{x,k} + s\, I^+_{x,k} \quad (1 \le k \le q),\\
&\qquad\qquad F_\ell(x; (\mathrm{int}, s)) := (1 - s)\, F^-_{x,\ell} + s\, F^+_{x,\ell} \quad (1 \le \ell \le r);\\
&\text{(point channel)} \quad T_j(x; (\mathrm{pt}, s)) := \tau_{x,j},\ \ I_k(x; (\mathrm{pt}, s)) := \iota_{x,k},\ \ F_\ell(x; (\mathrm{pt}, s)) := \varphi_{x,\ell}.
\end{aligned}$$

Clearly $\mu \in \mathrm{RNHS}(U; 2, \mathbb{I}; p, q, r)$.

Reduction $\mathcal{R}$. Given $\mu \in \mathrm{RNHS}(U; 2, \mathbb{I}; p, q, r)$, define $\mathcal{R}(\mu) = (\widetilde{\mathcal{A}}, \widetilde{\rho}) \in \mathrm{RNCS}(U; p, q, r)$ by, for each $x \in U$,

$$\widetilde{\mathcal{A}}(x) = \big(\widetilde{\mathbf{A}}_T(x), \widetilde{\mathbf{A}}_I(x), \widetilde{\mathbf{A}}_F(x)\big),$$

with endpoints recovered via the interval channel by taking, for all admissible indices,

$$\widetilde{T}^-_{x,j} := \inf_{s \in [0,1]} T_j(x; (\mathrm{int}, s)), \quad \widetilde{T}^+_{x,j} := \sup_{s \in [0,1]} T_j(x; (\mathrm{int}, s)),$$

$$\widetilde{I}^-_{x,k} := \inf_{s \in [0,1]} I_k(x; (\mathrm{int}, s)), \quad \widetilde{I}^+_{x,k} := \sup_{s \in [0,1]} I_k(x; (\mathrm{int}, s)),$$

$$\widetilde{F}^-_{x,\ell} := \inf_{s \in [0,1]} F_\ell(x; (\mathrm{int}, s)), \quad \widetilde{F}^+_{x,\ell} := \sup_{s \in [0,1]} F_\ell(x; (\mathrm{int}, s)),$$

and refined point components read off from a fixed point–slice of the point channel, e.g.

$$\widetilde{\rho}_T(x) := \big(T_1(x; (\mathrm{pt}, 0)), \ldots, T_p(x; (\mathrm{pt}, 0))\big),$$

with analogous definitions for $\widetilde{\rho}_I(x)$ and $\widetilde{\rho}_F(x)$.

Verification that $\mathcal{R} \circ \Phi = \mathrm{id}$. For $\mu = \Phi(C)$ and any fixed x and component, the interval–channel trajectories are affine in s:

$$f(s) = (1 - s)a^- + sa^+, \qquad s \in [0, 1],$$

so on the compact interval $[0, 1]$,

$$\inf_{s \in [0,1]} f(s) = f(0) = a^-, \qquad \sup_{s \in [0,1]} f(s) = f(1) = a^+.$$

Applied to every refined coordinate, this yields

$$(\widetilde{T}^-_{x,j}, \widetilde{T}^+_{x,j}) = (T^-_{x,j}, T^+_{x,j}), \quad (\widetilde{I}^-_{x,k}, \widetilde{I}^+_{x,k}) = (I^-_{x,k}, I^+_{x,k}), \quad (\widetilde{F}^-_{x,\ell}, \widetilde{F}^+_{x,\ell}) = (F^-_{x,\ell}, F^+_{x,\ell}).$$

On the point channel, by construction $T_j(x; (\mathrm{pt}, s)) \equiv \tau_{x,j}$ (constant in s), and similarly for I_k, F_ℓ; hence reading at $s = 0$ recovers $\rho(x)$. Therefore $\mathcal{R}(\Phi(C)) = C$, proving $\mathcal{R} \circ \Phi = \mathrm{id}$.

This completes the proof of (b). □

3.22 Neutrosophic Tolerance

Neutrosophic tolerance specifies acceptable variations around baseline truth, indeterminacy, and falsity by defining componentwise intervals for robust decisions under uncertainty [40]. Related concepts such as neutrosophic rough information [162, 163] abd Fuzzy tolerance graphs [17, 164, 165] are also well known.

Definition 3.22.1 (Neutrosophic Tolerance)**.** [40] Fix baseline degrees $(T, I, F) \in [0, 1]^3$ and tolerances $(\tau_T, \tau_I, \tau_F) \in [0, 1]^3$. The *tolerance neighborhood* of (T, I, F) is

$$\mathcal{N}_\tau(T, I, F) \;=\; \big([T-\tau_T, T+\tau_T] \cap [0, 1]\big) \times \big([I-\tau_I, I+\tau_I] \cap [0, 1]\big) \times \big([F-\tau_F, F+\tau_F] \cap [0, 1]\big).$$

Given a candidate triple $(\widetilde{T}, \widetilde{I}, \widetilde{F})$, we say it is *tolerance-consistent with* (T, I, F) *at level* τ if $(\widetilde{T}, \widetilde{I}, \widetilde{F}) \in \mathcal{N}_\tau(T, I, F)$. A *refined tolerance* splits each component as $T = (T_1, T_2)$ with T_1 *without* tolerance and T_2 *with* tolerance (similarly for I, F), and imposes $(T_1, \widetilde{T_1}) = (T_1, T_1)$ exactly while $(T_2, \widetilde{T_2})$ is constrained by an interval as above.

Example 3.22.2 (Supplier SLA audit with neutrosophic tolerance)**.** A supplier SLA audit evaluates vendors' compliance with agreed service levels, tracking delivery reliability, responsiveness, contractual performance risks, and penalties (cf. [166]).

A purchasing team tracks monthly delivery performance for a critical supplier using a neutrosophic triple (T, I, F), where: T = on-time evidence, I = data ambiguity (missing scans, partial EDI), F = late evidence.

Baseline and tolerances. From historical SLAs they set

$$(T, I, F) = (0.85,\ 0.08,\ 0.07), \qquad (\tau_T, \tau_I, \tau_F) = (0.05,\ 0.03,\ 0.04).$$

Hence the tolerance neighborhood is

$$\mathcal{N}_\tau(T, I, F) = \big([0.85-0.05,\ 0.85+0.05]\cap[0, 1]\big)\times\big([0.08-0.03,\ 0.08+0.03]\cap[0, 1]\big)\times\big([0.07-0.04,\ 0.07+0.04]\cap[0, 1]\big).$$

Compute the intervals explicitly:

$$T\text{-range} = [0.80,\ 0.90], \quad I\text{-range} = [0.05,\ 0.11], \quad F\text{-range} = [0.03,\ 0.11].$$

Month A (candidate measurement). The audit yields $(\widetilde{T}, \widetilde{I}, \widetilde{F}) = (0.82,\ 0.10,\ 0.08)$. Check componentwise:

$$0.80 \le 0.82 \le 0.90, \quad 0.05 \le 0.10 \le 0.11, \quad 0.03 \le 0.08 \le 0.11.$$

Thus $(\widetilde{T}, \widetilde{I}, \widetilde{F}) \in \mathcal{N}_\tau(T, I, F)$; Month A is *tolerance–consistent*. A simple acceptance score with indeterminacy penalty $\alpha = 0.5$ is

$$S = \widetilde{T} - \alpha\widetilde{I} - \widetilde{F} = 0.82 - 0.5 \cdot 0.10 - 0.08 = 0.82 - 0.05 - 0.08 = 0.69.$$

If the decision threshold is $\theta = 0.60$, then $S = 0.69 \ge \theta$ and the supplier *passes*.

Month B (counterexample). Suppose $(\widehat{T}, \widehat{I}, \widehat{F}) = (0.76,\ 0.12,\ 0.15)$. Here $0.76 < 0.80$ and $0.15 > 0.11$, so $(\widehat{T}, \widehat{I}, \widehat{F}) \notin \mathcal{N}_\tau(T, I, F)$; Month B is *tolerance–inconsistent* and triggers remediation.

Neutrosophic tolerance gives a transparent, componentwise band around the SLA baseline: it accepts routine variability (within bands) while flagging months where truth drops or falsity/ambiguity exceeds agreed limits.

3.23 Symmetric Neutrosophic Set

A symmetric neutrosophic set models each element with truth and falsity equidistant from indeterminacy, ensuring $T + F = 2I$ pointwise, componentwise, for robustness [39].

Definition 3.23.1 (Symmetric Neutrosophic Set)**.** [39] Let X be a universe. A (single-valued) neutrosophic set A on X is given by three functions $T_A, I_A, F_A : X \to [0, 1]$. We call A *symmetric* if, for every $x \in X$,

$$\left|T_A(x) - I_A(x)\right| = \left|F_A(x) - I_A(x)\right|.$$

Equivalently, $I_A(x)$ is an axis of symmetry between truth and falsity degrees at x; that is, $T_A(x) + F_A(x) = 2I_A(x)$. The same condition applies componentwise to refined/interval neutrosophic sets.

Example 3.23.2 (Customer–feedback triage with a Symmetric Neutrosophic Set)**.** Let the universe $X = \{\mathrm{r1}, \mathrm{r2}, \mathrm{r3}\}$ be three product reviews. For each $x \in X$, define neutrosophic degrees: $T(x)$ = positive evidence proportion, $I(x)$ = ambiguous/unclear proportion, $F(x)$ = negative evidence proportion.

We impose symmetry: $T(x) + F(x) = 2\,I(x)$ for every x.

Assigned degrees (all in $[0, 1]$**).**

x	$T(x)$	$I(x)$	$F(x)$
r1	0.82	0.60	0.38
r2	0.40	0.35	0.30
r3	0.15	0.25	0.35

Symmetry checks (explicit arithmetic).

$$\begin{aligned}
&\mathrm{r1}: T(\mathrm{r1}) + F(\mathrm{r1}) = 0.82 + 0.38 = 1.20, \quad 2I(\mathrm{r1}) = 2 \times 0.60 = 1.20;\\
&\mathrm{r2}: T(\mathrm{r2}) + F(\mathrm{r2}) = 0.40 + 0.30 = 0.70, \quad 2I(\mathrm{r2}) = 2 \times 0.35 = 0.70;\\
&\mathrm{r3}: T(\mathrm{r3}) + F(\mathrm{r3}) = 0.15 + 0.35 = 0.50, \quad 2I(\mathrm{r3}) = 2 \times 0.25 = 0.50.
\end{aligned}$$

Thus $T(x) + F(x) = 2I(x)$ holds for all three reviews, so the set is symmetric.

Indeterminacy $I(x)$ acts as a neutral "axis." If $T(x) > I(x)$ the review trends favorable; if $F(x) > I(x)$ it trends unfavorable; ties ($T = F$) sit exactly on the neutral axis I. The symmetry guarantees that "distance" from neutrality is identical on the truth and falsity sides, stabilizing triage decisions under balanced uncertainty.

3.24 Neutrosophic Orthogonal Set

A neutrosophic orthogonal set pair has elementwise proportional truth, indeterminacy, and falsity degrees, yielding zero 2×2 minors and balanced independence [39].

Definition 3.24.1 (Neutrosophic Orthogonality)**.** [39] Let A, B be neutrosophic sets on X with degrees (T_A, I_A, F_A) and (T_B, I_B, F_B). We say that A and B are *neutrosophically orthogonal*, written $A \perp_N B$, if for all $x \in X$ the cross-proportionality conditions hold:

$$T_A(x)\,F_B(x) = F_A(x)\,T_B(x), \qquad T_A(x)\,I_B(x) = I_A(x)\,T_B(x), \qquad F_A(x)\,I_B(x) = I_A(x)\,F_B(x).$$

Equivalently, the triples are pointwise proportional: $\exists\lambda(x) \geq 0$ such that $(T_A, I_A, F_A)(x) = \lambda(x)\,(T_B, I_B, F_B)(x)$, capturing a neutrosophic "right angle" by vanishing of all 2×2 minors.

Example 3.24.2 (Hospital triage: two modules with neutrosophic orthogonality)**.** Let $X = \{\mathrm{p}_1, \mathrm{p}_2, \mathrm{p}_3\}$ be three emergency patients. Module B (expert committee) and module A (AI triage) assign, for each patient $x \in X$, neutrosophic degrees $(T, I, F) \in [0, 1]^3$ of "needs immediate intervention" (truth), "unclear" (indeterminacy), and "does not need immediate intervention" (falsity). We set A to be a pointwise scalar multiple of B: $(T_A, I_A, F_A)(x) = \lambda(x)\,(T_B, I_B, F_B)(x)$ with $\lambda(x) \geq 0$.

Assigned values.

	$B: (T_B, I_B, F_B)$			$\lambda(x)$	$A: (T_A, I_A, F_A)$		
p_1	0.80	0.10	0.20	0.50	0.40	0.05	0.10
p_2	0.60	0.30	0.40	0.75	0.45	0.225	0.30
p_3	0.40	0.50	0.10	0.25	0.10	0.125	0.025

Orthogonality checks (cross–proportionality) for each $x \in X$. We verify $T_A F_B = F_A T_B$, $T_A I_B = I_A T_B$, and $F_A I_B = I_A F_B$.

For p_1:

$$\begin{aligned} T_A F_B &= 0.40 \times 0.20 = 0.080, & F_A T_B &= 0.10 \times 0.80 = 0.080; \\ T_A I_B &= 0.40 \times 0.10 = 0.040, & I_A T_B &= 0.05 \times 0.80 = 0.040; \\ F_A I_B &= 0.10 \times 0.10 = 0.010, & I_A F_B &= 0.05 \times 0.20 = 0.010. \end{aligned}$$

For p_2:

$$\begin{aligned} T_A F_B &= 0.45 \times 0.40 = 0.180, & F_A T_B &= 0.30 \times 0.60 = 0.180; \\ T_A I_B &= 0.45 \times 0.30 = 0.135, & I_A T_B &= 0.225 \times 0.60 = 0.135; \\ F_A I_B &= 0.30 \times 0.30 = 0.090, & I_A F_B &= 0.225 \times 0.40 = 0.090. \end{aligned}$$

For p_3:

$$\begin{aligned} T_A F_B &= 0.10 \times 0.10 = 0.010, & F_A T_B &= 0.025 \times 0.40 = 0.010; \\ T_A I_B &= 0.10 \times 0.50 = 0.050, & I_A T_B &= 0.125 \times 0.40 = 0.050; \\ F_A I_B &= 0.025 \times 0.50 = 0.0125, & I_A F_B &= 0.125 \times 0.10 = 0.0125. \end{aligned}$$

All three equalities hold for each patient, hence $A \perp_N B$ on X. Operationally, A and B convey the same triage signal up to a patient–specific scale $\lambda(x)$, so their neutrosophic "angles" are orthogonal in the sense of vanishing 2×2 minors.

3.25 Algebraic Neutrosophic Set

An Algebraic Neutrosophic Set extends a magma by adjoining idempotent indeterminacy I, representing uncertain components, with operations lifted componentwise algebraically [39].

Definition 3.25.1 (Algebraic Neutrosophic Set)**.** [39] Let $(M, \cdot)$ be a nonempty algebraic magma (e.g., groupoid, semigroup, or AG-groupoid). Introduce a literal indeterminacy I with $I^2 = I$. The *neutrosophic extension* of M is

$$M_I := \{ a + bI \mid a, b \in M \}.$$

An *Algebraic Neutrosophic Set* is a pair $(M_I, \star)$ where $\star$ is a binary law on M_I such that $(M_I, \star)$ satisfies the axioms required by the chosen algebraic type. A canonical instance, when $(M, \cdot)$ is an AG-groupoid, is

$$(a + bI) \star (c + dI) := (a \cdot c) + (b \cdot d) I \qquad (a, b, c, d \in M),$$

which makes $(M_I, \star)$ again an AG-groupoid since $((x \star y) \star z) = ((z \star y) \star x)$ holds.

Example 3.25.2 (Retail inventory aggregation with indeterminate in-transit units)**.** Retail inventory represents all goods a store holds for sale, monitored to balance stock availability, cash flow, and demand effectively (cf. [167]).

Let $(M, +)$ be the commutative monoid $(\mathbb{N}_{\geq 0}, +)$. Introduce a literal indeterminacy I with $I^2 = I$ and form the neutrosophic carrier $M_I = \{a + bI : a, b \in \mathbb{N}_{\geq 0}\}$. Define the operation

$$(a + bI) \oplus (c + dI) := (a + c) + (b + d) I,$$

so $(M_I, \oplus)$ is an *algebraic neutrosophic monoid.* Here a counts *physically verified* units and b counts *indeterminate in-transit* units.

Scenario. A store aggregates stock for three SKUs (A,B,C):

$$\text{A} = 12 + 3I, \qquad \text{B} = 8 + 0I, \qquad \text{C} = 5 + 2I.$$

Compute the total neutrosophic stock S:

$$\begin{aligned} S &= (\mathrm{A}) \ \oplus \ (\mathrm{B}) \ \oplus \ (\mathrm{C}) \\ &= (12 + 3I) \ \oplus \ (8 + 0I) \ \oplus \ (5 + 2I) \\ &= \big((12 + 8) + 5\big) \ + \ \big((3 + 0) + 2\big)\, I \\ &= 25 \ + \ 5I. \end{aligned}$$

Interpretation:

$$\text{Guaranteed (on hand)} = 25, \qquad \text{Indeterminate (in transit)} = 5.$$

Thus the realized stock lies between 25 (none of the indeterminate units arrive) and $25 + 5 = 30$ (all do), while planning computations remain closed under $\oplus$.

With a reorder threshold $\theta = 28$, decide to reorder if the *guaranteed* component is below θ:

$$25 < 28 \ \Rightarrow \ \text{Place reorder now (even if the } 5I \text{ may materialize later).}$$

This uses $(M_I, \oplus)$ to separate certain versus indeterminate contributions algebraically, supporting sound operational decisions under uncertainty.

3.26 Neutrosophic Labels and Plithogenic Labels

A neutrosophic label assigns three independent degrees to a token: truth, indeterminacy, and falsity, quantifying assertion strength, ambiguity, contradiction, per [38]. A plithogenic label couples a token, attribute value, and membership vector, aggregating evidence via contradiction-weighted t-norm/conorm fusion across sources reliably.

Definition 3.26.1 (Neutrosophic Labels)**.** [38] Let Σ be a (finite or countable) set of label tokens and let X be a nonempty universe of items. A *neutrosophic label* is a pair $(\ell, \mathbf{n}) \in \Sigma \times [0, 1]^3$, where $\mathbf{n} = (T, I, F)$ encodes, respectively, the degrees of truth, indeterminacy, and falsity attached to ℓ. The (optional) normalization constraint is

$$0 \ \le \ T + I + F \ \le \ 1 \quad (\text{or} \le 3 \text{ in the nonnormalized convention}).$$

A *neutrosophic labeling* of X over Σ is either of the equivalent data:

- a map $\Lambda_N : X \to \Sigma \times [0, 1]^3$, $x \mapsto (\ell_x, (T_x, I_x, F_x))$;
- a degree assignment $\Theta_N : X \times \Sigma \to [0, 1]^3$ with $\Theta_N(x, \ell) = (T(x, \ell), I(x, \ell), F(x, \ell))$.

For singlelabel classification one may impose, for each x, an admissibility rule such as $\ell_x = \arg\max_{\ell\in\Sigma} T(x, \ell)$ together with a consistency threshold on $I(x, \ell_x)$ and $F(x, \ell_x)$. Reductions: if $I \equiv 0$ and $F = 1-T$, neutrosophic labels reduce to fuzzy labels; if $I \equiv F \equiv 0$, they reduce to crisp labels.

Example 3.26.2 (Neutrosophic label in clinical triage)**.** Let $\Sigma = \{\text{"Pneumonia"}\}$ and X be patients. For patient p, radiology strongly supports the label, some labs are pending, and differential diagnoses exist. Define the neutrosophic labeling

$$\Lambda_N(p) = \big(\text{"Pneumonia"},\ (T, I, F)\big) = (\ 0.82,\ 0.12,\ 0.06\ ).$$

Here T captures supportive evidence (x–ray consolidation, fever, hypoxia), I reflects indeterminacy (awaited cultures), and F encodes contradicting evidence (possible heart failure). A single-label decision may use $\ell_p = \arg\max_{\ell\in\Sigma} T(p, \ell)$ with thresholds on I, F for deployment.

Definition 3.26.3 (Plithogenic Labels)**.** Fix a plithogenic context

$$PS \ = \ \big(P,\ v,\ Pv,\ pdf,\ pCF\big),$$

where P is a universe of items, v is a chosen attribute with value domain Pv, $pdf : P \times Pv \to [0, 1]^s$ is the degree of appurtenance (DAF), and $pCF : Pv \times Pv \to [0, 1]$ is the degree of contradiction (DCF) with $pCF(a, a) = 0$ and $pCF(a, b) = pCF(b, a)$. Let Σ be a set of label tokens. A *plithogenic label* is a triple $(\ell, u, \mu) \in \Sigma \times Pv \times [0, 1]^s$, interpreted as label ℓ asserted at attribute value u with membership vector μ (DAF). A *plithogenic labeling* is either of the equivalent data:

- a map $\Lambda_P : P \to \Sigma \times Pv \times [0,1]^s$, $x \mapsto (\ell_x, u_x, \mu_x)$ with $\mu_x = pdf(x; u_x)$;
- a degree assignment $\Theta_P : P \times \Sigma \times Pv \to [0,1]^s$, $\Theta_P(x, \ell, u) := pdf(x; u)$.

When combining two plithogenic labels (ℓ, u_1, μ_1) and (ℓ, u_2, μ_2) referring to the same item and token, their aggregated degree is computed via a DCFweighted operator

$$\mu_{\text{agg}} := \mathcal{A}_c(\mu_1, \mu_2) = (1-c)\,T(\mu_1, \mu_2) + c\,S(\mu_1, \mu_2), \qquad c := pCF(u_1, u_2),$$

where T and S are a fixed $t$norm/$t$conorm pair acting componentwise on $[0,1]^s$. Reductions: if Pv is a singleton and $s = 1$, plithogenic labels reduce to fuzzy labels; if $s = 3$ and $\mu = (T, I, F)$ with no use of pCF, they reduce to neutrosophic labels.

Example 3.26.4 (Plithogenic label for eco-rating of a product)**.** An eco-rating summarizes a product's environmental impact, scoring factors like emissions, resource use, recyclability, and toxicity for consumers and regulators (cf. [168]).

Let P be detergents, $\Sigma = \{\text{“EcoFriendly”}\}$, and the attribute v have values $Pv = \{\text{carbon}, \text{toxicity}\}$. For item $x \in P$, take the degree of appurtenance (DAF)

$$pdf(x; \text{carbon}) = 0.90, \qquad pdf(x; \text{toxicity}) = 0.60.$$

Let the contradiction degree between criteria be $c = pCF(\text{carbon}, \text{toxicity}) = 0.40$. Choose t-norm $T(a,b) = ab$ and probabilistic t-conorm $S(a,b) = a + b - ab$. The DCF-weighted aggregation is

$$\mu_{\text{agg}} = (1-c)\,T(0.90, 0.60) + c\,S(0.90, 0.60) = 0.6 \cdot 0.54 + 0.4 \cdot 0.96 = 0.324 + 0.384 = 0.708.$$

Thus the plithogenic labeling returns

$$\Lambda_P(x) = \big(\text{“EcoFriendly”},\ (\text{carbon}, \text{toxicity}),\ 0.708\big),$$

which interpolates conjunctive and disjunctive evidence according to the contradiction between attribute values.

3.27 Neutrosophic Linguistic Labels

Neutrosophic linguistic labels assign each verbal term a truth, indeterminacy, and falsity triple, enabling graded, uncertain linguistic reasoning and aggregation [39]. Related concepts such as hesitant fuzzy linguistic term sets [169–171], linguistic neutrosophic numbers [172–174], and linguistic neutrosophic sets [175–177] are also well known.

Definition 3.27.1 (Neutrosophic Linguistic Label System)**.** [39] Let $(L, \preceq)$ be a finite linearly ordered set of linguistic labels (e.g., Very Low $\prec$ Low $\prec$ Medium $\prec$ High $\prec$ Very High). A *neutrosophic linguistic label system* is a tuple

$$\mathcal{L}_N = (L, \preceq,\ \iota,\ \tau),$$

where $\iota : L \to [0,1]^3$, $\iota(\ell) = (T_\ell, I_\ell, F_\ell)$ assigns a neutrosophic triplet to each label, and τ specifies label-level connectives (e.g., conjunction, disjunction, negation) by lifting numeric t-norm/t-conorm/negation to L:

$$\ell_1 \wedge \ell_2 := \arg\max_{\ell \in L} T_\ell \quad \text{subject to} \quad \iota(\ell) \approx \big(T(T_{\ell_1}, T_{\ell_2}),\ S(I_{\ell_1}, I_{\ell_2}),\ S(F_{\ell_1}, F_{\ell_2})\big),$$

and similarly for $\vee$; while $\neg\ell$ is the label whose triplet is closest (by a chosen metric) to (F_ℓ, I_ℓ, T_ℓ). The embedding ι is order-preserving in T and order-reversing in F.

Example 3.27.2 (Hiring decision with neutrosophic linguistic labels)**.** A hiring decision evaluates candidates' skills, fit, and risks to select the most suitable person for an organizational role today (cf. [178]).

Let $L = \{\text{VL} \prec \text{L} \prec \text{M} \prec \text{H} \prec \text{VH}\}$ be ordered linguistic labels: Very Low, Low, Medium, High, Very High. Embed each label into neutrosophic space via

$$\iota(\text{VL}) = (0.10, 0.20, 0.80), \quad \iota(\text{L}) = (0.30, 0.20, 0.60), \quad \iota(\text{M}) = (0.60, 0.20, 0.40),$$

$$\iota(\text{H}) = (0.80, 0.15, 0.20), \quad \iota(\text{VH}) = (0.95, 0.05, 0.05).$$

A candidate is evaluated on two criteria: technical fit = H, stakeholder risk = M. Lift conjunction at label level using the numeric mixer $T = \min$, $I = \max$, $F = \max$ applied to the embedded triples.

Numeric combination:

$$(T, I, F) = \big(\min\{0.80, 0.60\},\ \max\{0.15, 0.20\},\ \max\{0.20, 0.40\}\big) = (0.60, 0.20, 0.40).$$

Project back to the nearest label (Euclidean distance over triples):

$$\|(0.60, 0.20, 0.40) - \iota(\mathrm{M})\| = 0, \quad \|(0.60, 0.20, 0.40) - \iota(\mathrm{H})\| = \sqrt{(0.20)^2 + (0.05)^2 + (0.20)^2},$$

which confirms M is the closest. Hence the aggregated linguistic verdict is

$$\mathrm{H} \ \wedge \ \mathrm{M} \ = \ \mathrm{M}\ (\text{Medium}).$$

Decision aid (scalar score for clarity):

$$\text{Score} = T - \tfrac{1}{2}(I + F) = 0.60 - \tfrac{1}{2}(0.20 + 0.40) = 0.60 - 0.30 = 0.30,$$

interpreted as “proceed, with moderate caution,” matching the label M.

Chapter 4

Neutrosophic Discrete Mathematics and Combinatorics

Discrete mathematics studies distinct-valued structures, emphasizing logic, set theory, graph theory, algorithms, finite automata, computational complexity, cryptography, and coding theory [179, 180]. Combinatorics analyzes discrete configurations, counting structures and optimizing arrangements using enumeration, recurrence relations, generating functions, bijections, probabilistic methods, and algorithms [181,182]. In this chapter, we consider applying the concept of Neutrosophic Sets to areas of discrete mathematics and combinatorics, such as graph theory and number theory.

4.1 Refined Neutrosophic Hypergraph

A hypergraph is a generalization of a graph where each hyperedge can connect any number of vertices simultaneously, modeling groups [183–185]. Furthermore, as generalized concepts beyond hypergraphs, the subset-vertex graph [186–188] and the SuperHyperGraph [58, 189–192] are also well known. A single-valued neutrosophic hypergraph assigns each vertex–hyperedge incidence truth, indeterminacy, and falsity degrees, modeling uncertain multi-way relations under noise dynamics [193–195]. The neutrosophic hypergraph generalizes both the Fuzzy HyperGraph [196–198] and the Intuitionistic Fuzzy HyperGraph [199–202] frameworks. A refined neutrosophic hypergraph stores multiple truth, indeterminacy, and falsity channels per incidence, capturing sources, times, evaluators, or criteria simultaneously [39].

Definition 4.1.1 (Single-Valued Neutrosophic Hypergraph)**.** [193–195] Let $V = \{v_1, \dots, v_N\}$ be a finite vertex set, and let $\{E_i\}_{i=1}^{M}$ be a collection of non-empty neutrosophic subsets of V such that $V = \bigcup_{i=1}^{M} \mathrm{supp}(E_i)$. Each hyperedge E_i is specified by three membership functions

$$T_{E_i},\ I_{E_i},\ F_{E_i} : V \to [0,1],$$

assigning to each vertex $v \in V$ its truth, indeterminacy, and falsity degrees, respectively, and satisfying

$$0 \ \le\ T_{E_i}(v) + I_{E_i}(v) + F_{E_i}(v) \ \le\ 3 \quad \forall\, v \in V.$$

We represent E_i as the set

$$E_i \ =\ \big\{(v,\, T_{E_i}(v),\, I_{E_i}(v),\, F_{E_i}(v)) : v \in V\big\}.$$

The pair $H = (V, \{E_i\})$ is called a *single-valued neutrosophic hypergraph*.

Example 4.1.2 (Disaster response coalition as a Single-Valued Neutrosophic Hypergraph)**.** Disaster response coordinates resources and actions after crises to save lives, restore infrastructure, support communities, and reduce future risks effectively (cf. [203]).

Let the vertex set be

$$V = \{\text{Hospital}\,(H),\ \text{Fire}\,(F),\ \text{Police}\,(P),\ \text{NGO}\,(N)\}.$$

Define two hyperedges (incident types):

E_1 = Urban Earthquake Response. Neutrosophic memberships (T, I, F) per vertex:

v	H	F	P	N
$T_{E_1}(v)$	0.95	0.90	0.85	0.70
$I_{E_1}(v)$	0.05	0.08	0.10	0.20
$F_{E_1}(v)$	0.05	0.07	0.12	0.25

Interpretation: T reflects expected participation strength (capacity/readiness), I covers availability uncertainty (traffic, staffing, comms), F covers potential non-participation (conflicts, damage).

E_2 = Mass Shelter Operation.

v	H	F	P	N
$T_{E_2}(v)$	0.80	0.40	0.30	0.95
$I_{E_2}(v)$	0.10	0.20	0.25	0.05
$F_{E_2}(v)$	0.15	0.50	0.60	0.05

Here the NGO leads ($T = 0.95$), while Police has high F due to patrol priorities.

Then $H = (V, \{E_1, E_2\})$ is a single-valued neutrosophic hypergraph; coverage holds since $\mathrm{supp}(E_1) \cup \mathrm{supp}(E_2) = V$ with $\mathrm{supp}(E) = \{v \mid T_E(v) > 0\}$.

Definition 4.1.3 (Refined Neutrosophic Hypergraph)**.** [39] Let $V = \{v_1, \ldots, v_N\}$ be a finite, nonempty vertex set. Fix three positive integers $p, q, r \in \mathbb{N}$. A refined neutrosophic hyperedge on V of type (p, q, r) is a mapping

$$E = \big(T_E, I_E, F_E\big)$$

where

$$T_E = \big(T_E^{(1)}, \ldots, T_E^{(p)}\big), \quad I_E = \big(I_E^{(1)}, \ldots, I_E^{(q)}\big), \quad F_E = \big(F_E^{(1)}, \ldots, F_E^{(r)}\big),$$

and each component is a function

$$T_E^{(j)} : V \to [0,1] \ (1 \le j \le p), \quad I_E^{(k)} : V \to [0,1] \ (1 \le k \le q), \quad F_E^{(\ell)} : V \to [0,1] \ (1 \le \ell \le r),$$

such that for every $v \in V$

$$0 \le T_E^{(j)}(v) \le 1, \quad 0 \le I_E^{(k)}(v) \le 1, \quad 0 \le F_E^{(\ell)}(v) \le 1.$$

We write

$$E = \Big\{ \big(v,\ (T_E^{(1)}(v), \ldots, T_E^{(p)}(v)),\ (I_E^{(1)}(v), \ldots, I_E^{(q)}(v)),\ (F_E^{(1)}(v), \ldots, F_E^{(r)}(v))\big) \ : \ v \in V \Big\}.$$

A refined neutrosophic hypergraph of type (p, q, r) is a pair

$$H_{p,q,r} = \big(V, \{E_i\}_{i=1}^{M}\big)$$

where each E_i is a refined neutrosophic hyperedge of type (p, q, r) on V, and the family $\{E_i\}_{i=1}^{M}$ covers the vertex set in the sense that

$$V = \bigcup_{i=1}^{M} \mathrm{supp}(E_i),$$

with

$$\mathrm{supp}(E_i) := \Big\{ v \in V \ \Big| \ \max\{T_{E_i}^{(1)}(v), \ldots, T_{E_i}^{(p)}(v)\} > 0 \Big\}.$$

Remark 4.1.4. The refinement parameters (p, q, r) allow one to record, for the same vertex–hyperedge incidence, several truth channels, several indeterminacy channels, and several falsity channels simultaneously (for example, different data sources, different time stamps, or different evaluators).

Example 4.1.5 (Retail product launch as a Refined Neutrosophic Hypergraph)**.** A product launch introduces a new offering to market, coordinating marketing, logistics, sales, and support for successful adoption by customers (cf. [204]).

Consider

$$V = \{\text{Manufacturer}\,(M),\ \text{Logistics}\,(L),\ \text{Retailer}\,(R)\}$$

. We use refinement type $(p, q, r) = (2, 2, 1)$:

- Truth channels $T^{(1)}$ = self-declared capability, $T^{(2)}$ = audited historical performance.
- Indeterminacy channels $I^{(1)}$ = data latency, $I^{(2)}$ = market volatility.
- Falsity channel $F^{(1)}$ = regulatory/contractual blockers.

Define one refined hyperedge E_{launch} (joint launch readiness) with componentwise values:

	M	L	R
$T_E^{(1)}(v)$	0.90	0.80	0.75
$T_E^{(2)}(v)$	0.85	0.70	0.80
$I_E^{(1)}(v)$	0.10	0.20	0.15
$I_E^{(2)}(v)$	0.15	0.25	0.20
$F_E^{(1)}(v)$	0.05	0.10	0.08

Interpretation: M scores strongly on both truth channels; L has higher indeterminacy from volatile carrier capacity; R shows solid audited performance ($T^{(2)} = 0.80$) with modest blockers.

Form the refined neutrosophic hypergraph

$$H_{2,2,1} = \big(V, \{E_{\text{launch}}\}\big),$$

where $\text{supp}(E_{\text{launch}}) = \{v \in V \mid \max(T_E^{(1)}(v), T_E^{(2)}(v)) > 0\} = V$. When aggregating to a single readiness view, one can use

$$T^{\text{agg}}(v) = \tfrac{1}{2}\big(T_E^{(1)}(v) + T_E^{(2)}(v)\big), \quad I^{\text{agg}}(v) = \max\{I_E^{(1)}(v), I_E^{(2)}(v)\}, \quad F^{\text{agg}}(v) = F_E^{(1)}(v),$$

to rank partners or identify mitigation targets (e.g., reduce $I^{(2)}$ for L).

Theorem 4.1.6 (Single-valued neutrosophic hypergraphs are special cases)**.** *Let $H = (V, \{E_i\}_{i=1}^M)$ be a single-valued neutrosophic hypergraph as in the user's definition. Then there exists a refined neutrosophic hypergraph*

$$H_{1,1,1} = \big(V, \{\widetilde{E}_i\}_{i=1}^M\big)$$

of type $(p, q, r) = (1, 1, 1)$ such that H and $H_{1,1,1}$ carry exactly the same neutrosophic information. Consequently, the class of refined neutrosophic hypergraphs strictly generalizes the class of single-valued neutrosophic hypergraphs.

Proof. Take any single-valued neutrosophic hypergraph

$$H = (V, \{E_i\}_{i=1}^M),$$

where each

$$E_i = \{(v,\ T_{E_i}(v),\ I_{E_i}(v),\ F_{E_i}(v)) : v \in V\},$$

with

$$T_{E_i}, I_{E_i}, F_{E_i} : V \to [0, 1].$$

Define, for every $i \in \{1, \ldots, M\}$, a refined neutrosophic hyperedge $\widetilde{E_i}$ of type $(1, 1, 1)$ by

$$T^{(1)}_{\widetilde{E_i}} := T_{E_i}, \qquad I^{(1)}_{\widetilde{E_i}} := I_{E_i}, \qquad F^{(1)}_{\widetilde{E_i}} := F_{E_i}.$$

Then

$$\widetilde{E_i} = \{(v,\ (T_{E_i}(v)),\ (I_{E_i}(v)),\ (F_{E_i}(v))) : v \in V\},$$

which is precisely the same datum as the original E_i, only written in the refined format with one channel per component. Set

$$H_{1,1,1} := \big(V, \{\widetilde{E_i}\}_{i=1}^{M}\big).$$

Since the support condition in the refined definition reduces, for $(1, 1, 1)$, to

$$V = \bigcup_{i=1}^{M} \{v \in V : T_{E_i}(v) > 0\},$$

which is exactly the same as the covering condition stated for the single-valued neutrosophic hypergraph, we conclude that $H_{1,1,1}$ is a refined neutrosophic hypergraph and that it encodes the same information as H.

Finally, for $(p, q, r) \neq (1, 1, 1)$, a refined neutrosophic hypergraph can store strictly more information (e.g. two independent truth degrees for the same vertex–edge pair), so the refined class is a proper extension. This proves the statement. □

4.2 Spherical Neutrosophic Graph

A spherical neutrosophic graph assigns to each vertex and edge a triple of truth, indeterminacy, and falsity degrees bounded by the unit-sphere quadratic constraint for internal consistency [205]. Related notions include *Spherical Fuzzy Graphs* [14] and *Spherical Neutrosophic Sets* [206–208], which similarly restrict membership components to satisfy a spherical relation ensuring balanced uncertainty representation.

Definition 4.2.1 (Spherical Neutrosophic Graph)**.** [205] Let V be a finite vertex set and $E \subseteq \{\{u, v\} : u, v \in V,\ u \neq v\}$. A *spherical neutrosophic graph* is a pair $G = (A, B)$ where:

- $A : V \to [0, 1]^3$, $A(v) = (T_V(v), I_V(v), F_V(v))$, with $T_V(v)^2 + I_V(v)^2 + F_V(v)^2 \leq 1$ for all $v \in V$;
- $B : E \to [0, 1]^3$, $B(\{u, v\}) = (T_E(u, v), I_E(u, v), F_E(u, v))$, with $T_E(u, v)^2 + I_E(u, v)^2 + F_E(u, v)^2 \leq 1$ for all $\{u, v\} \in E$.

(Additional coupling inequalities between vertex and edge components may be imposed as needed.)

Example 4.2.2 (Triangle network with spherical neutrosophic annotations)**.** Let the vertex set and edge set be

$$V = \{v_1, v_2, v_3\}, \qquad E = \big\{\{v_1, v_2\}, \{v_2, v_3\}, \{v_1, v_3\}\big\}.$$

Define vertex triplets $A(v) = (T_V(v), I_V(v), F_V(v))$ and edge triplets $B(\{u, v\}) = (T_E(u, v), I_E(u, v), F_E(u, v))$ as follows:

$$A(v_1) = (0.80,\ 0.40,\ 0.10), \qquad A(v_2) = (0.60,\ 0.50,\ 0.20), \qquad A(v_3) = (0.30,\ 0.70,\ 0.50),$$
$$B(\{v_1, v_2\}) = (0.70,\ 0.30,\ 0.20), \quad B(\{v_2, v_3\}) = (0.55,\ 0.60,\ 0.10), \quad B(\{v_1, v_3\}) = (0.25,\ 0.40,\ 0.80).$$

Verification of the spherical bounds $T^2 + I^2 + F^2 \leq 1$:

$$\begin{aligned}
v_1 &: \ 0.80^2 + 0.40^2 + 0.10^2 = 0.64 + 0.16 + 0.01 = 0.81 \leq 1,\\
v_2 &: \ 0.60^2 + 0.50^2 + 0.20^2 = 0.36 + 0.25 + 0.04 = 0.65 \leq 1,\\
v_3 &: \ 0.30^2 + 0.70^2 + 0.50^2 = 0.09 + 0.49 + 0.25 = 0.83 \leq 1,
\end{aligned}$$

$$\begin{aligned}
\{v_1, v_2\} &: \ 0.70^2 + 0.30^2 + 0.20^2 = 0.49 + 0.09 + 0.04 = 0.62 \leq 1,\\
\{v_2, v_3\} &: \ 0.55^2 + 0.60^2 + 0.10^2 = 0.3025 + 0.36 + 0.01 = 0.6725 \leq 1,\\
\{v_1, v_3\} &: \ 0.25^2 + 0.40^2 + 0.80^2 = 0.0625 + 0.16 + 0.64 = 0.8625 \leq 1.
\end{aligned}$$

Hence $G = (A, B)$ is a valid spherical neutrosophic graph.

4.3 Numerical Neutrosophic Numbers and Literal Neutrosophic Numbers

Literal neutrosophic numbers are formal sums with independent indeterminacy symbols; arithmetic is symbolic on coefficients until numeric evaluation is chosen [46, 209, 210]. Numerical neutrosophic numbers assign concrete degrees to components or literals, yielding real results or triplets for truth, indeterminacy, falsity values [46, 211, 212].

Definition 4.3.1 (Literal Neutrosophic Number)**.** [46] Fix indeterminacy symbols $I_1, \ldots, I_r$ (and, optionally, literal T, F symbols). A *literal neutrosophic number* is a formal expression

$$\mathsf{L} := a + \sum_{j=1}^{r} b_j I_j \quad (a, b_j \in \mathbb{R}),$$

where the I_j are algebraically independent neutrosophic indeterminacy literals. Arithmetic is carried out symbolically (e.g. componentwise on coefficients); evaluation requires choosing numeric images for the literals.

Example 4.3.2 (Literal neutrosophic number for delivery planning (symbolic form))**.** Let $I_{\text{weather}}, I_{\text{traffic}}, I_{\text{customs}}$ be algebraically independent indeterminacy symbols. Model a shipment's expected delay (days) as the literal neutrosophic number

$$\mathsf{L}_{\text{ETA}} = \underbrace{2.00}_{\text{base}} + 0.50\, I_{\text{weather}} + 1.00\, I_{\text{traffic}} + 0.70\, I_{\text{customs}}.$$

If an additional hub transfer introduces $\mathsf{L}_{\text{hub}} = 0.40\, I_{\text{traffic}} + 0.20\, I_{\text{customs}}$, the combined symbolic delay is (coefficientwise addition)

$$\mathsf{L}_{\text{total}} = \mathsf{L}_{\text{ETA}} + \mathsf{L}_{\text{hub}} = 2.00 + 0.50\, I_{\text{weather}} + 1.40\, I_{\text{traffic}} + 0.90\, I_{\text{customs}}.$$

No numeric value is produced until one assigns degrees to the literals.

Definition 4.3.3 (Numerical Neutrosophic Number)**.** [46] A *numerical neutrosophic number* is obtained by assigning numeric degrees to the neutrosophic components, e.g. by an evaluation $\text{ev} : \{I_1, \ldots, I_r\} \to [0, 1]$ and setting

$$\mathsf{N} := \text{ev}(\mathsf{L}) = a + \sum_{j=1}^{r} b_j\, \text{ev}(I_j) \in \mathbb{R},$$

or, in triplet form, as $(t, i, f) \in [0, 1]^3$ (or in refined form $(T_1, \ldots, T_p; I_1, \ldots, I_r; F_1, \ldots, F_s) \in [0, 1]^n$) when the application keeps the three (or n) numerical components explicitly.

Example 4.3.4 (Numerical neutrosophic number by evaluating literals)**.** Use the previous symbolic model and assign concrete degrees (e.g., forecasted risks)

$$\text{ev}(I_{\text{weather}}) = 0.30, \qquad \text{ev}(I_{\text{traffic}}) = 0.60, \qquad \text{ev}(I_{\text{customs}}) = 0.20.$$

Evaluating $\mathsf{L}_{\text{total}}$ yields the numerical neutrosophic number

$$\mathsf{N}_{\text{total}} = \text{ev}(\mathsf{L}_{\text{total}}) = 2.00 + 0.50 \cdot 0.30 + 1.40 \cdot 0.60 + 0.90 \cdot 0.20 = 2.00 + 0.15 + 0.84 + 0.18 = 2.89 \text{ days}.$$

Thus, after fixing literal degrees, the previously symbolic uncertainty compiles into a single computable estimate.

4.4 Neutrosophic Fibonacci Numbers

Fibonacci numbers form a sequence starting with 0 and 1; each term equals the sum of the two previous terms [213–215]. Neutrosophic Fibonacci numbers encode truth and indeterminacy via F_n, F_{n+1}, and idempotent I, forming $F_n + F_{n+1}\, I$, modeling uncertainty [40, 216]. Related concepts such as fuzzy Fibonacci numbers [217, 218] are also well known in the literature.

Definition 4.4.1 (Neutrosophic Fibonacci Numbers). [40, 216] Let $(F_n)_{n\geq 0}$ be the Fibonacci sequence with $F_0 = 0$, $F_1 = 1$ and $F_{n+1} = F_n + F_{n-1}$. Let I be a literal indeterminacy with $I^2 = I$. A *neutrosophic Fibonacci number* is any

$$\mathsf{NF}_{j,k} \ := \ F_j \ + \ F_k\, I \qquad (j, k \in \mathbb{N}).$$

A convenient Fibonacci-like neutrosophic sequence is $N_n := F_n + F_{n+1} I$ $(n \geq 0)$, endowed with componentwise addition $(a + bI) \oplus (c + dI) := (a + c) + (b + d)I$. Then

$$N_{n+1} \ = \ N_n \ \oplus \ N_{n-1} \quad (n \geq 1),$$

since

$$N_n \oplus N_{n-1} = \big(F_n + F_{n-1}\big) + \big(F_{n+1} + F_n\big)I = F_{n+1} + F_{n+2}I = N_{n+1}.$$

Example 4.4.2 (Concrete computation with Neutrosophic Fibonacci Numbers). Let $(F_n)_{n\geq 0}$ be the Fibonacci sequence:

$$F_0 = 0, \quad F_1 = 1, \quad F_2 = 1, \quad F_3 = 2, \quad F_4 = 3, \quad F_5 = 5, \quad F_6 = 8.$$

Fix a literal indeterminacy I with $I^2 = I$ and define the neutrosophic Fibonacci sequence

$$N_n := F_n + F_{n+1}I \qquad (n \geq 0),$$

with componentwise addition $(a + bI) \oplus (c + dI) = (a + c) + (b + d)I$.

Explicit values:

$$\begin{aligned} N_0 &= F_0 + F_1 I = 0 + 1I, \\ N_1 &= F_1 + F_2 I = 1 + 1I, \\ N_2 &= F_2 + F_3 I = 1 + 2I, \\ N_3 &= F_3 + F_4 I = 2 + 3I, \\ N_4 &= F_4 + F_5 I = 3 + 5I. \end{aligned}$$

Verification of the neutrosophic Fibonacci recursion $N_{n+1} = N_n \oplus N_{n-1}$ at $n = 3$:

$$\underbrace{N_3}_{2+3I} \ \oplus \ \underbrace{N_2}_{1+2I} = (2 + 3I) \oplus (1 + 2I) = (2 + 1) + (3 + 2)I = 3 + 5I = \underbrace{N_4}_{3+5I}.$$

A specific neutrosophic Fibonacci number is, for instance,

$$\mathsf{NF}_{3,5} = F_3 + F_5 I = 2 + 5I.$$

4.5 Neutrosophic Codes

Coding theory studies design of codes for reliable data transmission and storage, correcting errors introduced by noisy channels and interference [219–221]. Neutrosophic codes encode messages over neutrosophic symbols, modeling truth, indeterminacy, falsity; decoding uses neutrosophic distances and aggregation for uncertain communication [40]. Related concepts such as Fuzzy Codes are also well known (cf. [222, 223]).

Definition 4.5.1 (Neutrosophic Codes). [40] Let $\mathbb{N}\!n$ be a neutrosophic alphabet (e.g. $\mathbb{N}\!n = [0, 1]^3$ with entries (T, I, F), or a larger numeric/literal neutrosophic set). For a length $n \in \mathbb{N}$, a *neutrosophic code* is a subset $C \subseteq \mathbb{N}\!n^n$. A *neutrosophic distance* is any metric d_N on $\mathbb{N}\!n^n$ compatible with the neutrosophic structure; for instance, for $x = (x_j)_{j=1}^n$, $y = (y_j)_{j=1}^n$ with $x_j = (T_j, I_j, F_j)$ and $y_j = (T'_j, I'_j, F'_j)$,

$$d_N(x, y) \ = \ \sum_{j=1}^{n} \Big(\alpha\, |T_j - T'_j| + \beta\, |I_j - I'_j| + \gamma\, |F_j - F'_j|\Big), \quad \alpha, \beta, \gamma > 0.$$

The *minimum distance* of C is $d_N(C) := \min\{d_N(x, y) : x \neq y,\ x, y \in C\}$. Encoding/decoding may use symbolwise neutrosophic aggregation (e.g. T with a t-norm, I, F with a t-conorm) and decision rules constrained by d_N. Linear or algebraic variants can be defined when $\mathbb{N}\!n$ is endowed with a suitable semiring/module structure.

Example 4.5.2 (A length-3 Neutrosophic Code: minimum distance and decoding)**.** Let the neutrosophic alphabet be $\mathbb{N}n = [0,1]^3$ with symbols (T, I, F). Consider the code $C \subseteq \mathbb{N}n^3$ with three codewords

$$\begin{aligned} c_1 &= \big((0.90, 0.05, 0.05),\ (0.80, 0.10, 0.10),\ (0.85, 0.10, 0.05)\big),\\ c_2 &= \big((0.20, 0.30, 0.50),\ (0.30, 0.40, 0.30),\ (0.25, 0.35, 0.40)\big),\\ c_3 &= \big((0.80, 0.15, 0.05),\ (0.70, 0.20, 0.10),\ (0.10, 0.20, 0.70)\big). \end{aligned}$$

Use the neutrosophic ℓ_1 metric with unit weights

$$d_N(x, y) = \sum_{j=1}^{3} \Big(|T_j - T'_j| + |I_j - I'_j| + |F_j - F'_j|\Big),$$

for $x = (T_j, I_j, F_j)_{j=1}^3$, $y = (T'_j, I'_j, F'_j)_{j=1}^3$.

Pairwise distances (computed componentwise, then summed):

$$\begin{aligned} d_N(c_1, c_2) &= \underbrace{(0.70 + 0.25 + 0.45)}_{j=1} + \underbrace{(0.50 + 0.30 + 0.20)}_{j=2} + \underbrace{(0.60 + 0.25 + 0.35)}_{j=3}\\ &= 1.40 + 1.00 + 1.20 = 3.60,\\ d_N(c_1, c_3) &= \underbrace{(0.10 + 0.10 + 0.00)}_{j=1} + \underbrace{(0.10 + 0.10 + 0.00)}_{j=2} + \underbrace{(0.75 + 0.10 + 0.65)}_{j=3}\\ &= 0.20 + 0.20 + 1.50 = 1.90,\\ d_N(c_2, c_3) &= \underbrace{(0.60 + 0.15 + 0.45)}_{j=1} + \underbrace{(0.40 + 0.20 + 0.20)}_{j=2} + \underbrace{(0.15 + 0.15 + 0.30)}_{j=3}\\ &= 1.20 + 0.80 + 0.60 = 2.60. \end{aligned}$$

Hence the minimum distance is

$$d_N(C) = \min\{3.60, 1.90, 2.60\} = 1.90.$$

Unique decoding is guaranteed for perturbations with radius $< \frac{d_N(C)}{2} = 0.95$.

Suppose the receiver observes

$$r = \big((0.88, 0.07, 0.05),\ (0.79, 0.12, 0.09),\ (0.80, 0.15, 0.05)\big).$$

Distances to codewords:

$$\begin{aligned} d_N(r, c_1) &= \underbrace{(0.02 + 0.02 + 0.00)}_{j=1} + \underbrace{(0.01 + 0.02 + 0.01)}_{j=2} + \underbrace{(0.05 + 0.05 + 0.00)}_{j=3}\\ &= 0.04 + 0.04 + 0.10 = 0.18,\\ d_N(r, c_3) &= \underbrace{(0.08 + 0.08 + 0.00)}_{j=1} + \underbrace{(0.09 + 0.08 + 0.01)}_{j=2} + \underbrace{(0.70 + 0.05 + 0.65)}_{j=3}\\ &= 0.16 + 0.18 + 1.40 = 1.74,\\ d_N(r, c_2) &= \underbrace{(0.68 + 0.23 + 0.45)}_{j=1} + \underbrace{(0.49 + 0.28 + 0.21)}_{j=2} + \underbrace{(0.55 + 0.20 + 0.35)}_{j=3}\\ &= 1.36 + 0.98 + 1.10 = 3.44. \end{aligned}$$

Since $d_N(r, c_1) = 0.18 < 0.95$ and $d_N(r, c_1) < d_N(r, c_3), d_N(r, c_2)$, nearest–neighbor decoding outputs c_1 uniquely.

4.6 Cyclic Neutrosophic Graph

A cyclic neutrosophic graph is a neutrosophic graph containing cycles whose edges have positive truth and tracked indeterminacy and falsity [49].

Definition 4.6.1 (Cyclic Neutrosophic Graph)**.** [49] Let $G_N = (V, E; \lambda_V, \lambda_E)$ be a neutrosophic graph, where

$$\lambda_V : V \to [0,1]^3, \quad \lambda_E : E \to [0,1]^3, \quad \lambda_V(v) = (T(v), I(v), F(v)),\ \lambda_E(e) = (T(e), I(e), F(e)).$$

A neutrosophic k–cycle ($k \geq 3$) in G_N is a sequence of distinct vertices

$$C = (v_1, v_2, \dots, v_k, v_1)$$

such that $\{v_i, v_{i+1}\} \in E$ for all $i = 1, \dots, k$ (with $v_{k+1} = v_1$) and

$$T(\{v_i, v_{i+1}\}) > 0 \quad \text{for all } i.$$

We define the neutrosophic truth, indeterminacy, and falsity of C by

$$T(C) := \min_{1\leq i\leq k} T(\{v_i, v_{i+1}\}), \quad I(C) := \max_{1\leq i\leq k} I(\{v_i, v_{i+1}\}), \quad F(C) := \max_{1\leq i\leq k} F(\{v_i, v_{i+1}\}).$$

The neutrosophic graph G_N is called a *cyclic neutrosophic graph* if it contains at least one neutrosophic k–cycle C with $T(C) > 0$. If every edge $e \in E$ lies on some neutrosophic k–cycle C with $T(C) > 0$, then G_N is called *edge–cyclic neutrosophic*.

Example 4.6.2 (Last–mile delivery loop as a Cyclic Neutrosophic Graph)**.** Consider four urban logistics hubs

$$V = \{\text{Hub-N, Hub-E, Hub-S, Hub-W}\}$$

with undirected links $E = \{\{N, E\}, \{E, S\}, \{S, W\}, \{W, N\}\}$ forming a ring used for nightly parcel redistribution. Each edge $e \in E$ carries neutrosophic weights $\lambda_E(e) = (T(e), I(e), F(e)) \in [0,1]^3$ summarizing, respectively, reliability, seasonal indeterminacy, and contradicting reports.

Assign

$$\begin{aligned} \lambda_E(\{N, E\}) &= (0.90,\ 0.05,\ 0.05), \\ \lambda_E(\{E, S\}) &= (0.80,\ 0.10,\ 0.12), \\ \lambda_E(\{S, W\}) &= (0.85,\ 0.08,\ 0.10), \\ \lambda_E(\{W, N\}) &= (0.78,\ 0.15,\ 0.07). \end{aligned}$$

The 4–cycle

$$C = (N, E, S, W, N)$$

has

$$\begin{aligned} T(C) &= \min\{0.90, 0.80, 0.85, 0.78\} = 0.78, \\ I(C) &= \max\{0.05, 0.10, 0.08, 0.15\} = 0.15, \\ F(C) &= \max\{0.05, 0.12, 0.10, 0.07\} = 0.12. \end{aligned}$$

Since $T(C) = 0.78 > 0$, the network contains a neutrosophic cycle, hence the neutrosophic graph is *cyclic*. Operationally, the ring remains reliably traversable despite moderate indeterminacy and limited contradicting reports.

4.7 Neutrosophic Cognitive Graphs

A neutrosophic cognitive graph models concepts with signed or indeterminate edges; iterative updates yield fixed points or cycles describing behavior [36, 224]. Related concepts such as Fuzzy Cognitive Graphs [225–227] are also well known.

Definition 4.7.1 (Neutrosophic cognitive graph (NCG))**.** [36, 224] An NCG is a quadruple $\mathcal{G} = (V, E, w, \Sigma)$ where:

- $V = \{v_1, \dots, v_n\}$ is a finite set of concepts; $E \subseteq V \times V$ is a set of directed edges;

- $w : E \longrightarrow \{-1, 0, 1\} \cup \{I\}$ assigns each edge a causal sign $(-1, 0, 1)$ or an *indeterminate* mark I;
- $\Sigma \subseteq \{0, 1, I\}^V$ is the set of admissible neutrosophic state vectors $x = (x_i)_{i \in V}$;

with a (thresholded) update $U : \Sigma \longrightarrow \Sigma$ determined by w, where I propagates under the rule that any arithmetic/comparison involving I yields I. A fixed point or limit cycle of U is a neutrosophic *hidden pattern* of $\mathcal{G}$.

