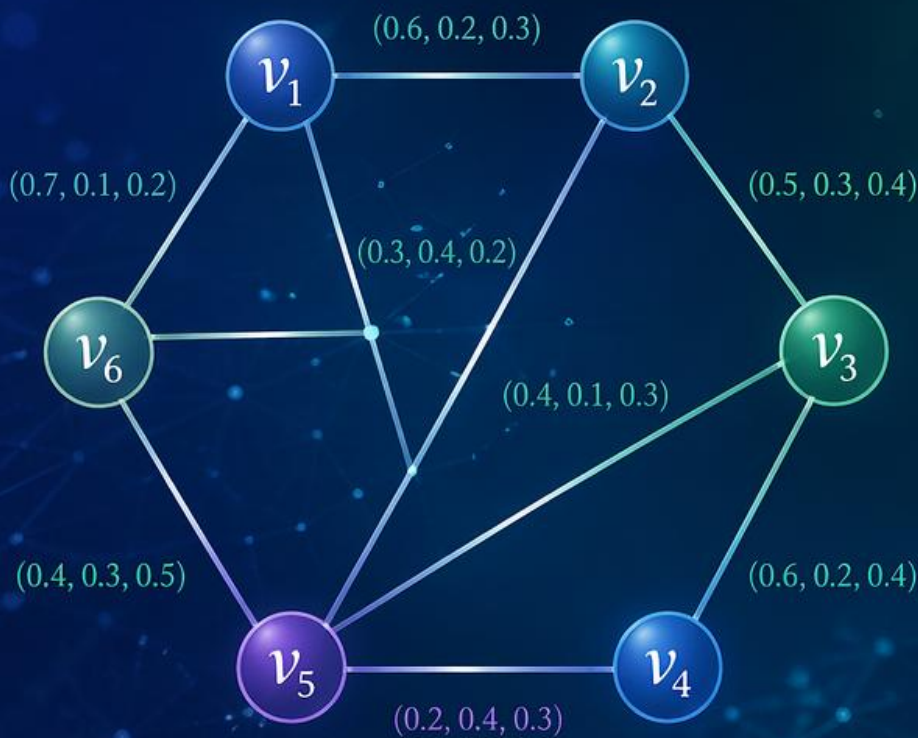


TOPOLOGICAL INDICES

ON

SINGLE-VALUED NEUTROSOPHIC GRAPHS



Wiener Index

$$W(G) = \sum_{u < v} d(u, v)$$



Connectivity Index

$$CI(G) = \sum_{uv \in E(G)} \frac{1}{\sqrt{s_u s_v}}$$



Sombor Index

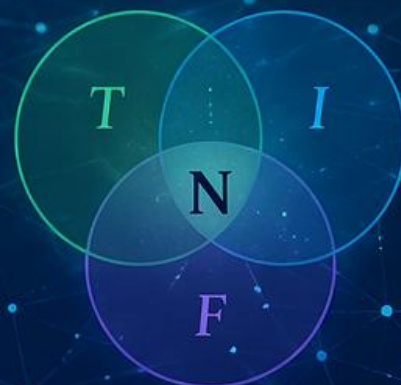
$$SO(G) = \sum_{uv \in E(G)} \sqrt{s_u^2 + s_v^2}$$

MASOUD GHODS (Semnan University)

ZAHRA ROSTAMI (Semnan University)

TAKA AKI FUJITA

FLORENTIN SMARANDACHE



$T + I + F$
indeterminacy

Masoud Ghods, Zahra Rostami, Takaaki Fujita, Florentin Smarandache

Topological indices on single-valued Neutrosophic Graphs

Masoud Ghods (Associate Professor at Faculty of Semnan University);

Zahra Rostami (Researcher at Semnan University);

Takaaki Fujita (Independent Researcher, Shinjuku, Shinjuku-ku, Tokyo, Japan)

and

Florentin Smarandache (Professor at the University of New Mexico)



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University of New Mexico

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University of Guayaquil

Av. Kennedy and Av. Delta

“Dr. Salvador Allende” University Campus

Guayaquil 090514, Ecuador



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Peer-Reviewers:

Le Hoang Son

VNU Univ. of Science, Vietnam National Univ. Hanoi, Vietnam
Email: sonlh@vnu.edu.vn

Francisco Chiclana

School of Computer Science and Informatics, De Montfort
University, Leicester, United Kingdom
Email: chiclana@dmu.ac.uk

Xiaohong Zhang

Department of Mathematics, Shaanxi University of Science &
Technology, Xian 710021, China
Email: zhangxh@shmtu.edu.cn

Madjid Tavanab

Business Information Systems Department, Faculty of Business
Administration and Economics, University of Paderborn,
D-33098 Paderborn, Germany
Email: tavana@lasalle.edu

Authors:

Masoud Ghods (Associate Professor at Faculty of Semnan University);
Zahra Rostami (Researcher at Semnan University);
Takaaki Fujita (Independent Researcher, Shinjuku, Shinjuku-ku, Tokyo, Japan)
and
Florentin Smarandache (Professor at the University of New Mexico, USA)

Foreword

We first express our heartfelt gratitude to the Merciful God, whose boundless generosity granted us the opportunity to devote ourselves to scientific inquiry and empowered us to contribute meaningfully to this field. His guidance has been our constant source of inspiration and strength throughout the development of this work.

The choice of the topic “Topological Indices on Single-Valued Neutrosophic Graphs” stems from our deep interest in novel mathematical structures and their wide-ranging applications in modeling complex real-world problems. In the current era, many phenomena and systems involve uncertainty, ambiguity, and inconsistency—from social networks and biological systems to medical, economic, and engineering data. Neutrosophic structures, particularly neutrosophic graphs, provide powerful tools for representing and analyzing these complexities by considering three components: truth, falsity, and indeterminacy. This approach enables more precise and realistic modeling compared to classical structures.

In this book, we have endeavored to define and investigate several key topological indices for the first time on single-valued neutrosophic graphs, including the degree-based index, the Wiener index, and the Sombor index. These indices play a crucial role in analyzing graph structures, measuring distances, complexities, and connections within networks. Our research in this area has been previously published in several scientific articles and is now presented here as a cohesive and systematic volume to provide a solid foundation for further development in the field.

Given the specialized nature of the subject, this book will serve graduate students, researchers, and professionals in mathematics, modeling, data science, and engineering who have a foundational understanding of neutrosophic concepts and complex graph theory. We recommend that readers possess a basic background in graph theory, topological indices, and neutrosophic logic to fully engage with the material.

The present work is organized into four chapters:

Chapter 1 introduces the fundamentals, basic definitions, and preliminary concepts.

Chapter 2 explores the degree-based index in neutrosophic graphs.

Chapter 3 presents and analyzes the Wiener index.

Chapter 4 defines and examines the Sombor index.

In each chapter, we present relevant theorems with rigorous mathematical proofs, accompanied by numerical and practical examples to clarify the concepts. Our objective in including these examples is to bridge theory and application and to facilitate the reader’s comprehension and engagement with the material.

Finally, we wish to extend our sincere gratitude to Semnan University and our academic colleagues for their unwavering support and encouragement throughout this scholarly journey. We regard this book as the starting point for future research that may lead to the development of new indices, the design of related algorithms, and the expansion of interdisciplinary applications

of neutrosophic graphs. It is our hope that this work will inspire further investigations in this emerging field and contribute meaningfully to the advancement of knowledge.

Masuod Ghods
Department of Mathematics, Semnan University
sep 2025

Chapter 1

Fundamentals of Graph Theory, Neutrosophic Sets, and Topological Indices

1.1 Introduction

The intricate structures and systems defining our world are fundamentally shaped by complex interrelationships among their constituent elements. Graph theory, with its elegant abstraction of entities as vertices (nodes) and their connections as edges, has served for decades as the universal language for modeling, analyzing, and understanding these relational networks. Its profound impact spans disciplines as diverse as mathematics, computer science, chemistry, engineering, sociology, and biology, providing indispensable tools to represent and interrogate connectivity patterns.

Yet, the real-world phenomena we seek to model are frequently imbued with inherent uncertainty, indeterminacy, inconsistency, and imprecision. While fuzzy logic pioneered the mathematical formalization of vagueness through partial membership, it often proves insufficient when confronted with scenarios involving simultaneous, independent degrees of truth, falsity, and indeterminacy arising from conflicting information sources or inherent ambiguity. This critical limitation is addressed by Neutrosophic Set Theory, a revolutionary framework introduced by Florentin Smarandache. Neutrosophic logic transcends its predecessors by characterizing each element with three independent membership functions: the degree of truth (T), the degree of falsity (F), and the degree of indeterminacy (I). This tripartite structure offers unparalleled flexibility in representing

the nuanced and often contradictory nature of complex systems.

The fusion of the robust modeling capabilities of graph theory with the inherent power of neutrosophic sets to handle three-dimensional uncertainty gives rise to Neutrosophic Graphs. In these advanced structures, vertices, edges, or both, can carry neutrosophic information. This enables a significantly more realistic representation of systems plagued by uncertainty, such as:

Social networks with ambiguous or evolving relationships.

Transportation or communication networks with unreliable links (varying T, F, I for connectivity).

Decision-making systems involving conflicting criteria or expert opinions.

Biological networks or molecular structures with uncertain interactions or bonds.

Topological Indices (TIs) are numerical graph invariants derived solely from a graph's structural topology. These powerful descriptors condense essential structural characteristics—such as size, branching, cyclicity, connectivity, and overall complexity—into single numerical values or vectors. Historically pivotal in mathematical chemistry (e.g., Wiener index for boiling points, Zagreb indices for molecular energy), TIs have evolved into indispensable tools across numerous fields, including drug discovery, materials science, complex network analysis (e.g., internet resilience, social network centrality), and bioinformatics (e.g., protein-protein interaction networks).

The Core Objective of Chapter 1 is to establish the rigorous theoretical bedrock essential for the development, analysis, and application of Neutrosophic Topological Indices explored in the subsequent chapters of this book. We meticulously construct this foundation through a structured exposition of the following critical components:

Fundamentals of Classical Graph Theory: A concise review of essential concepts, including graph types (simple, directed, weighted, etc.), key matrices (adjacency, incidence, distance), basic operations (union, join, product), and fundamental properties (degree, distance, connectivity).

Essentials of Fuzzy and Neutrosophic Set Theory: Precise definitions of fuzzy sets, intuitionistic fuzzy sets, and neutrosophic sets. Formalization of set operations (complement, union, intersection), relations, and functions within these frameworks, highlighting their axiomatic differences and

expressive capabilities.

Neutrosophic Graph Structures: Systematic introduction and formal definition of major neutrosophic graph models, including:

(Single-Valued) Neutrosophic Graphs (where T, F, I are single values in $[0,1]$).

Interval-Valued Neutrosophic Graphs (where T, F, I are intervals within $[0,1]$).

Simplified Neutrosophic Graphs (focusing on truth-membership only).

Potential distinctions between vertex-neutrosophic, edge-neutrosophic, and fully neutrosophic graphs.

Principles of Classical Topological Indices: Definition and categorization of key topological indices, emphasizing their structural basis:

Degree-Based Indices: (e.g., First Zagreb (M_1), Second Zagreb (M_2), Randić, Atom-Bond Connectivity (ABC), Geometric-Arithmetic (GA)).

Distance-Based Indices: (e.g., Wiener (W), Harary, Gutman).

Spectrum-Based Indices: (e.g., derived from the eigenvalues of adjacency or Laplacian matrices).

Basic properties and illustrative calculations.

Essential Theorems and Lemmas: Presentation and proof (or citation with context) of fundamental theorems and lemmas from classical graph theory and neutrosophic set/graph theory that underpin the derivations, generalizations, and proofs presented in Chapters 2, 3, and 4. This includes properties of graph operations under neutrosophic settings and inequalities relevant to index behavior.

This chapter serves as the indispensable catalyst for the entire book. A thorough comprehension of the concepts meticulously assembled and elucidated here is the mandatory prerequisite for venturing into the novel and fertile ground of neutrosophic topological indices. The definitions, notations, and theoretical results established in Chapter 1 provide the common language and logical scaffolding upon which the original contributions of the subsequent chapters—focusing on the generalization, computation, analysis, and application of topological indices within the rich and challenging context of neutrosophic graphs—are built. Our aim is to equip the reader with the con-

fidence and mastery necessary to navigate the compelling complexities and harness the significant potential of this rapidly evolving research frontier.

1.2 Simple Graph and some Concepts Its

Defintion 1.2.1. *a graph G consists of a set $V(G)$ of elements called vertices and a set $E(G)$ of elements called edges (arcs) together with a relation of incidence which associates with each member a pair of vertices, called its ends.*

Defintion 1.2.2. *Two graphs G_1 and G_2 are called isomorphic if there is a one to one correspondence between their vertex sets and adjacency is preserved. for example,*

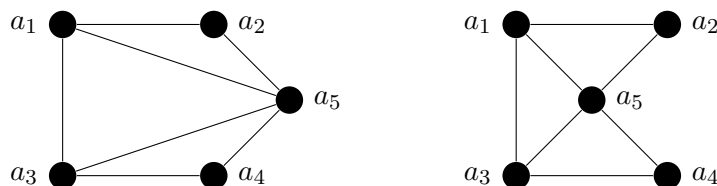


Figure 1.1: Isomorphic Graphs

Defintion 1.2.3. *Two vertex of graph are called adjacent if these nodes of a edge. A edge is called incident with a vertex if it is an end vertex of a edge.*

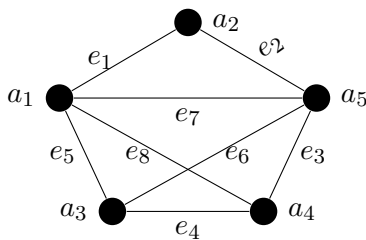


Figure 1.2: Simple graph G

for example in figure 1.2, two a_1 and a_4 are adjacent. Too, a_2 is incident with edge e_1 and e_2 .

Defintion 1.2.4. *A cycle graph is a graph consisting of a single cycle. Therefore C_k is a polygon with k members.*

Defintion 1.2.5. A path graph is a graph consisting of a single path. Hence P_k is a path with k nodes and $(k-1)$ members.

Defintion 1.2.6. A complete graph is a graph in which every two distinct nodes are connected by exactly one member. A complete graph with N nodes is denoted by K_N . For example:



Figure 1.3: Complete Graphs K_3 and K_4 .

1.3 Neutrosophic Set and Relation

In this section, we will cover some basic concepts about neutrosophic graphs.

Defintion 1.3.1. let A be a set of points. A Single-Valued neutrosophic set N on a nonempty set A is charactrized by a truth-membership function $T_N : A \rightarrow [0, 1]$, indeterminacy-membership function $I_N : A \rightarrow [0, 1]$, and a falsity-membership function $F_N : A \rightarrow [0, 1]$. So, $N = \{v, T_N(v), I_N(v), F_N(v) | v \in A\}$.

Example 1.3.2. Suppose $A = \{v_1, v_2, v_3, v_4\}$ is a reference collection, such that v_1 is the amount of red, v_2 is the amount of blue, v_3 is the amount of green, and v_4 is the amount of yellow that can be detected in a painting painted with a combination of these four colors. Then suppose N is a single-valued neutrosiphic set of A such that:

$$N = \left\{ (v_1, 0.3, 0.4, 0.2), (v_2, 0.5, 0.3, 0.2), (v_3, 0.3, 0.3, 0.5), (v_4, 0.1, 0, 0.8) \right\}.$$

1.4 Neutrosophic Graph Structures

Defintion 1.4.1. A single-valued neutrosophic graph on a nonempty V is a pair $G = (N, M)$.

Where N is single-valued neutrosophic set in V and M single-valued neutrosophic relation on V

such that

$$T_M(uv) \leq \min T_N(u), T_N(v),$$

$$I_M(uv) \leq \min I_N(u), I_N(v),$$

$$F_M(uv) \leq \max F_N(u), F_N(v),$$

For all $u, v \in V$. N is called single-valued neutrosophic vertex set of G and, M is called single-valued neutrosophic edge set of G , respectively.

Defintion 1.4.2. Let $G = (N, M)$ be a Single-Valued Neutrosophic Graph and P is a path in G . P is a collection of different vertices, $v_0, v_1, v_2, \dots, v_n$ such that $(T_M(v_{i-1}, v_i), I_M(v_{i-1}, v_i), F_M(v_{i-1}, v_i)) > 0$ for $0 \leq i \leq n$. P is a Neutrosophic cycle if $v_0 = v_n$ and $n \geq 3$.

Defintion 1.4.3. Let $G = (N, M)$ be the Neutrosophic Graph of G^* . If $H = (N', M')$ is a neutrosophic graph of G^* such that

$$T'(u) \leq T(u), \quad I'(u) \geq I(u), \quad F'(u) \geq F(u) \quad \forall u \in V;$$

$$T'(uv) \leq T(uv), \quad I'(uv) \geq I(uv), \quad F'(uv) \geq F(uv) \quad \forall uv \in E;$$

Then H is called a Neutrosophic subgraph of the Neutrosophic graph G .

Defintion 1.4.4. the order and the size of a neutrosophic graph $G = (N, M)$ are denoted by $O(G)$ and $S(G)$, respectively, and are defined as

$$O(G) = \left(\sum_{u \in V} T_N(u), \sum_{u \in V} I_N(u), \sum_{u \in V} F_N(u) \right),$$

And

$$S(G) = \left(\sum_{uv \in E} T_M(uv), \sum_{uv \in E} I_M(uv), \sum_{uv \in E} F_M(uv) \right).$$

Defintion 1.4.5. A neutrosophic graph $G = (N, M)$ is called a complete neutrosophic graph if the following conditions are satisfied:

$$T_M(uv) = \min T_N(u), T_N(v),$$

$$I_M(uv) = \min I_N(u), I_N(v),$$

$$F_M(uv) = \max F_N(u), F_N(v),$$

for all $u, v \in V$.

For example:

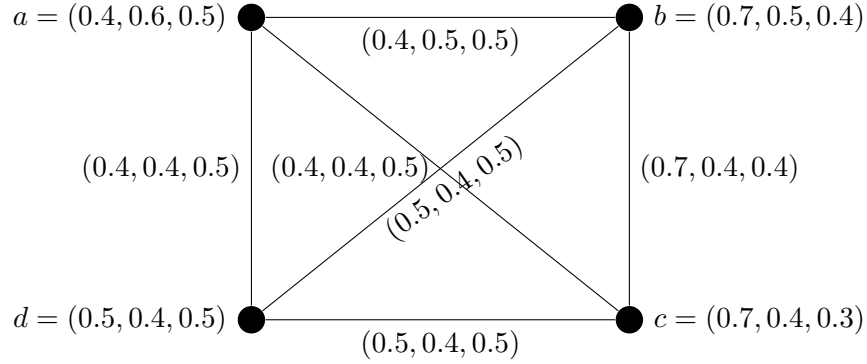


Figure 1.4: Complete Neutrosophic Graph

Defintion 1.4.6. Suppose $G = (N, M)$ a single-valued Neutrosophic graph. G is a connected Single-Valued Neutrosophic Graph if there exists no isolated vertex in G . ($v \in V_G$ is the isolated vertex, if there exists no incident edge to the vertex v .)

Defintion 1.4.7. Given $G = (N, M)$ is a single-valued neutrosophic graph, and $v \in V$ is vertex of G . the degree of the vertex v is the sum of the truth membership values, the sum of the indeterminacy membership values, and the sum of the falsity membership values of all the edges that are adjacent to vertex v , and is denoted by $d(v)$, that

$$d_G(u) = \left(\sum_{u \neq v} T_M(uv), \sum_{u \neq v} I_M(uv), \sum_{u \neq v} F_M(uv) \right),$$

$$Td_G(u) = \left(\sum_{u \neq v} T_M(uv) + T_N(u), \sum_{u \neq v} I_M(uv) + I_N(u), \sum_{u \neq v} F_M(uv) + F_N(u) \right)$$

Defintion 1.4.8. A neutrosophic graph $G_1 = (N_1, M_1)$ of the graph $G_1^* = (V_1, E_1)$ is isomorphic with neutrosophic graph $G_2 = (N_2, M_2)$ of the graph $G_2^* = (V_2, E_2)$ if we have f where $f : V_1 \rightarrow V_2$ is a bijection and following relations are satisfied

$$T_{N_1}(u) = T_{N_2}(f(u)), I_{N_1}(u) = I_{N_2}(f(u)), F_{N_1}(u) = F_{N_2}(f(u)),$$

For all $u \in V_1$ and

$$T_{M_1}(uv) = T_{M_2}(f(u)f(v)), I_{M_1}(uv) = I_{M_2}(f(u)f(v)), F_{M_1}(uv) = F_{M_2}(f(u)f(v)),$$

For all $uv \in E_1$.

Defintion 1.4.9. the m -barbell graph $B_{(m,m)}$ is the simple graph obtained by connecting two copies of a complete graph K_m by abridge.

Example 1.4.10. The figure below shows a m -barbell graph where each part is a complete graph k_4 .

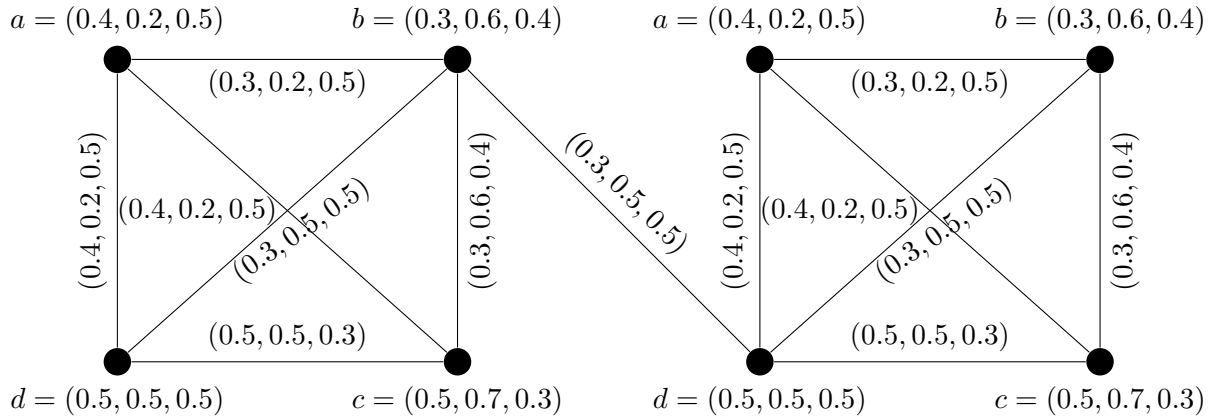


Figure 1.5: 4-barbell graph $B_{(4,4)}$

Defintion 1.4.11. Given $G = (N, M)$ is a single-valued neutrosophic graph, and the d_m - degree of any vertex v in G is denoted as $d_m(v)$ where

$$d_m(v) = \left(\sum_{u \neq v \in V} T_M^m(u, v), \sum_{u \neq v \in V} I_M^m(u, v), \sum_{u \neq v \in V} F_M^m(u, v) \right)$$

Here, the path $v = v_0, v_1, v_2, \dots, v_n = u$ is the shortest path between the vertices v and u , when the length of this path is m .

Defintion 1.4.12. Given $G = (N, M)$ is a single-valued neutrosophic graph, G is called a regular neutrosophic graph if any vertex has the same degree, that is, for all u in $V(G)$ we have $d_G(u) = (d_1, d_2, d_3)$.

Defintion 1.4.13. Given $G = (N, M)$ is a single-valued neutrosophic graph, G is called a totally regular neutrosophic graph if any vertex has the same degree, that is, for all u in $V(G)$ we have $Td_G(u) = (d_1, d_2, d_3)$.

Defintion 1.4.14. Given $G = (N, M)$ is a single-valued neutrosophic graph, G is a complement regular neutrosophic graph if it satisfies the following,

$$\sum_{v \neq u} T_M(v, u) = c, \quad \sum_{v \neq u} I_M(v, u) = c, \quad \sum_{v \neq u} F_M(v, u) = c,$$

Where c is a constant value.

Defintion 1.4.15. The maximum degree of a neutrosophic graph $G = (N, M)$ is defined as $\Delta(G) = (\Delta_T(G), \Delta_I(G), \Delta_F(G))$, such that,

$$\Delta_T(G) = \max\{d_T(u) : u \in V\},$$

$$\Delta_I(G) = \max\{d_I(u) : u \in V\},$$

$$\Delta_F(G) = \max\{d_F(u) : u \in V\}.$$

Defintion 1.4.16. The minimum degree of a neutrosophic graph $G = (N, M)$ is defined as $\delta(G) = (\delta_T(G), \delta_I(G), \delta_F(G))$, such that,

$$\delta_T(G) = \min\{d_T(u) : u \in V\},$$

$$\delta_I(G) = \min\{d_I(u) : u \in V\},$$

$$\delta_F(G) = \min\{d_F(u) : u \in V\}.$$

Example 1.4.17. suppose the neutrosophic graph G in figure 1.6 by direct calculations,

Now, according to Figure 1.6, we have:

$$\Delta(G) = (1.2, 1.6, 1.0),$$

$$\delta(G) = (0.5, 0.7, 0.9).$$

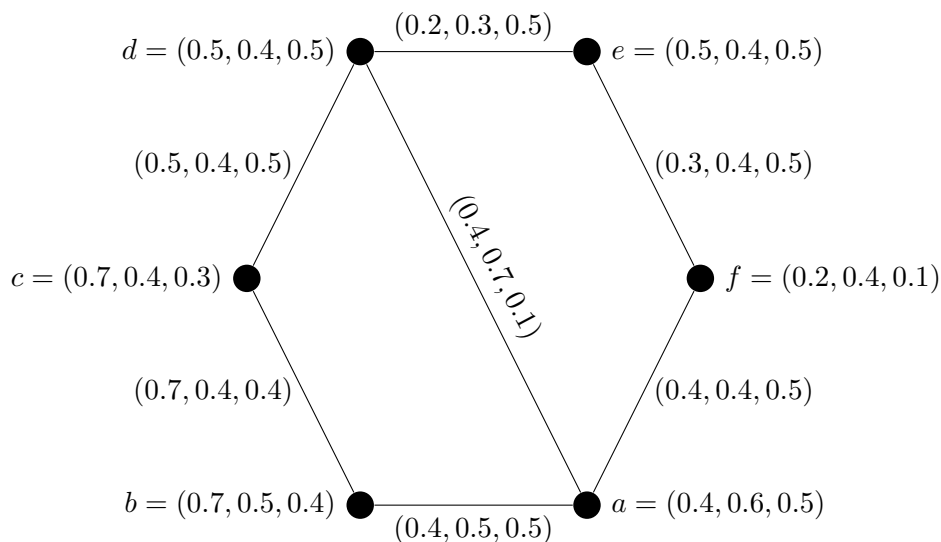


Figure 1.6: A Neutrosophic Graph

1.5 Topological Indices

A topological index is a quantitative characteristic of a graph that helps us to discuss and compare different graphs in terms of their structural properties. These indices can be used for chemical graphs, graphs related to social problems, and any other type of problem that can be modeled graphically.

In this book, we have used three widely used indices: Wiener Index, Connectivity Index, and Sombor index. The definition related to each of these indices is provided below.

1.5.1 Wiener Index

The Wiener Index is commonly considered as the most classic and widely used distance-based index in graph theory.

Defintion 1.5.1. *Given a graph $G = (V, E)$, the "Wiener index" of G is defied as:*

$$W(G) = \sum_{\{u,v\} \subseteq V(G)} d(u,v),$$

where $d(u,v)$ is the distance between vertices u and v in G .

Example 1.5.2. Suppose $G = (V, E)$ is a simple graph.

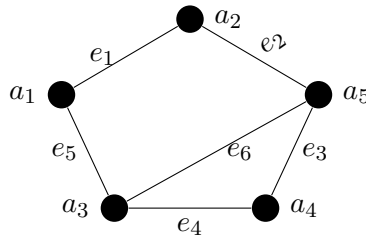


Figure 1.7: Simple graph G

then, we have:

$$W(G) = \sum_{\{u,v\} \subseteq V(G)} d(u,v) = 1 + 1 + 2 + 2 + 2 + 2 + 1 + 1 + 1 + 1 = 14.$$

1.5.2 Connectivity Index

The most natural parameter associated with a network is its connectivity. for example, in Internet, the possible bandwidth between two routers is the most significant parameter.

Defintion 1.5.3. Let $G = (V, E)$ be a simple graph. The "Connectivity Index" of G is defined by:

$$C(G) = \sum_{u,v \in V(G)} d_u d_v.$$

Example 1.5.4. Suppose $G = (V, E)$ is a connected graph.

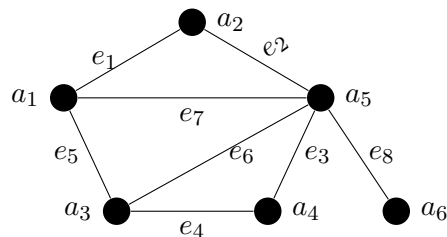


Figure 1.8: Simple graph G

then, we have:

$$\begin{aligned}
 C(G) &= \sum_{u,v \in V(G)} d_u d_v = (3 \times 2) + (3 \times 3) + (3 \times 2) + (3 \times 5) + (3 \times 1) \\
 &\quad + (2 \times 3) + (2 \times 2) + (2 \times 5) + (2 \times 1) + (3 \times 2) \\
 &\quad + (3 \times 5) + (3 \times 1) + (2 \times 5) + (2 \times 1) + (5 \times 1) \\
 &= 102.
 \end{aligned}$$

1.5.3 Sombor Index

The Sombor index is a degree-based topological index. This index was first introduced by Gutman in 2020. This index has different variants, which we have included here as needed for the definitions used in Chapter 4.

Defintion 1.5.5. *If $G = (V, E)$ is a simple graph, then "Sombor Index" defined as:*

$$SO = SO(G) = \sum_{e_{ij} \in E(G)} \sqrt{d_i^2 + d_j^2}.$$

Defintion 1.5.6. *In simple graph $G = (V, E)$, the "Average Sombor Index" defined as:*

$$SO_{Avr} = SO_{Avr}(G) = \sum_{e_{ij} \in E(G)} \sqrt{\left(d_i - \frac{2m}{n}\right)^2 + \left(d_j - \frac{2m}{n}\right)^2},$$

where, $m = |E(G)|$ and $n = |V(G)|$.

Defintion 1.5.7. *In simple graph $G = (V, E)$, the "Reduced Sombor Index" defined as:*

$$SO_{Red} = SO_{Red}(G) = \sum_{e_{ij} \in E(G)} \sqrt{(d_i - 1)^2 + (d_j - 1)^2}.$$

Example 1.5.8. *Suppose $G = (V, E)$ is a connected graph.*

then, we have:

$$\begin{aligned}
 SO = SO(G) &= \sum_{e_{ij} \in E(G)} \sqrt{d_i^2 + d_j^2} = \sqrt{4+4} + \sqrt{4+25} + \sqrt{4+25} \\
 &\quad + \sqrt{1+25} + \sqrt{1+25} + \sqrt{1+25} \\
 &= \sqrt{8} + \sqrt{29} + \sqrt{29} + 3 \times \sqrt{26} \\
 &= 28.89
 \end{aligned}$$

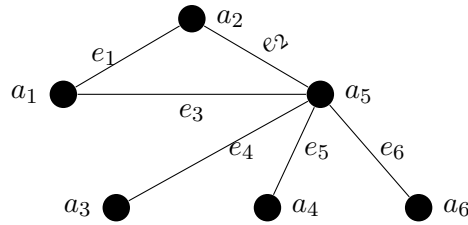


Figure 1.9: Simple graph G

Since $m = 6$ and $n = 6$, we have:

$$\begin{aligned}
 SO_{Avr} = SO_{Avr}(G) &= \sum_{e_{ij} \in E(G)} \sqrt{\left(d_i - \frac{2m}{n}\right)^2 + \left(d_j - \frac{2m}{n}\right)^2} = \sqrt{0+0} + \sqrt{0+9} \\
 &+ \sqrt{0+9} + \sqrt{1+9} + \sqrt{1+9} + \sqrt{1+9} \\
 &= 0 + \sqrt{9} + \sqrt{9} + 3 \times \sqrt{10} \\
 &= 15.49.
 \end{aligned}$$

1.6 Summary

This chapter has attempted to provide the basic definitions needed for the subsequent chapters. These definitions have been derived from authoritative sources cited at the end of the book, and an attempt has been made to provide the shortest and clearest definitions here. A number of examples have also been provided to clarify the definitions. In the second chapter, we will define and generalize the connectivity index on neutrosophic graphs.

Chapter 2

Connectivity Indices in Neutrosophic Graphs: Computation, Properties, and Theorems

2.1 Introduction

In this chapter, we first define the connected neutrosophic graph and connectivity index in the neutrosophic graphs. Note that definitions are provided for a connected neutrosophic graph in some references [5, 6], but the definition we use here will be based on connectivity. After providing some examples, the theorems related to the connectivity index are expressed and proved in neutrosophic graphs.

Connectivity stands as one of the most fundamental and operationally significant properties in graph theory. Classical measures like vertex connectivity ($V(G)$) and edge connectivity ($E(G)$) quantify the minimum number of vertices or edges whose removal disconnects a graph, directly reflecting its robustness, vulnerability, and structural cohesion. These concepts underpin critical analyses in network reliability, communication stability, social community detection, and biological pathway integrity.

However, as rigorously established in Chapter 1, neutrosophic graphs (NGs) model systems where entities and their relationships inherently possess simultaneous, independent degrees of truth (T), falsity (F), and indeterminacy (I). In such inherently uncertain environments, traditional binary notions of "connected" or "disconnected" become inadequate. The very existence of a vertex,

edge, or path may be partial, contradictory, or indeterminate. Consequently, classical connectivity measures fail to capture the nuanced, multi-dimensional nature of connection and disconnection within neutrosophic structures.

This chapter addresses this fundamental gap. Building upon the foundational concepts of neutrosophic sets and graphs presented in Chapter 1, we embark on the formal definition, theoretical development, and computational characterization of Neutrosophic Connectivity Indices.

The development of neutrosophic connectivity indices represents a critical advancement in analyzing uncertain networked systems. Unlike simply fuzzifying classical measures, our approach intrinsically respects the three-dimensional (T, F, I) nature of neutrosophic information.

2.2 Partial connectivity index in neutrosophic graphs

Here we first define the Partial and totally connectivity indices in neutrosophic graphs and provide examples to better understand it. And then in the next part we will present the boundaries for the Partial and totally connectivity indices in neutrosophic graphs.

Defintion 2.2.1. *Let $G = (N, M)$ be the connected Neutrosophic Graph. The partial connectivity index of G is defined as*

$$PCI_T(G) = \sum_{u,v \in N} T_N(u)T_N(v)CONN_{TG}(u, v),$$

$$PCI_I(G) = \sum_{u,v \in N} I_N(u)I_N(v)CONN_{IG}(u, v),$$

$$PCI_F(G) = \sum_{u,v \in N} F_N(u)F_N(v)CONN_{FG}(u, v),$$

Where $CONN_{TG}(u, v)$ is the strength of truth, $CONN_{IG}(u, v)$ the strength of indeterminacy and $CONN_{FG}(u, v)$ the strength of falsity between two vertices u and v . We have

$$CONN_{TG}(u, v) = \max\{\min\{T_M(e)|e \in P \text{ and } P \text{ is a path between } u \text{ and } v\}\},$$

$$CONN_{IG}(u, v) = \min\{\max\{I_M(e)|e \in P \text{ and } P \text{ is a path between } u \text{ and } v\}\},$$

$$CONN_{FG}(u, v) = \min\{\max\{F_M(e)|e \in P \text{ and } P \text{ is a path between } u \text{ and } v\}\}.$$

Also, the totally connectivity index of G is defined as

$$TCI(G) = \frac{4 + 2PCI_T(G) - 2PCI_F(G) - PCI_I(G)}{6}.$$

Defintion 2.2.2. Let $G = (N, M)$ be the Neutrosophic graph. G called a connected neutrosophic graph if for any two vertices $u, v \in N$,

$$CONN_{T_G}(u, v) > 0, \quad CONN_{I_G}(u, v) > 0, \quad CONN_{F_G}(u, v) > 0.$$

Example 2.2.3. Consider the Neutrosophic graph $G = (N, M)$ with $V = \{a, b, c, d\}$, that shown in figure 2.1. As can be seen, $(T_N, I_N, F_N)(a) = (0.4, 0.6, 0.5)$, $(T_N, I_N, F_N)(b) = (0.7, 0.5, 0.4)$, $(T_N, I_N, F_N)(c) = (0.7, 0.4, 0.3)$, $(T_N, I_N, F_N)(d) = (0.5, 0.4, 0.5)$, The edge set contains $(T_M, I_M, F_M)(b, c) = (0.7, 0.4, 0.4)$, $(T_M, I_M, F_M)(c, d) = (0.5, 0.4, 0.5)$, $(T_M, I_M, F_M)(a, d) = (0.4, 0.4, 0.5)$, $(T_M, I_M, F_M)(b, d) = (0.3, 0.5, 0.7)$, $(T_M, I_M, F_M)(a, b) = (0.4, 0.5, 0.5)$. By direct calculations, we have

	$CONN_{T_G}(u, v)$	$CONN_{I_G}(u, v)$	$CONN_{F_G}(u, v)$
a, b	0.4	0.5	0.5
a, c	0.4	0.4	0.5
a, d	0.4	0.4	0.5
b, c	0.7	0.4	0.4
b, d	0.5	0.4	0.5
c, d	0.5	0.4	0.5

Table 2.1: The strength of connectedness between each pair of vertices u and v .

Then the partial connectivity index of G is,

$$\begin{aligned} PCI_T(G) &= \sum_{u, v \in N} T_N(u)T_N(v)CONN_{T_G}(u, v) = (0.4)(0.7)(0.4) + (0.4)(0.7)(0.4) \\ &\quad + (0.4)(0.5)(0.4) + (0.7)(0.7)(0.7) + (0.7)(0.5)(0.5) \\ &\quad + (0.7)(0.5)(0.5) = 0.112 + 0.112 + 0.080 + 0.147 \\ &\quad + 0.245 + 0.245 = 0.941, \end{aligned}$$

$$\begin{aligned}
PCI_I(G) &= \sum_{u,v \in N} I_N(u)I_N(v)CONN_{I_G}(u,v) = (0.6)(0.5)(0.5) + (0.6)(0.4)(0.4) \\
&\quad + (0.6)(0.4)(0.4) + (0.5)(0.4)(0.4) + (0.5)(0.4)(0.4) \\
&\quad + (0.4)(0.4)(0.4) = 0.180 + 0.096 + 0.096 + 0.080 \\
&\quad + 0.080 + 0.064 = 0.596,
\end{aligned}$$

$$\begin{aligned}
PCI_F(G) &= \sum_{u,v \in N} F_N(u)F_N(v)CONN_{F_G}(u,v) = (0.5)(0.4)(0.5) + (0.5)(0.3)(0.5) \\
&\quad + (0.5)(0.5)(0.5) + (0.4)(0.3)(0.4) + (0.4)(0.5)(0.5) \\
&\quad + (0.3)(0.5)(0.5) = 0.100 + 0.075 + 0.125 + 0.048 \\
&\quad + 0.100 + 0.075 = 0.523.
\end{aligned}$$

Also by definition (2.2.1), we have

$$\begin{aligned}
TCI(G) &= \frac{4 + 2PCI_T(G) - 2PCI_F(G) - PCI_I(G)}{6} = \frac{4 + 2(0.941) - 2(0.523) - 0.596}{6} \\
&= 0.707.
\end{aligned}$$

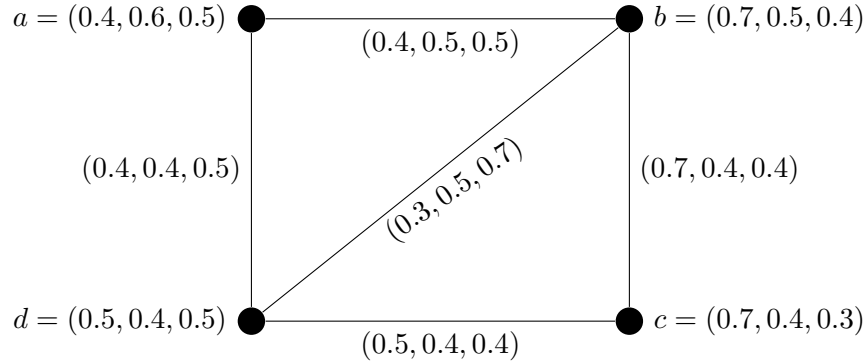


Figure 2.1: A neutrosophic graph with $V = \{a, b, c, d\}$

2.3 Examples of the connectivity index in neutrosophic special graphs

In this section, we calculate the connectivity index with examples for a number of neutrosophic graphs such as a neutrosophic graph with G^* of a cycle and a neutrosophic star.

Example 2.3.1. Suppose $G = (N, M)$ is a star neutrosophic as shown in Figure 2.2.

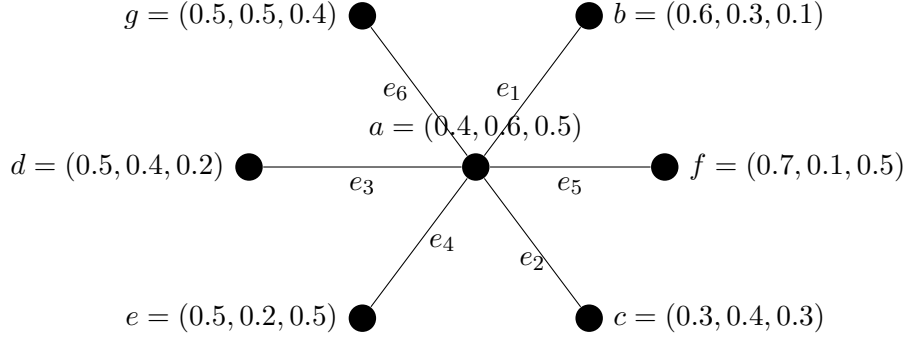


Figure 2.2: A neutrosophic Star

	(T_M, I_M, F_M)		(T_M, I_M, F_M)
e_1	$(0.4, 0.2, 0.5)$	e_4	$(0.7, 0.1, 0.2)$
e_2	$(0.3, 0.1, 0.4)$	e_5	$(0.4, 0.3, 0.5)$
e_3	$(0.7, 0.4, 0.4)$	e_6	$(0.2, 0.5, 0.3)$

Table 2.2: The edge between each pair of vertices u and v in G .

Then we have:

$$\begin{aligned}
 PCI_T(G) &= \sum_{u,v \in N} T_N(u)T_N(v)CONN_{T_G}(u,v) = (0.4)(0.6)(0.2) + (0.4)(0.3)(0.3) + (0.4)(0.5)(0.7) \\
 &\quad + (0.4)(0.5)(0.7) + (0.4)(0.7)(0.4) + (0.4)(0.5)(0.2) + (0.6)(0.3)(0.3) \\
 &\quad + (0.6)(0.5)(0.4) + (0.6)(0.5)(0.4) + (0.6)(0.7)(0.4) + (0.6)(0.5)(0.2) \\
 &\quad + (0.3)(0.5)(0.3) + (0.3)(0.5)(0.3) + (0.3)(0.7)(0.3) + (0.3)(0.5)(0.2) \\
 &\quad + (0.5)(0.5)(0.7) + (0.5)(0.7)(0.4) + (0.5)(0.5)(0.2) + (0.5)(0.7)(0.4) \\
 &\quad + (0.5)(0.5)(0.2) + (0.7)(0.5)(0.2) = 1.846,
 \end{aligned}$$

$$\begin{aligned}
 PCI_I(G) &= \sum_{u,v \in N} I_N(u)I_N(v)CONN_{I_G}(u,v) = (0.6)(0.3)(0.2) + (0.6)(0.4)(0.1) + (0.6)(0.4)(0.4) \\
 &\quad + (0.6)(0.2)(0.1) + (0.6)(0.1)(0.3) + (0.6)(0.5)(0.5) + (0.3)(0.4)(0.2) \\
 &\quad + (0.3)(0.4)(0.4) + (0.3)(0.2)(0.2) + (0.3)(0.1)(0.3) + (0.3)(0.5)(0.5) \\
 &\quad + (0.4)(0.4)(0.4) + (0.4)(0.2)(0.1) + (0.4)(0.1)(0.3) + (0.4)(0.5)(0.5) \\
 &\quad + (0.4)(0.2)(0.4) + (0.4)(0.1)(0.4) + (0.4)(0.5)(0.5) + (0.2)(0.1)(0.3) \\
 &\quad + (0.2)(0.5)(0.5) + (0.1)(0.5)(0.5) = 0.995,
 \end{aligned}$$

	$CONN_{T_G}(u, v)$	$CONN_{I_G}(u, v)$	$CONN_{F_G}(u, v)$
a, b	0.4	0.2	0.5
a, c	0.3	0.1	0.4
a, d	0.7	0.4	0.4
a, e	0.7	0.1	0.2
a, f	0.4	0.3	0.5
a, g	0.2	0.5	0.3
b, c	0.3	0.2	0.5
b, d	0.4	0.4	0.5
b, e	0.4	0.2	0.5
b, f	0.4	0.3	0.5
b, g	0.2	0.5	0.5
c, d	0.3	0.4	0.4
c, e	0.3	0.1	0.4
c, f	0.3	0.3	0.5
c, g	0.2	0.5	0.4
d, e	0.7	0.4	0.4
d, f	0.4	0.4	0.5
d, g	0.2	0.5	0.4
e, f	0.4	0.3	0.5
e, g	0.2	0.5	0.3
f, g	0.2	0.5	0.5

Table 2.3: The strength of connectedness between each pair of vertices u and v in Neutrosophic Star G .

$$\begin{aligned}
PCI_F(G) = \sum_{u,v \in N} F_N(u)F_N(v)CONN_{F_G}(u, v) &= (0.5)(0.1)(0.5) + (0.5)(0.3)(0.4) + (0.5)(0.2)(0.4) \\
&+ (0.5)(0.5)(0.2) + (0.5)(0.5)(0.5) + (0.5)(0.4)(0.3) + (0.1)(0.3)(0.5) \\
&+ (0.1)(0.2)(0.5) + (0.1)(0.5)(0.5) + (0.1)(0.5)(0.5) + (0.1)(0.4)(0.5) \\
&+ (0.3)(0.2)(0.4) + (0.3)(0.5)(0.4) + (0.3)(0.5)(0.5) + (0.3)(0.4)(0.4) \\
&+ (0.2)(0.5)(0.4) + (0.2)(0.5)(0.5) + (0.2)(0.4)(0.4) + (0.5)(0.5)(0.5) \\
&+ (0.5)(0.4)(0.3) + (0.5)(0.4)(0.5) = 1.069.
\end{aligned}$$

We have:

$$\begin{aligned}
TCI(G) &= \frac{4 + 2PCI_T(G) - 2PCI_F(G) - PCI_I(G)}{6} = \frac{4 + 2(1.846) - 2(1.069) - 0.995}{6} \\
&= 0.7598.
\end{aligned}$$

Example 2.3.2. Suppose $G = (N, M)$ is a neutrosophic graph with G^* is a cycle, as shown in Figure 2.3. Then we have:

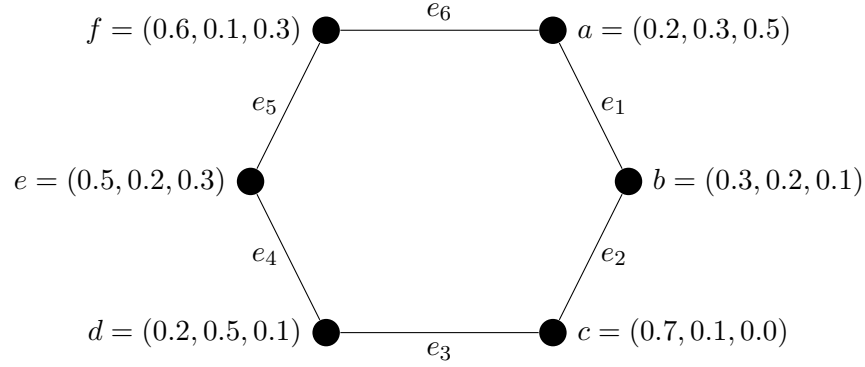


Figure 2.3: A neutrosophic Cycle

	(T_M, I_M, F_M)		(T_M, I_M, F_M)
e_1	$(0.2, 0.4, 0.3)$	e_4	$(0.2, 0.5, 0.3)$
e_2	$(0.3, 0.2, 0.1)$	e_5	$(0.5, 0.2, 0.3)$
e_3	$(0.2, 0.5, 0.1)$	e_6	$(0.2, 0.3, 0.1)$

Table 2.4: The edge between each pair of vertices u and v in G .

$$\begin{aligned}
 PCI_T(G) &= \sum_{u,v \in N} T_N(u)T_N(v)CONN_{T_G}(u, v) = (0.2)(0.3)(0.2) + (0.2)(0.7)(0.2) \\
 &\quad + (0.2)(0.2)(0.2) + (0.2)(0.5)(0.2) + (0.2)(0.6)(0.2) + (0.3)(0.7)(0.3) \\
 &\quad + (0.3)(0.2)(0.2) + (0.3)(0.5)(0.2) + (0.3)(0.6)(0.2) + (0.7)(0.2)(0.2) \\
 &\quad + (0.7)(0.5)(0.2) + (0.7)(0.6)(0.2) + (0.2)(0.5)(0.2) + (0.2)(0.6)(0.2) \\
 &\quad + (0.5)(0.6)(0.5) = 621,
 \end{aligned}$$

$$\begin{aligned}
 PCI_I(G) &= \sum_{u,v \in N} I_N(u)I_N(v)CONN_{I_G}(u, v) = (0.3)(0.2)(0.3) + (0.3)(0.1)(0.3) \\
 &\quad + (0.3)(0.5)(0.3) + (0.3)(0.2)(0.5) + (0.3)(0.1)(0.3) + (0.2)(0.1)(0.2) \\
 &\quad + (0.2)(0.5)(0.5) + (0.2)(0.2)(0.3) + (0.2)(0.1)(0.3) + (0.1)(0.5)(0.5) \\
 &\quad + (0.1)(0.1)(0.3) + (0.1)(0.1)(0.3) + (0.5)(0.2)(0.5) + (0.5)(0.1)(0.5) \\
 &\quad + (0.2)(0.1)(0.2) = 0.293,
 \end{aligned}$$

	$CONN_{T_G}(u, v)$	$CONN_{I_G}(u, v)$	$CONN_{F_G}(u, v)$
a, b	0.2	0.3	0.4
a, c	0.2	0.3	0.4
a, d	0.2	0.3	0.4
a, e	0.2	0.5	0.4
a, f	0.2	0.3	0.4
b, c	0.3	0.2	0.1
b, d	0.2	0.5	0.1
b, e	0.2	0.3	0.3
b, f	0.2	0.3	0.3
c, d	0.2	0.5	0.1
c, e	0.2	0.3	0.3
c, f	0.2	0.3	0.3
d, e	0.2	0.5	0.3
d, f	0.2	0.5	0.3
e, f	0.5	0.2	0.3

Table 2.5: The strength of connectedness between each pair of vertices u and v in Neutrosophic graph G .

$$\begin{aligned}
PCI_F(G) &= \sum_{u,v \in N} F_N(u)F_N(v)CONN_{F_G}(u, v) = (0.5)(0.1)(0.4) + (0.5)(0.0)(0.4) \\
&\quad + (0.5)(0.1)(0.4) + (0.5)(0.3)(0.4) + (0.5)(0.3)(0.4) + (0.1)(0.0)(0.1) \\
&\quad + (0.1)(0.1)(0.1) + (0.1)(0.3)(0.3) + (0.1)(0.3)(0.3) + (0.0)(0.1)(0.1) \\
&\quad + (0.0)(0.3)(0.3) + (0.0)(0.3)(0.3) + (0.1)(0.3)(0.3) + (0.1)(0.3)(0.3) \\
&\quad + (0.3)(0.3)(0.3) = 0.224,
\end{aligned}$$

Also by definition (2.2.1), we have

$$\begin{aligned}
TCI(G) &= \frac{4 + 2PCI_T(G) - 2PCI_F(G) - PCI_I(G)}{6} = \frac{4 + 2(0.621) - 2(0.224) - 0.293}{6} \\
&= 0.750.
\end{aligned}$$

2.4 Theories of connectivity index in neutrosophic graphs

In this section, a number of theorems related to the connectivity index on neutrosophic graphs are presented and proven.

Theorem 2.4.1. *Let $G = (N, M)$ be a connected neutrosophic graph and $H = (N', M')$ is a partial*

neutrosophic subgraph of G . then

$$PCI_T(H) \leq PCI_T(G), \quad PCI_I(H) \geq PCI_I(G), \quad PCI_F(H) \geq PCI_F(G)$$

Moreover, we have $TCI(H) \leq TCI(G)$.

Proof. Let $H = (N', M')$ is a partial neutrosophic subgraph of G , and $T_{(N')}(u) \leq T_N(u)$ for $u \in V$.

Since $T_{(M')}(uv) \leq T_M(uv)$ for uv , then $CONN_{TH}(u, v) \leq CONN_{TG}(u, v)$, thus we get

$$\begin{aligned} PCI_T(H) &= \sum_{u,v \in V} T_{N'}(u)T_{N'}(v)CONN_{TH}(u, v) \leq \sum_{u,v \in V} T_N(u)T_N(v)CONN_{TG}(u, v) \\ &= PCI_T(G). \end{aligned}$$

Using a similar proof, we can show that

$$\begin{aligned} PCI_I(H) &= \sum_{u,v \in V} I_{N'}(u)I_{N'}(v)CONN_{IH}(u, v) \geq \sum_{u,v \in V} I_N(u)I_N(v)CONN_{IG}(u, v) \\ &= PCI_I(G). \end{aligned}$$

And

$$\begin{aligned} PCI_F(H) &= \sum_{u,v \in V} F_{N'}(u)F_{N'}(v)CONN_{FH}(u, v) \geq \sum_{u,v \in V} F_N(u)F_N(v)CONN_{FG}(u, v) \\ &= PCI_F(G). \end{aligned}$$

Now, we show that

$$TCI(H) \leq TCI(G).$$

By definition totally connectivity index, and since $PCI_T(H) \leq PCI_T(G)$, $PCI_I(H) \geq PCI_I(G)$, $PCI_F(H) \geq PCI_F(G)$ we have

$$\begin{aligned} TCI(H) &= \frac{4 + 2PCI_T(H) - 2PCI_F(H) - PCI_I(H)}{6} \\ &\leq \frac{4 + 2PCI_T(G) - 2PCI_F(H) - PCI_I(H)}{6} \\ &\leq \frac{4 + 2PCI_T(G) - 2PCI_F(G) - PCI_I(G)}{6} \\ &= TCI(G). \end{aligned}$$

And, hence $TCI(H) \leq TCI(G)$. □

Example 2.4.2. Consider the neutrosophic graph $G = (N, M)$ whit

$$N = \{(a : 0.7, 0.3, 0.4), (b : 0.5, 0.2, 0.3), (c : 0.7, 0.3, 0.6), (d : 0.4, 0.3, 0.5)\},$$

$$M = \{(ab : 0.5, 0.2, 0.4), (ac : 0.7, 0.3, 0.6), (bc : 0.5, 0.2, 0.6), (cd : 0.4, 0.3, 0.6)\},$$

Also, let $H = (N', M')$ be a neutrosophic subgraph of G , whit

$$N' = \{(a : 0.6, 0.3, 0.5), (b : 0.4, 0.2, 0.4), (c : 0.6, 0.3, 0.7), (d : 0.3, 0.3, 0.6)\},$$

$$M' = \{(ab : 0.4, 0.2, 0.5), (ac : 0.5, 0.3, 0.7), (bc : 0.4, 0.2, 0.7), (cd : 0.3, 0.3, 0.7)\},$$

By direct calculations, we have

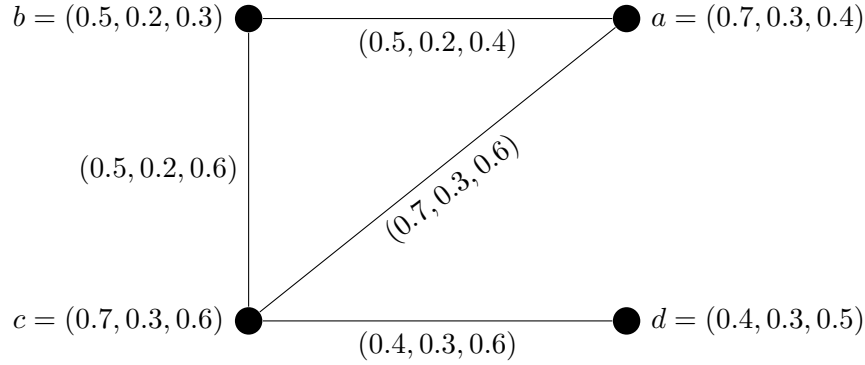


Figure 2.4: The neutrosophic graph G with $V = \{a, b, c, d\}$

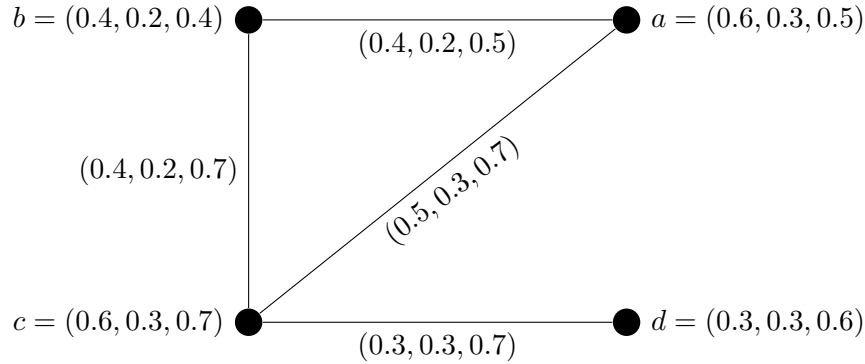


Figure 2.5: The neutrosophic subgraph H with $V = \{a, b, c, d\}$

$$PCI_T(G) = 0.997, \quad PCI_I(G) = 0.120, \quad PCI_F(G) = 0.690.$$

$$PCI_T(H) = 0.516, \quad PCI_I(H) = 0.120, \quad PCI_F(H) = 1.213.$$

Moreover

$$\begin{aligned} TCI(G) &= \frac{4 + 2PCI_T(G) - 2PCI_F(G) - PCI_I(G)}{6} \\ &= \frac{4 + 2(0.997) - 2(0.690) - (0.120)}{6} = 0.749. \end{aligned}$$

$$\begin{aligned} TCI(H) &= \frac{4 + 2PCI_T(H) - 2PCI_F(H) - PCI_I(H)}{6} \\ &= \frac{4 + 2(0.516) - 2(1.213) - (0.120)}{6} = 0.622. \end{aligned}$$

It is easy to see that $TCI(H) = 0.622 \leq TCI(G) = 0.749$.

Corollary 2.4.3. *Note that if $H = (N', M')$ is a partial neutrosophic subgraph of $G = (N, M)$ such that $N' = N \setminus \{v\}$ then*

$$PCI_T(H) < PCI_T(G), \quad PCI_I(H) > PCI_I(G), \quad PCI_F(H) > PCI_F(G).$$

Theorem 2.4.4. *Let $G_1 = (N_1, M_1)$ be isomorphic with $G_2 = (N_2, M_2)$. Then all of the following equation are established.*

$$PCI_T(G_1) = PCI_T(G_2), \quad PCI_I(G_1) = PCI_I(G_2), \quad PCI_F(G_1) = PCI_F(G_2).$$

Also, we have $TCI(G_1) = TCI(G_2)$.

Proof. Let $G_1 = (N_1, M_1)$ be isomorphic with $G_2 = (N_2, M_2)$, and $f : V_1 \rightarrow V_2$ be the bijection from V_1 to V_2 such that

$$T_{N_1}(u) = T_{N_2}(f(u)), \quad I_{N_1}(u) = I_{N_2}(f(u)), \quad F_{N_1}(u) = F_{N_2}(f(u)).$$

for all $uv \in E_1$. and

$$T_{M_1}(uv) = T_{M_2}(f(u)f(v)), \quad I_{M_1}(uv) = I_{M_2}(f(u)f(v)), \quad F_{M_1}(uv) = F_{M_2}(f(u)f(v)).$$

For all $uv \in E_1$. Since G_1 isomorphic with G_2 , the strength of any strongest path between u and v in G_1 is equal to that between $f(u)$ and $f(v)$ in G_2 . Hence

$$CONN_{T_{G_1}}(u, v) = CONN_{T_{G_2}}(f(u), f(v)),$$

$$CONN_{I_{G_1}}(u, v) = CONN_{I_{G_2}}(f(u), f(v)),$$

$$CONN_{F_{G_1}}(u, v) = CONN_{F_{G_2}}(f(u), f(v)).$$

Moreover

$$PCI_T(G_1) = PCI_T(G_2), \quad PCI_I(G_1) = PCI_I(G_2), \quad PCI_F(G_1) = PCI_F(G_2),$$

And

$$\begin{aligned} TCI(G_1) &= \frac{4 + 2PCI_T(G_1) - 2PCI_F(G_1) - PCI_I(G_1)}{6} \\ &= \frac{4 + 2PCI_T(G_2) - 2PCI_F(G_2) - PCI_I(G_2)}{6} = TCI(G_2). \end{aligned}$$

□

Theorem 2.4.5. *Let $G = (N, M)$ be a complete neutrosophic graph with $V = \{v_1, v_2, \dots, v_n\}$ such that $t_1 \leq t_2 \leq \dots \leq t_n$, $i_1 \leq i_2 \leq \dots \leq i_n$ and $f_1 \geq f_2 \geq \dots \geq f_n$ where $t_j = T_N(v_j)$, $i_j = I_N(v_j)$ and $f_j = F_N(v_j)$ for $j = 1, 2, \dots, n$. then*

$$\begin{aligned} PCI_T(G) &= \sum_{j=1}^{n-1} t_j^2 \sum_{k=j+1}^n t_k, \\ PCI_I(G) &= \sum_{j=1}^{n-1} i_j^2 \sum_{k=j+1}^n i_k, \\ PCI_F(G) &= \sum_{j=1}^{n-1} f_j^2 \sum_{k=j+1}^n f_k. \end{aligned}$$

Proof. Suppose v_1 is a vertex with the least Truth-membership value t_1 . In a complete neutrosophic graph, $CONN_{TG}(u, v) = T_M(u, v)$ for all $u, v \in V$. Therefore $T_M(v_1 v_k) = t_1$ for $k = 2, 3, \dots, n$ and hence $T_N(v_1)T_N(v_k)CONN_{TG}(v_1, v_k) = t_1^2 t_k$ for $k = 2, 3, \dots, n$. Then for v_1 , we have

$$\sum_{k=2}^n T_N(v_1)T_N(v_k)CONN_{TG}(v_1, v_k) = \sum_{k=2}^n t_1^2 t_k.$$

For v_2 , $T_N(v_2)T_N(v_k)CONN_{TG}(v_2, v_k) = t_2^2 t_k$ with $k = 3, 4, \dots, n$

$$\sum_{k=3}^n T_N(v_2)T_N(v_k)CONN_{TG}(v_2, v_k) = \sum_{k=3}^n t_2^2 t_k.$$

For v_{n-2} , $T_N(v_{n-2})T_N(v_k)CONN_{TG}(v_{n-2}, v_k) = t_{n-2}^2 t_k$ that $k = n-1, n$.

And for v_{n-1} , $T_N(v_{n-1})T_N(v_k)CONN_{TG}(v_{n-1}, v_k) = t_{n-1}^2 t_k$ for $k = n$.

Thus, by summing over v_j , $j = 1, 2, 3, \dots, n-1$, we get

$$PCI_T(G) = \sum_{k=2}^n t_1^2 t_k + \sum_{k=3}^n t_2^2 t_k + \dots + \sum_{k=n-1}^n t_{n-2}^2 t_k + \sum_{k=n}^n t_{n-1}^2 t_k = \sum_{j=1}^{n-1} t_j^2 \sum_{k=j+1}^n t_k.$$

Using the same argument, we can prove the other two cases. □

Theorem 2.4.6. Let $G = (N, M)$ be a neutrosophic graph whit $V = \{v_1, v_2, \dots, v_n\}$ such that $G^* = (V, E)$ is a complete bipartite graph and $T_M = (uv) = \min\{T_N(u), T_N(v)\}$, $I_M(uv) = \min\{I_N(u), I_N(v)\}$, $F_M(uv) = \max\{F_N(u), F_N(v)\}$ For all $u, v \in V$. Also, $V_1 = \{v_1, v_2, \dots, v_m\}$, and $V_2 = \{v_{(m+1)}, v_{(m+2)}, \dots, v_n\}$ whit $t_1 \leq t_2 \leq \dots \leq t_n$, $i_1 \leq i_2 \leq \dots \leq i_n$, and $f_1 \geq f_2 \geq \dots \geq f_n$ where $t_j = T_N(v_j)$, $i_j = I_N(v_j)$ and $f_j = F_N(v_j)$ for $j = 1, 2, \dots, n$. Then

$$\begin{aligned} PCI_T(G) &= \sum_{j=1}^m t_j^2 \sum_{k=j+1}^n t_k + t_m \sum_{j=m+1}^{n-1} t_j \sum_{k=j+1}^n t_k, \\ PCI_I(G) &= \sum_{j=1}^m i_j^2 \sum_{k=j+1}^n i_k + i_m \sum_{j=m+1}^{n-1} i_j \sum_{k=j+1}^n i_k, \\ PCI_F(G) &= \sum_{j=1}^m f_j^2 \sum_{k=j+1}^n f_k + f_m \sum_{j=m+1}^{n-1} f_j \sum_{k=j+1}^n f_k. \end{aligned}$$

Proof. Let $G = (N, M)$ be a neutrosophic graph whit $V = \{v_1, v_2, \dots, v_n\}$, and $G^* = K_{m,n}$ such that $t_1 \leq t_2 \leq \dots \leq t_n$, $i_1 \leq i_2 \leq \dots \leq i_n$ and $f_1 \geq f_2 \geq \dots \geq f_n$.

Here we prove $PCI_F(G)$, states $PCI_T(G)$ and $PCI_I(G)$ are similarly proved.

Using definition, we have

$$PCI_F(G) = \sum_{v_j, v_k \in V} F_N(v_j) F_N(v_k) CONN_{FG}(v_j, v_k).$$

Too, for $v_1, v_k \in V$ we have

$$CONN_{FG}(v_1, v_k) = \min\{\max\{f_1\}, \max\{f_1, f_2\}, \dots, \max\{f_1, f_m\}\} = \min\{f_1, f_1, \dots, f_1\} = f_1.$$

Accordingly for $v_1, v_k \in V$,

$$\sum_{v_k \neq v_1, v_k \in V} F_N(v_1) F_N(v_k) CONN_{FG}(v_1, v_k) = f_1 f_1 \sum_{k=2}^n f_k.$$

Similarly for $v_j, v_k \in V$ $j = 2, 3, \dots, m$:

$$\sum_{k=j+1}^n F_N(v_j) F_N(v_k) CONN_{FG}(v_j, v_k) = f_j f_j \sum_{k=j+1}^n f_k.$$

On the other hand, we have for $m < j < n$:

$$\sum_{k=j+1}^n F_N(v_j) F_N(v_k) CONN_{FG}(v_j, v_k) = f_m f_j \sum_{k=j+1}^n f_k.$$

Then

$$\begin{aligned}
 PCI_F(G) &= \sum_{v_j, v_k \in V} F_N(v_j)F_N(v_k)CONN_{FG}(v_j, v_k) \\
 &= f_1 f_1 \sum_{k=2}^n f_k + f_2 f_2 \sum_{k=3}^n f_k + \cdots + f_m f_m \sum_{k=m+1}^n f_k + f_m f_{m+1} \sum_{k=m+2}^n f_k + \cdots + f_m f_{n-1} f_n \\
 &= \sum_{j=1}^m f_j^2 \sum_{k=j+1}^n f_k + f_m \sum_{j=m+1}^{n-1} f_j \sum_{k=j+1}^n f_k.
 \end{aligned}$$

□

Corollary 2.4.7. *Clearly, in the above theorem it is enough to have*

$$\forall v \in V_1, \forall u \in V_2, \quad T_N(v) \leq T_N(u), \quad I_N(v) \geq I_N(u), \quad F_N(v) \geq F_N(u).$$

Then the case will be established. In the following example you can see the correctness of this claim.