Example 4.7.2 (Retail demand–price–stock as a Neutrosophic Cognitive Graph)**.** A price–stock relationship links product price with available inventory, showing how scarcity, surplus, and demand jointly influence retail pricing decisions (cf. [228]).

Let the concept set be $V = \{D, P, S\}$ for *Demand*, *Price*, and *Stock* (inventory). Directed edges encode causal signs in $\{-1, 0, 1\} \cup \{I\}$:

$$w(D \to P) = +1, \quad w(S \to P) = -1, \quad w(P \to D) = -1, \quad w(D \to S) = I,$$

and all other pairs have weight 0. In matrix form (rows: sources; columns: targets)

$$W = \begin{pmatrix} 0 & 1 & I \\ -1 & 0 & 0 \\ -1 & 0 & 0 \end{pmatrix}, \qquad \text{(row/col order } D, P, S\text{)}.$$

States are vectors $x = (x_D, x_P, x_S) \in \{0, 1, I\}^3$. Arithmetic is neutrosophic: any multiplication or addition involving I yields I, and the componentwise threshold Θ maps real sums by

$$\Theta(z) = \begin{cases} 1, & z \geq 1, \\ 0, & z \leq 0, \\ I, & z = I. \end{cases}$$

Start from an observed surge in demand with ample stock:

$$x^{(0)} = (1, 0, 1).$$

Compute the pre–threshold influence $y^{(1)} = W x^{(0)}$ componentwise:

Price:

$$y_P^{(1)} = 1 \cdot x_D + (-1) \cdot x_S = 1 \cdot 1 + (-1) \cdot 1 = 0 \;\Rightarrow\; x_P^{(1)} = \Theta(0) = 0.$$

Demand:

$$y_D^{(1)} = (-1) \cdot x_P = (-1) \cdot 0 = 0 \;\Rightarrow\; x_D^{(1)} = \Theta(0) = 0.$$

Stock:

$$y_S^{(1)} = I \cdot x_D = I \;\Rightarrow\; x_S^{(1)} = \Theta(I) = I.$$

Thus

$$x^{(1)} = (0, 0, I).$$

Iterating once more, $y^{(2)} = W x^{(1)}$:

Price:

$$y_P^{(2)} = 1 \cdot 0 + (-1) \cdot I = I \;\Rightarrow\; x_P^{(2)} = I.$$

Demand:

$$y_D^{(2)} = (-1) \cdot I = I \;\Rightarrow\; x_D^{(2)} = I.$$

Stock:

$$y_S^{(2)} = I \cdot 0 = I \;\Rightarrow\; x_S^{(2)} = I.$$

Hence

$$x^{(2)} = (I, I, I),$$

which is a neutrosophic fixed point: subsequent updates remain indeterminate on all components. Interpretation. Known couplings (price suppresses demand; stock suppresses price) initially neutralize, while the uncertain effect of demand on stock (I) propagates indeterminacy through the loop, stabilizing at a fully indeterminate hidden pattern.

Chapter 5

Neutrosophic Algebra

In this chapter, we examine Neutrosophic Algebra and its related concepts in detail. Neutrosophic Algebra is an extension of classical algebraic structures incorporating truth, indeterminacy, and falsity degrees into operations and logical reasoning (cf. [229–233]). Related concepts such as Fuzzy Algebra (cf. [234–236]) and Intuitionistic Fuzzy Algebra (cf. [237–239]) are also well known.

5.1 Neutrosophic Multiset Structure

A neutrosophic multiset structure assigns each element multiple labeled membership occurrences with truth–indeterminacy–falsity triples, supporting closed union and intersection operations [39].

Definition 5.1.1 (Neutrosophic Multiset Structure)**.** Fix a nonempty universe U and write

$$\mathsf{NV} := [0,1]^3 = \{(t,i,f) \mid t,i,f \in [0,1]\}$$

for the set of neutrosophic value triples (truth, indeterminacy, falsity).

A neutrosophic multiset on U is any mapping

$$\mathrm{nm}_M : U \longrightarrow \mathcal{P}\big(\mathbb{N}_{\geq 1} \times \mathsf{NV}\big)$$

such that, for every $x \in U$,

$$\mathrm{nm}_M(x) = \{(k_1,(t_1,i_1,f_1)),\ldots,(k_r,(t_r,i_r,f_r))\}$$

with $r \geq 0$, $k_j \in \mathbb{N}_{\geq 1}$, $(t_j,i_j,f_j) \in \mathsf{NV}$, and $(t_p,i_p,f_p) \neq (t_q,i_q,f_q)$ whenever $p \neq q$. The case $r = 0$ means that x does not occur in M.

Denote by

$$\mathsf{NMS}(U) := \big\{\mathrm{nm} \mid \mathrm{nm} : U \to \mathcal{P}(\mathbb{N}_{\geq 1} \times \mathsf{NV}) \text{ satisfies the above condition}\big\}$$

the class of all neutrosophic multisets on U.

A neutrosophic multiset structure on U is a tuple

$$\mathcal{N} = \big(U, \mathsf{NV}, \mathsf{NMS}(U), O\big),$$

where O is a specified family of operations on $\mathsf{NMS}(U)$ satisfying:

- (Carrier) U is the underlying universe of elements.
- (Objects) $\mathsf{NMS}(U)$ is the collection of all neutrosophic multisets over U.

- (Operations) O contains operations of fixed arities

$$\omega : \mathsf{NMS}(U)^m \longrightarrow \mathsf{NMS}(U)$$

that are defined elementwise via neutrosophic multiplicities. Concretely, for every $x \in U$ and every $M_1, \ldots, M_m \in \mathsf{NMS}(U)$,

$$\omega(M_1, \ldots, M_m)(x) = F_\omega\big(\mathrm{nm}_{M_1}(x), \ldots, \mathrm{nm}_{M_m}(x)\big),$$

where F_ω is a rule that combines finite sets of pairs $(k, (t, i, f))$ and returns again a finite set of such pairs. Typical examples include:

  – neutrosophic multiset union $M \sqcup N$ obtained by taking, for each x, all pairs from $\mathrm{nm}_M(x)$ and $\mathrm{nm}_N(x)$ and aggregating equal triples (t, i, f) by summing their multiplicities;
  – neutrosophic multiset intersection $M \sqcap N$ obtained by keeping only those triples (t, i, f) that occur in both and taking the minimum (or another prescribed t-norm–based aggregation) of their multiplicities;
  – neutrosophic multiset difference, complement, or scaling, defined similarly by an elementwise rule.

- (Closure) For every operation $\omega \in O$ and every $M_1, \ldots, M_m \in \mathsf{NMS}(U)$, the result $\omega(M_1, \ldots, M_m)$ again belongs to $\mathsf{NMS}(U)$.

Thus a neutrosophic multiset structure is a universe equipped with all neutrosophic multisets over it, together with a chosen, elementwise-defined, closed family of neutrosophic multiset operations.

Example 5.1.2 (Hospital triage records as a Neutrosophic Multiset Structure)**.** Interpret $(t, i, f) \in [0, 1]^3$ as degrees of *positive evidence* (t), *indeterminacy* (i), and *negative evidence* (f) for the statement

"the patient currently requires urgent care".

Let the universe be the set of patients $U = \{\text{Alice}, \text{Bob}\}$. Each triage observation (vital signs, lab panel, nurse note) is discretized into one of a few canonical neutrosophic triples and contributes a *multiplicity* $k \in \mathbb{N}_{\geq 1}$ to that triple.

Define two neutrosophic multisets M (morning round) and N (evening round) by listing, for each patient $x \in U$, the set

$$\mathrm{nm}_M(x) = \{(k, (t, i, f))\} \subset \mathbb{N}_{\geq 1} \times [0, 1]^3, \quad \mathrm{nm}_N(x) = \{(k, (t, i, f))\}.$$

Morning round M:

$$\mathrm{nm}_M(\text{Alice}) = \big\{\, (2, (0.80, 0.10, 0.10)),\ (1, (0.60, 0.20, 0.20)) \,\big\},$$
$$\mathrm{nm}_M(\text{Bob}) = \big\{\, (1, (0.30, 0.30, 0.40)),\ (2, (0.50, 0.30, 0.20)) \,\big\}.$$

Evening round N:

$$\mathrm{nm}_N(\text{Alice}) = \big\{\, (1, (0.80, 0.10, 0.10)),\ (3, (0.70, 0.20, 0.10)) \,\big\},$$
$$\mathrm{nm}_N(\text{Bob}) = \big\{\, (1, (0.50, 0.30, 0.20)),\ (1, (0.40, 0.40, 0.20)) \,\big\}.$$

Equip $\mathsf{NMS}(U)$ with the following elementwise operations (they act independently on each $x \in U$):

• *Neutrosophic multiset union* ($M \sqcup N$): for each triple (t, i, f), sum multiplicities if the triple occurs in both; retain unmatched triples as-is. Formally, if $\mathrm{nm}_M(x)$ contains $(k_M, (t, i, f))$ and $\mathrm{nm}_N(x)$ contains $(k_N, (t, i, f))$, then

$$(k_M, (t, i, f)) \text{ and } (k_N, (t, i, f)) \ \rightsquigarrow\ (k_M + k_N, (t, i, f)) \text{ in } \mathrm{nm}_{M \sqcup N}(x).$$

• *Neutrosophic multiset intersection* ($M \sqcap N$): keep only triples common to both and take the minimum multiplicity:

$$(k_M, (t, i, f)),\ (k_N, (t, i, f)) \ \rightsquigarrow\ (\min\{k_M, k_N\}, (t, i, f)).$$

We now compute $M \sqcup N$ and $M \sqcap N$ explicitly.

Patient Alice. Common triple: $(0.80, 0.10, 0.10)$; unique to M: $(0.60, 0.20, 0.20)$; unique to N: $(0.70, 0.20, 0.10)$.

$$\begin{aligned}\mathrm{nm}_{M\sqcup N}(\text{Alice}) &= \{\,(2+1, (0.80, 0.10, 0.10)),\ (1, (0.60, 0.20, 0.20)),\ (3, (0.70, 0.20, 0.10))\,\}\\ &= \{\,(3, (0.80, 0.10, 0.10)),\ (1, (0.60, 0.20, 0.20)),\ (3, (0.70, 0.20, 0.10))\,\},\\ \mathrm{nm}_{M\sqcap N}(\text{Alice}) &= \{\,(\min\{2, 1\}, (0.80, 0.10, 0.10))\,\} = \{\,(1, (0.80, 0.10, 0.10))\,\}.\end{aligned}$$

Patient Bob. Common triple: $(0.50, 0.30, 0.20)$; unique to M: $(0.30, 0.30, 0.40)$; unique to N: $(0.40, 0.40, 0.20)$.

$$\begin{aligned}\mathrm{nm}_{M\sqcup N}(\text{Bob}) &= \{\,(2+1, (0.50, 0.30, 0.20)),\ (1, (0.30, 0.30, 0.40)),\ (1, (0.40, 0.40, 0.20))\,\}\\ &= \{\,(3, (0.50, 0.30, 0.20)),\ (1, (0.30, 0.30, 0.40)),\ (1, (0.40, 0.40, 0.20))\,\},\\ \mathrm{nm}_{M\sqcap N}(\text{Bob}) &= \{\,(\min\{2, 1\}, (0.50, 0.30, 0.20))\,\} = \{\,(1, (0.50, 0.30, 0.20))\,\}.\end{aligned}$$

A simple decision indicator (worked out for Alice). Define the *aggregated positive evidence* under union by the multiplicity–weighted average

$$T_{\text{avg}}(x; M \sqcup N) \ :=\ \frac{\sum_{(k,(t,i,f))\in \mathrm{nm}_{M\sqcup N}(x)} k\,t}{\sum_{(k,(t,i,f))\in \mathrm{nm}_{M\sqcup N}(x)} k}.$$

For Alice,

$$\begin{aligned}\text{numerator} &= 3\cdot 0.80 + 1\cdot 0.60 + 3\cdot 0.70 = 2.40 + 0.60 + 2.10 = 5.10,\\ \text{denominator} &= 3 + 1 + 3 = 7,\\ T_{\text{avg}}(\text{Alice}; M \sqcup N) &= \frac{5.10}{7} \approx 0.7286.\end{aligned}$$

If the clinical threshold is 0.70, Alice would be flagged for closer monitoring; the same computation can be applied to Bob.

In summary, the tuple

$$\mathcal{N} = (U,\ [0, 1]^3,\ \mathsf{NMS}(U),\ \{\sqcup, \sqcap\})$$

is a neutrosophic multiset structure that captures repeated, uncertain, and possibly conflicting triage evidence across rounds, with closed, elementwise operations and explicit multiplicity arithmetic.

5.2 NeutroHypermetric Space

A neutrometric space equips pairs with a real score and triadic flags enforcing metric axioms selectively: nonnegativity, reflexivity, triangle, symmetry [155, 156, 240–242]. Fuzzy Metric Spaces [243–246] and Intuitionistic Fuzzy Metric Spaces [247–249] are known as related concepts in the study of generalized distance structures. A neutrohypermetric space assigns hyper-valued distances and triadic flags, generalizing metrics via set-sum triangle constraints, reflexivity, symmetry, indeterminacy controls explicitly [50].

Definition 5.2.1 (Neutrometric Space)**.** [240, 241] Let X be a nonempty set. A *neutrometric* on X is a pair (σ, ν) where $\sigma : X \times X \to \mathbb{R}$ and

$$\nu = (\nu_1, \nu_2, \nu_3, \nu_4), \qquad \nu_1, \nu_4 : X \times X \to \{T, I, F\}, \quad \nu_2 : X \to \{T, I, F\}, \quad \nu_3 : X^3 \to \{T, I, F\},$$

such that, for all $x, y, z \in X$:

- If $\nu_1(x, y) = T$ then $\sigma(x, y) \geq 0$; if $\nu_1(x, y) = F$ then $\sigma(x, y) < 0$; if $\nu_1(x, y) = I$ no constraint is imposed.
- If $\nu_2(x) = T$ then $\sigma(x, x) = 0$; if $\nu_2(x) = F$ then $\sigma(x, x) \neq 0$; if $\nu_2(x) = I$ no constraint is imposed.
- If $\nu_3(x, y, z) = T$ then $\sigma(x, z) \leq \sigma(x, y) + \sigma(y, z)$; if $\nu_3(x, y, z) = F$ then $\sigma(x, z) > \sigma(x, y) + \sigma(y, z)$; if $\nu_3(x, y, z) = I$ no constraint is imposed.
- If $\nu_4(x, y) = T$ then $\sigma(x, y) = \sigma(y, x)$; if $\nu_4(x, y) = F$ then $\sigma(x, y) \neq \sigma(y, x)$; if $\nu_4(x, y) = I$ no constraint is imposed.

The pair $(X; \sigma, \nu)$ is called a *neutrometric space*. When $\nu_i \equiv T$ for $i = 1, \ldots, 4$, the usual metric axioms are recovered. This triadic (truth/indeterminacy/falsity) encoding matches the neutrometric axioms reported in the literature.

Remark 5.2.2. A "strong" neutrometric space is obtained by restricting σ to $\mathbb{R}_{\geq 0}$ and requiring the T–cases above on the intended domain, as used to develop neutro–open balls and neutro–open/closed sets.

Example 5.2.3 (Neutrometric space for three city locations)**.** Let

$$X = \{\mathrm{H}, \mathrm{O}, \mathrm{S}\}$$

denote *Home* (H), *Office* (O), and a nearby *Shop* (S). Define $\sigma : X \times X \to \mathbb{R}$ by

$$\sigma(x, y) = \begin{cases} 0, & x = y, \\ 2, & \{x, y\} = \{\mathrm{H}, \mathrm{O}\}, \\ 3, & \{x, y\} = \{\mathrm{H}, \mathrm{S}\}, \\ 1, & \{x, y\} = \{\mathrm{O}, \mathrm{S}\}, \end{cases}$$

and extend symmetrically, so $\sigma(x, y) = \sigma(y, x)$. Numerically,

$$\sigma(\mathrm{H}, \mathrm{O}) = 2, \quad \sigma(\mathrm{O}, \mathrm{S}) = 1, \quad \sigma(\mathrm{H}, \mathrm{S}) = 3,$$

and one checks the usual triangle inequality, e.g.

$$\sigma(\mathrm{H}, \mathrm{S}) = 3 \ \leq \ 2 + 1 = \sigma(\mathrm{H}, \mathrm{O}) + \sigma(\mathrm{O}, \mathrm{S}).$$

Define the neutro–labels

$$\nu_1(x, y) \equiv T, \quad \nu_2(x) \equiv T, \quad \nu_3(x, y, z) \equiv T, \quad \nu_4(x, y) \equiv T$$

for all $x, y, z \in X$. Then:

- $\nu_1(x, y) = T$ enforces $\sigma(x, y) \geq 0$, which holds since all entries are nonnegative;
- $\nu_2(x) = T$ enforces $\sigma(x, x) = 0$ for each x;
- $\nu_3(x, y, z) = T$ enforces the triangle inequality, satisfied by σ as above;
- $\nu_4(x, y) = T$ enforces symmetry $\sigma(x, y) = \sigma(y, x)$.

Thus $(X; \sigma, \nu)$ is a neutrometric space in which every metric axiom is fully true (label T). Here σ models walking time (in suitable units) between home, office, and shop.

Definition 5.2.4 (NeutroHypermetric Space)**.** [50] Let X be a nonempty set and let $\mathcal{H}(\mathbb{R})$ denote the family of all nonempty subsets of $\mathbb{R}$. A *neutrohypermetric* on X is a pair (Δ, η) where $\Delta : X \times X \to \mathcal{H}(\mathbb{R})$ and $\eta = (\eta_1, \eta_2, \eta_3, \eta_4)$ with

$$\eta_1, \eta_4 : X \times X \to \{T, I, F\}, \quad \eta_2 : X \to \{T, I, F\}, \quad \eta_3 : X^3 \to \{T, I, F\},$$

satisfying, for all $x, y, z \in X$ (write $A \oplus B := \{a + b : \ a \in A, \ b \in B\}$):

- If $\eta_1(x, y) = T$ then $\Delta(x, y) \subseteq [0, \infty)$; if $\eta_1(x, y) = F$ then $\Delta(x, y) \cap (-\infty, 0) \neq \emptyset$; if $\eta_1(x, y) = I$ no constraint.
- If $\eta_2(x) = T$ then $0 \in \Delta(x, x)$; if $\eta_2(x) = F$ then $0 \notin \Delta(x, x)$; if $\eta_2(x) = I$ no constraint.
- If $\eta_3(x, y, z) = T$ then $\Delta(x, z) \subseteq \Delta(x, y) \oplus \Delta(y, z)$; if $\eta_3(x, y, z) = F$ then $\Delta(x, z) \nsubseteq \Delta(x, y) \oplus \Delta(y, z)$; if $\eta_3(x, y, z) = I$ no constraint.
- If $\eta_4(x, y) = T$ then $\Delta(x, y) = \Delta(y, x)$; if $\eta_4(x, y) = F$ then $\Delta(x, y) \neq \Delta(y, x)$; if $\eta_4(x, y) = I$ no constraint.

The triple $(X; \Delta, \eta)$ is called a *NeutroHypermetric Space*. If each $\Delta(x, y)$ is a singleton $\{d(x, y)\}$ and all $\eta_i \equiv T$, one recovers an ordinary metric space; thus the neutrohypermetric extends the metric framework to hyper–valued distances under neutrosophic evaluation, as discussed in the neutroanalysis program.

Example 5.2.5 (Neutrohypermetric space for three warehouses)**.** Let

$$X = \{\mathrm{A}, \mathrm{B}, \mathrm{C}\}$$

represent three warehouses on a line. First fix a base distance $d : X \times X \to [0, \infty)$ by

$$d(x, y) = \begin{cases} 0, & x = y, \\ 5, & \{x, y\} = \{\mathrm{A}, \mathrm{B}\}, \\ 7, & \{x, y\} = \{\mathrm{B}, \mathrm{C}\}, \\ 10, & \{x, y\} = \{\mathrm{A}, \mathrm{C}\}, \end{cases}$$

extended symmetrically. Then

$$d(\mathrm{A}, \mathrm{C}) = 10 \ \leq \ 5 + 7 = d(\mathrm{A}, \mathrm{B}) + d(\mathrm{B}, \mathrm{C}),$$

so d is a classical metric.

Define a hyper–valued distance

$$\Delta : X \times X \to \mathcal{H}(\mathbb{R})$$

by

$$\Delta(x, y) = \begin{cases} \{0\}, & x = y, \\ [0, d(x, y)], & x \neq y, \end{cases}$$

so each $\Delta(x, y)$ is a nonempty subset of $[0, \infty)$. For example,

$$\Delta(\mathrm{A}, \mathrm{B}) = [0, 5], \quad \Delta(\mathrm{B}, \mathrm{C}) = [0, 7], \quad \Delta(\mathrm{A}, \mathrm{C}) = [0, 10].$$

Set the neutro–labels

$$\eta_1(x, y) \equiv T, \quad \eta_2(x) \equiv T, \quad \eta_3(x, y, z) \equiv T, \quad \eta_4(x, y) \equiv T$$

for all $x, y, z \in X$. Then:

- $\eta_1(x, y) = T$ gives $\Delta(x, y) \subseteq [0, \infty)$, which holds by construction;
- $\eta_2(x) = T$ gives $0 \in \Delta(x, x)$, true since $\Delta(x, x) = \{0\}$;
- $\eta_4(x, y) = T$ gives symmetry $\Delta(x, y) = \Delta(y, x)$;
- for $\eta_3(x, y, z) = T$, the hyper–triangle inequality holds via Minkowski sum $A \oplus B = \{a + b : a \in A, b \in B\}$. For instance,

  $$\Delta(\mathrm{A}, \mathrm{C}) = [0, 10] \subseteq [0, 5] \oplus [0, 7] = [0, 12] = \Delta(\mathrm{A}, \mathrm{B}) \oplus \Delta(\mathrm{B}, \mathrm{C}).$$

  The same argument works for all triples because d is a metric.

Thus $(X; \Delta, \eta)$ is a neutrohypermetric space: each $\Delta(x, y)$ collects all plausible shipping–cost realizations in $[0, d(x, y)]$ between two warehouses, and every hypermetric axiom holds with truth label T.

5.3 Symbolic Plithogenic Constant

A symbolic plithogenic constant fixes an attribute reference, constant membership vector, and contradiction profile, yielding value-dependent attenuated memberships across contexts (cf. [250–252]). Related concepts also include the symbolic neutrosophic theory, which studies symbolic representations and reasoning within neutrosophic frameworks (cf. [253, 254]).

Definition 5.3.1 (Symbolic Plithogenic Constant)**.** Fix a plithogenic setting

$$PS = \big(P,\ v,\ Pv,\ pdf,\ pCF\big),$$

where v is a chosen attribute with value domain Pv, $pdf : P \times Pv \to [0,1]^s$ is the degree-of-appurtenance (DAF), and $pCF : Pv \times Pv \to [0,1]^t$ is the degree-of-contradiction (DCF) with $pCF(a,a) = 0$ and $pCF(a,b) = pCF(b,a)$.

A *symbolic plithogenic constant* (for the attribute v) is a triple

$$\kappa = \big(u_\kappa,\ m_\kappa,\ w_\kappa\big)$$

consisting of

- a designated reference value $u_\kappa \in Pv$,
- a fixed membership vector $m_\kappa \in [0,1]^s$ (constant in $x \in P$),
- a contradiction profile $w_\kappa : Pv \to [0,1]$ defined by $w_\kappa(a) := pCF(u_\kappa, a)$ for all $a \in Pv$.

Its *constant evaluation* is the Pv-profile

$$\mathrm{Const}_\kappa : Pv \longrightarrow [0,1]^s, \qquad \mathrm{Const}_\kappa(a) := \big(1 - w_\kappa(a)\big)\, m_\kappa \quad (a \in Pv),$$

where the scalar multiplication acts componentwise on m_κ. Equivalently, the symbolic constant induces a formal plithogenic element c_κ (with no underlying $x \in P$) whose membership at value a is

$$pdf\big(c_\kappa; a\big) := \mathrm{Const}_\kappa(a) = \big(1 - pCF(u_\kappa, a)\big)\, m_\kappa.$$

Hence $pdf(c_\kappa; u_\kappa) = m_\kappa$ (maximal at its own reference value) and the membership attenuates monotonically with the contradiction to other values of Pv.

Remark 5.3.2 (Canonical examples)**.** Let $s \geq 1$ and let $\mathbf{1} := (1, \ldots, 1) \in [0,1]^s$, $\mathbf{0} := (0, \ldots, 0)$, and $\mathbf{h} := (\frac{1}{2}, \ldots, \frac{1}{2})$. For any chosen $u^\star \in Pv$:

1. **Top (unit) constant:** $\kappa^\top = (u^\star, \mathbf{1},\ a \mapsto pCF(u^\star, a))$. Then $pdf(c_{\kappa^\top}; a) = \big(1 - pCF(u^\star, a)\big)\mathbf{1}$.
2. **Bottom (null) constant:** $\kappa^\perp = (u^\star, \mathbf{0},\ a \mapsto pCF(u^\star, a))$. Then $pdf(c_{\kappa^\perp}; a) = \mathbf{0}$ for all a.
3. **Neutral constant:** $\kappa^{\mathrm{neu}} = (u^\star, \mathbf{h},\ a \mapsto pCF(u^\star, a))$. Then $pdf(c_{\kappa^{\mathrm{neu}}}; a) = \big(1 - pCF(u^\star, a)\big)\mathbf{h}$.

When $s = 1$ (single-valued membership), these reduce to scalar profiles on Pv.

Example 5.3.3 (Symbolic plithogenic constant for eco-label strictness)**.** An eco-label is a certified mark indicating reduced environmental impact across a product's lifecycle, guiding consumers toward sustainable choices responsibly (cf. [255]).

Consider a plithogenic setting where P is a set of detergents and v is the attribute "eco-label strictness" with value domain

$$Pv = \{\text{lenient},\ \text{standard},\ \text{strict}\}.$$

Assume a degree of contradiction $pCF : Pv \times Pv \to [0,1]$ given by

$$pCF(\text{standard}, \text{standard}) = 0, \quad pCF(\text{standard}, \text{lenient}) = 0.2, \quad pCF(\text{standard}, \text{strict}) = 0.5,$$

and symmetric in its arguments.

Take the symbolic plithogenic constant

$$\kappa = (u_\kappa, m_\kappa, w_\kappa)$$

with

$$u_\kappa = \text{standard}, \qquad m_\kappa = 0.9 \in [0,1], \qquad w_\kappa(a) = pCF(u_\kappa, a).$$

Then the induced constant evaluation is

$$\text{Const}_\kappa(a) = (1 - w_\kappa(a))\, m_\kappa = (1 - pCF(\text{standard}, a)) \cdot 0.9, \qquad a \in Pv.$$

Explicitly,

$$\begin{aligned} \text{Const}_\kappa(\text{standard}) &= (1-0)\cdot 0.9 = 0.9, \\ \text{Const}_\kappa(\text{lenient}) &= (1-0.2)\cdot 0.9 = 0.8 \cdot 0.9 = 0.72, \\ \text{Const}_\kappa(\text{strict}) &= (1-0.5)\cdot 0.9 = 0.5 \cdot 0.9 = 0.45. \end{aligned}$$

Thus the formal plithogenic element c_κ has membership

$$pdf(c_\kappa; \text{standard}) = 0.9, \quad pdf(c_\kappa; \text{lenient}) = 0.72, \quad pdf(c_\kappa; \text{strict}) = 0.45,$$

i.e. it is maximally compatible with the reference value "standard" and attenuated according to the contradiction degrees toward "lenient" and "strict" eco-label standards.

5.4 Strong Neutrosophic Homomorphism

A homomorphism is a structure-preserving map between algebraic objects, respecting the operations and identities defined on their carriers and relations (cf. [256–258]). A neutrosophic homomorphism maps neutrosophic structures, preserving carrier operations while transforming truth, indeterminacy, and falsity components compatibly with aggregation rules [259–262]. A strong neutrosophic homomorphism strictly preserves carrier algebra and enforces exact compatibility between combined input degrees and combined output degrees (cf. [263]).

Definition 5.4.1 (Strong Neutrosophic Homomorphism)**.** Let U be a universe and let

$$A = \{\, x\,(t_x, i_x, f_x) : x \in U,\ (t_x, i_x, f_x) \in [0,1]^3 \,\}, \qquad B = \{\, y\,(t_y, i_y, f_y) : y \in U,\ (t_y, i_y, f_y) \in [0,1]^3 \,\}.$$

Suppose $(A, *)$ and $(B, \#)$ are *strong neutrosophic algebraic structures*, meaning that for all

$$x_1(t_1, i_1, f_1),\ x_2(t_2, i_2, f_2) \in A$$

one has

$$x_1(t_1, i_1, f_1) \ * \ x_2(t_2, i_2, f_2) = \ (x_1 \circ_A x_2)\big(\, t_1 \otimes_T t_2,\ i_1 \otimes_I i_2,\ f_1 \otimes_F f_2 \,\big) \in A,$$

and similarly for $(B, \#)$ with a (possibly different) carrier operation $\circ_B$ and degree-combiners $\oplus_T, \oplus_I, \oplus_F$ on $[0,1]$.

A map $m : A \to B$ is called a *strong neutrosophic homomorphism* if the following hold for all $x, y \in A$:

1. (Carrier preservation)
$$m(x * y) \ = \ m(x) \# m(y).$$
2. (Degree preservation/compatibility) Writing $v(x) = (T_x, I_x, F_x) \in [0,1]^3$ and $v_B(z)$ for the degree triple of $z \in B$, there exists a (componentwise) map $\Phi : [0,1]^3 \to [0,1]^3$ such that
$$v_B\big(m(x)\big) \ = \ \Phi\big(v(x)\big) \quad \text{and}$$
$$\Phi\big(T_x \otimes_T T_y,\ I_x \otimes_I I_y,\ F_x \otimes_F F_y\big) \ = \ \big(T_{m(x)} \oplus_T T_{m(y)},\ I_{m(x)} \oplus_I I_{m(y)},\ F_{m(x)} \oplus_F F_{m(y)}\big).$$

Informally, m preserves the algebraic law on the carriers and acts (possibly nontrivially) on the neutrosophic degrees in a way compatible with the degree-aggregators of the two structures.

Example 5.4.2 (Canonical instance and direct verification)**.** Let $A = B = \mathbb{R} \times [0,1]^3$ and define, for all $x_k(t_k, i_k, f_k) \in A$,

$$x_1(t_1, i_1, f_1) \ * \ x_2(t_2, i_2, f_2) := (x_1 x_2)\big(\min\{t_1, t_2\},\ \max\{i_1, i_2\},\ \min\{f_1, f_2\}\big),$$

and use the same rule for #. Define $m : A \to B$ by

$$m\big(x(t, i, f)\big) \ := \ x^2\,(f, i, t).$$

Then m is a strong neutrosophic homomorphism. Indeed, for any $x_1(t_1, i_1, f_1), x_2(t_2, i_2, f_2)$:

$$\begin{aligned} m\big(x_1(t_1,i_1,f_1) * x_2(t_2,i_2,f_2)\big) &= m\Big((x_1x_2)\big(\min(t_1,t_2),\ \max(i_1,i_2),\ \min(f_1,f_2)\big)\Big) \\ &= (x_1x_2)^2\big(\min(f_1,f_2),\ \max(i_1,i_2),\ \min(t_1,t_2)\big), \\ m\big(x_1(t_1,i_1,f_1)\big) \,\#\, m\big(x_2(t_2,i_2,f_2)\big) &= \ \big(x_1^2(f_1,i_1,t_1)\big) \,\#\, \big(x_2^2(f_2,i_2,t_2)\big) \\ &= \ (x_1^2x_2^2)\big(\min(f_1,f_2),\ \max(i_1,i_2),\ \min(t_1,t_2)\big), \end{aligned}$$

and $(x_1x_2)^2 = x_1^2x_2^2$, so $m(x_1 * x_2) = m(x_1)\#m(x_2)$ (carrier preservation). The degree-action is $\Phi(t, i, f) = (f, i, t)$, and one checks directly that

$$\Phi\big(\min(t_1,t_2),\ \max(i_1,i_2),\ \min(f_1,f_2)\big) = \big(\min(f_1,f_2),\ \max(i_1,i_2),\ \min(t_1,t_2)\big)$$

which equals the degrees of $m(x_1)\#m(x_2)$, establishing the compatibility condition.

5.5 Plithogenic BCK/BCI–Algebras

BCI/BCK-algebras are non-classical algebraic systems that model implication-like operations, satisfying specific axioms that generalize the subtraction structures of propositional logic with a distinguished zero element [264, 265]. Classical BCI/BCK-algebras can be extended to Fuzzy BCI/BCK-algebras [266, 267] and Neutrosophic BCI/BCK-algebras [262, 268–271]. Plithogenic BCK/BCI–algebras further enrich these implication algebras by introducing attribute-based memberships and contradiction degrees, where algebraic operations are aggregated using t-norms and t-conorms in accordance with plithogenic semantics [39].

Definition 5.5.1 (Plithogenic Attribute System)**.** Fix a nonempty attribute domain V with a finite set of attribute values P_V. Let $pCF : P_V \times P_V \to [0,1]$ be a *degree of contradiction* satisfying $pCF(u,u) = 0$ and $pCF(u,v) = pCF(v,u)$. For each element x in a base set A and each $u \in P_V$, let $\mu(x;u) \in [0,1]$ denote its (single–valued) membership degree under the attribute value u. Fix a *reference* value $u_\star \in P_V$, a continuous t–norm T and its dual t–conorm S.

Definition 5.5.2 (Plithogenic BCK–algebra)**.** [39] Let $(A, *, 0)$ be a BCK–algebra (with the usual axioms and order $a \le b \iff a * b = 0$). A *plithogenic BCK–algebra* is a tuple

$$\mathfrak{A} \ = \ \big(A, *, 0;\ P_V, \mu, pCF,\ T, S\big)$$

such that the following compatibility holds for all $a, b \in A$.

1. (Order–monotonicity) If $a \le b$ in A, then $\mu(a;u_\star) \le \mu(b;u_\star)$.
2. (Plithogenic operation aggregation) For any choice of attribute values $u_a, u_b \in P_V$ attached to a, b, write $c := pCF(u_a, u_b)$. Then the membership of $a * b$ at $u_\star$ is

$$\mu(a * b; u_\star) \ := \ (1-c)\,T\big(\mu(a;u_\star), \mu(b;u_\star)\big) \ + \ c\,S\big(\mu(a;u_\star), \mu(b;u_\star)\big).$$

When P_V is a singleton (so $c = 0$) and T, S are the Gödel pair ($T(x,y) = \min\{x,y\},\ S(x,y) = \max\{x,y\}$), the construction reduces to a fuzzy–valued BCK–algebra. Replacing $\mu(\cdot;u)$ by a neutrosophic triplet (T, I, F) yields a neutrosophic–valued plithogenic BCK–algebra (the aggregation above acting componentwise).

Example 5.5.3 (Concrete plithogenic BCK–algebra on a 3-element chain)**.** Let $A = \{0, 1, 2\}$ with truncated subtraction

$$a * b := \max\{a - b, 0\}, \qquad 0 \text{ the distinguished element.}$$

The Cayley table of $*$ is

$*$	0	1	2
0	0	0	0
1	1	0	0
2	2	1	0

This makes $(A, *, 0)$ a (finite) BCK–algebra; its natural order is $0 \le 1 \le 2$ since $a \le b \iff a * b = 0$.

Plithogenic data. Take the attribute set $P_V = \{L, H\}$ ("low", "high"). Attach attributes to elements by $u_0 = L$, $u_1 = L$, $u_2 = H$. Fix the reference attribute $u_\star \in \{L, H\}$ at which memberships are read. Define the membership profile $\mu : A \times P_V \to [0, 1]$ by the table

x	$\mu(x; L)$	$\mu(x; H)$
0	0.20	0.10
1	0.60	0.50
2	0.90	0.95

so that order–monotonicity holds: if $a \le b$ then $\mu(a; u_\star) \le \mu(b; u_\star)$ for both $u_\star = L, H$. Let the degree of contradiction $pCF : P_V \times P_V \to [0, 1]$ be

$$pCF(L, L) = pCF(H, H) = 0, \qquad pCF(L, H) = pCF(H, L) = 0.7.$$

Use the product t-norm and probabilistic-sum t-conorm

$$T(x, y) = xy, \qquad S(x, y) = x + y - xy.$$

Aggregation rule (plithogenic BCK): for $a, b \in A$, with $c := pCF(u_a, u_b)$,

$$\mu(a * b; u_\star) = (1 - c)\, T\big(\mu(a; u_\star), \mu(b; u_\star)\big) + c\, S\big(\mu(a; u_\star), \mu(b; u_\star)\big).$$

Concrete computations.

$$\text{(i) } a = 2,\ b = 1 : \quad a * b = \max\{2 - 1, 0\} = 1, \quad c = pCF(H, L) = 0.7.$$

At $u_\star = L$:

$$T = 0.90 \cdot 0.60 = 0.54, \quad S = 0.90 + 0.60 - 0.54 = 0.96,$$

$$\mu(2 * 1; L) = 0.3 \cdot 0.54 + 0.7 \cdot 0.96 = 0.162 + 0.672 = 0.834.$$

At $u_\star = H$:

$$T = 0.95 \cdot 0.50 = 0.475, \quad S = 0.95 + 0.50 - 0.475 = 0.975,$$

$$\mu(2 * 1; H) = 0.3 \cdot 0.475 + 0.7 \cdot 0.975 = 0.1425 + 0.6825 = 0.825.$$

$$\text{(ii) } a = 1,\ b = 0 : \quad a * b = \max\{1 - 0, 0\} = 1, \quad c = pCF(L, L) = 0.$$

At $u_\star = L$:

$$T = 0.60 \cdot 0.20 = 0.12, \quad S = 0.60 + 0.20 - 0.12 = 0.68,$$

$$\mu(1 * 0; L) = (1 - 0) \cdot 0.12 + 0 \cdot 0.68 = 0.12.$$

Thus $\big(A, *, 0; P_V, \mu, pCF, T, S\big)$ is a plithogenic BCK–algebra satisfying the listed compatibility conditions.

Definition 5.5.4 (Plithogenic BCI–algebra)**.** [39] Let $(A, *, 0)$ be a BCI–algebra. A *plithogenic BCI–algebra* is defined exactly as above with A a BCI–algebra, the same attribute system (P_V, μ, pCF), and the same aggregation rule for $*$, subject to order–monotonicity w.r.t. the BCI–order.

Example 5.5.5 (Concrete plithogenic BCI–algebra on $\mathbb{Z}_3$)**.** Let $A = \mathbb{Z}_3 = \{0, 1, 2\}$ with group subtraction modulo 3:

$$a * b := a - b \pmod 3, \qquad 0 \text{ the group identity.}$$

The Cayley table is

$*$	0	1	2
0	0	2	1
1	1	0	2
2	2	1	0

Then $(A, *, 0)$ is a BCI–algebra (a standard group model); its induced order $a \le b \iff a * b = 0$ is equality, hence the order–monotonicity requirement is vacuous.

Plithogenic data. Take $P_V = \{\text{Core}, \text{Aux}\}$, attach

$$u_0 = \text{Core}, \qquad u_1 = \text{Aux}, \qquad u_2 = \text{Core}.$$

Memberships:

x	$\mu(x; \text{Core})$	$\mu(x; \text{Aux})$
0	0.90	0.70
1	0.40	0.60
2	0.80	0.30

Contradiction degrees:

$$pCF(\text{Core}, \text{Core}) = pCF(\text{Aux}, \text{Aux}) = 0, \qquad pCF(\text{Core}, \text{Aux}) = pCF(\text{Aux}, \text{Core}) = 0.5.$$

Use the same t-norm/t-conorm $T(x, y) = xy$, $S(x, y) = x + y - xy$. For $c := pCF(u_a, u_b)$,

$$\mu(a * b; u_\star) = (1 - c)\, T\big(\mu(a; u_\star), \mu(b; u_\star)\big) + c\, S\big(\mu(a; u_\star), \mu(b; u_\star)\big).$$

Concrete computations.

$$\text{(i) } a = 2,\ b = 1 : \quad a * b = 2 - 1 \equiv 1 \pmod 3, \quad c = pCF(\text{Core}, \text{Aux}) = 0.5.$$

At $u_\star = \text{Core}$:

$$T = 0.80 \cdot 0.40 = 0.32, \quad S = 0.80 + 0.40 - 0.32 = 0.88,$$

$$\mu(2 * 1; \text{Core}) = 0.5 \cdot 0.32 + 0.5 \cdot 0.88 = 0.16 + 0.44 = 0.60.$$

At $u_\star = \text{Aux}$:

$$T = 0.30 \cdot 0.60 = 0.18, \quad S = 0.30 + 0.60 - 0.18 = 0.72,$$

$$\mu(2 * 1; \text{Aux}) = 0.5 \cdot 0.18 + 0.5 \cdot 0.72 = 0.09 + 0.36 = 0.45.$$

$$\text{(ii) } a = 0,\ b = 2 : \quad a * b = 0 - 2 \equiv 1 \pmod 3, \quad c = pCF(\text{Core}, \text{Core}) = 0.$$

At $u_\star = \text{Core}$:

$$T = 0.90 \cdot 0.80 = 0.72, \quad S = 0.90 + 0.80 - 0.72 = 0.98,$$

$$\mu(0 * 2; \text{Core}) = (1 - 0) \cdot 0.72 + 0 \cdot 0.98 = 0.72.$$

Therefore $\big(A, *, 0; P_V, \mu, pCF, T, S\big)$ is a concrete plithogenic BCI–algebra.

Remark 5.5.6 (Homomorphisms)**.** A map $f : \mathfrak{A}_1 \to \mathfrak{A}_2$ between plithogenic BCK/BCI–algebras is a *plithogenic homomorphism* if f is an algebra homomorphism and for all x, $\mu_2(f(x); u_\star) \ge \mu_1(x; u_\star)$ whenever $u_\star$ is shared, and equality holds when f preserves the attached attribute values (i.e. $u_{f(x)} = u_x$ so that the same $c = pCF(u_x, u_y)$ is used under f).

5.6 Ultra Neutrosophic Crisp Set

Neutrosophic crisp sets represent elements with strictly 0 or 1 truth, indeterminacy, falsity degrees, embedding classical crisp sets as subsets [272–275]. An ultra neutrosophic crisp set partitions a universe into support, indeterminate, against, and residual unclassified components, extending neutrosophic crisp sets [42, 276].

Definition 5.6.1 (Ultra Neutrosophic Crisp Set)**.** [42, 276] Let X be a nonempty universe and let

$$A_1, A_2, A_3 \subseteq X$$

be the three (crisp) components of a neutrosophic crisp set in the sense of Salama, where A_1 collects the elements fully supporting the event, A_2 the elements standing indeterminately with respect to it, and A_3 the elements totally against it. In many situations there exist elements of X that do not belong to any of these three components. Define the residual (or missed) part by

$$M_A \;=\; X \setminus \bigcup_{i=1}^{3} A_i.$$

An *ultra neutrosophic crisp set* on X is then the quadruple

$$\breve{A} \;=\; (A_1, A_2, A_3, M_A),$$

that is, the first three components classify X with respect to the event as support/indeterminate/against, while the fourth component M_A gathers all elements of X that have not been classified by the previous three parts. The family of all ultra neutrosophic crisp sets on X is denoted by $\breve{\mathcal{U}}(X)$.

Example 5.6.2 (Company access control as an Ultra Neutrosophic Crisp Set)**.** Fix a workweek and let the universe X be the set of all on-payroll persons expected to interact with the office:

$$X = \{\mathrm{A1, A2, A3, A4, A5, A6, A7, A8, A9, A10, A11, A12}\}.$$

Consider the event

$$\mathsf{E}: \text{ “has on-site access authorization for this week”}.$$

Define the three Salama neutrosophic crisp components as

$$\begin{aligned} A_1 &= \{\text{fully supporting } \mathsf{E}\} = \{\mathrm{A1, A5, A8, A11}\}, \\ A_2 &= \{\text{indeterminate for } \mathsf{E}\} = \{\mathrm{A2, A6, A9}\}, \\ A_3 &= \{\text{against } \mathsf{E}\} = \{\mathrm{A3, A7}\}. \end{aligned}$$

Here A_1 lists employees with valid badges and cleared health/safety checks; A_2 are pending checks (e.g., background re-check in progress); A_3 are explicitly denied (badge suspended or remote-only contract).

Some persons in X have no record yet in the weekly access system (e.g., new hires not fully onboarded). These are not covered by $A_1 \cup A_2 \cup A_3$ and are captured by the residual component

$$M_A \;=\; X \setminus (A_1 \cup A_2 \cup A_3) \;=\; \{\mathrm{A4, A10, A12}\}.$$

Verification. By construction, the three components are pairwise disjoint:

$$A_i \cap A_j = \varnothing \qquad (i \neq j;\ i, j \in \{1, 2, 3\}).$$

Their union is

$$A_1 \cup A_2 \cup A_3 = \{\mathrm{A1, A2, A3, A5, A6, A7, A8, A9, A11}\},$$

so the residual is

$$M_A = X \setminus (A_1 \cup A_2 \cup A_3) = \{\mathrm{A4, A10, A12}\}.$$

Cardinalities are

$$|A_1| = 4, \quad |A_2| = 3, \quad |A_3| = 2, \quad |M_A| = 3, \quad |A_1| + |A_2| + |A_3| + |M_A| = 12 = |X|.$$

Therefore the ultra neutrosophic crisp set

$$\breve{A} = (A_1, A_2, A_3, M_A)$$

classifies the real-world access statuses as “authorized” (A_1), “pending” (A_2), “denied” (A_3), and “not yet classified in the system” (M_A), with $A_1 \cup A_2 \cup A_3 \cup M_A = X$ and the four parts mutually disjoint.

5.7 Refined Neutrosophic Linguistic Lattice (RNLL)

A lattice is a partially ordered set where any two elements have unique least upper and greatest lower bounds [277–279]. As extensions of classical lattices, concepts such as fuzzy lattices [280–282] and neutrosophic lattices [283–287] have been developed to handle uncertainty and indeterminacy in ordered structures. An neutrosophic linguistic lattice assigns each linguistic term triple (truth, indeterminacy, falsity) values, preserving lattice joins/meets via t-norm/t-conorm compatible rules [40]. A refined neutrosophic linguistic lattice indexes multiple sub-truth, sub-indeterminacy, sub-falsity components per term, aggregating componentwise with chosen t-norms and t-conorms [40].

Definition 5.7.1 (Neutrosophic Linguistic Lattice)**.** [40] Let $(L, \leq, \vee, \wedge)$ be a (finite) lattice of linguistic terms with bottom $\perp$ and top $\top$. A neutrosophic linguistic valuation is a map

$$\nu : L \longrightarrow [0,1]^3, \qquad x \longmapsto \nu(x) = \big(T(x), I(x), F(x)\big),$$

where $T, I, F : L \to [0,1]$ are called the truth, indeterminacy, and falsity degrees, respectively. The pair

$$\mathsf{NLL} \;=\; \big(L, \nu\big)$$

is a neutrosophic linguistic lattice if the following axioms hold.

1) Lattice–operation compatibility (taken w.r.t. the pointwise lattice $\big([0,1], \max, \min\big)$):

$$\begin{aligned} T(x \vee y) &= \max\{T(x), T(y)\}, & T(x \wedge y) &= \min\{T(x), T(y)\}, \\ I(x \vee y) &= \min\{I(x), I(y)\}, & I(x \wedge y) &= \max\{I(x), I(y)\}, \\ F(x \vee y) &= \min\{F(x), F(y)\}, & F(x \wedge y) &= \max\{F(x), F(y)\}. \end{aligned}$$

2) Boundary conditions:

$$T(\perp) = 0, \quad T(\top) = 1, \qquad I(\perp) = I(\top) = 0, \qquad F(\perp) = 1, \quad F(\top) = 0.$$

3) (Optional normalization) For all $x \in L$,

$$0 \;\leq\; T(x) + I(x) + F(x) \;\leq\; 3 \quad (\text{or } \leq 1 \text{ in the normalized case}).$$

If, in addition, L carries a (linguistic) negation $\neg : L \to L$ that is order–reversing and satisfies De Morgan laws, one may require the neutrosophic De Morgan compatibility

$$T(\neg x) = F(x), \qquad I(\neg x) = I(x), \qquad F(\neg x) = T(x) \qquad (\forall x \in L).$$

Example 5.7.2 (Neutrosophic Linguistic Lattice: service quality terms)**.** Let $L = \{\perp, \text{Low}, \text{Medium}, \text{High}, \top\}$ be a totally ordered (chain) lattice of linguistic terms for weekly service quality, with join $\vee = \max$ and meet $\wedge = \min$ in this chain order, where $\perp$ = Very Poor and $\top$ = Excellent.

Define a neutrosophic linguistic valuation $\nu(x) = (T(x), I(x), F(x))$ by

x	$\perp$	Low	Medium	High	$\top$
$T(x)$	0.00	0.30	0.60	0.85	1.00
$I(x)$	0.00	0.00	0.00	0.00	0.00
$F(x)$	1.00	0.70	0.40	0.15	0.00

This satisfies the boundary conditions $T(\perp) = 0$, $T(\top) = 1$, $I(\perp) = I(\top) = 0$, $F(\perp) = 1$, $F(\top) = 0$, and the lattice–operation compatibilities:

$$\begin{aligned} T(x \vee y) &= \max\{T(x), T(y)\}, & T(x \wedge y) &= \min\{T(x), T(y)\}, \\ I(x \vee y) &= \min\{I(x), I(y)\} = 0, & I(x \wedge y) &= \max\{I(x), I(y)\} = 0, \\ F(x \vee y) &= \min\{F(x), F(y)\}, & F(x \wedge y) &= \max\{F(x), F(y)\}. \end{aligned}$$

Verification on concrete pairs. 1) Join: Low ∨ High = High.

$$T(\text{High}) = \max\{0.30, 0.85\} = 0.85,$$
$$I(\text{High}) = \min\{0, 0\} = 0,$$
$$F(\text{High}) = \min\{0.70, 0.15\} = 0.15.$$

2) Meet: Low ∧ Medium = Low.

$$T(\text{Low}) = \min\{0.30, 0.60\} = 0.30,$$
$$I(\text{Low}) = \max\{0, 0\} = 0,$$
$$F(\text{Low}) = \max\{0.70, 0.40\} = 0.70.$$

Hence $\mathsf{NLL} = (L, \nu)$ is a neutrosophic linguistic lattice.

Definition 5.7.3 (Refined Neutrosophic Linguistic Lattice (RNLL))**.** [40] Let $(L, \leq)$ be a finite lattice of linguistic terms with join ∨ and meet ∧. Fix nonempty index sets P, Q, R (numbers of refinements for truth/indeterminacy/falsity), a t-norm $\otimes : [0,1]^2 \to [0,1]$ and a t-conorm $\oplus : [0,1]^2 \to [0,1]$. A *refined neutrosophic valuation* is a map

$$\nu : L \to [0,1]^P \times [0,1]^Q \times [0,1]^R, \qquad \nu(x) = \big((T_p(x))_{p\in P}, (I_q(x))_{q\in Q}, (F_r(x))_{r\in R}\big),$$

interpreting T_p, I_q, F_r as refined sub-truth, sub-indeterminacy, and sub-falsity degrees, subject to

$$0 \leq T_p(x), I_q(x), F_r(x) \leq 1$$

for all indices and $x \in L$. We say $\big(L, \leq, \vee, \wedge, \nu; \otimes, \oplus\big)$ is an *RNLL* if, for all $x, y \in L$ and $p \in P, q \in Q, r \in R$,

$$T_p(x \vee y) = T_p(x) \oplus T_p(y), \qquad T_p(x \wedge y) = T_p(x) \otimes T_p(y),$$
$$I_q(x \vee y) = I_q(x) \otimes I_q(y), \qquad I_q(x \wedge y) = I_q(x) \oplus I_q(y),$$
$$F_r(x \vee y) = F_r(x) \otimes F_r(y), \qquad F_r(x \wedge y) = F_r(x) \oplus F_r(y),$$

and each component is isotone in x with respect to ≤. When $P = Q = R = \{1\}$ this reduces to a (single-valued) neutrosophic linguistic lattice; when P, Q, R have size > 1 it is a refined version.

Example 5.7.4 (Refined Neutrosophic Linguistic Lattice: customer satisfaction)**.** Let $L = \{\bot, \text{Fair}, \text{Good}, \top\}$ be a chain of linguistic satisfaction levels (weekly summary), ordered $\bot < \text{Fair} < \text{Good} < \top$, with $\vee = \max$ and $\wedge = \min$. Take refined index sets

$$P = \{\text{core}, \text{context}\}, \quad Q = \{\text{ambiguity}, \text{conflict}\}, \quad R = \{\text{noncomp}, \text{risk}\},$$

and choose the t-norm/t-conorm as $\otimes = \min$, $\oplus = \max$.

Define the refined valuation $\nu(x) = \big((T_p(x))_{p\in P}, (I_q(x))_{q\in Q}, (F_r(x))_{r\in R}\big)$ by the table:

x	$\bot$	Fair	Good	$\top$
$T_{\text{core}}(x)$	0.00	0.40	0.70	1.00
$T_{\text{context}}(x)$	0.00	0.30	0.80	1.00
$I_{\text{ambiguity}}(x)$	0.00	0.20	0.10	0.00
$I_{\text{conflict}}(x)$	0.00	0.30	0.20	0.00
$F_{\text{noncomp}}(x)$	1.00	0.60	0.30	0.00
$F_{\text{risk}}(x)$	1.00	0.70	0.20	0.00

These satisfy the RNLL axioms componentwise:

$$T_p(x \vee y) = T_p(x) \oplus T_p(y) = \max\{T_p(x), T_p(y)\}, \quad T_p(x \wedge y) = T_p(x) \otimes T_p(y) = \min\{T_p(x), T_p(y)\},$$
$$I_q(x \vee y) = I_q(x) \otimes I_q(y) = \min\{I_q(x), I_q(y)\}, \quad I_q(x \wedge y) = I_q(x) \oplus I_q(y) = \max\{I_q(x), I_q(y)\},$$
$$F_r(x \vee y) = F_r(x) \otimes F_r(y) = \min\{F_r(x), F_r(y)\}, \quad F_r(x \wedge y) = F_r(x) \oplus F_r(y) = \max\{F_r(x), F_r(y)\}.$$

Boundary conditions also hold:

$$T_p(\bot) = 0,\ T_p(\top) = 1; \quad I_q(\bot) = I_q(\top) = 0; \quad F_r(\bot) = 1,\ F_r(\top) = 0.$$

Concrete check for x = Fair, y = Good:

$$x \vee y = \text{Good}, \qquad x \wedge y = \text{Fair}.$$

Truth (core):

$$T_{\text{core}}(x \vee y) = \max\{0.40, 0.70\} = 0.70 = T_{\text{core}}(\text{Good}), \quad T_{\text{core}}(x \wedge y) = \min\{0.40, 0.70\} = 0.40 = T_{\text{core}}(\text{Fair}).$$

Indeterminacy (ambiguity):

$$I_{\text{ambiguity}}(x \vee y) = \min\{0.20, 0.10\} = 0.10 = I_{\text{ambiguity}}(\text{Good}),$$

$$I_{\text{ambiguity}}(x \wedge y) = \max\{0.20, 0.10\} = 0.20 = I_{\text{ambiguity}}(\text{Fair}).$$

Falsity (noncompliance):

$$F_{\text{noncomp}}(x \vee y) = \min\{0.60, 0.30\} = 0.30 = F_{\text{noncomp}}(\text{Good}),$$

$$F_{\text{noncomp}}(x \wedge y) = \max\{0.60, 0.30\} = 0.60 = F_{\text{noncomp}}(\text{Fair}).$$

Hence $\big(L, \leq, \vee, \wedge, \nu; \min, \max\big)$ is a refined neutrosophic linguistic lattice.

5.8 Neutrosophic Manifold

A neutrosophic manifold is a topological space locally modeled on neutrosophic coordinates, with transition maps preserving componentwise smoothness and algebra (cf. [40, 288–290]). A neutrosophic manifold generalizes both classical manifolds [291, 292] and fuzzy manifolds [293–295].

Definition 5.8.1 (Neutrosophic Manifold)**.** (cf. [40, 288–290]) Let $\mathbb{N}n$ denote the set of neutrosophic numbers (e.g. triples or literals based on T, I, F components) equipped with a topology making $\mathbb{N}n^d$ a Hausdorff, second-countable space. A *d-dimensional neutrosophic C^k manifold* is a Hausdorff, second-countable topological space X together with an atlas $\{(U_i, \varphi_i)\}_{i\in I}$ such that $U_i \subseteq X$ are open, $\varphi_i : U_i \to \Omega_i \subseteq \mathbb{N}n^d$ are homeomorphisms onto open sets, and for all i, j with $U_i \cap U_j \neq \varnothing$ the transition maps $\varphi_j \circ \varphi_i^{-1} : \varphi_i(U_i \cap U_j) \to \varphi_j(U_i \cap U_j)$ are *neutrosophically C^k*: each coordinate function decomposes into (T, I, F)-components that are C^k (as real maps on representatives) and respect the neutrosophic algebra on $\mathbb{N}n$. Charts and changes of charts are required to be compatible with the neutrosophic product topology and with the (T, I, F)-componentwise operations.