Example 2.4.8. *Consider the neutrosophic graph $G = (N, M)$ whit*

$$N = \{(a : 0.2, 0.6, 0.7), (b : 0.4, 0.6, 0.5), (c : 0.7, 0.5, 0.4), (d : 0.5, 0.3, 0.5), (e : 0.6, 0.4, 0.5)\}$$

$$\begin{aligned}
 M = \{(ac : 0.2, 0.6, 0.7), (ad : 0.2, 0.6, 0.7), (ae : 0.2, 0.6, 0.7), \\
 (bc : 0.4, 0.6, 0.5), (bd : 0.4, 0.6, 0.5), (be : 0.4, 0.6, 0.5)\}.
 \end{aligned}$$

By direct calculation, we have

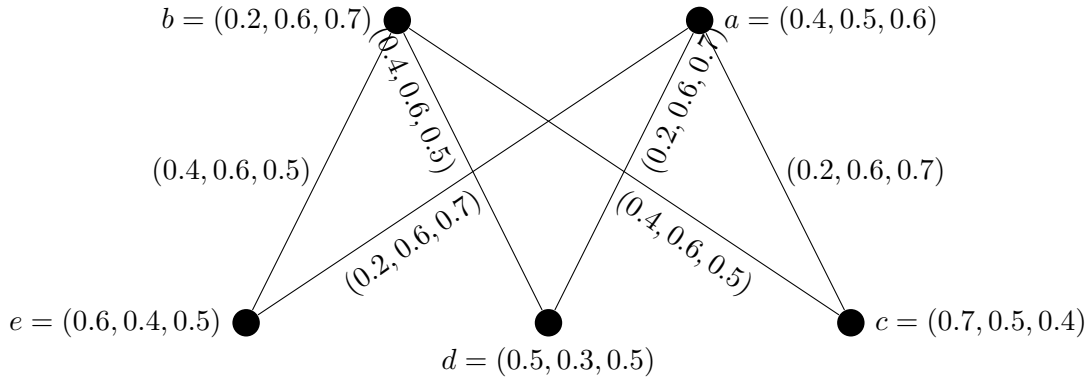


Figure 2.6: A complete bipartite neutrosophic graph whit $G^* = K_{2,3}$

$$CONN_{TG}(a, b) = CONN_{TG}(a, c) = CONN_{TG}(a, d) = CONN_{TG}(a, e) = 0.2 = T_N(a).$$

$$CONN_{TG}(b, c) = CONN_{TG}(b, d) = CONN_{TG}(b, e) = 0.4 = T_N(b)$$

$$CONN_{TG}(c, d) = CONN_{TG}(c, e) = 0.4 = T_N(b)$$

$$CONN_{TG}(d, e) = 0.4 = T_N(b).$$

$$\begin{aligned} PCI_T(G) &= \sum_{u,v \in N} T_N(u)T_N(v)CONN_{TG}(u, v) \\ &= (0.2)(0.4)(0.2) + (0.2)(0.7)(0.2) + (0.2)(0.5)(0.2) + (0.2)(0.6)(0.2) + (0.4)(0.4)(0.7) \\ &\quad + (0.4)(0.4)(0.5) + (0.4)(0.4)(0.6) + (0.7)(0.5)(0.4) + (0.7)(0.6)(0.4) + (0.5)(0.4)(0.6) \\ &= 0.804, \end{aligned}$$

Using Theorem 2.4.6,

$$\begin{aligned} PCI_T(G) &= \sum_{j=1}^m t_j^2 \sum_{k=j+1}^n t_k + t_m \sum_{j=m+1}^{n-1} t_j \sum_{k=j+1}^n t_k = \sum_{j=1}^2 t_j^2 \sum_{k=j+1}^5 t_k + t_m \sum_{j=3}^4 t_j \sum_{k=j+1}^5 t_k \\ &= (0.2)(0.2)(0.4 + 0.7 + 0.5 + 0.6) + (0.4)(0.4)(0.7 + 0.5 + 0.6) + (0.4)(0.7)(0.5 + 0.6) \\ &\quad + (0.4)(0.5)(0.6) = 0.804. \end{aligned}$$

As observed, the value of truth- partial connectivity index $PCI_T(G)$ is obtained from both methods equally.

Theorem 2.4.9. Let $G = (N, M)$ be a wheel neutrosophic graph whit $V = \{v_1, v_2, \dots, v_n\}$ such that G^* is a wheel graph and for all $uv \in M^*$:

$$T_M(uv) = \min\{T_N(u), T_N(v)\}, \quad I_M(uv) = \min\{I_N(u), I_N(v)\}, \quad F_M(uv) = \max\{F_N(u), F_N(v)\}.$$

If $t_1 \leq t_2 \leq \dots \leq t_n$ $i_1 \leq i_2 \leq \dots \leq i_n$ and $f_1 \geq f_2 \geq \dots \geq f_n$ where $t_j = T_N(v_j)$ $i_j = I_N(v_j)$ and $f_j = F_N(v_j)$ for $j = 1, 2, \dots, n$ and v_1 is the center vertex. Then

$$\begin{aligned} PCI_T(G) &= \sum_{j=1}^{n-1} t_j^2 \sum_{k=j+1}^n t_k, \\ PCI_I(G) &= \sum_{j=1}^{n-1} i_j^2 \sum_{k=j+1}^n i_k, \\ PCI_F(G) &= \sum_{j=1}^{n-1} f_j^2 \sum_{k=j+1}^n f_k. \end{aligned}$$

Proof. Let $G = (N, M)$ be a wheel neutrosophic graph with the conditions stated in the theorem.

Here we prove $PCI_I(G)$, states $PCI_T(G)$ and $PCI_F(G)$ are similarly proved. Then

Suppose v_1 is the center vertex. Using definition, for $v_1, v_k \in V$ we have:

$$\begin{aligned} CONN_{IG}(v_1, v_2) &= \min\{\max\{i_1\}, \max\{i_1, i_2\}, \max\{i_1, i_2, i_3\}, \dots, \max\{i_1, \dots, i_n\}\} \\ &= \min\{i_1, i_2, \dots, i_n\} = i_1, \end{aligned}$$

Then

$$\sum_{k=2}^n I_N(v_1)I_N(v_k)CONN_{IG}(v_1, v_k) = i_1i_1i_2 + i_1i_1i_3 + \dots + i_1i_1i_{n-1} + i_1i_1i_n = \sum_{k=2}^n i_1^2i_k.$$

Similarly for $v_j, v_k \in V$, $j = 2, 3, \dots, n-1$

$$CONN_{IG}(v_j, v_k) = \sum_{k=j+1}^n I_N(v_j)I_N(v_k)CONN_{IG}(v_j, v_k) = \sum_{k=j+1}^n i_j^2i_k,$$

This shows that

$$\begin{aligned} PCI_I(G) &= \sum_{v_j, v_k \in V} I_N(v_j)I_N(v_k)CONN_{IG}(v_j, v_k) = \sum_{k=2}^n i_1^2i_k + \sum_{k=3}^n i_2^2i_k \\ &+ \dots + \sum_{k=j+1}^n i_j^2i_k + \dots + i_{n-1}i_{n-1}i_n = \sum_{j=1}^{n-1} i_j^2 \sum_{k=j+1}^n i_k. \end{aligned}$$

□

Theorem 2.4.10. Let $G = (N, M)$ be a complete neutrosophic graph of $G^* = (V, E)$, and $B_c(m, m)$ is a m -barbell graph of G . if $t_1 \leq t_2 \leq \dots \leq t_n$, $i_1 \geq i_2 \geq \dots \geq i_n$ and $f_1 \geq f_2 \geq \dots \geq f_n$ where $t_j = T_N(v_j)$, $i_j = I_N(v_j)$ and $f_j = F_N(v_j)$ for $j = 1, 2, \dots, n$. and uv a I -Strong edge with $M(uv) = (T_M(uv), I_M(uv), F_M(uv))$ where $T_M(uv) \leq t_1$, $I_M(uv) \geq i_1$, $F_M(uv) \geq f_1$ and uv connecting two copies of complete neutrosophic graphs. Then

$$\begin{aligned} PCI_T(B_{m,m}) &= 2 \sum_{j=1}^{n-1} t_j^2 \sum_{k=j+1}^n t_k + T_M(uv) \sum_{j=1}^n t_j \sum_{k=j}^n t_k, \\ PCI_I(B_{m,m}) &= 2 \sum_{j=1}^{n-1} i_j^2 \sum_{k=j+1}^n i_k + I_M(uv) \sum_{j=1}^n i_j \sum_{k=j}^n i_k, \\ PCI_F(B_{m,m}) &= 2 \sum_{j=1}^{n-1} f_j^2 \sum_{k=j+1}^n f_k + F_M(uv) \sum_{j=1}^n f_j \sum_{k=j}^n f_k. \end{aligned}$$

Proof. Let $G = (N, M)$ be a wheel neutrosophic graph with the conditions stated in the theorem. By definition 5, here we have two copies of the complete graph K_m . Also using Theorem 3, for a complete neutrosophic graph

$$\begin{aligned} PCI_T(G) &= \sum_{j=1}^{n-1} t_j^2 \sum_{k=j+1}^n t_k, \\ PCI_I(G) &= \sum_{j=1}^{n-1} i_j^2 \sum_{k=j+1}^n i_k, \\ PCI_F(G) &= \sum_{j=1}^{n-1} f_j^2 \sum_{k=j+1}^n f_k. \end{aligned}$$

Now it suffices to obtain the connectivity between two vertices from two copies of K_m . Suppose vertex v_j is from one of the two copies of K_m and vertex v_k is from another copy, in which case we have

$$CONN_{TG}(v_j, v_k) = \max\{\min\{T_M(uv) \wedge \min\{t_k | t_k \in P(v_j - v_k)\}\}\} = T_M(uv),$$

Then

$$\begin{aligned} PCI_T(B_{m,m}) &= \sum_{v_j, v_k \in V} T_N(v_j) T_N(v_k) CONN_{TG}(v_j, v_k) \\ &= \sum_{j=1}^{n-1} t_j^2 \sum_{k=j+1}^n t_k + \sum_{j=1}^{n-1} t_j^2 \sum_{k=j+1}^n t_k + t_1 t_1 T_M(uv) + t_1 t_2 T_M(uv) + \cdots + t_n t_n T_M(uv) \\ &= 2 \sum_{j=1}^{n-1} t_j^2 \sum_{k=j+1}^n t_k + T_M(uv) \sum_{j=1}^n t_j \sum_{k=j}^n t_k. \end{aligned}$$

The proof will be the same for the other two cases. \square

Example 2.4.11. Consider the neutrosophic graph $G = K_4 = (N, M)$ with

$$N = \{(a : 0.2, 0.6, 0.8), (b : 0.3, 0.5, 0.7), (c : 0.3, 0.4, 0.7), (d : 0.4, 0.4, 0.5)\}$$

$$M = \{(ab : 0.2, 0.6, 0.8), (ac : 0.2, 0.6, 0.8), (ad : 0.2, 0.6, 0.8), (bc : 0.3, 0.5, 0.7), (cd : 0.3, 0.4, 0.7)\}$$

Now suppose that the edge that connects the two complete graphs does not hold true. As shown in figure 2.7, for example, if we want to go from vertex b in the right graph to vertex a in the left graph, there are paths with different connectivity.

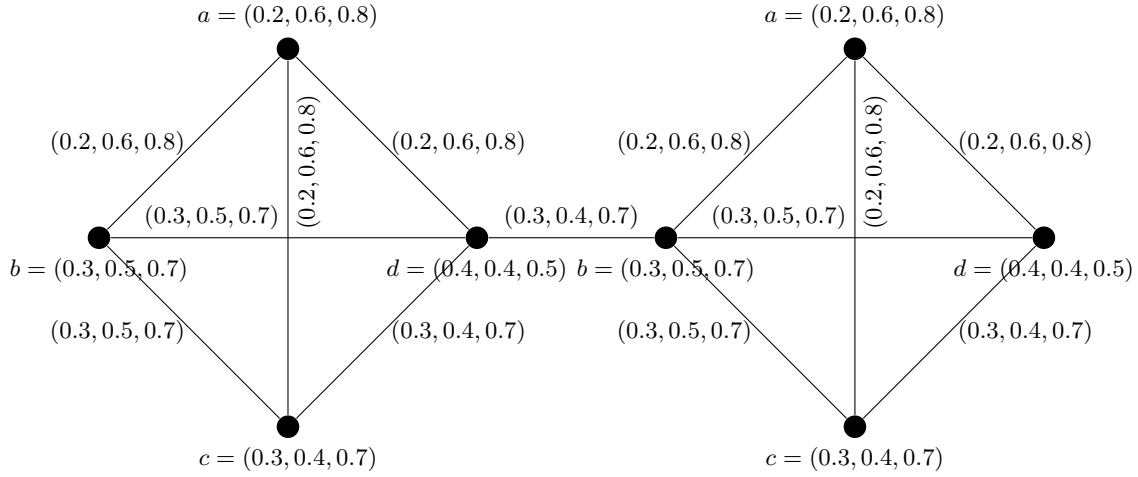


Figure 2.7: A m-barbell neutrosophic graph whit $G^* = K_4$

2.5 Bounds for connectivity index

In this section, we discuss bunds for partial connectivity index (PCI) and totally connectivity index (TCI). We show that, among all neutrosophic graphs whit a same support, the complete neutrosophic graph will have maximum totally connectivity index.

Theorem 2.5.1. *Let $G = (N, M)$ be a neutrosophic graph whit $|N| = n$, and $G' = (N', M')$ is the complete neutrosophic graph spanned by the vertex set of G . Then,*

$$0 \leq PCI_T(G) \leq PCI_T(G'),$$

$$0 \leq PCI_I(G) \leq PCI_I(G'),$$

$$0 \leq PCI_F(G) \leq PCI_F(G').$$

Also if $I_M(uv) = I_{(M')}(uv)$, and $F_M(uv) = F_{(M')}(uv)$, for all $uv \in E$ then $0 \leq TCI_F(G) \leq TCI_F(G')$.

Proof. Consider the neutrosophic graph $G = (N, M)$ whit $|N| = n$. If $|E| = 0$ clearly, $PCI_T(G) = PCI_I(G) = PPCI_F(G) = TCI(G) = 0$. Let $|E| > 0$ and $G' = (N', M')$ is the complete neutrosophic graph whit $|N'| = n$. Suppose $(T_N(u), I_N(u), F_N(u)) = (T_{(N')}(u), I_{(N')}(u), F_{(N')}(u))$ for all $u \in X$. Since

$$T_M(uv) \leq T_{M'}; \quad I_M(uv) \leq I_{M'}(uv); \quad F_M(uv) \leq F_{M'}(uv); \quad \forall uv \in E.$$

Therefore, we have $CONN_{TG}(u, v) \leq CONN_{TG'}(u, v)$ $CONN_{IG}(u, v) \leq CONN_{IG'}(u, v)$ and $CONN_{FG}(u, v) \leq CONN_{FG'}(u, v)$. Then

$$\begin{aligned} 0 \leq PCI_T(G) &\leq \sum_{u,v \in X} T_N(u)T_N(v)CONN_{TG}(u, v) \\ &\leq \sum_{u,v \in X} T_{N'}(u)T_{N'}(v)CONN_{TG'}(u, v) = PCI_T(G'). \end{aligned}$$

Using a similar proof we can show that

$$0 \leq PCI_I(G) \leq PCI_I(G'), \quad 0 \leq PCI_F(G) \leq PCI_F(G').$$

Also, according to definition $TCI(G)$, if $I_M(uv) = I_{(M')}(uv)$, and $F_M(uv) = F_{(M')}(uv)$, for all $uv \in E$, then

$$\begin{aligned} TCI(G) &= \frac{4 + 2PCI_T(G) - 2PCI_F(G) - PCI_I(G)}{6} \\ &\leq \frac{4 + 2PCI_T(G') - 2PCI_F(G') - PCI_I(G')}{6} = TCI(G'). \end{aligned}$$

□

2.6 Applications

Neutrosophic graphs are one of the most practical branches of graph theory, and various applications of them have been studied so far. For further reading, you can refer to the items mentioned in the references section at the end of the book. Here we will mention another application.

Behavioral sciences, which is one of the branches of humanities, is one of the most extensive sciences in our time. Every day, many theorists in this field create new theories and cause them to expand more and more. So every day they are faced with a lot of new data and information. Mathematics has always been one of the best tools for modeling and categorizing this data and information. Among these, graphic models are among the most appropriate models that come with the help of behavioral sciences and with proper modeling, provide the conditions for a more accurate analysis of these complex problems. What is very important in behavioral sciences is the existence of a relationship, the relationship between individuals, groups, communities, organizations and institutions, and, so on. Studying and discovering these relationships, categorizing them, and then

examining and studying the extent and impact of these relationships on each other is a complex task. Neutrosophic graph models can help with these problems and help answer some of the questions. Questions such as: Which relationship is most effective? Which relationship should end? Which person is more influential in a relationship? And many other questions.

Here we are dealing with the relationship between several families. Information related to this problem is data from a real study obtained from a behavioral science study clinic. Of course, given the limitations we had, we have provided a small sample of that data in this article.

In this problem, we studied 5 families that are related. First, each family was studied separately and the behavior of each family member was studied by experts, and then we obtained an average of the behaviors and traits studied in family members. These features were classified into three categories. Good qualities include the ability to communicate, cooperate, be honest, etc; Bad traits include jealousy, misconceptions, lack of anger control, personal aggression, etc; Neutral behaviors include behaviors that do not involve any behavioral actions. The experts then assigned a numerical value to each of these behaviors, which we named T, F, and I, respectively. Experts then studied the relationships between families and the extent of each family's impact on another family and the type of impact of each family. The effect of each family on other families was evaluated using behavioral science criteria. The experts coded these relationships into three categories: good, neutral, and bad, and obtained a numerical quantity for each category based on the coding results.

Here we present a neutrosophic graph model related to 5 families from 137 families surveyed. By direct calculations Then

$$\begin{aligned} PCI_T(G) &= \sum_{u,v \in N} T_N(u)T_N(v)CONN_{TG}(u, v) = 1.3845, \\ PCI_I(G) &= \sum_{u,v \in N} I_N(u)I_N(v)CONN_{IG}(u, v) = 0.519, \\ PCI_F(G) &= \sum_{u,v \in N} F_N(u)F_N(v)CONN_{FG}(u, v) = 0.118. \end{aligned}$$

Also, we have

$$\begin{aligned} TCI(G) &= \frac{4 + 2PCI_T(G) - 2PCI_F(G) - PCI_I(G)}{6} \\ &\leq \frac{4 + 2(1.3845) - 2(0.118) - 0.519}{6} = 1.002. \end{aligned}$$

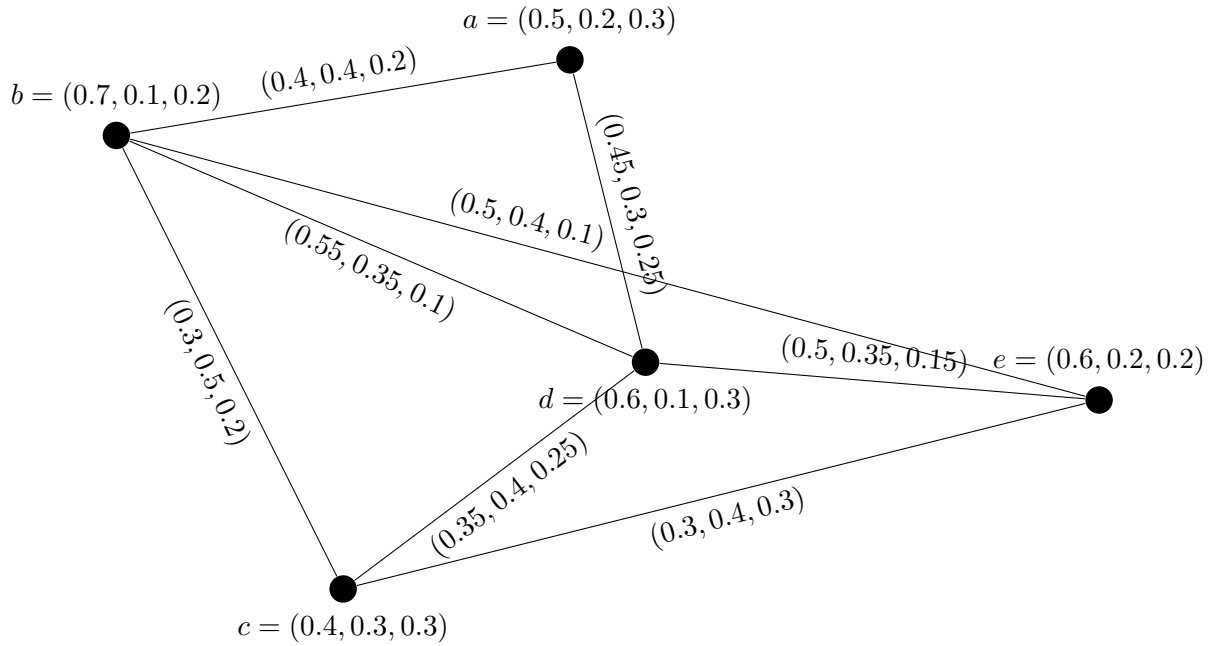


Figure 2.8: A neutrosophic graph model corresponding to 5 families

	$CONN_{TG}(u, v)$	$CONN_{IG}(u, v)$	$CONN_{FG}(u, v)$
a, b	0.45	0.35	0.2
a, c	0.35	0.4	0.2
a, d	0.45	0.3	0.2
a, e	0.45	0.35	0.2
b, c	0.35	0.4	0.2
b, d	0.55	0.35	0.1
b, e	0.5	0.35	0.1
c, d	0.35	0.4	0.2
c, e	0.35	0.4	0.2
d, e	0.5	0.35	0.1

Table 2.6: The strength of connectedness between each pair of vertices u and v .

The connectivity index is used as a numerical index in evaluating the interactions of these five families. Note that the analysis of this problem will be done by behavioral science experts.

2.7 Neutrosophic tree

In this section, the types of edges are first classified and defined in terms of edge strength. Then we will provide some other definitions depending on the type of edges. Based on the strength of connectivity between the end vertices of an edge, edges of neutrosophic graphs can be divided into two categories as given below.

Defintion 2.7.1. An edge uv in a neutrosophic graph $G = (N, M)$ is called

- A weak edge if $CONN_{(G-uv)}(u, v) = CONN_G(u, v)$ and $CONN_G(u, v) \neq M(uv)$.
- A neutral edge if $CONN_{(G-uv)}(u, v) = CONN_G(u, v)$ and $CONN_G(u, v) = M(uv)$.
- A -strong edge if

$$CONN_{(G-uv)}(u, v) \leq CONN_G(u, v), \quad CONN_G(u, v) = (T_M(uv), I_M(uv), F_M(uv)) = M(uv),$$

- A -strong edge if $CONN_{(G-uv)}(u, v) \leq CONN_G(u, v)$, $CONN_G(u, v) \neq M(uv)$.

Example 2.7.2. Consider the neutrosophic graph $G = (N, M)$ on $V = \{a, b, c, d, e, f\}$ as shown in figure 2.9.

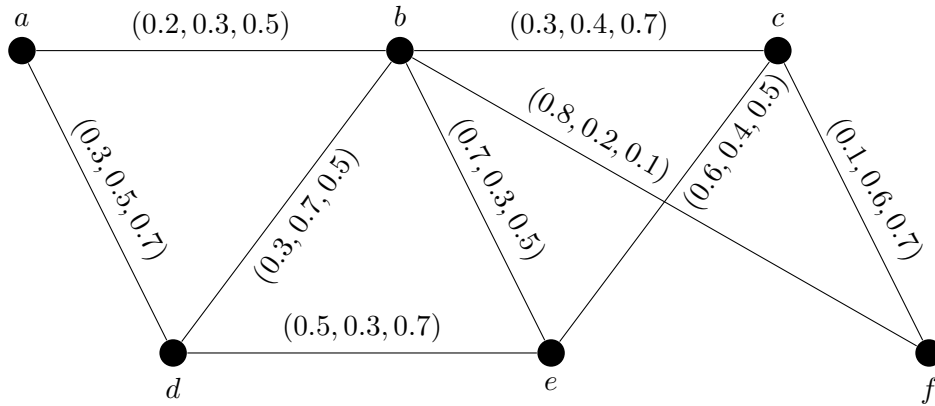


Figure 2.9: Neutrosophic graph G

As can be seen in Table 2.7, edge bc and cf are weak, be , bf and ce are I-strong edges, and ac , ad , bd and de are II-strong edge.

	$CONN_G(u, v)$	$CONN_{G-uv}(u, v)$	$M(uv)$
a, b	(0.3, 0.3, 0.5)	(0.3, 0.5, 0.7)	(0.2, 0.3, 0.5)
a, d	(0.3, 0.3, 0.5)	(0.2, 0.3, 0.5)	(0.3, 0.5, 0.7)
b, c	(0.6, 0.4, 0.5)	(0.6, 0.4, 0.5)	(0.3, 0.4, 0.7)
b, d	(0.5, 0.3, 0.5)	(0.5, 0.3, 0.7)	(0.3, 0.7, 0.5)
b, e	(0.7, 0.3, 0.5)	(0.3, 0.4, 0.7)	(0.7, 0.3, 0.5)
b, f	(0.8, 0.2, 0.1)	(0.1, 0.6, 0.7)	(0.8, 0.2, 0.1)
c, e	(0.6, 0.4, 0.5)	(0.3, 0.4, 0.7)	(0.6, 0.4, 0.5)
c, f	(0.6, 0.4, 0.5)	(0.6, 0.4, 0.5)	(0.1, 0.6, 0.7)
d, e	(0.5, 0.3, 0.5)	(0.3, 0.5, 0.5)	(0.5, 0.3, 0.7)

Table 2.7: The strength of connectedness between each pair of vertices u and v .

Defintion 2.7.3. A path in a neutrosophic graph is called a *I-strong path* if all its edges are *I-strong* and called a *II-strong path* if all its edges are *II-strong*. Also is said to be a *strong path* if all its edges are either *I-strong edge* or *II-strong edge*.

Defintion 2.7.4. Let $G = (N, M)$ be a neutrosophic graph and C be a cycle in G . C called *strong cycle* if all its edges are either *I-strong edge* or *II-strong edge*.

Defintion 2.7.5. Let $G = (N, M)$ be a neutrosophic graph. G called a *neutrosophic tree* if it has no strong cycle.

Example 2.7.6. Consider a neutrosophic graph $G = (N, M)$ and $H = (A, B)$ as shown in figure 2.10.

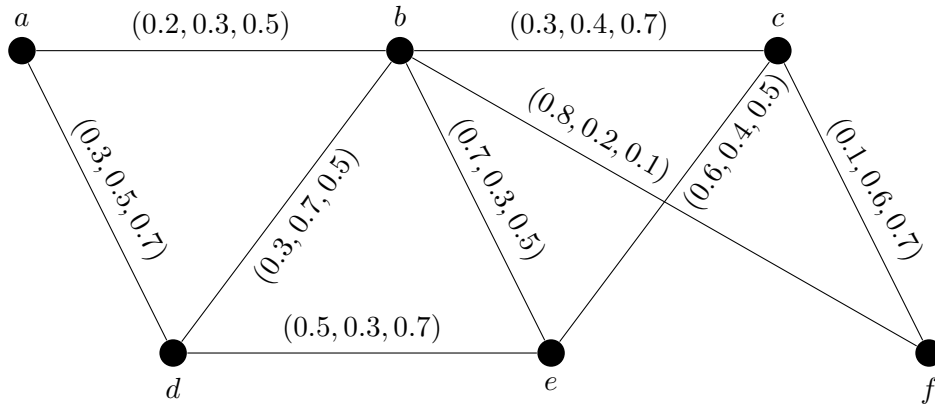


Figure 2.10: G is not a neutrosophic tree

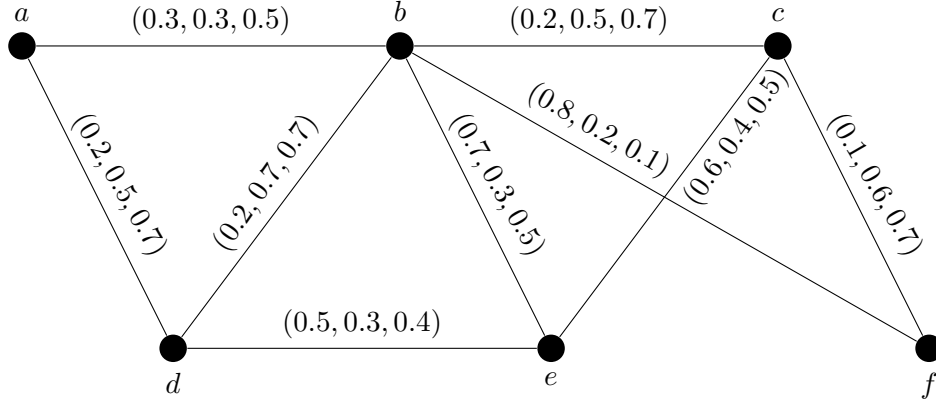


Figure 2.11: H is a neutrosophic tree

It is clear from figure 2.11 that G is not a neutrosophic tree. Since G contains strong neutrosophic cycles. Cycles such as $abda$, $abeda$, $abcda$, ect. are strong neutrosophic cycles in G . But H is a neutrosophic tree, H has no strong neutrosophic cycle.

Defintion 2.7.7. Let $G = (N, M)$ be a connected neutrosophic graph and T , is a neutrosophic spanning subgraph of G that T spanned by the vertex set of G and T^* is a tree. If the edges of T are selected from G such that for each edge uv of T , uv is either I-strong edge or II-strong edge. Then T called a strong spanning tree and denoted by (SST).

Defintion 2.7.8. Let $G = (N, M)$ be a connected neutrosophic graph with at least one strong spanning tree. Then the strength of strong spanning tree in G is defined and denoted by

$$S(T) = \sum_{uv \in T} S(uv) = \sum_{uv \in T} \frac{4 + 2T_M(uv) - 2F_M(uv) - I_M(uv)}{6}.$$

Also, F called maximum spanning tree if $S(F) \geq S(T)$ for any strong spanning tree T .

Theorem 2.7.9. Let $G = (N, M)$ be a connected neutrosophic graph. Then G is a neutrosophic tree if and only if the following conditions are equivalent for any $u, v \in V$

- uv is a I-strong edge;
- $(CONN_{TG}(u, v), CONN_{IG}(u, v), CONN_{FG}(u, v)) = (T_M(uv), I_M(uv), F_M(uv))$.

Proof. This theorem can be easily proved by defining a strong edge. □

2.7.1 Maximum spanning tree

In this section, a version of the maximum spanning tree discussed on a graph by strength of edges. In the following, we propose a neutrosophic maximum spanning tree algorithm, whose computing steps are described below. Note that the strength function $S(uv) = \frac{4+2T_M(uv)-2F_M(uv)-I_M(uv)}{6}$ is used to label here.

The algorithm for finding the maximum spanning tree (MST)

Here, the input is adjacency matrix $M = [(T_M(u_i u_j), I_M(u_i u_j), F_M(u_i u_j))]_{n \times n}$ of the neutrosophic graph $G = (N, M)$, and output is a tree F with weighted edges.

Step 1. Input matrix M ;

Step 2. Using the strength function $S(u_i u_j) = \frac{4+2T_M(u_i u_j)-2F_M(u_i u_j)-I_M(u_i u_j)}{6}$, convert the neutrosophic matrix into a strength matrix $S = [S(u_i u_j)]_{(n \times n)}$;

Step 3. Iterate steps 4 and 5 until all $n - 1$ elements of S are either labeled to 0 or all the nonzero elements of the matrix are labeled;

Step 4. Find the M either column or row to compute the unlabeled maximum element $S(u_i u_j)$, which is the value of the corresponding are $e(u_i u_j) \in M$;

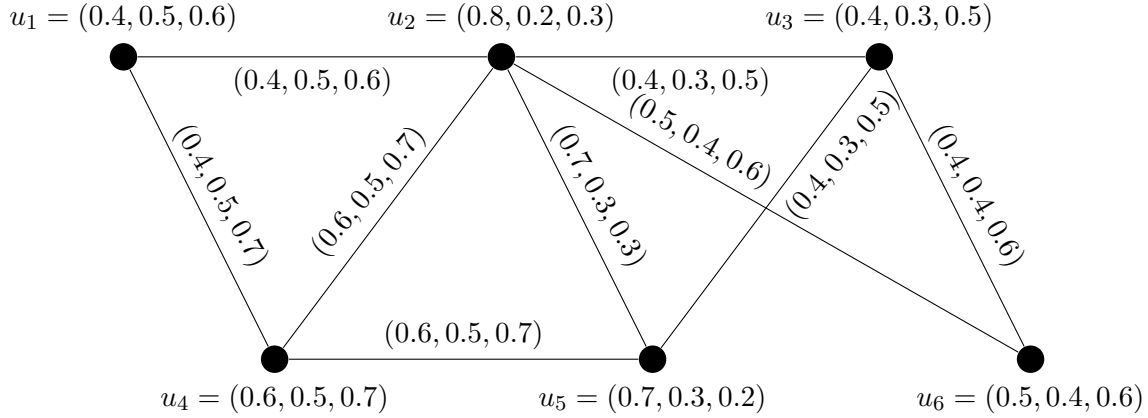
Step 5. If the corresponding edge $e(u_i u_j) \in M$ of chosen S produce a cycle whit the previous labeled entries of the strength matrix S than set $S(u_i u_j) = 0$ else label $S(u_i u_j)$;

Step 6. Design the tree F including only the labeled elements from the S which will be computed MST of G ;

Step 7. Stop (end algorithm).