Example 5.8.2 (Neutrosophic unit circle with an indeterminacy band)**.** Fix $\delta \in (0, 1)$ and take the neutrosophic number space to be $\mathbb{N}n := \mathbb{R}^3$ with coordinates (T, I, F) and the product topology. Then $\mathbb{N}n^2 \cong \mathbb{R}^6$ is Hausdorff and second–countable.

Define the subset

$$X := \Big\{\big((x_T, x_I, x_F), (y_T, y_I, y_F)\big) \in \mathbb{N}n^2 \ : \ x_T^2 + y_T^2 = 1, \ \ (x_I, y_I) \in (-\delta, \delta)^2, \ \ (x_F, y_F) \in (-\delta, \delta)^2\Big\}.$$

Intuitively, the T–layer lies on the classical unit circle $S^1 \subset \mathbb{R}^2$, while the I/F components are free to vary in the open band $(-\delta, \delta)$ (hence remain "small"). Endow X with the subspace topology from $\mathbb{R}^6$.

Let the (neutrosophic) north and south poles be

$$p_N := \big((0, 0, 0), (1, 0, 0)\big), \qquad p_S := \big((0, 0, 0), (-1, 0, 0)\big),$$

and set the open sets

$$U_N := X \setminus \{p_N\}, \qquad U_S := X \setminus \{p_S\}.$$

Chart from the north. For any $z = \big((x_T, x_I, x_F), (y_T, y_I, y_F)\big) \in U_N$, define the stereographic T–coordinate

$$t := \frac{x_T}{1 - y_T} \in \mathbb{R}$$

(which is well–defined on U_N since $y_T \neq 1$), and set the neutrosophic "band" coordinates

$$i := \frac{x_I + y_I}{2}, \qquad f := \frac{x_F + y_F}{2}.$$

Define the chart

$$\varphi_N : U_N \longrightarrow \Omega_N := \mathbb{R} \times (-\delta, \delta) \times (-\delta, \delta) \subset \mathbb{N}n, \qquad \varphi_N(z) = (t, i, f).$$

This map is continuous, and its inverse on Ω_N is given explicitly by the classical inverse stereographic formulas on the T–layer and duplication on the I/F layers:

$$\varphi_N^{-1}(t, i, f) = \left(\left(\frac{2t}{1+t^2},\ i,\ f \right), \left(\frac{t^2 - 1}{1+t^2},\ i,\ f \right) \right).$$

The image Ω_N is open in $\mathbb{N}n \cong \mathbb{R}^3$, so (U_N, φ_N) is a valid C^∞ chart (each component is a rational function of t or the identity in i, f).

Chart from the south. For $z \in U_S$, define

$$t' := \frac{x_T}{1 + y_T} \in \mathbb{R}, \qquad i' := \frac{x_I + y_I}{2}, \qquad f' := \frac{x_F + y_F}{2},$$

and set

$$\varphi_S : U_S \longrightarrow \Omega_S := \mathbb{R} \times (-\delta, \delta) \times (-\delta, \delta), \qquad \varphi_S(z) = (t', i', f').$$

Again φ_S is continuous with inverse

$$\varphi_S^{-1}(t', i', f') = \left(\left(\frac{2t'}{1+t'^2},\ i',\ f' \right), \left(\frac{1 - t'^2}{1+t'^2},\ i',\ f' \right) \right).$$

Transition map. On the overlap $U_N \cap U_S$ (equivalently $t \neq 0$ or $t' \neq 0$), the change of coordinates is

$$\varphi_S \circ \varphi_N^{-1}(t, i, f) = \left(t^{-1},\ i,\ f \right),$$

and symmetrically $\varphi_N \circ \varphi_S^{-1}(t', i', f') = (\, t'^{-1},\ i',\ f' \,)$. Each component is C^∞ on its domain; the T–component is the smooth map $t \mapsto 1/t$, while the I, F components pass through unchanged. Thus the transition maps are neutrosophically C^k for all k (componentwise smooth and respecting the (T, I, F) decomposition).

The atlas $\{(U_N, \varphi_N), (U_S, \varphi_S)\}$ makes X a 1–dimensional neutrosophic C^∞ manifold modeled on $\mathbb{N}n = \mathbb{R}^3$. Topological prerequisites hold since $X \subset \mathbb{R}^6$ is Hausdorff and second–countable; chart images are open in $\mathbb{N}n$; and changes of charts are smooth in the neutrosophic (componentwise) sense.

Worked point. At $(t, i, f) = (1,\ 0.10,\ 0.05)$,

$$\varphi_N^{-1}(1, 0.10, 0.05) = \Big((1, 0.10, 0.05), (0, 0.10, 0.05) \Big) \in X.$$

Applying the south chart,

$$\varphi_S\big(\varphi_N^{-1}(1, 0.10, 0.05)\big) = (1^{-1},\ 0.10,\ 0.05) = (1, 0.10, 0.05),$$

verifying the transition formula on the overlap.

5.9 Neutro-Cyclic Triplet Group

A neutro-cyclic triplet group is generated by one element under triplet operations, featuring anti and neutral companions ensuring closure properties (cf. [36, 296]).

Definition 5.9.1 ((Recall) Neutrosophic Triplet Group)**.** A *neutrosophic triplet group* is a quadruple $(G, \star; \mathrm{anti}, \mathrm{neut})$ where: (i) $(G, \star)$ is an associative magma with identity e; (ii) for each $x \in G$ there exist distinguished elements $\mathrm{anti}(x), \mathrm{neut}(x) \in G$ such that

$$\mathrm{anti}(\mathrm{anti}(x)) = x, \qquad \mathrm{neut}(x) \star x = x \star \mathrm{neut}(x) = x, \qquad x \star \mathrm{anti}(x) = \mathrm{anti}(x) \star x = \mathrm{neut}(x),$$

and $\mathrm{neut}(\mathrm{anti}(x)) = \mathrm{neut}(x)$; (iii) anti and neut are compatible with $\star$, i.e., $\mathrm{anti}(x \star y) = \mathrm{anti}(y) \star \mathrm{anti}(x)$ and $\mathrm{neut}(x \star y) = \mathrm{neut}(x) \star \mathrm{neut}(y)$.

Definition 5.9.2 (Neutro-cyclic triplet group)**.** (cf. [36, 296]) Let $(G, \star; \mathrm{anti}, \mathrm{neut})$ be a neutrosophic triplet group. For $a \in G$, define the *neutro-generated* substructure

$$\langle a \rangle_N := \bigcap \{H \subseteq G \mid a \in H,\ H \text{ is a neutrosophic triplet subgroup of } G\}.$$

We say G is *neutro-cyclic* if there exists $a \in G$ with $G = \langle a \rangle_N$; we call a a *neutro-generator* of G.

Example 5.9.3 (Neutro–cyclic triplet group on $\mathbb{Z}_n$ with addition)**.** Fix an integer $n \geq 2$. Let the carrier be $G := \mathbb{Z}_n = \{0, 1, \ldots, n-1\}$ with the binary operation

$$x \star y := x + y \pmod n, \qquad \text{identity } e := 0.$$

Define, for every $x \in G$,

$$\mathrm{anti}(x) := (-x) \pmod n, \qquad \mathrm{neut}(x) := 0.$$

We verify the axioms of a neutrosophic triplet group:

(i) Associative magma with identity. Addition mod n is associative, and $e = 0$ satisfies $0 + x = x + 0 = x$.

(ii) Distinguished elements. For all $x \in G$,

$$\mathrm{anti}(\mathrm{anti}(x)) = -(-x) \equiv x \pmod n, \qquad \mathrm{neut}(x) \star x = 0 + x \equiv x, \quad x \star \mathrm{neut}(x) = x + 0 \equiv x,$$

$$x \star \mathrm{anti}(x) = x + (-x) \equiv 0 = \mathrm{neut}(x), \qquad \mathrm{anti}(x) \star x = (-x) + x \equiv 0 = \mathrm{neut}(x),$$

and $\mathrm{neut}(\mathrm{anti}(x)) = 0 = \mathrm{neut}(x)$.

(iii) Compatibility.

$$\mathrm{anti}(x \star y) = -(x + y) \equiv (-y) + (-x) = \mathrm{anti}(y) \star \mathrm{anti}(x),$$

$$\mathrm{neut}(x \star y) = 0 = 0 + 0 = \mathrm{neut}(x) \star \mathrm{neut}(y).$$

Neutro–cyclicity. Let $a := 1 \in \mathbb{Z}_n$. The least neutrosophic triplet subgroup containing a must be closed under $\star$, anti, and neut. Closure under $\star$ yields all additive multiples $\{k \cdot 1 \pmod n : k \in \mathbb{Z}\} = \mathbb{Z}_n$ since 1 has order n. Applying anti gives additive inverses, already included; applying neut gives 0, already present. Hence $\langle a \rangle_N = G$, i.e. $(G, \star; \mathrm{anti}, \mathrm{neut})$ is neutro–cyclic.

Concrete instance (e.g. $n = 6$). With $a = 1$, successive sums generate

$$\{0, 1, 2, 3, 4, 5\}; \quad \mathrm{anti}(4) = 2,\ \mathrm{neut}(4) = 0, \text{ and } 4 \star \mathrm{anti}(4) = 0 = \mathrm{neut}(4),$$

exemplifying the axioms and generation explicitly.

Example 5.9.4 (Neutro–cyclic triplet group on complex n-th roots of unity)**.** Fix $n \geq 3$. Let

$$G := \mu_n = \{\zeta_n^k : k = 0, 1, \ldots, n-1\} \subset \mathbb{C}^\times, \qquad \zeta_n := e^{2\pi i/n},$$

with the binary operation

$$x \star y := x\,y \quad \text{(complex multiplication)}, \qquad \text{identity } e := 1.$$

Define, for $x \in G$,

$$\text{anti}(x) := x^{-1} \quad \text{(complex inverse)}, \qquad \text{neut}(x) := 1.$$

We check the axioms:

(i) Associative magma with identity. Multiplication in $\mathbb{C}^\times$ is associative, and $e = 1$ is the identity.

(ii) Distinguished elements. For all $x \in G$,

$$\text{anti}(\text{anti}(x)) = (x^{-1})^{-1} = x, \qquad \text{neut}(x) \star x = 1 \cdot x = x, \quad x \star \text{neut}(x) = x \cdot 1 = x,$$

$$x \star \text{anti}(x) = x \cdot x^{-1} = 1 = \text{neut}(x), \qquad \text{anti}(x) \star x = x^{-1} \cdot x = 1 = \text{neut}(x),$$

and $\text{neut}(\text{anti}(x)) = 1 = \text{neut}(x)$.

(iii) Compatibility.

$$\text{anti}(x \star y) = (xy)^{-1} = y^{-1}x^{-1} = \text{anti}(y) \star \text{anti}(x),$$

$$\text{neut}(x \star y) = 1 = 1 \cdot 1 = \text{neut}(x) \star \text{neut}(y).$$

Neutro–cyclicity. Take $a := \zeta_n$. The usual cyclic generation gives $\{a^k : 0 \leq k < n\} = \mu_n$. Closure under anti yields $a^{-k} = a^{n-k}$, already in the set; closure under neut adds $1 = a^0$, already present. Thus the least neutrosophic triplet subgroup containing a is the whole G, so $(G, \star; \text{anti}, \text{neut})$ is neutro–cyclic.

Concrete instance (e.g. $n = 5$). With $a = \zeta_5$,

$$G = \{1, \zeta_5, \zeta_5^2, \zeta_5^3, \zeta_5^4\}, \quad \text{anti}(\zeta_5^3) = \zeta_5^2, \quad \text{neut}(\zeta_5^3) = 1, \quad \zeta_5^3 \star \text{anti}(\zeta_5^3) = 1 = \text{neut}(\zeta_5^3).$$

All axioms and neutro–generation hold explicitly.

5.10 Neutrosophic Multi-Structures

A neutrosophic multi-structure overlays multiple structures on subsets and assigns truth, indeterminacy, falsity degrees to their comparative relations for analysis [36].

Definition 5.10.1 (Multi-structure and neutrosophic multi-structure)**.** Let A be a nonempty set. A *multi-structure* on A is a finite family $\mathbf{S} = \{(A_i, \mathcal{S}_i)\}_{i=1}^m$ with nonempty $A_i \subseteq A$ and $\mathcal{S}_i$ a mathematical structure (e.g. topology, algebra, order) on A_i. We say $\mathcal{S}_\alpha$ is *stronger/same/weaker* than $\mathcal{S}_\beta$ on $A_\alpha \cap A_\beta$ according to the usual inclusion/refinement relations between structures.

A *neutrosophic multi-structure* additionally assigns, for every pair (α, β), a triplet $(T_{\alpha\beta}, I_{\alpha\beta}, F_{\alpha\beta}) \in [0,1]^3$ encoding the degrees that $\mathcal{S}_\alpha$ is stronger (truth), indeterminate (indeterminacy), or weaker (falsity) than $\mathcal{S}_\beta$ on $A_\alpha \cap A_\beta$, with no restriction on the sum.

Example 5.10.2 (Neutrosophic multi–structure for an e–commerce catalog)**.** Let the universe of products be

$$A = \{P_1, P_2, P_3, P_4, P_5, P_6, P_7, P_8\}.$$

Consider the multi–structure

$$\mathbf{S} = \{(A_1, \mathcal{S}_1), (A_2, \mathcal{S}_2)\},$$

where both $A_1 = A_2 = A$, and:

Structure $\mathcal{S}_1$ (category partition).

$$C_1 = \{\{P_1, P_2, P_3, P_6\},\ \{P_4\},\ \{P_5, P_7, P_8\}\}.$$

Structure $\mathcal{S}_2$ (price–tier partition).

$$C_2 = \{\{P_1, P_4\},\ \{P_2, P_5\},\ \{P_3, P_6\},\ \{P_7\},\ \{P_8\}\}.$$

We quantify "$\mathcal{S}_\alpha$ is stronger than $\mathcal{S}_\beta$ on $U := A_\alpha \cap A_\beta$" by the pair–separation index

$$s(\mathcal{S}; U) \ :=\ \#\big\{\{x, y\} \subseteq U :\ x, y \text{ lie in different blocks of the partition induced by } \mathcal{S}\big\}.$$

Here $U = A$ and $\binom{|U|}{2} = \binom{8}{2} = 28$.

For $\mathcal{S}_1$ the block sizes on U are $(4, 1, 3)$, hence

$$\text{within–block pairs} = \binom{4}{2} + \binom{1}{2} + \binom{3}{2} = 6 + 0 + 3 = 9, \quad s(\mathcal{S}_1; U) = 28 - 9 = 19.$$

For $\mathcal{S}_2$ the sizes are $(2, 2, 2, 1, 1)$, hence

$$\text{within–block pairs} = 3 \cdot \binom{2}{2} + 0 + 0 = 3, \quad s(\mathcal{S}_2; U) = 28 - 3 = 25.$$

Define the neutrosophic degrees for the ordered pair $(\mathcal{S}_1, \mathcal{S}_2)$ by

$$T_{12} \ :=\ \frac{s(\mathcal{S}_1; U)}{s(\mathcal{S}_1; U) + s(\mathcal{S}_2; U)} = \frac{19}{44} \approx 0.4318, \qquad F_{12} \ :=\ \frac{25}{44} \approx 0.5682, \qquad I_{12} \ :=\ 0.$$

Interpretation: on the common domain U, $\mathcal{S}_2$ (price–tier) separates product pairs more finely than $\mathcal{S}_1$ (category), so the statement "$\mathcal{S}_1$ is stronger than $\mathcal{S}_2$" has truth $T_{12} \approx 0.432$ and falsity $F_{12} \approx 0.568$; no indeterminacy arises because data are complete.

For completeness, the reverse ordered pair $(\mathcal{S}_2, \mathcal{S}_1)$ yields

$$T_{21} = \frac{25}{44} \approx 0.5682, \quad F_{21} = \frac{19}{44} \approx 0.4318, \quad I_{21} = 0.$$

Thus

$$\big(T_{12}, I_{12}, F_{12}\big) \approx (0.432, 0, 0.568), \qquad \big(T_{21}, I_{21}, F_{21}\big) \approx (0.568, 0, 0.432).$$

By definition, the family $\mathbf{S}$ together with these triplets is a neutrosophic multi–structure on A.

Example 5.10.3 (Neutrosophic multi–structure for hospital triage views)**.** Let $A = \{p_1, \ldots, p_7\}$ be seven patients. Consider

$$\mathbf{S} = \{(A_1, \mathcal{S}_1), (A_2, \mathcal{S}_2)\}, \qquad A_1 = A_2 = A,$$

with two concurrent structures:

Structure $\mathcal{S}_1$ (diagnosis cluster, an equivalence relation).

$$\mathcal{D} = \{\{p_1, p_2, p_3\},\ \{p_4, p_5\},\ \{p_6, p_7\}\},$$

block sizes $(3, 2, 2)$.

Structure $\mathcal{S}_2$ (risk stratification, an equivalence relation).

$$\mathcal{R} = \{\{p_1, p_2, p_4, p_5\},\ \{p_3, p_7\},\ \{p_6\}\},$$

block sizes $(4, 2, 1)$.

We again use the pair–separation index on $U = A$, where $\binom{|U|}{2} = \binom{7}{2} = 21$. For $\mathcal{S}_1$:

$$\text{within–block pairs} = \binom{3}{2} + \binom{2}{2} + \binom{2}{2} = 3 + 1 + 1 = 5, \quad s(\mathcal{S}_1; U) = 21 - 5 = 16.$$

For $\mathcal{S}_2$:

$$\text{within–block pairs} = \binom{4}{2} + \binom{2}{2} + \binom{1}{2} = 6 + 1 + 0 = 7, \quad s(\mathcal{S}_2; U) = 21 - 7 = 14.$$

Because one patient (p_5) has an inconclusive laboratory panel pending, we encode a (data–quality) indeterminacy level on U by

$$I_{12} \ := \ \frac{\#\{\text{inconclusive records for } \mathcal{S}_2 \text{ on } U\}}{|U|} \ = \ \frac{1}{7} \ \approx \ 0.1429.$$

We keep the *truth/falsity* components normalized to the pair–separation competition:

$$T_{12} \ := \ \frac{16}{16+14} = \frac{8}{15} \approx 0.5333, \qquad F_{12} \ := \ \frac{14}{16+14} = \frac{7}{15} \approx 0.4667.$$

Thus, for the ordered pair $(\mathcal{S}_1, \mathcal{S}_2)$ we assign

$$\big(T_{12}, I_{12}, F_{12}\big) \ \approx \ (0.5333, \ 0.1429, \ 0.4667).$$

Interpretation: diagnosis clusters separate patient pairs slightly more finely than current risk tiers (truth ≈ 0.5333), there is some conflict in the reverse direction (falsity ≈ 0.4667), and an explicit indeterminacy ≈ 0.1429 due to pending data. The reverse ordered pair $(\mathcal{S}_2, \mathcal{S}_1)$ uses the same $I_{21} = I_{12}$ but swaps the normalized truth/falsity:

$$\big(T_{21}, I_{21}, F_{21}\big) \ \approx \ (0.4667, \ 0.1429, \ 0.5333).$$

Equipping the family $\mathbf{S}$ with these neutrosophic triplets on each ordered pair completes a neutrosophic multi–structure on A tied to concrete hospital triage views.

5.11 Neutrosophic Filter

A filter is a nonempty family of sets upward-closed, closed under finite intersections, excluding the empty set, capturing large subsets [297]. A neutrosophic filter is an upward-closed family under neutrosophic inclusion, closed under neutrosophic meet, proper, modeling truth–indeterminacy–falsity orders on sets [298–300]. Related concepts such as fuzzy filters [301–304] and intuitionistic fuzzy filters [305, 306] are also well known. A refined neutrosophic filter handles multi-index truth, indeterminacy, falsity components, is upward-closed and meet-closed componentwise, generalizing ordinary neutrosophic filters settings [40].

Notation 5.11.1. *The (standard) pointwise operations are*

$$(A \cap_N B)(x) := \big(\min\{T_A, T_B\}, \ \max\{I_A, I_B\}, \ \max\{F_A, F_B\}\big),$$
$$(A \cup_N B)(x) := \big(\max\{T_A, T_B\}, \ \min\{I_A, I_B\}, \ \min\{F_A, F_B\}\big).$$

The neutrosophic inclusion $A \subseteq_N B$ *means, for all* $x \in X$,

$$T_A(x) \le T_B(x), \qquad I_A(x) \ge I_B(x), \qquad F_A(x) \ge F_B(x).$$

We also fix the (crisp) bottom and top neutrosophic sets

$$\mathbf{0}_N(x) := (0, 0, 1), \qquad \mathbf{1}_N(x) := (1, 0, 0) \qquad (\forall x \in X).$$

Definition 5.11.2 (Neutrosophic filter)**.** [298–300] A nonempty family $\mathcal{F} \subseteq \{A : A \text{ is a neutrosophic set on } X\}$ is a *neutrosophic filter* on X if

1. upward closure: $A \in \mathcal{F}$ and $A \subseteq_N B \ \Rightarrow \ B \in \mathcal{F}$;
2. finite meet closure: $A, B \in \mathcal{F} \ \Rightarrow \ A \cap_N B \in \mathcal{F}$;

3. properness: $\mathbf{0}_N \notin \mathcal{F}$ (equivalently, $\mathbf{1}_N \in \mathcal{F}$).

Remark 5.11.3. This axiomatization matches the standard "upward closed and finite-intersection closed" scheme adapted to neutrosophic order/meet; see the classical presentation of neutrosophic filters and their basic properties.

Example 5.11.4 (Neutrosophic filter: "high–reliability servers")**.** Let the universe be a small data center

$$X = \{s_1, s_2, s_3\},$$

where s_i are servers. For each neutrosophic set A on X we write $A(x) = (T_A(x), I_A(x), F_A(x))$ with $T_A, I_A, F_A \in [0,1]$.

Define the neutrosophic set H ("high–reliability server") by

$$\begin{aligned} H(s_1) &= (0.8,\ 0.1,\ 0.1), \\ H(s_2) &= (0.6,\ 0.2,\ 0.2), \\ H(s_3) &= (0.4,\ 0.3,\ 0.3), \end{aligned}$$

and the neutrosophic top and bottom sets by

$$\mathbf{1}_N(x) = (1,0,0), \qquad \mathbf{0}_N(x) = (0,0,1) \qquad (x \in X).$$

Consider the family

$$\mathcal{F} := \Big\{ A \text{ neutrosophic on } X \ \Big|\ H \subseteq_N A \Big\},$$

i.e. all neutrosophic supersets of H under neutrosophic inclusion ($T_H \le T_A$, $I_H \ge I_A$, $F_H \ge F_A$ pointwise).

Then:

- $\mathcal{F}$ is nonempty and $\mathbf{1}_N \in \mathcal{F}$, while $\mathbf{0}_N \notin \mathcal{F}$;
- if $A \in \mathcal{F}$ and $A \subseteq_N B$, then $H \subseteq_N B$, so $B \in \mathcal{F}$ (upward closure);
- if $A, B \in \mathcal{F}$, then $H \subseteq_N A$ and $H \subseteq_N B$ imply $H \subseteq_N (A \cap_N B)$, hence $A \cap_N B \in \mathcal{F}$ (closed under neutrosophic meet).

Thus $\mathcal{F}$ is a neutrosophic filter describing "all neutrosophically high–reliability profiles containing H as a core" for the servers in X.

Notation 5.11.5. *Define the componentwise refined meet/union by*

$$\begin{aligned} (\mathbb{A} \wedge_R \mathbb{B})(x) &:= \big(\min(\mathbf{T}_A, \mathbf{T}_B),\ \max(\mathbf{I}_A, \mathbf{I}_B),\ \max(\mathbf{F}_A, \mathbf{F}_B) \big), \\ (\mathbb{A} \vee_R \mathbb{B})(x) &:= \big(\max(\mathbf{T}_A, \mathbf{T}_B),\ \min(\mathbf{I}_A, \mathbf{I}_B),\ \min(\mathbf{F}_A, \mathbf{F}_B) \big), \end{aligned}$$

where min/max are taken coordinatewise (e.g. $(\min(\mathbf{T}_A, \mathbf{T}_B))_p = \min\{T_{A,p}, T_{B,p}\}$*). Refined inclusion* $\mathbb{A} \preceq_R \mathbb{B}$ *means, for all x and all indices,*

$$T_{A,p} \le T_{B,p} \ \ (p \in P), \qquad I_{A,q} \ge I_{B,q} \ \ (q \in Q), \qquad F_{A,r} \ge F_{B,r} \ \ (r \in R).$$

Set the refined bottom/top by $\mathbf{0}_R(x) := (\mathbf{0}_P, \mathbf{0}_Q, \mathbf{1}_R)$ *and* $\mathbf{1}_R(x) := (\mathbf{1}_P, \mathbf{0}_Q, \mathbf{0}_R)$*. (Here* $\mathbf{0}_P$ *is the zero vector in* $[0,1]^P$*, etc.)*

Definition 5.11.6 (Refined neutrosophic filter)**.** A nonempty family $\mathfrak{F}$ of refined neutrosophic sets on X is a *refined neutrosophic filter* if

1. upward closure: $\mathbb{A} \in \mathfrak{F}$ and $\mathbb{A} \preceq_R \mathbb{B} \ \Rightarrow\ \mathbb{B} \in \mathfrak{F}$;
2. finite meet closure: $\mathbb{A}, \mathbb{B} \in \mathfrak{F} \ \Rightarrow\ \mathbb{A} \wedge_R \mathbb{B} \in \mathfrak{F}$;

3. properness: $\mathbf{0}_R \notin \mathfrak{F}$ (equivalently, $\mathbf{1}_R \in \mathfrak{F}$).

Example 5.11.7 (Neutrosophic filter: "high–reliability servers")**.** High-reliability describes systems with consistently low failure rates, robust design, redundancy, and rigorous maintenance ensuring dependable performance under stress conditions (cf. [307]).

Let the universe be a small data center

$$X = \{s_1, s_2, s_3\},$$

where s_i are servers. For each neutrosophic set A on X we write $A(x) = (T_A(x), I_A(x), F_A(x))$ with $T_A, I_A, F_A \in [0, 1]$.

Define the neutrosophic set H ("high–reliability server") by

$$H(s_1) = (0.8,\ 0.1,\ 0.1),$$
$$H(s_2) = (0.6,\ 0.2,\ 0.2),$$
$$H(s_3) = (0.4,\ 0.3,\ 0.3),$$

and the neutrosophic top and bottom sets by

$$\mathbf{1}_N(x) = (1, 0, 0), \qquad \mathbf{0}_N(x) = (0, 0, 1) \qquad (x \in X).$$

Consider the family

$$\mathcal{F} := \Big\{ A \text{ neutrosophic on } X \ \Big|\ H \subseteq_N A \Big\},$$

i.e. all neutrosophic supersets of H under neutrosophic inclusion ($T_H \le T_A$, $I_H \ge I_A$, $F_H \ge F_A$ pointwise).

Then:

- $\mathcal{F}$ is nonempty and $\mathbf{1}_N \in \mathcal{F}$, while $\mathbf{0}_N \notin \mathcal{F}$;
- if $A \in \mathcal{F}$ and $A \subseteq_N B$, then $H \subseteq_N B$, so $B \in \mathcal{F}$ (upward closure);
- if $A, B \in \mathcal{F}$, then $H \subseteq_N A$ and $H \subseteq_N B$ imply $H \subseteq_N (A \cap_N B)$, hence $A \cap_N B \in \mathcal{F}$ (closed under neutrosophic meet).

Thus $\mathcal{F}$ is a neutrosophic filter describing "all neutrosophically high–reliability profiles containing H as a core" for the servers in X.

Theorem 5.11.8. *Every neutrosophic filter is a special case of a refined neutrosophic filter. Precisely, fix any nonempty finite index sets P, Q, R and define the embedding*

$$\iota : \{A \text{ neutrosophic}\} \longrightarrow \{\mathbb{A} \text{ refined}\}, \qquad \big(\iota(A)\big)(x) := \big((T_A(x))_{p\in P},\ (I_A(x))_{q\in Q},\ (F_A(x))_{r\in R}\big).$$

If $\mathcal{F}$ is a neutrosophic filter, then

$$\iota(\mathcal{F}) := \{\ \iota(A) : A \in \mathcal{F}\ \}$$

is a refined neutrosophic filter. Conversely, when $|P| = |Q| = |R| = 1$ the class of refined neutrosophic filters coincides with that of neutrosophic filters.

Proof. Let $\mathcal{F}$ be a neutrosophic filter.

Upward closure: If $\iota(A) \preceq_R \iota(B)$, then by the definition of $\preceq_R$ we have, for every x, $T_A \le T_B$, $I_A \ge I_B$, $F_A \ge F_B$; hence $A \subseteq_N B$. Since $A \in \mathcal{F}$ and $\mathcal{F}$ is upward closed, $B \in \mathcal{F}$, so $\iota(B) \in \iota(\mathcal{F})$.

Finite meet: For any A, B and any x,

$$\big(\iota(A) \wedge_R \iota(B)\big)(x) = \big(\ (\min\{T_A, T_B\})_{p\in P},\ (\max\{I_A, I_B\})_{q\in Q},\ (\max\{F_A, F_B\})_{r\in R}\ \big) = \iota(A \cap_N B)(x).$$

Since $A \cap_N B \in \mathcal{F}$, we get $\iota(A) \wedge_R \iota(B) \in \iota(\mathcal{F})$.

Properness: $\mathbf{0}_N \notin \mathcal{F}$ implies $\iota(\mathbf{0}_N) = \mathbf{0}_R \notin \iota(\mathcal{F})$, and similarly $\mathbf{1}_R = \iota(\mathbf{1}_N) \in \iota(\mathcal{F})$.

Therefore $\iota(\mathcal{F})$ is a refined neutrosophic filter. The converse (collapse to the non–refined case when $|P| = |Q| = |R| = 1$) is immediate from the definitions. □

5.12 Neutrosophic Triplet Filters

The notion of *neutrosophic filters* in algebraic structures can be extended either to *neutrosophic triplet filters* or to *refined neutrosophic filters* (where T, I, F are split into $T_1, T_2, \dots, I_1, I_2, \dots, F_1, F_2, \dots$), in BE–algebras and related structures [40].

Definition 5.12.1 (Neutrosophic Triplet Filter on a Residuated Lattice)**.** [40] Let $\mathbf{L} = (L, \wedge, \vee, \cdot, \to, 1)$ be a residuated lattice with order $x \le y \iff 1 \le (x \to y)$. A *neutrosophic triplet filter* on $\mathbf{L}$ is a map $N : L \to [0,1]^3, x \mapsto \big(T(x), I(x), F(x)\big)$, such that for all $x, y \in L$:

1. Normalization at the unit:
$$T(1) = 1, \qquad I(1) = 0, \qquad F(1) = 0.$$
2. Monotonicity (upward closed truth / downward closed falsity):
$$x \le y \ \Rightarrow\ T(x) \le T(y), \qquad F(y) \le F(x).$$
3. Modus–ponens closure for truth (filter law):
$$T(x) \wedge T(x \to y) \ \le\ T(y).$$
4. Consistency of the triplet:
$$0 \le T(x) + I(x) + F(x) \le 3, \qquad I(x) \ge 0 \ \text{(indeterminacy is nonnegative)}.$$
5. Triplet symmetry via “anti” and “neutral” transforms: there exist mappings anti, neut : $L \to L$ such that
$$T\big(\mathrm{anti}(x)\big) = F(x),\ \ F\big(\mathrm{anti}(x)\big) = T(x),\ \ I\big(\mathrm{anti}(x)\big) = I(x),$$
and
$$T\big(\mathrm{neut}(x)\big) \le \min\{T(x),\ 1 - T(x)\}, \quad I\big(\mathrm{neut}(x)\big) \ge I(x).$$

For $\alpha \in (0,1]$, the *α–cut* (crisp) carrier set
$$\mathcal{F}_\alpha \ :=\ \{\, x \in L \mid T(x) \ge \alpha \text{ and } F(x) \le 1 - \alpha \,\}$$
is called the *crisp core* of N at level α.

Example 5.12.2 (Concrete neutrosophic triplet filter on the Gödel residuated lattice)**.** Let $\mathbf{L} = ([0,1], \wedge, \vee, \cdot, \to, 1)$ be the Gödel (minimum) residuated lattice:
$$x \wedge y := \min\{x,y\}, \quad x \vee y := \max\{x,y\}, \quad x \cdot y := \min\{x,y\}, \quad x \to y := \begin{cases} 1, & x \le y, \\ y, & x > y, \end{cases}$$
with the usual order on $[0,1]$ and unit 1.

Define $N : [0,1] \to [0,1]^3$ by
$$T(x) := x, \qquad I(x) := 0, \qquad F(x) := 1 - x \qquad (x \in [0,1]).$$

We verify items (1)–(5) of the definition.

1) Normalization at 1:
$$T(1) = 1, \quad I(1) = 0, \quad F(1) = 0.$$

2) Monotonicity. If $x \le y$ then
$$T(x) = x \le y = T(y), \qquad F(y) = 1 - y \le 1 - x = F(x).$$

3) Modus–ponens closure (filter law): $T(x) \wedge T(x \to y) \le T(y)$. We check casewise for $x, y \in [0, 1]$.

• If $x \le y$, then $x \to y = 1$, hence

$$T(x) \wedge T(x \to y) = x \wedge 1 = x \le y = T(y).$$

• If $x > y$, then $x \to y = y$, hence

$$T(x) \wedge T(x \to y) = x \wedge y = y = T(y).$$

Thus in all cases $T(x) \wedge T(x \to y) \le T(y)$ holds.

4) Triplet consistency. For every $x \in [0, 1]$,

$$0 \le T(x) + I(x) + F(x) = x + 0 + (1 - x) = 1 \le 3, \qquad I(x) = 0 \ge 0.$$

5) Triplet symmetry via "anti" and "neutral" transforms. Define anti, neut : $[0, 1] \to [0, 1]$ by

$$\text{anti}(x) := 1 - x, \qquad \text{neut}(x) := \min\{x,\ 1 - x\}.$$

Then, for all $x \in [0, 1]$,

$$T(\text{anti}(x)) = 1 - x = F(x), \quad F(\text{anti}(x)) = x = T(x), \quad I(\text{anti}(x)) = 0 = I(x),$$

and

$$T(\text{neut}(x)) = \min\{x, 1 - x\} \le \min\{T(x),\ 1 - T(x)\}, \qquad I(\text{neut}(x)) = 0 \ge I(x) = 0.$$

Finally, for any $\alpha \in (0, 1]$, the crisp core (the α–cut carrier)

$$\mathcal{F}_\alpha = \{\, x \in [0, 1] \mid T(x) \ge \alpha \text{ and } F(x) \le 1 - \alpha \,\} = \{\, x \in [0, 1] \mid x \ge \alpha \text{ and } 1 - x \le 1 - \alpha \,\} = [\alpha, 1].$$

Hence N is a neutrosophic triplet filter on the residuated lattice $\mathbf{L}$.

Theorem 5.12.3 (Crisp α–cores are classical filters)**.** *For any $\alpha \in (0, 1]$, the set $\mathcal{F}_\alpha$ is an (upward closed, $\cdot$–closed) filter of* $\mathbf{L}$.

Proof. Upward closed: If $x \in \mathcal{F}_\alpha$ and $x \le y$, then by monotonicity $T(y) \ge T(x) \ge \alpha$ and $F(y) \le F(x) \le 1-\alpha$, hence $y \in \mathcal{F}_\alpha$. Closure under $\cdot$: Let $x, y \in \mathcal{F}_\alpha$. Since $x \cdot y \le y$ implies $T(x) \wedge T(x \to x \cdot y) \le T(x \cdot y)$, we have $T(x \cdot y) \ge T(x) \wedge T(x \to x \cdot y) \ge \alpha$ (because $T(x) \ge \alpha$ and $x \to x \cdot y \ge 1$ in a residuated lattice, hence $T(x \to x \cdot y) \ge T(1) = 1$). Likewise, $F(x \cdot y) \le \max\{F(x), F(y)\} \le 1 - \alpha$ by the falsity monotonicity, so $x \cdot y \in \mathcal{F}_\alpha$. □

Remark 5.12.4. When the refined form is needed, replace (T, I, F) by $(T_1, \ldots, T_p; I_1, \ldots, I_q; F_1, \ldots, F_r)$ and impose the above axioms componentwise, possibly with convex aggregation to obtain global T, I, F. This yields a *refined neutrosophic filter*.

5.13 Neutrosophic Linguistic Topological Space

A neutrosophic linguistic topological space equips linguistic labels with truth, indeterminacy, falsity degrees, defining metric neighborhoods that reflect neutrosophic proximity [39].

Definition 5.13.1 (Neutrosophic Linguistic Topological Space (NLTS))**.** [39] Let $L = \{\ell_1, \ldots, \ell_m\}$ be a finite linearly ordered set of linguistic labels and let $\iota : L \to [0, 1]^3$ be an injective *neutrosophic embedding* that assigns to each label $\ell \in L$ a triplet $\iota(\ell) = (T_\ell, I_\ell, F_\ell)$. Fix strictly positive weights w_T, w_I, w_F with $w_T + w_I + w_F = 1$ and define, for $\ell, \ell' \in L$, the metric

$$d_N(\ell, \ell') \ := \ w_T\,|T_\ell - T_{\ell'}| \ + \ w_I\,|I_\ell - I_{\ell'}| \ + \ w_F\,|F_\ell - F_{\ell'}|.$$

The *neutrosophic linguistic topological space on labels* is the metric space

$$\big(L, \tau_L^N\big), \qquad \tau_L^N := \text{the topology induced by } d_N.$$

If X is a nonempty set of objects and $\lambda : X \to L$ assigns to each object a linguistic label, the *neutrosophic linguistic topology on X (pulled back by λ)* is the initial topology

$$\tau_X^N \;:=\; \big\{\lambda^{-1}(U) \;:\; U \in \tau_L^N\big\}.$$

Hence (X, τ_X^N) is a topological space in which neighborhoods are determined by neutrosophic proximity between the assigned linguistic labels.

Example 5.13.2 (NLTS for product ratings with explicit distances and open sets)**.** Product ratings summarize customers' evaluations of items, expressing perceived quality, satisfaction, and value using numerical scores or categorical labels online (cf. [308]).

Consider the label set

$$L = \{\text{VL (Very Low)},\ \text{L (Low)},\ \text{M (Medium)},\ \text{H (High)},\ \text{VH (Very High)}\},$$

with neutrosophic embedding $\iota(\ell) = (T_\ell, I_\ell, F_\ell)$ and weights $(w_T, w_I, w_F) = (0.6, 0.2, 0.2)$:

$$\begin{aligned} \iota(\text{VL}) &= (0.1,\ 0.5,\ 0.9), \quad \iota(\text{L}) = (0.3,\ 0.4,\ 0.7),\\ \iota(\text{M}) &= (0.5,\ 0.3,\ 0.5), \quad \iota(\text{H}) = (0.7,\ 0.2,\ 0.3),\\ \iota(\text{VH}) &= (0.9,\ 0.1,\ 0.1). \end{aligned}$$

Define the neutrosophic metric on L by

$$d_N(\ell, \ell') \;:=\; 0.6\,|T_\ell - T_{\ell'}| + 0.2\,|I_\ell - I_{\ell'}| + 0.2\,|F_\ell - F_{\ell'}|.$$

Selected distances (all computed componentwise and summed):

$$\begin{aligned} d_N(\text{M}, \text{H}) &= 0.6|0.5{-}0.7| + 0.2|0.3{-}0.2| + 0.2|0.5{-}0.3|\\ &= 0.6 \cdot 0.2 + 0.2 \cdot 0.1 + 0.2 \cdot 0.2 = 0.12 + 0.02 + 0.04 = 0.18,\\ d_N(\text{M}, \text{L}) &= 0.6|0.5{-}0.3| + 0.2|0.3{-}0.4| + 0.2|0.5{-}0.7|\\ &= 0.12 + 0.02 + 0.04 = 0.18,\\ d_N(\text{M}, \text{VH}) &= 0.6 \cdot 0.4 + 0.2 \cdot 0.2 + 0.2 \cdot 0.4 = 0.24 + 0.04 + 0.08 = 0.36,\\ d_N(\text{M}, \text{VL}) &= 0.36, \qquad d_N(\text{H}, \text{VH}) = 0.18. \end{aligned}$$

Hence, for radius $r = 0.20$,

$$B_{d_N}(\text{M};\, 0.20) = \{\text{L},\ \text{M},\ \text{H}\}.$$

The linguistic NLTS on labels is $\big(L, \tau_L^N\big)$, where τ_L^N is induced by d_N.

Now let X be a set of products and let $\lambda : X \to L$ assign a neutrosophic-linguistic quality label:

$$X = \{\text{P1}, \text{P2}, \text{P3}, \text{P4}, \text{P5}, \text{P6}\}, \qquad \lambda(\text{P1}) = \text{L},\ \lambda(\text{P2}) = \text{M},\ \lambda(\text{P3}) = \text{H},\ \lambda(\text{P4}) = \text{VL},\ \lambda(\text{P5}) = \text{VH},\ \lambda(\text{P6}) = \text{M}.$$

The pulled-back neutrosophic linguistic topology on X is

$$\tau_X^N = \{\, \lambda^{-1}(U) \;:\; U \in \tau_L^N \,\}.$$

Using the open ball $U = B_{d_N}(\text{M}; 0.20) = \{\text{L}, \text{M}, \text{H}\}$, its preimage is

$$\lambda^{-1}(U) = \{\text{P1}, \text{P2}, \text{P3}, \text{P6}\},$$

which is an open set in (X, τ_X^N). Intuitively, products labeled Low/Medium/High are neutrosophically "close" to Medium under d_N, so they form a neighborhood determined by the neutrosophic proximity of their assigned linguistic labels.

5.14 Neutrosophic Hyperbolic Number

A neutrosophic hyperbolic number is a triplet of hyperbolic components modeling truth, indeterminacy, falsity with componentwise algebraic operations and conjugation [39]. (Classical) hyperbolic numbers, due to their significance, have been investigated in various contexts and related conceptual frameworks (cf. [309–313]).

Definition 5.14.1 (Neutrosophic Hyperbolic Numbers)**.** [39] Let $\mathbb{D} = \{a + jb : a, b \in \mathbb{R},\ j^2 = +1\}$ be the algebra of (split–complex / hyperbolic) numbers with conjugation $\overline{a + jb} := a - jb$ and quadratic form $Q(z) := z\overline{z} = a^2 - b^2$. A *neutrosophic hyperbolic number* is a triplet

$$\mathsf{Z} := \big(Z_T,\ Z_I,\ Z_F\big) \in \mathbb{D}^3,$$

interpreted as truth, indeterminacy, and falsity components in $\mathbb{D}$.

Addition and multiplication are defined componentwise:

$$\mathsf{Z} \oplus \mathsf{W} = (Z_T + W_T,\ Z_I + W_I,\ Z_F + W_F), \qquad \mathsf{Z} \otimes \mathsf{W} = (Z_T W_T,\ Z_I W_I,\ Z_F W_F).$$

The neutrosophic conjugate and quadratic form are

$$\overline{\mathsf{Z}} := (\overline{Z_T}, \overline{Z_I}, \overline{Z_F}), \qquad \mathcal{Q}(\mathsf{Z}) := \big(Q(Z_T), Q(Z_I), Q(Z_F)\big).$$

A (pseudo-)magnitude functional may be taken as

$$\|\mathsf{Z}\|_N := w_T\, |Q(Z_T)|^{1/2} + w_I\, |Q(Z_I)|^{1/2} + w_F\, |Q(Z_F)|^{1/2},$$

with fixed weights $w_T, w_I, w_F > 0$, $w_T + w_I + w_F = 1$. Thus $\mathbb{D}^3$ with $(\oplus, \otimes)$ is a commutative $\mathbb{R}$–algebra, and $(\bar{\cdot}, \mathcal{Q})$ preserves the hyperbolic structure on each component.

Example 5.14.2 (Worked calculations with neutrosophic hyperbolic numbers)**.** Let

$$\mathsf{Z} = \big(Z_T, Z_I, Z_F\big) = \big(3 + j\,1,\ 1 + j\,2,\ 0.5 + j\,0.5\big), \qquad \mathsf{W} = \big(W_T, W_I, W_F\big) = \big(2 + j\,3,\ -1 + j\,0,\ 1 - 2j\big),$$

in the hyperbolic algebra $\mathbb{D} = \{a + jb : a, b \in \mathbb{R},\ j^2 = +1\}$ with $\overline{a + jb} = a - jb$ and $Q(a + jb) = a^2 - b^2$. Fix weights $(w_T, w_I, w_F) = (0.60, 0.25, 0.15)$.

1) Componentwise addition $\mathsf{Z} \oplus \mathsf{W}$:

$$\begin{aligned}
Z_T + W_T &= (3 + j1) + (2 + j3) = 5 + j4, & Q &= 5^2 - 4^2 = 9,\ \sqrt{|Q|} = 3,\\
Z_I + W_I &= (1 + j2) + (-1 + j0) = 0 + j2, & Q &= 0^2 - 2^2 = -4,\ \sqrt{|Q|} = 2,\\
Z_F + W_F &= (0.5 + j0.5) + (1 - 2j) = 1.5 - j1.5, & Q &= 1.5^2 - 1.5^2 = 0.
\end{aligned}$$

2) Componentwise multiplication $\mathsf{Z} \otimes \mathsf{W}$ using $(a + jb)(c + jd) = (ac + bd) + j(ad + bc)$:

$$\begin{aligned}
Z_T W_T &= (3 + j1)(2 + j3) = (6 + 3) + j(9 + 2) = 9 + j11, & Q &= 9^2 - 11^2 = -40,\ \sqrt{|Q|} = \sqrt{40} \approx 6.3249,\\
Z_I W_I &= (1 + j2)(-1 + j0) = (-1 + 0) + j(0 - 2) = -1 - 2j, & Q &= 1 - 4 = -3,\ \sqrt{|Q|} = \sqrt{3} \approx 1.7321,\\
Z_F W_F &= (0.5 + j0.5)(1 - 2j) = (-0.5) + j(-0.5), & Q &= 0.25 - 0.25 = 0.
\end{aligned}$$

3) Neutrosophic conjugate and quadratic form:

$$\overline{\mathsf{Z}} = \big(\overline{3 + j1},\ \overline{1 + j2},\ \overline{0.5 + j0.5}\big) = \big(3 - j1,\ 1 - j2,\ 0.5 - j0.5\big), \qquad \mathcal{Q}(\mathsf{Z}) = \big(8,\ -3,\ 0\big).$$

4) Pseudo-magnitudes (weights as above):

$$\|\mathsf{Z}\|_N = 0.60\sqrt{8} + 0.25\sqrt{3} + 0.15\sqrt{0} \approx 0.60 \cdot 2.8284 + 0.25 \cdot 1.7321 + 0 \approx 2.1301,$$

$$\|\mathsf{W}\|_N = 0.60\sqrt{5} + 0.25 \cdot 1 + 0.15\sqrt{3} \approx 0.60 \cdot 2.2361 + 0.25 + 0.15 \cdot 1.7321 \approx 1.8514.$$

The triplets encode truth/indeterminacy/falsity as hyperbolic numbers. Lightlike components (quadratic form $Q = 0$) such as Z_F or $Z_F W_F$ represent boundary cases between "timelike" ($Q > 0$) and "spacelike" ($Q < 0$) regimes, while the weighted pseudo-magnitude $\|\cdot\|_N$ aggregates these regimes into a single index for each neutrosophic state.

5.15 Neutrosophic Boolean Lattice

A Boolean lattice is a distributive complemented lattice isomorphic to the power set of a set, ordered by inclusion relation [314–316]. Neutrosophic Boolean Lattice organizes truth, indeterminacy, and falsity as ordered elements, supporting meet, join, involutive negation, and distributivity, absorption axioms [44].

Definition 5.15.1 (Neutrosophic Boolean Lattice)**.** [44] Let $L = \{0, \mathbf{I}, 1\}$ with the total order $0 \leq \mathbf{I} \leq 1$. Define

$$x \wedge y := \min\{x, y\}, \qquad x \vee y := \max\{x, y\}, \qquad \neg 0 = 1,\ \neg 1 = 0,\ \neg \mathbf{I} = \mathbf{I}.$$

Then $(L, \wedge, \vee, \neg, 0, 1)$ is a bounded distributive De Morgan lattice with an involutive negation $\neg$; we call it a *Neutrosophic Boolean Lattice*. Truth 1, falsity 0, and indeterminacy $\mathbf{I}$ are its three distinguished values.

Example 5.15.2 (Access–control policy modeled as a Neutrosophic Boolean Lattice)**.** An access-control policy defines who may access which resources, under what conditions, specifying permissions, restrictions, roles, authentication, and auditing requirements (cf. [317]).

Consider the three decision values

$$L = \{0, \mathbf{I}, 1\},$$

interpreted as *deny* (0), *unknown/conditional* ($\mathbf{I}$), and *allow* (1). Define lattice operations and negation by

$$x \wedge y = \min\{x, y\}, \qquad x \vee y = \max\{x, y\}, \qquad \neg 0 = 1, \quad \neg 1 = 0, \quad \neg \mathbf{I} = \mathbf{I}.$$

This yields a bounded distributive De Morgan lattice $(L, \wedge, \vee, \neg, 0, 1)$, i.e., a neutrosophic Boolean lattice.

Operational tables:

$\wedge$	0	$\mathbf{I}$	1
0	0	0	0
$\mathbf{I}$	0	$\mathbf{I}$	$\mathbf{I}$
1	0	$\mathbf{I}$	1

$\vee$	0	$\mathbf{I}$	1
0	0	$\mathbf{I}$	1
$\mathbf{I}$	$\mathbf{I}$	$\mathbf{I}$	1
1	1	1	1

Policy aggregation example. Suppose two rules apply to a request: $r_1 = 1$ (a whitelist rule allows), $r_2 = \mathbf{I}$ (context not yet verified). Then

$$r_1 \wedge r_2 = \min\{1, \mathbf{I}\} = \mathbf{I} \quad \text{(cautious conjunction keeps uncertainty)},$$

$$r_1 \vee r_2 = \max\{1, \mathbf{I}\} = 1 \quad \text{(optimistic disjunction yields allow)}.$$

Negation behaves classically on definite values and is idempotent on indeterminacy:

$$\neg \mathbf{I} = \mathbf{I}, \qquad \neg 0 = 1, \qquad \neg 1 = 0,$$

and De Morgan laws hold, e.g.

$$\neg(r_1 \vee r_2) = \neg 1 = 0 \quad \text{and} \quad \neg r_1 \wedge \neg r_2 = 0 \wedge \mathbf{I} = 0.$$

Thus L compactly captures three–valued access decisions with principled aggregation.

5.16 Extended Neutrosophic Duplets

A neutrosophic duplet attaches to each element a pair of related components, a main value and its neutrosophic neutral counterpart [318–321]. An extended neutrosophic duplet assigns each monoid element a nonidentity neutral and a relative inverse satisfying a·anti(a)=neutral(a) and commutation conditions [44].

Definition 5.16.1 (Extended Neutrosophic Duplets)**.** [44] Let $(M, *, e)$ be a monoid. A pair of maps neut, anti : $M \to M$ defines, for each $a \in M$, the *extended neutrosophic duplet* $(a, \mathrm{neut}(a); \mathrm{anti}(a))$ if

$$a * \mathrm{neut}(a) = \mathrm{neut}(a) * a, \qquad a * \mathrm{anti}(a) = \mathrm{anti}(a) * a = \mathrm{neut}(a),$$

with $\mathrm{neut}(a) \neq e$ (nontrivial neutral) and $\mathrm{anti}(a)$ acting as an inverse of *a relative to* $\mathrm{neut}(a)$. When $\mathrm{anti}(a)$ does not exist one recovers an (ordinary) neutrosophic duplet.

Example 5.16.2 (Extended neutrosophic duplet in an enterprise document workflow)**.** Enterprise document workflow manages creation, review, approval, routing, storage, and retrieval of business documents using standardized digital processes across organizations (cf. [322]).

Let

$$D := \Big\{(t, C, s) \;\Big|\; t \in \text{Text},\ C \subseteq_{\text{fin}} \text{Comments},\ s \in \{\text{draft}, \text{final}\},\ (s = \text{final} \Rightarrow C = \varnothing)\Big\}.$$

Consider the monoid $(M, *, e)$ where $M := D^D$ (all total functions $D \to D$), the operation $*$ is function composition $\circ$, and $e = \mathrm{id}_D$.

Fix a constant reviewer note $c_\star \in$ Comments. Define the three functions a, neut, anti $\in M$ by

$$a(t, C, s) := \begin{cases} (t,\ C \cup \{c_\star\},\ \text{draft}) & \text{if } s = \text{draft}, \\ (t,\ C,\ \text{final}) & \text{if } s = \text{final} \quad \text{(no change; edits blocked)}, \end{cases}$$

$$\text{neut}(t, C, s) := (t,\ \varnothing,\ \text{final}) \quad \text{for all } (t, C, s) \in D,$$

and set anti := neut. Intuitively, a “adds a reviewer comment when the document is a draft,” neut “finalizes and hard-masks the document (no comments, final),” and anti is the same hard mask (a relative inverse toward the neutral state).

We verify the extended neutrosophic duplet axioms for the element $a \in M$.

1) Commutation with the neutral: for any $d = (t, C, s) \in D$,

$$(a * \text{neut})(d) = a\big(t, \varnothing, \text{final}\big) = \big(t, \varnothing, \text{final}\big) = \text{neut}(d) = \text{neut}\big(a(t, C, s)\big) = (\text{neut} * a)(d).$$

Hence $a * \text{neut} = \text{neut} * a = \text{neut}$.

2) Relative inverse w.r.t. the neutral: since anti = neut, the computation above yields

$$a * \text{anti} \;=\; a * \text{neut} \;=\; \text{neut} \quad \text{and} \quad \text{anti} * a \;=\; \text{neut} * a \;=\; \text{neut}.$$

3) Nontrivial neutral and monoid identity: neut $\neq e$ because $\text{neut}(t, C, s) = (t, \varnothing, \text{final}) \neq (t, C, s)$ whenever $s = \text{draft}$ or $C \neq \varnothing$. The carrier $(M, *, e)$ is a monoid by construction (all endofunctions under composition).

Therefore $(M, *, e)$ together with the maps neut, anti exhibits an *extended neutrosophic duplet* for the real-life “document review $\to$ finalization” pipeline: applying the editing action a and then its relative inverse anti (or in the reverse order) deterministically yields the neutral “finalized” state,

$$a * \text{anti} = \text{anti} * a = \text{neut}, \qquad a * \text{neut} = \text{neut} * a = \text{neut}, \qquad \text{neut} \neq e.$$

Chapter 6

NeutroTech (Neutrosophic Technology)

Neutrosophic sets are also applied in fields such as computer science (Neutrosophic computer science [323, 324]), engineering (Neutrosophic engineering [325–327]), and technology (NeutroTech [328, 329]). This chapter examines several concepts related to NeutroTech (Neutrosophic Technology), exploring how neutrosophic principles can enhance computational models, decision systems, and technological architectures dealing with uncertainty, inconsistency, and indeterminacy.

6.1 Neutrosophic 3D-Image Processing

Image Processing is the discipline that acquires, transforms, analyzes, and interprets digital images for enhancement, segmentation, recognition, and measurement tasks [330–333]. Related approaches such as Fuzzy Image Processing [334–337] and Neutrosophic Image Processing (cf. [338–341]) are also well known. 3D image processing analyzes volumetric data or image stacks to reconstruct, enhance, segment, and visualize three-dimensional structures for medical applications [342–344]. Neutrosophic 3D-image processing represents each voxel by truth, indeterminacy, and falsity components, applying componentwise operators for robust denoising and segmentation [44].

Definition 6.1.1 (Neutrosophic 3D-Image Processing)**.** [44] Let $V \subset \mathbb{Z}^3$ be a finite voxel grid and let a *neutrosophic 3D image* be a mapping

$$\mathcal{N} : V \longrightarrow [0,1]^3, \qquad \mathcal{N}(v) = (T(v), I(v), F(v)),$$

where T (objectness/foreground), I (indeterminacy/noise), F (background) are independent components. A *neutrosophic 3D operator* of finite radius $r \in \mathbb{N}$ is any mapping

$$\Phi : \ [0,1]^{3V} \to [0,1]^{3V}, \qquad \Phi(\mathcal{N})(v) = \mathbf{A}\big(\{\mathcal{N}(u) : \|u - v\|_\infty \le r\}\big),$$

where $\mathbf{A}$ is built componentwise from continuous t-norm/t-conorm aggregations and pointwise transformations on $[0,1]$ (e.g. contrast, smoothing). A *neutrosophic 3D segmentation* is a decision rule $\mathrm{seg}(v) \in \{\mathrm{FG}, \mathrm{UNK}, \mathrm{BG}\}$ defined by thresholds on (T, I, F).

Example 6.1.2 (Brain MRI tumor segmentation in a 3D neutrosophic volume)**.** Brain MRI tumor segmentation automatically delineates tumor regions in brain scans to aid diagnosis, treatment planning, monitoring, and research effectively (cf. [345]).

Let $V \subset \mathbb{Z}^3$ be the voxel grid and $I_{\mathrm{raw}} : V \to \mathbb{R}_{\ge 0}$ the MR intensity. Define a neutrosophic 3D image $\mathcal{N}(v) = (T(v), I(v), F(v)) \in [0,1]^3$ by

$$\tilde{I}(v) := \mathrm{clip}_{[0,1]}\left(\frac{I_{\mathrm{raw}}(v) - \mu_{\mathrm{bg}}}{\mu_{\mathrm{fg}} - \mu_{\mathrm{bg}}}\right), \quad T(v) := \tilde{I}(v),$$

$$I(v) := \mathrm{clip}_{[0,1]}\left(\frac{\mathrm{std}\big(I_{\mathrm{raw}}(u) : u \in \mathcal{N}_r(v)\big)}{\kappa}\right), \quad F(v) := 1 - T(v),$$

where $\mu_{\mathrm{bg}}, \mu_{\mathrm{fg}}$ are background/foreground reference means, $\kappa > 0$ is a noise scale, and $\mathcal{N}_r(v) := \{u \in V : \|u - v\|_\infty \le r\}$.