Example 2.7.10. Consider a neutrosophic graph $G = (N, M)$ on $V = \{u_1, u_2, u_3, u_4, u_5, u_6\}$ as shown in Figure 2.12.

and

Figure 2.12: a neutrosophic graph G on V

$$\begin{bmatrix} 0 & (0.4, 0.5, 0.6) & 0 & (0.4, 0.5, 0.7) & 0 & 0 \\ (0.4, 0.5, 0.6) & 0 & (0.4, 0.3, 0.5) & (0.6, 0.5, 0.7) & (0.7, 0.3, 0.3) & (0.5, 0.4, 0.6) \\ 0 & (0.4, 0.3, 0.5) & 0 & 0 & (0.4, 0.3, 0.5) & (0.4, 0.4, 0.6) \\ (0.4, 0.5, 0.7) & (0.6, 0.5, 0.7) & 0 & 0 & (0.7, 0.3, 0.2) & 0 \\ 0 & (0.7, 0.3, 0.3) & (0.4, 0.3, 0.5) & (0.6, 0.5, 0.7) & 0 & 0 \\ 0 & (0.5, 0.4, 0.6) & (0.4, 0.4, 0.6) & 0 & 0 & 0 \end{bmatrix}$$

Using the strength function $S(u_i u_j) = \frac{4+2T_M(u_i u_j)-2F_M(u_i u_j)-I_M(u_i u_j)}{6}$ we have

$$S(u_i u_j) = \begin{bmatrix} 0 & 0.517 & 0 & 0.483 & 0 & 0 \\ 0.517 & 0 & 0.583 & 0.550 & 0.750 & 0.567 \\ 0 & 0.583 & 0 & 0 & 0.583 & 0.533 \\ 0.483 & 0.550 & 0 & 0 & 0.550 & 0 \\ 0 & 0.750 & 0.583 & 0.550 & 0 & 0 \\ 0 & 0.567 & 0.533 & 0 & 0 & 0 \end{bmatrix}$$

Now search the matrix S to find the maximum value and select the edge corresponding to the row and column of that element. The following figure edge $u_2 u_5$ is highlighted.

The next maximum element 0.583 is marked and corresponding edges $u_2 u_3$ and $u_3 u_5$, but the simultaneous selection of these two edges causes the formation of a cycle, so we choose one of these two edges arbitrarily and ignore the other.

Continuing this process, edges $u_2 u_6$, $u_2 u_4$, and $u_2 u_1$ are selected, respectively. The maximum spanning tree is obtained as figure 2.18.

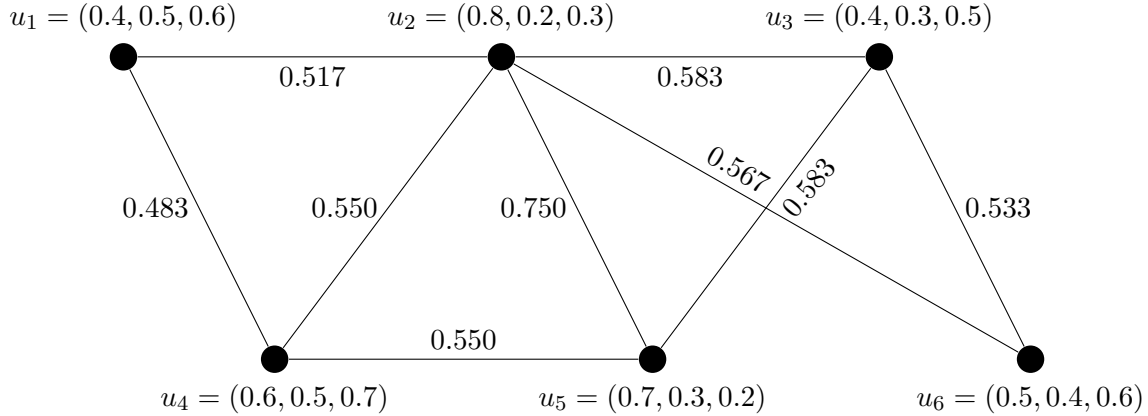


Figure 2.13: A neutrosophic graph G with strength of edges

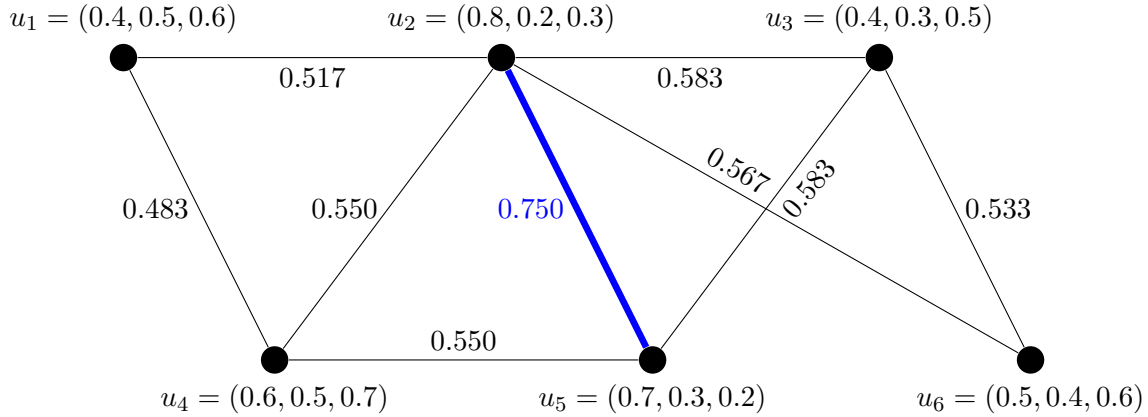


Figure 2.14: An edge u_2u_5 is highlighted

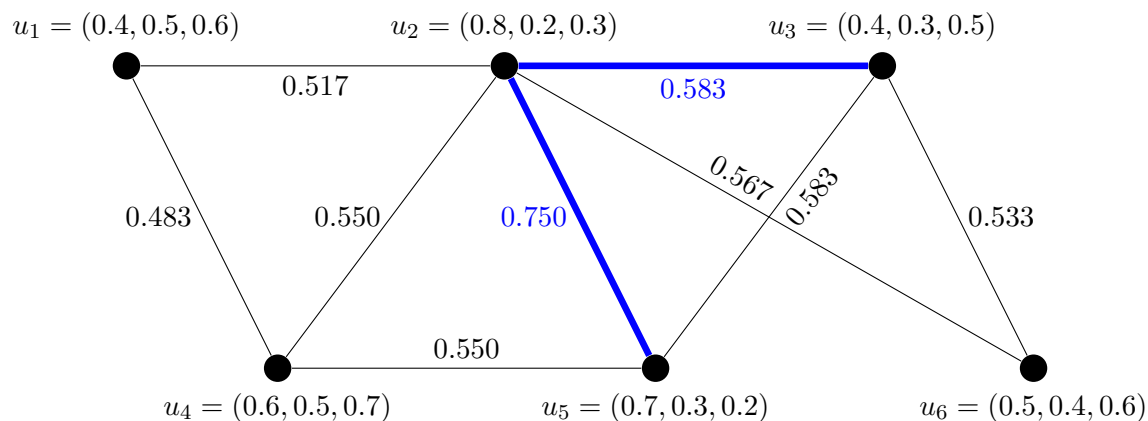
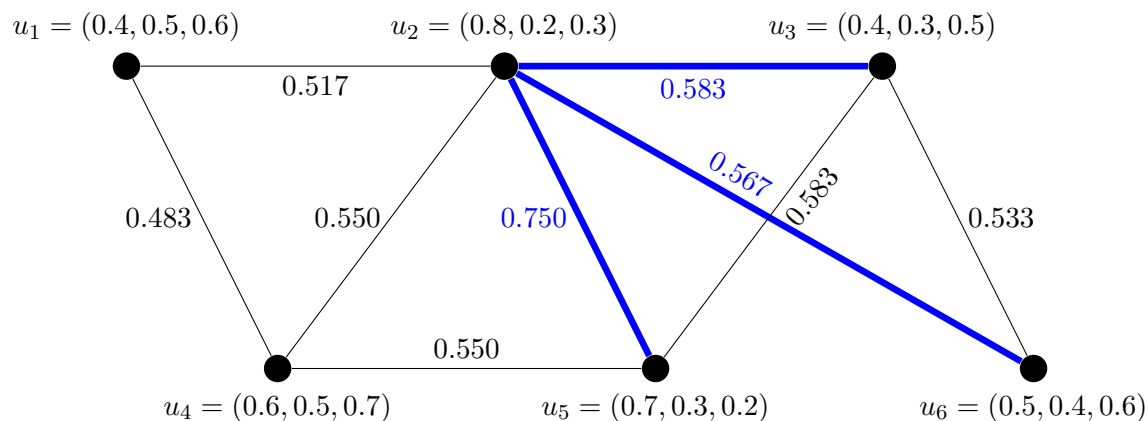
As it was observed, the selection of the maximum spanning tree was not unique, so neutrosophic graph $G = (N, M)$ is not a neutrosophic tree, also G contains a strong neutrosophic cycle.

Corollary 2.7.11. *Obviously, if $G = (N, M)$ has a unique strong spanning tree, it will also have a unique maximum spanning tree, but the conversely is not necessarily true.*

2.7.2 Partial connectivity index in the neutrosophic tree

In this section, the results of examining the Partial connectivity index and totally connectivity index on the neutrosophic trees are presented and proved.

Theorem 2.7.12. *Let $G = (N, M)$ be a neutrosophic graph. Then $TCI(G - uv) = TCI(G)$ if and*

Figure 2.15: An edge u_2u_3 is highlightedFigure 2.16: An edge u_2u_6 is highlighted

only if either uv is a weak edge or neutral edge.

Proof. The proof of this theorem is clear using definition 2.2.2. \square

Corollary 2.7.13. Let $G = (N, M)$ be a neutrosophic graph and, uv is an edge in G , uv is a bridge if and only if uv is either I-strong edge or II-strong edge.

Corollary 2.7.14. Let $G = (N, M)$ be a neutrosophic graph. Then for any uv , $TCI(G - uv) \neq TCI(G)$ if G^* is a tree.

Theorem 2.7.15. Let $G = (N, M)$ be a connected neutrosophic graph whit strong spanning tree

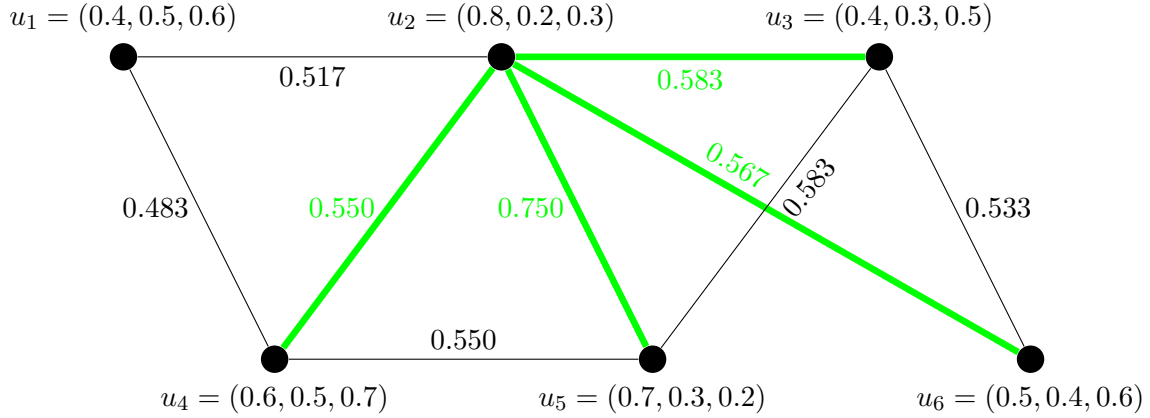


Figure 2.17: An edge u_2u_6 is highlighted

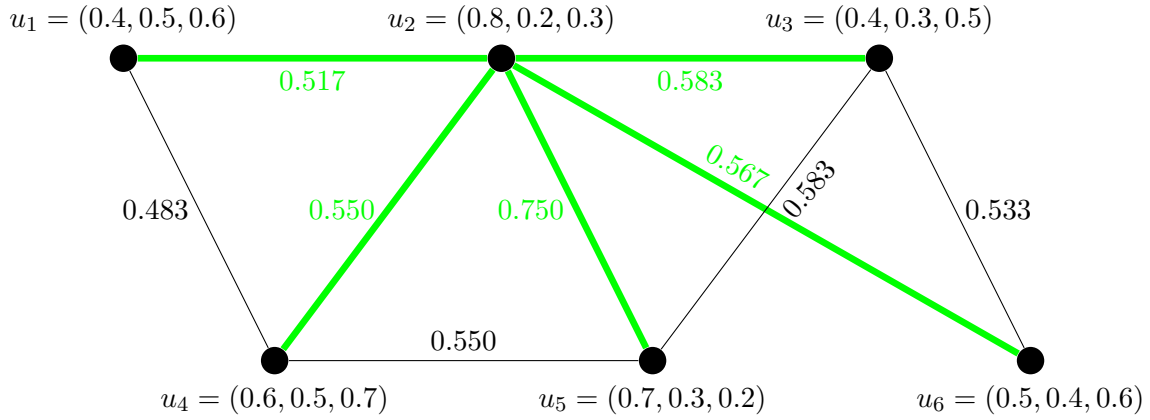


Figure 2.18: Maximum spanning tree MST

(SST) T . for any $uv \in M$, where uv is an edge of T , then either

$$PCI_T(G - uv) < PCI_T(G)$$

or

$$(PCI_I(G - uv) > PCI_I(G)) \vee (PCI_F(G - uv) > PCI_F(G))$$

Hence we have $TCI(G - uv) < TCI(G)$.

Proof. Suppose $G = (N, M)$ be a connected neutrosophic graph whit strong spanning tree (SST) T . Since T is SST then any edge of T is either I-strong edge or II-strong edge. By Corollary 1, for

each $uv \in M$, uv is a bridge. Then $PCI_T(G - uv) < PCI_T(G)$ or

$$[(PCI_I(G - uv) > PCI_I(G)) \vee (PCI_F(G - uv) > PCI_F(G))].$$

□

Theorem 2.7.16. *Let $G = (N, M)$ be a connected neutrosophic tree and G^* is not a tree. Then there exists at least one edge $uv \in M^*$ such that $TCI(G - uv) = TCI(G)$.*

Proof. Let $G = (N, M)$ be a neutrosophic tree and G^* is not a tree. Hence there is at least one cycle in G^* . As respects a tree is a connected forest, there exist $uv \in M^*$ so that at least one of the following

$$T_M(uv) < CONN_{T(G-uv)}(u, v), I_M(uv) > CONN_{I(G-uv)}(u, v), F_M(uv) > CONN_{F(G-uv)}(u, v),$$

Then $PCI_T(G - uv) = PCI_T(G)$ and $PCI_I(G - uv) = PCI_I(G)$ and $PCI_F(G - uv) = PCI_F(G)$

Therefore $TCI(G - uv) = TCI(G)$. □

Theorem 2.7.17. *Let $G = (N, M)$ be a connected neutrosophic graph then G is a neutrosophic tree if and only if G has a unique strong spanning tree.*

Proof. Suppose $G = (N, M)$ is a connected neutrosophic graph with only one strong spanning tree T . Then G has no strong edges except the edges of T . hence G has no strong cycle. Therefore by definition 6, G is a neutrosophic tree. Conversely, assume that G is a neutrosophic tree. Again according to definition 6, G lacks a strong circle. Therefore, there is only one strong path between the two arbitrary vertices of G . then the strong spanning tree of G is unique. □

Theorem 2.7.18. *Let $G = (N, M)$ be a connected neutrosophic graph and T the corresponding SST of G . Then $TCI(T) = TCI(G)$ if and only if T is the unique strong spanning tree of G .*

Proof. Suppose $G = (N, M)$ is a connected neutrosophic graph and T the corresponding SST of G . And $TCI(T) = TCI(G)$. Now, shown that T is a unique strong spanning tree of G . Proof of this is easily possible using Theorem 5. Conversely, assume that T is the unique strong spanning tree of G . It is clear that to obtain the connectivity index of G , only the strong paths will be the same paths of T . then $TCI(T) = TCI(G)$. □

Corollary 2.7.19. *Let $G = (N, M)$ be a neutrosophic tree with the unique strong spanning tree (T) and the unique maximum spanning tree (F). Then $TCI(T) = TCI(G) = TCI(F)$.*

Theorem 2.7.20. *Let $G = (N, M)$ be a connected neutrosophic graph and $uv \in M^*$. Then $TCI(G - uv) < TCI(G)$ for any uv and $uv \in TCI(G - uv) < TCI(G)$ and*

$$(CONN_{TG}(u, v), CONN_{IG}(u, v), CONN_{FG}(u, v)) = (T_M(uv), I_M(uv), F_M(uv))$$

if and only if G^ is a tree.*

Proof. Suppose $G = (N, M)$ is a connected neutrosophic graph and G^* is a tree. It is clear $TCI(G - uv) < TCI(G)$. Since G^* is a tree, for any $uv \in M^*$, $G - uv$ is not connected. Also for any $uv \in G$ we have $(CONN_{TG}(u, v), CONN_{IG}(u, v), CONN_{FG}(u, v)) = (T_M(uv), I_M(uv), F_M(uv))$. Conversely assume that for each uv , $TCI(G - uv) < TCI(G)$ and

$$(CONN_{TG}(u, v), CONN_{IG}(u, v), CONN_{FG}(u, v)) = (T_M(uv), I_M(uv), F_M(uv)),$$

Then both uv is a neutrosophic bridge and a I-strong edge. By theorem 1, G is a tree. Since, for each uv , $TCI(G - uv) < TCI(G)$, G^* is a tree. \square

Theorem 2.7.21. *Let $G = (N, M)$ be a connected neutrosophic graph such that G^* is a star graph. If v_1 is the center vertex and for any $uv \in M^*$*

$$T_M(uv) = \min\{T_N(u), T_N(v)\}, \quad I_M(uv) = \min\{I_N(u), I_N(v)\}, \quad F_M(uv) = \max\{F_N(u), F_N(v)\}.$$

Also $\forall j \geq 2, t_1 \leq t_j, i_1 \leq i_j, f_1 \geq f_j$ where $t_j = T_N(v_j)$, $i_j = I_N(v_j)$ and $f_j = F_N(v_j)$ for $j = 1, 2, \dots, n$. then

$$\begin{aligned} PCI_T(G) &= t_1 \sum_{j=1}^{n-1} t_j \sum_{k=j+1}^n t_k, \\ PCI_I(G) &= i_1 \sum_{j=1}^{n-1} i_j \sum_{k=j+1}^n i_k, \\ PCI_F(G) &= f_1 \sum_{j=1}^{n-1} f_j \sum_{k=j+1}^n f_k. \end{aligned}$$

Proof. Let $G = (N, M)$ be a neutrosophic graph such that G^* is a star graph and v_1 is the center vertex. Therefore for any vertex v_j , we have

$$CONN_{TG}(v_1, v_j) = T_M(v_1 v_j) = \min\{T_N(v_1), T_N(v_j)\} = T_N(v_1),$$

$$CONN_{IG}(v_1, v_j) = I_M(v_1 v_j) = \min\{I_N(v_1), I_N(v_j)\} = I_N(v_1),$$

$$CONN_{FG}(v_1, v_j) = F_M(v_1 v_j) = \max\{F_N(v_1), F_N(v_j)\} = F_N(v_1).$$

Then

$$\sum_{k=2}^n T_N(v_1) T_N(v_k) CONN_{TG}(v_1, v_k) = (T_N(v_1))^2 \sum_{k=2}^n T_N(v_k) = t_1^2 \sum_{k=2}^n t_k,$$

Too for any $j, k \neq 1$ we have $CONN_{TG}(v_j, v_k) = T_N(v_1) = t_1$. Hence

$$\begin{aligned} PCI_T(G) &= \sum_{u,v \in N} T_N(u) T_N(v) CONN_{TG}(u, v) \\ &= \sum_{k=2}^n T_N(v_1) T_N(v_k) CONN_{TG}(v_1, v_k) + \sum_{k=3}^n T_N(v_2) T_N(v_k) CONN_{TG}(v_2, v_k) + \cdots \\ &\quad + T_N(v_{n-1}) T_N(v_n) CONN_{TG}(v_{n-1}, v_n) \\ &= (T_N(v_1))^2 \sum_{k=2}^n T_N(v_k) + T_N(v_1) \sum_{k=3}^n T_N(v_2) T_N(v_k) + \cdots + T_N(v_1) T_N(v_{n-1}) T_N(v_n) \\ &= (T_N(v_1))^2 \sum_{k=2}^n T_N(v_k) + T_N(v_1) \sum_{j=2}^{n-1} T_N(v_j) \sum_{k=j+1}^n T_N(v_k) = t_1 \sum_{j=1}^{n-1} t_j \sum_{k=j+1}^n t_k. \end{aligned}$$

□

Theorem 2.7.22. *Let $G = (N, M)$ be a connected neutrosophic graph such that $G^* = C_n$. Then the following are equivalent.*

- $TCI(G - uv) = TCI(G)$ for any uv ;
- M is a constant function;
- G has n strong spanning tree whit $S(T) = \gamma$ that γ is a constant value.

Proof. Suppose $G = (N, M)$ be a neutrosophic graph with $G^* = C_n$.

$a \rightarrow b$ Assume that $TCI(G - uv) = TCI(G)$ for any uv . This means that deleting each edge will not change the value of the connectivity index. Therefore, the membership function will be the same for all edges.

$b \rightarrow c$ Assume that M is a constant function. Hence all the edges of G are I-strong edge. Since removing each edge from the cycle will result a new tree of G . then the number of strong spanning trees of G will be n and strength of any strong spanning tree is a constant value.

$c \rightarrow a$ Assume that G has n strong spanning tree whit $S(T) = \gamma$ that γ is a constant value. It is clear for each edge of G we have $TCI(G - uv) = TCI(G)$. \square

2.8 Summary

In the second chapter, we defined the connectivity index on neutrosophic graphs. We also presented an algorithm for finding spanning trees and maximum spanning trees. In this chapter, an example of how to implement the algorithm is presented. We hope that by examining more examples, the respected readers will be able to master the steps of the algorithm and use it in real cases.

Chapter 3

Wiener index in Neutrosophic Graph

In this chapter, we will introduce the Wiener index in neutrosophic graphs. The Wiener index is a distance-based index that is widely used in symmetric graphs. Like the connectivity index, we divide the Wiener index into a Totally and Partial Wiener index and define it as follows.

3.1 Introduction

The intricate interplay between graph theory and the inherent ambiguity of the universe has long presented a captivating and fundamental challenge for researchers. Graph theory, through its mathematical language, deciphers discrete structures and relationships within complex systems, ranging from molecular networks to global communication infrastructures. Within this domain, Topological Indices (TIs) serve as invaluable quantitative metrics, providing a compact and insightful map of global graph properties such as compactness, branching, and transmission efficiency. Foremost among these influential and widely used indices is the Wiener Index (WI). This seminal index, first introduced by the eminent chemist Harry Wiener in 1947 within the context of studying hydrocarbon boiling points, rapidly transcended the boundaries of theoretical chemistry. It evolved into an essential tool in discrete mathematics, computer science, drug design, and network science. The seemingly simple definition of the Wiener Index – the sum of the shortest-path distances between all pairs of distinct vertices in a connected graph – encapsulates profound structural information that predicts the topological and dynamical behavior of the system.

However, the real world that graph theory seeks to model is replete with Uncertainty, Am-

biguity, and Inconsistency. Can we definitively and precisely determine how "close" two nodes are in a social network? Is the weight of a connection in a biological neural network or an urban transportation network a completely fixed and immutable value? Neutrosophic Set Theory (NT), pioneered by Florentin Smarandache in the 1990s, provides a revolutionary and novel response to these challenges. By introducing three independent membership functions – Truth-membership (T), Indeterminacy-membership (I), and Falsity-membership (F) – to describe each element, this theory offers a comprehensive and flexible framework for modeling phenomena that are inherently vague, incomplete, contradictory, or influenced by non-deterministic parameters. Neutrosophic Graphs (NGs), as a powerful generalization of classical graphs and even fuzzy graphs, possess the unique capability to simultaneously incorporate multi-dimensional uncertainty both in vertices (representing ambiguous entities) and in edges (representing non-deterministic, variable, or inconsistent relationships) within the graph structure itself.

Chapter 3 of this book marks the convergence of these two rich worlds – classical topological indices and neutrosophic theory – focusing specifically on the generalization and analysis of the Wiener Index within the complex domain of Neutrosophic Graphs. Our aim in this chapter extends beyond merely proposing a new definition; it involves a deep exploration of the fundamental concepts of "distance" and "sum of distances" under conditions where the vertices and edges themselves are characterized by degrees of truth, indeterminacy, and falsity. The core challenges we face are significant:

Defining Neutrosophic Distance: How can we define the concept of the "shortest path" in a graph where the length (or even the existence) of each edge is not a crisp number, but a neutrosophic triple (T, I, F)? Do different paths exhibit varying levels of truth and indeterminacy in their lengths? How do we compare these paths and select the "shortest" one?

Defining the Neutrosophic Wiener Index: Given a meaningful definition of neutrosophic distance between two vertices, how do we compute the sum of these distances for all vertex pairs? Will the result be a real number, a neutrosophic set, or an interval? Which arithmetic operations are most appropriate for aggregating neutrosophic distances?

Interpretation and Application: What novel topological information about the graph's structure and its internal levels of uncertainty does the computed Neutrosophic Wiener Index convey? How can this index be used for ranking graphs, analyzing real-world uncertain networks, or predicting properties in neutrosophic systems?

Relationship to Classical Concepts: Does the Neutrosophic Wiener Index reduce to the classical Wiener Index under specific conditions (e.g., when uncertainty vanishes)? What relationship exists between the value of this index and the overall degrees of truth, indeterminacy, and falsity governing the graph?

In this chapter, leveraging the rich legacy of the Wiener Index in deterministic and fuzzy graphs, and employing the principles of neutrosophic set theory and calculus, we will establish a rigorous and coherent framework for defining, computing, and analyzing the Wiener Index within Neutrosophic Graphs. We begin by reviewing and solidifying the fundamental definitions of neutrosophic graphs and essential concepts such as neutrosophic paths and neutrosophic distance. Subsequently, by examining various approaches to defining neutrosophic distance (e.g., expected-value-based approaches, deterministic transformation approaches, single-valued number approaches, etc.), we will present comprehensive and meaningful definitions for the Neutrosophic Wiener Index. The fundamental theorems in this chapter will explore the mathematical properties of this generalized index (such as bounds, relationships with other neutrosophic topological indices, behavior under graph operations). Furthermore, practical computation of this index will be demonstrated for important classes of neutrosophic graphs (such as neutrosophic paths, cycles, stars, and trees) to enhance the understanding of its applicability and interpretability.

The significance of this chapter lies not only in extending a crucial classical index to a more indeterminate space but also in opening a new window for the topological analysis of complex real-world systems intrinsically intertwined with uncertainty. Social networks with ambiguous friendships, biological networks with unknown or variable interactions, logistical systems with uncertain travel times, and electronic circuits with noisy resistances are just a few examples of domains where neutrosophic modeling and indices like the Neutrosophic Wiener Index can provide powerful analytical

tools for understanding their structure and behavior. This chapter constitutes a fundamental step in enriching "Neutrosophic Topology" and expanding its broad applications in sciences dealing with complexity and uncertainty.

Chapter Structure: Following this analytical introduction, the chapter proceeds as follows:

Theoretical Foundations: A concise yet precise review of essential concepts of neutrosophic graphs, definitions of neutrosophic paths and distance (emphasizing the definitions adopted in this book).

Defining the Neutrosophic Wiener Index: Formal presentation of definitions for the Neutrosophic Wiener Index (NWI) based on the selected approach(es) to neutrosophic distance. Discussion on the advantages and limitations of each definition.

Properties and Theorems: Derivation and proof of the most important mathematical properties of NWI, including upper and lower bounds, behavior in specific graph classes, relationships with other parameters, and effects of graph operations.

Computations and Examples: Explicit calculation of NWI for canonical classes of neutrosophic graphs (paths, cycles, stars, trees) and illustrative numerical examples.

Advanced Topics (if space permits): A brief overview of other generalizations (e.g., vertex-based Wiener index), connections to centrality concepts, and potential application domains.

The ultimate goal of this chapter is to equip the reader with the knowledge and tools necessary to understand, compute, and apply the Wiener Index as a key topological metric within the imprecise and ambiguous world of Neutrosophic Graphs. It is hoped that these discussions will lay a cornerstone for future research in this emerging and dynamic field.

3.2 Partial and totally Wiener Index

In this section, we will introduce the Wiener index in neutrosophic graphs. The Wiener index is a distance-based index that is widely used in symmetric graphs.

Like the connectivity index, we divide the Wiener index into a Totally and Partial Wiener index and define it as follows.

Defintion 3.2.1. Let $G = (N, M)$ be the Neutrosophic Graph and $v_1, v_2 \in V(G)$. A strong path P from v_1 to v_2 is called a neutrosophic geodesic if there is no strong shorter path between v_1 and v_2 .

Note that in the above definition, the shortest strong path must be calculated separately for each of truth (T), indeterminacy (I), and falsity (F) states.

Defintion 3.2.2. Let $G = (N, M)$ be the Neutrosophic Graph. The Partial Wiener Index (PWI) of G is defined as

$$\begin{aligned}
 PWI_T(G) &= \sum_{u,v \in N} T_N(u)T_N(v)d_{ST}(u,v), \\
 PWI_I(G) &= \sum_{u,v \in N} I_N(u)I_N(v)d_{SI}(u,v), \\
 PWI_F(G) &= \sum_{u,v \in N} F_N(u)F_N(v)d_{SF}(u,v),
 \end{aligned}$$

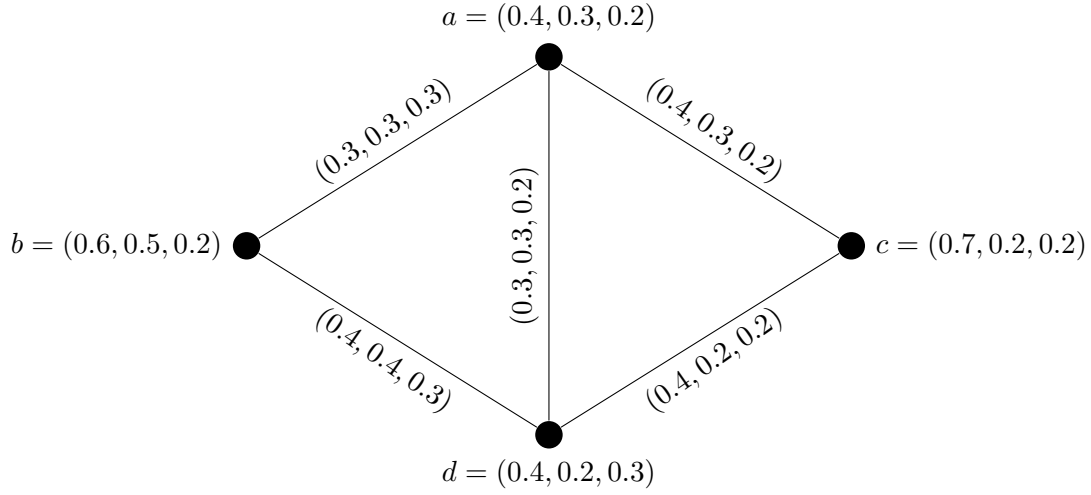
when $d_s(u, v)$ is the minimum, the sum of the weights of the edges in geodesic between u and v . Also, the Totally Wiener Index (TWI) of G is defined by

$$TWI(G) = \frac{(4 + 2PWI_T(G) - 2PWI_F(G) - PWI_I(G))}{6}.$$

Example 3.2.3. Consider the Neutrosophic Graph $G = (N, M)$ as shown in figure 1, with the vertex set $V = \{a, b, c, d\}$ where $(T_N, I_N, F_N)(a) = (0.4, 0.3, 0.2), (T_N, I_N, F_N)(b) = (0.6, 0.5, 0.2), (T_N, I_N, F_N)(c) = (0.7, 0.2, 0.2),$ and $(T_N, I_N, F_N)(d) = (0.4, 0.2, 0.3)$, whit the edge set $(T_M, I_M, F_M)(a, b) = (0.3, 0.3, 0.3), (T_M, I_M, F_M)(a, c) = (0.4, 0.3, 0.2), (T_M, I_M, F_M)(a, d) = (0.3, 0.3, 0.2), (T_M, I_M, F_M)(b, d) = (0.4, 0.4, 0.3), (T_M, I_M, F_M)(c, d) = (0.4, 0.2, 0.2)$, We have,

	$d_{ST}(u, v)$	$d_{SI}(u, v)$	$d_{SF}(u, v)$
a, b	$0.4 + 0.4 + 0.4 = 1.2$	0.3	0.3
a, c	0.4	0.3	0.2
a, d	$0.4 + 0.4 = 0.8$	0.3	0.2
b, c	$0.4 + 0.4 = 0.8$	$0.3 + 0.3 = 0.6$	$0.2 + 0.3 = 0.5$
b, d	0.4	$0.3 + 0.3 = 0.6$	0.3
c, d	0.4	0.2	0.2

Table 3.1: The sum of the weights of the edges in geodesic between each pair of vertices u and v .