A neutrosophic 3D operator Φ (radius r) smooths/denoises componentwise via a t-norm/t-conorm scheme:

$$\Phi(\mathcal{N})(v) = \big(T'(v), I'(v), F'(v)\big) \text{ with } \begin{cases} T'(v) = \max_{u \in \mathcal{N}_r(v)} \min\{T(u),\ 1 - I(u)\}, \\ I'(v) = \min_{u \in \mathcal{N}_r(v)} I(u), \\ F'(v) = \max_{u \in \mathcal{N}_r(v)} F(u). \end{cases}$$

A segmentation rule declares

$$\mathrm{seg}(v) = \begin{cases} \mathrm{FG} & \text{if } T'(v) \ge \tau_T \text{ and } I'(v) \le \tau_I, \\ \mathrm{UNK} & \text{if } I'(v) > \tau_I, \\ \mathrm{BG} & \text{if } F'(v) \ge \tau_F \text{ (otherwise BG by default)}, \end{cases} \qquad \text{with } (\tau_T, \tau_I, \tau_F) = (0.60, 0.40, 0.50).$$

Concrete numeric illustration. For a tumor-core voxel v_0 suppose

$$T(v_0) = 0.65, \quad I(v_0) = 0.30, \quad F(v_0) = 0.35,$$

and, over $\mathcal{N}_1(v_0)$, the aggregated values yield

$$T'(v_0) = 0.70, \quad I'(v_0) = 0.25, \quad F'(v_0) = 0.15.$$

Then $T'(v_0) \ge 0.60$ and $I'(v_0) \le 0.40$, so $\mathrm{seg}(v_0) = \mathrm{FG}$ (foreground tumor). Indeterminate boundary voxels with locally high variance (e.g., $I'(v) > 0.40$) are labeled UNK for targeted review or adaptive reprocessing.

6.2 Plithogenic Image Processing

Plithogenic image processing aggregates multi-attribute pixel memberships via contradiction-aware t-norm/conorm mixing to robustly filter, segment, enhance, and rank image regions [39].

Definition 6.2.1 (Plithogenic Image and Operator)**.** [39] Fix a pixel domain $D \subset \mathbb{Z}^2$. Let $\mathcal{A}$ be a finite set of attributes (e.g., intensity, color, texture), and for each $a \in \mathcal{A}$ let V_a be its value set and $pCF_a : V_a \times V_a \to [0, 1]$ a *contradiction degree* between attribute values. A *plithogenic image* is a tuple

$$\mathsf{PI} = \big(D,\ \{(V_a, pCF_a)\}_{a \in \mathcal{A}},\ \{\mu_a(\cdot \mid x)\}_{a \in \mathcal{A},\, x \in D}\big),$$

where, for each pixel $x \in D$ and attribute a, the mapping $\mu_a(\cdot \mid x) : V_a \to [0, 1]$ assigns (possibly normalized) membership degrees to attribute values observed at x.

Fix a continuous t-norm T and its dual t-conorm S. For $u, v \in [0, 1]$ and a contradiction level $p \in [0, 1]$, define the *plithogenic mixer*

$$u \mathbin{\widetilde{\wedge}_p} v \ :=\ (1 - p)\, T(u, v) \ +\ p\, S(u, v).$$

Given a pixel-wise *dominant attribute profile* $\mathbf{v}^\star(x) = \big(v_a^\star(x) \in V_a\big)_{a \in \mathcal{A}}$, the *plithogenic aggregation score* of pixel x is

$$\mathrm{Agg}(x) \ :=\ \widetilde{\wedge}_{a \in \mathcal{A}} \Big(\ \mu_a\big(v_a^\star(x) \bigm| x\big) \mathbin{\widetilde{\wedge}}_{pCF_a(\cdot, v_a^\star(x))} \sup_{v \in V_a} \mu_a(v \mid x)\ \Big),$$

where each mixer uses its corresponding contradiction value pCF_a. A *plithogenic image processing operator* is any mapping $\mathcal{O}$ of the form

$$\mathcal{O}(\mathsf{PI})(x) \ =\ G\big(\mathrm{Agg}(x)\big),$$

with $G : [0, 1] \to \mathbb{R}$ a measurable transfer (e.g., thresholding, ranking, filtering).

Example 6.2.2 (Plithogenic crack detection in grayscale imagery)**.** Let $D \subset \mathbb{Z}^2$ be the pixel grid and $I_{\mathrm{raw}} : D \to [0, 255]$ a grayscale image (cf. [346]). Normalize $I(x) := \frac{I_{\mathrm{raw}}(x)-I_{\min}}{I_{\max}-I_{\min}} \in [0, 1]$ and let $G(x) \in [0, 1]$ be the gradient magnitude (e.g., Sobel, normalized).

Attributes and value-sets.

$$\mathcal{A} = \{\text{intensity}, \text{texture}\}, \qquad V_{\mathrm{int}} = \{\text{dark}, \text{mid}, \text{bright}\}, \quad V_{\mathrm{tex}} = \{\text{smooth}, \text{edge}\}.$$

Memberships (per pixel $x \in D$).

$$\mu_{\mathrm{int}}(\text{dark} \mid x) = 1 - I(x), \qquad \mu_{\mathrm{int}}(\text{mid} \mid x) = \max\{0,\ 1 - 2|I(x) - \tfrac{1}{2}|\},$$
$$\mu_{\mathrm{int}}(\text{bright} \mid x) = I(x),$$
$$\mu_{\mathrm{tex}}(\text{edge} \mid x) = G(x), \qquad \mu_{\mathrm{tex}}(\text{smooth} \mid x) = 1 - G(x).$$

Contradiction degrees. For intensity categories

$$pCF_{\mathrm{int}}(\text{dark}, \text{bright}) = 1, \quad pCF_{\mathrm{int}}(\text{dark}, \text{mid}) = pCF_{\mathrm{int}}(\text{mid}, \text{bright}) = \tfrac{1}{2}, \quad pCF_{\mathrm{int}}(v, v) = 0,$$

and for texture

$$pCF_{\mathrm{tex}}(\text{smooth}, \text{edge}) = 1, \qquad pCF_{\mathrm{tex}}(v, v) = 0.$$

Plithogenic mixer. Fix $T(u, v) = \min\{u, v\}$ and $S(u, v) = \max\{u, v\}$. For $p \in [0, 1]$ set

$$u \mathbin{\widetilde{\wedge}_p} v \ := \ (1 - p)\, T(u, v) \ + \ p\, S(u, v).$$

Dominant profile for cracks.

$$\mathbf{v}^{\star}(x) = \big(\text{dark for intensity},\ \text{edge for texture}\big).$$

For each attribute $a \in \mathcal{A}$ define

$$u_a(x) := \mu_a\big(v_a^{\star} \mid x\big), \qquad v_a(x) := \sup_{v \in V_a} \mu_a(v \mid x), \qquad p_a(x) := pCF_a\big(v_a^{\star},\ \arg\max_{v \in V_a} \mu_a(v \mid x)\big).$$

The *per-attribute plithogenic score* is $s_a(x) := u_a(x) \mathbin{\widetilde{\wedge}_{p_a(x)}} v_a(x)$, and the *pixel aggregation* is conservatively

$$\mathrm{Agg}(x) := \min\big\{\, s_{\mathrm{int}}(x),\ s_{\mathrm{tex}}(x) \,\big\}.$$

A binary crack mask is obtained by $\chi_{\mathrm{crack}}(x) = \mathbf{1}\{\mathrm{Agg}(x) \geq \tau\}$ with, say, $\tau = 0.65$.

Concrete numeric illustration. Let x_0 be a pixel on a crack with $I(x_0) = 0.15$ and $G(x_0) = 0.80$. Then

$$\mu_{\mathrm{int}}(\text{dark} \mid x_0) = 0.85,\ \mu_{\mathrm{int}}(\text{mid} \mid x_0) = 1 - 2|0.15 - 0.5| = 0.30,\ \mu_{\mathrm{int}}(\text{bright} \mid x_0) = 0.15,$$

so $v_{\mathrm{int}}(x_0) = 0.85$ attained at "dark", hence $p_{\mathrm{int}}(x_0) = 0$ and

$$s_{\mathrm{int}}(x_0) = 0.85 \mathbin{\widetilde{\wedge}_0} 0.85 = \min\{0.85, 0.85\} = 0.85.$$

For texture, $\mu_{\mathrm{tex}}(\text{edge} \mid x_0) = 0.80$ and $\mu_{\mathrm{tex}}(\text{smooth} \mid x_0) = 0.20$, so $v_{\mathrm{tex}}(x_0) = 0.80$ at "edge", $p_{\mathrm{tex}}(x_0) = 0$, and

$$s_{\mathrm{tex}}(x_0) = 0.80 \mathbin{\widetilde{\wedge}_0} 0.80 = \min\{0.80, 0.80\} = 0.80.$$

Thus $\mathrm{Agg}(x_0) = \min\{0.85, 0.80\} = 0.80 \geq \tau$, hence $\chi_{\mathrm{crack}}(x_0) = 1$ (detected).

Consider a non-crack pixel y_0 with $I(y_0) = 0.75$ and $G(y_0) = 0.10$. Then $u_{\mathrm{int}}(y_0) = \mu_{\mathrm{int}}(\text{dark} \mid y_0) = 0.25$, while $v_{\mathrm{int}}(y_0) = \max\{0.25,\ 0.50,\ 0.75\} = 0.75$ attained at "bright", so $p_{\mathrm{int}}(y_0) = pCF_{\mathrm{int}}(\text{dark}, \text{bright}) = 1$ and

$$s_{\mathrm{int}}(y_0) = 0.25 \mathbin{\widetilde{\wedge}_1} 0.75 = \max\{0.25, 0.75\} = 0.75.$$

For texture, $u_{\mathrm{tex}}(y_0) = \mu_{\mathrm{tex}}(\text{edge} \mid y_0) = 0.10$, $v_{\mathrm{tex}}(y_0) = \max\{0.10, 0.90\} = 0.90$ at "smooth", so $p_{\mathrm{tex}}(y_0) = 1$ and

$$s_{\mathrm{tex}}(y_0) = 0.10 \mathbin{\widetilde{\wedge}_1} 0.90 = \max\{0.10, 0.90\} = 0.90.$$

Consequently $\mathrm{Agg}(y_0) = \min\{0.75, 0.90\} = 0.75$ would be high; to enforce conjunction across attributes we instead use the *target-consistency score*

$$\mathrm{Agg}'(x) := \min\{\, u_{\mathrm{int}}(x),\ u_{\mathrm{tex}}(x) \,\}.$$

For y_0, $\mathrm{Agg}'(y_0) = \min\{0.25, 0.10\} = 0.10 < \tau$, so $\chi_{\mathrm{crack}}(y_0) = 0$. In practice, Agg can guide ranking while Agg' enforces strict target matching for segmentation.

6.3 Neutrosophic training set

Neutrosophic training set labels each sample with truth, indeterminacy, and falsity degrees, weighting supervised learning losses under unreliable annotations sources [39]. Related concepts such as the Classical Training Set [347–349] and Fuzzy Training Set [350, 351] are also well known.

Definition 6.3.1 (Neutrosophic training set)**.** [39] Let $X \subseteq \mathbb{R}^d$ be a feature space and Y a label set. A neutrosophic training set is a finite family

$$\mathcal{D}_\mathrm{N} = \{(x_i, y_i, T_i, I_i, F_i)\}_{i=1}^n$$

with $x_i \in X$, $y_i \in Y$, and $(T_i, I_i, F_i) \in [0,1]^3$, where (T_i, I_i, F_i) expresses the neutrosophic reliability of the annotation (x_i, y_i): T_i is the degree to which x_i truly has label y_i, I_i is the annotation indeterminacy (uncertain/contradictory sources), and F_i is the degree to which the annotation may be false. Learning aims at finding a model M minimizing a neutrosophic loss, e.g.

$$\mathcal{L}(M) = \sum_{i=1}^n \Big(T_i\,\ell_i^{(+)} + I_i\,\ell_i^{(\mathrm{ind})} + F_i\,\ell_i^{(-)}\Big),$$

where $\ell_i^{(+)}, \ell_i^{(\mathrm{ind})}, \ell_i^{(-)}$ are task–dependent partial losses.

Example 6.3.2 (Binary classification with noisy annotations)**.** Let $X \subset \mathbb{R}^2$, $Y = \{0,1\}$, and suppose a classifier outputs $p_i := \Pr_\theta(y_i = 1 \mid x_i) \in [0,1]$. A neutrosophic training set $\mathcal{D}_\mathrm{N} = \{(x_i, y_i, T_i, I_i, F_i)\}_{i=1}^3$ is

i	x_i	y_i	(T_i, I_i, F_i)	p_i	comment
1	(0.2, 0.8)	1	(0.90, 0.05, 0.05)	0.85	highly reliable positive
2	(0.6, 0.4)	1	(0.50, 0.40, 0.10)	0.55	uncertain positive
3	(0.9, 0.1)	0	(0.80, 0.10, 0.10)	0.20	reliable negative

Define a neutrosophic loss (logarithms are natural)

$$\ell_i^{\mathrm{CE}} := \begin{cases} -\ln p_i, & y_i = 1, \\ -\ln(1-p_i), & y_i = 0, \end{cases}$$

$$\ell_i^{\mathrm{opp}} := \begin{cases} -\ln(1-p_i), & y_i = 1, \\ -\ln p_i, & y_i = 0, \end{cases}$$

$$H(p_i) := -\big(p_i \ln p_i + (1-p_i)\ln(1-p_i)\big),$$

and aggregate

$$\mathcal{L}(\theta) = \sum_{i=1}^3 \Big(T_i\,\ell_i^{\mathrm{CE}} + I_i\,H(p_i) + F_i\,\ell_i^{\mathrm{opp}}\Big).$$

Numerical evaluation:

$$\text{For } i = 1: \quad \ell_1^{\mathrm{CE}} = -\ln 0.85 = 0.1625, \quad \ell_1^{\mathrm{opp}} = -\ln 0.15 = 1.8971, \quad H(p_1) = 0.4227,$$
$$T_1\ell_1^{\mathrm{CE}} + I_1 H(p_1) + F_1\ell_1^{\mathrm{opp}} = 0.9(0.1625) + 0.05(0.4227) + 0.05(1.8971) \approx 0.2623.$$

$$\text{For } i = 2: \quad \ell_2^{\mathrm{CE}} = -\ln 0.55 = 0.5978, \quad \ell_2^{\mathrm{opp}} = -\ln 0.45 = 0.7985, \quad H(p_2) = 0.6881,$$
$$T_2\ell_2^{\mathrm{CE}} + I_2 H(p_2) + F_2\ell_2^{\mathrm{opp}} = 0.5(0.5978) + 0.4(0.6881) + 0.1(0.7985) \approx 0.6540.$$

$$\text{For } i = 3: \quad \ell_3^{\mathrm{CE}} = -\ln 0.8 = 0.2231, \quad \ell_3^{\mathrm{opp}} = -\ln 0.2 = 1.6094, \quad H(p_3) = 0.5004,$$
$$T_3\ell_3^{\mathrm{CE}} + I_3 H(p_3) + F_3\ell_3^{\mathrm{opp}} = 0.8(0.2231) + 0.1(0.5004) + 0.1(1.6094) \approx 0.3895.$$

Hence $\mathcal{L}(\theta) \approx 0.2623 + 0.6540 + 0.3895 = 1.3058$. This setup downweights dubious labels via I_i and allows "flip-aware" penalization via F_i when annotations may be wrong.

6.4 Neutrosophic Soft Code and Refined Variant

A neutrosophic soft code represents each object by parameterized triples of truth, indeterminacy, and falsity, enabling thresholded, distance-based decoding decisions [39]. A related concept known as the Fuzzy Soft Code is also well established [352–354].

Definition 6.4.1 (Neutrosophic Soft Code)**.** [39] Let Ω be a universe of objects and E a finite set of parameters (code positions). For each $e \in E$ let $(T_e, I_e, F_e) : \Omega \to [0,1]^3$ be neutrosophic components. The *neutrosophic soft code* of $\omega \in \Omega$ is the length-$|E|$ tuple

$$\mathsf{C}(\omega) := \big(\langle T_e(\omega), I_e(\omega), F_e(\omega)\rangle\big)_{e\in E} \in \big([0,1]^3\big)^E.$$

Given thresholds $\theta_e = (\tau_e, \iota_e, \varphi_e) \in [0,1]^3$, a basic decoder accepts ω at position e iff $T_e(\omega) \geq \tau_e$ and $F_e(\omega) \leq \varphi_e$; global acceptance can be defined via majority or weighted rules over E. A neutrosophic distance between two codewords is

$$d\big(\mathsf{C}(\omega), \mathsf{C}(\omega')\big) := \sum_{e\in E}\big(|T_e(\omega) - T_e(\omega')| + |I_e(\omega) - I_e(\omega')| + |F_e(\omega) - F_e(\omega')|\big).$$

Example 6.4.2 (Neutrosophic Soft Code with explicit decoding)**.** Let the universe be $\Omega = \{\omega_A, \omega_B\}$ and parameters $E = \{e_1, e_2, e_3\}$. For each ω and $e \in E$, its code symbol is a triplet $\langle T_e(\omega), I_e(\omega), F_e(\omega)\rangle \in [0,1]^3$.

Codewords (rows are parameters):

	$\langle T, I, F\rangle$ for ω_A	$\langle T, I, F\rangle$ for ω_B
e_1	$\langle 0.82, 0.10, 0.08\rangle$	$\langle 0.60, 0.25, 0.15\rangle$
e_2	$\langle 0.68, 0.20, 0.16\rangle$	$\langle 0.74, 0.12, 0.22\rangle$
e_3	$\langle 0.91, 0.05, 0.04\rangle$	$\langle 0.66, 0.10, 0.34\rangle$

Decision rule (per position e): accept if $T_e(\omega) \geq \tau_e$ and $F_e(\omega) \leq \varphi_e$ with thresholds

$$\tau_{e_1} = \tau_{e_2} = \tau_{e_3} = 0.70, \qquad \varphi_{e_1} = \varphi_{e_2} = \varphi_{e_3} = 0.30.$$

Global decision: accept ω if at least two of three positions accept.

Positionwise decisions:

$\omega_A : e_1 :\ 0.82 \geq 0.70,\ 0.08 \leq 0.30$ (pass); $\quad e_2 :\ 0.68 < 0.70$ (fail); $\quad e_3 :\ 0.91 \geq 0.70,\ 0.04 \leq 0.30$ (pass);
$\Rightarrow$ two passes $\Rightarrow$ **ACCEPT**.

$\omega_B : e_1 :\ 0.60 < 0.70$ (fail); $\quad e_2 :\ 0.74 \geq 0.70,\ 0.22 \leq 0.30$ (pass); $\quad e_3 :\ 0.66 < 0.70$ or $0.34 > 0.30$ (fail);
$\Rightarrow$ one pass $\Rightarrow$ **REJECT**.

Neutrosophic ℓ^1 distance between codewords:

$$\begin{aligned} d\big(\mathsf{C}(\omega_A), \mathsf{C}(\omega_B)\big) &= \sum_{j=1}^{3}\big(|\Delta T_{e_j}| + |\Delta I_{e_j}| + |\Delta F_{e_j}|\big) \\ &= (0.22{+}0.15{+}0.07) + (0.06{+}0.08{+}0.06) + (0.25{+}0.05{+}0.30) \\ &= 0.44 + 0.20 + 0.60 = \mathbf{1.24}. \end{aligned}$$

This distance can be used for nearest-neighbor decoding or clustering under neutrosophic uncertainty.

A neutrosophic refined soft code stores multi-component sub-truth, sub-indeterminacy, and sub-falsity per parameter, supporting blockwise aggregation and robust decoding strategies [39].

Definition 6.4.3 (Neutrosophic Refined Soft Code)**.** Fix integers $p, r, s \geq 1$ and $n := p + r + s$. For each $e \in E$ let $\mathbf{T}_e \in [0,1]^p$, $\mathbf{I}_e \in [0,1]^r$, $\mathbf{F}_e \in [0,1]^s$ be the n–refined blocks, and define

$$\mathsf{C}_n(\omega) \ := \ \big(\langle \mathbf{T}_e(\omega) \mid \mathbf{I}_e(\omega) \mid \mathbf{F}_e(\omega)\rangle\big)_{e\in E} \ \in \ \big([0,1]^n\big)^E.$$

Blockwise t–norm/t–conorm mixers yield refined encoders/decoders. A refined distance can be taken as the ℓ^1 sum over all blocks and positions.

Example 6.4.4 (Refined code with blockwise thresholds and distance)**.** Let parameters $E = \{\text{quality}, \text{safety}\}$ and choose refinement sizes $(p, r, s) = (2, 1, 1)$. For each ω and $e \in E$,

$$\langle \mathbf{T}_e(\omega) \mid \mathbf{I}_e(\omega) \mid \mathbf{F}_e(\omega)\rangle = \langle (T_{e,1}, T_{e,2}) \mid (I_e) \mid (F_e)\rangle \in [0,1]^4.$$

Consider two products ω_1, ω_2 with codewords

	$T_{e,1}$	$T_{e,2}$	I_e	F_e
quality(ω_1)	0.82	0.76	0.10	0.12
safety(ω_1)	0.74	0.71	0.12	0.18
quality(ω_2)	0.69	0.73	0.14	0.21
safety(ω_2)	0.67	0.70	0.20	0.25

Blockwise aggregation per position e:

$$T_e^\star := \min(T_{e,1}, T_{e,2}), \qquad I_e^\star := I_e, \qquad F_e^\star := F_e.$$

Thresholds: $T_e^\star \geq 0.70$ and $F_e^\star \leq 0.20$ for acceptance at e. Global decision: accept ω only if both positions accept.

Decisions:

$$\begin{aligned}
\omega_1 : \ & \text{quality: } T^\star = \min(0.82, 0.76) = 0.76 \ (\geq 0.70),\ F^\star = 0.12 \ (\leq 0.20) \Rightarrow \text{pass;}\\
& \text{safety: } T^\star = \min(0.74, 0.71) = 0.71 \ (\geq 0.70),\ F^\star = 0.18 \ (\leq 0.20) \Rightarrow \text{pass;}\\
& \Rightarrow \textbf{ACCEPT}.\\
\omega_2 : \ & \text{quality: } T^\star = \min(0.69, 0.73) = 0.69 \ (< 0.70) \text{ or } F^\star = 0.21 \ (> 0.20) \Rightarrow \text{fail;}\\
& \text{safety: } T^\star = \min(0.67, 0.70) = 0.67 \ (< 0.70) \Rightarrow \text{fail;}\\
& \Rightarrow \textbf{REJECT}.
\end{aligned}$$

Refined ℓ^1 distance (sum of absolute componentwise differences across positions):

$$\begin{aligned}
d_{\text{ref}}\big(\mathsf{C}_n(\omega_1), \mathsf{C}_n(\omega_2)\big) &= \underbrace{\big(|0.82{-}0.69| + |0.76{-}0.73| + |0.10{-}0.14| + |0.12{-}0.21|\big)}_{\text{quality} = 0.13+0.03+0.04+0.09=0.29}\\
&\quad + \underbrace{\big(|0.74{-}0.67| + |0.71{-}0.70| + |0.12{-}0.20| + |0.18{-}0.25|\big)}_{\text{safety} = 0.07+0.01+0.08+0.07=0.23}\\
&= 0.29 + 0.23 \ = \ \mathbf{0.52}.
\end{aligned}$$

This refined metric supports nearest-neighbor decoding while preserving sub-component evidence in (T, I, F) blocks.

6.5 Neutrosophic network denoising

Network denoising removes noise from graph or neural network data, preserving essential structure and signals while improving robustness and performance (cf. [355]). Neutrosophic network denoising separates inputs into truth, indeterminacy, falsity channels and reweights features, suppressing uncertain/adversarial noise while preserving reliable signal.

Definition 6.5.1 (Neutrosophic network denoising)**.** Let $\mathcal{N}$ be a (trained or untrained) neural network parameterized by θ, and let an observed input x admit a neutrosophic description

$$x \rightsquigarrow \big(x_T, x_I, x_F\big),$$

where x_T is the trusted component of x, x_I the indeterminate/noisy part, and x_F the adversarial/false part. A neutrosophic network denoising operator is a map

$$\mathcal{D}_{\mathrm{N}} : \big(\mathcal{N}, x_T, x_I, x_F\big) \longmapsto \big(\mathcal{N}', x'_T\big)$$

such that:

- $x'_T = g_T(x_T, x_I, x_F)$ suppresses or attenuates the x_I and x_F components using their neutrosophic degrees;
- $\mathcal{N}'$ optionally reweights internal features/activations by neutrosophic masks $m = (m_T, m_I, m_F) \in [0,1]^3$, giving higher weight to T–parts and lower to I, F–parts;
- for all inputs, the denoised output satisfies

  $$\|\mathcal{N}'(x'_T) - \mathcal{N}(x_T)\| \le \varepsilon$$

  for a prescribed tolerance $\varepsilon > 0$.

In words, neutrosophic network denoising is denoising carried out in three coordinated channels (truth, indeterminacy, falsity), instead of a single noise channel.

Example 6.5.2 (Image classifier with neutrosophic masks and explicit numeric attenuation)**.** Let an input feature vector decompose neutrosophically as

$$x \rightsquigarrow (x_T, x_I, x_F) \quad \text{with} \quad x_T = (1.00,\ 0.50,\ 0.00),\ x_I = (0.20,\ 0.10,\ 0.40),\ x_F = (0.10,\ 0.00,\ 0.30).$$

Choose attenuation coefficients for indeterminacy and falsity

$$\mu_I = 0.60, \qquad \mu_F = 0.90,$$

and define the denoised input by suppressing x_I, x_F:

$$x'_T \ :=\ x_T \ +\ (1-\mu_I)\,x_I \ +\ (1-\mu_F)\,x_F \ =\ x_T \ +\ 0.4\,x_I \ +\ 0.1\,x_F.$$

Compute each coordinate explicitly:

$$\begin{aligned}
0.4\,x_I &= (0.08,\ 0.04,\ 0.16),\\
0.1\,x_F &= (0.01,\ 0.00,\ 0.03),\\
x'_T &= (1.00{+}0.08{+}0.01,\ 0.50{+}0.04{+}0.00,\ 0.00{+}0.16{+}0.03)\\
&= (1.09,\ 0.54,\ 0.19).
\end{aligned}$$

Let the first layer be the identity map $h(x) = Wx$ with $W = I_3$ for clarity. Baseline clean activation:

$$h(x_T) = x_T = (1.00,\ 0.50,\ 0.00).$$

Denoised activation:

$$h(x'_T) = x'_T = (1.09,\ 0.54,\ 0.19).$$

The deviation from the clean activation is

$$\Delta h \ =\ h(x'_T) - h(x_T) \ =\ (0.09,\ 0.04,\ 0.19),$$

with Euclidean norm

$$\|\Delta h\|_2 \ =\ \sqrt{0.09^2 + 0.04^2 + 0.19^2} \ =\ \sqrt{0.0081 + 0.0016 + 0.0361} \ =\ \sqrt{0.0458} \ \approx\ 0.214.$$

Hence the denoising tolerance constraint

$$\|\,\mathcal{N}'(x'_T) - \mathcal{N}(x_T)\|_2 \ \le\ \varepsilon$$

holds for, e.g., $\varepsilon = 0.25$ in this toy layer.

A simple neutrosophic feature reweighting inside the network can be written as

$$h'(x) \ =\ m_T \odot h(x_T) \ +\ m_I \odot h(x_I) \ +\ m_F \odot h(x_F),$$

with masks $m_T = (1,1,1)$, $m_I = (0.4, 0.4, 0.4)$, $m_F = (0.1, 0.1, 0.1)$. This yields the same x'_T above when $h = I_3$, explicitly demonstrating triple–channel suppression of indeterminacy and falsity while preserving truth.

6.6 Neutrosophic Data

Needless to say, fuzzy sets and neutrosophic sets play a major role in the field of data science (e.g. [356–358]). Here, we focus on neutrosophic data. Neutrosophic data represent each datum by degrees of truth, indeterminacy, and falsity, explicitly encoding reliability, ambiguity, potential error, and noise levels [49].

Definition 6.6.1 (Neutrosophic Data)**.** [49] Let U be a nonempty universe and let d be the data (a measurement, record, feature vector, symbol, or image) attached to an item $x \in U$. A neutrosophic datum is a quadruple

$$\mathsf{ND}(x) = \big(d, T_x, I_x, F_x\big),$$

where

$$T_x, I_x, F_x \in [0, 1], \qquad 0 \le T_x + I_x + F_x \le 3,$$

and

- T_x is the degree to which d is accepted/true/reliable;
- I_x is the degree to which d is indeterminate/incomplete/ambiguous;
- F_x is the degree to which d is rejected/false/noisy.

A neutrosophic dataset is a finite family

$$\mathcal{D}_N = \big\{\mathsf{ND}(x_i) = (d_i, T_i, I_i, F_i) \;\big|\; i = 1, \dots, n\big\}.$$

Example 6.6.2 (Clinical lab record as neutrosophic datum with explicit decision rule)**.** Consider a patient item $x = \text{P123}$. The attached datum is the serum potassium

$$d = \text{“K}^+ = 5.4 \text{ mmol/L”}.$$

Based on instrumentation logs and sampling notes, assign neutrosophic degrees

$$(T_x, I_x, F_x) = (0.78,\ 0.12,\ 0.10),$$

meaning: reliability 78%, indeterminacy 12% (possible hemolysis, storage delay), and potential falsehood 10% (transcription/device risk).

Define a simple neutrosophic credibility score with indeterminacy penalty $\lambda = 0.5$:

$$\text{Score}(x) := T_x - F_x - \lambda I_x.$$

Compute explicitly:

$$\text{Score}(x) = 0.78 - 0.10 - 0.5 \times 0.12 = 0.78 - 0.10 - 0.06 = 0.62.$$

Use the triage rule “escalate if Score ≥ 0.50 and $I_x \le 0.20$.” Since $0.62 \ge 0.50$ and $0.12 \le 0.20$, the datum is flagged as a credible high-K^+ alert for clinician review.

A neutrosophic dataset then records entries $\mathsf{ND}(x_i) = (d_i, T_i, I_i, F_i)$ across patients, supporting downstream learning or decision algorithms that weight (T, I, F) explicitly.

6.7 Neutrosophic Wireless Application

A wireless application is software or service accessed over wireless networks, supporting communication, data transfer, sensing, and mobile computing tasks [359–361]. Neutrosophic wireless applications model network decisions using truth, indeterminacy, falsity degrees for links and services, enabling robust routing under uncertainty [39].

Definition 6.7.1 (Neutrosophic Wireless Application)**.** Let $X = \{x_1, \dots, x_m\}$ be candidate paths (routes/links) and $C = \{c_1, \dots, c_n\}$ be service parameters (e.g. bandwidth, delay, loss). A *neutrosophic wireless application model* consists of three functions

$$\mathrm{T},\ \mathrm{I},\ \mathrm{F}:\ X \times C \longrightarrow [0,1],$$

with the intended semantics, for all $x \in X$ and $c \in C$:

$$\mathrm{T}(x,c) = \text{degree that path } x \text{ meets the demand of parameter } c, \quad \mathrm{F}(x,c) = \text{degree that } x \text{ is denied by } c,$$

$$\mathrm{I}(x,c) = \text{degree that } x \text{ neither meets nor is denied by } c.$$

The triple $\big(\mathrm{T}(x,c), \mathrm{I}(x,c), \mathrm{F}(x,c)\big)$ obeys $0 \le \mathrm{T} + \mathrm{I} + \mathrm{F} \le 1, 2$, or 3 according to the modelled dependence among components.

Example 6.7.2 (Wireless route selection under uncertain QoS)**.** Wireless route selection chooses network paths while Quality of Service ensures bandwidth, delay, loss, and reliability meet application performance requirements (cf. [362]).

Let the candidate paths be $X = \{x_1, x_2\}$ and the service parameters be $C = \{\text{bandwidth, delay, loss}\}$ with weights

$$w_{\text{bw}} = 0.5, \qquad w_{\text{delay}} = 0.3, \qquad w_{\text{loss}} = 0.2, \qquad \sum w_c = 1.$$

For each (x, c) we are given neutrosophic degrees $(\mathrm{T}, \mathrm{I}, \mathrm{F}) \in [0,1]^3$:

	bandwidth	delay	loss
$\mathrm{T}(x_1, \cdot)$	0.80	0.60	0.70
$\mathrm{I}(x_1, \cdot)$	0.10	0.20	0.10
$\mathrm{F}(x_1, \cdot)$	0.10	0.20	0.20
$\mathrm{T}(x_2, \cdot)$	0.60	0.80	0.60
$\mathrm{I}(x_2, \cdot)$	0.10	0.10	0.20
$\mathrm{F}(x_2, \cdot)$	0.30	0.10	0.20

Define a per-parameter neutrosophic score with indeterminacy penalty $\lambda = 0.5$:

$$s(x,c) \ := \ \mathrm{T}(x,c) \ - \ \lambda\, \mathrm{I}(x,c) \ - \ \mathrm{F}(x,c).$$

Compute s explicitly.

For x_1:

$$\begin{aligned} s(x_1, \text{bw}) &= 0.80 - 0.5 \cdot 0.10 - 0.10 = 0.80 - 0.05 - 0.10 = \underline{0.65}, \\ s(x_1, \text{delay}) &= 0.60 - 0.5 \cdot 0.20 - 0.20 = 0.60 - 0.10 - 0.20 = \underline{0.30}, \\ s(x_1, \text{loss}) &= 0.70 - 0.5 \cdot 0.10 - 0.20 = 0.70 - 0.05 - 0.20 = \underline{0.45}. \end{aligned}$$

For x_2:

$$\begin{aligned} s(x_2, \text{bw}) &= 0.60 - 0.5 \cdot 0.10 - 0.30 = 0.60 - 0.05 - 0.30 = \underline{0.25}, \\ s(x_2, \text{delay}) &= 0.80 - 0.5 \cdot 0.10 - 0.10 = 0.80 - 0.05 - 0.10 = \underline{0.65}, \\ s(x_2, \text{loss}) &= 0.60 - 0.5 \cdot 0.20 - 0.20 = 0.60 - 0.10 - 0.20 = \underline{0.30}. \end{aligned}$$

Aggregate by the weighted sum

$$S(x) \ := \ \sum_{c \in C} w_c\, s(x,c).$$

Then

$$\begin{aligned} S(x_1) &= 0.5 \cdot 0.65 \ + \ 0.3 \cdot 0.30 \ + \ 0.2 \cdot 0.45 \\ &= 0.325 \ + \ 0.090 \ + \ 0.090 \ = \ 0.505, \end{aligned}$$

$$\begin{aligned} S(x_2) &= 0.5 \cdot 0.25 \ + \ 0.3 \cdot 0.65 \ + \ 0.2 \cdot 0.30 \\ &= 0.125 \ + \ 0.195 \ + \ 0.060 \ = \ 0.380. \end{aligned}$$

Decision: choose the path with maximal neutrosophic score. Since $S(x_1) = 0.505 > S(x_2) = 0.380$, the application selects x_1. A deployment may also require $s(x,c) \ge 0$ for all c as a feasibility check.

6.8 Neutrosophic Cloud Computing System

Cloud computing delivers on-demand computing resources and services over networks, enabling scalable storage, processing, and applications without local infrastructure management (cf. [363–365]). Neutrosophic cloud systems model resource–job assignments with truth, indeterminacy, falsity degrees, enabling robust multi-criteria QoS-aware scheduling under dynamic uncertainty constraints [39].

Definition 6.8.1 (Neutrosophic Cloud Computing System)**.** [39] Let R be a set of cloud resources (VMs, storage, links), J a set of jobs/tasks, and K a set of QoS attributes (e.g. availability, latency, cost). A *neutrosophic cloud computing system* is a structure

$$\mathsf{C} = \big(R, J, K;\ \mathrm{T}, \mathrm{I}, \mathrm{F}\big),$$

where

$$\mathrm{T},\ \mathrm{I},\ \mathrm{F}:\ R \times J \times K \ \longrightarrow\ [0,1]$$

assign, for each triple (r, j, k), the degrees of suitability/truth T, indeterminacy I, and unsuitability/falsity F of allocating job j to resource r with respect to attribute k. Given an aggregator A (e.g. a t-norm/conorm–based operator), one may define a neutrosophic score for an assignment $\pi : J \to R$ by

$$\mathrm{Score}(\pi) = \Big(A_{j\in J,k\in K}\mathrm{T}\big(\pi(j), j, k\big),\ A_{j,k}\mathrm{I}\big(\pi(j), j, k\big),\ A_{j,k}\mathrm{F}\big(\pi(j), j, k\big)\Big),$$

and seek π^* that maximizes truth while controlling indeterminacy and falsity under the same dependence bounds $0 \le \mathrm{T} + \mathrm{I} + \mathrm{F} \le 1, 2, 3$ as above.

Example 6.8.2 (VM allocation under uncertain QoS with explicit scoring)**.** VM allocation assigns virtual machines to physical hosts or cloud resources, balancing performance, capacity, costs, reliability, and energy usage efficiently (cf. [366]).

Let resources $R = \{r_1, r_2\}$, jobs $J = \{j_1, j_2\}$, and QoS attributes $K = \{\text{availability, latency, cost}\}$ with weights

$$w_{\text{avail}} = 0.4, \qquad w_{\text{lat}} = 0.3, \qquad w_{\text{cost}} = 0.3, \qquad \sum_{k\in K} w_k = 1.$$

For each triple (r, j, k) we are given neutrosophic degrees $(T, I, F) \in [0,1]^3$. Define the per-pair score with penalties $\lambda = 0.5,\ \mu = 1$ by

$$s(r, j) \ :=\ \sum_{k\in K} w_k \,\big(T(r, j, k) \ - \ \lambda\, I(r, j, k) \ - \ \mu\, F(r, j, k)\big).$$

The total score of an assignment $\pi : J \to R$ is $S(\pi) = \sum_{j\in J} s(\pi(j), j)$; we choose $\pi^\star = \arg\max_\pi S(\pi)$.

Given neutrosophic QoS estimates:

pair	k	T	I	F
(r_1, j_1)	availability	0.95	0.03	0.02
(r_1, j_1)	latency	0.75	0.10	0.15
(r_1, j_1)	cost	0.55	0.10	0.35
(r_2, j_1)	availability	0.85	0.07	0.08
(r_2, j_1)	latency	0.65	0.15	0.20
(r_2, j_1)	cost	0.70	0.10	0.20
(r_1, j_2)	availability	0.80	0.10	0.10
(r_1, j_2)	latency	0.60	0.20	0.20
(r_1, j_2)	cost	0.65	0.10	0.25
(r_2, j_2)	availability	0.88	0.06	0.06
(r_2, j_2)	latency	0.78	0.10	0.12
(r_2, j_2)	cost	0.55	0.10	0.20

Compute $s(r, j)$ explicitly using $T - \lambda I - \mu F$ (with $\lambda = 0.5,\ \mu = 1$):

$$\begin{aligned}
s(r_1, j_1) &= 0.4\,(0.95 - 0.5 \cdot 0.03 - 0.02) + 0.3\,(0.75 - 0.5 \cdot 0.10 - 0.15) + 0.3\,(0.55 - 0.5 \cdot 0.10 - 0.35) \\
&= 0.4\,(0.915) + 0.3\,(0.55) + 0.3\,(0.15) = 0.366 + 0.165 + 0.045 = 0.576.
\end{aligned}$$

$$\begin{aligned}
s(r_2, j_1) &= 0.4\,(0.85 - 0.5 \cdot 0.07 - 0.08) + 0.3\,(0.65 - 0.5 \cdot 0.15 - 0.20) + 0.3\,(0.70 - 0.5 \cdot 0.10 - 0.20) \\
&= 0.4\,(0.735) + 0.3\,(0.375) + 0.3\,(0.45) = 0.294 + 0.1125 + 0.135 = 0.5415.
\end{aligned}$$

$$\begin{aligned}
s(r_1, j_2) &= 0.4\,(0.80 - 0.5 \cdot 0.10 - 0.10) + 0.3\,(0.60 - 0.5 \cdot 0.20 - 0.20) + 0.3\,(0.65 - 0.5 \cdot 0.10 - 0.25) \\
&= 0.4\,(0.65) + 0.3\,(0.30) + 0.3\,(0.35) = 0.26 + 0.09 + 0.105 = 0.455.
\end{aligned}$$

$$\begin{aligned}
s(r_2, j_2) &= 0.4\,(0.88 - 0.5 \cdot 0.06 - 0.06) + 0.3\,(0.78 - 0.5 \cdot 0.10 - 0.12) + 0.3\,(0.55 - 0.5 \cdot 0.10 - 0.20) \\
&= 0.4\,(0.79) + 0.3\,(0.61) + 0.3\,(0.30) = 0.316 + 0.183 + 0.09 = 0.589.
\end{aligned}$$

Evaluate two assignments:

$$\begin{aligned}
\pi_1 : j_1 \mapsto r_1,\ j_2 \mapsto r_2 \quad &\Rightarrow \quad S(\pi_1) = s(r_1, j_1) + s(r_2, j_2) = 0.576 + 0.589 = 1.165. \\
\pi_2 : j_1 \mapsto r_2,\ j_2 \mapsto r_1 \quad &\Rightarrow \quad S(\pi_2) = s(r_2, j_1) + s(r_1, j_2) = 0.5415 + 0.455 = 0.9965.
\end{aligned}$$

Decision: since $S(\pi_1) > S(\pi_2)$, the neutrosophic scheduler selects π_1 (assign $j_1 \to r_1,\ j_2 \to r_2$). Feasibility constraints may further require $T - \lambda I - \mu F \geq 0$ for each (r, j, k).

Chapter 7

Neutrosophic Applied Science

The Neutrosophic Set has been applied in many other fields beyond those discussed so far. This chapter examines its applications across diverse domains.

7.1 Neutrosophic Risk and MultiRisk

Neutrosophic Risk models loss under uncertainty using truth, indeterminacy, and falsity degrees for likelihood and impact, enabling aggregation and policies [36]. Research on Neutrosophic Risk Assessment [367–369] and Neutrosophic Risk Management [370, 371] has also been actively conducted. Neutrosophic MultiRisk treats portfolios of correlated hazards via vectors of truth, indeterminacy, falsity triplets, supporting multivariate aggregation, diversification, and prioritization.

Definition 7.1.1 (Neutrosophic Risk)**.** [36] Let X be a set of scenarios (or states), and fix three qualitative risk classes

$$C \ := \ \{\text{Optimistic},\ \text{Neutral},\ \text{Pessimistic}\} \ = \ \{\mathrm{O}, \mathrm{N}, \mathrm{P}\}.$$

A *neutrosophic risk structure* on X is a family of functions

$$\big\{(T_c, I_c, F_c) : X \to [0,1]^3\big\}_{c \in C},$$

where for each scenario $x \in X$ and class $c \in C$, the triple

$$(T_c(x),\ I_c(x),\ F_c(x)) \in [0,1]^3$$

denotes, respectively, the degrees of *truth/support*, *indeterminacy*, and *falsity* of the statement "x exhibits risk class c". The *neutrosophic risk descriptor* of x is the 3×3 array

$$\mathrm{NR}(x) \ = \ \begin{pmatrix} T_{\mathrm{O}}(x) & T_{\mathrm{N}}(x) & T_{\mathrm{P}}(x) \\ I_{\mathrm{O}}(x) & I_{\mathrm{N}}(x) & I_{\mathrm{P}}(x) \\ F_{\mathrm{O}}(x) & F_{\mathrm{N}}(x) & F_{\mathrm{P}}(x) \end{pmatrix} \in [0,1]^{3\times 3}.$$

No normalizing constraint is required in general; if a partition-type interpretation is desired, one may additionally assume

$$\sum_{c \in C} T_c(x) \le 1, \qquad \sum_{c \in C} I_c(x) \le 1, \qquad \sum_{c \in C} F_c(x) \le 1 \quad (x \in X).$$

Optionally, if $E \subseteq X$ is a risk factor/event, a *neutrosophic probability* is a map $P^N(E) = (P_T(E), P_I(E), P_F(E)) \in [0,1]^3$ capturing the chances that E occurs, does not occur, or is indeterminate; one can then form expected (or weighted) neutrosophic risk indices by combining NR with P^N.

Example 7.1.2 (Neutrosophic Risk — Outdoor Marathon Under Heat Advisory)**.** Consider the single scenario x = "hold the marathon this Sunday". Its neutrosophic risk descriptor $\mathrm{NR}(x) \in [0,1]^{3\times 3}$ over classes $\{\mathrm{O}, \mathrm{N}, \mathrm{P}\} = \{\text{Optimistic}, \text{Neutral}, \text{Pessimistic}\}$ is

$$\mathrm{NR}(x) = \begin{pmatrix} T_{\mathrm{O}} & T_{\mathrm{N}} & T_{\mathrm{P}} \\ I_{\mathrm{O}} & I_{\mathrm{N}} & I_{\mathrm{P}} \\ F_{\mathrm{O}} & F_{\mathrm{N}} & F_{\mathrm{P}} \end{pmatrix} = \begin{pmatrix} 0.62 & 0.28 & 0.10 \\ 0.18 & 0.22 & 0.30 \\ 0.20 & 0.50 & 0.60 \end{pmatrix}.$$

Interpretation: the optimistic label is strongly supported ($T_{\mathrm{O}} = 0.62$), while the pessimistic label has high falsity ($F_{\mathrm{P}} = 0.60$) but also nontrivial indeterminacy ($I_{\mathrm{P}} = 0.30$) due to uncertain weather.

Let E = "heat index exceeds 35°C on race day" with neutrosophic probability

$$P^N(E) = (P_T(E), P_I(E), P_F(E)) = (0.40,\ 0.40,\ 0.20).$$

A componentwise expected *pessimistic* triplet is

$$\big(P_T(E)\,T_{\mathrm{P}},\ \ P_I(E)\,I_{\mathrm{P}},\ \ P_F(E)\,F_{\mathrm{P}}\big) = \big(0.40\times 0.10,\ 0.40\times 0.30,\ 0.20\times 0.60\big) = (0.04,\ 0.12,\ 0.12).$$

If a scalar summary is desired, one admissible policy uses weights $(w_T, w_I, w_F) = (0.60, 0.30, 0.10)$:

$$\mathrm{Index}_{\mathrm{P}} \;=\; w_T(0.04) + w_I(0.12) + w_F(0.12) = 0.60\cdot 0.04 + 0.30\cdot 0.12 + 0.10\cdot 0.12 = 0.072.$$

This yields a low aggregated pessimistic index, while retaining the full neutrosophic triplet $(0.04, 0.12, 0.12)$ for policy-sensitive interpretation.

Definition 7.1.3 (Neutrosophic MultiRisk)**.** Let $R = \{r_1, \dots, r_m\}$ be $m \ge 1$ distinct risk factors (e.g. economic, safety, compliance). A *neutrosophic multirisk structure* on X is a collection

$$\big\{\mathrm{NR}^{(j)} : X \to [0,1]^{3\times 3}\big\}_{j=1}^{m},$$

where each $\mathrm{NR}^{(j)}(x)$ is a neutrosophic risk descriptor (as above) for factor r_j. Equivalently, for every class $c \in \{\mathrm{O}, \mathrm{N}, \mathrm{P}\}$ and component $\sigma \in \{T, I, F\}$ we have functions $\sigma_c^{(j)} : X \to [0,1]$ and

$$\mathrm{NR}^{(j)}(x) = \begin{pmatrix} T_{\mathrm{O}}^{(j)}(x) & T_{\mathrm{N}}^{(j)}(x) & T_{\mathrm{P}}^{(j)}(x) \\ I_{\mathrm{O}}^{(j)}(x) & I_{\mathrm{N}}^{(j)}(x) & I_{\mathrm{P}}^{(j)}(x) \\ F_{\mathrm{O}}^{(j)}(x) & F_{\mathrm{N}}^{(j)}(x) & F_{\mathrm{P}}^{(j)}(x) \end{pmatrix}.$$

Given nonnegative weights $w_1, \dots, w_m$ with $\sum_{j=1}^m w_j = 1$, the *aggregate multirisk descriptor* is defined entrywise by

$$\overline{\sigma}_c(x) \;:=\; \sum_{j=1}^{m} w_j\, \sigma_c^{(j)}(x) \quad (\sigma \in \{T, I, F\},\ c \in \{\mathrm{O}, \mathrm{N}, \mathrm{P}\}),$$

and

$$\overline{\mathrm{NR}}(x) \;:=\; \begin{pmatrix} \overline{T}_{\mathrm{O}}(x) & \overline{T}_{\mathrm{N}}(x) & \overline{T}_{\mathrm{P}}(x) \\ \overline{I}_{\mathrm{O}}(x) & \overline{I}_{\mathrm{N}}(x) & \overline{I}_{\mathrm{P}}(x) \\ \overline{F}_{\mathrm{O}}(x) & \overline{F}_{\mathrm{N}}(x) & \overline{F}_{\mathrm{P}}(x) \end{pmatrix}.$$

If neutrosophic probabilities $P^{N,(j)}$ are specified for each factor/event, one may analogously form weighted expectations of $\mathrm{NR}^{(j)}$ under $P^{N,(j)}$ before aggregating.

Example 7.1.4 (Neutrosophic MultiRisk — Smartphone Launch (Supply, Market, Regulatory))**.** Let x = "launch a mid-tier smartphone in Q3" and consider $m = 3$ risk factors r_1 = Supply Chain, r_2 = Market Demand, r_3 = Regulation with weights $(w_1, w_2, w_3) = (0.40,\ 0.35,\ 0.25)$. Each factor provides a 3×3 descriptor $\mathrm{NR}^{(j)}(x)$:

$$\mathrm{NR}^{(1)}(x) = \begin{pmatrix} 0.50 & 0.30 & 0.20 \\ 0.20 & 0.25 & 0.35 \\ 0.30 & 0.45 & 0.45 \end{pmatrix},\quad \mathrm{NR}^{(2)}(x) = \begin{pmatrix} 0.55 & 0.25 & 0.20 \\ 0.25 & 0.30 & 0.30 \\ 0.20 & 0.45 & 0.50 \end{pmatrix},\quad \mathrm{NR}^{(3)}(x) = \begin{pmatrix} 0.40 & 0.35 & 0.25 \\ 0.30 & 0.30 & 0.35 \\ 0.30 & 0.35 & 0.40 \end{pmatrix}.$$

The entrywise weighted aggregate $\overline{\mathrm{NR}}(x) = \sum_{j=1}^3 w_j\, \mathrm{NR}^{(j)}(x)$ is

$$\overline{\mathrm{NR}}(x) = \begin{pmatrix} 0.4925 & 0.2950 & 0.2125 \\ 0.2425 & 0.2800 & 0.3325 \\ 0.2650 & 0.4250 & 0.4550 \end{pmatrix}.$$

For instance, the aggregated *pessimistic-truth* entry is

$$\overline{T}_{\mathrm{P}}(x) = 0.40\cdot 0.20 + 0.35\cdot 0.20 + 0.25\cdot 0.25 = 0.0800 + 0.0700 + 0.0625 = 0.2125.$$

This multirisk view makes explicit that pessimistic support is moderate (0.2125), indeterminacy on pessimism is sizable (0.3325), and falsity of pessimism is relatively high (0.4550), with contributions transparently decomposed by factor.

7.2 Neutrosophic Military

Research has also explored the application of Fuzzy Sets and Neutrosophic Sets within the field of Military Science (e.g. [372–376]). Neutrosophic Military models forces, missions, and theaters using truth, indeterminacy, falsity evaluations for readiness, capability, threats, enabling admissible, optimized planning [51].

Definition 7.2.1 (Neutrosophic Military System (NMS))**.** [51] Fix finite index sets of *units* U, *missions* M, *capability types* K (e.g. firepower, mobility, ISR, sustainment), and *resources* R. Let the *theater graph* be $G = (V, E)$ (locations and traversable links), and the *command–control network* be $H = (U, E_{C2})$.

A *neutrosophic military system* is a tuple

$$\mathbf{N} := \big(U, M, K, R,\ G, H,\ \rho,\ \kappa,\ \mu,\ \sigma,\ \theta,\ \mathsf{RoE},\ \mathbb{P}^N,\ \mathcal{A}\big),$$

whose components are:

- **Readiness** $\rho : U \to [0,1]^3$, $\rho(u) = (\rho_T, \rho_I, \rho_F)$ for each unit u;
- **Capabilities** $\kappa : U \times K \to [0,1]^3$, with $\kappa(u,k) = (\kappa_T, \kappa_I, \kappa_F)$;
- **Mobility** $\mu : U \times E \to [0,1]^3$ giving traversal degrees on each edge for each unit;
- **Sustainment** $\sigma : U \to [0,1]^3$ (logistics/maintenance support degrees);
- **Threat** $\theta : (V \cup E) \to [0,1]^3$ encoding friendly truth/indeterminacy/falsity w.r.t. safety (e.g. θ_T high = safe, θ_F high = dangerous);
- **Rules/constraints** $\mathsf{RoE} : U \times \mathcal{A}ct \to [0,1]^3$ mapping a unit and admissible action to (T, I, F) (legal/permitted, uncertain, prohibited);
- **Neutrosophic probability space** $(\Omega, \mathcal{F}, \mathbb{P}^N)$ with $\mathbb{P}^N = (P_T, P_I, P_F)$ three countably additive probability measures on $(\Omega, \mathcal{F})$ capturing, respectively, supportive, indeterminate, and adversarial randomness;
- **Aggregator family** $\mathcal{A} = (A_T, A_I, A_F)$ where each $A_\bullet : [0,1]^n \to [0,1]$ ($n \geq 1$ arbitrary) is continuous, monotone in each argument, and satisfies boundary conditions (e.g. A_T a t-norm, A_I, A_F t-conorms).

Example 7.2.2 (Instantiating an NMS)**.** Let $U = \{u_1, u_2\}$ with u_1 = ISR drone and u_2 = MEDEVAC vehicle. Let $M = \{m_1, m_2\}$ with m_1 = ISR over Town B, m_2 = Evacuate to Clinic C. Capability types $K = \{\text{ISR}, \text{Evac}\}$ and resources $R = \{\text{fuel}, \text{battery}\}$. The theater graph $G = (V, E)$ has $V = \{A, B, C\}$ (base, town, clinic) and $E = \{e_{AB}, e_{BC}\}$; the C2 network $H = (U, E_{C2})$ is fully connected.

Readiness and sustainment triplets:

$$\rho(u_1) = (0.85, 0.10, 0.05), \quad \rho(u_2) = (0.75, 0.15, 0.10), \qquad \sigma(u_1) = (0.80, 0.10, 0.10), \quad \sigma(u_2) = (0.70, 0.20, 0.10).$$

Capabilities:

$$\kappa(u_1, \text{ISR}) = (0.90, 0.07, 0.03), \qquad \kappa(u_2, \text{Evac}) = (0.88, 0.08, 0.04).$$

Mobility on edges:

$$\mu(u_1, e_{AB}) = (0.95, 0.03, 0.02), \quad \mu(u_2, e_{AB}) = (0.80, 0.10, 0.10), \quad \mu(u_2, e_{BC}) = (0.70, 0.20, 0.10).$$

Scenario set $\Omega = \{\omega_1, \omega_2\}$ with neutrosophic probabilities

$$P_T = (0.7, 0.3), \quad P_I = (0.5, 0.5), \quad P_F = (0.2, 0.8).$$

Threat/safety (higher T = safer) on path locations:

	ω_1 (clear)	ω_2 (jam/ambush)
$\theta(A)$	(0.95, 0.10, 0.05)	(0.90, 0.20, 0.20)
$\theta(B)$	(0.85, 0.10, 0.07)	(0.60, 0.30, 0.40)
$\theta(C)$	(0.80, 0.10, 0.10)	(0.55, 0.30, 0.45)
$\theta(e_{AB})$	(0.90, 0.10, 0.05)	(0.70, 0.30, 0.30)
$\theta(e_{BC})$	(0.80, 0.10, 0.10)	(0.50, 0.30, 0.50)

Rules of engagement (RoE) for used actions:

$$\mathsf{RoE}(u_1, \text{recon}) = (0.90, 0.05, 0.05), \qquad \mathsf{RoE}(u_2, \text{evac}) = (0.85, 0.10, 0.05).$$

Aggregators: $A_T = \min$ (supportive t-norm), $A_I = \max$, $A_F = \max$.

Definition 7.2.3 (Plans and assignment)**.** A *plan* specifies for each mission $m \in M$:

- a nonempty assigned unit set $U_m \subseteq U$;
- for each $u \in U_m$, a path $p_{u,m} \in \mathsf{Paths}(G)$ from origin to the mission area;
- an action profile $a_{u,m} \in \mathcal{A}ct$ (maneuver, engage, support, etc.).

The set of all plans is denoted Π.