Figure 3.1: A neutrosophic graph G

use table 3.1, we have:

$$\begin{aligned}
 PWI_T(G) &= \sum_{u,v \in N} T_N(u)T_N(v)d_{ST}(u,v) = (0.4)(0.6)(1.2) + (0.4)(0.7)(0.4) \\
 &\quad + (0.4)(0.4)(0.8) + (0.6)(0.7)(0.8) + (0.6)(0.4)(0.4) + (0.7)(0.4)(0.4) \\
 &= 0.288 + 0.112 + 0.128 + 0.336 + 0.096 + 0.112 = 1.072
 \end{aligned}$$

$$\begin{aligned}
 PWI_I(G) &= \sum_{u,v \in N} I_N(u)I_N(v)d_{SI}(u,v) = (0.3)(0.5)(0.3) + (0.3)(0.2)(0.3) \\
 &\quad + (0.3)(0.2)(0.3) + (0.5)(0.2)(0.6) + (0.5)(0.2)(0.6) + (0.2)(0.2)(0.2) \\
 &= 0.045 + 0.018 + 0.018 + 0.060 + 0.060 + 0.008 = 0.209
 \end{aligned}$$

$$\begin{aligned}
 PWI_F(G) &= \sum_{u,v \in N} F_N(u)F_N(v)d_{SF}(u,v) = (0.2)(0.3)(0.3) + (0.2)(0.2)(0.2) \\
 &\quad + (0.2)(0.3)(0.2) + (0.2)(0.2)(0.5) + (0.2)(0.3)(0.3) + (0.2)(0.3)(0.2) \\
 &= 0.012 + 0.008 + 0.012 + 0.020 + 0.018 + 0.012 = 0.082
 \end{aligned}$$

too, totally wiener index is:

$$\begin{aligned}
 TWI(G) &= \frac{4 + 2PWI_T(G) - 2PWI_F(G) - PWI_I(G)}{6} \\
 &= \frac{4 + 2(1.072) - 2(0.082) - 0.209}{6} = 0.941.
 \end{aligned}$$

3.3 Examples of the Wiener Index in neutrosophic special graphs

In this section, we calculate the Wiener index with examples for a number of neutrosophic graphs such as a neutrosophic graph with G^* of a path and a neutrosophic star.

3.3.1 Wiener Index in neutrosophic Graph with G^* path

Example 3.3.1. Let $G = (N, M)$ is a neutrosophic graph with G^* is a path.

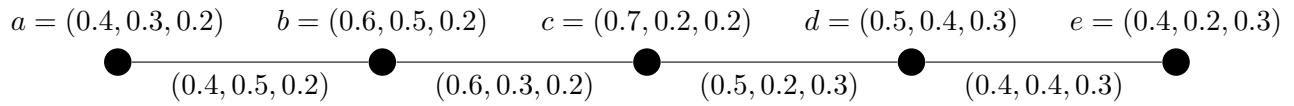


Figure 3.2: A neutrosophic graph G

	$d_{ST}(u, v)$	$d_{SI}(u, v)$	$d_{SF}(u, v)$
a, b	0.4	0.5	0.2
a, c	$0.4 + 0.6 = 1$	$0.5 + 0.3 = 0.9$	$0.2 + 0.2 = 0.4$
a, d	$0.4 + 0.6 + 0.5 = 1.5$	$0.5 + 0.3 + 0.2 = 1$	$0.2 + 0.2 + 0.3 = 0.7$
a, e	$0.4 + 0.6 + 0.5 + 0.4 = 1.9$	$0.5 + 0.3 + 0.2 + 0.4 = 1.4$	$0.2 + 0.2 + 0.3 + 0.3 = 1.0$
b, c	0.6	0.3	0.2
b, d	$0.6 + 0.5 = 1.1$	$0.3 + 0.2 = 0.5$	$0.2 + 0.3 = 0.5$
b, e	$0.6 + 0.5 + 0.4 = 1.5$	$0.3 + 0.2 + 0.4 = 0.9$	$0.2 + 0.3 + 0.3 = 0.8$
c, d	0.5	0.2	0.3
c, e	$0.5 + 0.4 = 0.9$	$0.2 + 0.4 = 0.6$	$0.3 + 0.3 = 0.6$
d, e	0.4	0.4	0.3

Table 3.2: $d_S(u, v)$ of the edges in geodesic between each pair of vertices u and v .

Now, we have:

$$\begin{aligned}
 PWI_T(G) &= \sum_{u,v \in N} T_N(u)T_N(v)d_{ST}(u, v) = (0.4)(0.6)(0.4) + (0.4)(0.7)(1.0) \\
 &\quad + (0.4)(0.5)(1.5) + (0.4)(0.4)(1.9) + (0.6)(0.7)(0.6) + (0.6)(0.5)(1.1) \\
 &\quad + (0.6)(0.4)(1.5) + (0.7)(0.5)(0.5) + (0.7)(0.4)(0.9) + (0.5)(0.4)(0.4) \\
 &= 2.429,
 \end{aligned}$$

$$\begin{aligned}
PWI_I(G) &= \sum_{u,v \in N} I_N(u)I_N(v)d_{SI}(u,v) = (0.3)(0.5)(0.5) + (0.3)(0.2)(0.9) \\
&+ (0.3)(0.4)(1.1) + (0.3)(0.2)(1.4) + (0.5)(0.2)(0.3) + (0.5)(0.4)(0.5) \\
&+ (0.5)(0.2)(0.9) + (0.2)(0.4)(0.2) + (0.2)(0.2)(0.6) + (0.4)(0.2)(0.4) \\
&= 0.637,
\end{aligned}$$

$$\begin{aligned}
PWI_F(G) &= \sum_{u,v \in N} F_N(u)F_N(v)d_{SF}(u,v) = (0.2)(0.2)(0.2) + (0.2)(0.2)(0.4) \\
&+ (0.2)(0.3)(0.7) + (0.2)(0.3)(1.0) + (0.2)(0.2)(0.2) + (0.2)(0.3)(0.5) \\
&+ (0.2)(0.3)(0.8) + (0.2)(0.3)(0.3) + (0.2)(0.3)(0.6) + (0.3)(0.3)(0.3) \\
&= 0.293,
\end{aligned}$$

too, totally wiener index is:

$$\begin{aligned}
TWI(G) &= \frac{4 + 2PWI_T(G) - 2PWI_F(G) - PWI_I(G)}{6} \\
&= \frac{4 + 2(2.429) - 2(0.293) - 0.637}{6} = 1.2725.
\end{aligned}$$

3.3.2 Wiener Index in neutrosophic Star

Example 3.3.2. Let $G = (N, M)$ ia a neutrosophic graph with G^* is a phat. Now, we have:

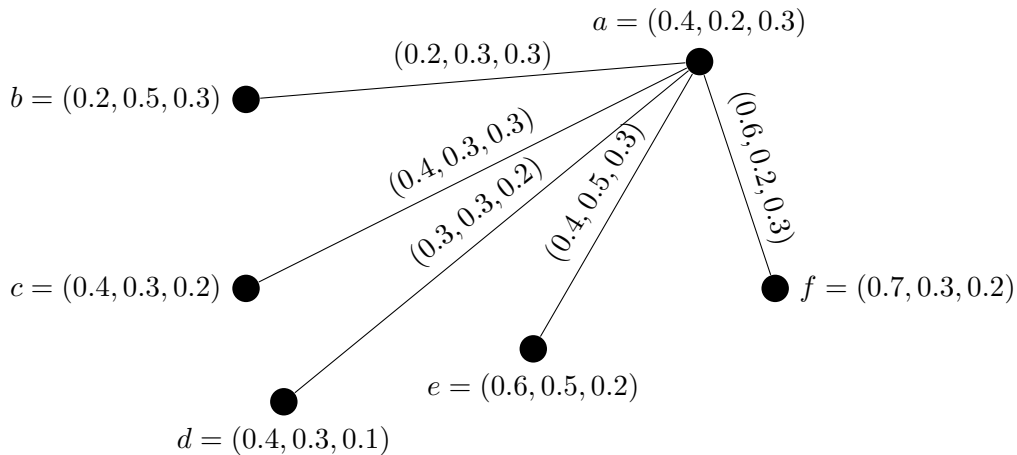


Figure 3.3: A neutrosophic graph G

	$d_{ST}(u, v)$	$d_{SI}(u, v)$	$d_{SF}(u, v)$
a, b	0.2	0.3	0.3
a, c	0.4	0.3	0.3
a, d	0.3	0.3	0.2
a, e	0.4	0.5	0.3
a, f	0.6	0.2	0.3
b, c	$0.2 + 0.4 = 0.6$	$0.3 + 0.3 = 0.6$	$0.3 + 0.3 = 0.6$
b, d	$0.2 + 0.3 = 0.5$	$0.3 + 0.3 = 0.6$	$0.2 + 0.3 = 0.5$
b, e	$0.2 + 0.4 = 0.6$	$0.3 + 0.5 = 0.8$	$0.3 + 0.3 = 0.6$
b, f	$0.6 + 0.2 = 0.8$	$0.3 + 0.2 = 0.5$	$0.3 + 0.3 = 0.6$
c, d	$0.4 + 0.3 = 0.7$	$0.3 + 0.3 = 0.6$	$0.3 + 0.2 = 0.5$
c, e	$0.4 + 0.4 = 0.8$	$0.3 + 0.5 = 0.8$	$0.3 + 0.3 = 0.6$
c, f	$0.6 + 0.4 = 1.0$	$0.2 + 0.3 = 0.5$	$0.3 + 0.3 = 0.6$
d, e	$0.3 + 0.4 = 0.7$	$0.3 + 0.5 = 0.8$	$0.3 + 0.2 = 0.5$
d, f	$0.3 + 0.6 = 0.9$	$0.3 + 0.2 = 0.5$	$0.3 + 0.2 = 0.5$
e, f	$0.4 + 0.6 = 1.0$	$0.5 + 0.2 = 0.7$	$0.3 + 0.3 = 0.6$

Table 3.3: $d_S(u, v)$ of the edges in geodesic between each pair of vertices u and v .

$$\begin{aligned}
PWI_T(G) &= \sum_{u,v \in N} T_N(u)T_N(v)d_{ST}(u, v) = (0.4)(0.2)(0.2) + (0.4)(0.4)(0.4) \\
&+ (0.4)(0.4)(0.3) + (0.4)(0.6)(0.4) + (0.4)(0.7)(0.6) + (0.2)(0.4)(0.6) \\
&+ (0.2)(0.4)(0.5) + (0.2)(0.6)(0.6) + (0.2)(0.7)(0.8) + (0.4)(0.4)(0.7) \\
&+ (0.4)(0.6)(0.8) + (0.4)(0.7)(1.0) + (0.4)(0.6)(0.7) + (0.4)(0.7)(0.9) \\
&+ (0.6)(0.7)(1.0) = 2.088,
\end{aligned}$$

$$\begin{aligned}
PWI_I(G) &= \sum_{u,v \in N} I_N(u)I_N(v)d_{SI}(u, v) = (0.2)(0.5)(0.3) + (0.2)(0.3)(0.3) \\
&+ (0.2)(0.3)(0.3) + (0.2)(0.5)(0.5) + (0.2)(0.3)(0.2) + (0.5)(0.3)(0.6) \\
&+ (0.5)(0.3)(0.6) + (0.5)(0.5)(0.8) + (0.5)(0.3)(0.5) + (0.3)(0.3)(0.6) \\
&+ (0.3)(0.5)(0.8) + (0.3)(0.3)(0.5) + (0.3)(0.5)(0.8) + (0.3)(0.3)(0.5) \\
&+ (0.5)(0.3)(0.7) = 1.072,
\end{aligned}$$

$$\begin{aligned}
PWI_F(G) &= \sum_{u,v \in N} F_N(u)F_N(v)d_{SF}(u,v) = (0.3)(0.3)(0.3) + (0.3)(0.2)(0.3) \\
&+ (0.3)(0.1)(0.2) + (0.3)(0.2)(0.3) + (0.3)(0.2)(0.3) + (0.3)(0.2)(0.6) \\
&+ (0.3)(0.1)(0.5) + (0.3)(0.2)(0.6) + (0.3)(0.2)(0.6) + (0.2)(0.1)(0.5) \\
&+ (0.2)(0.2)(0.6) + (0.2)(0.2)(0.6) + (0.1)(0.2)(0.5) + (0.1)(0.2)(0.5) \\
&+ (0.2)(0.2)(0.6) = 0.312,
\end{aligned}$$

too, totally wiener index is:

$$\begin{aligned}
TWI(G) &= \frac{4 + 2PWI_T(G) - 2PWI_F(G) - PWI_I(G)}{6} \\
&= \frac{4 + 2(2.088) - 2(0.312) - 1.072}{6} = 1.08.
\end{aligned}$$

3.4 Theories of Wiener index in neutrosophic graphs

Theorem 3.4.1. Let $G = (N, M)$ be a complete neutrosophic graph with $V = [v_1, v_2, \dots, v_n]$ such that $t_1 \leq t_2 \leq \dots \leq t_n$, $i_1 \leq i_2 \leq \dots \leq i_n$, $f_1 \geq f_2 \geq \dots \geq f_n$ where $t_i = T_N(v_j)$, $i_j = I_N(v_j)$ and $f_j = F_N(v_j)$ for $j = 1, 2, \dots, n$ then:

$$PWI_T(G) = \sum_{j=1}^{n-1} t_j^2 \sum_{k=j+1}^n t_k, \quad PWI_I(G) = \sum_{j=1}^{n-1} i_j^2 \sum_{k=j+1}^n i_k, \quad PWI_F(G) = \sum_{j=1}^{n-1} f_j^2 \sum_{k=j+1}^n f_k.$$

Proof. Consider neutrosophic graph $G = (N, M)$ with the conditions given in the theorem. According to the definition of the Wiener index

$$PWI_T(G) = \sum_{u,v \in N} T_N(u)T_N(v)d_{ST}(u,v)$$

Since G is a complete neutrosophic graph, there is a path of length one between the two vertices. We show that the path is geodesic. Let $u = v_1$. Then for any $2 \leq i \leq n$, we have $t_1 \leq t_i$, it is easy to see that

$$d_{ST}(v_1, v_i) = t_1, \quad 2 \leq i \leq n,$$

now, we have for v_2 :

$$d_{ST}(v_2, v_i) = t_2, \quad 3 \leq i \leq n,$$

for v_k :

$$d_{ST}(v_k, v_i) = t_k, \quad k + 1 \leq i \leq n,$$

and, we have for v_{n-1} :

$$d_{ST}(v_{n-1}, v_n) = t_{n-1}.$$

now, we get by placing the above relation $PWI_T(G)$:

$$\begin{aligned} PWI_T(G) &= \sum_{u,v \in N} T_N(u)T_N(v)d_{ST}(u,v) = T_N(v_1)T_N(v_2)t_1 + \cdots + T_N(v_1)T_N(v_n)t_1 \\ &+ T_N(v_2)T_N(v_3)t_2 + \cdots + T_N(v_2)T_N(v_n)t_2 + \cdots + T_N(v_k)T_N(v_{k+1})t_k + \cdots \\ &+ T_N(v_k)T_N(v_n)t_k + \cdots + T_N(v_{n-1})T_N(v_n)t_{n-1} = t_1t_2t_1 + \cdots + t_1t_nt_1 \\ &+ t_2t_3t_2 + \cdots + t_2t_nt_2 + \cdots + t_kt_{k+1}t_k + \cdots + t_kt_nt_k + \cdots + t_{n-1}t_nt_{n-1} \\ &= t_1^2(t_2 + \cdots + t_n) + t_2^2(t_3 + \cdots + t_n) + \cdots + t_k^2(t_{k+1} + \cdots + t_n) + \cdots + t_{n-1}^2t_n \\ &= \sum_{j=1}^{n-1} t_j^2 \sum_{k=j+1}^n t_k. \end{aligned}$$

Similarly, $PWI_I(G)$ and $PWI_F(G)$ can be proved. \square

Corollary 3.4.2. *Consider the complete neutrosophic graph $G = (N, M)$ with the above theorem conditions, then*

$$PWI_T(G) = PCI_T(G), \quad PWI_I(G) = PCI_I(G), \quad PWI_F(G) = PCI_F(G).$$

Also, $TWI(G) = TCI(G)$.

Proof. According to theorem 2.6, and the above theorem is clear. \square

Theorem 3.4.3. *Let $G = (N, M)$ be a neutrosophic graph with $|N^*| = n$, such that G^* is a tree. If for each $uv \in M$, $G - uv$ has two connecting components w_1 and w_2 , it has l and k vertices,*

respectively such that $l + k = n$. Then

$$\begin{aligned} PWI_T(G) &= \sum_{uv \in G} T_M(uv) \sum_{i=1}^l T_N(u_i) \sum_{j=1}^k T_N(v_j), \\ PWI_I(G) &= \sum_{uv \in G} I_M(uv) \sum_{i=1}^l I_N(u_i) \sum_{j=1}^k I_N(v_j), \\ PWI_F(G) &= \sum_{uv \in G} F_M(uv) \sum_{i=1}^l F_N(u_i) \sum_{j=1}^k F_N(v_j). \end{aligned}$$

Proof. Let $G = (N, M)$ be a neutrosophic graph with $|N^*| = n$, and G^* is a tree. Now suppose we remove the desired edge $uv, uv \in M$, from G . Graph G is divided into two connecting components w_1 and w_2 , so that w_1 will contain l vertices and w_2 will contain $k = n - l$ vertices. If $l = 1$ and $k = n - 1$, and $v_1 \in w_1$ then

$$\begin{aligned} PWI_T(G) &= \sum_{u,v \in N} T_N(u)T_N(v)d_{ST}(u,v) = T_N(v_1)T_N(v_2)T_M(uv) + T_N(v_1)T_N(v_3)(T_M(uv) + e_1) \\ &\quad + \cdots + T_N(v_1)T_N(v_n)(T_M(uv) + e_1 + \cdots + e_m) + \sum_{u,v \in N-v_1} T_N(u)T_N(v)d_{ST}(u,v) \end{aligned}$$

where $e_i \in M$ and $e_i \neq uv$.

Repeat the same process for $\sum_{(u,v \in N-v_1)} T_N(u)T_N(v)d_{ST}(u,v)$. We continue this until only one vertex remains in w_2 . Then, by factoring and summing the number of vertices of the two components, we reach the desired result. Similarly, $PWI_I(G)$ and $PWI_F(G)$ can be proved. \square

Theorem 3.4.4. *Let $G = (N, M)$ be a connected neutrosophic graph with the unique strong spanning tree T . then*

$$PWI_T(G) = PWI_T(T), \quad PWI_I(G) = PWI_I(T), \quad PWI_F(G) = PWI_F(T),$$

Hence $TWI(G) = TWI(T)$.

Proof. Let G be a connected neutrosophic graph and T is the unique strong spanning tree of G . By definition of strong spanning tree, if u and v are two vertices of G , we have

$$d_{ST}(u,v)(G) = d_{ST}(u,v)(T), \quad d_{SI}(u,v)(G) = d_{SI}(u,v)(T), \quad d_{SF}(u,v)(G) = d_{SF}(u,v)(T).$$

Since, it is clear from the above relation that

$$PWI_T(G) = PWI_T(T), \quad PWI_I(G) = PWI_I(T), \quad PWI_F(G) = PWI_F(T),$$

Therefore $TWI(G) = TWI(T)$. □

Theorem 3.4.5. *Let $G = (N, M)$ be a neutrosophic graph with $G^* = C_n$. Let M be a constant function. Then*

1. For $n = 2m$, $m \in \mathbb{N}$

$$\begin{aligned} PWI_T(G) &= \sum_{k=1}^{\frac{n}{2}-1} kt \left(\sum_{j=1}^n T_N(u_j) T_N(u_{j+k}) \right) + \frac{n}{2} t \sum_{l=1}^{\frac{n}{2}} T_N(u_l) T_N(u_{l+\frac{n}{2}}) \\ PWI_I(G) &= \sum_{k=1}^{\frac{n}{2}-1} ki \left(\sum_{j=1}^n I_N(u_j) I_N(u_{j+k}) \right) + \frac{n}{2} i \sum_{l=1}^{\frac{n}{2}} I_N(u_l) I_N(u_{l+\frac{n}{2}}) \\ PWI_F(G) &= \sum_{k=1}^{\frac{n}{2}-1} kf \left(\sum_{j=1}^n F_N(u_j) F_N(u_{j+k}) \right) + \frac{n}{2} f \sum_{l=1}^{\frac{n}{2}} F_N(u_l) F_N(u_{l+\frac{n}{2}}) \end{aligned}$$

2. For $n = 2m + 1$, $m \in \mathbb{N}$

$$\begin{aligned} PWI_T(G) &= \sum_{k=1}^{\frac{n-1}{2}} kt \left(\sum_{j=1}^n T_N(u_j) T_N(u_{j+k}) \right) \\ PWI_I(G) &= \sum_{k=1}^{\frac{n-1}{2}} ki \left(\sum_{j=1}^n I_N(u_j) I_N(u_{j+k}) \right) \\ PWI_F(G) &= \sum_{k=1}^{\frac{n-1}{2}} kf \left(\sum_{j=1}^n F_N(u_j) F_N(u_{j+k}) \right) \end{aligned}$$

Note that for $j + k > n$, $u_{j+k} = u_d$, this is, $j + k \equiv d \pmod{n}$.

Also for $G - uv$, we have

$$\begin{aligned} PWI_T(G) &= \sum_{k=1}^{n-1} kt \left(\sum_{j=1}^{n-k} T_N(u_j) T_N(u_{j+k}) \right) \\ PWI_I(G) &= \sum_{k=1}^{n-1} ki \left(\sum_{j=1}^{n-k} I_N(u_j) I_N(u_{j+k}) \right) \\ PWI_F(G) &= \sum_{k=1}^{n-1} kf \left(\sum_{j=1}^{n-k} F_N(u_j) F_N(u_{j+k}) \right) \end{aligned}$$

Where $M = (t, i, f)$ is a constant function.

Proof. First, we assume that G^* is a cycle of even length, and $M = (t, i, f)$ is a constant function. Hence each edge of G is a neutral edge. Then, the maximum length of a neutrosophic geodesic in G is $\frac{n}{2}$. Now consider a case where the distance between two vertices is less than $\frac{n}{2}$. Suppose the distance between u and v is equal to k , where k is less than $\frac{n}{2}$. In that case, we define the geodesic length between the two vertices u and v as follows

$$P_k = \{(u, v) \in N^* \times N^*, k \text{ is equal to the geodetic length between } u \text{ and } v\}$$

On the other hand, we know that there are $\frac{n}{2}$ pairs of vertices (u, v) such that the geodesic length between them is exactly equal to $\frac{n}{2}(t, i, f)$, for these $\frac{n}{2}$ pairs of vertices, it is sufficient to obtain a product of $T_N(u)$ in $T_N(v)$ [Similarly, $I_N(u)$ in $I_N(v)$, and $F_N(u)$ in $F_N(v)$]. And then sum on u and v . Then we get

$$\frac{n}{2}t \sum_{l=1}^{\frac{n}{2}} T_N(u_l)T_N(u_{l+\frac{n}{2}}), \quad (3.1)$$

[Similarly for I and F]. Now back to the state that $1 \leq k < \frac{n}{2}$. For each vertex such as u on the cycle C_n , there is a vertex with distance kt from it. Suppose $k = 1$, so we have

$$T_N(u_1)T_N(u_2) + T_N(u_2)T_N(u_3) + \cdots + T_N(u_j)T_N(u_{j+1}) + \cdots + T_N(u_n)T_N(u_{n+1})$$

since $n + 1 \equiv 1 \pmod{n}$, hence $T_N(u_n)T_N(u_{n+1}) = T_N(u_n)T_N(u_1)$. Then

for $k = 1$, we have

$$1 \times t \times \sum_{j=1}^n T_N(u_j)T_N(u_{j+1}),$$

for $k = 2$:

$$2 \times t \times \sum_{j=1}^n T_N(u_j)T_N(u_{j+2}),$$

for $k = m$, $m < \frac{n}{2}$:

$$m \times t \times \sum_{j=1}^n T_N(u_j)T_N(u_{j+m}),$$

by continuing this process and summing on k , we get

$$\sum_{k=1}^{\frac{n}{2}-1} kt \left(\sum_{j=1}^n T_N(u_j)T_N(u_{j+k}) \right) \quad (3.2)$$

use from (3.1) and (3.2),

$$PWI_T(G) = (3.1) + (3.2) = \sum_{l=1}^{\frac{n}{2}} T_N(u_l)T_N(u_{l+\frac{n}{2}}) + \sum_{k=1}^{\frac{n}{2}-1} kt \left(\sum_{j=1}^n T_N(u_j)T_N(u_{j+k}) \right).$$

To prove that n is odd, note that the maximum distance between the vertices u , and v is $\frac{(n-1)}{2}$.

The continuation of the proof is similar to the case where n is even. \square

Theorem 3.4.6. *Let $G = (N, M)$ be a neutrosophic tree $|N^*| \geq 3$. Then*

$$PCI_T(G) < PWI_T(G), \quad PCI_I(G) < PWI_I(G), \quad PCI_F(G) < PWI_F(G).$$

But, $TCI(G)$ need not be less than or equal to $TWI(G)$.

Proof. Let $G = (N, M)$ be a neutrosophic tree and $|N^*| \geq 3$. Since in the neutrosophic tree, there is a unique strong path between vertices u and v , for any u and v . hence this path is the unique strongest path from u to v . then, $d_{ST}(u, v)$, for each u and v , is equal the sum of the truth-membership values of edges where those edges belong to the strong path from u to v . In other hands, $CONN_{TG}(u, v)$, is truth-membership values of the weakest edge of the $(u - v)$ -path. It follows that

$$CONN_{TG}(u, v) \leq d_{ST}(u, v),$$

In the above relation, equality occurs when uv is a strong edge. Otherwise

$$CONN_{TG}(u, v) < d_{ST}(u, v),$$

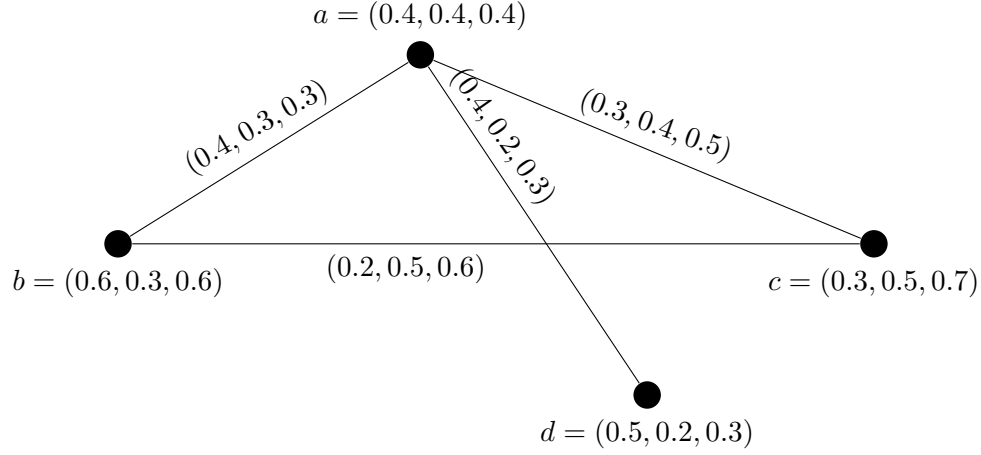
then, we have

$$PCI(G) < PWI(G).$$

Similarly, $PWI_I(G)$ and $PWI_F(G)$ can be proved. \square

Here we show with an example that $TCI(G)$ dose not always have to be less than $TWI(G)$.

Example 3.4.7. *Consider the Neutrosophic tree $G = (N, M)$ as shown in figure (3.4), Note that here bc is a weak edge.*

Figure 3.4: A neutrosophic tree G

	$CONN_{TG}(u, v)$	$CONN_{IG}(u, v)$	$CONN_{FG}(u, v)$	$d_{ST}(u, v)$	$d_{SI}(u, v)$	$d_{SF}(u, v)$
a, b	0.4	0.3	0.3	0.4	0.3	0.3
a, c	0.3	0.4	0.5	0.3	0.4	0.5
a, d	0.4	0.2	0.3	0.4	0.2	0.3
b, c	0.3	0.4	0.5	0.7	0.7	0.8
d, d	0.4	0.3	0.3	0.8	0.5	0.6
c, d	0.3	0.4	0.5	0.7	0.6	0.8

Table 3.4: The strength of connectedness and the geodesic between each pair of vertices u and v .

By direct calculations, we have

$$\begin{aligned}
 PCI_T(G) &= \sum_{u,v \in N} T_N(u)T_N(v)CONN_{TG}(u, v) \\
 &= (0.4)(0.6)(0.4) + (0.4)(0.3)(0.3) + (0.4)(0.5)(0.4) + (0.6)(0.3)(0.3) + (0.6)(0.5)(0.4) \\
 &\quad + (0.3)(0.5)(0.3) = 0.096 + 0.036 + 0.080 + 0.054 + 0.120 + 0.045 = 0.431,
 \end{aligned}$$

$$PCI_I(G) = \sum_{u,v \in N} I_N(u)I_N(v)CONN_{IG}(u, v) = 0.036 + 0.080 + 0.016 + 0.060 + 0.018 + 0.040 = 0.25,$$

$$PCI_F(G) = \sum_{u,v \in N} F_N(u)F_N(v)CONN_{FG}(u, v) = 0.072 + 0.14 + 0.036 + 0.21 + 0.054 + 0.105 = 0.617,$$

$$TCI(G) = \frac{4 + 2PCI_T(G) - 2PCI_F(G) - PCI_I(G)}{6} = \frac{3.378}{6} = 0.563.$$

$$PWI_T(G) = \sum_{u,v \in N} T_N(u)T_N(v)d_{ST}(u,v) = 0.096 + 0.036 + 0.08 + 0.126 + 0.24 + 0.105 = 0.683,$$

$$PWI_I(G) = \sum_{u,v \in N} I_N(u)I_N(v)d_{SI}(u,v) = 0.036 + 0.08 + 0.016 + 0.105 + 0.03 + 0.060 = 0.327,$$

$$PWI_F(G) = \sum_{u,v \in N} F_N(u)F_N(v)d_{SF}(u,v) = 0.072 + 0.14 + 0.036 + 0.336 + 0.108 + 0.168 = 0.86,$$

$$TWI(G) = \frac{4 + 2PWI_T(G) - 2PWI_F(G) - PWI_I(G)}{6} = \frac{3.319}{6} = 0.553.$$

As seen in this example

$$PCI_T(G) = 0.431 < PWI_T(G) = 0.683,$$

$$PCI_I(G) = 0.25 < PWI_I(G) = 0.327,$$

$$PCI_F(G) = 0.617 < PWI_F(G) = 0.86.$$

but, we have: $TCI(G) = 0.563 > TWI(G) = 0.553$.

The neutrosophic graph shown in the figure below is also a tree in which

$$PCI_T(G) < PWI_T(G), \quad PCI_I(G) < PWI_I(G), \quad PCI_F(G) < PWI_F(G).$$

and, $TCI(G) < TWI(G)$.

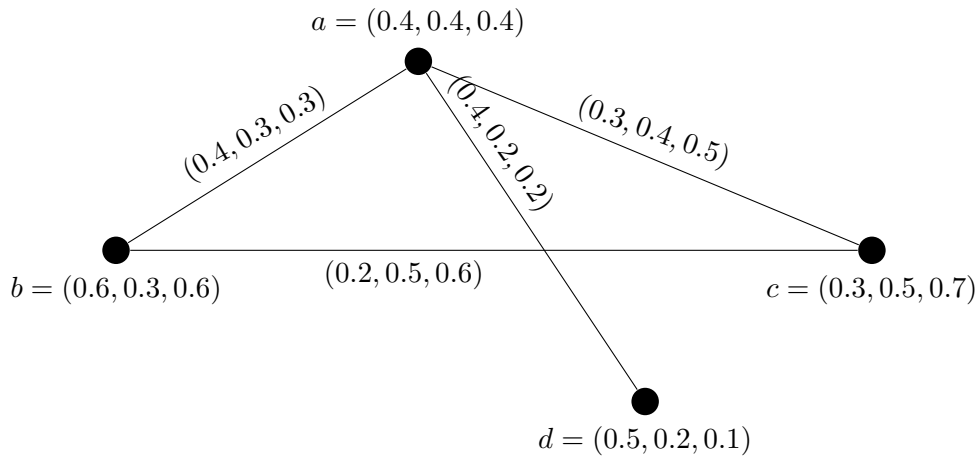


Figure 3.5: A neutrosophic tree with $V = \{a, b, c, d\}$

Theorem 3.4.8. Let $G = (N, M)$ be a neutrosophic tree $|N^*| \geq 3$, With G^* is a star. Let M be a constant function. if v_1 is the center vertex and v_2, v_3, \dots, v_n are the vertices adjacent to vertex

v_1 , then

$$\begin{aligned} PWI_T(G) &= 2t \sum_{j=1}^{n-1} T_N(v_j) \sum_{k=j+1}^n T_N(v_k) - tT_N(v_1) \sum_{j=2}^n T_N(v_j) \\ PWI_I(G) &= 2i \sum_{j=1}^{n-1} I_N(v_j) \sum_{k=j+1}^n I_N(v_k) - iT_N(v_1) \sum_{j=2}^n I_N(v_j) \\ PWI_F(G) &= 2f \sum_{j=1}^{n-1} F_N(v_j) \sum_{k=j+1}^n F_N(v_k) - fF_N(v_1) \sum_{j=2}^n F_N(v_j) \end{aligned}$$

where $M = (t, i, f)$.

Proof. Let $G = (N, M)$ be a neutrosophic tree $|N^*| \geq 3$, With G^* is a star. Since $M = (t, i, f)$ is a constant function and v_i is the center vertex, for each $v_i, 2 \leq i \leq n$, we have

$$d_{ST}(v_1, v_i) = t, \quad d_{SI}(v_1, v_i) = i, \quad d_{SF}(v_1, v_i) = f.$$

Also, for v_i and $v_j, i, j \neq 1$, then

$$d_{ST}(v_j, v_i) = 2t, \quad d_{SI}(v_j, v_i) = 2i, \quad d_{SF}(v_j, v_i) = 2f.$$

Then

$$\begin{aligned} PWI_T(G) &= \sum_{v_j, v_i \in N} T_N(v_i)T_N(v_j)d_{ST}(v_i, v_j) \\ &= \sum_{v_j \in N} T_N(v_1)T_N(v_j)d_{ST}(v_1, v_j) + \sum_{\substack{v_j, v_i \in N \\ i \neq 1}} T_N(v_i)T_N(v_j)d_{ST}(v_i, v_j) \\ &= tT_N(v_1) \sum_{j=2}^n T_N(v_j) + 2t \sum_{\substack{v_j, v_i \in N \\ i \neq 1}} T_N(v_i)T_N(v_j)d_{ST}(v_i, v_j) \\ &= tT_N(v_1) \sum_{j=2}^n T_N(v_j) + \left[2t \sum_{j=1}^{n-1} T_N(v_j) \sum_{k=j+1}^n T_N(v_k) - 2tT_N(v_1) \sum_{j=2}^n T_N(v_j) \right] \\ &= 2t \sum_{j=1}^{n-1} T_N(v_j) \sum_{k=j+1}^n T_N(v_k) - tT_N(v_1) \sum_{j=2}^n T_N(v_j). \end{aligned}$$

Similarly, $PWI_I(G)$ and $PWI_F(G)$ can be proved. □

3.5 Applications

One of the most important topics is the use of neutrosophic sets in other sciences and also the use of these assemblies to model various problems. Many applications have been discussed by experts

so far. Which can be referred to as application of neutrosophic in graphs [12, 17-19], application in algebraic topics [11, 14], application in intelligent systems and optimization [3, 4].