Example 7.2.4 (Plan and assignment)**.** Define a plan π by

$$U_{m_1} = \{u_1\}, \quad p_{u_1,m_1} : A \to B \ (e_{AB}), \quad a_{u_1,m_1} = \text{recon};$$

$$U_{m_2} = \{u_2\}, \quad p_{u_2,m_2} : A \to B \to C \ (e_{AB}, e_{BC}), \quad a_{u_2,m_2} = \text{evac}.$$

Definition 7.2.5 (Mission viability components (scenario–wise))**.** Fix $\mathbf{N}$ and a plan $\pi \in \Pi$. For $m \in M$ and $\omega \in \Omega$ define three scenario–wise components in $[0, 1]$:

$$\begin{aligned}
\mathsf{Sup}_{m,\pi}(\omega) &:= A_T\Big(\{ \rho_T(u),\ \sigma_T(u),\ \kappa_T(u, k_m),\ \mu_T(u, e),\ \theta_T(x) \}\Big),\\
\mathsf{Ind}_{m,\pi}(\omega) &:= A_I\Big(\{ \rho_I(u),\ \sigma_I(u),\ \kappa_I(u, k_m),\ \mu_I(u, e),\ \theta_I(x) \}\Big),\\
\mathsf{Opp}_{m,\pi}(\omega) &:= A_F\Big(\{ \rho_F(u),\ \sigma_F(u),\ \kappa_F(u, k_m),\ \mu_F(u, e),\ \theta_F(x) \}\Big),
\end{aligned}$$

where the multisets range over $u \in U_m$, over edges e along $p_{u,m}$, and locations/links $x \in p_{u,m}$; $k_m \in K$ is the capability type emphasized by mission m (e.g. strike, ISR, airlift). Intuitively, Sup aggregates supportive factors, Ind aggregates indeterminacies, and Opp aggregates opposing risks.

Example 7.2.6 (Mission viability components under scenarios)**.** For m_1 (ISR via u_1) and scenario ω_1:

$$\mathsf{Sup}_{m_1,\pi}(\omega_1) = \min\{0.85, 0.80, 0.90, 0.95, 0.95, 0.90, 0.85\} = 0.80,$$

$$\mathsf{Ind}_{m_1,\pi}(\omega_1) = \max\{0.10, 0.10, 0.07, 0.03, 0.10, 0.10, 0.10\} = 0.10,$$

$$\mathsf{Opp}_{m_1,\pi}(\omega_1) = \max\{0.05, 0.10, 0.03, 0.02, 0.05, 0.05, 0.07\} = 0.10.$$

For scenario ω_2 along the same path:

$$\mathsf{Sup}_{m_1,\pi}(\omega_2) = \min\{0.85, 0.80, 0.90, 0.95, 0.90, 0.70, 0.60\} = 0.60,$$

$$\mathsf{Ind}_{m_1,\pi}(\omega_2) = \max\{0.10, 0.10, 0.07, 0.03, 0.20, 0.30, 0.30\} = 0.30,$$

$$\mathsf{Opp}_{m_1,\pi}(\omega_2) = \max\{0.05, 0.10, 0.03, 0.02, 0.20, 0.30, 0.40\} = 0.40.$$

For m_2 (Evac via u_2) and ω_1:

$$\mathsf{Sup}_{m_2,\pi}(\omega_1) = \min\{0.75, 0.70, 0.88, 0.80, 0.70, 0.95, 0.90, 0.85, 0.80, 0.80\} = 0.70,$$

$$\mathsf{Ind}_{m_2,\pi}(\omega_1) = \max\{0.15, 0.20, 0.08, 0.10, 0.20, 0.10, 0.10, 0.10, 0.10, 0.10\} = 0.20,$$

$$\mathsf{Opp}_{m_2,\pi}(\omega_1) = \max\{0.10, 0.10, 0.04, 0.10, 0.10, 0.05, 0.05, 0.07, 0.10, 0.10\} = 0.10.$$

For ω_2:

$$\mathsf{Sup}_{m_2,\pi}(\omega_2) = \min\{0.75, 0.70, 0.88, 0.80, 0.70, 0.90, 0.70, 0.60, 0.50, 0.55\} = 0.50,$$

$$\mathsf{Ind}_{m_2,\pi}(\omega_2) = \max\{0.15, 0.20, 0.08, 0.10, 0.20, 0.20, 0.30, 0.30, 0.30, 0.30\} = 0.30,$$

$$\mathsf{Opp}_{m_2,\pi}(\omega_2) = \max\{0.10, 0.10, 0.04, 0.10, 0.10, 0.20, 0.30, 0.40, 0.50, 0.45\} = 0.50.$$

Definition 7.2.7 (Neutrosophic mission success and legality)**.** The *neutrosophic success triplet* of mission m under plan π is

$$\mathrm{Succ}^N(m,\pi) \;:=\; \Big(\mathbb{E}_{P_T}[\mathsf{Sup}_{m,\pi}],\ \mathbb{E}_{P_I}[\mathsf{Ind}_{m,\pi}],\ \mathbb{E}_{P_F}[\mathsf{Opp}_{m,\pi}]\Big) \;\in [0,1]^3.$$

The *neutrosophic legality/feasibility* of π is

$$\mathrm{Feas}^N(\pi) \;:=\; \Big(\, A_T(\{\mathsf{RoE}_T(u,a_{u,m})\}),\ A_I(\{\mathsf{RoE}_I(u,a_{u,m})\}),\ A_F(\{\mathsf{RoE}_F(u,a_{u,m})\})\,\Big),$$

aggregated over all (u,m) used by the plan. Given thresholds $\tau_T \in (0,1]$, $\tau_F \in [0,1)$, a plan is *admissible* if $\mathrm{Feas}^N_T(\pi) \ge \tau_T$ and $\mathrm{Feas}^N_F(\pi) \le \tau_F$.

Example 7.2.8 (Neutrosophic mission success and plan legality)**.** Neutrosophic success triplets:

$$\mathrm{Succ}^N(m_1,\pi) = \Big(\underbrace{0.7\cdot 0.80 + 0.3\cdot 0.60}_{=0.74},\ \underbrace{0.5\cdot 0.10 + 0.5\cdot 0.30}_{=0.20},\ \underbrace{0.2\cdot 0.10 + 0.8\cdot 0.40}_{=0.34}\Big) = (0.74, 0.20, 0.34),$$

$$\mathrm{Succ}^N(m_2,\pi) = \Big(\underbrace{0.7\cdot 0.70 + 0.3\cdot 0.50}_{=0.64},\ \underbrace{0.5\cdot 0.20 + 0.5\cdot 0.30}_{=0.25},\ \underbrace{0.2\cdot 0.10 + 0.8\cdot 0.50}_{=0.42}\Big) = (0.64, 0.25, 0.42).$$

Legality/feasibility aggregation over used actions:

$$\mathrm{Feas}^N(\pi) = \big(\min\{0.90, 0.85\},\ \max\{0.05, 0.10\},\ \max\{0.05, 0.05\}\big) = (0.85, 0.10, 0.05).$$

With thresholds $\tau_T = 0.80$, $\tau_F = 0.20$, the plan is admissible since $0.85 \ge 0.80$ and $0.05 \le 0.20$.

Definition 7.2.9 (Neutrosophic military decision rule)**.** For weights $\alpha, \beta, \gamma \ge 0$ with $\alpha + \beta + \gamma = 1$ define the scalar score of m under π by

$$\mathsf{Score}_{\alpha,\beta,\gamma}(m,\pi) \;:=\; \alpha\,\mathrm{Succ}^N_T(m,\pi) \;-\; \beta\,\mathrm{Succ}^N_F(m,\pi) \;+\; \gamma\,(1 - \mathrm{Succ}^N_I(m,\pi)).$$

An *optimal admissible plan* solves

$$\pi^\star \;\in\; \arg \max_{\pi\in\Pi \text{ admissible}} \sum_{m\in M} \mathsf{Score}_{\alpha,\beta,\gamma}(m,\pi).$$

Example 7.2.10 (Decision rule scoring)**.** Choose $(\alpha,\beta,\gamma) = (0.50, 0.30, 0.20)$. Then

$$\mathsf{Score}(m_1,\pi) = 0.50\cdot 0.74 \;-\; 0.30\cdot 0.34 \;+\; 0.20\cdot(1 - 0.20) = 0.37 - 0.102 + 0.160 = 0.428,$$

$$\mathsf{Score}(m_2,\pi) = 0.50\cdot 0.64 \;-\; 0.30\cdot 0.42 \;+\; 0.20\cdot(1 - 0.25) = 0.320 - 0.126 + 0.150 = 0.344.$$

Total plan score: $0.428 + 0.344 = 0.772$. Mission m_1 is prioritized under these policy weights, while the plan remains RoE-admissible.

7.3 Refined Neutrosophic Sentiment Analysis

Sentiment analysis classifies text, speech, or images into polarity and emotion categories using features, lexicons, and machine-learning models at scale (cf. [377–382]). Neutrosophic sentiment analysis represents opinions by truth, indeterminacy, and falsity degrees, enabling robust classification under inconsistent, incomplete, or ambiguous annotations (cf. [383, 384]). Refined neutrosophic sentiment analysis splits truth, indeterminacy, falsity into subcomponents per aspect or source, supporting fine-grained aggregation and explainable decisions.

Definition 7.3.1 (Refined Neutrosophic Sentiment Analysis)**.** Fix a document (or utterance) set X and an aspect set A. Let $p, q, r \in \mathbb{N}$. A *refined neutrosophic sentiment* on $(x,a) \in X \times A$ is

$$\mathsf{S}(x,a) \;:=\; \big(\mathbf{T}(x,a),\ \mathbf{I}(x,a),\ \mathbf{F}(x,a)\big) \in [0,1]^p \times [0,1]^q \times [0,1]^r,$$

where the coordinates of $\mathbf{T}$ (e.g. joy, trust, approval), $\mathbf{I}$ (ambiguity, sarcasm, missing context), and $\mathbf{F}$ (anger, disgust, disapproval) are independent. Given nonnegative weights $w \in \mathbb{R}^{p+q+r}_{\ge 0}$, the *refined neutrosophic score* is

$$\mathrm{Score}(x,a) \;:=\; \langle w_T, \mathbf{T}\rangle \;-\; \langle w_F, \mathbf{F}\rangle \;-\; \langle w_I, \mathbf{I}\rangle,$$

used for classification or ranking; no normalization such as sum-to-one is required.

Example 7.3.2 (Single document, single aspect)**.** Consider the review text (document x) and the target aspect a = Battery Life:

"Great, my phone died in two hours — amazing battery."

We take refined neutrosophic sentiment with $p = 3$ truth subcomponents $\mathbf{T} = $ (joy, trust, approval), $q = 2$ indeterminacy subcomponents $\mathbf{I} = $ (ambiguity, sarcasm), and $r = 3$ falsity subcomponents $\mathbf{F} = $ (anger, disgust, disapproval).

Assign component intensities in [0, 1] (e.g., from a classifier or human coding):

$$\mathbf{T}(x, a) = \big(0.10,\ 0.05,\ 0.15\big), \quad \mathbf{I}(x, a) = \big(0.10,\ 0.80\big), \quad \mathbf{F}(x, a) = \big(0.60,\ 0.35,\ 0.70\big).$$

Choose nonnegative weights $w_T = (0.20, 0.30, 0.50)$, $w_I = (0.40, 0.60)$, $w_F = (0.30, 0.20, 0.50)$ to emphasize approval and sarcasm; the refined neutrosophic score is

$$\mathrm{Score}(x, a) = \langle w_T, \mathbf{T}\rangle - \langle w_F, \mathbf{F}\rangle - \langle w_I, \mathbf{I}\rangle.$$

We compute each term explicitly:

$$\langle w_T, \mathbf{T}\rangle = 0.10 \cdot 0.20 + 0.05 \cdot 0.30 + 0.15 \cdot 0.50 = 0.020 + 0.015 + 0.075 = 0.110,$$

$$\langle w_F, \mathbf{F}\rangle = 0.60 \cdot 0.30 + 0.35 \cdot 0.20 + 0.70 \cdot 0.50 = 0.180 + 0.070 + 0.350 = 0.600,$$

$$\langle w_I, \mathbf{I}\rangle = 0.10 \cdot 0.40 + 0.80 \cdot 0.60 = 0.040 + 0.480 = 0.520.$$

Hence

$$\mathrm{Score}(x, a) = 0.110 - 0.600 - 0.520 = -1.010.$$

Interpretation. The strong *sarcasm* ($I_2 = 0.80$) and high *disapproval/anger* ($F_3 = 0.70$, $F_1 = 0.60$) dominate, yielding a large negative score. Thus the sentiment toward the aspect *Battery Life* is classified as negative, while refined components make explicit *why* it is negative (sarcastic tone and disapproval rather than simple low joy/trust).

7.4 Plithogenic Sentiment Analysis

Plithogenic Sentiment Analysis models aspect sentiments via attribute values, aggregates memberships using contradiction aware fusion, generalizing neutrosophic and refined variants (cf. [385, 386]).

Definition 7.4.1 (Plithogenic Sentiment Space and Profile)**.** Let X be a set of documents/utterances and A a set of aspects. Fix a plithogenic attribute v with value domain (sentiment facets) P_v, a degree-of-appurtenance map

$$\mathsf{pdf} : (X \times A) \times P_v \longrightarrow [0, 1], \qquad ((x, a), \varphi) \longmapsto \mathsf{pdf}\big((x, a); \varphi\big),$$

and a symmetric degree-of-contradiction

$$pCF : P_v \times P_v \to [0, 1], \qquad pCF(\varphi, \varphi) = 0, \quad pCF(\varphi, \psi) = pCF(\psi, \varphi).$$

Assume a (disjoint) trichotomy of facets

$$P_v^+ \,\dot\cup\, P_v^0 \,\dot\cup\, P_v^- \;=\; P_v,$$

interpreted as positive, indeterminate/neutral, and negative refined sub-facets. Given *refinement weights*

$\alpha \in \Delta^{|P_v^+|-1}$, $\beta \in \Delta^{|P_v^0|-1}$, $\gamma \in \Delta^{|P_v^-|-1}$ (each a simplex vector, entries nonnegative and summing to 1),

define the aggregated plithogenic components for $(x, a) \in X \times A$ by

$$T_P(x, a) := \sum_{\varphi \in P_v^+} \alpha(\varphi)\, \mathsf{pdf}\big((x, a); \varphi\big),$$

$$I_P(x, a) := \sum_{\varphi \in P_v^0} \beta(\varphi)\, \mathsf{pdf}\big((x, a); \varphi\big),$$

$$F_P(x, a) := \sum_{\varphi \in P_v^-} \gamma(\varphi)\, \mathsf{pdf}\big((x, a); \varphi\big).$$

The *plithogenic sentiment profile* is

$$\mathsf{S}_P(x, a) \;:=\; \big(T_P(x, a),\ I_P(x, a),\ F_P(x, a)\big) \in [0, 1]^3.$$

Example 7.4.2 (Plithogenic Sentiment Space and Profile)**.** Let $X = \{x^*\}$ (a review), $A = \{\text{battery}\}$, and $P_v = P_v^+ \dot{\cup} P_v^0 \dot{\cup} P_v^- = \{\text{joy}, \text{trust}\} \dot{\cup} \{\text{uncert}\} \dot{\cup} \{\text{anger}, \text{disgust}\}$. Suppose

$$\mathsf{pdf}((x^*, \text{battery}); \text{joy}) = 0.70,\ \text{trust} = 0.50,\ \text{uncert} = 0.20,\ \text{anger} = 0.30,\ \text{disgust} = 0.10.$$

Take refinement weights $\alpha(\text{joy}, \text{trust}) = (0.6, 0.4)$, $\beta(\text{uncert}) = 1$, $\gamma(\text{anger}, \text{disgust}) = (0.7, 0.3)$. Then

$$\begin{aligned} T_P &= 0.6 \cdot 0.70 + 0.4 \cdot 0.50 = 0.42 + 0.20 = 0.62, \\ I_P &= 1 \cdot 0.20 = 0.20, \\ F_P &= 0.7 \cdot 0.30 + 0.3 \cdot 0.10 = 0.21 + 0.03 = 0.24, \end{aligned}$$

so $\mathsf{S}_P(x^*, \text{battery}) = (0.62, 0.20, 0.24)$.

Definition 7.4.3 (Plithogenic Fusion (evidence aggregation))**.** Let $\mathcal{E}$ be a finite multiset of evidence sources for (x, a) (e.g. token cues, annotators, models). For each $\varphi \in P_v$ let $m_e(\varphi) \in [0, 1]$ be the source-e membership to facet φ. Fix a t-norm $\top$ and a t-conorm $\perp$ on $[0, 1]$ (e.g. product and probabilistic-sum). For $\varphi, \psi \in P_v$ define the plithogenic pairwise fusion

$$m_e(\varphi) \mathbin{\widetilde{\otimes}_{pCF}} m_f(\psi) \ :=\ (1 - pCF(\varphi, \psi))\ \top\big(m_e(\varphi), m_f(\psi)\big)\ +\ pCF(\varphi, \psi)\ \perp\big(m_e(\varphi), m_f(\psi)\big).$$

A global fused membership per facet is obtained, for instance, by folding over pairs:

$$\widetilde{m}(\varphi)\ :=\ \operatorname*{Fold}_{e \in \mathcal{E}} \Big(m_e(\varphi)\Big) \quad \text{with} \quad m_{e \cup f}(\varphi) := m_e(\varphi) \mathbin{\widetilde{\otimes}_{pCF}} m_f(\varphi),$$

(or any associative extension thereof). Then replace $\mathsf{pdf}((x, a); \varphi)$ by $\widetilde{m}(\varphi)$ in the previous definition to obtain fused T_P, I_P, F_P.

Example 7.4.4 (Plithogenic Fusion (evidence aggregation))**.** Two sources $\mathcal{E} = \{e_1, e_2\}$ (lexicon, classifier). Let memberships

	joy	trust	uncert	anger	disgust
m_{e_1}	0.70	0.40	0.30	0.20	0.10
m_{e_2}	0.50	0.60	0.10	0.30	0.20

Use t-norm $\top(a, b) = ab$, t-conorm $\perp(a, b) = a + b - ab$, and $pCF(\varphi, \varphi) = 0$. For same-facet folding,

$$\widetilde{m}(\varphi) = m_{e_1}(\varphi) \mathbin{\widetilde{\otimes}_{pCF}} m_{e_2}(\varphi) = (1 - 0)\, ab + 0 \cdot (a + b - ab) = ab.$$

Hence

$$\widetilde{m}(\text{joy}) = 0.70 \cdot 0.50 = 0.35, \quad \widetilde{m}(\text{trust}) = 0.40 \cdot 0.60 = 0.24, \quad \widetilde{m}(\text{uncert}) = 0.30 \cdot 0.10 = 0.03,$$

$$\widetilde{m}(\text{anger}) = 0.20 \cdot 0.30 = 0.06, \quad \widetilde{m}(\text{disgust}) = 0.10 \cdot 0.20 = 0.02.$$

With the same (α, β, γ) as above,

$$\begin{aligned} T_P^{\text{fused}} &= 0.6 \cdot 0.35 + 0.4 \cdot 0.24 = 0.21 + 0.096 = 0.306, \\ I_P^{\text{fused}} &= 1 \cdot 0.03 = 0.03, \\ F_P^{\text{fused}} &= 0.7 \cdot 0.06 + 0.3 \cdot 0.02 = 0.042 + 0.006 = 0.048. \end{aligned}$$

Thus the fused profile is $\mathsf{S}_P^{\text{fused}} = (0.306, 0.03, 0.048)$.

Definition 7.4.5 (Plithogenic Sentiment Score)**.** For nonnegative policy weights $(\lambda_T, \lambda_I, \lambda_F)$ with $\lambda_T + \lambda_I + \lambda_F = 1$ define

$$\text{Score}_P(x, a)\ :=\ \lambda_T\, T_P(x, a)\ -\ \lambda_F\, F_P(x, a)\ -\ \lambda_I\, I_P(x, a).$$

This scalar is used for ranking/decision; the triplet $\big(T_P, I_P, F_P\big)$ is used for explanation.

Example 7.4.6 (Plithogenic Sentiment Score)**.** Choose policy weights $(\lambda_T, \lambda_I, \lambda_F) = (0.6, 0.1, 0.3)$. From the Example of Plithogenic Sentiment Space and Profile,

$$\text{Score}_P = 0.6 \cdot 0.62 - 0.3 \cdot 0.24 - 0.1 \cdot 0.20 = 0.372 - 0.072 - 0.020 = 0.280.$$

From the fused profile in Example 2,

$$\text{Score}_P^{\text{fused}} = 0.6 \cdot 0.306 - 0.3 \cdot 0.048 - 0.1 \cdot 0.03 = 0.1836 - 0.0144 - 0.003 = 0.1662.$$

Theorem 7.4.7 (Plithogenic Sentiment generalizes Neutrosophic Sentiment)**.** *Let* $P_v = \{\tau, \iota, \varphi\}$ *with* $P_v^+ = \{\tau\}$, $P_v^0 = \{\iota\}$, $P_v^- = \{\varphi\}$, *and let* pCF *be arbitrary with* $pCF(\varphi, \varphi) = 0$. *Define*

$$\mathsf{pdf}\big((x,a);\tau\big) = T(x,a), \quad \mathsf{pdf}\big((x,a);\iota\big) = I(x,a), \quad \mathsf{pdf}\big((x,a);\varphi\big) = F(x,a).$$

With the (unique) refinement weights $\alpha = \beta = \gamma = (1)$ *one has*

$$T_P(x,a) = T(x,a), \quad I_P(x,a) = I(x,a), \quad F_P(x,a) = F(x,a),$$

hence $\mathsf{S}_P(x,a) = \big(T(x,a), I(x,a), F(x,a)\big)$ *and, for any* $(\lambda_T, \lambda_I, \lambda_F)$,

$$\mathrm{Score}_P(x,a) = \lambda_T T(x,a) - \lambda_F F(x,a) - \lambda_I I(x,a),$$

which is exactly the neutrosophic sentiment score.

Proof. By construction,

$$T_P(x,a) = \sum_{\varphi \in P_v^+} \alpha(\varphi)\, \mathsf{pdf}((x,a);\varphi) = \mathsf{pdf}((x,a);\tau) = T(x,a).$$

The same algebra with the singleton sums gives $I_P(x,a) = I(x,a)$ and $F_P(x,a) = F(x,a)$. Substituting these equalities into Score_P yields the claimed identity. No use of fusion is needed (single facet per block), so the pCF values are irrelevant. □

Theorem 7.4.8 (Plithogenic Sentiment generalizes Refined Neutrosophic Sentiment)**.** *Let* $p, q, r \in \mathbb{N}$. *Take a disjoint union*

$$P_v^+ = \{\tau_1, \ldots, \tau_p\}, \quad P_v^0 = \{\iota_1, \ldots, \iota_q\}, \quad P_v^- = \{\varphi_1, \ldots, \varphi_r\}.$$

Given refined neutrosophic components

$$\mathbf{T}(x,a) = (T_1, \ldots, T_p), \quad \mathbf{I}(x,a) = (I_1, \ldots, I_q), \quad \mathbf{F}(x,a) = (F_1, \ldots, F_r),$$

define the plithogenic memberships

$$\mathsf{pdf}\big((x,a);\tau_j\big) = T_j, \quad \mathsf{pdf}\big((x,a);\iota_k\big) = I_k, \quad \mathsf{pdf}\big((x,a);\varphi_\ell\big) = F_\ell.$$

For any refinement weights $\alpha \in \Delta^{p-1}$, $\beta \in \Delta^{q-1}$, $\gamma \in \Delta^{r-1}$, *the plithogenic aggregates satisfy*

$$T_P(x,a) = \sum_{j=1}^{p} \alpha_j T_j, \quad I_P(x,a) = \sum_{k=1}^{q} \beta_k I_k, \quad F_P(x,a) = \sum_{\ell=1}^{r} \gamma_\ell F_\ell,$$

and consequently

$$\mathrm{Score}_P(x,a) = \lambda_T \sum_{j=1}^{p} \alpha_j T_j - \lambda_F \sum_{\ell=1}^{r} \gamma_\ell F_\ell - \lambda_I \sum_{k=1}^{q} \beta_k I_k,$$

which is the refined neutrosophic score with per-block convex aggregation.

Proof. Substituting the definitions of pdf into the blockwise sums gives

$$T_P(x,a) = \sum_{\varphi \in P_v^+} \alpha(\varphi)\, \mathsf{pdf}((x,a);\varphi) = \sum_{j=1}^{p} \alpha(\tau_j)\, T_j.$$

Identical computations yield the formulas for I_P and F_P. Finally, inserting these expressions into Score_P results in the displayed equality. This reproduces the refined neutrosophic scoring rule with blockwise convex weights (α, β, γ). □

7.5 Neutrosophic Divergence Measures

A divergence measure is a nonnegative functional quantifying dissimilarity between probability distributions, often asymmetric, generalizing distance-like concepts and information content [387–390]. A fuzzy divergence measure quantifies dissimilarity between fuzzy sets or fuzzy distributions, aggregating membership differences, uncertainties, and possibly nonmembership information [391–394]. Neutrosophic divergence measures quantify dissimilarity between neutrosophic distributions by aggregating divergences over truth, indeterminacy, falsity components with adjustable per-component weights.

Definition 7.5.1 (Neutrosophic Probability Model)**.** (cf. [395–397]) Let $(\Omega, \mathcal{F}, \mu)$ be a measurable space. A *neutrosophic distribution* is a triplet $P = (p_T, p_I, p_F)$ of nonnegative μ–integrable functions on Ω with

$$\int_\Omega p_T\, d\mu = \int_\Omega p_I\, d\mu = \int_\Omega p_F\, d\mu = 1.$$

We write $\mathcal{P}_N$ for the set of all such P. (In the discrete finite case, integrals are replaced by sums and the same normalization holds.)

Definition 7.5.2 (Neutrosophic f–Divergence)**.** Let $f : (0, \infty) \to \mathbb{R}$ be convex with $f(1) = 0$, and let $w_T, w_I, w_F > 0$, $w_T + w_I + w_F = 1$. For $P = (p_T, p_I, p_F)$, $Q = (q_T, q_I, q_F) \in \mathcal{P}_N$ with $q_\bullet > 0$ a.e., the *neutrosophic f–divergence* is

$$D_f^{(N)}(P\|Q) \ := \sum_{c\in\{T,I,F\}} w_c \int_\Omega q_c(\omega)\, f\!\left(\frac{p_c(\omega)}{q_c(\omega)}\right) d\mu(\omega) \ \in [0, \infty].$$

It satisfies nonnegativity $D_f^{(N)}(P\|Q) \geq 0$ and $D_f^{(N)}(P\|Q) = 0$ iff $p_c = q_c$ μ–a.e. for $c \in \{T, I, F\}$.

Example 7.5.3 (Weighted neutrosophic KL divergence on a two-point space)**.** Let the sample space be $\Omega = \{\omega_1, \omega_2\}$ (counting measure). Consider two neutrosophic distributions $P = (p_T, p_I, p_F)$ and $Q = (q_T, q_I, q_F)$ with

$$\begin{aligned} p_T &= (0.5,\ 0.5), & q_T &= (0.6,\ 0.4), \\ p_I &= (0.2,\ 0.8), & q_I &= (0.5,\ 0.5), \\ p_F &= (0.1,\ 0.9), & q_F &= (0.2,\ 0.8), \end{aligned} \qquad \sum_j p_c(\omega_j) = \sum_j q_c(\omega_j) = 1\ (c \in \{T, I, F\}).$$

Take $f(u) = u \ln u$, so $D_f(p\|q) = \sum_j p_j \ln \frac{p_j}{q_j}$ (natural log), and weights $(w_T, w_I, w_F) = (0.6, 0.2, 0.2)$.

Component KLs:

$$\begin{aligned} D_T &= \tfrac{1}{2} \ln \frac{0.5}{0.6} + \tfrac{1}{2} \ln \frac{0.5}{0.4} \ = \ 0.5\big(\ln 0.833333 + \ln 1.25\big) \ \approx \ 0.0204115, \\ D_I &= 0.2 \ln \frac{0.2}{0.5} + 0.8 \ln \frac{0.8}{0.5} \ = \ 0.2 \ln 0.4 + 0.8 \ln 1.6 \ \approx \ 0.1927448, \\ D_F &= 0.1 \ln \frac{0.1}{0.2} + 0.9 \ln \frac{0.9}{0.8} \ = \ 0.1 \ln 0.5 + 0.9 \ln 1.125 \ \approx \ 0.0362651. \end{aligned}$$

Weighted neutrosophic divergence:

$$D_f^{(N)}(P\|Q) = w_T D_T + w_I D_I + w_F D_F \approx 0.6\cdot 0.0204115 + 0.2\cdot 0.1927448 + 0.2\cdot 0.0362651 \approx 0.0580489 \text{ nats.}$$

Thus, the aggregated discrepancy across truth/indeterminacy/falsity channels is about 5.80×10^{-2} nats for this example.

7.6 Plithogenic derivatives

Plithogenic derivatives extend classical derivatives to plithogenic functions, weighting variations by attribute contradiction and aggregating via t-norm/t-conorm operators in analysis (cf. [250, 252]).

Definition 7.6.1 (Plithogenic setting and DCF–weighted aggregator)**.** Fix a plithogenic context

$$PS = \big(P,\ v,\ Pv,\ pdf,\ pCF\big),$$

where v is an attribute with value domain Pv, $pdf : P \times Pv \to [0,1]^s$ is the degree-of-appurtenance (DAF), and $pCF : Pv \times Pv \to [0,1]$ is the degree-of-contradiction (DCF), satisfying $pCF(a,a) = 0$ and $pCF(a,b) = pCF(b,a)$. Let $T, S : [0,1]^s \times [0,1]^s \to [0,1]^s$ be a fixed t-norm/t-conorm pair acting componentwise. For $c \in [0,1]$ define the DCF–weighted aggregator

$$\mathcal{A}_c(u,v) := (1-c)\,T(u,v) + c\,S(u,v) \qquad (u,v \in [0,1]^s).$$

Endow Pv with a C^1 curve structure (e.g. a smooth manifold or an open set of $\mathbb{R}^d$) and fix a metric d_{Pv} on Pv. Equip $[0,1]^s$ with the norm $\|\xi\|_1 := \sum_{j=1}^{s} |\xi_j|$.

Definition 7.6.2 (Plithogenic increment)**.** Let $F : X \times Pv \to [0,1]^s$ be a plithogenic map, $F(x,a) := pdf(x;a)$, where $X \subseteq \mathbb{R}^n$ is open. For $(x,a) \in X \times Pv$, a state increment $h \in \mathbb{R}^n$, and a C^1 curve $\gamma : (-\epsilon,\epsilon) \to Pv$ with $\gamma(0) = a$, define the DCF–weighted plithogenic increment

$$\Delta^{(P)} F(x,a;\, h,\gamma,t) := \mathcal{A}_{c(t)}\big(F(x+h,\gamma(t)),\ F(x,a)\big) - F(x,a), \qquad c(t) := pCF\big(a,\gamma(t)\big).$$

Definition 7.6.3 (Plithogenic derivatives)**.** With notation as above, assume all limits exist componentwise in $[0,1]^s$.

1) Plithogenic state (spatial) derivative at fixed attribute a:

$$D_x^{(P)} F(x,a)\,[h] := \lim_{t\to 0} \frac{F(x+th,\ a) - F(x,a)}{t}.$$

This is the classical Fréchet derivative of $x \mapsto F(x,a)$.

2) Plithogenic attribute (value) directional derivative at fixed state x along $u \in T_a Pv$: let γ be any C^1 curve in Pv with $\gamma(0) = a$, $\gamma'(0) = u$. Set

$$D_a^{(P)} F(x,a)\,[u] := \lim_{t\to 0} \frac{\mathcal{A}_{c(t)}\big(F(x,\gamma(t)),\ F(x,a)\big) - F(x,a)}{d_{Pv}(\gamma(t),a)}, \quad c(t) := pCF\big(a,\gamma(t)\big),$$

and require the limit to be independent of the chosen γ with $\gamma(0) = a$, $\gamma'(0) = u$.

3) Plithogenic total derivative (joint variation in state and attribute): for $h \in \mathbb{R}^n$ and $u \in T_a Pv$, choose any C^1 curve γ with $\gamma(0) = a$, $\gamma'(0) = u$ and define

$$D^{(P)} F(x,a)\,[h,u] := \lim_{t\to 0} \frac{\mathcal{A}_{c(t)}\big(F(x+th,\gamma(t)),\ F(x,a)\big) - F(x,a)}{\sqrt{\|th\|_2^2 + d_{Pv}(\gamma(t),a)^2}}, \quad c(t) := pCF\big(a,\gamma(t)\big).$$

When both partial derivatives exist and $\mathcal{A}_c$ is C^1 in (c,u,v), one has the first-order expansion

$$\mathcal{A}_{c(t)}\big(F(x+th,\gamma(t)),\ F(x,a)\big) = F(x,a) + \partial_1\mathcal{A}_0(F,F)\,D_xF(x,a)[th] + \partial_c\mathcal{A}_0(F,F)\,\dot c(0)\,t + o\big(\|th\| + t\big),$$

where $F = F(x,a)$, ∂_1 denotes the partial derivative of $\mathcal{A}_c(\cdot,\cdot)$ w.r.t. its first $[0,1]^s$-argument at $c = 0$, and $\dot c(0) = \mathrm{d}(pCF(a,\cdot))_a[u]$.

Example 7.6.4 (Concrete plithogenic derivatives with an explicit computation)**.** Let $X = \mathbb{R}$ and $Pv = (0,1)$ (attribute = sentiment polarity). Take $s = 1$, plithogenic map $F : X \times Pv \to [0,1]$ defined by

$$F(x,a) := a\,\sigma(x), \qquad \sigma(x) := \frac{1}{1+e^{-x}}.$$

Use the DCF $pCF(a,b) = (a-b)^2$ and the DCF–weighted convex aggregator

$$\mathcal{A}_c(u,v) = (1-c)\,u + c\,v \qquad (u,v \in [0,1]).$$

State (spatial) derivative at fixed a is classical:

$$D_x^{(P)} F(x,a)[h] = \frac{\partial}{\partial x}(a\,\sigma(x))\,h = a\,\sigma(x)\big(1-\sigma(x)\big)\,h.$$

Attribute (value) directional derivative at fixed x along $u \in \mathbb{R}$: choose $\gamma(t) = a + tu$ so $c(t) = pCF(a, \gamma(t)) = t^2u^2$. Then

$$\frac{\mathcal{A}_{c(t)}\big(F(x,\gamma(t)), F(x,a)\big) - F(x,a)}{t} = \frac{(1-t^2u^2)\,F(x,a+tu) + t^2u^2\,F(x,a) - F(x,a)}{t} \to \frac{\partial F}{\partial a}(x,a)\,u = \sigma(x)\,u,$$

hence $D_a^{(P)}F(x,a)[u] = \sigma(x)\,u$.

Numerical instance at $(x, a) = (0,\ 0.8)$:

$$\sigma(0) = \tfrac{1}{2}, \quad \sigma'(0) = \tfrac{1}{4}, \quad D_x^{(P)}F(0,0.8)[h] = 0.8 \cdot \tfrac{1}{4}\,h = 0.2\,h, \quad D_a^{(P)}F(0,0.8)[u] = \tfrac{1}{2}\,u.$$

7.7 Neutrosophic Quaternions

Neutrosophic quaternions extend classical quaternions by allowing each real coefficient to be a neutrosophic number capturing truth, indeterminacy, falsity intervals [398, 399].

Definition 7.7.1 (Neutrosophic quaternions)**.** Let $I_0, I_1, I_2, I_3 \subseteq \mathbb{R}$ be nonempty indeterminacy carriers. Define the set of *neutrosophic quaternions* by

$$\mathbb{H}_N \ := \ \big\{ q \ = \ (a_0 + b_0I_0) \ + \ (a_1 + b_1I_1)\,\mathbf{i} \ + \ (a_2 + b_2I_2)\,\mathbf{j} \ + \ (a_3 + b_3I_3)\,\mathbf{k} \ \big|\ a_k, b_k \in \mathbb{R} \big\}.$$

Each $q \in \mathbb{H}_N$ represents the *set* of classical quaternions obtained by choosing $u_k \in I_k$ and substituting $a_k + b_ku_k$ for the k-th coefficient. Addition and multiplication in $\mathbb{H}_N$ are *pointwise set-lifts* of the classical quaternion operations:

$$(q \oplus r) \ := \ \{\, q(u) + r(v) : u \in I_\bullet,\ v \in J_\bullet \,\}, \qquad (q \otimes r) \ := \ \{\, q(u)\,r(v) : u \in I_\bullet,\ v \in J_\bullet \,\}.$$

The *neutrosophic conjugate* and *neutrosophic norm set* are

$$\overline{q} \ = \ (a_0 + b_0I_0) - (a_1 + b_1I_1)\,\mathbf{i} - (a_2 + b_2I_2)\,\mathbf{j} - (a_3 + b_3I_3)\,\mathbf{k}, \qquad \|q\|^2 \ := \ \{\, q(u)\,\overline{q(u)} : u \in I_\bullet \,\} \subseteq \mathbb{R}_{\geq 0}.$$

Example 7.7.2 (A neutrosophic quaternion with numeric bounds)**.** Let $I_0 = I_2 = [-0.01,\ 0.01]$ and $I_1 = I_3 = \{0\}$. Define

$$q \ = \ (1 + 1 \cdot I_0) \ + \ (0 + 0 \cdot I_1)\,\mathbf{i} \ + \ (2 + 1 \cdot I_2)\,\mathbf{j} \ + \ (0 + 0 \cdot I_3)\,\mathbf{k} \ \in \ \mathbb{H}_N.$$

Thus, for $u_0, u_2 \in [-0.01, 0.01]$,

$$q(u_0, u_2) \ = \ (1 + u_0) \ + \ 0 \cdot \mathbf{i} \ + \ (2 + u_2)\,\mathbf{j} \ + \ 0 \cdot \mathbf{k},$$

so the scalar part ranges in $[0.99, 1.01]$ and the $\mathbf{j}$-part in $[1.99, 2.01]$.

Neutrosophic conjugate and norm set. For each (u_0, u_2),

$$\overline{q(u_0, u_2)} = (1 + u_0) - (2 + u_2)\,\mathbf{j}, \qquad \|q(u_0, u_2)\|^2 = (1 + u_0)^2 + (2 + u_2)^2.$$

Since $(1 + u_0)^2 \in [0.99^2, 1.01^2] = [0.9801,\ 1.0201]$ and $(2 + u_2)^2 \in [1.99^2, 2.01^2] = [3.9601,\ 4.0401]$,

$$\|q\|^2 \ \in \ [\,0.9801 + 3.9601,\ 1.0201 + 4.0401\,] \ = \ [\,4.9402,\ 5.0602\,].$$

A concrete instance. Take $(u_0, u_2) = (0.008, -0.006)$. Then

$$q(0.008, -0.006) = 1.008 + 1.994\,\mathbf{j}, \quad \overline{q} = 1.008 - 1.994\,\mathbf{j}, \quad \|q\|^2 = 1.008^2 + 1.994^2 = 1.016064 + 3.976036 = 4.992100.$$

Neutrosophic addition with another uncertain quaternion. Let $J_0 = [-0.02, 0.03]$, $J_2 = [-0.02, 0.03]$ and

$$r = (0.5 + 0.5J_0) \ + \ 1 \cdot \mathbf{i} \ + \ (0 + 0.2J_2)\,\mathbf{j} \ + \ 0 \cdot \mathbf{k}.$$

Componentwise interval sums give

$$\Re(q \oplus r) \in [0.99, 1.01] + [0.5 - 0.01,\ 0.5 + 0.015] = [1.48,\ 1.525],$$

$$\mathbf{i}\text{-part} \in [0,0] + [1,1] = [1,1], \qquad \mathbf{j}\text{-part} \in [1.99, 2.01] + [-0.004, 0.006] = [1.986,\ 2.016],$$
$$\mathbf{k}\text{-part} \in [0,0] + [0,0] = [0,0].$$

Neutrosophic multiplication at center values (for reference). At the midpoints $u_0 = u_2 = 0$ and $J_0 = J_2 = 0$,

$$q_0 = 1 + 2\,\mathbf{j}, \qquad r_0 = 0.5 + 1\,\mathbf{i},$$

and using $\mathbf{i}^2 = \mathbf{j}^2 = \mathbf{k}^2 = \mathbf{ijk} = -1$ with $\mathbf{ji} = -\mathbf{k}$,

$$q_0\, r_0 = (1)(0.5) + (1)(1)\mathbf{i} + (2\mathbf{j})(0.5) + (2\mathbf{j})(1\mathbf{i}) = 0.5 + \mathbf{i} + 1\,\mathbf{j} - 2\,\mathbf{k}.$$

Small deviations of (u_0, u_2) and (J_0, J_2) perturb these coefficients by $O(|u_0| + |u_2| + |J_0| + |J_2|)$, yielding tight interval enclosures if needed via standard interval arithmetic.

7.8 Neutrosophic Quantum Computer

A quantum computer performs computation using qubits, exploiting superposition and entanglement to sometimes solve problems faster than comparable classical machines [400–402]. A neutrosophic quantum computer represents quantum states with truth, indeterminacy, and falsity components, modeling inconsistent or incomplete information and operations (cf. [403, 404]).

Fix a finite-dimensional Hilbert space $\mathcal{H}$. A classical quantum state is a density operator $\rho \in \mathcal{D}(\mathcal{H}) = \{\rho \succeq 0 : \operatorname{Tr}\rho = 1\}$. A neutrosophic state separates *truth/known*, *indeterminate/unknown*, and *false/contradictory* evidences:

Definition 7.8.1 (Neutrosophic density triple)**.** A *neutrosophic state* on $\mathcal{H}$ is a triple of positive semidefinite operators

$$\boldsymbol{\rho} \;=\; (\rho_T, \rho_I, \rho_F), \qquad \rho_T, \rho_I, \rho_F \succeq 0,$$

with *component traces* $t := \operatorname{Tr}\rho_T$, $i := \operatorname{Tr}\rho_I$, $f := \operatorname{Tr}\rho_F$ lying in $[0,1]$ (no normalization constraint across t, i, f). A *neutrosophic quantum channel* is a triple $\boldsymbol{\Phi} = (\Phi_T, \Phi_I, \Phi_F)$ of completely positive trace-nonincreasing maps on $\mathcal{B}(\mathcal{H})$, acting componentwise: $\boldsymbol{\Phi}(\boldsymbol{\rho}) := (\Phi_T(\rho_T), \Phi_I(\rho_I), \Phi_F(\rho_F))$.

Definition 7.8.2 (Measurement)**.** Given a POVM $\{M_k\}_k$ on $\mathcal{H}$, the *neutrosophic outcome triple* is

$$\mathbf{p}(k) \;=\; \big(\operatorname{Tr}(M_k\rho_T),\ \operatorname{Tr}(M_k\rho_I),\ \operatorname{Tr}(M_k\rho_F)\big) \in [0,1]^3.$$

One may extract a crisp probability by a policy (e.g., redistribute i proportionally to t vs. f or retain it as undecided mass), but the primitive output is $\mathbf{p}(k)$.

Example 7.8.3 (Neutrosophic qubit)**.** For $\mathcal{H} = \mathbb{C}^2$, let $\rho_T = \left(\begin{smallmatrix}0.6 & 0\\ 0 & 0.2\end{smallmatrix}\right)$, $\rho_I = \left(\begin{smallmatrix}0.1 & 0\\ 0 & 0.0\end{smallmatrix}\right)$, $\rho_F = \left(\begin{smallmatrix}0.0 & 0\\ 0 & 0.1\end{smallmatrix}\right)$. Measuring in the computational basis gives outcome triples

$$\mathbf{p}(0) = (0.6, 0.1, 0.0), \qquad \mathbf{p}(1) = (0.2, 0.0, 0.1).$$

A gate U (unitary) acts as $(\rho_T, \rho_I, \rho_F) \mapsto (U\rho_T U^\dagger, U\rho_I U^\dagger, U\rho_F U^\dagger)$.

7.9 Neutrosophic Movie Ratings

Movie ratings summarize viewers' opinions about films using numerical or star scales, supporting recommendation systems, comparisons, and quality assessment processes (cf. [405–407]). Neutrosophic movie ratings assign each film three scores: like, indeterminacy, and dislike, capturing conflicting reviews, uncertainty, and ambiguity in opinions. Let $\mathcal{U}$ be a set of users and $\mathcal{M}$ a set of movies [51].

Definition 7.9.1 (Rating and aggregation)**.** A *neutrosophic rating* is a map $R : \mathcal{U} \times \mathcal{M} \to [0,1]^3$, $(u, m) \mapsto R(u,m) = (t_{u,m}, i_{u,m}, f_{u,m})$. Given weights $w_u \ge 0$ with $\sum_{u \in U_m} w_u = 1$ for the rater cohort $U_m := \{u \in \mathcal{U} : (u, m) \text{ rated}\}$, define the *consensus rating*

$$\overline{R}(m) \;=\; \Big(\sum_{u \in U_m} w_u\, t_{u,m},\ \sum_{u \in U_m} w_u\, i_{u,m},\ \sum_{u \in U_m} w_u\, f_{u,m}\Big).$$

Definition 7.9.2 (Ranking score (one admissible policy))**.** For nonnegative policy weights (α, β, γ) with $\alpha + \beta + \gamma = 1$, define the score

$$\mathrm{Score}_{\alpha,\beta,\gamma}(m) \;:=\; \alpha\,\bar{t}(m) \;-\; \beta\,\overline{f}(m) \;+\; \gamma\big(1 - \bar{i}(m)\big),$$

where $(\bar{t}, \bar{i}, \overline{f}) = \overline{R}(m)$. Movies are ranked by decreasing $\mathrm{Score}_{\alpha,\beta,\gamma}$; ties may be broken by smaller $\bar{i}(m)$ then larger $\bar{t}(m)$.

Example 7.9.3 (Rating)**.** Suppose three users rate a movie m with equal weights. User triples are $(0.8, 0.1, 0.1)$, $(0.7, 0.2, 0.1)$, $(0.4, 0.4, 0.2)$. Then

$$\overline{R}(m) = (0.633\overline{3},\ 0.233\overline{3},\ 0.133\overline{3}).$$

With $(\alpha, \beta, \gamma) = (\frac{1}{2}, \frac{1}{2}, 0)$ one gets Score $= 0.25$; with $(\alpha, \beta, \gamma) = (\frac{1}{2}, \frac{1}{3}, \frac{1}{6})$ one gets Score $\approx 0.316\overline{6}$.

7.10 Neutrosophic Product Ratings

Product ratings summarize customers' opinions on items [408–411]. A *neutrosophic product rating* uses three components: like (truth), indeterminacy, and dislike (falsity) for each product.

Definition 7.10.1 (Neutrosophic product rating)**.** Let $\mathcal{U}$ be a set of users and $\mathcal{P}$ a set of products. A neutrosophic product rating is a map

$$R : \mathcal{U} \times \mathcal{P} \to [0, 1]^3, \qquad (u, p) \longmapsto R(u, p) = (t_{u,p}, i_{u,p}, f_{u,p}),$$

where $t_{u,p}$ is the user's like degree for p, $i_{u,p}$ is the indeterminacy degree, and $f_{u,p}$ is the dislike degree. For a fixed product p, let $U_p := \{u \in \mathcal{U} \mid (u, p) \text{ rated}\}$ and choose weights $w_u^{(p)} \geq 0$ with $\sum_{u \in U_p} w_u^{(p)} = 1$. The aggregated rating of p is

$$\overline{R}(p) = \big(\bar{t}(p), \bar{i}(p), \overline{f}(p)\big) := \Big(\sum_{u \in U_p} w_u^{(p)} t_{u,p},\ \sum_{u \in U_p} w_u^{(p)} i_{u,p},\ \sum_{u \in U_p} w_u^{(p)} f_{u,p}\Big).$$

For policy weights $(\alpha, \beta, \gamma) \geq 0$ with $\alpha + \beta + \gamma = 1$, define the score

$$\mathrm{Score}(p) := \alpha\,\bar{t}(p) - \beta\,\overline{f}(p) + \gamma\big(1 - \bar{i}(p)\big).$$

Products are ranked by decreasing $\mathrm{Score}(p)$.

Example 7.10.2 (Computation)**.** Let $\mathcal{U} = \{u_1, u_2\}$ and $\mathcal{P} = \{p\}$. Suppose

$$R(u_1, p) = (0.9, 0.05, 0.05), \qquad R(u_2, p) = (0.7, 0.20, 0.10),$$

with equal weights $w_{u_1}^{(p)} = w_{u_2}^{(p)} = \frac{1}{2}$. Then

$$\bar{t}(p) = \tfrac{1}{2}(0.9 + 0.7) = 0.8, \quad \bar{i}(p) = \tfrac{1}{2}(0.05 + 0.20) = 0.125, \quad \overline{f}(p) = \tfrac{1}{2}(0.05 + 0.10) = 0.075.$$

Take $(\alpha, \beta, \gamma) = (0.5, 0.3, 0.2)$. Since $1 - \bar{i}(p) = 0.875$, we obtain

$$\mathrm{Score}(p) = 0.5 \cdot 0.8 - 0.3 \cdot 0.075 + 0.2 \cdot 0.875 = 0.4 - 0.0225 + 0.175 = 0.5525.$$

Thus the neutrosophic product rating combines likes, dislikes, and uncertainty into a single scalar score 0.5525 for p.

7.11 Neutrosophic Customer Satisfaction Index

Customer Satisfaction Index measures customer satisfaction with products or services using survey scores, aggregated into a standardized quantitative performance indicator (cf. [412–414]). Neutrosophic Customer Satisfaction Index models satisfaction, dissatisfaction, and indeterminacy degrees, aggregating them into an uncertainty-aware quantitative measure of customer experience.

Definition 7.11.1 (Neutrosophic customer satisfaction index)**.** Let $\mathcal{U}$ be a set of customers and $\mathcal{S}$ a set of services or products. A neutrosophic satisfaction rating is a map

$$R : \mathcal{U} \times \mathcal{S} \to [0,1]^3, \qquad (u,s) \longmapsto R(u,s) = (t_{u,s}, i_{u,s}, f_{u,s}),$$

where, for service s and customer u, $t_{u,s}$ is the satisfaction (truth) degree, $i_{u,s}$ is the indeterminacy degree, and $f_{u,s}$ is the dissatisfaction (falsity) degree.

For fixed $s \in \mathcal{S}$, let $U_s := \{u \in \mathcal{U} \mid (u,s) \text{ has a recorded rating}\}$, and choose weights $w_u^{(s)} \geq 0$ with $\sum_{u \in U_s} w_u^{(s)} = 1$. Define the aggregated neutrosophic triple

$$\big(\bar{t}(s), \bar{i}(s), \overline{f}(s)\big) := \Big(\sum_{u \in U_s} w_u^{(s)} t_{u,s}, \; \sum_{u \in U_s} w_u^{(s)} i_{u,s}, \; \sum_{u \in U_s} w_u^{(s)} f_{u,s} \Big).$$

For policy weights $(\alpha, \beta, \gamma) \geq 0$ with $\alpha + \beta + \gamma = 1$, the *neutrosophic customer satisfaction index (NCSI)* of s is

$$\mathrm{NCSI}(s) := \alpha\, \bar{t}(s) - \beta\, \overline{f}(s) + \gamma\big(1 - \bar{i}(s)\big).$$

Example 7.11.2 (Explicit computation)**.** Let one service s be rated by three customers $\mathcal{U} = \{u_1, u_2, u_3\}$ with equal weights $w_{u_1}^{(s)} = w_{u_2}^{(s)} = w_{u_3}^{(s)} = \frac{1}{3}$. Assume

$$R(u_1, s) = (0.9, 0.05, 0.05), \quad R(u_2, s) = (0.7, 0.20, 0.10), \quad R(u_3, s) = (0.5, 0.30, 0.25).$$

Aggregated components:

$$\begin{aligned} \bar{t}(s) &= \tfrac{1}{3}(0.9 + 0.7 + 0.5) = \tfrac{2.1}{3} = 0.7, \\ \bar{i}(s) &= \tfrac{1}{3}(0.05 + 0.20 + 0.30) = \tfrac{0.55}{3} \approx 0.1833\overline{3}, \\ \overline{f}(s) &= \tfrac{1}{3}(0.05 + 0.10 + 0.25) = \tfrac{0.40}{3} \approx 0.1333\overline{3}. \end{aligned}$$

Choose policy weights

$$(\alpha, \beta, \gamma) = (0.5, 0.3, 0.2), \qquad \alpha + \beta + \gamma = 1.$$

Then

$$1 - \bar{i}(s) \approx 1 - 0.1833\overline{3} = 0.8166\overline{6},$$

and the neutrosophic customer satisfaction index is

$$\begin{aligned} \mathrm{NCSI}(s) &= 0.5 \cdot 0.7 - 0.3 \cdot 0.1333\overline{3} + 0.2 \cdot 0.8166\overline{6} \\ &= 0.35 - 0.04 + 0.1633\overline{3} \\ &\approx 0.4733\overline{3}. \end{aligned}$$

Equivalently, in exact fractional form,

$$\mathrm{NCSI}(s) = \frac{71}{150} \approx 0.4733\overline{3}.$$

Thus s achieves a neutrosophic customer satisfaction index of about 0.47 on the $[0,1]$ scale, combining satisfaction, dissatisfaction, and uncertainty.

7.12 Neutrosophic Randomness

Randomness describes outcomes governed by chance, where individual results are unpredictable, though long-run frequencies may follow probabilistic patterns over time [415–417]. Neutrosophic randomness models chance using three probability components for supportive, indeterminate, and contradictory behaviors, enabling richer uncertainty-aware expectations and analysis [51, 418].

Let $(\Omega, \mathcal{F})$ be a measurable space. A *neutrosophic probability triplet* on $(\Omega, \mathcal{F})$ is

$$\mathbf{P} \;=\; (P_T, P_I, P_F),$$

where each $P_\bullet : \mathcal{F} \to [0,1]$ is a (countably additive) probability measure. No coupling constraint is imposed on P_T, P_I, P_F; thus for $A \in \mathcal{F}$ one may have $0 \leq P_T(A) + P_I(A) + P_F(A) \leq 3$. Intuitively, P_T measures supportive randomness (truth), P_I models unresolved or contextual uncertainty (indeterminacy), and P_F encodes contradictory/adversarial randomness (falsity).

Definition 7.12.1 (Neutrosophic random variable and its law)**.** [418–420] Let $(S, \mathcal{S})$ be a measurable state space. A *neutrosophic random variable* is a measurable map $X : (\Omega, \mathcal{F}) \to (S, \mathcal{S})$ together with a neutrosophic probability triplet $\mathbf{P}$. Its *neutrosophic distribution* (law) is the triplet of pushforward measures

$$\mathbf{P}_X \;=\; (P_T \circ X^{-1},\; P_I \circ X^{-1},\; P_F \circ X^{-1}) \;\in\; \mathcal{P}(S)^3.$$

Example 7.12.2 (Neutrosophic random variable — morning commute time)**.** Consider a commuter's morning travel time (in minutes) on a fixed route (cf. [421]). Let the state space be $S = \{20, 30, 40, 60\}$ and define a neutrosophic probability triplet $\mathbf{P} = (P_T, P_I, P_F)$ with discrete pmf's

x	20	30	40	60
$P_T(x)$	0.15	0.50	0.25	0.10
$P_I(x)$	0.10	0.30	0.35	0.25
$P_F(x)$	0.05	0.20	0.25	0.50

Here, P_T summarizes reliable historical weekdays (truth), P_I encodes unusual conditions with unresolved causes (indeterminacy), and P_F collects contradictory reports from a faulty traffic feed (falsity). The neutrosophic random variable is $X : \Omega \to S$ together with $\mathbf{P}$. Its component expectations are

$$\begin{aligned}
\mathbb{E}_{P_T}[X] &= 20(0.15) + 30(0.50) + 40(0.25) + 60(0.10) \\
&= 3 + 15 + 10 + 6 = 34 \text{ min}, \\
\mathbb{E}_{P_I}[X] &= 20(0.10) + 30(0.30) + 40(0.35) + 60(0.25) \\
&= 2 + 9 + 14 + 15 = 40 \text{ min}, \\
\mathbb{E}_{P_F}[X] &= 20(0.05) + 30(0.20) + 40(0.25) + 60(0.50) \\
&= 1 + 6 + 10 + 30 = 47 \text{ min}.
\end{aligned}$$

Component variances (with $E[X^2] = \sum_x x^2 p(x)$) are

$$\begin{aligned}
E_{P_T}[X^2] &= 400(0.15) + 900(0.50) + 1600(0.25) + 3600(0.10) = 1270, \\
\mathrm{Var}_{P_T}[X] &= 1270 - 34^2 = 1270 - 1156 = 114, \\
E_{P_I}[X^2] &= 400(0.10) + 900(0.30) + 1600(0.35) + 3600(0.25) = 1770, \\
\mathrm{Var}_{P_I}[X] &= 1770 - 40^2 = 1770 - 1600 = 170, \\
E_{P_F}[X^2] &= 400(0.05) + 900(0.20) + 1600(0.25) + 3600(0.50) = 2400, \\
\mathrm{Var}_{P_F}[X] &= 2400 - 47^2 = 2400 - 2209 = 191.
\end{aligned}$$

If a single forecast is needed, use policy weights $(w_T, w_I, w_F) = (0.7, 0.2, 0.1)$:

$$\mathbb{E}^{(w)}[X] = 0.7 \cdot 34 + 0.2 \cdot 40 + 0.1 \cdot 47 = 23.8 + 8 + 4.7 = 36.5 \text{ min}.$$

Thus the neutrosophic summary retains all three facets $(\mathbb{E}_{P_T}, \mathbb{E}_{P_I}, \mathbb{E}_{P_F}) = (34, 40, 47)$ while also yielding a policy–aggregated 36.5 minutes when required.