Here the Wiener index is calculated for a neutrosophic graph associated with a real-time example. You can see this issue and its explanation on the website www.pantechsolutions.net. The neutrosophic graph of this issue is also given in [5]. There, the author examines energy, Laplacian energy, and signless Laplacian energy. We also use the modeling used in [5] here. This neutrosophic graph is intended for four different time periods. According to each time period, we define a neutrosophic graph in the following order:

G_1 from 16 January 2018 to 15 February 2018 (figure 3.6);

G_2 from 16 February 2018 to 15 March 2018 (figure 3.7);

G_3 from 16 March 2018 to 15 April 2018 (figure 3.8);

G_4 from 16 April 2018 to 15 May 2018 (figure 3.9);

We now calculate the Wiener index (partial Wiener index and totally Wiener index) for each of the above time periods.

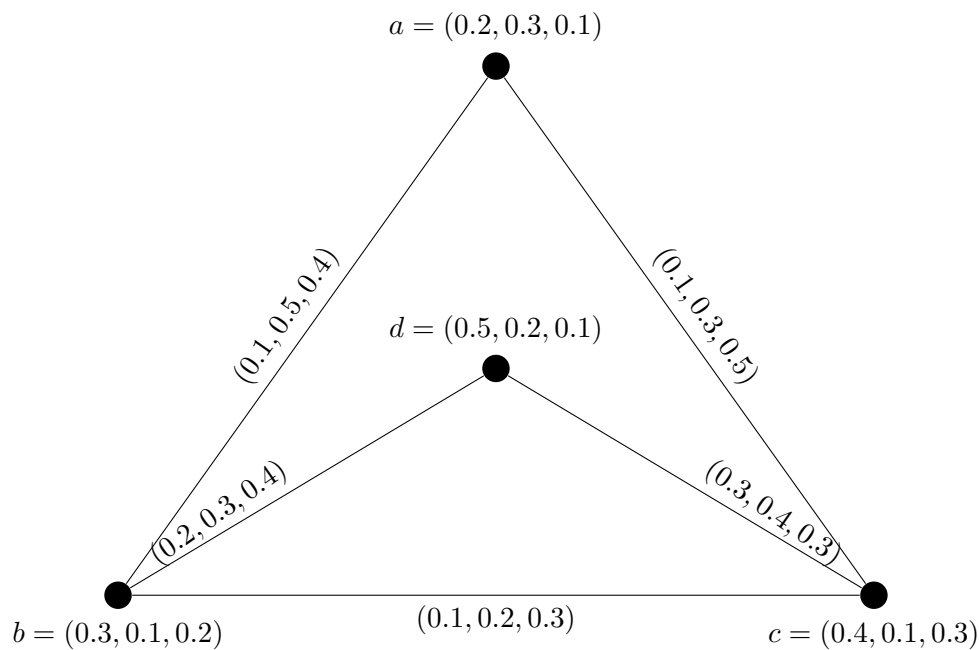


Figure 3.6: Neutrosophic graph G_1

	$d_{ST}(u, v)$	$d_{SI}(u, v)$	$d_{SF}(u, v)$
a, b	0.1	$0.3 + 0.2 = 0.5$	0.4
a, c	0.1	0.3	$0.4 + 0.3 = 0.7$
a, d	$0.1 + 0.2 = 0.3$	$0.3 + 0.4 = 0.7$	$0.4 + 0.4 = 0.8$
b, c	$0.2 + 0.3 = 0.5$	0.2	$0.2 + 0.3 = 0.5$
b, d	0.2	0.3	$0.3 + 0.3 = 0.6$
c, d	0.3	$0.2 + 0.3 = 0.5$	0.3

Table 3.5: The sum of the weights of the edges in geodesic between each pair of vertices u and v .

$$\begin{aligned}
 PWI_T(G_1) &= \sum_{u,v \in N} T_N(u)T_N(v)d_{ST}(u, v) \\
 &= (0.2)(0.3)(0.1) + (0.2)(0.4)(0.1) + (0.2)(0.5)(0.3) + (0.3)(0.4)(0.5) \\
 &\quad + (0.3)(0.5)(0.2) + (0.4)(0.5)(0.3) = 0.194,
 \end{aligned}$$

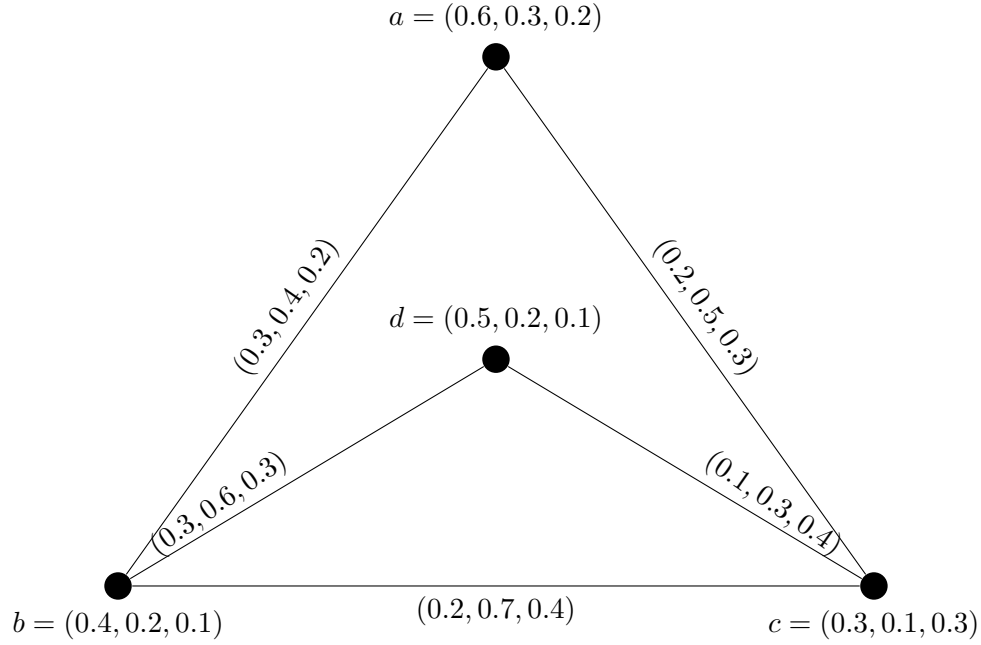
$$\begin{aligned}
 PWI_I(G_1) &= \sum_{u,v \in N} I_N(u)I_N(v)d_{SI}(u, v) \\
 &= (0.3)(0.1)(0.5) + (0.3)(0.1)(0.3) + (0.3)(0.2)(0.7) + (0.1)(0.1)(0.2) \\
 &\quad + (0.1)(0.2)(0.3) + (0.1)(0.2)(0.5) = 0.084,
 \end{aligned}$$

$$\begin{aligned}
 PWI_F(G_1) &= \sum_{u,v \in N} F_N(u)F_N(v)d_{SF}(u, v) \\
 &= (0.1)(0.2)(0.4) + (0.1)(0.3)(0.7) + (0.1)(0.1)(0.8) + (0.2)(0.3)(0.3) \\
 &\quad + (0.2)(0.1)(0.6) + (0.3)(0.1)(0.3) = 0.076,
 \end{aligned}$$

$$\begin{aligned}
 TWI(G_1) &= \frac{4 + 2PWI_T(G_1) - 2PWI_F(G_1) - PWI_I(G_1)}{6} \\
 &= \frac{4 + 2(0.196) - 2(0.076) - (0.084)}{6} = \frac{4.152}{6} = 0.629.
 \end{aligned}$$

	$d_{ST}(u, v)$	$d_{SI}(u, v)$	$d_{SF}(u, v)$
a, b	0.3	0.4	0.2
a, c	0.2	0.5	0.3
a, d	0.3	$0.3 + 0.5 = 0.8$	$0.2 + 0.3 = 0.5$
b, c	0.2	$0.4 + 0.5 = 0.9$	$0.2 + 0.3 = 0.5$
b, d	0.3	$0.4 + 0.5 + 0.3 = 1.2$	0.3
c, d	$0.2 + 0.3 = 0.5$	0.3	$0.3 + 0.2 + 0.3 = 0.8$

Table 3.6: The sum of the weights of the edges in geodesic between each pair of vertices u and v .

Figure 3.7: Neutrosophic graph G_2

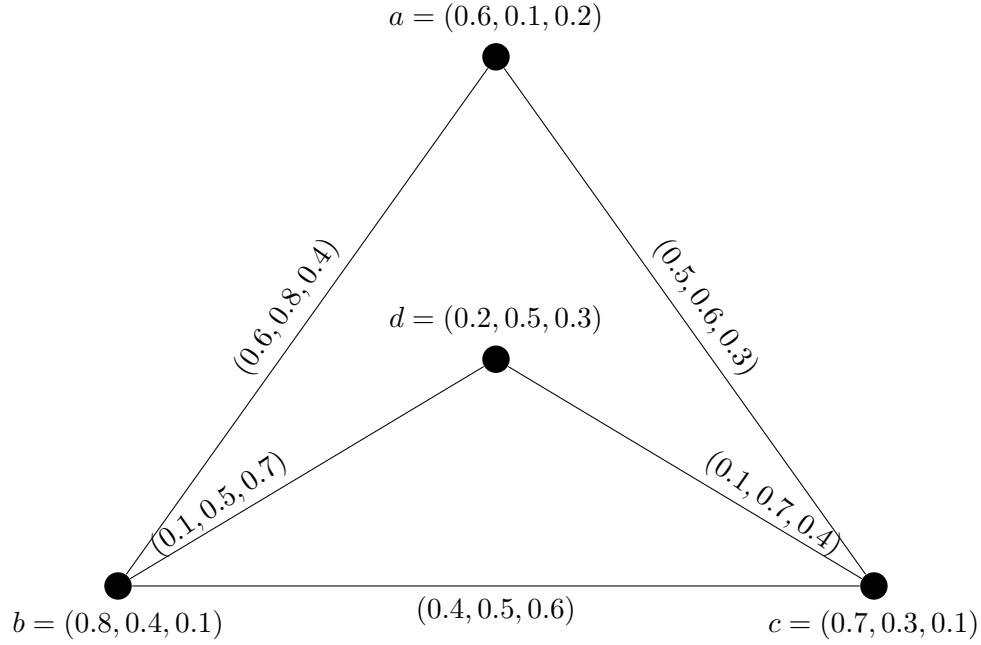
$$\begin{aligned}
 PWI_T(G_2) &= \sum_{u,v \in N} T_N(u)T_N(v)d_{ST}(u,v) \\
 &= 0.072 + 0.036 + 0.090 + 0.024 + 0.060 + 0.075 = 0.357,
 \end{aligned}$$

$$\begin{aligned}
 PWI_I(G_1) &= \sum_{u,v \in N} I_N(u)I_N(v)d_{SI}(u,v) \\
 &= 0.024 + 0.015 + 0.048 + 0.018 + 0.048 + 0.006 = 0.159,
 \end{aligned}$$

$$\begin{aligned}
 PWI_F(G_1) &= \sum_{u,v \in N} F_N(u)F_N(v)d_{SF}(u,v) \\
 &= 0.004 + 0.018 + 0.010 + 0.015 + 0.003 + 0.024 = 0.074,
 \end{aligned}$$

$$\begin{aligned}
 TWI(G_1) &= \frac{4 + 2PWI_T(G_1) - 2PWI_F(G_1) - PWI_I(G_1)}{6} \\
 &= \frac{4 + 2(0.357) - 2(0.074) - (0.159)}{6} = \frac{4.307}{6} = 0.718.
 \end{aligned}$$

$$\begin{aligned}
 PWI_T(G_2) &= \sum_{u,v \in N} T_N(u)T_N(v)d_{ST}(u,v) \\
 &= 0.288 + 0.210 + 0.072 + 0.616 + 0.016 + 0.014 = 1.216,
 \end{aligned}$$

Figure 3.8: Neutrosophic graph G_3

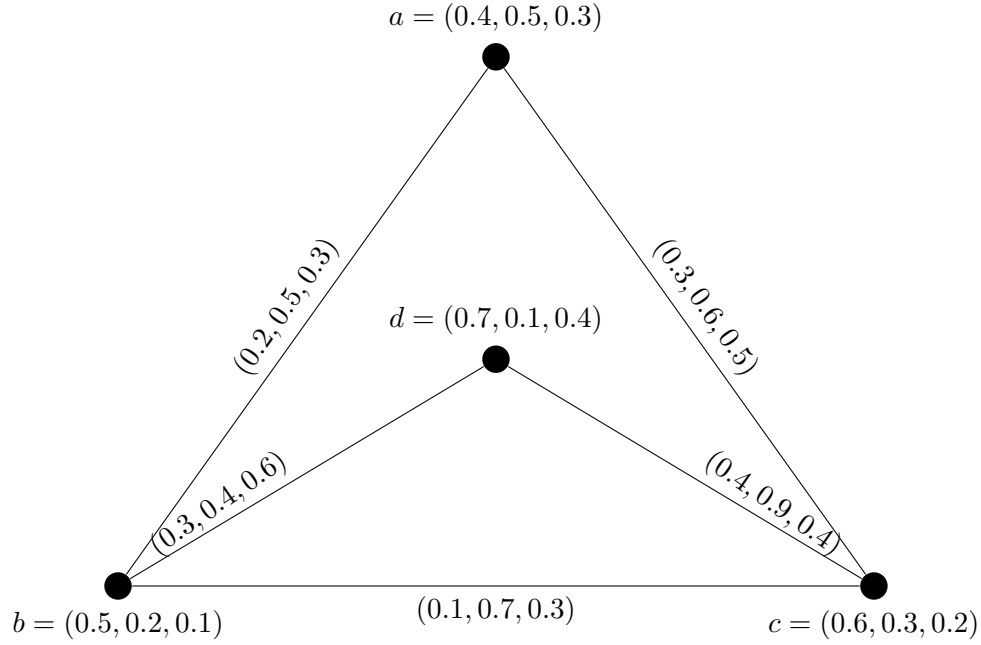
	$d_{ST}(u, v)$	$d_{SI}(u, v)$	$d_{SF}(u, v)$
a, b	0.6	$0.5 + 0.6 = 1.1$	0.4
a, c	0.5	0.6	0.3
a, d	$0.5 + 0.1 = 0.6$	$0.6 + 0.5 + 0.5 = 1.6$	$0.3 + 0.4 = 0.7$
b, c	$0.6 + 0.5 = 1.1$	0.5	$0.4 + 0.3 = 0.7$
b, d	0.1	0.5	$0.4 + 0.3 + 0.4 = 1.1$
c, d	0.1	$0.5 + 0.5 = 1$	0.4

Table 3.7: The sum of the weights of the edges in geodesic between each pair of vertices u and v in G_3 .

$$\begin{aligned}
 PWI_I(G_1) &= \sum_{u,v \in N} I_N(u)I_N(v)d_{SI}(u, v) \\
 &= 0.044 + 0.018 + 0.080 + 0.060 + 0.1 + 0.15 = 0.452,
 \end{aligned}$$

$$\begin{aligned}
 PWI_F(G_1) &= \sum_{u,v \in N} F_N(u)F_N(v)d_{SF}(u, v) \\
 &= 0.008 + 0.006 + 0.042 + 0.007 + 0.033 + 0.012 = 0.108
 \end{aligned}$$

$$\begin{aligned}
 TWI(G_1) &= \frac{4 + 2PWI_T(G_1) - 2PWI_F(G_1) - PWI_I(G_1)}{6} \\
 &= \frac{4 + 2(1.216) - 2(0.108) - (0.452)}{6} = \frac{5.548}{6} = 0.925.
 \end{aligned}$$

Figure 3.9: Neutrosophic graph G_4

	$d_{ST}(u, v)$	$d_{SI}(u, v)$	$d_{SF}(u, v)$
a, b	$0.3 + 0.4 + 0.3 = 1$	0.5	0.3
a, c	0.3	0.6	$0.3 + 0.3 = 0.6$
a, d	$0.3 + 0.4 = 0.7$	$0.5 + 0.4 = 0.9$	$0.5 + 0.4 = 0.9$
b, c	$0.3 + 0.4 = 0.7$	$0.25 + 0.6 = 1.1$	0.3
b, d	0.3	0.4	$0.3 + 0.4 = 0.7$
c, d	0.4	$0.6 + 0.5 = 1.1$	0.4

Table 3.8: The sum of the weights of the edges in geodesic between each pair of vertices u and v in G_4 .

$$\begin{aligned}
 PWI_T(G_2) &= \sum_{u,v \in N} T_N(u)T_N(v)d_{ST}(u, v) \\
 &= 0.20 + 0.072 + 0.196 + 0.210 + 0.105 + 0.168 = 0.951,
 \end{aligned}$$

$$\begin{aligned}
 PWI_I(G_1) &= \sum_{u,v \in N} I_N(u)I_N(v)d_{SI}(u, v) \\
 &= 0.050 + 0.180 + 0.045 + 0.066 + 0.008 + 0.033 = 0.382,
 \end{aligned}$$

$$\begin{aligned}
 PWI_F(G_1) &= \sum_{u,v \in N} F_N(u)F_N(v)d_{SF}(u, v) \\
 &= 0.009 + 0.036 + 0.108 + 0.006 + 0.028 + 0.032 = 0.219,
 \end{aligned}$$

$$\begin{aligned}
 TWI(G_1) &= \frac{4 + 2PWI_T(G_1) - 2PWI_F(G_1) - PWI_I(G_1)}{6} \\
 &= \frac{4 + 2(0.951) - 2(0.219) - (0.382)}{6} = \frac{5.082}{6} = 0.847.
 \end{aligned}$$

Now, using the Wiener index obtained for each of the neutrosophic graphs G_1 , G_2 , G_3 , and G_4 , we can compare these four components in the time intervals given in the problem. As shown

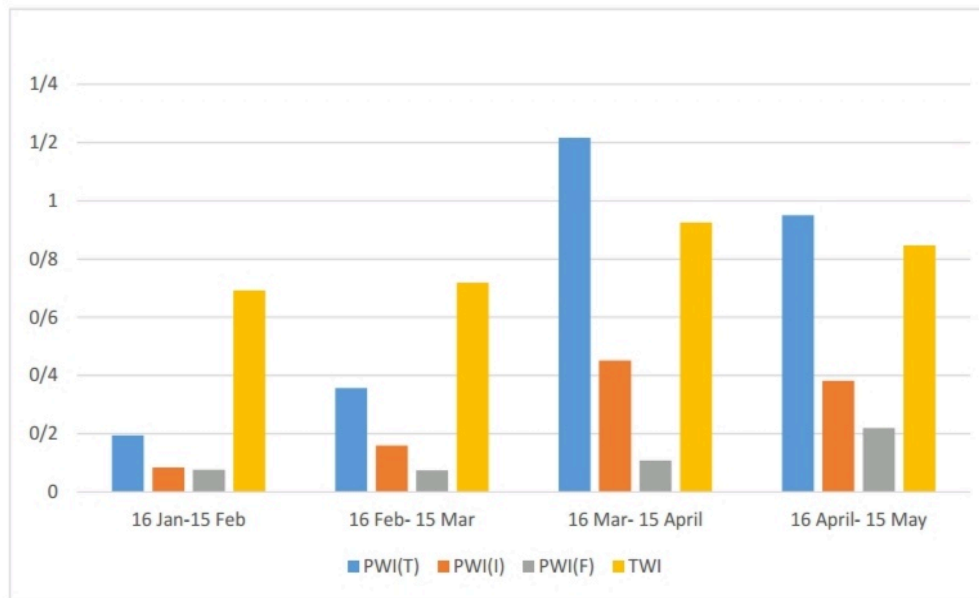


Figure 3.10: Wiener index comparison chart in G_1 , G_2 , G_3 , and G_4

in Figure 3.10, they can be easily studied using the Wiener index and assigning a logical value to each of the neutrosophic graphs.

3.6 Summary

In this chapter, we examine the Wiener index in neutrosophic graphs. First, this index was defined for this group of graphs and then it was calculated for certain modes of neutrosophic graphs. In the following, we provide an example of the application of this index in real problems. As you can see here, this index, which is one of the most important topological indices based on distance, can be a good criterion for comparing neutrosophic graphs under the same conditions. This index can also be studied and used for bipolar and interval valued neutrosophic graphs.

Chapter 4

Sombor index in Neutrosophic Graph

4.1 Introduction

In this chapter, we first define the types Sombor indices on neutrosophic graphs. The Sombor index is a degree-based index introduced by Gutman. This index is defined as

$$SO(G) = \sum_{e_{ij} \in E(G)} \sqrt{\deg(v_i)^2 + \deg(v_j)^2}.$$

The Sombor index for neutrosophic graphs is defined as the following triad.

The Sombor index is a degree-based index introduced by Gutman in 2020 [44]. After the introduction of this index, many researchers tried to introduce and use this index and in a short time, several articles were presented about this index. In the final references section, some of the articles published in this field are included for further study by those interested.

4.2 Partial and totally Sombor Index

Defintion 4.2.1. *let $G = (N, M)$ be a neutrosophic graph. the Sombor index (SO) of the graph G is defined by,*

$$SO_T(G) = \frac{1}{2} \sum_{uv \in E(G)} \left(T_N^2(u)d_{2_T}(u) + T_N^2(v)d_{2_T}(v) \right)^{\frac{1}{2}},$$

$$SO_I(G) = \frac{1}{2} \sum_{uv \in E(G)} \left(I_N^2(u)d_{2_I}(u) + I_N^2(v)d_{2_I}(v) \right)^{\frac{1}{2}},$$

$$SO_F(G) = \frac{1}{2} \sum_{uv \in E(G)} \left(F_N^2(u)d_{2_F}(u) + F_N^2(v)d_{2_F}(v) \right)^{\frac{1}{2}},$$

$$SO(G) = (SO_T(G), SO_I(G), SO_F(G)), \quad |SO(G)| = \frac{4 + 2SO_T(G) - 2SO_F(G) - SO_I(G)}{6}.$$

Too, we have:

$$d_2(u) = (d_{2_T}, d_{2_I}, d_{2_F}) = \left(\sum_{\substack{u,v \in V \\ u \neq v}} T_M^2(u, v), \sum_{\substack{u,v \in V \\ u \neq v}} I_M^2(u, v), \sum_{\substack{u,v \in V \\ u \neq v}} F_M^2(u, v) \right).$$

Defintion 4.2.2. Sombor index whit the totally degree of the graph G is defined as:

$$\begin{aligned} SO_T(G_T) &= \frac{1}{2} \sum_{uv \in E(G)} \left(T_N^2(u) T d_{2_T}(u) + T_N^2(v) T d_{2_T}(v) \right)^{\frac{1}{2}}, \\ SO_I(G_T) &= \frac{1}{2} \sum_{uv \in E(G)} \left(I_N^2(u) T d_{2_I}(u) + I_N^2(v) T d_{2_I}(v) \right)^{\frac{1}{2}}, \\ SO_F(G_T) &= \frac{1}{2} \sum_{uv \in E(G)} \left(F_N^2(u) T d_{2_F}(u) + F_N^2(v) T d_{2_F}(v) \right)^{\frac{1}{2}}, \\ SO(G_T) &= (SO_T(G_T), SO_I(G_T), SO_F(G_T)), \\ |SO(G_T)| &= \frac{4 + 2SO_T(G_T) - 2SO_F(G_T) - SO_I(G_T)}{6}. \end{aligned}$$

Too, we have:

$$\begin{aligned} Td_2(u) &= (Td_{2_T}, Td_{2_I}, Td_{2_F}) \\ &= \left(\sum_{\substack{u,v \in V \\ u \neq v}} T_M^2(u, v) + T_N^2(u), \sum_{\substack{u,v \in V \\ u \neq v}} I_M^2(u, v) + I_N^2(u), \sum_{\substack{u,v \in V \\ u \neq v}} F_M^2(u, v) + F_N^2(u) \right). \end{aligned}$$

Defintion 4.2.3. The Reduced Sombor index of the graph G is defined as:

$$\begin{aligned} SO_{RedT}(G) &= \frac{1}{2} \sum_{uv \in E(G)} \left(T_N^2(u) T d_{2_{RedT}}(u) + T_N^2(v) T d_{2_{RedT}}(v) \right)^{\frac{1}{2}}, \\ SO_{RedI}(G) &= \frac{1}{2} \sum_{uv \in E(G)} \left(I_N^2(u) T d_{2_{RedI}}(u) + I_N^2(v) T d_{2_{RedI}}(v) \right)^{\frac{1}{2}}, \\ SO_{RedF}(G) &= \frac{1}{2} \sum_{uv \in E(G)} \left(F_N^2(u) T d_{2_{RedF}}(u) + F_N^2(v) T d_{2_{RedF}}(v) \right)^{\frac{1}{2}}, \\ SO_{Red}(G) &= (SO_{RedT}(G), SO_{RedI}(G), SO_{RedF}(G)), \\ |SO_{Red}(G)| &= \frac{4 + 2SO_{RedT}(G) - 2SO_{RedF}(G) - SO_{RedI}(G)}{6}. \end{aligned}$$

Too, we have:

$$d_{2Red}(u) = \left(d_{2Red_T}(u), d_{2Red_I}(u), d_{2Red_F}(u) \right) \\ = \left(\left| \sum_{\substack{u,v \in V \\ u \neq v}} T_M^2(u,v) - T_N^2(u) \right|, \left| \sum_{\substack{u,v \in V \\ u \neq v}} I_M^2(u,v) - I_N^2(u) \right|, \left| \sum_{\substack{u,v \in V \\ u \neq v}} F_M^2(u,v) - F_N^2(u) \right| \right).$$

Defintion 4.2.4. The Average Sombor index of the graph G is defined as:

$$SO_{AvgT}(G) = \frac{1}{2} \sum_{uv \in E(G)} \left(T_N^2(u) T d_{2AvgT}(u) + T_N^2(v) T d_{2AvgT}(v) \right)^{\frac{1}{2}},$$

$$SO_{AvgI}(G) = \frac{1}{2} \sum_{uv \in E(G)} \left(I_N^2(u) T d_{2AvgI}(u) + I_N^2(v) T d_{2AvgI}(v) \right)^{\frac{1}{2}},$$

$$SO_{AvgF}(G) = \frac{1}{2} \sum_{uv \in E(G)} \left(F_N^2(u) T d_{2AvgF}(u) + F_N^2(v) T d_{2AvgF}(v) \right)^{\frac{1}{2}},$$

$$SO_{Avg}(G) = (SO_{AvgT}(G), SO_{AvgI}(G), SO_{AvgF}(G)),$$

$$|SO_{Avg}(G)| = \frac{4 + 2SO_{AvgT}(G) - 2SO_{AvgF}(G) - SO_{AvgI}(G)}{6}.$$

Too, we have:

$$d_{2Avg}(u) = \left(d_{2Avg_T}(u), d_{2Avg_I}(u), d_{2Avg_F}(u) \right) \\ = \left(\left| \sum_{\substack{u,v \in V \\ u \neq v}} T_M^2(u,v) - \frac{2 \sum_{uv \in E(G)} T_M(uv)}{\sum_{u \in V(G)} T_N(u)} \right|, \left| \sum_{\substack{u,v \in V \\ u \neq v}} I_M^2(u,v) - \frac{2 \sum_{uv \in E(G)} I_M(uv)}{\sum_{u \in V(G)} I_N(u)} \right|, \right. \\ \left. \left| \sum_{\substack{u,v \in V \\ u \neq v}} F_M^2(u,v) - \frac{2 \sum_{uv \in E(G)} F_M(uv)}{\sum_{u \in V(G)} F_N(u)} \right| \right).$$

Example 4.2.5. Let $G = (N, M)$ be the neutrosophic graph with $V = \{a, b, c, d, e\}$ where:

$(T_N, I_N, F_N)(u_1) = (0.1, 0.2, 0.5)$, $(T_N, I_N, F_N)(u_2) = (0.2, 0.7, 0.3)$, $(T_N, I_N, F_N)(u_3) = (0.4, 0.3, 0.7)$,
 $(T_N, I_N, F_N)(u_4) = (0.5, 0.2, 0.4)$, and $(T_N, I_N, F_N)(u_5) = (0.4, 0.5, 0.3)$, The edge set contains
 $(T_M, I_M, F_M)(u_1, u_2) = (0.1, 0.2, 0.5)$, $(T_M, I_M, F_M)(u_3, u_2) = (0.1, 0.2, 0.6)$, $(T_M, I_M, F_M)(u_3, u_4) =$
 $(0.2, 0.3, 0.8)$, and $(T_M, I_M, F_M)(u_5, u_2) = (0.2, 0.5, 0.3)$. (Figure 4.1)

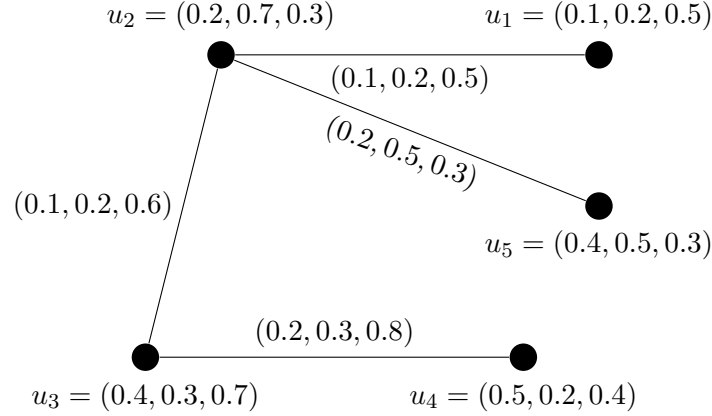


Figure 4.1: G is a neutrosophic graph with $V = \{a, b, c, d, e\}$

By direct calculations,

$$d(u_1) = (0.1, 0.2, 0.5),$$

$$d(u_2) = (0.1 + 0.2 + 0.1, 0.2 + 0.2 + 0.5, 0.5 + 0.3 + 0.6) = (0.4, 0.9, 1.4),$$

$$d(u_3) = (0.1 + 0.2, 0.2 + 0.3, 0.6 + 0.8) = (0.3, 0.5, 1.4),$$

$$d(u_4) = (0.2, 0.3, 0.8),$$

$$d(u_5) = (0.2, 0.5, 0.3),$$

By definition of $d_m(v)$, we have

$$d_2(v) = \left(\sum_{\substack{u,v \in V \\ u \neq v}} T_M^2(u, v), \sum_{\substack{u,v \in V \\ u \neq v}} I_M^2(u, v), \sum_{\substack{u,v \in V \\ u \neq v}} F_M^2(u, v) \right)$$

Therefore

$$d_2(u_1) = (0.01, 0.04, 0.25),$$

$$d_2(u_2) = (0.06, 0.33, 0.7),$$

$$d_2(u_3) = (0.05, 0.13, 1.0),$$

$$d_2(u_4) = (0.04, 0.09, 0.64),$$

$$d_2(u_5) = (0.04, 0.25, 0.09),$$

Also, for $u_i u_j \in E(G)$:

$$SO_T(G) = \frac{1}{2} \sum_{uv \in E(G)} \left(T_N^2(u) d_{2T}(u) + T_N^2(v) d_{2T}(v) \right)^{\frac{1}{2}} = 0.148,$$

$$SO_I(G) = \frac{1}{2} \sum_{uv \in E(G)} \left(I_N^2(u)d_{2_I}(u) + I_N^2(v)d_{2_I}(v) \right)^{\frac{1}{2}} = 0.771,$$

$$SO_F(G) = \frac{1}{2} \sum_{uv \in E(G)} \left(F_N^2(u)d_{2_F}(u) + F_N^2(v)d_{2_F}(v) \right)^{\frac{1}{2}} = 0.987,$$

$$SO(G) = (SO_T(G), SO_I(G), SO_F(G)) = (0.148, 0.771, 0.987),$$

$$\begin{aligned} |SO(G)| &= \frac{4 + 2SO_T(G) - 2SO_F(G) - SO_I(G)}{6} \\ &= \frac{4 + 2(0.148) - 2(0.987) - (0.771)}{6} = 0.2585. \end{aligned}$$

4.3 Examples of the Sombor Index in neutrosophic special graphs

In this section, to gain a deeper understanding of the Sombor index in neutrosophic graphs, we examine two specific examples under particular conditions. The first example explores various types of Sombor indices in a neutrosophic graph with a cycle as the underlying graph, while the second example investigates different types of Sombor indices in a neutrosophic star graph.

4.3.1 Somor Index in neutrosophic Graph with G^* Cycle

Example 4.3.1. Suppose that $G = (N, M)$ is a neutrosophic graph whose underlying graph is a cycle, as illustrated in figure 4.2.