Definition 7.12.3 (Neutrosophic expectation and variance)**.** Let $X : \Omega \to \mathbb{R}$ be integrable w.r.t. each component. The *neutrosophic expectation* and *neutrosophic variance* are

$$\mathbb{E}^N[X] \;:=\; \big(\mathbb{E}_{P_T}[X],\; \mathbb{E}_{P_I}[X],\; \mathbb{E}_{P_F}[X]\big), \qquad \mathrm{Var}^N[X] \;:=\; \big(\mathrm{Var}_{P_T}[X],\; \mathrm{Var}_{P_I}[X],\; \mathrm{Var}_{P_F}[X]\big).$$

A scalar *policy aggregation* may be used when a single summary is needed: for fixed weights $w_T, w_I, w_F \ge 0$ with $w_T + w_I + w_F = 1$,

$$\mathbb{E}^{(w)}[X] \;:=\; w_T\, \mathbb{E}_{P_T}[X] \;+\; w_I\, \mathbb{E}_{P_I}[X] \;+\; w_F\, \mathbb{E}_{P_F}[X].$$

Definition 7.12.4 (Neutrosophic independence)**.** Neutrosophic r.v.'s $X, Y : (\Omega, \mathcal{F}) \to (S, \mathcal{S})$ are *neutrosophically independent* if they are independent under each component measure, i.e. for all $A, B \in \mathcal{S}$ and for each $c \in \{T, I, F\}$,

$$P_c(X \in A,\ Y \in B) \ = \ P_c(X \in A)\, P_c(Y \in B).$$

Definition 7.12.5 (Neutrosophic randomness index via entropy (discrete case))**.** Let X take values in a finite set $S = \{s_1, \ldots, s_m\}$ with component pmf's $p_c(j) := P_c(X = s_j)$ for $c \in \{T, I, F\}$. For a fixed log base $b > 1$, define

$$H_c(X) \ := \ -\sum_{j=1}^{m} p_c(j)\ \log_b p_c(j) \quad \text{(Shannon entropy of component } c\text{)},$$

and the *neutrosophic randomness index*

$$\mathcal{R}_w(X) \ := \ w_T\, H_T(X) \ + \ w_I\, H_I(X) \ + \ w_F\, H_F(X),$$

with $w_T, w_I, w_F \geq 0$ and $w_T + w_I + w_F = 1$.

Example 7.12.6 (Neutrosophic randomness — fraud triage entropy index)**.** [51,418] An e–commerce platform classifies a payment as $S = \{\text{Legit}, \text{Review}, \text{Fraud}\} = \{L, R, F\}$. Under three evidence channels we observe pmf's

$$p_T = (0.7,\ 0.2,\ 0.1), \qquad p_I = (0.4,\ 0.3,\ 0.3), \qquad p_F = (0.2,\ 0.2,\ 0.6),$$

representing supportive signals (truth), unresolved ambiguity (indeterminacy), and contradictory/adversarial cues (falsity). With base-2 Shannon entropy $H(\,p\,) = -\sum_j p_j \log_2 p_j$, the component entropies are

$$\begin{aligned}
H_T &= -\big[0.7\log_2 0.7 + 0.2\log_2 0.2 + 0.1\log_2 0.1\big]\\
&\approx -\big[0.7(-0.5146) + 0.2(-2.3219) + 0.1(-3.3219)\big]\\
&\approx 1.1568 \text{ bits},\\
H_I &= -\big[0.4\log_2 0.4 + 0.3\log_2 0.3 + 0.3\log_2 0.3\big]\\
&\approx -\big[0.4(-1.3219) + 2\times 0.3(-1.7370)\big]\\
&\approx 1.5710 \text{ bits},\\
H_F &= -\big[0.2\log_2 0.2 + 0.2\log_2 0.2 + 0.6\log_2 0.6\big]\\
&\approx -\big[2\times 0.2(-2.3219) + 0.6(-0.7370)\big]\\
&\approx 1.3710 \text{ bits}.
\end{aligned}$$

A neutrosophic randomness index with policy weights $(w_T, w_I, w_F) = (0.6, 0.3, 0.1)$ is

$$\mathcal{R}_w = 0.6\, H_T + 0.3\, H_I + 0.1\, H_F \approx 0.6(1.1568) + 0.3(1.5710) + 0.1(1.3710) \approx 1.3025 \text{ bits}.$$

Operationally, H_I being largest flags that most uncertainty comes from the ambiguous channel; mitigation should reduce p_I's spread (e.g., request stronger KYC signals) to lower $\mathcal{R}_w$ while preserving true positives.

7.13 Neutrosophic Chess Games

Neutrosophic chess games model positions and moves with truth, indeterminacy, falsity degrees, capturing uncertain, fuzzy, or partially known strategies better [48]. A related concept known as Neutrosophic Games has also been introduced. Neutrosophic games model strategic interactions where payoffs encode independent truth, indeterminacy, and falsity, capturing ambiguous, contradictory, evolving behaviors and uncertainties (cf. [422–424]).

Definition 7.13.1 (Neutrosophic chess position)**.** Let $B = \{1, \ldots, 8\} \times \{1, \ldots, 8\}$ be the chessboard squares, $\mathsf{Side} = \{\text{White}, \text{Black}\}$, and Piece the set of piece types $\{\text{K}, \text{Q}, \text{R}, \text{B}, \text{N}, \text{P}\}$. A *neutrosophic chess position* is a map

$$\Sigma : \ B \times \mathsf{Side} \times \mathsf{Piece} \longrightarrow [0,1]^3, \qquad (s, u, p) \longmapsto \big(T_\Sigma(s, u, p),\ I_\Sigma(s, u, p),\ F_\Sigma(s, u, p)\big),$$

where T (truth) is the degree that square s is occupied by side u with piece p, F the degree that it is not so, and I an indeterminacy degree (sensor noise, occlusion, incomplete information). Per triple we require $0 \leq T + I + F \leq 3$. For each (s, u) we may also define an "emptiness" triplet $E_\Sigma(s, u) = (T^E, I^E, F^E)$ for convenience (e.g. T^E is the degree that s is empty of side u's piece).

Definition 7.13.2 (Neutrosophic legality of a move)**.** Fix a t-norm $\otimes$ and a t-conorm $\oplus$ on $[0,1]$ (e.g. product and probabilistic-sum). Given a position Σ and a pseudo-legal move m (in algebraic chess sense), define its *neutrosophic legality* as a triplet

$$\mathrm{Leg}_{\Sigma}(m) \;:=\; \big(T_{\mathrm{leg}}(m),\; I_{\mathrm{leg}}(m),\; F_{\mathrm{leg}}(m)\big) \in [0,1]^3,$$

obtained by aggregating the rule preconditions that make m legal:

$$T_{\mathrm{leg}}(m) \;=\; \bigotimes_{j=1}^{J} T(\mathrm{premise}_j), \qquad F_{\mathrm{leg}}(m) \;=\; \bigoplus_{k=1}^{K} F(\mathrm{forbidden}_k),$$

and $I_{\mathrm{leg}}(m)$ chosen to satisfy a dependence model (e.g. $I_{\mathrm{leg}} := \max\{\text{premise } I\text{'s}, \text{forbidden } I\text{'s}\}$). Typical premises include: origin square contains the right piece, transit squares are empty, target square is empty (or occupied by opponent), castling path is clear and not in check, etc., each taken with the corresponding (T, I, F) from Σ.

Definition 7.13.3 (Neutrosophic chess game)**.** Let $M(\Sigma)$ be the finite set of pseudo-legal moves from position Σ. A *neutrosophic chess game* is a tuple

$$\mathcal{G} = \big(\mathcal{S},\, M,\, \mathsf{turn},\, \mathrm{Leg},\, \mathcal{T},\, U\big),$$

where $\mathcal{S}$ is the set of neutrosophic positions, $M : \mathcal{S} \to 2^{\mathsf{Moves}}$, $\mathsf{turn} : \mathcal{S} \to \{\text{White}, \text{Black}\}$ identifies side to move, Leg is as above, $\mathcal{T}$ is a (possibly stochastic) transition kernel $\mathcal{T}(\Sigma, m, \cdot)$ on $\mathcal{S}$ concentrating on successors of Σ after m, and $U : \mathcal{S} \to [-1,1]^3$ is a neutrosophic evaluation $U(\Sigma) = (T_{\mathrm{win}}, I_{\mathrm{unc}}, F_{\mathrm{loss}})$ from the side-to-move perspective. A neutrosophic move-selection rule chooses

$$m^* \;\in\; \arg\max_{m \in M(\Sigma)} \; \Phi\big(\mathrm{Leg}_{\Sigma}(m),\; \mathbb{E}_{\Sigma' \sim \mathcal{T}(\Sigma,m,\cdot)}[\, U(\Sigma')\,]\big)\,,$$

where Φ is an aggregator (e.g. weighted lexicographic over the $T/I/F$ components).

Example 7.13.4 (Kingside castling under uncertainty)**.** Consider White to move in a position Σ where:

$$\text{White K on } e1 : \; (1,0,0), \quad \text{White R on } h1 : \; (0.90, 0.10, 0),$$

$$f1 \text{ empty} : \; (0.85, 0.10, 0.05), \quad g1 \text{ empty} : \; (0.80, 0.15, 0.05).$$

Ignoring check-related conditions for brevity, castling $O{-}O$ requires "rook present on $h1$" and "$f1, g1$ empty". With $\otimes = \min$ and $\oplus = \max$,

$$T_{\mathrm{leg}}(O{-}O) = \min\{0.90, 0.85, 0.80\} = 0.80, \quad F_{\mathrm{leg}}(O{-}O) = \max\{0, 0.05, 0.05\} = 0.05,$$

and we may set $I_{\mathrm{leg}}(O{-}O) = \max\{0.10, 0.10, 0.15\} = 0.15$, hence $\mathrm{Leg}_{\Sigma}(O{-}O) = (0.80, 0.15, 0.05)$.

7.14 Neutrosophic Voting

Voting is a collective decision-making process where individuals express preferences over alternatives, typically selecting representatives, policies, or actions during elections [425–427]. Neutrosophic voting extends classical voting by assigning each option truth, indeterminacy, and falsity degrees, modeling contradictory, incomplete preferences and uncertainty.

Definition 7.14.1 (Ballots and neutrosophic tallies)**.** Let $C = \{c_1, \ldots, c_m\}$ be a finite candidate set and V the finite set of registered voters, $|V| = N$. Each voter $v \in V$ submits one of the following ballots:

- **For-**c: vote in favor of a single $c \in C$;
- **Against-**c: vote explicitly against a single $c \in C$;
- **White** (blank): no candidate chosen;
- **Black**: vote against all candidates simultaneously;
- **Abstention**: not coming to voting.

For each candidate $c \in C$, define the neutrosophic tallies

$$T(c) = \frac{1}{N}\sum_{v\in V} \mathbf{1}\{v \text{ casts For-}c\}, \quad F(c) = \frac{1}{N}\sum_{v\in V}\Big(\mathbf{1}\{v \text{ casts Against-}c\} + \mathbf{1}\{v \text{ casts Black}\}\Big),$$

$$I(c) = \frac{1}{N}\sum_{v\in V} \mathbf{1}\{v \text{ casts White}\},$$

so that, for every c, $0 \le T(c) + I(c) + F(c) \le 1$ and abstentions contribute 0 to all three. Collect the election outcome as the map

$$\mathsf{Vote}_N : C \longrightarrow [0,1]^3, \qquad c \longmapsto \big(T(c), I(c), F(c)\big).$$

Definition 7.14.2 (Winner selection (neutrosophic scoring))**.** Fix weights $w_T, w_I, w_F > 0$ with $w_T + w_I + w_F = 1$ and define the scalar neutrosophic score

$$\mathrm{Score}(c) := w_T\, T(c) - w_I\, I(c) - w_F\, F(c).$$

A set of winners is

$$\mathsf{Win} := \arg\max_{c\in C} \mathrm{Score}(c),$$

with ties resolved by a prescribed tie-break (e.g. higher $T(c)$, then lower $F(c)$). Vector-wise (multi-criteria) selections are also possible, e.g. lexicographic on $(T, -F, -I)$.

Example 7.14.3 (Two-candidate election with white/black votes)**.** Let $C = \{A, B\}$ and $N = 10$ voters. The ballots are:

For-A: 4, For-B: 3, Against-A: 1, Black: 1, White: 1, Abstention: 0.

Then

$$T(A) = \tfrac{4}{10} = 0.40, \quad F(A) = \tfrac{1+1}{10} = 0.20, \quad I(A) = \tfrac{1}{10} = 0.10,$$
$$T(B) = \tfrac{3}{10} = 0.30, \quad F(B) = \tfrac{0+1}{10} = 0.10, \quad I(B) = \tfrac{1}{10} = 0.10.$$

With weights $(w_T, w_I, w_F) = (0.6, 0.2, 0.2)$,

$$\mathrm{Score}(A) = 0.6\cdot0.40 - 0.2\cdot0.10 - 0.2\cdot0.20 = 0.18, \qquad \mathrm{Score}(B) = 0.6\cdot0.30 - 0.2\cdot0.10 - 0.2\cdot0.10 = 0.14,$$

so A wins under this neutrosophic scoring.

Remark 7.14.4. The model explicitly accommodates "white" (indeterminacy mass shared by all candidates), "black" (global opposition against all), and classical for/against votes, while keeping abstentions neutral. Normalization by N keeps the triplets comparable across elections.

7.15 Neutrosophic and Plithogenic Rhotrices (including Linguistic variants)

A neutrosophic rhotrix is a heart-based matrix whose entries are neutrosophic triplets (truth, indeterminacy, falsity), aggregated componentwise by admissible operators ensuring closure [50, 428, 429]. It is known that related concepts such as Fuzzy Rhotrix [430] have also been studied. A plithogenic rhotrix stores attribute-parameterized memberships in each entry and fuses them using contradiction-weighted t-norm/conorm mixtures during heart-based composition [50, 431]. A neutrosophic linguistic rhotrix assigns linguistic labels mapped to neutrosophic triplets per entry, combining channels componentwise and optionally de-quantizing results into labels [50, 432]. A plithogenic linguistic rhotrix binds linguistic labels to attribute values, storing plithogenic degrees and merging entries via contradiction-weighted fusion in heart-based multiplication [50, 431].

Definition 7.15.1 (Rhotrix (order 3) and heart-based product)**.** A (order-3) *rhotrix* is the rhomboid array

$$R = \begin{array}{ccc} & a & \\ b & h(R) & d \\ & e & \end{array} \qquad (a, b, d, e, h(R) \in \mathbb{R} \text{ or a specified domain}),$$

with *heart* $h(R)$ at the center. The *heart-based multiplication* $R \circ Q$ is defined by

$$R \circ Q = \begin{array}{ccc} & af + dg & \\ bf + eg & h(R)\,h(Q) & aj + dk \\ & bj + ek & \end{array} \quad \text{for} \quad Q = \begin{array}{ccc} & f & \\ g & h(Q) & j \\ & k & \end{array}.$$

Example 7.15.2 (Urban traffic merging (order-3 rhotrix with heart-based product))**.** Interpret a rhotrix as a 3-arm junction: a (north inflow), b (west inflow), d (east inflow), e (south inflow), and heart $h(\cdot)$ as the junction's signal capacity factor. Morning profile (R) and mid-day profile (Q) are

$$R = \begin{array}{ccc} & 2 & \\ 1 & 5 & 3 \\ & 4 & \end{array}, \qquad Q = \begin{array}{ccc} & 1 & \\ 2 & 2 & 3 \\ & 0 & \end{array}.$$

The heart-based product yields

$$R \circ Q = \begin{array}{ccc} & af + dg & \\ bf + eg & h(R)h(Q) & aj + dk \\ & bj + ek & \end{array} = \begin{array}{ccc} & 2 \cdot 1 + 3 \cdot 2 & \\ 1 \cdot 1 + 4 \cdot 2 & 5 \cdot 2 & 2 \cdot 3 + 3 \cdot 0 \\ & 1 \cdot 3 + 4 \cdot 0 & \end{array} = \begin{array}{ccc} & 8 & \\ 9 & 10 & 6 \\ & 3 & \end{array}.$$

Thus, combining morning inflows with mid-day signal settings predicts a merged profile of (8; 9, 10, 6; 3) (same units as inflow/capacity).

Definition 7.15.3 (Neutrosophic Rhotrix)**.** [50, 428, 429] Let $\mathcal{N} = [0,1]^3$ denote the set of neutrosophic triplets (T, I, F). A *neutrosophic rhotrix* is a rhotrix

$$R_N = \begin{array}{ccc} & \alpha & \\ \beta & \eta & \delta \\ & \varepsilon & \end{array} \in \mathcal{N}^5,$$

whose entries are neutrosophic values. Addition is componentwise: $(T, I, F) \oplus (T', I', F') = (T + T', I + I', F + F')$ (with a chosen clipping/normalization if needed). A neutrosophic heart-based product $R_N \circ Q_N$ is obtained by applying the heart-based pattern entrywise together with a fixed per-component binary operator (e.g. t-norm on T, and suitable operators on I, F), yielding again a neutrosophic rhotrix.

Example 7.15.4 (Diagnostic evidence fusion (neutrosophic rhotrix))**.** A clinician combines prior and symptom evidence (R_N) with lab panels (Q_N). Entries are neutrosophic triplets (T, I, F).

$$R_N = \begin{array}{ccc} & (0.70, 0.20, 0.10) & \\ (0.60, 0.25, 0.15) & (0.80, 0.10, 0.10) & (0.50, 0.30, 0.20) \\ & (0.40, 0.40, 0.20) & \end{array},$$

$$Q_N = \begin{array}{ccc} & (0.65, 0.25, 0.10) & \\ (0.50, 0.30, 0.20) & (0.90, 0.05, 0.05) & (0.30, 0.50, 0.20) \\ & (0.20, 0.60, 0.20) & \end{array}.$$

Use the componentwise Hadamard product $\odot$ and componentwise sum $\oplus$ (with clipping to $[0, 1]$ if needed). Then the top entry of $R_N \circ Q_N$ is

$$\alpha \odot f \oplus \delta \odot g = (0.70, 0.20, 0.10) \odot (0.65, 0.25, 0.10) \oplus (0.50, 0.30, 0.20) \odot (0.50, 0.30, 0.20),$$

$$= (0.455, 0.050, 0.010) \oplus (0.250, 0.090, 0.040) = (0.705, 0.140, 0.050).$$

The heart (center) combines as $\eta \odot h = (0.80, 0.10, 0.10) \odot (0.90, 0.05, 0.05) = (0.72, 0.005, 0.005)$, quantifying consolidated truth/indeterminacy/falsity for the overall diagnostic state.

Definition 7.15.5 (Plithogenic Rhotrix)**.** [50, 431] Fix a plithogenic setting (P, a, V, d, c) where P is a set of objects, a a (leading) trait with value set V, $d : P \times V \to [0, 1]$ a grade of appurtenance, and $c : V \times V \to [0, 1]$ a contradiction function. A *plithogenic rhotrix* is a rhotrix whose entries are of the form $d(x, v)$ for selected $x \in P$ and $v \in V$:

$$R_P = \begin{array}{ccc} & d(x_1, v_1) & \\ d(x_2, v_2) & d(x_3, v_3) & d(x_4, v_4) \\ & d(x_5, v_5) & \end{array}.$$

When multiplying $R_P \circ Q_P$, the heart-based pattern is used as in matrices, while each scalar combination is performed by a *plithogenic fusion operator* that mixes a t-norm and a t-conorm according to the contradiction degree $c(\cdot, \cdot)$ between the involved trait values; the result remains a plithogenic rhotrix. (Plithogenic fuzzy/intuitionistic-fuzzy/neutrosophic rhotrices arise by choosing d in the corresponding scale.)

Example 7.15.6 (Smartphone selection under contradictory traits (plithogenic rhotrix))**.** Let the leading trait be *battery endurance* with values $V = \{\text{low}, \text{med}, \text{high}\}$. Appurtenance $d(x, v) \in [0, 1]$ measures how well phone

x satisfies v. Contradiction function c obeys $c(v, v) = 0$, $c(\text{low}, \text{high}) = 1$, $c(\text{low}, \text{med}) = c(\text{med}, \text{high}) = \frac{1}{2}$. Define the DCF–weighted fusion

$$\mathcal{A}_c(u, v) = (1 - c)\ \min(u, v) + c\ \max(u, v).$$

Two product profiles R_P, Q_P (rows: west/east models; heart: baseline preference) are

$$R_P = \begin{array}{c} d(x_1, \text{high}) = 0.90 \\ 0.70\ \ 0.80\ \ 0.60 \\ 0.20 \end{array}, \qquad Q_P = \begin{array}{c} d(y_1, \text{med}) = 0.75 \\ 0.55\ \ 0.85\ \ 0.30 \\ 0.40 \end{array}.$$

Top entry of $R_P \circ Q_P$ fuses a with f (high vs med, $c = \frac{1}{2}$) and d with g (high vs med, $c = \frac{1}{2}$):

$$\mathcal{A}_{1/2}(0.90, 0.75) = \tfrac{1}{2} \cdot 0.75 + \tfrac{1}{2} \cdot 0.90 = 0.825, \quad \mathcal{A}_{1/2}(0.60, 0.55) = \tfrac{1}{2} \cdot 0.55 + \tfrac{1}{2} \cdot 0.60 = 0.575.$$

Aggregating these two contributions (e.g., by max) gives the top entry $\max(0.825, 0.575) = 0.825$. The center heart can be fused analogously, attenuating or amplifying by contradiction c between involved values.

Definition 7.15.7 (Neutrosophic Linguistic Rhotrix)**.** Let L be a finite set of linguistic labels (e.g. $\{\text{VL}, \text{L}, \text{M}, \text{H}, \text{VH}\}$). A *neutrosophic linguistic rhotrix* is a linguistic rhotrix whose entries are neutrosophic representations of labels, i.e. each entry is a triplet (ℓ_T, ℓ_I, ℓ_F) with $\ell_\bullet \in L$ (or their numeric quantifications). With a fixed injective quantification $\iota : L \to [0, 1]^3$, the heart-based product $R \circ Q$ is computed by the heart pattern together with a per-component linguistic aggregation (e.g. max∘ min on the quantified channels) and then (optionally) de-quantified back to labels.

Example 7.15.8 (Course evaluation with linguistic evidence (neutrosophic linguistic rhotrix))**.** Let labels $L = \{\text{L}, \text{M}, \text{H}\}$ with neutrosophic quantification $\iota(\text{L}) = (0.30, 0.40, 0.30)$, $\iota(\text{M}) = (0.60, 0.30, 0.10)$, $\iota(\text{H}) = (0.85, 0.10, 0.05)$. Instructor survey ($R$) and exam moderation ($Q$) are linguistic rhotrices:

$$R = \begin{array}{c} \text{H} \\ \text{M}\ \text{M}\ \text{L} \\ \text{M} \end{array}, \qquad Q = \begin{array}{c} \text{M} \\ \text{L}\ \text{H}\ \text{H} \\ \text{L} \end{array}.$$

Map entries via ι, combine by the heart pattern using componentwise min on T and max on I, F (one admissible policy), then (optionally) de-quantify by nearest label. For the center:

$$\iota(\text{M})\,\text{heart}\,\iota(\text{H}) \rightsquigarrow (T, I, F) = (\min(0.60, 0.85),\ \max(0.30, 0.10),\ \max(0.10, 0.05)) = (0.60, 0.30, 0.10) \approx \text{M}.$$

Thus, the integrated heart suggests an overall *medium* consolidated evaluation.

Definition 7.15.9 (Plithogenic Linguistic Rhotrix)**.** Let L be linguistic labels and let $q : L \to V$ map labels to trait values of a plithogenic model (P, a, V, d, c). A *plithogenic linguistic rhotrix* is a linguistic rhotrix whose entry at each position is the plithogenic appurtenance $d(x, q(\ell)) \in [0, 1]$ (for a designated object $x \in P$ and label $\ell \in L$). The heart-based product uses the rhotrix heart pattern; each scalar combination is a plithogenic fusion governed by the contradiction $c\big(q(\ell_1), q(\ell_2)\big)$ between the involved labels.

Example 7.15.10 (Service quality with plithogenic linguistic ratings (plithogenic linguistic rhotrix))**.** Service quality describes how well a service meets customer expectations, covering reliability, responsiveness, assurance, empathy, tangible aspects, and outcomes (cf. [433]).

Let labels $L = \{\text{Poor}, \text{Fair}, \text{Good}\}$ mapped to trait values $q(L) \subset V = \{\text{low}, \text{med}, \text{high}\}$ and appurtenance $d(x, q(\ell)) \in [0, 1]$ for a restaurant x. With contradiction $c(\text{low}, \text{high}) = 1$, $c(\text{low}, \text{med}) = c(\text{med}, \text{high}) = \frac{1}{2}$, consider

$$R = \begin{array}{c} \text{Good} \\ \text{Fair}\ \ \text{Good}\ \ \text{Fair} \\ \text{Poor} \end{array}, \qquad Q = \begin{array}{c} \text{Fair} \\ \text{Fair}\ \ \text{Good}\ \ \text{Poor} \\ \text{Poor} \end{array},$$

quantified by $d(x, q(\cdot))$ as, e.g., $d(x, \text{Good}) = 0.82$, $d(x, \text{Fair}) = 0.60$, $d(x, \text{Poor}) = 0.25$. Fuse scalars in the heart pattern by $\mathcal{A}_c(u, v) = (1 - c)\min(u, v) + c\max(u, v)$. For the top entry (Good vs Fair, $c = \frac{1}{2}$) and (Fair vs Fair, $c = 0$):

$$\mathcal{A}_{1/2}(0.82, 0.60) = 0.71, \qquad \mathcal{A}_0(0.60, 0.60) = \min(0.60, 0.60) = 0.60,$$

so the aggregated top entry (e.g., max) is 0.71. This yields a plithogenic linguistic rhotrix consistent with label contradictions.

7.16 Plithogenic Divergence Measures

Plithogenic divergence measures quantify dissimilarity between plithogenic component distributions weighted by contradiction degrees and dominance, generalizing f-divergences over attributes sets.

Definition 7.16.1 (Plithogenic Attribute Model)**.** Let $\mathcal{A}$ be a finite set of attributes. For each $a \in \mathcal{A}$, let V_a be a finite value set and let $pCF_a : V_a \times V_a \to [0, 1]$ be a symmetric *contradiction degree* with $pCF_a(v, v) = 0$. Fix a *dominant value* $v_a^\star \in V_a$ for each a.

Definition 7.16.2 (Plithogenic Component Distributions)**.** For a measurable space $(\Omega, \mathcal{F}, \mu)$, a *plithogenic distribution* is a family $P = \{p_{a,v}\}_{a\in\mathcal{A},\, v\in V_a}$ of nonnegative densities on Ω such that, for each fixed a, $\sum_{v\in V_a} \int_\Omega p_{a,v}\, d\mu = 1$. Similarly $Q = \{q_{a,v}\}$ with $q_{a,v} > 0$ a.e.

Definition 7.16.3 (Contradiction–Aware Weights)**.** For each $a \in \mathcal{A}$ and $v \in V_a$, define the raw weight

$$\tilde{\omega}_{a,v} \ := \ 1 - pCF_a\big(v, v_a^\star\big) \ \in [0, 1], \qquad \Omega_a \ := \sum_{u\in V_a} \tilde{\omega}_{a,u} \ (> 0),$$

and the normalized weights $\omega_{a,v} := \tilde{\omega}_{a,v}/\Omega_a$ so that $\sum_{v\in V_a} \omega_{a,v} = 1$.

Definition 7.16.4 (Plithogenic f–Divergence)**.** Let f be as above. The *plithogenic f–divergence* between $P = \{p_{a,v}\}$ and $Q = \{q_{a,v}\}$ is

$$D_f^{(\mathrm{Pl})}(P\|Q) \ := \sum_{a\in\mathcal{A}} \sum_{v\in V_a} \omega_{a,v} \int_\Omega q_{a,v}(\omega)\, f\!\left(\frac{p_{a,v}(\omega)}{q_{a,v}(\omega)}\right) d\mu(\omega).$$

It satisfies $D_f^{(\mathrm{Pl})} \geq 0$ and vanishes iff $p_{a,v} = q_{a,v}$ μ–a.e. for all a, v. If every V_a is a singleton, $D_f^{(\mathrm{Pl})}$ reduces to a standard f–divergence.

Example 7.16.5 (KL-based plithogenic f–divergence on a two-point domain)**.** Attributes: $\mathcal{A} = \{\text{color}, \text{texture}\}$. For color $V_{\text{col}} = \{\text{red}, \text{blue}\}$ with dominant $v_{\text{col}}^\star = \text{red}$ and $pCF_{\text{col}}(\text{blue}, \text{red}) = 0.4$; thus $\tilde{\omega}_{\text{red}} = 1$, $\tilde{\omega}_{\text{blue}} = 0.6$, $\Omega_{\text{col}} = 1.6$, giving $\omega_{\text{red}} = 1/1.6 = 0.625$, $\omega_{\text{blue}} = 0.6/1.6 = 0.375$.

For texture $V_{\text{tex}} = \{\text{smooth}, \text{rough}\}$ with dominant $v_{\text{tex}}^\star = \text{smooth}$ and $pCF_{\text{tex}}(\text{rough}, \text{smooth}) = 0.3$; thus $\tilde{\omega}_{\text{smooth}} = 1$, $\tilde{\omega}_{\text{rough}} = 0.7$, $\Omega_{\text{tex}} = 1.7$, giving $\omega_{\text{smooth}} = 1/1.7 \approx 0.588235$, $\omega_{\text{rough}} = 0.7/1.7 \approx 0.411765$.

Sample space: $\Omega = \{\omega_1, \omega_2\}$ with counting measure. Choose component densities $P = \{p_{a,v}\}$ and $Q = \{q_{a,v}\}$ (all entries > 0), with equal total mass per value:

	$p_{a,v}$		$q_{a,v}$	
pair (a, v)	ω_1	ω_2	ω_1	ω_2
(color, red)	0.30	0.20	0.28	0.22
(color, blue)	0.20	0.30	0.22	0.28
(texture, smooth)	0.42	0.18	0.40	0.20
(texture, rough)	0.16	0.24	0.18	0.22

Let $f(u) = u \ln u$ (natural log), so $D_f(p\|q) = \sum_i p_i \ln \frac{p_i}{q_i}$. The plithogenic divergence is

$$D_f^{(\mathrm{Pl})}(P\|Q) = \sum_{a\in\{\text{color},\text{texture}\}} \sum_{v\in V_a} \omega_{a,v} \sum_{j=1}^{2} p_{a,v}(\omega_j)\, \ln \frac{p_{a,v}(\omega_j)}{q_{a,v}(\omega_j)}.$$

Compute each component (two explicit terms shown; values in "nats"):

Color, red:

$$\begin{aligned} D_{\text{red}} &= 0.30 \ln \frac{0.30}{0.28} + 0.20 \ln \frac{0.20}{0.22} \\ &= 0.30 \ln(1.0714286) + 0.20 \ln(0.9090909) \\ &\approx 0.30(0.069014) + 0.20(-0.095310) \\ &\approx 0.020704 - 0.019062 = 0.001642 \text{ nats.} \end{aligned}$$

Color, blue (by symmetry with swapped entries):

$$D_{\text{blue}} \approx 0.001636 \text{ nats}.$$

Weighted color contribution:

$$\omega_{\text{red}} D_{\text{red}} + \omega_{\text{blue}} D_{\text{blue}} = 0.625(0.001636) + 0.375(0.001636) \approx 0.001636.$$

Texture, smooth:

$$\begin{aligned} D_{\text{smooth}} &= 0.42 \ln \frac{0.42}{0.40} + 0.18 \ln \frac{0.18}{0.20} \\ &= 0.42 \ln(1.05) + 0.18 \ln(0.9) \\ &\approx 0.42(0.048790) + 0.18(-0.105361) \\ &\approx 0.020492 - 0.018965 = 0.001527. \end{aligned}$$

Texture, rough:

$$\begin{aligned} D_{\text{rough}} &= 0.16 \ln \frac{0.16}{0.18} + 0.24 \ln \frac{0.24}{0.22} \\ &= 0.16 \ln(0.8888889) + 0.24 \ln(1.0909091) \\ &\approx 0.16(-0.117783) + 0.24(0.087011) \\ &\approx -0.018845 + 0.020883 = 0.002037. \end{aligned}$$

Weighted texture contribution:

$$\omega_{\text{smooth}} D_{\text{smooth}} + \omega_{\text{rough}} D_{\text{rough}} \approx 0.588235(0.001527) + 0.411765(0.002037) \approx 0.001737.$$

Therefore, the plithogenic divergence (with these weights and components) is

$$D_f^{(\text{Pl})}(P\|Q) \approx 0.001636 + 0.001737 = 0.003373 \text{ nats}.$$

7.17 Neutrosophic Life

A neutrosophic life models events by additive truth, indeterminacy, and falsity impacts, capturing beneficial, unknown, and harmful outcomes over time [44].

Definition 7.17.1 (Neutrosophic Life)**.** Let (Ω, Σ) be a measurable space of an agent's life events. A *neutrosophic life measure* is a map $\mathcal{L} = (T, I, F) : \Sigma \to [0,1]^3$ such that for disjoint $E_1, E_2 \in \Sigma$,

$$\mathcal{L}(E_1 \cup E_2) = \mathcal{L}(E_1) + \mathcal{L}(E_2) \quad \text{(componentwise)}, \qquad \mathcal{L}(\varnothing) = (0,0,0),$$

where T quantifies beneficial/positive impact, F harmful/negative impact, and I indeterminacy/unknown. A time-indexed *neutrosophic life trajectory* is $t \mapsto (T_t, I_t, F_t) \in [0,1]^3$, $t \in \mathbb{R}$, describing instantaneous degrees that need not sum to 1.

Example 7.17.2 (Year plan as a neutrosophic life measure)**.** Let $\Omega = \{E_{\text{degree}}, E_{\text{job}}, E_{\text{illness}}\}$ and $\Sigma = 2^{\Omega}$. Define a neutrosophic life measure $\mathcal{L} = (T, I, F) : \Sigma \to [0,1]^3$ by its values on atoms:

$$\mathcal{L}(\{E_{\text{degree}}\}) = (0.50,\ 0.20,\ 0.10), \quad \mathcal{L}(\{E_{\text{job}}\}) = (0.30,\ 0.10,\ 0.10), \quad \mathcal{L}(\{E_{\text{illness}}\}) = (0.10,\ 0.10,\ 0.60).$$

Since the atoms are disjoint, finite additivity gives, e.g.,

$$\mathcal{L}(\{E_{\text{degree}}, E_{\text{job}}\}) = \mathcal{L}(\{E_{\text{degree}}\}) + \mathcal{L}(\{E_{\text{job}}\}) = (0.80,\ 0.30,\ 0.20) \in [0,1]^3,$$

and

$$\mathcal{L}(\Omega) = (0.50 + 0.30 + 0.10,\ 0.20 + 0.10 + 0.10,\ 0.10 + 0.10 + 0.60) = (0.90,\ 0.40,\ 0.80).$$

A time-indexed neutrosophic life trajectory on months $t \in [0,12]$ can be modeled piecewise as

$$(T_t, I_t, F_t) = \begin{cases} (0.50,\ 0.40,\ 0.10), & 0 \le t \le 4 \quad \text{(coursework uncertain but mostly positive)}, \\ (0.70,\ 0.20,\ 0.10), & 4 < t \le 8 \quad \text{(degree progress and job offer increase benefit)}, \\ (0.60,\ 0.25,\ 0.15), & 8 < t \le 12 \quad \text{(illness raises risk yet positives remain)}. \end{cases}$$

This illustrates how beneficial (truth), unknown (indeterminacy), and harmful (falsity) impacts aggregate over disjoint events and evolve over time.

7.18 Neutrosophic Permittivity

Permittivity quantifies a material's ability to permit electric field lines, determining capacitance, polarization response, and electromagnetic wave propagation speed within (cf. [434–436]). A neutrosophic permittivity assigns a propagation weight from truth, indeterminacy, and falsity degrees, attenuating noisy components and amplifying reliable evidence [38, 437].

Definition 7.18.1 (Neutrosophic Permittivity)**.** [38, 437] A *neutrosophic permittivity* is a function

$$\varepsilon_N : [0,1]^3 \longrightarrow [0,\infty), \qquad (t,i,f) \longmapsto \varepsilon_N(t,i,f),$$

that modulates the propagation/weight of neutrosophic information and satisfies:

1. Monotonicity: ε_N is nondecreasing in t and nonincreasing in f; its dependence on i is application-specific (often nonincreasing).
2. Normalization: $\varepsilon_N(0,i,1) = 0$ and $\varepsilon_N(1,0,0) = \varepsilon_{\max}$ (some fixed maximum).
3. Lipschitz (stability): there exists $L > 0$ with $|\varepsilon_N(\mathbf{u}) - \varepsilon_N(\mathbf{v})| \le L\|\mathbf{u} - \mathbf{v}\|_1$.

Typical choices include

$$\varepsilon_N(t,i,f) = \frac{t}{1+f} \quad \text{or} \quad \varepsilon_N(t,i,f) = \frac{t}{1+i+f+\delta}\ (\delta > 0),$$

used as adaptive gains in filters, similarity measures, or decision weights.

Example 7.18.2 (Sensor fusion with neutrosophic permittivity)**.** Define

$$\varepsilon_N(t,i,f) \ = \ \frac{t}{1+i+f+\delta}, \qquad \delta = 0.01.$$

This is nondecreasing in t and nonincreasing in f; larger i, f reduce the weight.

Two sensors observe a scalar quantity with readings

$$y_1 = 72, \quad (t_1,i_1,f_1) = (0.80, 0.10, 0.10), \qquad y_2 = 68, \quad (t_2,i_2,f_2) = (0.50, 0.30, 0.20).$$

Compute permittivities:

$$\varepsilon_1 = \frac{0.80}{1+0.10+0.10+0.01} = \frac{0.80}{1.21} \approx 0.661157,$$
$$\varepsilon_2 = \frac{0.50}{1+0.30+0.20+0.01} = \frac{0.50}{1.51} \approx 0.331126.$$

Normalize to weights

$$w_1 = \frac{\varepsilon_1}{\varepsilon_1+\varepsilon_2} = \frac{0.661157}{0.661157+0.331126} \approx \frac{0.661157}{0.992283} \approx 0.6663,$$
$$w_2 = 1 - w_1 \approx 0.3337.$$

The fused estimate is the permittivity–weighted average:

$$\hat{y} = w_1 y_1 + w_2 y_2 \approx 0.6663 \times 72 + 0.3337 \times 68 \approx 47.9712 + 22.6940 \approx 70.6652.$$

Monotonicity check (decreasing falsity increases weight): replacing (t_1,i_1,f_1) by $(0.80, 0.10, 0.05)$ gives

$$\varepsilon_1' = \frac{0.80}{1+0.10+0.05+0.01} = \frac{0.80}{1.16} \approx 0.689655 > \varepsilon_1 \approx 0.661157.$$

7.19 Neutrosophic Quantum Statistics

Quantum statistics studies ensembles of quantum systems, describing particle distributions, thermodynamic properties, measurement outcomes, underlying quantum probabilities, and nonclassical correlations [438–440]. Neutrosophic quantum statistics models quantum measurements with triplet probabilities, capturing truth, indeterminacy, and falsity, enabling uncertainty aware expectations and inference [36].

Definition 7.19.1 ((Recall) Neutrosophic probability space)**.** A *neutrosophic probability space* is a triple $(\Omega, \mathcal{F}, \nu)$ where Ω is a sample space, $\mathcal{F}$ a σ-algebra, and $\nu : \mathcal{F} \to [0,1]^3$, $A \mapsto \nu(A) = (T(A), I(A), F(A))$ assigns truth/indeterminacy/falsity degrees to events.

Definition 7.19.2 (Neutrosophic quantum statistics)**.** Let $\mathcal{H}$ be a separable Hilbert space and ρ a density operator. A *neutrosophic observable* is a projection-valued (or POVM) map $E : \mathcal{B}(\mathbb{R}) \to \mathcal{L}(\mathcal{H})$ together with a neutrosophic probability $\nu_\rho(B) = (T_\rho(B), I_\rho(B), F_\rho(B))$ for each Borel set B, where $T_\rho(B)$ reflects the degree $\operatorname{tr}(\rho E(B))$ (possibly interval-extended), $F_\rho(B)$ the degree assigned to the complement, and $I_\rho(B)$ captures indeterminacy due to hidden/contextual factors. Neutrosophic expectations are triples

$$\mathbb{E}^N_\rho[X] := \left(\int x\, dT_\rho,\ \int x\, dI_\rho,\ \int x\, dF_\rho \right),$$

with analogous variance/covariance triplets defined componentwise.

Example 7.19.3 (Qubit measurement with neutrosophic triplet probabilities)**.** Qubit measurement projects a quantum state onto basis outcomes, yielding classical bits probabilistically and irreversibly, disturbing superposition and entanglement information (cf. [441]).

Let $\mathcal{H} = \mathbb{C}^2$ with computational basis $\{|0\rangle, |1\rangle\}$. Consider the observable Z with POVM $\{E_+ = |0\rangle\langle 0|,\ E_- = |1\rangle\langle 1|\}$ (eigenvalues $+1, -1$). Take a mixed state

$$\rho = (1-p)\, |0\rangle\langle 0| + p\, |1\rangle\langle 1|, \qquad p = 0.35.$$

Then $\operatorname{tr}(\rho E_+) = 1 - p = 0.65$ and $\operatorname{tr}(\rho E_-) = p = 0.35$.

Assign neutrosophic triplets to each outcome $B \in \{+, -\}$:

$$\nu_\rho(B) = (T_\rho(B), I_\rho(B), F_\rho(B)).$$

We set

$$(T_\rho(+), I_\rho(+), F_\rho(+)) = (0.65,\ 0.08,\ 0.35), \qquad (T_\rho(-), I_\rho(-), F_\rho(-)) = (0.35,\ 0.08,\ 0.65).$$

Here $T_\rho(\pm)$ equals the Born probabilities, $F_\rho(\pm)$ equals the complement outcome's probability, and $I_\rho(\pm)$ encodes contextual indeterminacy (sensor drift, timing jitter).

Neutrosophic expectation of Z is the triplet

$$\mathbb{E}^N_\rho[Z] = \Big(\underbrace{\sum_{x \in \{+1,-1\}} x\, T_\rho(x),}_{\text{truth component}} \quad \underbrace{\sum_x x\, I_\rho(x)}_{\text{indeterminacy component}}, \quad \underbrace{\sum_x x\, F_\rho(x)}_{\text{falsity component}} \Big).$$

Compute each component explicitly:

$$\mathbb{E}_T[Z] = (+1) \cdot 0.65 + (-1) \cdot 0.35 = 0.65 - 0.35 = 0.30,$$

$$\mathbb{E}_I[Z] = (+1) \cdot 0.08 + (-1) \cdot 0.08 = 0.08 - 0.08 = 0.00,$$

$$\mathbb{E}_F[Z] = (+1) \cdot 0.35 + (-1) \cdot 0.65 = 0.35 - 0.65 = -0.30.$$

Thus

$$\mathbb{E}^N_\rho[Z] = (0.30,\ 0.00,\ -0.30).$$

Interpretation: the truth-expectation matches the ordinary mean $\langle Z \rangle = 0.30$; an indeterminacy-balanced assignment yields zero net I-expectation; the falsity-expectation mirrors the complement, here -0.30.

7.20 Neutrosophic Megagraph

A neutrosophic megagraph models uncertain multiway transformations via vertices and hyperarrows with truth, indeterminacy, falsity degrees on ordered endpoint sequences [40].

Definition 7.20.1 (Neutrosophic Megagraph)**.** Let V be a (finite or countable) set of vertices and let

$$\mathrm{Seq}(V) \;:=\; \bigsqcup_{n\geq 0} V^n$$

denote the set of all finite sequences (ordered lists, allowing repetitions) of elements of V. A *megagraph* is a triple $M = (V, E, s, t)$ where E is a set of *hyperarrows* and $s, t : E \to \mathrm{Seq}(V)$ assign to each $e \in E$ a (possibly empty) input list $s(e)$ and output list $t(e)$. Thus e can connect an ordered tuple of inputs to an ordered tuple of outputs; multiplicities and loops are allowed.

A *neutrosophic megagraph* is a pair

$$M_N \;=\; \big(M,\; \lambda_V, \lambda_E\big)$$

where $\lambda_V : V \to [0,1]^3$ and $\lambda_E : E \to [0,1]^3$ are neutrosophic valuations,

$$\lambda_V(v) = (T_V(v), I_V(v), F_V(v)), \qquad \lambda_E(e) = (T_E(e), I_E(e), F_E(e)),$$

with $T, I, F \in [0,1]$ and (optionally) $0 \leq T + I + F \leq 3$. One may impose the incidence compatibilities

$$T_E(e) \;\leq\; \min\{\, T_V(v) \;:\; v \in s(e) \cup t(e)\}, \qquad I_E(e) \;\geq\; \max\{\, I_V(v) \;:\; v \in s(e) \cup t(e)\},$$

$$F_E(e) \;\geq\; \max\{\, F_V(v) \;:\; v \in s(e) \cup t(e)\},$$

expressing that an arrow is not more certainly true than its endpoints but may be more indeterminate or more false.

Remark 7.20.2. If every $e \in E$ satisfies $|s(e)| = |t(e)| = 1$ and we forget order, we recover a (neutrosophic) multigraph; if $s(e) = \varnothing$ and only $t(e)$ is used (or vice versa), we obtain a (directed) neutrosophic hypergraph. Hence neutrosophic megagraphs strictly generalize neutrosophic graphs and hypergraphs by allowing multiple ordered inputs and outputs.

Example 7.20.3 (Healthcare triage megagraph with compatibility checks)**.** Let

$$V = \{\mathrm{ECG}, \mathrm{SpO_2}, \mathrm{Risk}, \mathrm{Monitor}, \mathrm{Alarm}\}$$

. Define hyperarrows $E = \{e_1, e_2\}$ with ordered incidence

$$s(e_1) = (\mathrm{ECG}, \mathrm{SpO_2}),\; t(e_1) = (\mathrm{Risk}), \qquad s(e_2) = (\mathrm{Risk}),\; t(e_2) = (\mathrm{Monitor}, \mathrm{Alarm}).$$

Assign neutrosophic valuations $\lambda_V(v) = (T_V(v), I_V(v), F_V(v))$:

v	T_V	I_V	F_V
ECG	0.88	0.06	0.06
$\mathrm{SpO_2}$	0.80	0.10	0.10
Risk	0.75	0.15	0.10
Monitor	0.70	0.20	0.10
Alarm	0.65	0.25	0.10

and edge valuations $\lambda_E(e) = (T_E(e), I_E(e), F_E(e))$:

e	T_E	I_E	F_E
e_1	0.72	0.18	0.12
e_2	0.62	0.27	0.12

Compatibility (arrow not more "true" than endpoints, and possibly more indeterminate/false) holds numerically.

For e_1 with endpoints $\{\mathrm{ECG}, \mathrm{SpO_2}, \mathrm{Risk}\}$:

$$\min T_V = \min\{0.88, 0.80, 0.75\} = 0.75 \;\Rightarrow\; T_E(e_1) = 0.72 \leq 0.75,$$

$$\max I_V = \max\{0.06, 0.10, 0.15\} = 0.15 \Rightarrow I_E(e_1) = 0.18 \geq 0.15,$$
$$\max F_V = \max\{0.06, 0.10, 0.10\} = 0.10 \Rightarrow F_E(e_1) = 0.12 \geq 0.10.$$

For e_2 with endpoints {Risk, Monitor, Alarm}:

$$\min T_V = \min\{0.75, 0.70, 0.65\} = 0.65 \Rightarrow T_E(e_2) = 0.62 \leq 0.65,$$
$$\max I_V = \max\{0.15, 0.20, 0.25\} = 0.25 \Rightarrow I_E(e_2) = 0.27 \geq 0.25,$$
$$\max F_V = \max\{0.10, 0.10, 0.10\} = 0.10 \Rightarrow F_E(e_2) = 0.12 \geq 0.10.$$

Interpretation. e_1 fuses vital signs (ECG, SpO_2) into a risk score; e_2 branches Risk into (Monitor, Alarm). Triplet degrees encode reliability (T), ambiguity/missingness (I), and contradiction/noise (F).

7.21 Neutrosophic Physical Law

A physical law is a concise mathematical or conceptual rule describing consistent patterns in nature, empirically verified and universally applicable (cf. [442–444]). A neutrosophic physical law assigns each state truth, indeterminacy, and falsity degrees quantifying validity, uncertainty, and violation from measurements contextually (cf. [49, 445]).

Definition 7.21.1 (Neutrosophic Physical Law)**.** [49] Let $\mathcal{S}$ be a nonempty set of physical states (space–time events, configurations, or experimental situations). Let L be a classical physical law written as a predicate

$$L : \mathcal{S} \to \{\text{true}, \text{false}\},$$

or equivalently as an equation/inequality $F(s) = 0$, $G(s) \leq 0$, etc., that is intended to hold for all $s \in \mathcal{S}$.

A neutrosophic physical law associated with L is a mapping

$$\mathcal{L}_N : \mathcal{S} \longrightarrow [0,1]^3, \qquad s \longmapsto \big(T_L(s),\ I_L(s),\ F_L(s)\big),$$

where, for every state $s \in \mathcal{S}$,

- $T_L(s) \in [0, 1]$ is the degree to which L is observed/confirmed/valid at s;
- $I_L(s) \in [0, 1]$ is the degree of indeterminacy of L at s (due to measurement noise, incomplete modeling, or coexistence of alternative theories);
- $F_L(s) \in [0, 1]$ is the degree to which L is violated/contradicted at s.

No normalization is mandatory, but often

$$0 \leq T_L(s) + I_L(s) + F_L(s) \leq 3$$

(or ≤ 1 in the normalized case) is imposed. The pair $(L, \mathcal{L}_N)$ is called a neutrosophic physical law. A family $\{\, (L_\alpha, \mathcal{L}_{N,\alpha}) \,\}_{\alpha \in A}$ forms a neutrosophic physical theory when defined on the same state space $\mathcal{S}$.

Example 7.21.2 (Ohm's law under sensor noise and heating)**.** Classical law: $L(s)$ is Ohm's law with nominal resistance $R_0 = 10\,\Omega$, i.e., $V - R_0 I = 0$ for state $s = (V, I, \text{noise})$ (cf. [446]).

Define the residual

$$r(s) \ := \ \frac{|V - R_0 I|}{|V| + |R_0 I|} \ \in [0, 1],$$

and a neutrosophic evaluation

$$T_L(s) = e^{-5r(s)}, \qquad I_L(s) = \eta \text{ (measured noise level } \eta \in [0, 1]), \qquad F_L(s) = r(s).$$

Concrete measurement (heated resistor, moderate sensor noise):

$$V = 9.6\text{ V},\quad I = 1.05\text{ A},\quad \eta = 0.15,\quad R_0 = 10\,\Omega.$$

Step–by–step values:

$$R_0 I = 10.5,\quad |V - R_0 I| = |9.6 - 10.5| = 0.9,\quad |V| + |R_0 I| = 9.6 + 10.5 = 20.1,$$

$$r = \frac{0.9}{20.1} = 0.04478\ (\text{approx}).$$

Hence

$$T_L = e^{-5\cdot 0.04478} = e^{-0.2239} = 0.7994,\qquad I_L = 0.15,\qquad F_L = 0.0448.$$

Interpretation: at this operating point, Ohm's law holds with high truth (≈ 0.80), moderate indeterminacy from noise (0.15), and small violation (≈ 0.045).

7.22 Neutrosophic signature

A neutrosophic signature assigns each symbol a truth, indeterminacy, and falsity triple, encoding uncertain, conflicting, and reliable evidence for interpretation [41, 447].

Definition 7.22.1 (Neutrosophic signature)**.** (cf. [41, 447]) Let $\Sigma = \{s_1, \ldots, s_m\}$ be a finite set of symbols. A neutrosophic signature on Σ is a mapping

$$\mathrm{NSig} : \Sigma \to [0,1]^3,\qquad \mathrm{NSig}(s_j) = \big(T_{s_j}, I_{s_j}, F_{s_j}\big),$$

where $T_{s_j}, I_{s_j}, F_{s_j} \in [0,1]$ denote, respectively, the truth, indeterminacy, and falsity degrees attached to the symbol s_j. Optionally one may require $T_{s_j} + I_{s_j} + F_{s_j} \le 3$ (unrestricted form) or ≤ 1 (normalized form).

Example 7.22.2 (Authorship-attribution signature over stylistic symbols)**.** Authorship-attribution analyzes linguistic and stylistic features of texts to infer which author most likely wrote a given document or passage (cf. [448]).

Let the symbol set be

$$\Sigma = \{\mathrm{token}_A, \mathrm{token}_B, \mathrm{token}_C\}$$

, and define a neutrosophic signature

$$\mathrm{NSig}(\mathrm{token}_A) = (0.82, 0.10, 0.08),\quad \mathrm{NSig}(\mathrm{token}_B) = (0.55, 0.30, 0.15),\quad \mathrm{NSig}(\mathrm{token}_C) = (0.20, 0.50, 0.30).$$

Given a document with observed counts

$$n_A = 12,\quad n_B = 5,\quad n_C = 3,\quad N = n_A + n_B + n_C = 20,$$

its neutrosophic match score is the occurrence–weighted average

$$(T, I, F) = \frac{1}{N}\Big(n_A\,\mathrm{NSig}(\mathrm{token}_A) + n_B\,\mathrm{NSig}(\mathrm{token}_B) + n_C\,\mathrm{NSig}(\mathrm{token}_C)\Big).$$

Compute each component explicitly:

$$T = \frac{12\cdot 0.82 + 5\cdot 0.55 + 3\cdot 0.20}{20} = \frac{9.84 + 2.75 + 0.60}{20} = \frac{13.19}{20} = 0.6595,$$

$$I = \frac{12\cdot 0.10 + 5\cdot 0.30 + 3\cdot 0.50}{20} = \frac{1.20 + 1.50 + 1.50}{20} = \frac{4.20}{20} = 0.21,$$

$$F = \frac{12\cdot 0.08 + 5\cdot 0.15 + 3\cdot 0.30}{20} = \frac{0.96 + 0.75 + 0.90}{20} = \frac{2.61}{20} = 0.1305.$$

A simple decision rule "accept if $T - \max\{I, F\} \ge \tau$" with $\tau = 0.20$ yields

$$T - \max\{I, F\} = 0.6595 - \max\{0.21, 0.1305\} = 0.6595 - 0.21 = 0.4495 \ge 0.20,$$

so the document is attributed to the target author under this neutrosophic signature.

7.23 Neutrosophic decision function

A neutrosophic decision function maps each input to label-wise triples (truth, indeterminacy, falsity), enabling decisions under uncertain, conflicting evidence conditions. The neutrosophic function is known to generalize both the fuzzy decision function [449–452] and the intuitionistic fuzzy decision function.

Definition 7.23.1 (Neutrosophic decision function)**.** [41] Let $X \subseteq \mathbb{R}^d$ be a feature space and let Y be a finite set of decision labels. A neutrosophic decision function is a mapping

$$f_{\mathrm{N}} : X \to \big([0,1]^3\big)^Y$$

such that, for every $x \in X$ and every $y \in Y$,

$$f_{\mathrm{N}}(x)(y) = \big(T_{x,y}, I_{x,y}, F_{x,y}\big) \in [0,1]^3,$$

interpreted as "when the input is x, the decision label y holds with truth $T_{x,y}$, is indeterminate with $I_{x,y}$, and is rejected with $F_{x,y}$". A crisp neutrosophic decision can be obtained by

$$\arg\max_{y\in Y}\ T_{x,y} - F_{x,y} \quad \text{or} \quad \arg\max_{y\in Y}\ (T_{x,y}, -I_{x,y}, -F_{x,y}).$$

Example 7.23.2 (Email spam filtering with a neutrosophic decision function)**.** Email spam filtering automatically detects and blocks unwanted, suspicious, or malicious messages using content analysis, sender reputation, and machine-learning models (cf. [453]).

Let $X \subseteq \mathbb{R}^d$ be a feature space and $Y = \{\mathrm{spam}, \mathrm{ham}\}$. For a message $x \in X$, suppose

$$f_{\mathrm{N}}(x)(\mathrm{spam}) = (T, I, F) = (0.72,\ 0.15,\ 0.13), \qquad f_{\mathrm{N}}(x)(\mathrm{ham}) = (0.28,\ 0.20,\ 0.60).$$

Use the score $s_y := T_{x,y} - F_{x,y}$ and predict $\arg\max_{y\in Y} s_y$.

Compute explicitly:

$$s_{\mathrm{spam}} = 0.72 - 0.13 = 0.59, \qquad s_{\mathrm{ham}} = 0.28 - 0.60 = -0.32.$$

Since $0.59 > -0.32$, the decision is spam. Optionally, impose an abstention if indeterminacy is too high. Here $I_{\mathrm{spam}} = 0.15$ and $I_{\mathrm{ham}} = 0.20$; both are below a typical threshold (e.g. 0.5), so no abstention is triggered.

7.24 Neutrosophic Validity

Neutrosophic validity assigns each statement triple (truth, indeterminacy, falsity) over model class via plausibility, enabling graded, gappy semantics and thresholds [36].

Definition 7.24.1 (Neutrosophic semantic validity)**.** [36] Let $\mathfrak{M}$ be a class of structures for a language $\mathcal{L}$ and let μ be a plausibility measure on $\mathfrak{M}$. For a sentence φ define

$$\mathrm{Val}_N(\varphi) := \big(T(\varphi), I(\varphi), F(\varphi)\big) := \big(\mu(\{M \in \mathfrak{M} \mid M \models \varphi\}),\ \mu(\{M \mid M \not\models \varphi \wedge M \not\models \neg\varphi\}),\ \mu(\{M \mid M \models \neg\varphi\})\big).$$

We say φ is *neutrosophically valid at level* (τ, ι, φ) if $T(\varphi) \geq \tau$, $I(\varphi) \leq \iota$, and $F(\varphi) \leq \varphi$.

Example 7.24.2 (Computing neutrosophic validity for $\varphi := p \to q$)**.** Let the language be $\{p, q\}$ and let $\mathfrak{M}$ be four three-valued (Kleene K3) structures with values $\{0, I, 1\}$. Define implication by $p \to q \equiv \neg p \vee q$ using strong Kleene tables. Equip $\mathfrak{M}$ with the uniform plausibility $\mu(M) = \frac{1}{4}$ for each $M \in \mathfrak{M}$.

Valuations:

M	p	q	value of $\varphi = \neg p \vee q$
M_1	1	1	1
M_2	1	0	0
M_3	1	I	I
M_4	I	0	I

Hence

$$\{M \models \varphi\} = \{M_1\}, \qquad \{M \models \neg\varphi\} = \{M_2\}, \qquad \{M \not\models \varphi\ \wedge\ M \not\models \neg\varphi\} = \{M_3, M_4\}.$$

By Definition (neutrosophic semantic validity),

$$\mathrm{Val}_N(\varphi) = \big(T(\varphi), I(\varphi), F(\varphi)\big) = \Big(\underbrace{\mu(\{M_1\})}_{\frac{1}{4}},\ \underbrace{\mu(\{M_3, M_4\})}_{\frac{1}{2}},\ \underbrace{\mu(\{M_2\})}_{\frac{1}{4}}\Big) = \Big(\tfrac{1}{4}, \tfrac{1}{2}, \tfrac{1}{4}\Big).$$

For thresholds $(\tau, \iota, \varphi) = (0.2, 0.6, 0.3)$ the sentence is neutrosophically valid ($0.25 \geq 0.2$, $0.5 \leq 0.6$, $0.25 \leq 0.3$); for $(\tau, \iota, \varphi) = (0.6, 0.3, 0.2)$ it is not.

7.25 Refined Neutrosophic Clustering

Intuitively, refined neutrosophic clustering extends neutrosophic clustering by splitting the three components T, I, F into multiple refined subcomponents $T_1, T_2, \ldots, I_1, I_2, \ldots, F_1, F_2, \ldots$ and clustering with respect to all of them simultaneously. This approach generalizes Fuzzy Clustering [454–456], Intuitionistic Fuzzy Clustering [457–459], and Neutrosophic Clustering [460–462].

Definition 7.25.1 (Refined Neutrosophic Clustering (RNC))**.** [40] Let $X = \{x_i\}_{i=1}^n \subset \mathbb{R}^d$ be a dataset and $K \in \mathbb{N}$ the number of clusters. Fix integers $p, q, r \geq 1$ (the numbers of refined components) and fuzzifier $m > 1$. An *RNC model* consists of:

- prototypes $V = \{v_k\}_{k=1}^K \subset \mathbb{R}^d$,
- refined membership tensors

$$T = \big(t_{ik}^{(a)}\big) \in [0,1]^{n\times K\times p}, \quad I = \big(i_{ik}^{(b)}\big) \in [0,1]^{n\times K\times q}, \quad F = \big(f_{ik}^{(c)}\big) \in [0,1]^{n\times K\times r},$$

subject to the per–component normalizations, for each i,

$$\sum_{k=1}^K t_{ik}^{(a)} = 1\ (a = 1, \ldots, p), \qquad \sum_{k=1}^K i_{ik}^{(b)} = 1\ (b = 1, \ldots, q), \qquad \sum_{k=1}^K f_{ik}^{(c)} = 1\ (c = 1, \ldots, r),$$

and bounds $0 \leq t_{ik}^{(a)}, i_{ik}^{(b)}, f_{ik}^{(c)} \leq 1$.

Let $d_{ik} = \|x_i - v_k\|^2$ and let $w_T, w_I, w_F > 0$ with $w_T + w_I + w_F = 1$ be component weights. Define the loss

$$\mathcal{J}(V, T, I, F) = \sum_{i=1}^n \sum_{k=1}^K \Bigg[w_T \sum_{a=1}^p \big(t_{ik}^{(a)}\big)^m d_{ik} + w_I \sum_{b=1}^q \big(i_{ik}^{(b)}\big)^m \big|d_{ik} - \bar{d}_i\big| + w_F \sum_{c=1}^r \big(f_{ik}^{(c)}\big)^m \frac{1}{d_{ik} + \varepsilon} \Bigg],$$

where $\bar{d}_i = \frac{1}{K} \sum_{k=1}^K d_{ik}$ and $\varepsilon > 0$ avoids division by zero.

An RNC solution is any (V, T, I, F) minimizing $\mathcal{J}$ under the above constraints. The aggregated (non–refined) neutrosophic degrees for (i, k) are

$$T_{ik} = \sum_{a=1}^p \alpha_a\, t_{ik}^{(a)}, \qquad I_{ik} = \sum_{b=1}^q \beta_b\, i_{ik}^{(b)}, \qquad F_{ik} = \sum_{c=1}^r \gamma_c\, f_{ik}^{(c)},$$

for fixed convex weights $(\alpha_a), (\beta_b), (\gamma_c)$ on each refined family.

Remark 7.25.2. The T-term pulls points toward nearby prototypes, the I-term emphasizes boundary/ambiguous assignments via deviation from $\bar{d}_i$, and the F-term penalizes allocating a point to far prototypes (since $1/(d_{ik}+\varepsilon)$ is small when d_{ik} is large). When $p = q = r = 1$ the model reduces to a standard neutrosophic clustering objective.

Example 7.25.3 (Hospital triage (ED) with refined components)**.** We cluster emergency-department episodes into care pathways: C_1 = "home–care", C_2 = "observation", C_3 = "admission". Each visit x_i = (acuity, risk) $\in \mathbb{R}^2$ is described by refined neutrosophic degrees: truth $T = (T_1, T_2)$ with T_1 = clinical criteria match, T_2 = resource suitability; indeterminacy $I = (I_1, I_2)$ with I_1 = missing/incomplete data, I_2 = conflicting evidence; falsity $F = (F_1, F_2)$ with F_1 = protocol violation, F_2 = outlier/anomaly suspicion. We use $p = q = r = 2$ refined components and $K = 3$ clusters.

Data and prototypes

$$X = \begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{bmatrix} = \begin{bmatrix} (1.0,\ 0.2) \\ (0.9,\ 0.8) \\ (0.2,\ 0.9) \\ (0.7,\ 0.4) \end{bmatrix}, \qquad V = \{v_1, v_2, v_3\} = \{(0.2,\ 0.2),\ (0.7,\ 0.7),\ (1.0,\ 1.0)\}.$$

Squared distances $d_{ik} = \|x_i - v_k\|_2^2$:

$$[d_{ik}]_{i,k} = \begin{bmatrix} 0.64 & 0.34 & 0.64 \\ 0.85 & 0.05 & 0.05 \\ 0.49 & 0.29 & 0.65 \\ 0.29 & 0.09 & 0.45 \end{bmatrix}.$$

Refined memberships are constrained, for each i and each refinement index, $\sum_{k=1}^{3} t_{ik}^{(a)} = 1$ ($a = 1, 2$), $\sum_{k=1}^{3} i_{ik}^{(b)} = 1$ ($b = 1, 2$), $\sum_{k=1}^{3} f_{ik}^{(c)} = 1$ ($c = 1, 2$), with values in $[0, 1]$. We illustrate one visit ($i = 2$) whose point $x_2 = (0.9, 0.8)$ lies close to v_2 and v_3:

refined component	C_1	C_2	C_3
$T_1 : t_{2k}^{(1)}$	0.05	0.475	0.475
$T_2 : t_{2k}^{(2)}$	0.10	0.80	0.10
$I_1 : i_{2k}^{(1)}$	0.10	0.30	0.60
$I_2 : i_{2k}^{(2)}$	0.30	0.20	0.50
$F_1 : f_{2k}^{(1)}$	0.70	0.20	0.10
$F_2 : f_{2k}^{(2)}$	0.60	0.30	0.10

(Clinically, C_2 is highly suitable by resources T_2, while C_3 suffers larger indeterminacy I due to conflicting evidence.)

With convex refinement weights $\alpha = (\frac{1}{2}, \frac{1}{2})$, $\beta = (\frac{1}{2}, \frac{1}{2})$, $\gamma = (\frac{1}{2}, \frac{1}{2})$, the aggregated (non–refined) degrees for $i = 2$ are

$$(T_{2k})_{k=1}^{3} = \tfrac{1}{2}(0.05+0.10,\ 0.475+0.80,\ 0.475+0.10) = (0.075,\ 0.6375,\ 0.2875),$$
$$(I_{2k})_{k=1}^{3} = \tfrac{1}{2}(0.10+0.30,\ 0.30+0.20,\ 0.60+0.50) = (0.20,\ 0.25,\ 0.55),$$
$$(F_{2k})_{k=1}^{3} = \tfrac{1}{2}(0.70+0.60,\ 0.20+0.30,\ 0.10+0.10) = (0.65,\ 0.25,\ 0.10).$$

Hence C_2 dominates in truth and keeps both indeterminacy and falsity moderate.

For a concrete RNC objective (fuzzifier $m = 2$; component weights $w_T = 0.60$, $w_I = 0.25$, $w_F = 0.15$; $\varepsilon = 0.01$), recall

$$\mathcal{J} = \sum_{i,k} \left[w_T \sum_{a=1}^{2} (t_{ik}^{(a)})^2\, d_{ik} + w_I \sum_{b=1}^{2} (i_{ik}^{(b)})^2 \left| d_{ik} - \bar{d}_i \right| + w_F \sum_{c=1}^{2} (f_{ik}^{(c)})^2\, \frac{1}{d_{ik}+\varepsilon} \right],$$

with $\bar{d}_i = \frac{1}{3} \sum_k d_{ik}$. For $i = 2$ one has $\bar{d}_2 = \frac{0.85+0.05+0.05}{3} = 0.316\overline{6}$ and

$$\begin{aligned} \text{term}(i{=}2, k{=}2) &= 0.60\,(0.475^2{+}0.80^2) \cdot 0.05 + 0.25\,(0.30^2{+}0.20^2) \cdot |0.05{-}0.316\overline{6}| \\ &\quad + 0.15\,(0.20^2{+}0.30^2) \cdot \frac{1}{0.05{+}0.01} \\ &\approx 0.60 \cdot 0.865625 \cdot 0.05 + 0.25 \cdot 0.13 \cdot 0.2667 + 0.15 \cdot 0.13 \cdot 16.6667 \\ &\approx 0.02597\ +\ 0.00867\ +\ 0.32500\ =\ 0.3596, \end{aligned}$$

which is lower than the corresponding value for $k = 3$ (equal proximity but higher indeterminacy), thus favoring C_2 for x_2. In practice, optimizing (V, T, I, F) over the whole dataset yields refined clusters that explicitly separate "why" a visit belongs (clinical vs. resource reasons), "where" uncertainty lies (missing vs. conflicting data), and "what" contradicts the assignment (protocol vs. anomaly), improving downstream triage and resource planning.

7.26 Plithogenic Clustering

Plithogenic clustering groups data using attribute contradictions to weight fuzzy/neutrosophic memberships, fusing t-norm/conorm components into contradiction-aware cluster assignments with robustness.

Notation 7.26.1. *Let $X = \{x_i\}_{i=1}^n \subset \mathbb{R}^d$ be a dataset and $K \in \mathbb{N}$ the number of clusters. Fix a* plithogenic setting

$$PS = \big(P,\ v,\ Pv,\ pdf,\ pCF\big),$$

where P is a universe, v is an attribute with value domain Pv, $pdf : P \times Pv \to [0,1]^s$ is a degree of appurtenance (DAF), and $pCF : Pv \times Pv \to [0,1]$ is a degree of contradiction (DCF) with $pCF(a,a) = 0$ and $pCF(a,b) = pCF(b,a)$. We assume a disjoint block partition

$$Pv \ = \ V_T \,\dot\cup\, V_I \,\dot\cup\, V_F, \qquad |V_T| = p,\ |V_I| = q,\ |V_F| = r,$$

intended to refine the neutrosophic channels T, I, F. For a monotone decreasing $g : [0,1] \to [0,1]$ we use the DCF–weights

$$w_k^{\bullet}(a) \ := \ \frac{g\big(pCF(u_k^{\bullet}, a)\big)}{\sum_{a' \in V_\bullet} g\big(pCF(u_k^{\bullet}, a')\big)} \quad \big(\bullet \in \{T, I, F\},\ a \in V_\bullet,\ k = 1, \ldots, K\big),$$

based on anchor values $u_k^{\bullet} \in V_\bullet$ *selected for cluster k (a canonical choice is $g(c) = 1 - c$).*

Definition 7.26.2 (Plithogenic Clustering (PLC))**.** Fix fuzzifier $m > 1$, component weights $w_T, w_I, w_F > 0$ with $w_T + w_I + w_F = 1$, and $\varepsilon > 0$. A PLC model consists of:

- prototypes $V = \{v_k\}_{k=1}^K \subset \mathbb{R}^d$,
- anchors $\{u_k^T \in V_T,\ u_k^I \in V_I,\ u_k^F \in V_F\}_{k=1}^K$,
- refined memberships

$$T = \big(t_{ik}^{(a)}\big)_{i,k,a \in V_T}, \quad I = \big(i_{ik}^{(b)}\big)_{i,k,b \in V_I}, \quad F = \big(f_{ik}^{(c)}\big)_{i,k,c \in V_F},$$

subject to the normalizations, for each i and each refinement index,

$$\sum_{k=1}^K t_{ik}^{(a)} = 1 \ (a \in V_T), \qquad \sum_{k=1}^K i_{ik}^{(b)} = 1 \ (b \in V_I), \qquad \sum_{k=1}^K f_{ik}^{(c)} = 1 \ (c \in V_F),$$

with $0 \le t_{ik}^{(a)}, i_{ik}^{(b)}, f_{ik}^{(c)} \le 1$. Let $d_{ik} := \|x_i - v_k\|_2^2$ and $\bar{d}_i := \frac{1}{K}\sum_{k=1}^K d_{ik}$. The *PLC objective* is

$$\begin{aligned}\mathcal{J}_{\text{PLC}}(V, U; T, I, F) = \sum_{i=1}^n \sum_{k=1}^K \Bigg[& w_T \sum_{a \in V_T} w_k^T(a)\, \big(t_{ik}^{(a)}\big)^m\, d_{ik} \\ & + w_I \sum_{b \in V_I} w_k^I(b)\, \big(i_{ik}^{(b)}\big)^m\, \big|d_{ik} - \bar{d}_i\big| \\ & + w_F \sum_{c \in V_F} w_k^F(c)\, \big(f_{ik}^{(c)}\big)^m\, \frac{1}{d_{ik} + \varepsilon} \Bigg].\end{aligned}$$

Any $(V, U; T, I, F)$ minimizing $\mathcal{J}_{\text{PLC}}$ under the constraints is a *PLC solution*. The *plithogenically fused* (non–refined) degrees for (i,k) are the DCF–weighted averages

$$\widetilde{T}_{ik} := \sum_{a \in V_T} w_k^T(a)\, t_{ik}^{(a)}, \quad \widetilde{I}_{ik} := \sum_{b \in V_I} w_k^I(b)\, i_{ik}^{(b)}, \quad \widetilde{F}_{ik} := \sum_{c \in V_F} w_k^F(c)\, f_{ik}^{(c)}.$$

Remark 7.26.3 (Role of contradiction)**.** The DCF–weights $w_k^{\bullet}(\cdot)$ down–weight refined components that contradict the cluster's anchor value. With $g(c) = 1 - c$ one has $w_k^{\bullet}(u_k^{\bullet}) = \max$ and $w_k^{\bullet}(a)$ decreases as $pCF(u_k^{\bullet}, a)$ increases.

Example 7.26.4 (Tiny PLC demo with contradiction-weighted fusion)**.** Consider two refined truth attributes for clustering, $V_T = \{\text{accuracy}, \text{cost}\}$. Cluster k has an *anchor* $u_k^T \in V_T$, and contradiction degrees $pCF(\text{accuracy}, \text{cost}) = pCF(\text{cost}, \text{accuracy}) = 0.6$, while $pCF(a,a) = 0$. Use the standard DCF weight $w_k^T(a) \propto 1 - pCF(u_k^T, a)$.

Anchors: u_1^T = accuracy, u_2^T = cost. Thus

$$w_1^T(\text{accuracy}) = \frac{1-0}{(1-0)+(1-0.6)} = \frac{5}{7}, \quad w_1^T(\text{cost}) = \frac{1-0.6}{(1-0)+(1-0.6)} = \frac{2}{7},$$

$$w_2^T(\text{cost}) = \frac{5}{7}, \qquad w_2^T(\text{accuracy}) = \frac{2}{7}.$$

A data point x has refined truth-memberships to clusters ($k = 1, 2$):

$$t_{x,1}^{(\text{accuracy})} = 0.80, \; t_{x,1}^{(\text{cost})} = 0.30; \qquad t_{x,2}^{(\text{accuracy})} = 0.50, \; t_{x,2}^{(\text{cost})} = 0.70.$$

Plithogenic (DCF–weighted) fused truths:

$$\widetilde{T}_{x,1} = \tfrac{5}{7}\cdot 0.80 + \tfrac{2}{7}\cdot 0.30 = \tfrac{4.6}{7} \approx 0.6571, \qquad \widetilde{T}_{x,2} = \tfrac{5}{7}\cdot 0.70 + \tfrac{2}{7}\cdot 0.50 = \tfrac{4.5}{7} \approx 0.6429.$$

Let prototype distances be $d_{x,1} = 0.40$, $d_{x,2} = 0.30$ (e.g., squared Euclidean). A simple score (larger is better) is

$$S_k := \frac{\widetilde{T}_{x,k}}{d_{x,k}} \quad \Rightarrow \quad S_1 = \frac{0.6571}{0.40} \approx 1.6429, \qquad S_2 = \frac{0.6429}{0.30} \approx 2.1429.$$

Even though x has slightly higher fused truth for cluster 1, the smaller distance to cluster 2 makes S_2 larger, so x is assigned to cluster 2. This illustrates how contradiction-aware fusion (via pCF) interacts with geometry to yield robust, attribute-sensitive cluster decisions.

Theorem 7.26.5 (Reduction to Neutrosophic Clustering (NC))**.** *Take $p = q = r = 1$, i.e. $V_T = \{\tau\}$, $V_I = \{\iota\}$, $V_F = \{\varphi\}$. Choose anchors $u_k^T = \tau$, $u_k^I = \iota$, $u_k^F = \varphi$ and any g. Then $w_k^T(\tau) = w_k^I(\iota) = w_k^F(\varphi) = 1$, and $\mathcal{J}_{\text{PLC}}$ reduces to*

$$\sum_{i,k}\Big[w_T\,(t_{ik})^m d_{ik} + w_I\,(i_{ik})^m|d_{ik} - \bar{d}_i| + w_F\,(f_{ik})^m/(d_{ik}+\varepsilon)\Big],$$

which is the standard neutrosophic clustering objective with single (T, I, F) per (i, k).

Proof. With singletons V_T, V_I, V_F, the inner sums collapse and all DCF–weights equal 1 by construction; the stated expression follows directly. The membership constraints also reduce to $\sum_k t_{ik} = \sum_k i_{ik} = \sum_k f_{ik} = 1$, which are the usual NC constraints. □

Theorem 7.26.6 (Reduction to Refined Neutrosophic Clustering (RNC))**.** *Fix V_T, V_I, V_F with sizes (p, q, r) as in RNC, and set anchors so that $pCF(u_k^\bullet, a)$ is independent of k and equal to a constant $c_\bullet(a) \in [0, 1)$ for $a \in V_\bullet$. Let $g(c) = 1 - c$ and define*

$$\alpha_a := \frac{1 - c_T(a)}{\sum_{a'\in V_T}\big(1 - c_T(a')\big)}, \quad \beta_b := \frac{1 - c_I(b)}{\sum_{b'\in V_I}\big(1 - c_I(b')\big)}, \quad \gamma_c := \frac{1 - c_F(c)}{\sum_{c'\in V_F}\big(1 - c_F(c')\big)}.$$

Then $w_k^T(a) = \alpha_a$, $w_k^I(b) = \beta_b$, $w_k^F(c) = \gamma_c$ for all k, and $\mathcal{J}_{\text{PLC}}$ becomes exactly

$$\sum_{i,k}\left[w_T \sum_{a\in V_T} \alpha_a\,(t_{ik}^{(a)})^m d_{ik} + w_I \sum_{b\in V_I} \beta_b\,(i_{ik}^{(b)})^m\big|d_{ik} - \bar{d}_i\big| + w_F \sum_{c\in V_F} \gamma_c\,(f_{ik}^{(c)})^m \frac{1}{d_{ik}+\varepsilon}\right],$$

which is the RNC objective with fixed convex refinement weights (α_a), (β_b), (γ_c).

Proof. Under the hypothesis, $w_k^\bullet(\cdot)$ does not depend on k and equals the stated convex weights by normalization, whence termwise identity with the RNC objective. Constraints are identical in PLC and RNC. □

Definition 7.26.7 (Clusterwise plithogenic membership)**.** For fixed cluster k, define the map

$$pdf_k : X \times Pv \longrightarrow [0, 1]^3, \qquad pdf_k\big(x_i; a\big) := \Big(\,t_{ik}^{(a)},\; i_{ik}^{(a)},\; f_{ik}^{(a)}\,\Big),$$

where on each block we understand $i_{ik}^{(a)} \equiv 0$ for $a \in V_T$, etc., or we keep the natural triplet when each entry carries all three channels. For $a, b \in Pv$ set the plithogenic fusion

$$\mathcal{A}_{pCF(a,b)}\big(pdf_k(x_i; a),\, pdf_k(x_i; b)\big) := \;(1 - c)\,T\big(\cdot,\cdot\big) \;+\; c\,S\big(\cdot,\cdot\big), \quad c := pCF(a, b),$$

where T, S are a fixed t-norm/t-conorm acting componentwise on $[0, 1]^3$.

Theorem 7.26.8 (PLC induces a plithogenic set)**.** *For each cluster k, the pair $\big(P := X,\ v,\ Pv,\ pdf_k,\ pCF\big)$ is a plithogenic set in the sense that: (i) $pdf_k(x_i;\cdot)$ maps Pv into $[0,1]^3$; (ii) pCF is reflexive–zero and symmetric; (iii) for all $a, b \in Pv$ and $x_i \in X$, the fused evaluation $\mathcal{A}_{pCF(a,b)}\big(pdf_k(x_i;a), pdf_k(x_i;b)\big)$ is again in $[0,1]^3$ and varies monotonically with $c = pCF(a,b)$ between the t-norm and t-conorm outcomes.*

Proof. (i) and (ii) hold by construction. For (iii), T and S map $[0,1]^3 \times [0,1]^3 \to [0,1]^3$ and are monotone. Convex mixing $(1-c)T + cS$ remains in $[0,1]^3$, is continuous and monotone in $c \in [0,1]$, and yields $\mathcal{A}_0 = T$, $\mathcal{A}_1 = S$, as required by the plithogenic fusion principle. □

7.27 Neutrosophic Transdisciplinarity

Transdisciplinarity integrates knowledge across disciplines and stakeholders, creating shared frameworks that transcend boundaries to address complex real-world problems and uncertainty [463–467]. Neutrosophic transdisciplinarity integrates multi-disciplinary truth, indeterminacy, and falsity assessments, aggregating heterogeneous evidence to support decisions under uncertainty and conflict contexts(cf. [468–470]).

Definition 7.27.1 (Neutrosophic transdisciplinary system)**.** [468] Let $\mathcal{D}$ be a finite set of disciplines and $\mathcal{S}$ a set of propositions. A *neutrosophic transdisciplinary system* is a tuple

$$\mathbf{T} = (\mathcal{D}, \mathcal{S}, \{\mathrm{Val}_d\}_{d\in\mathcal{D}},\ \otimes_T, \otimes_I, \otimes_F),$$

where each discipline d provides a valuation $\mathrm{Val}_d : \mathcal{S} \to [0,1]^3$, $\mathrm{Val}_d(\sigma) = (T_d(\sigma), I_d(\sigma), F_d(\sigma))$, and $(\otimes_T, \otimes_I, \otimes_F)$ are associative/commutative aggregators (e.g. a t-norm on T, a t-conorm on I and F) producing an integrated valuation

$$\mathrm{Val}_{\mathrm{trans}}(\sigma) \ :=\ \bigotimes_{d\in\mathcal{D}} \mathrm{Val}_d(\sigma) \ =\ \left(\bigotimes_T T_d(\sigma),\ \bigotimes_I I_d(\sigma),\ \bigotimes_F F_d(\sigma)\right).$$

The triple $\mathrm{Val}_{\mathrm{trans}}(\sigma)$ quantifies, respectively, cross-disciplinary support, indeterminacy, and opposition to σ, enabling principled integration across levels and knowledge domains.

Example 7.27.2 (Cross-disciplinary pandemic policy integration)**.** Let the disciplines be

$$\mathcal{D} = \{\text{Medicine},\ \text{Economics},\ \text{Ethics}\}.$$

Consider the proposition σ: "Prioritize boosters for immunocompromised adults now."

Each discipline $d \in \mathcal{D}$ provides a neutrosophic valuation $\mathrm{Val}_d(\sigma) = (T_d, I_d, F_d) \in [0,1]^3$:

$\mathrm{Val}_{\mathrm{Medicine}}(\sigma) = (0.92,\ 0.06,\ 0.02)$, $\mathrm{Val}_{\mathrm{Economics}}(\sigma) = (0.75,\ 0.18,\ 0.20)$, $\mathrm{Val}_{\mathrm{Ethics}}(\sigma) = (0.88,\ 0.12,\ 0.05)$.

Aggregate across disciplines by

$$T_{\mathrm{trans}} = \min_d T_d, \qquad I_{\mathrm{trans}} = \max_d I_d, \qquad F_{\mathrm{trans}} = \max_d F_d,$$

which corresponds to a cautious t-norm for support and t-conorms for indeterminacy/opposition.

Concrete computation:

$$\begin{aligned} T_{\mathrm{trans}} &= \min\{0.92,\ 0.75,\ 0.88\} = 0.75,\\ I_{\mathrm{trans}} &= \max\{0.06,\ 0.18,\ 0.12\} = 0.18,\\ F_{\mathrm{trans}} &= \max\{0.02,\ 0.20,\ 0.05\} = 0.20. \end{aligned}$$

Decision rule (one option): endorse if $T_{\mathrm{trans}} - F_{\mathrm{trans}} \geq \tau$ with $\tau = 0.5$. Here $0.75 - 0.20 = 0.55 \geq 0.5$, so the integrated transdisciplinary stance is "approve now," while acknowledging non-negligible economic opposition ($F = 0.20$) and uncertainty ($I = 0.18$).

7.28 Neutrosophic Macroeconomics Variable

Macroeconomics studies aggregate behavior of economies, analyzing growth, inflation, unemployment, fiscal policy, monetary policy, and international economic interactions over time [471–473]. A neutrosophic macroeconomics variable records time-indexed degrees of stability, indeterminacy, and instability, optionally set-valued, supporting uncertain equilibria and policy analysis [39].

Definition 7.28.1 (Neutrosophic Macroeconomics Variable). [39] Let $\mathcal{T} \subseteq \mathbb{R}$ be a time axis. A *neutrosophic macroeconomics variable* is a mapping

$$\mathsf{M} : \mathcal{T} \longrightarrow \mathcal{P}([0,1])^3, \qquad t \longmapsto \big(S(t),\, I(t),\, U(t)\big),$$

where $S(t), I(t), U(t) \subseteq [0,1]$ denote, respectively, the (set-valued) degrees of stability/equilibrium, indeterminacy (unclear whether equilibrium or disequilibrium), and instability/disequilibrium at time t. A single-valued specialization uses measurable functions $(s,u,i) : \mathcal{T} \to [0,1]^3$ such that, depending on the modelled dependence among components,

$$0 \le s(t)+i(t)+u(t) \le 1 \quad \text{(fully dependent)}, \qquad 0 \le s(t)+i(t)+u(t) \le 2 \quad \text{(partially independent)},$$

$$0 \le s(t)+i(t)+u(t) \le 3 \quad \text{(pairwise independent)}.$$

Intervals such as $S(t) = [0.8, 1]$ encode "near-to-equilibrium" states; analogous intervals can be used for $U(t)$ to express "near-to-disequilibrium".

Example 7.28.2 (Quarterly inflation–employment stability index). Let the time axis be quarters

$$t \in \{2025\text{Q1}, 2025\text{Q2}, 2025\text{Q3}, 2025\text{Q4}\}$$

. Define the single–valued specialization

$$\mathsf{M}(t) = (s(t),\, i(t),\, u(t)) \in [0,1]^3, \qquad s(t)+i(t)+u(t) \le 1,$$

where s measures closeness to macro–equilibrium (inflation near target and unemployment near natural rate), u measures disequilibrium pressure (e.g., stagflation or overheating), and i captures data/model indeterminacy.

Concrete quarterly assessments:

Quarter t	$s(t)$	$i(t)$	$u(t)$	$s(t)+i(t)+u(t)$
2025Q1	0.72	0.18	0.10	1.00
2025Q2	0.58	0.27	0.15	1.00
2025Q3	0.40	0.25	0.35	1.00
2025Q4	0.65	0.20	0.15	1.00

Interpretation. In 2025Q3 the higher disequilibrium $u(2025\text{Q3}) = 0.35$ reflects a widening output gap and sticky core inflation, while indeterminacy i remains moderate due to noisy labor participation data. By 2025Q4, stabilization policies raise s to 0.65 and reduce u to 0.15, indicating improved macro–balance with still moderate uncertainty.

7.29 Neutrosophic Sports Game

Applications in sports have also been investigated using Neutrosophic Sets [474–477]. A Neutrosophic Sports Game models teams, neutral officials, and events with truth, indeterminacy, and falsity degrees to represent rule uncertainty [39].

Definition 7.29.1 (Neutrosophic Sports Game). [39] A *neutrosophic sports game* is a tuple

$$\mathsf{G} = \big(A,\ \text{anti}\,A,\ \text{neut}\,A,\ \mathcal{E},\ \nu\big),$$

where A and anti A are the two opposing teams, neut A represents the neutral authority (e.g. referees), $\mathcal{E}$ is a set of event/proposition labels (e.g. "A wins", "rule r is satisfied"), and

$$\nu : \mathcal{E} \longrightarrow [0,1]^3, \qquad e \longmapsto \big(\mathrm{T}(e),\, \mathrm{I}(e),\, \mathrm{F}(e)\big)$$

assigns to each event e a neutrosophic valuation, where $\mathrm{T}(e)$ is the degree supporting A's side of e, $\mathrm{F}(e)$ the degree supporting anti A's side, and $\mathrm{I}(e)$ the neutral/indeterminate degree (uncertainty or referee-mediated neutrality). Admissible triples satisfy $0 \le \mathrm{T}(e)+\mathrm{I}(e)+\mathrm{F}(e) \le 1, 2,$ or 3 according to the chosen dependence policy.

Example 7.29.2 (Soccer match with VAR)**.** VAR is a video assistant referee system that reviews match incidents to correct clear errors in goals, penalties, red cards (cf. [478]).

Let A = Falcons, B = Tigers, N = Referees. Let $\mathcal{E} = \{\mathsf{Goal\_A}, \mathsf{Offside\_A}, \mathsf{A\_wins}\}$. Define the game tuple $\mathsf{G} = (A, B, N, \mathcal{E}, \nu)$ with neutrosophic valuations

$$\nu(\mathsf{Goal\_A}) = (0.82,\ 0.12,\ 0.06), \quad \nu(\mathsf{Offside\_A}) = (0.10,\ 0.15,\ 0.75), \quad \nu(\mathsf{A\_wins}) = (0.65,\ 0.20,\ 0.15).$$

A simple decision score for team A is

$$S_A := T(\mathsf{Goal\_A}) + T(\mathsf{A\_wins}) - F(\mathsf{Offside\_A}) = 0.82 + 0.65 - 0.75 = 0.72 > 0.$$

Interpretation. VAR evidence reduces indeterminacy for the goal ($I = 0.12$); the pro-A truth mass outweighs the anti-A falsity, so A is favored.

Chapter 8

Other Science

In this chapter, we examine concepts that are not directly related to Neutrosophic Sets.

8.1 Lucky Calculation

A Lucky Calculation is an incorrect method that, despite invalid steps, coincidentally yields the exact correct answer through error cancellation [42].

Definition 8.1.1 (Lucky Calculation)**.** [42] Let $\mathcal{P}$ be a mathematical (or scientific) problem with correct value $a^\star$, and let C be a concrete calculation (method, algorithm, procedure) that is *not* valid for $\mathcal{P}$, i.e. C uses a wrong step, rule, or formula. If nevertheless

$$C(\mathcal{P}) = a^\star,$$

then C is called a *Lucky Calculation* (or Lucky Method/Algorithm/Operation). Equivalently: a Lucky Calculation is a formally incorrect or funny/similar–to–correct computation that, by chance or internal compensation of errors, outputs the correct result.

Example 8.1.2 (Anomalous digit cancellation)**.** Consider

$$\frac{16}{64}.$$

An incorrect step "cancels the common digit 6":

$$\frac{16}{64} \overset{\text{wrong}}{\Longrightarrow} \frac{1\,\not{6}}{\not{6}\,4} = \frac{1}{4}.$$

This is invalid because digits are not algebraic factors. Nevertheless the outcome equals the true value by coincidence:

$$\frac{16}{64} = \frac{16}{16 \cdot 4} = \frac{1}{4} \quad \Rightarrow \quad 0.25 = 0.25.$$

Thus the faulty procedure returns the exact answer due to hidden error cancellation, hence a *Lucky Calculation*.

8.2 Continuous Linguistic Set

A Continuous Linguistic Set is a continuum of blended linguistic terms formed by convex combinations, interpolating discrete labels into semantics [49]. Various other linguistic concepts have also been explored [479–483].

Definition 8.2.1 (Continuous Linguistic Set)**.** [49] Let $L = \{l_0, l_1, \ldots, l_n\}$ be an ordered discrete linguistic term set, where l_0 is the lowest and l_n is the highest term. A continuous linguistic set generated by L is a family

$$\mathcal{L}_c(L) = \left\{ \ell(p_1, \ldots, p_n) \;\middle|\; p_i \in [0,1], \sum_{i=1}^{n} p_i = 1 \right\},$$

whose generic element $\ell(p_1, \ldots, p_n)$ is interpreted as "p_1 part of l_0 and $(1-p_1)$ part of l_1", or more generally a convex linguistic blend of two consecutive terms. In the simplest two–term case $L = \{l_0, l_1\}$ we write

$$[l_0,\ p\, l_0 + (1-p)\, l_1,\ l_1], \qquad p \in (0,1),$$

which continuously fills in all intermediate linguistic meanings between l_0 and l_1. Thus a continuous linguistic set is the closure of a discrete linguistic set under all such convex linguistic interpolations.

Example 8.2.2 (Blending discrete terms into a continuum)**.** Let the ordered linguistic term set be

$$L = \{\text{Cold},\ \text{Warm},\ \text{Hot}\},$$

with numeric anchors $\varphi(\text{Cold}) = 0$, $\varphi(\text{Warm}) = \frac{1}{2}$, $\varphi(\text{Hot}) = 1$. A continuous linguistic element is any convex blend

$$\ell(w_{\mathrm{C}}, w_{\mathrm{W}}, w_{\mathrm{H}}) = w_{\mathrm{C}} \cdot \text{Cold} + w_{\mathrm{W}} \cdot \text{Warm} + w_{\mathrm{H}} \cdot \text{Hot}, \quad w_{\mathrm{C}}, w_{\mathrm{W}}, w_{\mathrm{H}} \in [0,1],\ \sum w = 1.$$

Choose $(w_{\mathrm{C}}, w_{\mathrm{W}}, w_{\mathrm{H}}) = (0.10, 0.20, 0.70)$. The semantic score (for interpretation/thresholding) is the convex image

$$s(\ell) = 0.10 \cdot 0 \ +\ 0.20 \cdot \tfrac{1}{2}\ +\ 0.70 \cdot 1 =\ 0.80,$$

which we may label "mostly Hot". Varying $(w_{\mathrm{C}}, w_{\mathrm{W}}, w_{\mathrm{H}})$ continuously fills all intermediate meanings between the discrete labels.

8.3 IndetermSoft Set, IndetermHyperSoft Set, TreeSoft Set, and ForestSoft Set

An N-IndetermSoft Set assigns each object a single N-level grade per parameter, allowing an indeterminate symbol when information is missing. An N-IndetermHyperSoft Set grades objects for multi-argument parameter tuples, returning one N-level value or an indeterminate mark for incomplete cases. An N-TreeSoft Set assigns exactly one N-level grade to each object at every node of a hierarchical parameter tree structure. An N-ForestSoft Set extends tree-based grading to a disjoint union of attribute trees, evaluating each object consistently across parallel hierarchies. Each of these represents a generalization of the IndetermSoft Set, IndetermHyperSoft Set, TreeSoft Set, and ForestSoft Set through the framework of an N-Soft Set. As a related concept, the N-HyperSoft Set is also known in the literature [484]. The related concepts and their formal definitions are presented below.

Definition 8.3.1 (N-Soft Set)**.** [485–487] Let Ω be a nonempty universe, E a set of parameters, and $\zeta \subseteq E$ a nonempty subset. Let $R = \{0, 1, \ldots, N-1\}$ be a finite ordered grade set with $N \in \{2, 3, \ldots\}$. An *N-Soft Set* on Ω w.r.t. ζ and R is a triple

$$(\nabla, \zeta, N),$$

where

$$\nabla : \zeta \longrightarrow \mathcal{P}(\Omega \times R)$$

satisfies the single–grade condition

$$\forall \varepsilon \in \zeta\ \forall \omega \in \Omega\ \ \exists!\, r_\varepsilon(\omega) \in R :\ (\omega, r_\varepsilon(\omega)) \in \nabla(\varepsilon).$$

Equivalently, one may write

$$\nabla(\varepsilon)(\omega) = r_\varepsilon(\omega) \in R,$$

so that every object ω receives exactly one ordered grade for every active parameter ε.

Definition 8.3.2 (IndetermSoft Set). [488–491] Let U be a universe of discourse, $H \subseteq U$ a nonempty subset, and let A be a set of attribute values. A mapping

$$F : A \longrightarrow \mathcal{P}(H)$$

is called an *IndetermSoft Set* if at least one of the following holds:

1. the attribute domain A contains indeterminate/uncertain values;
2. the codomain $\mathcal{P}(H)$ contains indeterminate/partially known subsets;

3. there exists $a \in A$ such that $F(a)$ itself is indeterminate (not uniquely specified);

4. any two or all three of the above.

We write (F, A) for such a structure.

Definition 8.3.3 (IndetermHyperSoft Set)**.** [492–494] Let U be a universe of discourse, $H \subseteq U$ a nonempty subset, and let $a_1, \dots, a_n$ ($n \geq 1$) be pairwise distinct attributes with value-sets $A_1, \dots, A_n$ such that $A_i \cap A_j = \emptyset$ for $i \neq j$. A pair

$$\big(F,\, A_1 \times \cdots \times A_n\big), \qquad F : A_1 \times \cdots \times A_n \longrightarrow \mathcal{P}(H),$$

is an *IndetermHyperSoft Set* if (at least) one of the following holds:

1. some A_i is itself indeterminate or incompletely specified;

2. $\mathcal{P}(H)$ contains indeterminate/partial subsets;

3. for some $(a_1, \dots, a_n)$ the value $F(a_1, \dots, a_n)$ is indeterminate.

Definition 8.3.4 (TreeSoft Set)**.** [495–498] Let U be a universe, $H \subseteq U$ a nonempty subset, and let $A = \{A_1, \dots, A_n\}$ be a (possibly multi-level) hierarchical family of attributes whose nodes form a tree, denoted $\mathrm{Tree}(A)$. A *TreeSoft Set* is a mapping

$$F : \mathcal{P}(\mathrm{Tree}(A)) \longrightarrow \mathcal{P}(H),$$

which assigns to every subset of tree-nodes (attributes, subattributes, leaves) a subset of H.

Definition 8.3.5 (ForestSoft Set)**.** [499–501] Let $\{\, \mathrm{Tree}(A(t)) : t \in T \,\}$ be a finite (or countable) family of attribute trees and let

$$\mathrm{Forest}(\{A(t)\}_{t\in T}) \;=\; \bigsqcup_{t\in T} \mathrm{Tree}(A(t))$$

be their disjoint union. A *ForestSoft Set* is a mapping

$$F : \mathcal{P}\big(\mathrm{Forest}(\{A(t)\}_{t\in T})\big) \longrightarrow \mathcal{P}(H)$$

such that, for every $X \subseteq \mathrm{Forest}(\{A(t)\}_{t\in T})$,

$$F(X) \;=\; \bigcup_{\substack{t\in T \\ X\cap \mathrm{Tree}(A(t))\neq\emptyset}} F_t\big(X \cap \mathrm{Tree}(A(t))\big),$$

where each $F_t : \mathcal{P}(\mathrm{Tree}(A(t))) \to \mathcal{P}(H)$ is a TreeSoft Set on the t-th tree.

Definition 8.3.6 (N-IndetermSoft Set)**.** Let Ω, E, ζ, R be as above and let $\perp$ be a distinguished symbol meaning "indeterminate / not-available". An *N-IndetermSoft Set* on Ω is a triple

$$(\nabla^{\mathrm{ind}}, \zeta, N),$$

where

$$\nabla^{\mathrm{ind}} : \zeta \longrightarrow \mathcal{P}\big(\Omega \times (R \cup \{\perp\})\big)$$

satisfies the at-most-one condition

$$\forall \varepsilon \in \zeta \;\forall \omega \in \Omega : \; \#\big\{x \in R \cup \{\perp\} : (\omega, x) \in \nabla^{\mathrm{ind}}(\varepsilon)\big\} \leq 1.$$

If $(\omega, \perp) \in \nabla^{\mathrm{ind}}(\varepsilon)$, then the value of ω under parameter ε is present but indeterminate; if $(\omega, r) \in \nabla^{\mathrm{ind}}(\varepsilon)$ with $r \in R$, we recover the ordinary graded assignment.

Example 8.3.7 (N-IndetermSoft Set — Public Library Acquisition with Missing Vendor Data)**.** Public library acquisition selects, budgets, and evaluates new materials for collections, balancing community needs, diversity, relevance, and long-term value effectively (cf. [502]). A city library evaluates three new titles for acquisition, but vendor reliability is missing for some titles due to delayed reports.

Universe and grades:

$$\Omega = \{\text{BookA}, \text{BookB}, \text{BookC}\}, \quad R = \{0, 1, 2\}\ (0 = \text{low},\ 1 = \text{medium},\ 2 = \text{high}).$$

Active parameters $\zeta = \{\text{Urgency}, \text{BudgetFit}, \text{VendorReliability}\}$.

N-IndetermSoft assignment $\nabla^{\text{ind}} : \zeta \to \mathcal{P}(\Omega \times (R \cup \{\perp\}))$:

$$\begin{aligned} \nabla^{\text{ind}}(\text{Urgency}) &= \{(\text{BookA}, 2), (\text{BookB}, 1), (\text{BookC}, 0)\}, \\ \nabla^{\text{ind}}(\text{BudgetFit}) &= \{(\text{BookA}, 1), (\text{BookB}, 2), (\text{BookC}, 1)\}, \\ \nabla^{\text{ind}}(\text{VendorReliability}) &= \{(\text{BookB}, 2), (\text{BookA}, \perp), (\text{BookC}, \perp)\}. \end{aligned}$$

Each (ω, x) pair appears at most once per parameter; $\perp$ flags "present but indeterminate" information (awaiting vendor checks). Decisions can proceed on known grades while tracking the indeterminate entries for later update.

Definition 8.3.8 (N-IndetermHyperSoft Set)**.** Let Ω be a universe. Let $E_1, \ldots, E_k$ be pairwise disjoint parameter–classes and let $F_i \subseteq E_i$ be nonempty subparameter sets. Put

$$\Xi \ := \ F_1 \times \cdots \times F_k, \qquad R = \{0, 1, \ldots, N-1\}.$$

An *N-IndetermHyperSoft Set* on Ω is a triple

$$(\nabla^{\text{ih}}, \Xi, N),$$

where

$$\nabla^{\text{ih}} : \Xi \longrightarrow \mathcal{P}\big(\Omega \times (R \cup \{\perp\})\big),$$

and for every $q = (f_1, \ldots, f_k) \in \Xi$ and $\omega \in \Omega$ we have

$$\#\big\{x \in R \cup \{\perp\} : (\omega, x) \in \nabla^{\text{ih}}(q)\big\} \leq 1.$$

Thus each multiargument parameter q may grade an object by an N–level value or mark it as indeterminate. For $k = 1$ and no $\perp$ we return to an N-Soft Set.

Example 8.3.9 (N-IndetermHyperSoft Set — Affordable Housing Allocation with Multi-Attribute Intake)**.** Affordable housing allocation distributes limited subsidized units to eligible households fairly, using criteria like income, family size, vulnerability, and location (cf. [503]). A municipal housing office screens applicant households using three parameter classes with tupled queries; some dossiers are incomplete.

Universe and grades:

$$\Omega = \{\text{H1}, \text{H2}, \text{H3}\}, \quad R = \{0, 1, 2\}\ (0 = \text{reject},\ 1 = \text{waitlist},\ 2 = \text{priority}).$$

Parameter classes (pairwise disjoint):

$$E_1 = \{\text{Single}, \text{Family}\},\ E_2 = \{\text{Low}, \text{Mid}\},\ E_3 = \{\text{None}, \text{Registered}\}.$$

Let $F_1 = \{\text{Single}, \text{Family}\}$, $F_2 = \{\text{Low}, \text{Mid}\}$, $F_3 = \{\text{None}, \text{Registered}\}$, and $\Xi = F_1 \times F_2 \times F_3$.

N-IndetermHyperSoft assignment $\nabla^{\text{ih}} : \Xi \to \mathcal{P}(\Omega \times (R \cup \{\perp\}))$ (sample entries):

$$\begin{aligned} \nabla^{\text{ih}}(\text{Single}, \text{Low}, \text{Registered}) &= \{(\text{H1}, 2), (\text{H3}, \perp)\}, \\ \nabla^{\text{ih}}(\text{Family}, \text{Mid}, \text{None}) &= \{(\text{H2}, 1)\}, \\ \nabla^{\text{ih}}(\text{Family}, \text{Low}, \text{Registered}) &= \{(\text{H2}, 2), (\text{H3}, 1)\}. \end{aligned}$$

For each tuple $q \in \Xi$ and each household $\omega \in \Omega$ there is *at most one* grade, which may be $\perp$ if documents are incomplete. This multi-argument view supports consistent, explainable intake decisions across heterogeneous criteria.

Definition 8.3.10 (N-TreeSoft Set)**.** Let T be a finite rooted tree whose nodes are (possibly hierarchical) parameters; write $\mathrm{Node}(T)$ for the set of nodes. Fix a universe Ω and an ordered grade set $R = \{0, 1, \ldots, N-1\}$. An *N-TreeSoft Set* on Ω is a pair

$$(\nabla^{\mathrm{tr}}, T),$$

where

$$\nabla^{\mathrm{tr}} : \mathrm{Node}(T) \longrightarrow \mathcal{P}(\Omega \times R)$$

and for every node $a \in \mathrm{Node}(T)$ and $\omega \in \Omega$ there is exactly one grade $r_a(\omega) \in R$ such that $(\omega, r_a(\omega)) \in \nabla^{\mathrm{tr}}(a)$. If T degenerates into a one–level star (root + flat children all treated as parameters), then $(\nabla^{\mathrm{tr}}, T)$ is exactly an N-Soft Set.

Example 8.3.11 (N-TreeSoft Set — Hiring Evaluation with Hierarchical Criteria)**.** A hiring committee evaluates candidates using a criterion tree.

Universe and grades:

$$\Omega = \{\mathrm{CandA}, \mathrm{CandB}\}, \quad R = \{0, 1, 2\} \ (0 = \text{weak},\ 1 = \text{adequate},\ 2 = \text{strong}).$$

Parameter tree T (root "OverallFit"; children "Skills", "Experience", "Culture"; "Skills" has leaves "Programming", "Statistics"):

$$\mathrm{Node}(T) = \{\mathrm{OverallFit}, \mathrm{Skills}, \mathrm{Programming}, \mathrm{Statistics}, \mathrm{Experience}, \mathrm{Culture}\}.$$

N-TreeSoft assignment $\nabla^{\mathrm{tr}} : \mathrm{Node}(T) \to \mathcal{P}(\Omega \times R)$:

$$\begin{aligned}
\nabla^{\mathrm{tr}}(\mathrm{Programming}) &= \{(\mathrm{CandA}, 2), (\mathrm{CandB}, 1)\},\\
\nabla^{\mathrm{tr}}(\mathrm{Statistics}) &= \{(\mathrm{CandA}, 1), (\mathrm{CandB}, 2)\},\\
\nabla^{\mathrm{tr}}(\mathrm{Skills}) &= \{(\mathrm{CandA}, 2), (\mathrm{CandB}, 2)\},\\
\nabla^{\mathrm{tr}}(\mathrm{Experience}) &= \{(\mathrm{CandA}, 1), (\mathrm{CandB}, 2)\},\\
\nabla^{\mathrm{tr}}(\mathrm{Culture}) &= \{(\mathrm{CandA}, 2), (\mathrm{CandB}, 1)\},\\
\nabla^{\mathrm{tr}}(\mathrm{OverallFit}) &= \{(\mathrm{CandA}, 2), (\mathrm{CandB}, 2)\}.
\end{aligned}$$

Each node assigns exactly one grade per candidate, enabling transparent aggregation at different levels (leaf, subcategory, overall).

Definition 8.3.12 (N-ForestSoft Set)**.** Let $\{T_j\}_{j=1}^m$ be a finite family of disjoint attribute trees and put

$$\mathcal{F} := \bigsqcup_{j=1}^{m} \mathrm{Node}(T_j).$$

An *N-ForestSoft Set* on Ω with grade set $R = \{0, 1, \ldots, N-1\}$ is a pair

$$(\nabla^{\mathrm{fo}}, \{T_j\}_{j=1}^m),$$

where

$$\nabla^{\mathrm{fo}} : \mathcal{F} \longrightarrow \mathcal{P}(\Omega \times R)$$

and, for every $x \in \mathcal{F}$ and $\omega \in \Omega$, there exists exactly one $r_x(\omega) \in R$ with $(\omega, r_x(\omega)) \in \nabla^{\mathrm{fo}}(x)$. When $m = 1$ we get an N-TreeSoft Set; when every T_j has only one level, we get an ordinary N-Soft Set.

Example 8.3.13 (N-ForestSoft Set — City Project Funding with Parallel Policy Trees)**.** Project funding allocates financial resources to specific initiatives, evaluating feasibility, impact, risk, and alignment with organizational or governmental priorities effectively (cf. [504]). A city council ranks infrastructure projects using two independent policy trees: Economic Impact and Social Equity. The overall attribute space is their disjoint union (a forest).

Universe and grades:

$$\Omega = \{\mathrm{ProjX}, \mathrm{ProjY}\}, \quad R = \{0, 1, 2\} \ (0 = \text{low},\ 1 = \text{moderate},\ 2 = \text{high}).$$

Trees:

$$T_1 = \text{EconomicImpact tree with nodes } \{\text{EconOverall}, \text{LocalGDP}, \text{JobCreation}\},$$
$$T_2 = \text{SocialEquity tree with nodes } \{\text{EquityOverall}, \text{Accessibility}, \text{DisadvantagedAreas}\}.$$

Forest node set $\mathcal{F} = \text{Node}(T_1) \sqcup \text{Node}(T_2)$.

N-ForestSoft assignment $\nabla^{\text{fo}} : \mathcal{F} \to \mathcal{P}(\Omega \times R)$:

$$\begin{aligned}
\nabla^{\text{fo}}(\text{LocalGDP}) &= \{(\text{ProjX}, 2), (\text{ProjY}, 1)\},\\
\nabla^{\text{fo}}(\text{JobCreation}) &= \{(\text{ProjX}, 1), (\text{ProjY}, 2)\},\\
\nabla^{\text{fo}}(\text{EconOverall}) &= \{(\text{ProjX}, 2), (\text{ProjY}, 2)\},\\
\nabla^{\text{fo}}(\text{Accessibility}) &= \{(\text{ProjX}, 1), (\text{ProjY}, 2)\},\\
\nabla^{\text{fo}}(\text{DisadvantagedAreas}) &= \{(\text{ProjX}, 2), (\text{ProjY}, 1)\},\\
\nabla^{\text{fo}}(\text{EquityOverall}) &= \{(\text{ProjX}, 2), (\text{ProjY}, 2)\}.
\end{aligned}$$

Each forest node carries exactly one grade per project. Policymakers can reason within each tree (domain-specific priorities) and then synthesize across the forest for a balanced funding decision.

Theorem 8.3.14 (Four N-extensions generalize N-Soft Sets)**.** *Every N-Soft Set* (∇, ζ, N) *can be seen as*

1. *an N-IndetermSoft Set with no occurrence of* $\perp$*;*
2. *an N-IndetermHyperSoft Set with* $k = 1$ *and no occurrence of* $\perp$*;*
3. *an N-TreeSoft Set whose tree has a single level consisting of the parameters in* ζ*;*
4. *an N-ForestSoft Set whose forest consists of a single one-level tree.*

Proof. (1) Given (∇, ζ, N), define

$$\nabla^{\text{ind}}(\varepsilon) := \nabla(\varepsilon) \subseteq \Omega \times R \subseteq \Omega \times (R \cup \{\perp\}),$$

so no pair uses $\perp$. All conditions of the N-IndetermSoft Set are satisfied, hence (∇, ζ, N) is a special case.

(2) Take $k = 1$, $F_1 = \zeta$, and $\Xi = F_1 = \zeta$. Define $\nabla^{\text{ih}}(q) := \nabla(q)$ for $q \in \zeta$, and do not use $\perp$. Then the definition of N-IndetermHyperSoft Set reduces exactly to that of N-Soft Set.

(3) Build a tree T with one root and $|\zeta|$ children, each child identified with one parameter $\varepsilon \in \zeta$. Put

$$\nabla^{\text{tr}}(\varepsilon) := \nabla(\varepsilon).$$

Since every node now has exactly one grade for each $\omega \in \Omega$, the condition for N-TreeSoft Set holds, and the structure is identical to (∇, ζ, N).

(4) Make a forest with a single tree T constructed as in (3), and define ∇^{fo} on the nodes of T by $\nabla^{\text{fo}} := \nabla^{\text{tr}}$. Then an N-ForestSoft Set specializes to (∇, ζ, N). Hence in all four cases N-Soft Sets sit inside the corresponding N–extensions as particular, strictly more determinate, non-hierarchical, non-multiargument instances. □

8.4 Subindeterminacy

Subindeterminacy denotes a distinguished symbol representing uncertainty contributed by a specific source, component, or context within overall indeterminacy degree measure [38, 505, 506].

Definition 8.4.1 (Subindeterminacy)**.** [38] Let R be a commutative ring with identity and let $Z(R)$ denote the set of zero divisors of R. Define a family $\{I_s : s \in Z(R) \cup \{0\}\}$ of formal *indeterminacy symbols*. The *neutrosophic extension* of R is

$$R_I := R \cup \{ I_s : s \in Z(R) \cup \{0\} \}.$$

Each I_s is called a *subindeterminacy* and abstracts the indeterminacy generated by expressions of the form r/s (with $s = 0$ or a zero divisor) in R; distinct s induce distinct subindeterminacies I_s. For example, in $\mathbb{Z}_n$ every zero divisor s yields a symbol I_s and $s = 0$ yields I_0.

Example 8.4.2 (Zero–divisor–induced subindeterminacy in $\mathbb{Z}_{12}$)**.** Let $R = \mathbb{Z}_{12}$. The nonzero zero divisors are $\{2, 3, 4, 6, 8, 9, 10\}$. Introduce a subindeterminacy symbol I_6 to mark uncertainty arising from (illegitimate) division by 6.

Consider the congruence

$$6x \equiv 6 \pmod{12}.$$

Naively "dividing by 6" would claim $x \equiv 1$, but 6 is a zero divisor, so cancellation is invalid. Solving correctly,

$$6(x-1) \equiv 0 \pmod{12} \implies x-1 \equiv 0 \pmod{2} \implies x \in \{1, 3, 5, 7, 9, 11\}.$$

Assess the specific claim "$x \equiv 1$" neutrosophically by attributing the residual multiplicity to the subindeterminacy generated by 6:

$$\text{Truth } T = \frac{1}{6}, \qquad \text{Indeterminacy } I = \frac{5}{6} \text{ (tagged as } I_6), \qquad \text{Falsity } F = 0.$$

Here, T counts the one valid value among six solutions; I captures the ambiguity caused precisely by the zero–divisor 6, recorded as the subindeterminacy I_6.

8.5 Discrete Indeterminate Set

A Discrete Indeterminate Set collects formal numbers whose indeterminacy takes finitely many values, realized as corresponding finite value sets precisely [45].

Definition 8.5.1 (Discrete Indeterminate Set)**.** [45] Fix a base field/ring $\mathbb{K}$ and a finite nonempty set $I_{\text{disc}} \subset \mathbb{K}$ of discrete indeterminacy values. A *discrete indeterminate (neutrosophic) number* is any formal pair

$$N = a + b\, I_{\text{disc}} \quad (a, b \in \mathbb{K}),$$

which represents the realized *discrete* value set

$$\text{Realize}(N) := \{\, a + b\,\iota \mid \iota \in I_{\text{disc}} \,\}.$$

A *Discrete Indeterminate Set* is any collection $\mathcal{S} \subseteq \{\, a + b\, I_{\text{disc}} \,\}$, equipped with the evaluation map Realize : $\mathcal{S} \to \mathcal{P}(\mathbb{K})$ above.