By direct calculations,

$$d(a) = (0.2 + 0.6, 0.3 + 0.2, 0.3 + 0.3) = (0.8, 0.5, 0.6),$$

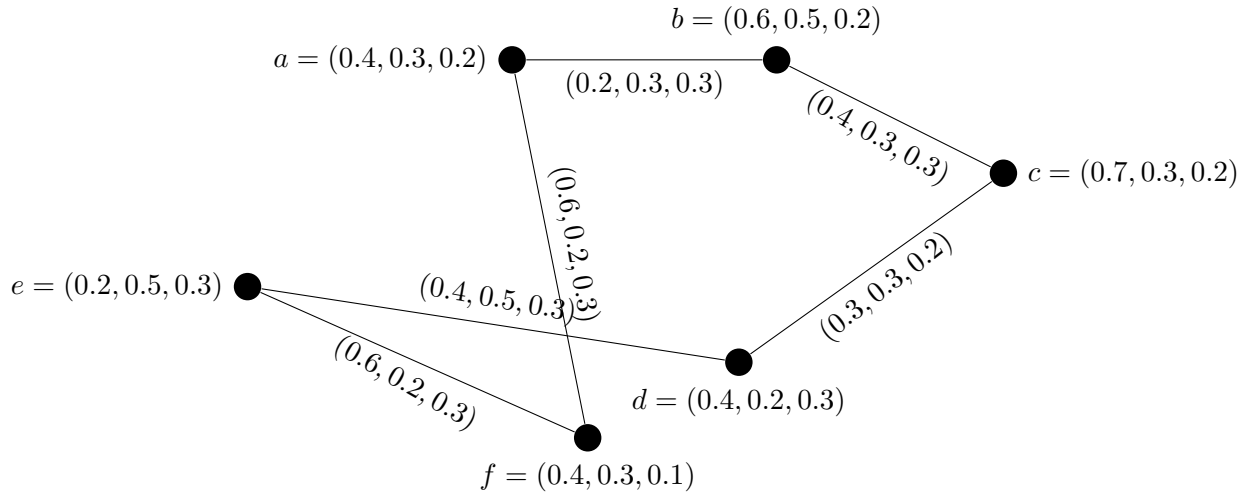
$$d(b) = (0.4 + 0.2, 0.3 + 0.3, 0.3 + 0.3) = (0.6, 0.6, 0.6),$$

$$d(c) = (0.4 + 0.3, 0.3 + 0.3, 0.3 + 0.2) = (0.7, 0.6, 0.5)$$

$$d(d) = (0.3 + 0.4, 0.3 + 0.5, 0.2 + 0.3) = (0.7, 0.8, 0.5),$$

$$d(e) = (0.4 + 0.6, 0.5 + 0.2, 0.3 + 0.3) = (1.0, 0.7, 0.6),$$

$$d(f) = (0.6 + 0.6, 0.2 + 0.2, 0.3 + 0.3) = (1.2, 0.4, 0.6),$$

Figure 4.2: A neutrosophic graph G

Then:

$$d_2(a) = (0.4, 0.19, 0.18),$$

$$d_2(b) = (0.2, 0.18, 0.18),$$

$$d_2(c) = (0.27, 0.18, 0.13),$$

$$d_2(d) = (0.27, 0.34, 0.13),$$

$$d_2(e) = (0.52, 0.29, 0.18),$$

$$d_2(f) = (0.72, 0.04, 0.18),$$

Also, for $uv \in E(G)$:

$$\begin{aligned} SO_T(G) &= \frac{1}{2} \sum_{uv \in E(G)} \left(T_N^2(u) d_{2T}(u) + T_N^2(v) d_{2T}(v) \right)^{\frac{1}{2}} \\ &= \frac{1}{2} \left[\left((0.4)^2(0.4) + (0.6)^2(0.2) \right) + \left((0.6)^2(0.2) + (0.7)^2(0.27) \right) \right. \\ &\quad + \left((0.7)^2(0.27) + (0.4)^2(0.27) \right) + \left((0.4)^2(0.27) + (0.2)^2(0.52) \right) \\ &\quad \left. + \left((0.2)^2(0.52) + (0.4)^2(0.72) \right) + \left((0.4)^2(0.72) + (0.4)^2(0.4) \right) \right] = 0.4475, \end{aligned}$$

$$\begin{aligned}
SO_I(G) &= \frac{1}{2} \sum_{uv \in E(G)} \left(I_N^2(u)d_{2_I}(u) + I_N^2(v)d_{2_I}(v) \right)^{\frac{1}{2}} \\
&= \frac{1}{2} \left[\left((0.3)^2(0.19) + (0.5)^2(0.18) \right) + \left((0.5)^2(0.18) + (0.3)^2(0.18) \right) \right. \\
&\quad + \left((0.3)^2(0.18) + (0.2)^2(0.34) \right) + \left((0.2)^2(0.34) + (0.5)^2(0.29) \right) \\
&\quad \left. + \left((0.5)^2(0.29) + (0.3)^2(0.04) \right) + \left((0.3)^2(0.04) + (0.3)^2(0.19) \right) \right] = 0.168,
\end{aligned}$$

$$\begin{aligned}
SO_F(G) &= \frac{1}{2} \sum_{uv \in E(G)} \left(F_N^2(u)d_{2_F}(u) + F_N^2(v)d_{2_F}(v) \right)^{\frac{1}{2}} \\
&= \frac{1}{2} \left[\left((0.2)^2(0.18) + (0.2)^2(0.18) \right) + \left((0.2)^2(0.18) + (0.2)^2(0.13) \right) \right. \\
&\quad + \left((0.2)^2(0.13) + (0.3)^2(0.13) \right) + \left((0.3)^2(0.13) + (0.3)^2(0.18) \right) \\
&\quad \left. + \left((0.3)^2(0.18) + (0.1)^2(0.18) \right) + \left((0.1)^2(0.18) + (0.2)^2(0.18) \right) \right] = 0.0493,
\end{aligned}$$

$$SO(G) = (SO_T(G), SO_I(G), SO_F(G)) = (0.4475, 0.168, 0.0493).$$

$$\begin{aligned}
|SO(G)| &= \frac{4 + 2SO_T(G) - 2SO_F(G) - SO_I(G)}{6} \\
&= \frac{4 + 2(0.4475) - 2(0.0493) - (0.168)}{6} = 0.7714.
\end{aligned}$$

$$\begin{aligned}
Td_2(u) &= (Td_{2_T}, Td_{2_I}, Td_{2_F}) \\
&= \left(\sum_{\substack{u,v \in V \\ u \neq v}} T_M^2(u,v) + T_N^2(u), \sum_{\substack{u,v \in V \\ u \neq v}} I_M^2(u,v) + I_N^2(u), \sum_{\substack{u,v \in V \\ u \neq v}} F_M^2(u,v) + F_N^2(u) \right),
\end{aligned}$$

Therefore,

$$Td_2(a) = (0.4 + 0.16, 0.19 + 0.09, 0.18 + 0.04) = (0.56, 0.28, 0.22),$$

$$Td_2(b) = (0.2 + 0.36, 0.18 + 0.25, 0.18 + 0.04) = (0.56, 0.43, 0.22),$$

$$Td_2(c) = (0.27 + 0.49, 0.18 + 0.09, 0.13 + 0.04) = (0.76, 0.27, 0.17),$$

$$Td_2(d) = (0.27 + 0.16, 0.34 + 0.04, 0.13 + 0.09) = (0.43, 0.38, 0.22),$$

$$Td_2(e) = (0.52 + 0.04, 0.29 + 0.25, 0.18 + 0.09) = (0.56, 0.54, 0.27),$$

$$Td_2(f) = (0.72 + 0.16, 0.04 + 0.09, 0.18 + 0.01) = (0.88, 0.13, 0.19).$$

$$\begin{aligned}
SO_T(G_T) &= \frac{1}{2} \sum_{uv \in E(G)} \left(T_N^2(u)Td_{2_T}(u) + T_N^2(v)Td_{2_T}(v) \right)^{\frac{1}{2}} \\
&= \frac{1}{2} \left[\left((0.4)^2(0.56) + (0.6)^2(0.56) \right) + \left((0.6)^2(0.56) + (0.7)^2(0.76) \right) \right. \\
&\quad + \left((0.7)^2(0.76) + (0.4)^2(0.43) \right) + \left((0.4)^2(0.43) + (0.2)^2(0.56) \right) \\
&\quad \left. + \left((0.2)^2(0.56) + (0.4)^2(0.88) \right) + \left((0.4)^2(0.88) + (0.4)^2(0.56) \right) \right] = 0.8956,
\end{aligned}$$

$$\begin{aligned}
SO_I(G_T) &= \frac{1}{2} \sum_{uv \in E(G)} \left(I_N^2(u)Td_{2_I}(u) + I_N^2(v)Td_{2_I}(v) \right)^{\frac{1}{2}} \\
&= \frac{1}{2} \left[\left((0.3)^2(0.28) + (0.5)^2(0.43) \right) + \left((0.5)^2(0.43) + (0.3)^2(0.27) \right) \right. \\
&\quad + \left((0.3)^2(0.27) + (0.2)^2(0.38) \right) + \left((0.2)^2(0.38) + (0.5)^2(0.54) \right) \\
&\quad \left. + \left((0.5)^2(0.54) + (0.3)^2(0.13) \right) + \left((0.3)^2(0.13) + (0.3)^2(0.28) \right) \right] = 0.3189,
\end{aligned}$$

$$\begin{aligned}
SO_F(G_T) &= \frac{1}{2} \sum_{uv \in E(G)} \left(F_N^2(u)Td_{2_F}(u) + F_N^2(v)Td_{2_F}(v) \right)^{\frac{1}{2}} \\
&= \frac{1}{2} \left[\left((0.2)^2(0.22) + (0.2)^2(0.22) \right) + \left((0.2)^2(0.22) + (0.2)^2(0.17) \right) \right. \\
&\quad + \left((0.2)^2(0.17) + (0.3)^2(0.22) \right) + \left((0.3)^2(0.22) + (0.3)^2(0.27) \right) \\
&\quad \left. + \left((0.3)^2(0.27) + (0.1)^2(0.19) \right) + \left((0.1)^2(0.19) + (0.2)^2(0.22) \right) \right] = 0.0704,
\end{aligned}$$

$$SO(G_T) = (SO_T(G_T), SO_I(G_T), SO_F(G_T))$$

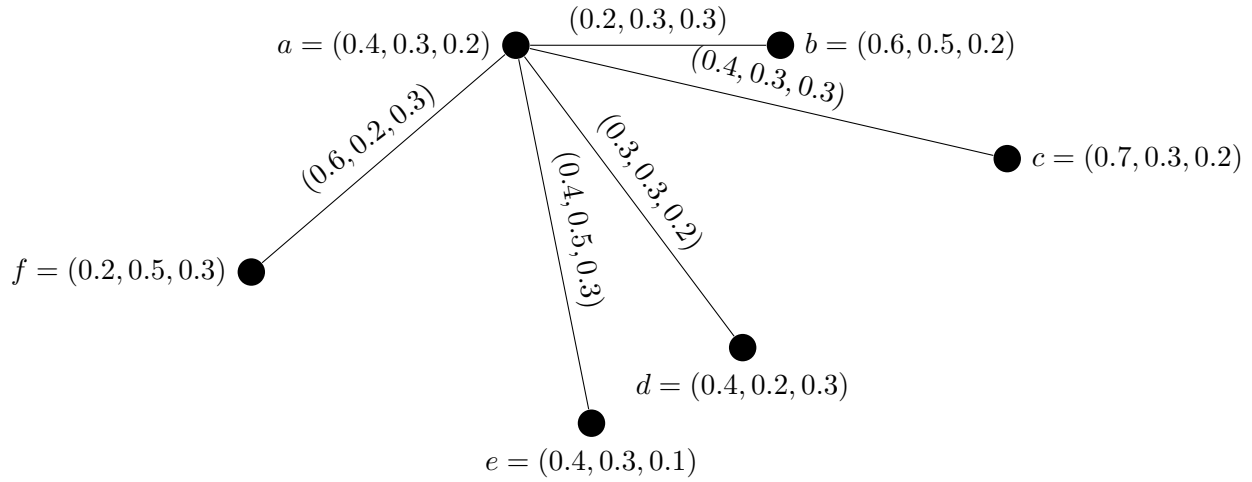
$$= (0.8956, 0.3189, 0.0704),$$

$$|SO(G_T)| = \frac{4 + 2SO_T(G_T) - 2SO_F(G_T) - SO_I(G_T)}{6}$$

$$= \frac{4 + 2(0.8956) - 2(0.0704) - (0.3189)}{6} = 0.8886.$$

4.3.2 Sombor Index in neutrosophic Star

Example 4.3.2. Let $G = (N, M)$ is a neutrosophic Star, as illustrated in figure 4.3. we have:

Figure 4.3: A neutrosophic graph G

By direct calculations,

$$\begin{aligned}
 d(a) &= (0.2 + 0.4 + 0.3 + 0.4 + 0.6, 0.3 + 0.3 + 0.3 + 0.5 + 0.2, 0.3 + 0.3 + 0.2 + 0.3 + 0.3) \\
 &= (1.9, 1.6, 1.4), \\
 d(b) &= (0.2, 0.3, 0.3), \\
 d(c) &= (0.4, 0.3, 0.3), \\
 d(d) &= (0.3, 0.3, 0.2), \\
 d(e) &= (0.4, 0.5, 0.3), \\
 d(f) &= (0.6, 0.2, 0.3),
 \end{aligned}$$

Then:

$$\begin{aligned}
 d_2(a) &= (0.81, 0.56, 0.4), \\
 d_2(b) &= (0.04, 0.09, 0.09), \\
 d_2(c) &= (0.16, 0.09, 0.09), \\
 d_2(d) &= (0.09, 0.09, 0.04), \\
 d_2(e) &= (0.16, 0.25, 0.09), \\
 d_2(f) &= (0.36, 0.04, 0.09),
 \end{aligned}$$

Also, for $uv \in E(G)$:

$$\begin{aligned} SO_T(G) &= \frac{1}{2} \sum_{uv \in E(G)} \left(T_N^2(u)d_{2_T}(u) + T_N^2(v)d_{2_T}(v) \right)^{\frac{1}{2}} \\ &= \left[\left((0.4)^2(0.81) + (0.6)^2(0.04) \right) + \left((0.4)^2(0.81) + (0.7)^2(0.16) \right) \right. \\ &\quad + \left((0.4)^2(0.81) + (0.4)^2(0.09) \right) + \left((0.4)^2(0.81) + (0.4)^2(0.16) \right) \\ &\quad \left. + \left((0.4)^2(0.81) + (0.2)^2(0.36) \right) \right] = \frac{0.7952}{2} = 0.3976, \end{aligned}$$

$$\begin{aligned} SO_I(G) &= \frac{1}{2} \sum_{uv \in E(G)} \left(I_N^2(u)d_{2_I}(u) + I_N^2(v)d_{2_I}(v) \right)^{\frac{1}{2}} \\ &= \left[\left((0.3)^2(0.56) + (0.5)^2(0.09) \right) + \left((0.3)^2(0.56) + (0.3)^2(0.09) \right) \right. \\ &\quad + \left((0.3)^2(0.56) + (0.2)^2(0.09) \right) + \left((0.3)^2(0.56) + (0.3)^2(0.25) \right) \\ &\quad \left. + \left((0.3)^2(0.56) + (0.5)^2(0.04) \right) \right] = \frac{0.3187}{2} = 0.15935, \end{aligned}$$

$$\begin{aligned} SO_F(G) &= \frac{1}{2} \sum_{uv \in E(G)} \left(F_N^2(u)d_{2_F}(u) + F_N^2(v)d_{2_F}(v) \right)^{\frac{1}{2}} \\ &= \left[\left((0.2)^2(0.4) + (0.2)^2(0.09) \right) + \left((0.2)^2(0.4) + (0.2)^2(0.09) \right) \right. \\ &\quad + \left((0.2)^2(0.4) + (0.3)^2(0.04) \right) + \left((0.2)^2(0.4) + (0.1)^2(0.09) \right) \\ &\quad \left. + \left((0.2)^2(0.4) + (0.2)^2(0.09) \right) \right] = \frac{0.0953}{2} = 0.04765, \end{aligned}$$

$$SO(G) = (SO_T(G), SO_I(G), SO_F(G)) = (0.3976, 0.15935, 0.04765),$$

$$\begin{aligned} |SO(G)| &= \frac{4 + 2SO_T(G) - 2SO_F(G) - SO_I(G)}{6} \\ &= \frac{4 + 2(0.3976) - 2(0.04765) - (0.15935)}{6} = 0.75676. \end{aligned}$$

$$\begin{aligned} d_{2Red}(u) &= \left(d_{2Red_T}(u), d_{2Red_I}(u), d_{2Red_F}(u) \right) \\ &= \left(\left| \sum_{\substack{u,v \in V \\ u \neq v}} T_M^2(u,v) - T_N^2(u) \right|, \left| \sum_{\substack{u,v \in V \\ u \neq v}} I_M^2(u,v) - I_N^2(u) \right|, \left| \sum_{\substack{u,v \in V \\ u \neq v}} F_M^2(u,v) - F_N^2(u) \right| \right). \end{aligned}$$

Then:

$$d_{2Red}(a) = (|0.81 - 0.16|, |0.56 - 0.09|, |0.4 - 0.04|) = (0.65, 0.47, 0.36),$$

$$d_{2Red}(b) = (|0.04 - 0.36|, |0.09 - 0.25|, |0.09 - 0.04|) = (0.32, 0.16, 0.05),$$

$$d_{2Red}(c) = (|0.16 - 0.49|, |0.09 - 0.09|, |0.09 - 0.04|) = (0.33, 0.0, 0.05),$$

$$d_{2Red}(d) = (|0.09 - 0.16|, |0.09 - 0.04|, |0.04 - 0.09|) = (0.07, 0.05, 0.05),$$

$$d_{2Red}(e) = (|0.16 - 0.16|, |0.25 - 0.09|, |0.09 - 0.01|) = (0.0, 0.16, 0.08),$$

$$d_{2Red}(f) = (|0.36 - 0.04|, |0.04 - 0.25|, |0.09 - 0.09|) = (0.32, 0.21, 0.0),$$

$$\begin{aligned} SO_{RedT}(G) &= \frac{1}{2} \sum_{uv \in E(G)} \left(T_N^2(u) T d_{2Red_T}(u) + T_N^2(v) T d_{2Red_T}(v) \right)^{\frac{1}{2}} \\ &= \left[\left((0.4)^2(0.65) + (0.6)^2(0.32) \right) + \left((0.4)^2(0.65) + (0.7)^2(0.33) \right) \right. \\ &\quad + \left((0.4)^2(0.65) + (0.4)^2(0.07) \right) + \left((0.4)^2(0.65) + (0.4)^2(0.0) \right) \\ &\quad \left. + \left((0.4)^2(0.65) + (0.2)^2(0.32) \right) \right] = \frac{0.8209}{2} = 0.41045, \end{aligned}$$

$$\begin{aligned} SO_{RedI}(G) &= \frac{1}{2} \sum_{uv \in E(G)} \left(I_N^2(u) T d_{2Red_I}(u) + I_N^2(v) T d_{2Red_I}(v) \right)^{\frac{1}{2}} \\ &= \left[\left((0.3)^2(0.47) + (0.5)^2(0.16) \right) + \left((0.3)^2(0.47) + (0.3)^2(0.00) \right) \right. \\ &\quad + \left((0.3)^2(0.47) + (0.2)^2(0.05) \right) + \left((0.3)^2(0.47) + (0.3)^2(0.16) \right) \\ &\quad \left. + \left((0.3)^2(0.47) + (0.5)^2(0.2) \right) \right] = \frac{0.3179}{2} = 0.15895, \end{aligned}$$

$$\begin{aligned} SO_{RedF}(G) &= \frac{1}{2} \sum_{uv \in E(G)} \left(F_N^2(u) T d_{2Red_F}(u) + F_N^2(v) T d_{2Red_F}(v) \right)^{\frac{1}{2}} \\ &= \left[\left((0.2)^2(0.36) + (0.2)^2(0.05) \right) + \left((0.2)^2(0.36) + (0.2)^2(0.05) \right) \right. \\ &\quad + \left((0.2)^2(0.36) + (0.3)^2(0.05) \right) + \left((0.2)^2(0.36) + (0.1)^2(0.08) \right) \\ &\quad \left. + \left((0.2)^2(0.36) + (0.3)^2(0.0) \right) \right] = \frac{0.0813}{2} = 0.04065, \end{aligned}$$

$$\begin{aligned}
SO_{Red}(G) &= (SO_{RedT}(G), SO_{RedI}(G), SO_{RedF}(G)) \\
&= (0.41045, 0.15895, 0.04065) \\
|SO_{Red}(G)| &= \frac{4 + 2SO_{RedT}(G) - 2SO_{RedF}(G) - SO_{RedI}(G)}{6} \\
&= \frac{4 + 2(0.41045) - 2(0.04065) - (0.15895)}{6} \\
&= 0.763441667.
\end{aligned}$$

4.4 Theories of Sombor index in neutrosophic graphs

Theorem 4.4.1. *let G be the Neutrosophic Graph, and H is the Neutrosophic subgraph of G such that $H = G - u$ then*

$$SO(H) < SO(G).$$

Proof. Given that by omitting a vertex of G , a positive value, the sum is lost, so the proof is obvious. \square

Theorem 4.4.2. *Let $G_1 = (N_1, M_1)$ be isomorphic with $G_2 = (N_2, M_2)$. Then all of the following equations are established.*

$$SO_T(G_1) = SO_T(G_2), \quad SO_I(G_1) = SO_I(G_2), \quad SO_F(G_1) = SO_F(G_2)$$

Also, we have $SO(G_1) = SO(G_2)$.

Proof. Let $G_1 = (N_1, M_1)$ be isomorphic with $G_2 = (N_2, M_2)$, and $f : V_1 \rightarrow V_2$ be the bijection from V_1 to V_2 such that

$$T_{N_1}(u) = T_{N_2}(f(u)), \quad I_{N_1}(u) = I_{N_2}(f(u)), \quad F_{N_1}(u) = F_{N_2}(f(u))$$

For all $u \in V_1$, and

$$T_{M_1}(uv) = T_{M_2}(f(u)f(v)), \quad I_{M_1}(uv) = I_{M_2}(f(u)f(v)), \quad F_{M_1}(uv) = F_{M_2}(f(u)f(v))$$

For all $uv \in E_1$. Since G_1 isomorphic with G_2 , any edge between two vertices, for example, u and v in G_1 are equal to that between $f(u)$ and $f(v)$ in G_2 . Hence

$$d_{2T_{G_1}}(u) = d_{2T_{G_2}}(f(u)), \quad d_{2I_{G_1}}(u) = d_{2I_{G_2}}(f(u)), \quad d_{2F_{G_1}}(u) = d_{2F_{G_2}}(f(u)),$$

for $u, v \in N_1^*$. Therefore

$$SO_T(G_1) = SO_T(G_2), \quad SO_I(G_1) = SO_I(G_2), \quad SO_F(G_1) = SO_F(G_2),$$

and

$$\begin{aligned} |SO(G_1)| &= \frac{4 + 2SO_T(G_1) - 2SO_F(G_1) - SO_I(G_1)}{6} \\ &= \frac{4 + 2SO_T(G_2) - 2SO_F(G_2) - SO_I(G_2)}{6} = |SO(G_2)|. \end{aligned}$$

□

Theorem 4.4.3. Let $G = (N, M)$ be a complete neutrosophic graph with $V = \{v_1, v_2, \dots, v_n\}$ such that $t_1 \leq t_2 \leq \dots \leq t_n$, $i_1 \leq i_2 \leq \dots \leq i_n$ and $f_1 \geq f_2 \geq \dots \geq f_n$ where $t_j = T_N(v_j)$, $i_j = I_N(v_j)$, $f_j = F_N(v_j)$ for $j = 1, 2, \dots, n$. Also, G is the regular neutrosophic graph of the second rank (that is, for each vertex u in $V(G)$, $d_{2G}(v_j) = (d_1, d_2, d_3)$). Then

$$\begin{aligned} SO_T(G) &= \frac{\sqrt{d_1}}{2} \sum_{\substack{1 \leq j < k \\ 2 \leq k \leq n}} (t_j^2 + t_k^2)^{\frac{1}{2}}, \\ SO_I(G) &= \frac{\sqrt{d_2}}{2} \sum_{\substack{1 \leq j < k \\ 2 \leq k \leq n}} (i_j^2 + i_k^2)^{\frac{1}{2}}, \\ SO_F(G) &= \frac{\sqrt{d_3}}{2} \sum_{\substack{1 \leq j < k \\ 2 \leq k \leq n}} (f_j^2 + f_k^2)^{\frac{1}{2}}. \end{aligned}$$

Too, we have for u in $V(G)$:

$$d_2(v_j) = (d_{2T}(v_j), d_{2I}(v_j), d_{2F}(v_j)).$$

Proof. Suppose G is a complete neutrosophic graph, we have

$$\begin{aligned} SO_T(G) &= \frac{1}{2} \sum_{uv \in E(G)} \left(T_N^2(u) d_{2T}(u) + T_N^2(v) d_{2T}(v) \right)^{\frac{1}{2}} = \frac{1}{2} \sum_{uv \in E(G)} \left(T_N^2(u) d_1 + T_N^2(v) d_1 \right)^{\frac{1}{2}} \\ &= \frac{1}{2} \sum_{uv \in E(G)} \sqrt{d_1} \left(T_N^2(u) + T_N^2(v) \right)^{\frac{1}{2}} = \frac{\sqrt{d_1}}{2} \sum_{uv \in E(G)} \left(T_N^2(u) + T_N^2(v) \right)^{\frac{1}{2}} \\ &= \frac{\sqrt{d_1}}{2} \left(\sqrt{t_1^2 + t_2^2} + \sqrt{t_1^2 + t_3^2} + \dots + \sqrt{t_1^2 + t_n^2} + \sqrt{t_2^2 + t_3^2} + \dots + \sqrt{t_2^2 + t_n^2} + \dots \right. \\ &\quad \left. + \sqrt{t_{n-2}^2 + t_{n-1}^2} + \sqrt{t_{n-2}^2 + t_n^2} + \sqrt{t_{n-1}^2 + t_n^2} \right) = \frac{\sqrt{d_1}}{2} \sum_{\substack{1 \leq j < k \\ 2 \leq k \leq n}} (t_j^2 + t_k^2)^{\frac{1}{2}}. \end{aligned}$$

Using the same argument, we can prove the other two cases. □

Theorem 4.4.4. *Let $G = (N, M)$ be a complete neutrosophic graph with $|V| = n$. Then,*

If M is a constant function, that is, $M(uv) = (t_m, i_m, f_m)$, for $uv \in E(G)$. Now, we have:

$$SO_T(G) = \frac{\sqrt{n-1}}{2} t_m \sum_{uv \in E(G)} \left(T_N^2(u) + T_N^2(v) \right)^{\frac{1}{2}},$$

$$SO_I(G) = \frac{\sqrt{n-1}}{2} i_m \sum_{uv \in E(G)} \left(I_N^2(u) + I_N^2(v) \right)^{\frac{1}{2}},$$

$$SO_F(G) = \frac{\sqrt{n-1}}{2} f_m \sum_{uv \in E(G)} \left(F_N^2(u) + F_N^2(v) \right)^{\frac{1}{2}}.$$

If M and N are the constant function, that is, $M(uv) = (t_m, i_m, f_m)$, for $uv \in E(G)$, and $N(u) = (t_n, i_n, f_n)$, for $u \in V(G)$. Now, we have:

$$SO_T(G) = \frac{n(n-1)^{\frac{3}{2}}}{2\sqrt{2}} t_m \cdot t_n,$$

$$SO_I(G) = \frac{n(n-1)^{\frac{3}{2}}}{2\sqrt{2}} i_m \cdot i_n,$$

$$SO_F(G) = \frac{n(n-1)^{\frac{3}{2}}}{2\sqrt{2}} f_m \cdot f_n.$$

Also,

$$\left| SO(G) \right| = \frac{n(n-1)^{\frac{3}{2}}}{12\sqrt{2}} (2t_m t_n - i_m i_n - 2f_m f_n) + \frac{2}{3}.$$

If M and N are the constant and the same function, that is, $M(uv) = (t, i, f)$, for $uv \in E(G)$, and $N(u) = (t, i, f)$, for $u \in V(G)$. Now, we have:

$$SO_T(G) = \frac{n(n-1)^{\frac{3}{2}}}{2\sqrt{2}} t^2,$$

$$SO_I(G) = \frac{n(n-1)^{\frac{3}{2}}}{2\sqrt{2}} i^2,$$

$$SO_F(G) = \frac{n(n-1)^{\frac{3}{2}}}{2\sqrt{2}} f^2.$$

Also,

$$\left| SO(G) \right| = \frac{n(n-1)^{\frac{3}{2}}}{12\sqrt{2}} (2t^2 - i^2 - 2f^2) + \frac{2}{3}.$$

Proof. In the first case, suppose G is a complete neutrosophic graph, with $|V| = n$. Also, $uv \in E(G)$, we have $M(uv) = (t_m, i_m, f_m)$. Therefore,

$$\begin{aligned} d_2(u) &= (d_{2T}(u), d_{2I}(u), d_{2F}(u)) = \left(\sum_{\substack{u,v \in V \\ u \neq v}} T_M^2(u, v), \sum_{\substack{u,v \in V \\ u \neq v}} I_M^2(u, v), \sum_{\substack{u,v \in V \\ u \neq v}} F_M^2(u, v), \right) \\ &= ((n-1)t_m^2, (n-1)i_m^2, (n-1)f_m^2). \end{aligned}$$

Now, by placing $d_2(u)$ in relation to the Sombor index we get:

$$\begin{aligned} SO_T(G) &= \frac{1}{2} \sum_{uv \in E(G)} \left(T_N^2(u)d_{2T}(u) + T_N^2(v)d_{2T}(v) \right)^{\frac{1}{2}} \\ &= \frac{1}{2} \sum_{uv \in E(G)} \left(T_N^2(u)(n-1)t_m^2 + T_N^2(v)(n-1)t_m^2 \right)^{\frac{1}{2}} \\ &= \frac{1}{2} \sum_{uv \in E(G)} \left((n-1)t_m^2 \right)^{\frac{1}{2}} \left(T_N^2(u) + T_N^2(v) \right)^{\frac{1}{2}} \\ &= \frac{1}{2} \left((n-1)t_m^2 \right)^{\frac{1}{2}} \sum_{uv \in E(G)} \left(T_N^2(u) + T_N^2(v) \right)^{\frac{1}{2}} \\ &= \frac{\sqrt{n-1}}{2} t_m \sum_{uv \in E(G)} \left(T_N^2(u) + T_N^2(v) \right)^{\frac{1}{2}}. \end{aligned}$$

In the second case, consider M and N are the constant functions, such that, $M(uv) = (t_m, i_m, f_m)$, and $N(u) = (t_n, i_n, f_n)$. In this state, we have for $u \in V(G)$,

$$\begin{aligned} d_2(u) &= (d_{2T}(u), d_{2I}(u), d_{2F}(u)) = \left(\sum_{\substack{u,v \in V \\ u \neq v}} T_M^2(u, v), \sum_{\substack{u,v \in V \\ u \neq v}} I_M^2(u, v), \sum_{\substack{u,v \in V \\ u \neq v}} F_M^2(u, v), \right) \\ &= ((n-1)t_m^2, (n-1)i_m^2, (n-1)f_m^2). \end{aligned}$$

by placing $d_2(u)$,

$$\begin{aligned} SO_T(G) &= \frac{1}{2} \sum_{uv \in E(G)} \left(T_N^2(u)d_{2T}(u) + T_N^2(v)d_{2T}(v) \right)^{\frac{1}{2}} \\ &= \frac{1}{2} \sum_{uv \in E(G)} \left(T_N^2(u)(n-1)t_m^2 + T_N^2(v)(n-1)t_m^2 \right)^{\frac{1}{2}} \\ &= \frac{1}{2} \sum_{uv \in E(G)} \left(t_n^2(n-1)t_m^2 + t_n^2(n-1)t_m^2 \right)^{\frac{1}{2}} = \frac{n(n-1)^{\frac{3}{2}}}{2\sqrt{2}} t_m t_n. \end{aligned}$$

The same can be proved for $SO_I(G)$ and $SO_F(G)$. Also, by direct placing in $|SO(G)|$,

$$\begin{aligned} |SO(G)| &= \frac{4 + 2SO_T(G) - 2SO_F(G) - SO_I(G)}{6} \\ &= \frac{4 + 2\frac{n(n-1)^{\frac{3}{2}}}{2\sqrt{2}}t_m t_n - 2\frac{n(n-1)^{\frac{3}{2}}}{2\sqrt{2}}f_m f_n - \frac{n(n-1)^{\frac{3}{2}}}{2\sqrt{2}}i_m i_n}{6} \\ &= \frac{n(n-1)^{\frac{3}{2}}}{12\sqrt{2}}(2t_m t_n - i_m i_n - 2f_m f_n) + \frac{2}{3}. \end{aligned}$$

To prove the third case, it is enough to consider $d_2(u)$ like the previous parts and then replace the value of $M(uv) = (t, i, f)$, and $N(u) = (t, i, f)$. The proofs for $SO_I(G)$ and $SO_F(G)$ are similar to the case for $SO_T(G)$. \square

Theorem 4.4.5. . Let $G = (N, M)$ be a complete neutrosophic graph with $|V| = n$. Then, If M and N are the constant functions, that is, $M(uv) = (t_m, i_m, f_m)$, for $uv \in E(G)$, and $N(u) = (t_n, i_n, f_n)$, for $u \in V(G)$. Now, we have

$$\begin{aligned} SO_T(G_T) &= \frac{n(n-1)}{2\sqrt{2}}t_n \left((n-1)t_m^2 + t_n^2 \right)^{\frac{1}{2}}, \\ SO_I(G_T) &= \frac{n(n-1)}{2\sqrt{2}}i_n \left((n-1)i_m^2 + i_n^2 \right)^{\frac{1}{2}}, \\ SO_F(G_T) &= \frac{n(n-1)}{2\sqrt{2}}f_n \left((n-1)f_m^2 + f_n^2 \right)^{\frac{1}{2}}. \end{aligned}$$

Also,

$$|SO(G_T)| = \frac{n(n-1)}{12\sqrt{2}} \left(2t_n \left((n-1)t_m^2 + t_n^2 \right)^{\frac{1}{2}} - i_n \left((n-1)i_m^2 + i_n^2 \right)^{\frac{1}{2}} - 2f_n \left((n-1)f_m^2 + f_n^2 \right)^{\frac{1}{2}} \right) + \frac{2}{3}.$$

If M and N are the constant and the same functions, that is, $M(uv) = (t, i, f)$, for $uv \in E(G)$, and $N(u) = (t, i, f)$, for $u \in V(G)$. Now, we have:

$$\begin{aligned} SO_T(G_T) &= \frac{(n-1)(n)^{\frac{3}{2}}}{2\sqrt{2}}t^2, \\ SO_I(G_T) &= \frac{(n-1)(n)^{\frac{3}{2}}}{2\sqrt{2}}i^2, \\ SO_F(G_T) &= \frac{(n-1)(n)^{\frac{3}{2}}}{2\sqrt{2}}f^2. \end{aligned}$$