Example 8.5.2 (Finite discrete indeterminacy over $\mathbb{Z}$)**.** Let $\mathbb{K} = \mathbb{Z}$ and $I_{\text{disc}} = \{-1, 0, 1\}$. Consider the collection

$$\mathcal{S} = \Big\{ 2 + 1 \cdot I_{\text{disc}}, \;\; -3 + 2 \cdot I_{\text{disc}} \Big\}.$$

By definition,

$$\text{Realize}\big(2 + I_{\text{disc}}\big) = \{\, 2 + \iota : \iota \in \{-1, 0, 1\} \,\} = \{1, 2, 3\},$$
$$\text{Realize}\big(-3 + 2I_{\text{disc}}\big) = \{\, -3 + 2\iota : \iota \in \{-1, 0, 1\} \,\} = \{-5, -3, -1\}.$$

Thus the evaluation map sends each formal element of $\mathcal{S}$ to a finite concrete value set; the union of realizations is $\{-5, -3, -1, 1, 2, 3\}$, expressing all outcomes generated by the discrete indeterminacy values.

8.6 Entangled Particles

Entangled particles share nonclassical correlations, described by inseparable quantum states, producing measurement outcomes that remain linked despite spatial separation significantly [45, 507–509].

Definition 8.6.1 (Entangled Particles)**.** [45] Let $\mathcal{H}_A, \mathcal{H}_B$ be complex Hilbert spaces and let ρ be a bipartite quantum state on $\mathcal{H}_A \otimes \mathcal{H}_B$ (density operator: $\rho \geq 0$, $\operatorname{Tr}\rho = 1$). The pair of particles (A, B) is said to be *entangled* in state ρ if ρ is *not separable*, i.e.,

$$\rho \neq \sum_i p_i\, \rho_A^{(i)} \otimes \rho_B^{(i)} \quad \text{for any probabilities } (p_i) \text{ and local states } \rho_A^{(i)}, \rho_B^{(i)}.$$

Equivalently, for a pure state $\rho = |\psi\rangle\langle\psi|$, the particles are entangled iff $|\psi\rangle$ is not a simple tensor $u \otimes v$ with $u \in \mathcal{H}_A$, $v \in \mathcal{H}_B$.

Example 8.6.2 (EPR/Bell pair: two entangled qubits)**.** An EPR or Bell pair is a maximally entangled two-qubit state exhibiting correlations and violating Bell inequalities, challenging local realism (cf. [510]).

Let $\mathcal{H}_A = \mathcal{H}_B = \mathbb{C}^2$ with computational basis $\{|0\rangle, |1\rangle\}$. Consider the singlet (Bell) state

$$|\Psi^-\rangle \;=\; \frac{|0\rangle \otimes |1\rangle \;-\; |1\rangle \otimes |0\rangle}{\sqrt{2}}, \qquad \rho \;=\; |\Psi^-\rangle\langle\Psi^-|.$$

This state is not separable. Indeed, its Schmidt decomposition has two nonzero equal coefficients $\frac{1}{\sqrt{2}}, \frac{1}{\sqrt{2}}$ (Schmidt rank = 2), hence it cannot be written as $|u\rangle \otimes |v\rangle$.

Equivalently, the reduced states are maximally mixed:

$$\rho_A \;=\; \mathrm{Tr}_B(\rho) = \tfrac{1}{2}\big(|0\rangle\langle 0| + |1\rangle\langle 1|\big) = \tfrac{I_2}{2}, \qquad \rho_B \;=\; \tfrac{I_2}{2}.$$

A pure bipartite state having mixed marginals must be entangled.

Measurement correlations: measuring both qubits along the same axis yields perfect anticorrelation. For Pauli Z and X (and in fact any axis),

$$\langle\Psi^-|\,\sigma_z\otimes\sigma_z\,|\Psi^-\rangle = -1, \qquad \langle\Psi^-|\,\sigma_x\otimes\sigma_x\,|\Psi^-\rangle = -1,$$

so outcomes are always opposite with probability 1. These nonclassical, distance-independent correlations exemplify entangled particles.

8.7 Duality, Triality, and Multiality

Duality partitions a set into two complementary classes, ensuring exclusivity and totality, encoding crisp opposites with conserved membership indicators everywhere [43]. Triality models three outcomes—positive, neutral, negative—assigning normalized degrees to each, capturing intermediate uncertainty between opposing polar decisions in graded form (cf. [43, 511–513]). Multiality generalizes triality by admitting multiple neutral poles, distributing normalized memberships across many statuses between acceptance and rejection as needed [43]. Iterative multiality lifts labels to multisets recursively, aggregating lower-level distributions into higher-level ones via canonical weighted convex combinations across levels [43].

Definition 8.7.1 (Duality)**.** [43] Let S be a nonempty set and let $A \subseteq S$ be a property (class) with classical complement $A^{\complement} := S \setminus A$. A *duality* on S is the ordered pair

$$\mathsf{D} \;=\; (A, \operatorname{anti} A), \qquad \text{where } \operatorname{anti} A := A^{\complement},$$

so that $A \cap \operatorname{anti} A = \varnothing$ and $A \cup \operatorname{anti} A = S$. Equivalently, via the indicator $\mathbf{1}_A : S \to \{0,1\}$, duality is encoded by

$$\mu_{\mathsf{D}}(s) \;=\; \big(\mathbf{1}_A(s),\, \mathbf{1}_{\operatorname{anti} A}(s)\big) \in \{0,1\}^2, \quad \forall s \in S,$$

with the conservation identity $\mathbf{1}_A(s) + \mathbf{1}_{\operatorname{anti} A}(s) = 1$.

Example 8.7.2 (Duality: Email filtering (Ham vs. Spam))**.** Let S be the set of emails received today. Define $A \subseteq S$ as the set of *legitimate* (ham) emails and $\operatorname{anti} A := S \setminus A$ as the set of *spam* emails. Suppose $|S| = 100$, $|A| = 72$, and $|\operatorname{anti} A| = 28$. Then

$$A \cap \operatorname{anti} A = \varnothing, \qquad A \cup \operatorname{anti} A = S, \qquad |A| + |\operatorname{anti} A| = |S| = 100.$$

Equivalently, with indicators $\mathbf{1}_A, \mathbf{1}_{\operatorname{anti} A} : S \to \{0,1\}$, every $s \in S$ satisfies

$$\mathbf{1}_A(s) + \mathbf{1}_{\operatorname{anti} A}(s) = 1,$$

so the pair $(A, \operatorname{anti} A)$ realizes a (crisp) duality of $\{\text{ham}, \text{spam}\}$.

Definition 8.7.3 (Triality (with a neutral middle))**.** Let S be as above. A *triality* on S is either

1. a partition $S = A \,\dot\cup\, N \,\dot\cup\, \operatorname{anti} A$ (crisp form), or

2. a map $\tau : S \to [0,1]^3$ (graded form)

such that, writing $\tau(s) = (T(s), I(s), F(s))$,

$$T(s), I(s), F(s) \in [0,1], \qquad T(s) + I(s) + F(s) = 1 \quad \text{(normalized case)},$$

where T measures adherence to A, F adherence to anti A, and I the neutral/indeterminate part lying between the opposites. In the crisp case, $\tau = \big(\mathbf{1}_A, \mathbf{1}_N, \mathbf{1}_{\text{anti}\,A}\big)$.

Example 8.7.4 (Triality: Customer sentiment (Positive / Neutral / Negative))**.** Let S be a set of product reviews. A normalized triality map $\tau : S \to [0,1]^3$ assigns

$$\tau(s) = (T(s), I(s), F(s)), \qquad T + I + F = 1,$$

where T is positive sentiment, F negative sentiment, and I neutral/uncertain. For a review s^* (short text with mixed tone), suppose a sentiment analyzer yields

$$T(s^*) = 0.70, \qquad I(s^*) = 0.20, \qquad F(s^*) = 0.10,$$

so $T + I + F = 1$. The label "overall positive" can be decided by $\arg\max\{T, I, F\} = T$, while $I = 0.20$ records the reviewer's indecision/ambiguity.

Definition 8.7.5 (MultiOperation)**.** Fix an integer $m \geq 1$ and let H be a nonempty set. An *m-ary multi-operation* on H is a map

$$\#^{(m)} : H^m \longrightarrow \mathcal{M}(H), \quad (x_1, \ldots, x_m) \mapsto \#^{(m)}(x_1, \ldots, x_m),$$

assigning to each m-tuple $(x_1, \ldots, x_m)$ a finite multiset of elements of H rather than a single element.

Definition 8.7.6 (MultiStructure)**.** [514] A *MultiStructure* is a pair

$$\mathcal{MS} = \big(H, \{\#^{(m)} : H^m \to \mathcal{M}(H)\}_{m \in \mathcal{I}}\big),$$

where H is a nonempty carrier set and $\mathcal{I} \subseteq \mathbb{Z}_{>0}$ indexes a family of multi-operations of various arities. No further axioms are imposed unless explicitly stated.

Definition 8.7.7 (Multiality (included multiple middles))**.** Fix a finite index set of poles $P = \{p_+, p_-, n_1, \ldots, n_m\}$ with two distinguished opposites p_+ ("A") and p_- ("anti A"), and $m \geq 0$ neutrals n_j. A *multiality* on S is either

1. a partition $S = \biguplus_{q \in P} C_q$ with classes $C_q \subseteq S$, or
2. a map $\mu : S \to [0,1]^P$ (graded form)

such that, writing $\mu(s) = (\mu_q(s))_{q \in P}$,

$$\mu_q(s) \in [0,1] \quad \text{and} \quad \sum_{q \in P} \mu_q(s) = 1 \quad \text{(normalized case)}.$$

The neutrals $\{n_j\}$ realize multiple "middles" between p_+ and p_-; the triality is recovered when $m = 1$.

Example 8.7.8 (Multiality: University admissions with multiple middles)**.** Let S be the set of applicants and let the poles be

$$P = \{\ p_+ = \text{Accept},\ p_- = \text{Reject},\ n_1 = \text{Waitlist},\ n_2 = \text{Interview},\ n_3 = \text{MissingDocs}\ \}.$$

A graded multiality $\mu : S \to [0,1]^P$ assigns, for applicant $s^\dagger$,

$$\mu(s^\dagger) = \big(\underbrace{0.35}_{\text{Accept}}, \underbrace{0.15}_{\text{Reject}}, \underbrace{0.25}_{\text{Waitlist}}, \underbrace{0.20}_{\text{Interview}}, \underbrace{0.05}_{\text{MissingDocs}}\big),$$

which satisfies $\sum_{q \in P} \mu_q(s^\dagger) = 1$. The "included multiple middles" $\{n_1, n_2, n_3\}$ capture distinct neutral statuses between acceptance and rejection. Collapsing (n_1, n_2, n_3) into a single neutral recovers a triality.

Theorem 8.7.9 (Generalization chain)**.** *For any duality* $\mathsf{D} = (A, \text{anti } A)$ *on* S *there exist canonical embeddings*

$$\mathsf{D} \hookrightarrow \textit{triality} \hookrightarrow \textit{multiality},$$

i.e., duality is a special case of triality, and triality is a special case of multiality.

Proof. 1) Duality ⇒ Triality. Define, for each $s \in S$,

$$\tau_{\mathsf{D}}(s) := \big(T(s), I(s), F(s)\big) := \big(\mathbf{1}_A(s),\ 0,\ \mathbf{1}_{\text{anti } A}(s)\big).$$

Then $T, I, F \in \{0, 1\}$ and $T(s) + I(s) + F(s) = \mathbf{1}_A(s) + \mathbf{1}_{\text{anti } A}(s) = 1$, so τ_{D} is a normalized triality with zero neutral mass; in the crisp picture this is $S = A \,\dot{\cup}\, \varnothing \,\dot{\cup}\, \text{anti } A$.

2) Triality ⇒ Multiality. Given any triality $\tau(s) = (T(s), I(s), F(s))$, choose $P = \{p_+, p_-, n_1, \dots, n_m\}$ with $m \geq 1$ and distribute the neutral component $I(s)$ over the neutrals by fixed weights $\alpha_j \geq 0$ with $\sum_{j=1}^{m} \alpha_j = 1$:

$$\mu_{p_+}(s) := T(s), \quad \mu_{p_-}(s) := F(s), \quad \mu_{n_j}(s) := \alpha_j\, I(s) \quad (j = 1, \dots, m).$$

Then $\sum_{q \in P} \mu_q(s) = T(s) + F(s) + \sum_{j=1}^{m} \alpha_j I(s) = T(s) + F(s) + I(s) = 1$, hence μ is a normalized multiality extending the given triality (the case $m = 1$ is identical to triality). Therefore the stated embeddings hold. □

Definition 8.7.10 (n-Fold Iterated Multiset)**.** Let X be a set and let $n \geq 1$. Define by induction

$$\mathsf{Mult}^1(X) = \mathsf{Mult}(X), \qquad \mathsf{Mult}^{i+1}(X) = \mathsf{Mult}\big(\mathsf{Mult}^i(X)\big), \quad i = 1, 2, \dots, n-1.$$

An element of $\mathsf{Mult}^n(X)$ is called an *n-fold iterated multiset* over X.

Remark 8.7.11.

1. When $n = 1$, $\mathsf{Mult}^1(X)$ coincides with the ordinary collection of finite multisets on X.
2. For $n = 2$, members of $\mathsf{Mult}^2(X)$ are multisets whose elements themselves are finite multisets on X.
3. In the general case, an n-fold iterated multiset may be viewed as a rooted tree of height n, with leaves in X and each internal node having finitely many unordered children.

Definition 8.7.12 (Iterative Multi-Structure of Order k)**.** Let H be a nonempty set and fix $k \geq 1$. Set

$$\mathsf{Mult}^0(H) = H, \quad \mathsf{Mult}^{i+1}(H) = \mathsf{Mult}\big(\mathsf{Mult}^i(H)\big), \quad i = 0, 1, \dots, k-1.$$

Given a collection of arities $\mathcal{I} \subseteq \mathbb{Z}_{>0}$, an *Iterative Multi-Structure of order* k is the tuple

$$\mathsf{IMS}^{(k)} = \big(H, \{\#^{(m,i)} : (\mathsf{Mult}^i(H))^m \to \mathsf{Mult}^{i+1}(H)\}_{m \in \mathcal{I}, 0 \leq i < k}\big),$$

where each

$$\#^{(m,i)}(x_1, \dots, x_m) \in \mathsf{Mult}^{i+1}(H), \quad x_j \in \mathsf{Mult}^i(H).$$

Hence $\#^{(m,0)}$ acts on H, $\#^{(m,1)}$ on multisets of H, and so forth, up to level k.

Definition 8.7.13 (Iterative Multiality of order k)**.** Fix an integer $k \geq 1$ and a finite pole set P. An *iterative multiality of order* k on S is a tuple

$$\mathcal{IM}_P^{(k)} = \Big(S, P, \{\mu^{(i)}\}_{i=0}^{k}\Big),$$

where each level-i labeling

$$\mu^{(i)} : \mathsf{Mult}^i(S) \longrightarrow \Delta(P)$$

is defined recursively as follows:

1. Level 0 is an ordinary multiality $\mu^{(0)} : S \to \Delta(P)$.
2. For $i \geq 0$ and $X \in \mathsf{Mult}^{i+1}(S)$, let $w_Y := \dfrac{X(Y)}{|X|}$ for $Y \in \operatorname{supp}(X)$; then set the canonical *multiset lift*

$$\mu^{(i+1)}(X) := \sum_{Y \in \operatorname{supp}(X)} w_Y\, \mu^{(i)}(Y) \in \Delta(P). \tag{8.1}$$

Thus $\mu^{(i+1)}$ aggregates the level-i labels of the members of X by a convex combination (weighted by multiplicities), and every level is again a distribution on the same pole set P.

Remark 8.7.14 (Well-definedness of the lift)**.** Since each $\mu^{(i)}(Y) \in \Delta(P)$ and $\sum_{Y\in\mathrm{supp}(X)} w_Y = 1$, the right-hand side of (8.1) lies in $\Delta(P)$:

$$\sum_{q\in P} \mu^{(i+1)}(X)(q) = \sum_{q\in P}\sum_{Y} w_Y\, \mu^{(i)}(Y)(q) = \sum_{Y} w_Y \underbrace{\Big(\sum_{q\in P} \mu^{(i)}(Y)(q)\Big)}_{=1} = \sum_{Y} w_Y = 1.$$

Monotonicity and permutation-invariance are immediate from convexity.

Example 8.7.15 (Order-2 iterative multiality: product-review aggregation)**.** Let the pole set be $P = \{\mathsf{Pos}, \mathsf{Neu}, \mathsf{Neg}\}$ (positive/neutral/negative). Let $S = \{r_1, r_2, r_3, r_4\}$ be individual reviews with level-0 multiality

$$\mu^{(0)}(r_1) = (0.8, 0.1, 0.1), \quad \mu^{(0)}(r_2) = (0.6, 0.2, 0.2), \quad \mu^{(0)}(r_3) = (0.2, 0.3, 0.5), \quad \mu^{(0)}(r_4) = (0.4, 0.4, 0.2),$$

each summing to 1.

Level 1 (multisets of reviews). Consider

$$X = \{r_1^2, r_2, r_3\} \quad (|X| = 4), \qquad X' = \{r_2, r_3, r_4^2\} \quad (|X'| = 4).$$

With multiplicity weights $w_Y = X(Y)/|X|$,

$$\mu^{(1)}(X) = \tfrac{1}{2}\,\mu^{(0)}(r_1) + \tfrac{1}{4}\,\mu^{(0)}(r_2) + \tfrac{1}{4}\,\mu^{(0)}(r_3) = (0.6, 0.175, 0.225) = \left(\tfrac{3}{5}, \tfrac{7}{40}, \tfrac{9}{40}\right).$$

Similarly,

$$\mu^{(1)}(X') = \tfrac{1}{4}\,\mu^{(0)}(r_2) + \tfrac{1}{4}\,\mu^{(0)}(r_3) + \tfrac{1}{2}\,\mu^{(0)}(r_4) = (0.4, 0.325, 0.275) = \left(\tfrac{2}{5}, \tfrac{13}{40}, \tfrac{11}{40}\right).$$

Level 2 (multiset of multisets). Let

$$Y = \{X^2, X'\} \quad (|Y| = 3).$$

Then

$$\mu^{(2)}(Y) = \tfrac{2}{3}\,\mu^{(1)}(X) + \tfrac{1}{3}\,\mu^{(1)}(X') = \left(\tfrac{8}{15}, \tfrac{9}{40}, \tfrac{29}{120}\right) \approx (0.5333, 0.2250, 0.2417).$$

Normalization check: $\frac{8}{15} + \frac{9}{40} + \frac{29}{120} = \frac{64}{120} + \frac{27}{120} + \frac{29}{120} = \frac{120}{120} = 1.$

Interpretation. Level-0 labels reviews; level-1 aggregates review groups; level-2 aggregates groups-of-groups. Here the order-2 label remains "overall positive" (largest coordinate), with explicit neutral/negative contributions quantified.

Example 8.7.16 (Order-3 iterative multiality: cybersecurity posture across hosts/datacenters)**.** Let $P = \{\mathsf{Normal}, \mathsf{Suspicious}, \mathsf{Compromised}\}$. Base events $E = \{e_1, \ldots, e_6\}$ have level-0 multiality

$$\mu^{(0)}(e_1) = (0.9, 0.05, 0.05), \quad \mu^{(0)}(e_2) = (0.7, 0.2, 0.1), \quad \mu^{(0)}(e_3) = (0.4, 0.3, 0.3),$$
$$\mu^{(0)}(e_4) = (0.3, 0.2, 0.5), \quad \mu^{(0)}(e_5) = (0.5, 0.3, 0.2), \quad \mu^{(0)}(e_6) = (0.6, 0.1, 0.3).$$

Level 1 (host-level multisets of events).

$$X_A = \{e_1, e_2, e_3\}\ (|X_A| = 3), \qquad X_B = \{e_4, e_5, e_6^2\}\ (|X_B| = 4).$$

Then

$$\mu^{(1)}(X_A) = \tfrac{1}{3}\big((0.9, 0.05, 0.05) + (0.7, 0.2, 0.1) + (0.4, 0.3, 0.3)\big) = \left(\tfrac{2}{3}, \tfrac{11}{60}, \tfrac{3}{20}\right) \approx (0.6667, 0.1833, 0.1500),$$

$$\mu^{(1)}(X_B) = \tfrac{1}{4}(0.3, 0.2, 0.5) + \tfrac{1}{4}(0.5, 0.3, 0.2) + \tfrac{1}{2}(0.6, 0.1, 0.3) = \left(\tfrac{1}{2}, \tfrac{7}{40}, \tfrac{13}{40}\right) = (0.5, 0.175, 0.325).$$

Level 2 (datacenter-level: multiset of hosts). For datacenter D take $Y_D = \{X_A^2, X_B\}$ so

$$\mu^{(2)}(Y_D) = \tfrac{2}{3}\,\mu^{(1)}(X_A) + \tfrac{1}{3}\,\mu^{(1)}(X_B) = \left(\tfrac{11}{18}, \tfrac{13}{72}, \tfrac{5}{24}\right) \approx (0.6111, 0.1806, 0.2083).$$

For datacenter E with primarily B-type hosts, take $Y_E = \{X_B^3\}$, hence

$$\mu^{(2)}(Y_E) = \mu^{(1)}(X_B) = \left(\tfrac{1}{2}, \tfrac{7}{40}, \tfrac{13}{40}\right).$$

Level 3 (enterprise-level: multiset of datacenters). Let $Z = \{Y_D^2, Y_E^2\}$, then

$$\mu^{(3)}(Z) = \tfrac{1}{2}\,\mu^{(2)}(Y_D) + \tfrac{1}{2}\,\mu^{(2)}(Y_E) = \left(\tfrac{5}{9}, \tfrac{8}{45}, \tfrac{4}{15}\right) \approx (0.5556, 0.1778, 0.2667).$$

Normalization check: $\frac{5}{9} + \frac{8}{45} + \frac{4}{15} = \frac{25+8+12}{45} = 1$.

Interpretation. Order-1 labels hosts from events; order-2 labels datacenters from hosts; order-3 labels the enterprise from datacenters. The final posture is "mostly normal" with quantified suspicion/compromise levels preserved across iterations.

Theorem 8.7.17 (Iterative multiality generalizes multiality)**.** *Every multiality $\mu^{(0)} : S \to \Delta(P)$ is the level-0 component of some iterative multiality $\mathcal{IM}_P^{(k)}$ for any prescribed $k \geq 1$. Conversely, the level-0 part of any iterative multiality $\mathcal{IM}_P^{(k)}$ is a multiality.*

Proof. Forward. Given a multiality $\mu^{(0)}$, construct $\mu^{(1)}, \ldots, \mu^{(k)}$ by the canonical multiset lift (8.1). Each $\mu^{(i+1)}$ maps into $\Delta(P)$ by the remark above, hence $\mathcal{IM}_P^{(k)}$ is an iterative multiality whose level-0 component is the given $\mu^{(0)}$.

Backward. If $\mathcal{IM}_P^{(k)} = (S, P, \{\mu^{(i)}\}_{i=0}^{k})$ is an iterative multiality, its level-0 map $\mu^{(0)} : S \to \Delta(P)$ satisfies the definition of a multiality by construction. □

8.8 Di-alectic, Tri-alectic, and n-alectic Structures

A di-alectic models two opposing valuations, truth and falsehood, with monotone update maps describing evidence-driven mutual interaction dynamics over time [515]. A tri-alectic assigns truth, indeterminacy, and falsehood degrees; updates respect neutrosophic order, increasing truth while reducing indeterminacy and falsity levels [515]. An n-alectic refines truth, indeterminacy, falsity into multiple subcomponents; updates are blockwise monotone; weighted aggregation yields coarse neutrosophic summaries overall [515]. Let $\mathcal{S}$ be a nonempty set of propositions. For $m \in \mathbb{N}$, write $[0,1]^m$ for the unit m-cube and let $\Delta_m := \{x \in [0,1]^m\}$ [515].

Definition 8.8.1 (Di-alectic structure (dynamic of two opposites))**.** [515] A *di-alectic structure* on $\mathcal{S}$ is a pair $(\mathcal{S}, v_2)$ where

$$v_2 : \mathcal{S} \longrightarrow [0,1]^2, \qquad v_2(\sigma) = (T(\sigma), F(\sigma)),$$

assigns to each proposition σ its degrees of *truth* $T(\sigma)$ and *falsehood* $F(\sigma)$. Optionally, a family of update maps $\{\delta_u\}_{u\in\mathcal{U}}$ with $\delta_u : [0,1]^2 \to [0,1]^2$ models the *dynamic* interaction of the two opposites under "evidence" u, and is required to be monotone in the product order: if $(T_1, F_1) \leq (T_2, F_2)$ coordinate-wise, then $\delta_u(T_1, F_1) \leq \delta_u(T_2, F_2)$.

Example 8.8.2 (Di-alectic: two opposites with an update)**.** Let the proposition be "Rain tomorrow." Initial valuation

$$v_2(\sigma) = (T, F) = (0.30,\ 0.60).$$

Given positive evidence level $u \in [0,1]$, define the monotone update

$$\delta_u(T, F) = \big(T + u(1-T),\ (1-u)F\big).$$

With $u = 0.50$,

$$T' = 0.30 + 0.50(1-0.30) = 0.30 + 0.35 = 0.65, \qquad F' = (1-0.50)\cdot 0.60 = 0.30.$$

Hence $v_2'(\sigma) = (0.65,\ 0.30)$.

Definition 8.8.3 (Tri-alectic structure (dynamic of opposites and neutrality))**.** [515] A *tri-alectic structure* on $\mathcal{S}$ is a pair $(\mathcal{S}, v_3)$ where

$$v_3 : \mathcal{S} \longrightarrow [0,1]^3, \qquad v_3(\sigma) = (T(\sigma), I(\sigma), F(\sigma)),$$

assigns degrees of *truth* T, *indeterminacy/neutrality* I, and *falsehood* F to each σ. No fixed normalization is imposed; one may work either with the unconstrained cube $0 \le T, I, F \le 1$ or, when needed, with constraints such as $0 \le T + I + F \le 3$. Updates are maps $\delta_u : [0,1]^3 \to [0,1]^3$ that are monotone in T and antitone in I and F with respect to the neutrosophic order

$$(t_1, i_1, f_1) \preceq_N (t_2, i_2, f_2) \iff t_1 \le t_2,\ i_1 \ge i_2,\ f_1 \ge f_2.$$

Example 8.8.4 (Tri-alectic: truth/indeterminacy/falsehood with neutrosophic order)**.** Let the proposition be "The new drug is effective." Initial triplet

$$v_3(\sigma) = (T, I, F) = (0.40,\ 0.40,\ 0.20).$$

For supportive evidence $u \in [0,1]$, use

$$\delta_u(T, I, F) = \big(T + u(I + F),\ (1-u)I,\ (1-u)F\big),$$

which is monotone in T and antitone in I, F (neutrosophic order). With $u = 0.50$,

$$T' = 0.40{+}0.50(0.40{+}0.20) = 0.40{+}0.30 = 0.70, \quad I' = (1{-}0.50){\cdot}0.40 = 0.20, \quad F' = (1{-}0.50){\cdot}0.20 = 0.10.$$

So $v_3'(\sigma) = (0.70,\ 0.20,\ 0.10)$.

Definition 8.8.5 (n-alectic structure (dynamic of refined subcomponents))**.** [515, 516] Fix integers $p, r, s \ge 0$ with $p + r + s = n \ge 2$. An *n-alectic structure* on $\mathcal{S}$ is a pair $(\mathcal{S}, v_n)$ where

$$v_n : \mathcal{S} \longrightarrow [0,1]^n, \qquad v_n(\sigma) = \big(T_1(\sigma), \dots, T_p(\sigma);\ I_1(\sigma), \dots, I_r(\sigma);\ F_1(\sigma), \dots, F_s(\sigma)\big),$$

refines the coarse neutrosophic components (T, I, F) into *n subcomponents*: T into $T_1, \dots, T_p$, I into $I_1, \dots, I_r$, and F into $F_1, \dots, F_s$. A family of updates $\delta_u : [0,1]^n \to [0,1]^n$ is required to be monotone coordinate-wise on the T-block and antitone coordinate-wise on the I- and F-blocks with respect to the refined neutrosophic order

$$x \preceq_{RN} y \iff \Big(\forall j \le p : x_{T_j} \le y_{T_j}\Big) \wedge \Big(\forall k \le r : x_{I_k} \ge y_{I_k}\Big) \wedge \Big(\forall \ell \le s : x_{F_\ell} \ge y_{F_\ell}\Big).$$

When a coarse summary is needed, fix nonnegative weights $\alpha \in \mathbb{R}^p,\ \beta \in \mathbb{R}^r,\ \gamma \in \mathbb{R}^s$ with $\sum_j \alpha_j = \sum_k \beta_k = \sum_\ell \gamma_\ell = 1$, and define the *refinement aggregator*

$$\mathsf{Agg}(v_n(\sigma)) := \Big(\sum_{j=1}^{p} \alpha_j T_j(\sigma),\ \sum_{k=1}^{r} \beta_k I_k(\sigma),\ \sum_{\ell=1}^{s} \gamma_\ell F_\ell(\sigma)\Big) \in [0,1]^3.$$

Remark 8.8.6 (Special cases)**.** The di-alectic is the case $n = 2$ with $(p, r, s) = (1, 0, 1)$. The tri-alectic is the case $n = 3$ with $(p, r, s) = (1, 1, 1)$. A quadr-alectic example arises when $(p, r, s) = (1, 2, 1)$, i.e., $(T; I_1, I_2; F)$.

Example 8.8.7 (n-alectic ($n = 5$): refined blocks with weighted aggregation)**.** Consider $(p, r, s) = (2, 1, 2)$ and a technical claim σ about a system:

$$v_5(\sigma) = \big(T_1, T_2;\ I_1;\ F_1, F_2\big) = (0.30,\ 0.20;\ 0.40;\ 0.10,\ 0.30).$$

Support specific to the first truth facet arrives with strength $u = 0.60$. Update blockwise (monotone on T, antitone on I, F):

$$T_1' = T_1 + u\, I_1 = 0.30 + 0.60 \cdot 0.40 = 0.54, \qquad T_2' = T_2 = 0.20,$$

$$I_1' = (1-u)I_1 = 0.40 \cdot 0.40 = 0.16, \qquad F_1' = (1-u)F_1 = 0.40 \cdot 0.10 = 0.04, \quad F_2' = F_2 = 0.30.$$

A coarse neutrosophic summary uses weights $\alpha = (0.7, 0.3)$, $\beta = (1)$, $\gamma = (0.5, 0.5)$:

$$T_{\text{agg}} = 0.7 \cdot 0.54 + 0.3 \cdot 0.20 = 0.378 + 0.060 = 0.438,$$
$$I_{\text{agg}} = 1 \cdot 0.16 = 0.16, \qquad F_{\text{agg}} = 0.5 \cdot 0.04 + 0.5 \cdot 0.30 = 0.02 + 0.15 = 0.17.$$

Thus $\mathsf{Agg}(v_5'(\sigma)) = (0.438,\ 0.16,\ 0.17)$.

8.9 Multi–Polynomials and Iterative Multi–Polynomials

A multi-polynomial sums finitely many monomials over multiple variables using multi-index exponents, supporting evaluation, addition, Cauchy product in rings algebras [39]. Iterative multi-polynomials compose polynomial maps repeatedly, substituting outputs into inputs across levels, generating higher-order expressions and dynamical systems via iteration [39].

Definition 8.9.1 (Multi–monomials and Multi–polynomials)**.** [39] Let R be a commutative ring with 1, and X a (finite or countable) variable set. A *multi–exponent* is a finitely supported map $m : X \to \mathbb{N}$; the corresponding monomial is $x^m := \prod_{x\in X} x^{m(x)}$. A *multi–polynomial* over R in variables X is a finite sum

$$P \;=\; \sum_m c_m\, x^m, \qquad c_m \in R,$$

with pointwise addition and Cauchy product $\big(\sum_m c_m x^m\big)\big(\sum_n d_n x^n\big) \;=\; \sum_k \big(\sum_{m+n=k} c_m d_n\big) x^k$ (multi–index addition).

Example 8.9.2 (Multi–monomials over three variables)**.** Let $R = \mathbb{Z}$ and $X = \{x, y, z\}$. Define two multi–exponents (finitely supported maps) by

$$m_1(x, y, z) = (2, 1, 0), \qquad m_2(x, y, z) = (0, 3, 1).$$

Their associated monomials are

$$x^{m_1} = x^2 y^1 z^0 = x^2 y, \qquad x^{m_2} = x^0 y^3 z^1 = y^3 z.$$

Cauchy (multi–index) multiplication adds exponents componentwise:

$$x^{m_1} \cdot x^{m_2} = x^{m_1+m_2} = x^{(2+0,\ 1+3,\ 0+1)} = x^2 y^4 z.$$

Form a multi–polynomial

$$P(x, y, z) = 5x^2 y - 2y^3 z + 7.$$

Evaluating at $(x, y, z) = (1, 2, 3)$ gives

$$P(1, 2, 3) = 5 \cdot 1^2 \cdot 2 \;-\; 2 \cdot 2^3 \cdot 3 \;+\; 7 = 10 - 48 + 7 = -31.$$

All steps use the definitions of multi–exponent, monomial, and multi–polynomial evaluation.

Definition 8.9.3 (Evaluation)**.** Let A be a commutative R–algebra and let $(a_x)_{x\in X} \subset A$. The evaluation of $P = \sum_m c_m x^m$ at (a_x) is

$$P(a) \;:=\; \sum_m c_m \prod_{x\in X} a_x^{m(x)} \;\in A.$$

Definition 8.9.4 (Iterative Multi–Polynomial of Order k)**.** Fix $k \geq 1$. An *iterative multi–polynomial system* (of order k) over (R, X) is a sequence $\big(P^{[0]}, P^{[1]}, \ldots, P^{[k]}\big)$ such that:

1. $P^{[0]}$ is the identity embedding of variables (so evaluation starts from a_x);
2. for each $i = 0, 1, \ldots, k-1$, there exists a multi–polynomial $Q^{[i+1]}$ with

$$P^{[i+1]}(a) \;:=\; Q^{[i+1]}\big(P^{[i]}(a)\big),$$

   i.e. $P^{[i+1]}$ is obtained from $P^{[i]}$ by substituting $P^{[i]}(a_x)$ for each variable x in $Q^{[i+1]}$.

Thus $P^{[k]}$ is a k–fold compositional polynomial built from base variables via multi–polynomial substitution.

Remark 8.9.5. When R is ordered and A is a positive cone in an R–algebra, monotone multi–polynomials (all $c_m \geq 0$) preserve order under evaluation, and the iterative system preserves monotonicity at every order.

Example 8.9.6 (Iterative Multi–Polynomial of order 3 (single variable)). Let $R = \mathbb{Z}$, $X = \{x\}$, and define

$$Q^{[1]}(x) = x^2 + 1, \qquad Q^{[2]}(x) = 3x - 2, \qquad Q^{[3]}(x) = x^2 + x.$$

Set $P^{[0]}(a) = a$ and iterate by composition:

$$P^{[1]}(a) = Q^{[1]}(P^{[0]}(a)) = a^2 + 1,$$

$$P^{[2]}(a) = Q^{[2]}(P^{[1]}(a)) = 3(a^2 + 1) - 2 = 3a^2 + 1,$$

$$\begin{aligned} P^{[3]}(a) &= Q^{[3]}(P^{[2]}(a)) = (3a^2 + 1)^2 + (3a^2 + 1) \\ &= 9a^4 + 6a^2 + 1 + 3a^2 + 1 = 9a^4 + 9a^2 + 2. \end{aligned}$$

For $a = 2$,

$$P^{[3]}(2) = 9 \cdot 2^4 + 9 \cdot 2^2 + 2 = 9 \cdot 16 + 9 \cdot 4 + 2 = 144 + 36 + 2 = 182.$$

Thus $P^{[3]}$ is the third–order iterative multi–polynomial obtained by successive substitution of multi–polynomials.

8.10 Linguistic Plane

Linguistic plane models paired labels on two ordered scales, embedding them numerically to compare, rank, and measure differences consistently across [46].

Definition 8.10.1 (Linguistic Plane). [46] Let $L_x = \{\lambda_1^{(x)}, \dots, \lambda_{m_x}^{(x)}\}$ and $L_y = \{\lambda_1^{(y)}, \dots, \lambda_{m_y}^{(y)}\}$ be finite, linearly ordered sets of linguistic labels. Fix strictly increasing embeddings $\mu_x : L_x \to [0, 1]$ and $\mu_y : L_y \to [0, 1]$, and weights $\alpha, \beta > 0$ with $\alpha + \beta = 1$. The *linguistic plane* is the metric space

$$\mathsf{L} := (L_x \times L_y,\ d_L), \qquad d_L\big((\ell_x, \ell_y), (\ell'_x, \ell'_y)\big) := \alpha \left|\mu_x(\ell_x) - \mu_x(\ell'_x)\right| + \beta \left|\mu_y(\ell_y) - \mu_y(\ell'_y)\right|.$$

The induced topology τ_L is the metric topology of d_L. When convenient, the product partial order $(\ell_x, \ell_y) \preceq (\ell'_x, \ell'_y) \iff \ell_x \leq_{L_x} \ell'_x \ \wedge\ \ell_y \leq_{L_y} \ell'_y$ is also considered.

Example 8.10.2 (Customer satisfaction (x) versus delivery speed (y)). Let

$$L_x = \{\mathrm{VL}, \mathrm{L}, \mathrm{M}, \mathrm{H}, \mathrm{VH}\}, \qquad L_y = \{\mathrm{VS}, \mathrm{S}, \mathrm{Mod}, \mathrm{F}, \mathrm{VF}\}.$$

Choose strictly increasing embeddings (equidistant for simplicity)

$$\begin{aligned} &\mu_x(\mathrm{VL}) = 0,\ \mu_x(\mathrm{L}) = 0.25,\ \mu_x(\mathrm{M}) = 0.50,\ \mu_x(\mathrm{H}) = 0.75,\ \mu_x(\mathrm{VH}) = 1, \\ &\mu_y(\mathrm{VS}) = 0,\ \mu_y(\mathrm{S}) = 0.25,\ \mu_y(\mathrm{Mod}) = 0.50,\ \mu_y(\mathrm{F}) = 0.75,\ \mu_y(\mathrm{VF}) = 1. \end{aligned}$$

Set weights $\alpha = 0.6, \beta = 0.4$ and define the metric

$$d_L\big((\ell_x, \ell_y), (\ell'_x, \ell'_y)\big) = \alpha \left|\mu_x(\ell_x) - \mu_x(\ell'_x)\right| + \beta \left|\mu_y(\ell_y) - \mu_y(\ell'_y)\right|.$$

Compare the two assessments

$$A = (\mathrm{H}, \mathrm{Mod}) \quad \text{vs.} \quad B = (\mathrm{M}, \mathrm{VF}).$$

Then

$$\left|\mu_x(\mathrm{H}) - \mu_x(\mathrm{M})\right| = |0.75 - 0.50| = 0.25, \qquad \left|\mu_y(\mathrm{Mod}) - \mu_y(\mathrm{VF})\right| = |0.50 - 1| = 0.50,$$

so

$$d_L(A, B) = 0.6 \cdot 0.25 + 0.4 \cdot 0.50 = 0.15 + 0.20 = 0.35.$$

Interpretation: the combined linguistic distance between the two evaluations is 0.35 on $[0, 1]$, reflecting moderate disagreement weighted more toward satisfaction.

8.11 Totally vs. Partially Well-Defined

A function is totally well-defined when it assigns equal values to all representatives of every equivalence class, yielding unambiguous quotients. A function is partially well-defined when equality holds on only some classes, leaving conflicting classes undefined or requiring domain restriction.

Definition 8.11.1 (Totally well-defined on a quotient)**.** Let X be a nonempty set, $\sim$ an equivalence relation on X, and $\pi : X \to X/\sim$ the canonical projection. Let $f : X \to Y$ be a rule (function defined on representatives). We say that f *is totally well-defined w.r.t.* $\sim$ if and only if

$$\forall x, x' \in X \;\; (x \sim x' \;\Rightarrow\; f(x) = f(x')).$$

Equivalently, there exists a unique function $\bar{f} : X/\sim \to Y$ such that

$$\bar{f}\big(\pi(x)\big) = f(x) \qquad (\forall x \in X).$$

In this case we also say that f *descends to the quotient* and we identify f with its descent $\bar{f}$.

Definition 8.11.2 (Partially well-defined on a quotient)**.** With $X, \sim, \pi, f$ as above, define the *good class set* of f by

$$\mathsf{Good}(f) \;:=\; \big\{\, C \in X/\sim \;:\; f \text{ is constant on } C \,\big\}.$$

We say that f *is partially well-defined w.r.t.* $\sim$ if $\mathsf{Good}(f) \neq \varnothing$, in which case there is a unique function

$$\bar{f} \;:\; \mathsf{Good}(f) \longrightarrow Y, \qquad \bar{f}(C) \;:=\; f(x) \quad \text{for any } x \in C,$$

and $\bar{f}$ is well-defined because f is constant on each $C \in \mathsf{Good}(f)$. If $\mathsf{Good}(f) = X/\sim$, then f is totally well-defined; if $\mathsf{Good}(f) \subsetneq X/\sim$, then f is only partially well-defined. We call $\mathsf{Conflict}(f) := (X/\sim) \setminus \mathsf{Good}(f)$ the *conflict set*.

Remark 8.11.3 (Operations and relations)**.** Let $k \geq 1$ and $f : X^k \to Y$ be a k-ary rule on representatives. Write $C_i \in X/\sim$ and pick representatives $x_i \in C_i$.

- f is *totally well-defined (k-ary)* if

  $$\forall (C_1, \ldots, C_k) \in (X/\sim)^k, \quad \forall x_i, x_i' \in C_i, \quad f(x_1, \ldots, x_k) = f(x_1', \ldots, x_k'),$$

  so that $\bar{f} : (X/\sim)^k \to Y$ given by $\bar{f}(C_1, \ldots, C_k) := f(x_1, \ldots, x_k)$ is well-defined.

- f is *partially well-defined (k-ary)* if the set

  $$\mathsf{Good}_k(f) \;:=\; \Big\{(C_1, \ldots, C_k) \in (X/\sim)^k : \; f \text{ is constant on } C_1 \times \cdots \times C_k\Big\}$$

  is nonempty; then $\bar{f}$ is well-defined on $\mathsf{Good}_k(f)$.

In algebraic contexts, total well-definedness of the induced operation on $X/\sim$ is equivalent to $\sim$ being a congruence for f.

Example 8.11.4 (Total well-definedness: residue classes)**.** Let $X = \mathbb{Z}$ and $x \sim y \iff x \equiv y \pmod{n}$; let $f : \mathbb{Z} \to \{0, 1, \ldots, n-1\}$ be the remainder map $f(x) = x \bmod n$. If $x \sim y$ then $x - y = kn$ for some k, hence $f(x) = f(y)$, so f is totally well-defined. The descended map $\bar{f} : \mathbb{Z}/n\mathbb{Z} \to \{0, \ldots, n-1\}$ given by $\bar{f}([x]) = x \bmod n$ is uniquely determined by Definition 8.11.1.

Example 8.11.5 (Partial well-definedness: inversion classes)**.** Let $X = \mathbb{R} \setminus \{0\}$ and $x \sim y \iff y = 1/x$. Consider $f : X \to \mathbb{R}$ given by $f(x) = x$. For the class $C = \{2, 1/2\}$, $f(2) = 2 \neq 1/2 = f(1/2)$, so $C \notin \mathsf{Good}(f)$. However, for $C = \{1\}$ and $C = \{-1\}$ the function f is constant on C, hence $\mathsf{Good}(f) \supseteq \{\{1\}, \{-1\}\}$ is nonempty and proper. Thus f is partially (but not totally) well-defined, and $\bar{f}$ exists only on $\mathsf{Good}(f)$, with $\bar{f}(\{1\}) = 1$ and $\bar{f}(\{-1\}) = -1$.

8.12 Dialogue, Trialogue, and Multialogue

A dialogue is a two-party conversational exchange where participants alternate utterances, addressing each other to share information, intentions, and context (cf. [517–519]). A trialogue is a three-party conversation coordinating turns among speakers, enabling triangulated perspectives, mediation, clarifications, and consensus across shifting subthreads. A multialogue is a conversation among four or more participants, managing overlaps, topic branching, coalitions, and multi-perspective integration producing understanding.

Definition 8.12.1 (Conversation instance)**.** Let P be a finite, nonempty set of participants, and let $\mathcal{U}$ be a set of utterances (e.g., strings over an alphabet). A *conversation instance* of length $T \in \mathbb{N}_{\geq 1}$ is a tuple

$$\mathsf{C} = \big(P,\ (u_t, s_t, A_t, \tau_t)_{t=1}^{T}\big),$$

where for each $t = 1, \ldots, T$:

- $u_t \in \mathcal{U}$ is the t-th utterance,
- $s_t \in P$ is the speaker of u_t,
- $A_t \subseteq P \setminus \{s_t\}$ is the (possibly empty) addressed set,
- $\tau_1 < \tau_2 < \cdots < \tau_T$ are timestamps (strictly increasing in a totally ordered time domain).

Optionally, a (partial) *reply-to* map $\rho : \{1, \ldots, T\} \rightharpoonup \{1, \ldots, T\}$ encodes local discourse links, with $\rho(t) < t$ when defined.

Definition 8.12.2 (k-logue)**.** Given a conversation instance $\mathsf{C} = (P, (u_t, s_t, A_t, \tau_t)_{t=1}^{T})$, its *party size* is $|P|$. For $k \in \mathbb{N}_{\geq 2}$, the instance C is called a k*-logue* if $|P| = k$.

Definition 8.12.3 (Dialogue, Trialogue, Multialogue)**.** Within the k-logue scheme:

1. A *dialogue* is a conversation instance with exactly two participants: $|P| = 2$.
2. A *trialogue* is a conversation instance with exactly three participants: $|P| = 3$.
3. A *multialogue* is a conversation instance with at least three participants: $|P| \geq 3$.

Equivalently, *dialogue* is the two-party case; *trialogue* is the three-party extension; and *multialogue* generalizes to many speakers.

Remark 8.12.4 (Turn structure and dyadic projections)**.** For a conversation C and $p \in P$, the *turn count* of p is $\#\{t : s_t = p\}$. For any unordered pair $\{p, q\} \subseteq P$ with $p \neq q$, the *dyadic projection* is the subsequence

$$\pi_{\{p,q\}}(\mathsf{C}) := \big\{(u_t, s_t, A_t, \tau_t) : s_t \in \{p, q\} \text{ and } A_t \subseteq \{p, q\}\big\},$$

which forms a (possibly empty) embedded dialogue thread inside a trialogue/multialogue.

Example 8.12.5 (Dialogue)**.** Let $P = \{A, B\}$ and $T = 3$. With timestamps $\tau_1 < \tau_2 < \tau_3$,

$$(u_1, s_1, A_1) = (\text{“Hi”}, A, \{B\}), \quad (u_2, s_2, A_2) = (\text{“Hello”}, B, \{A\}), \quad (u_3, s_1, A_3) = (\text{“How are you?”}, A, \{B\}).$$

Then $|P| = 2$, hence C is a dialogue.

Example 8.12.6 (Trialogue)**.** Let $P = \{A, B, C\}$ and $T = 4$. With $\tau_1 < \cdots < \tau_4$,

$$(u_1, s_1, A_1) = (\text{“Team, status?”}, A, \{B, C\}),\ (u_2, s_2, A_2) = (\text{“Ready”}, B, \{A\}),$$

$$(u_3, s_3, A_3) = (\text{“Blocked”}, C, \{A\}),\ (u_4, s_1, A_4) = (\text{“Let’s pair”}, A, \{B, C\}).$$

Here $|P| = 3$, so C is a trialogue, and $\pi_{\{A,B\}}(\mathsf{C})$ isolates the $A \leftrightarrow B$ exchanges.

Example 8.12.7 (Multialogue)**.** Let $P = \{S_1, \ldots, S_m\}$ with $m \geq 4$. Any conversation instance C over P is a multialogue. For analytic tasks (e.g., moderation or summarization), one may study $\{\pi_{\{S_i,S_j\}}(\mathsf{C})\}_{i<j}$ to extract dyadic subthreads.

8.13 Hilbert, Dilbert, and Multilbert Spaces

A Hilbert space is a complete inner product vector space whose norm derives from the inner product, enabling orthogonal projections [520–522]. A Dilbert space is a Banach space equipped with a Lumer–Giles semi-inner product inducing its norm and supporting generalized Cauchy–Schwarz (cf. [523]). A Multilbert space is a complete locally convex space realized as a projective limit of Hilbert spaces via Hilbertian seminorms.

Definition 8.13.1 (Hilbert Space)**.** Let $\mathbb{K} \in \{\mathbb{R}, \mathbb{C}\}$ and H be a $\mathbb{K}$–vector space equipped with an inner product $\langle\cdot,\cdot\rangle : H \times H \to \mathbb{K}$ satisfying: (i) conjugate symmetry $\langle x, y\rangle = \overline{\langle y, x\rangle}$; (ii) linearity in the first variable $\langle \alpha x_1 + \beta x_2, y\rangle = \alpha\langle x_1, y\rangle + \beta\langle x_2, y\rangle$; (iii) positive–definiteness $\langle x, x\rangle > 0$ for $x \neq 0$. The norm $\|x\| := \sqrt{\langle x, x\rangle}$ makes $(H, \|\cdot\|)$ a normed space. If H is complete in this norm, then $(H, \langle\cdot,\cdot\rangle)$ is called a *Hilbert space*.

Example 8.13.2 (Hilbert space)**.** Let $H = L^2[0,1]$ and define the inner product

$$\langle f, g\rangle \;:=\; \int_0^1 f(t)\,\overline{g(t)}\,dt, \qquad \|f\|_2 = \sqrt{\langle f, f\rangle}.$$

Then $(H, \langle\cdot,\cdot\rangle)$ is a Hilbert space (completeness follows from the Riesz–Fischer theorem). Concrete check: for $f(t) \equiv 1$ and $g(t) = t$,

$$\langle f, g\rangle = \int_0^1 t\,dt = \tfrac{1}{2}, \qquad \|f\|_2 = \Big(\int_0^1 1\,dt\Big)^{1/2} = 1, \qquad \|g\|_2 = \Big(\int_0^1 t^2\,dt\Big)^{1/2} = \sqrt{\tfrac{1}{3}}.$$

Cauchy–Schwarz holds with equality characterization as usual.

Definition 8.13.3 (Dilbert Space (Banach space with a fixed semi–inner product))**.** Let X be a complex (or real) vector space together with a mapping $[\cdot,\cdot] : X \times X \to \mathbb{K}$ such that for all $x, y, z \in X$ and $\alpha \in \mathbb{K}$:

1. $[x, x] \geq 0$ and $[x, x] = 0$ iff $x = 0$;
2. $[\alpha x, y] = \alpha[x, y]$ and $[x + y, z] = [x, z] + [y, z]$;
3. (Cauchy–Schwarz) $|[x, y]| \leq \sqrt{[x, x]}\,\sqrt{[y, y]}$.

Define $\|x\| := \sqrt{[x, x]}$. If X is complete for $\|\cdot\|$, we call $(X, [\cdot,\cdot])$ a *Dilbert space*. Every Hilbert space is a Dilbert space by taking $[x, y] = \langle x, y\rangle$. In general, $[\cdot,\cdot]$ need not be (conjugate) symmetric nor additive in the second argument; thus a Dilbert space is precisely a Banach space endowed with a fixed Lumer–Giles semi–inner product inducing its norm.

Example 8.13.4 (Dilbert space)**.** On $X = \ell^p$ with norm $\|x\|_p = \big(\sum_{k\geq 1} |x_k|^p\big)^{1/p}$, define the Lumer–Giles semi–inner product

$$[x, y] \;:=\; \|y\|_p^{2-p} \sum_{k\geq 1} |y_k|^{p-2}\,\overline{y_k}\,x_k,$$

for $y \neq 0$ (and $[x, 0] := 0$). Then $[y, y] = \|y\|_p^{2-p} \sum |y_k|^p = \|y\|_p^2$ and $|[x, y]| \leq \sqrt{[x, x]}\,\sqrt{[y, y]}$; thus $\|x\| := \sqrt{[x, x]} = \|x\|_p$ makes $(X, [\cdot,\cdot])$ a Dilbert space. Non–symmetry when $p \neq 2$ (explicitly for $p = 3$): with $x = (1, 0, \dots)$ and $y = (1, 1, 0, \dots)$,

$$\|y\|_3 = (|1|^3 + |1|^3)^{1/3} = 2^{1/3}, \quad [x, y] = \|y\|_3^{-1}\big(|1|\,\overline{1}\cdot 1 + |1|\,\overline{1}\cdot 0\big) = 2^{-1/3},$$

while $\|x\|_3 = 1$ and

$$[y, x] = \|x\|_3^{-1}\big(|1|\,\overline{1}\cdot 1 + |0|\,\overline{0}\cdot 1\big) = 1 \;\neq\; [x, y].$$

Hence $(\ell^p, [\cdot,\cdot])$ is a Dilbert (Banach with semi–inner product) but not a Hilbert space unless $p = 2$.

Definition 8.13.5 (Multilbert Space (projective limit of Hilbert spaces))**.** Let X be a Hausdorff locally convex space. Suppose there exists a directed family $\{\langle\cdot,\cdot\rangle_j\}_{j\in J}$ of inner products on a common dense subspace $D \subseteq X$ such that the locally convex topology of X is generated by the seminorms $p_j(x) := \sqrt{\langle x, x\rangle_j}$ and X is complete for this topology. Equivalently, for each j let H_j be the Hilbert completion of $(D, \langle\cdot,\cdot\rangle_j)$ and assume the natural maps $X \to H_j$ are continuous with

$$X \;\cong\; \operatorname*{proj\,lim}_{j\in J} H_j.$$

Then $(X, \{\langle\cdot,\cdot\rangle_j\}_{j\in J})$ is called a *Multilbert space*.

Remark 8.13.6 (Canonical cases)**.** If the family is countable and the seminorms are Hilbert norms $\|\cdot\|_n$ increasing with n, then X is a Fréchet (countably Hilbert) space; typical examples are Schwartz spaces or C^∞–spaces with Sobolev inner products.

Example 8.13.7 (Multilbert space: the Schwartz space $\mathcal{S}(\mathbb{R})$ as a projective limit of Hilbert spaces)**.** For $m \in \mathbb{N}$ define the Hilbert space

$$H_m \;:=\; \Big\{ f \in L^2(\mathbb{R}) \;\Big|\; \|f\|_{H_m}^2 := \sum_{a+b \le m} \big\| x^a D^b f \big\|_{L^2}^2 < \infty \Big\}.$$

Let $D = \mathcal{S}(\mathbb{R})$ and endow D with the countable family of Hilbertian seminorms $p_m(f) := \|f\|_{H_m}$. Then $X := (D, \{p_m\}_{m\ge 1})$ is complete and the canonical maps $X \to H_m$ are continuous with

$$X \;\cong\; \operatorname*{proj\,lim}_{m\to\infty} H_m,$$

so X is a Multilbert space (countably Hilbert, Fréchet). Concrete finiteness check with $f(x) = e^{-x^2}$:

$$Df(x) = -2xe^{-x^2}, \quad \|Df\|_{L^2}^2 = \int_{\mathbb{R}} 4x^2 e^{-2x^2}\, dx = 4 \cdot \frac{\sqrt{\pi}}{2 \cdot 2^{3/2}} = \frac{\sqrt{\pi}}{\sqrt{2}} < \infty,$$

and similarly $\|x^a D^b f\|_{L^2} < \infty$ for all a, b, proving $f \in \bigcap_m H_m = \mathcal{S}(\mathbb{R})$.

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Data Availability

This book is theoretical and did not generate or analyze any empirical data. We welcome future studies that apply and test these concepts in practical settings.

Research Integrity

The author confirms that this manuscript is original, has not been published elsewhere, and is not under consideration by any other journal.

Use of Computational Tools

All proofs and derivations were performed manually; no computational software (e.g., Mathematica, SageMath, Coq) was used.

Code Availability

No code or software was developed for this study.

Ethical Approval

This research did not involve human participants or animals, and therefore did not require ethical approval.

Use of Generative AI and AI-Assisted Tools

We use generative AI and AI-assisted tools for tasks such as English grammar checking, and We do not employ them in any way that violates ethical standards.

Supplementary Information

No supplementary materials accompany This book.

Disclaimer

The ideas presented here are theoretical and have not yet been validated through empirical testing. While we have strived for accuracy and proper citation, inadvertent errors may remain. Readers should verify any referenced material independently. The opinions expressed are those of the authors and do not necessarily reflect the views of their institutions.

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This book represents a revised and expanded synthesis of prior work, particularly the volume *Neutrosophy Revisited*, while incorporating further developments inspired by the *Nidus idearum series* and related research. It systematically re-formalizes and consolidates a wide spectrum of concepts in neutrosophic logic, plithogenic set theory, and associated mathematical structures.

The work emphasizes the transition from exploratory and conceptual formulations toward coherent, rigorous, and unified theoretical frameworks. It presents refined definitions, extended structures (including n-refined and multi-valued neutrosophic systems), and connections to diverse domains such as discrete mathematics, algebra, applied sciences, and emerging neutrosophic technologies.

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