Also,

$$|SO(G_T)| = \frac{(n-1)(n)^{\frac{3}{2}}}{12\sqrt{2}}(2t^2 - i^2 - 2f^2) + \frac{2}{3}.$$

Proof. Suppose G is a complete neutrosophic graph, with $|V| = n$. Also, $uv \in E(G)$. The first case, we have $M(uv) = (t_m, i_m, f_m)$, for $uv \in E(G)$, and $N(u) = (t_n, i_n, f_n)$, for $u \in V(G)$. Therefore,

$$\begin{aligned} Td_2(u) &= (Td_{2T}(u), Td_{2I}(u), Td_{2F}(u)) \\ &= \left(\sum_{\substack{u,v \in V \\ u \neq v}} T_M^2(u, v) + T_N^2(u), \sum_{\substack{u,v \in V \\ u \neq v}} I_M^2(u, v) + I_N^2(u), \sum_{\substack{u,v \in V \\ u \neq v}} F_M^2(u, v) + F_N^2(u), \right) \\ &= \left((n-1)t_m^2 + t_n^2, (n-1)i_m^2 + i_n^2, (n-1)f_m^2 + f_n^2 \right). \end{aligned}$$

Now, by placing $Td_2(u)$ in relation to the Sombor index we get:

$$\begin{aligned} SO_T(G_T) &= \frac{1}{2} \sum_{uv \in E(G)} \left(T_N^2(u)Td_{2T}(u) + T_N^2(v)Td_{2T}(v) \right)^{\frac{1}{2}} \\ &= \frac{1}{2} \sum_{uv \in E(G)} \left(T_N^2(u)((n-1)t_m^2 + t_n^2) + T_N^2(v)((n-1)t_m^2 + t_n^2) \right)^{\frac{1}{2}} \\ &= \frac{1}{2} \sum_{uv \in E(G)} \left(((n-1)t_m^2 + t_n^2)(T_N^2(u) + T_N^2(v)) \right)^{\frac{1}{2}} \\ &= \frac{1}{2} \sum_{uv \in E(G)} \left(((n-1)t_m^2 + t_n^2)(t_n^2 + t_n^2) \right)^{\frac{1}{2}} = \frac{n(n-1)}{2\sqrt{2}} t_n \left((n-1)t_m^2 + t_n^2 \right)^{\frac{1}{2}}. \end{aligned}$$

It is similarly possible for the other two modes.

$$\begin{aligned} |SO(G)| &= \frac{4 + 2SO_T(G) - 2SO_F(G) - SO_I(G)}{6} \\ &= \frac{4 + 2\frac{n(n-1)}{2\sqrt{2}}t_n \left((n-1)t_m^2 + t_n^2 \right)^{\frac{1}{2}} - 2\frac{n(n-1)}{2\sqrt{2}}f_n \left((n-1)f_m^2 + f_n^2 \right)^{\frac{1}{2}}}{6} \\ &\quad \times \frac{-\frac{n(n-1)}{2\sqrt{2}}i_n \left((n-1)i_m^2 + i_n^2 \right)^{\frac{1}{2}}}{6} \\ &= \frac{4 + \frac{n(n-1)}{2\sqrt{2}} \left(2 \left((n-1)t_m^2 + t_n^2 \right)^{\frac{1}{2}} - 2 \left((n-1)f_m^2 + f_n^2 \right)^{\frac{1}{2}} - \left((n-1)i_m^2 + i_n^2 \right)^{\frac{1}{2}} \right)}{6} \\ &= \frac{n(n-1)}{12\sqrt{2}} \left(2t_n \left((n-1)t_m^2 + t_n^2 \right)^{\frac{1}{2}} - i_n \left((n-1)i_m^2 + i_n^2 \right)^{\frac{1}{2}} - 2f_n \left((n-1)f_m^2 + f_n^2 \right)^{\frac{1}{2}} \right) + \frac{2}{3}. \end{aligned}$$

For case 2: we have $M(uv) = (t, i, f)$, for $uv \in E(G)$, and $N(u) = (t, i, f)$, for $u \in V(G)$, hence:

$$\begin{aligned} Td_2(u) &= (Td_{2T}(u), Td_{2I}(u), Td_{2F}(u)) \\ &= \left(\sum_{\substack{u,v \in V \\ u \neq v}} T_M^2(u, v) + T_N^2(u), \sum_{\substack{u,v \in V \\ u \neq v}} I_M^2(u, v) + I_N^2(u), \sum_{\substack{u,v \in V \\ u \neq v}} F_M^2(u, v) + F_N^2(u), \right) \\ &= \left((n-1)t^2 + t^2, (n-1)i^2 + i^2, (n-1)f^2 + f^2 \right) = (nt^2, ni^2, nf^2). \end{aligned}$$

$$\begin{aligned}
SO_T(G_T) &= \frac{1}{2} \sum_{uv \in E(G)} \left(T_N^2(u)Td_{2_T}(u) + T_N^2(v)Td_{2_T}(v) \right)^{\frac{1}{2}} \\
&= \frac{1}{2} \sum_{uv \in E(G)} \left(T_N^2(u)(nt^2) + T_N^2(v)(nt^2) \right)^{\frac{1}{2}} \\
&= \frac{1}{2} \sum_{uv \in E(G)} \left((nt^2)(t^2 + t^2) \right)^{\frac{1}{2}} = \frac{1}{2}(nt^2)^{\frac{1}{2}} \sum_{uv \in E(G)} \left(t^2 + t^2 \right)^{\frac{1}{2}} \\
&= \frac{(n-1)(n)^{\frac{3}{2}}}{2\sqrt{2}} t^2.
\end{aligned}$$

It is similarly proved for the $SO_I(G_T)$ and $SO_F(G_T)$. Also,

$$\begin{aligned}
|SO(G)| &= \frac{4 + 2SO_T(G) - 2SO_F(G) - SO_I(G)}{6} \\
&= \frac{4 + 2\left(\frac{(n-1)(n)^{\frac{3}{2}}}{2\sqrt{2}} t^2\right) - 2\left(\frac{(n-1)(n)^{\frac{3}{2}}}{2\sqrt{2}} f^2\right) - \left(\frac{(n-1)(n)^{\frac{3}{2}}}{2\sqrt{2}} i^2\right)}{6} \\
&= \frac{4 + \left(\frac{(n-1)(n)^{\frac{3}{2}}}{2\sqrt{2}}\right) \left(2t^2 - 2f^2 - i^2\right)}{6} = \frac{(n-1)(n)^{\frac{3}{2}}}{12\sqrt{2}} (2t^2 - i^2 - 2f^2) + \frac{2}{3}.
\end{aligned}$$

The proof is complete. □

4.5 Summary

In this chapter, the Sombor index was examined in the context of neutrosophic graphs. First, different types of Sombor indices for neutrosophic graphs were defined, then their mathematical properties were analyzed, and theorems related to these indices were presented and proven.

The main goal of this book is to expand the theory of topological indices in the neutrosophical space and to provide a basis for future applications in data science, social networks, and decision-making systems under uncertainty. We have tried to study and review a number of the most important and widely used topological indices along with related theorems. We hope that this collection will be welcomed by those interested.

Chapter 5

Introduction to topological indices in Neutrosophic Super Hyper Graph and Neutrosophic Super Hyper Power Graphs

5.1 Introduction

A hypergraph generalizes a classical graph by introducing hyperedges, which can connect any number of vertices—not just two—making it suitable for modeling complex, multi-way relationships. A SuperHyperGraph takes this concept even further. Recently introduced and actively studied in a growing body of literature, the SuperHyperGraph incorporates recursive structures into hypergraphs through iterated applications of the power set operation. Conceptually, a SuperHyperGraph can be viewed as a hierarchical generalization of a hypergraph, in which both vertices and hyperedges are drawn from higher-order powersets of a base vertex set. The formal definition is presented below.

In this chapter, we first present some important definitions from various sources, and then, after defining superhypergraphs and Neutrosophic Super Hyper Power Graphs, we generalize topological indices and define them for this class of neutrosophic graphs for the first time.

Superhypergraphs have applications in computer science, data science, and chemistry. They can also be used in social network analysis with large and diverse data sets.

See the basic definitions below.

Defintion 5.1.1. *Let A be a set. We say that A has Helly property if and only if, For every non-empty set S such that $S \subseteq A$ and for all sets x, y such that $x, y \in S$ holds x meets y holds $\cap S \neq \emptyset$.*

Proposition 5.1.2. *Let T be a tree and X be a finite set such that for every set x such that $x \in X$ there exists a subtree t of T such that x is equal the vertices of t . Then X has Helly property.*

Defintion 5.1.3 (Powerset). *Let S be any set. The powerset of S , denoted $P(S)$, is the collection of all subsets of S :*

$$P(S) = \{A | A \subseteq S\}.$$

In particular, $\emptyset \in P(S)$ and $S \in P(S)$.

Defintion 5.1.4 (Hypergraph). *A hypergraph $H = (V(H), E(H))$ consists of:*

- *A nonempty set $V(H)$ of vertices.*
- *A set $E(H)$ of hyperedges, where each hyperedge is a nonempty subset of $V(H)$, thereby allowing connections among multiple vertices. Unlike standard graphs, hypergraphs are well-suited to represent higher-order relationships. In this paper, we restrict ourselves to the case where both $V(H)$ and $E(H)$ are finite.*

Defintion 5.1.5 (n-th Powerset). *The n -th powerset of a set H , denoted $P_n(H)$, is defined iteratively, starting with the standard powerset. The recursive construction is given by:*

$$P_1(H) = P(H), \quad P_{n+1}(H) = P(P_n(H)), \quad \text{for } n \geq 1.$$

Similarly, the n -th non-empty powerset, denoted $P_n^(H)$, is defined recursively as:*

$$P_1^*(H) = P^*(H), \quad P_{n+1}^*(H) = P^*(P_n^*(H)).$$

Here, $P^(H)$ represents the powerset of H with the empty set removed.*

Defintion 5.1.6 (n-SuperHyperGraph). *. Let V_0 be a finite base set of vertices. For each integer $k \geq 0$, define the iterative powerset by*

$$P_0(V_0) = V_0, \quad P_{k+1}(V_0) = P(P_k(V_0)),$$

where $P(\cdot)$ denotes the usual powerset operation. An n -SuperHyperGraph is then a pair

$$SHT(n) = (V, E),$$

with

$$V \subseteq P_n(V_0) \quad \text{and} \quad E \subseteq P_n(V_0).$$

Each element of V is called an n -supervertex and each element of E an n -superedge

Defintion 5.1.7 (Directed n -Superhypergraph). Let V_0 be a nonempty finite set and, for each integer $k \geq 0$, define

$$P^0(V_0) = V_0, \quad P^{k+1}(V_0) = P(P^k(V_0)).$$

A directed n -Superhypergraph is a pair

$$N^n = (V^n, E^n)$$

where

$$V^{(n)} \subseteq P^n(V_0), \quad E^{(n)} \subseteq V^{(n)} \times V^{(n)}.$$

Each element of $V^{(n)}$ is called an n -supervertex, and each ordered pair $(u, v) \in E^{(n)}$ is called an n -superedge

Defintion 5.1.8 (Fuzzy n -Superhypergraph). Let $SHT(n) = (V, E)$ be an n -Superhypergraph. A fuzzy n -Superhypergraph is a quadruple

$$(V, E, \sigma, \mu,)$$

where

- $\sigma : V \rightarrow [0, 1]$ assigns to each n -supervertex v a membership degree $\sigma(v)$.
- $\mu : E \rightarrow [0, 1]$ assigns to each n -superedge e a membership degree $\mu(e)$.

These functions satisfy the appurtenance constraint

$$\mu(e) \leq \min_{v \in e} \sigma(v), \forall e \in E$$

5.2 Neutrosophic Super Hyper Power Graphs

In this section, we provide a modified definition of Super Hyper Power Graphs (SHP-graph), and then generalize this definition to neutrosophic graphs.

Defintion 5.2.1 (Super Hyper Power Graph (SHP- Graph)). *A Super Hyper Power Graph (SHPG) is an ordered pair $SHPG = (X \subseteq (V) \setminus \emptyset, E \subseteq P(V) \times P(V))$, where*

1. $V = \{v_1, v_2, \dots, v_n\}$ is a finite set of $n \geq 0$ vertices, or an infinite set.
2. $P(V)$ is the power set of V (all subset of V). therefore, an SHPG-vertex may be a single (classical) vertex (V_{si}), or a super-vertex (V_{su}) (a subset of many vertices) that represents a group (organization), or even an indeterminate-vertex (V_I) (unclear, unknown vertex);
3. $E = \{e_1, e_2, \dots, e_m\}$, for $m \geq 1$, is a family of subsets of $V \times V$, and each e_i is an SHPG-edge, $e_i \in P(V) \times P(V)$. An SHPG-edge may be a (classical) edge, or a super-edge (edge between super vertices) that represents connections between two groups (organizations), or hyper-super-edge that represents connections between three or more groups (organizations), or even an indeterminate-edge (unclear, unknown edge); then we have:
 - (a) Single Edges (E_{Si}), as in classical graphs.
 - (b) Hyper Edges (E_H), edges connecting three or more single- vertices.
 - (c) Super Edges (E_{Su}), edges connecting only two SHG- vertices and at least one vertex is super Vertex.
 - (d) Hyper Super Edges (E_{HS}), edges connecting three or more single- vertices (and at least one vertex is super vertex).
 - (e) Indeterminate Edges (E_I), either we do not know their value, or we do not know what vertices they might connect.

Then, $G = (X, E)$ where $X = (V_{Si}, V_{Su}, V_I) \subseteq P(V) \setminus \emptyset$, and $E = (E_{Si}, E_H, E_{Su}, E_{HS}, E_I) \subseteq P(V) \times (V)$.

Defintion 5.2.2 (Neutrosophic Super Hyper Power Graphs). *Let $G = (X, E)$ be a super hyper power graph. If all vertices and edges of a SHP-graph G belong to the neutrosophic set, then the SHP-graph is a Neutrosophic Super Hyper Power Graphs (NSHP-Graphs). If x is a neutrosophic super vertex containing vertices $\{v_1, v_2, \dots, v_k\}$, where $v_i \in V$ for $1 \leq i \leq k$, then*

$$T_X(x) = \min\{T_X(v_i), 1 \leq i \leq k\},$$

$$I_X(x) = \min\{I_X(v_i), 1 \leq i \leq k\},$$

$$F_X(x) = \max\{F_X(v_i), 1 \leq i \leq k\}.$$

Defintion 5.2.3. *Let $G = (X, E)$ be a super hyper power graph, with $X = (V_{Si}, V_{Su}, V_I) \subseteq P(V) \setminus \emptyset$, and $E = (E_{Si}, E_H, E_{Su}, E_{HS}, E_I) \subseteq P(V) \times P(V)$. Then, the adjacency matrix $A(G) = (a_{ij})$ of G is defined as a square matrix which columns and rows its, is shown by the vertices of G and for each $v_i, v_j \in X$,*

$$a_{ij} = \begin{cases} 0 & \text{there should be no edge between vertices } v_i \text{ and } v_j; \\ 1 & \text{there is a single edge between vertices } v_i \text{ and } v_j; \\ S & \text{there is a super edge between vertices } v_i \text{ and } v_j; \\ H & \text{there is a hyper edge between vertices } v_i \text{ and } v_j; \\ SH & \text{there is a super hyper edge between vertices } v_i \text{ and } v_j. \end{cases}$$

Note that in the adjacency matrix A , a value of one can be placed instead of non-numeric values (S , H and SH) if necessary for calculations. So that, since A is a symmetric and values of A is positive, eigenvalues of A are real.

Defintion 5.2.4. *Let $G = (X, E)$ be a super hyper power graph, with $X = (V_{Si}, V_{Su}, V_I) \subseteq P(V) \setminus \emptyset$, and $E = (E_{Si}, E_H, E_{Su}, E_{HS}, E_I) \subseteq P(V) \times P(V)$. If $E = (e_1, e_2, \dots, e_m)$ then an incidence matrix $B(G) = (b_{ij})$ define as*

$$b_{ij} = \begin{cases} 1 & \text{if } v_i \in e_j; \\ 0 & \text{otherwise.} \end{cases}$$

Defintion 5.2.5. *Let $G = (X, E)$ be a super hyper power graph, with $X = (V_{Si}, V_{Su}, V_I) \subseteq P(V) \setminus \emptyset$, and $E = (E_{Si}, E_H, E_{Su}, E_{HS}, E_I) \subseteq P(V) \times P(V)$. If $D = \text{diag}(D(v_1), D(v_2), \dots, D(v_n))$ where $D(v_i) = \sum_{(v_j \in X)} a_{(v_i v_j)}$, then, a laplacian matrix define as*

$$L(G) = D - A(G).$$

Example 5.2.6. Consider $G = (X, E)$ as shown in figure ?? (This figure is selected from reference [1]). Where $X = \{V_1, V_2, V_3, V_4, V_5, V_6, V_7, V_8, Iv_9, Sv_{4,5}, Sv_{1,2,3}\}$ and $E = \{SiE_{5,6}, IE_{7,8}, SE_{123,45}, HE_{459,3}, HSE_{123,7,8}\}$. We now obtain the superhypergraph – related matrices in figure 1 using the above definitions.

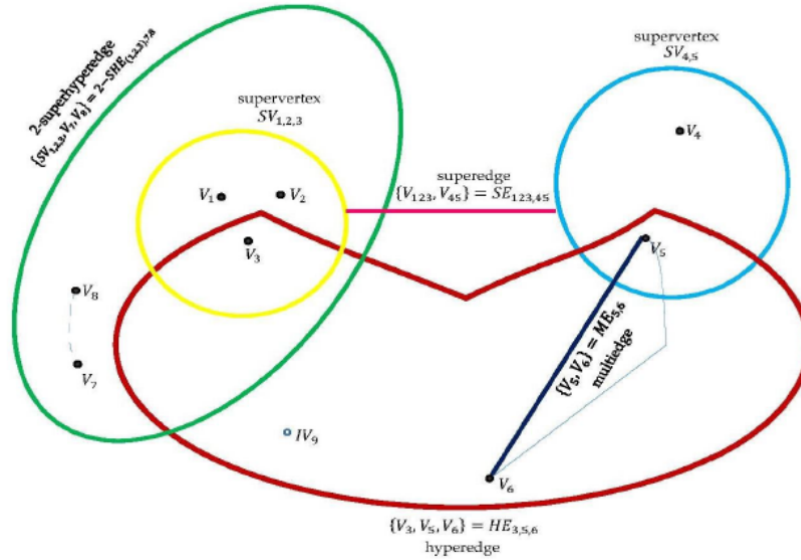


Figure 5.1: Super hyper power graph $G = (X, E)$

a: Adjacency matrix

$$\mathbb{A} = \begin{matrix} & v_1 & v_2 & v_3 & v_4 & v_5 & v_6 & v_7 & v_8 & Iv_9 & Sv_{4,5} & Sv_{1,2,3} \\ \begin{matrix} v_1 \\ v_2 \\ v_3 \\ v_4 \\ v_5 \\ v_6 \\ v_7 \\ v_8 \\ Iv_9 \\ Sv_{4,5} \\ Sv_{1,2,3} \end{matrix} & \left(\begin{array}{cccccccccccc} 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & H & H & 0 & 0 & H & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & H & 0 & 0 & 2, H & 0 & 0 & H & 0 & 0 & 0 \\ 0 & 0 & H & 0 & 2, H & 0 & 0 & 0 & H & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & I & 0 & 0 & SH \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & I & 0 & 0 & 0 & SH \\ 0 & 0 & H & 0 & H & H & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & S \\ 0 & 0 & 0 & 0 & 0 & 0 & SH & SH & 0 & S & 0 & 0 \end{array} \right) \end{matrix}$$

b: incidence matrix

c: laplacian matrix

$$B = \begin{matrix} E_{5,6} \\ SE_{123,45} \\ HE_{3,5,6,9} \\ SHE_{123,7,8} \\ IE_{7,8} \end{matrix} \begin{pmatrix} v_1 & v_2 & v_3 & v_4 & v_5 & v_6 & v_7 & v_8 & Iv_9 & Sv_{4,5} & Sv_{1,2,3} \\ 0 & 0 & 0 & 0 & 2 & 2 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 1 \\ 0 & 0 & 1 & 0 & 1 & 1 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 1 & 1 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 & 0 & 0 & 1 & 1 & 0 & 0 & 0 \end{pmatrix}$$

To calculate the Laplacian matrix, we first obtain the diameter matrix D , in which the vertices on the principal diameter, the degree of vertices, and the other vertices are 0. Then its Laplacian matrix is calculated as follows.

$$L = \begin{pmatrix} 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 3 & 0 & -1 & -1 & 0 & 0 & -1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & -1 & 0 & 5 & -3 & 0 & 0 & -1 & 0 & 0 \\ 0 & 0 & -1 & 0 & -3 & 5 & 0 & 0 & -1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 2 & -1 & 0 & 0 & -1 \\ 0 & 0 & 0 & 0 & 0 & 0 & -1 & 2 & 0 & 0 & -1 \\ 0 & 0 & -1 & 0 & -1 & -1 & 0 & 0 & 3 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & -1 \\ 0 & 0 & 0 & 0 & 0 & 0 & -1 & -1 & 0 & -1 & 3 \end{pmatrix}$$

5.3 Neutrosophic Super Hyper Power trees

In this section, we first provide a definition of Neutrosophic Super Hyper Power Tree. We then define the subtree for Neutrosophic Super Hyper Power graphs. In the following, we will examine the Helly property in this type of power graphs.

Defintion 5.3.1. Let $G = (X, E)$ be a Neutrosophic Super Hyper Power Graph. Then G is called a Neutrosophic Super Hyper Power Tree (NSHP-tree) if G be a connected NSHP-Graph without a neutrosophic cycle.

Defintion 5.3.2. Let $H = (A, B)$ be a NSHP-Graph. Then H is called a subtree NSHP-Graph if there exists a tree T with the same vertex set V such that each hyperedge, superedge, or hypersuperedge $e \in E$ induces a subtree in T .

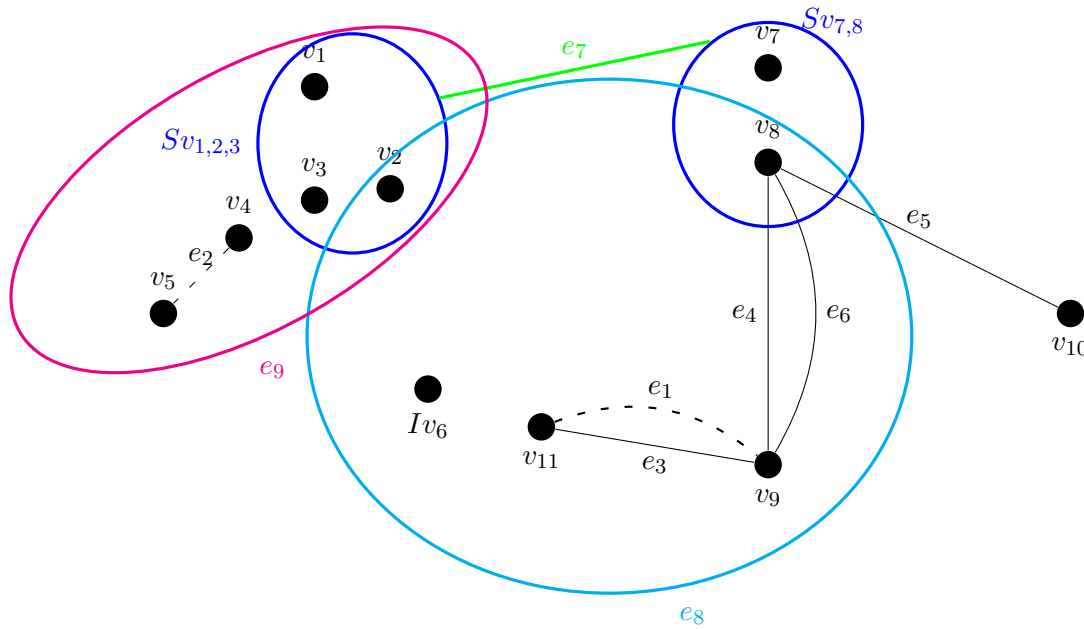


Figure 5.2: Super hyper power graph

Here we consider the underlying graph H^* to find the subtree of NSHP-graph.

Example 5.3.3. Consider $G = (X, E)$ a Super Hyper Power Graph as shown in figure 2.

As you can see, since G contains the cycle, so that G is not a Super Hyper Power tree. An SHP-subgraph induced by the subset $\{e_7, e_8, e_9, e_5\}$ of X , is a SHP-tree.

Example 5.3.4. Consider $G = (X, E)$ a Neutrosophic Super Hyper Power Graph as shown in figure 3. Note that in this example all vertices and edges belong to the neutrosophic sets. As you can see, G is a super hyper power tree.

Now we find a subtree according to definition 7 for G .

Now, let $T = (A, C)$ be a tree, that is, T is a connected neutrosophic graph without cycle. Then, we build a connected NSHP-Graph H in the following way:

1. The set of vertices of H is the set of vertices of T ;
2. The set of edges (hyperedges, superedges or superhyperedges) are a family E of subset V such that induced subgraph T_i is a subtree of T where T_i is produced by vertices located on edge

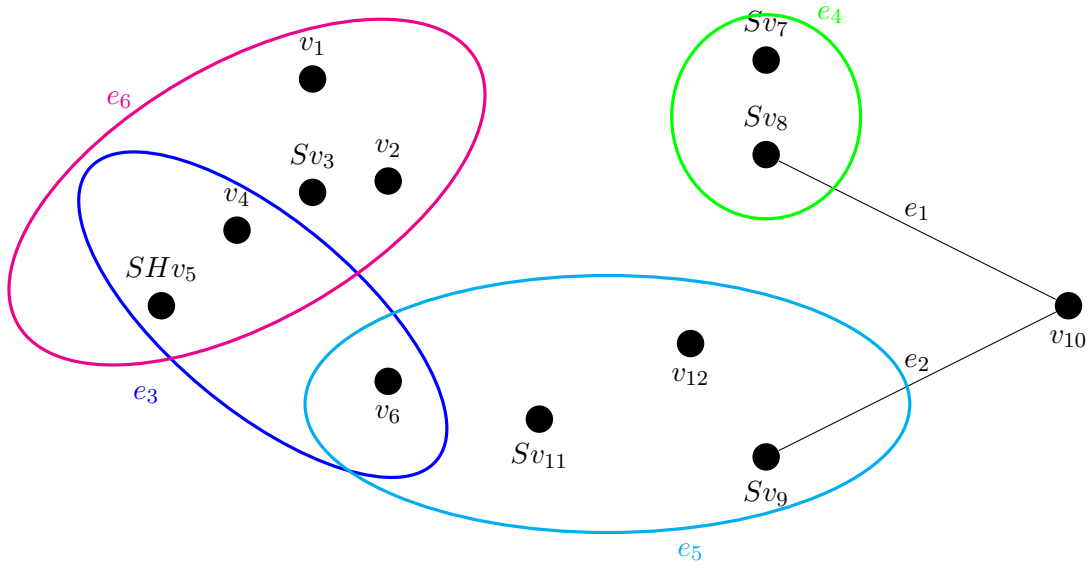
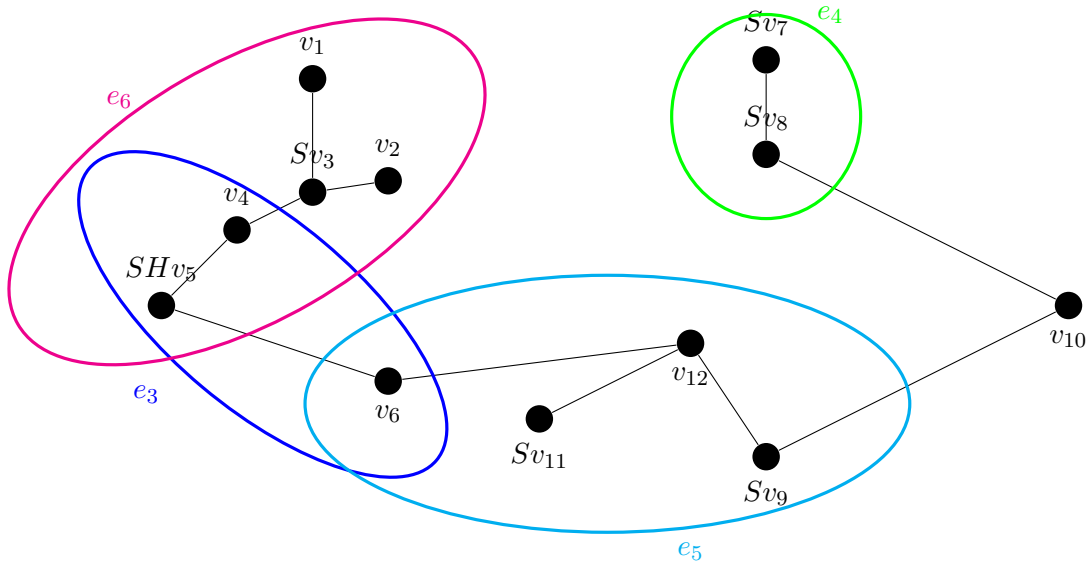


Figure 5.3: A neutrosophic super hyper tree G



$e_i \in E$. so that subgraph T_i is a tree.

Theorem 5.3.5. *Let $T = (V, E')$ be a tree. Also, H is a subtree super hyper power graph according to T . Then H has the Helly property.*

Proof. Since for each tree there exist exactly one path between the two vertices v_i, v_j . The path between two vertices v_i, v_j denoted $P[v_i, v_j]$. suppose that, v_i, v_j and v_k are three vertices of H . The

paths $P[v_i, v_j], P[v_j, v_k]$ and $P[v_k, v_i]$ have one common vertex. Now, using theorem 1, for each family of edges (hyperedges, superedges and superhyperedges) where the edge contains at least two of the vertices v_i, v_j and v_k have a non-empty intersection. \square

Theorem 5.3.6. *Let $T = (V, E')$ be a tree. Also, H is a subtree super hyper power graph according to T . Then $L(H)$ is a chordal graph.*

Proof. Consider $T = (V, E')$ is a tree. Suppose H is a subtree SHP-graph according to T . If $|V| = 1$, then H include exactly one vertex and one hyperedge, so that, the linegraph of H has only one vertex hence H is a clique. It turns out that H is a chordal graph. Next, assume that the assertion is true for each tree with $|V| = n - 1, n \geq 1$. Now we have to show that the problem assumption is valid for n vertices as well. For that, suppose $v \in V$ is a vertex leaf on H . remember that in a tree with at least two vertices there exist at least two leaves. If $T_1 = (V - \{v\}, E'_1)$, where T_1 is the subgraph on $V - \{v\}$, and

$$H_1(V - \{v\}) = (V - \{v\}, E_1), \quad |V| \geq 1.$$

The $T_1 = (V - \{v\}, E'_1)$ is a tree moreover $H_1 = (V - \{v\}, E_1)$ is an induced subtree SHP-graph associated with T_1 . Hence $L(H_1)$ is chordal. Now, if the number of edges should be the same, that is, $|E'| = |E'_1|$ then we have $L(H) \approx L(H_1)$ so that $L(H)$ is a chordal graph. If $|E'| \neq |E'_1|$ then we have

$$\{v\} \in E' \quad \text{and} \quad |E'| \geq |E'_1|.$$

It is easy to show that a neighborhood from $\{v\}$ in $L(H)$ is a clique. Hence any cycle passing through $\{v\}$ is chordal in $L(H)$ and so $L(H)$ is chordal. \square

5.4 Topological indices in Neutrosophic Super Hyper Power graphs

As we have seen, we can obtain the Laplacian matrix for a Neutrosophic Super Hyper Power graph. Now, using it, we can define the topological index of the Laplacian energy given the eigenvalues of the Laplacian matrix for a Neutrosophic Super Hyper Power graphs.

Defintion 5.4.1. Let $G = (N, M)$ is a NSHP-Graph, and L is its associated Laplacian matrix and the λ_i 's are the eigenvalues of this matrix, then the Laplacian energy index of G is defined as:

$$LE_T(G) = \sum_{i=1}^n \left| \lambda_i(L) - \frac{2 \sum_{ij} T_M(v_i, v_j)}{\sum_{i=1} T_N(v_i)} \right|,$$

$$LE_I(G) = \sum_{i=1}^n \left| \lambda_i(L) - \frac{2 \sum_{ij} I_M(v_i, v_j)}{\sum_{i=1} I_N(v_i)} \right|,$$

$$LE_F(G) = \sum_{i=1}^n \left| \lambda_i(L) - \frac{2 \sum_{ij} F_M(v_i, v_j)}{\sum_{i=1} F_N(v_i)} \right|,$$

too;

$$LE(G) = \frac{4 + 2LE_T(G) - 2LE_F(G) - LE_I(G)}{6}.$$

5.5 Summary

In this chapter, after giving important definitions about superhypergraphs, we defined the topological index of the Laplacian energy. Note that this definition can be generalized to other indices and other functions can be used instead of the Laplacian energy. We will follow this generalization in the following articles and examine its applications.

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ADVANCING GRAPH THEORY IN THE PRESENCE OF UNCERTAINTY

This book presents a comprehensive study of topological indices in single-valued neutrosophic graphs, integrating uncertainty, indeterminacy, and inconsistency into classical graph theory. It develops the connectivity index, Wiener index, and Sombor index within neutrosophic settings, providing definitions, examples, theorems, bounds, and applications. Topics include neutrosophic trees, cycles, stars, and super hyper power graphs, offering a rigorous foundation for uncertainty-aware network analysis.



Foundations

Graph theory, neutrosophic sets, and topological indices



Key Indices

Connectivity, Wiener, and Sombor indices



Theorems & Bounds

Properties, bounds, and proofs in neutrosophic graphs



Applications

Networks, decision-making, and complex systems

$$W(G) = \sum_{u < v} d(u, v)$$

$$CI(G) = \sum_{uv \in E(G)} \frac{1}{\sqrt{s_u s_v}}$$

$$SO(G) = \sum_{uv \in E(G)} \sqrt{s_u^2 + s_v^2}$$

