NEUTROSOPHIC SCIENCE INTERNATIONAL ASSOCIATION | UNIVERSITY OF NEW MEXICO

From Classical Measure Theory

to Refined Uncertain

Over / Under / Off-Measure

Definitions . Theorems . Examples . Applications

Part I — Classical Measure Theory
Part II — Uncertain Measure — Fuzzy & Extensions
Part III — Refined Uncertain Measure
Part IV — Uncertain Over-/Under-/Off-Measure
Part V — Refined Uncertain Over-/Under-/Off-Measure

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Scientist, writer, philosopher, and artist. Wrote in four languages: English, Romanian, French, and Spanish.

Academic Formation

Graduated first of his class from the Department of Mathematics and Computer Science, University of Craiova, Romania (1979). Earned a Ph.D. in Mathematics from the State University of Moldova at Kishinev (1997). Postdoctoral research at the University of Texas at Austin, Arizona State University, New Mexico State University, and Los Alamos National Laboratory.

Academic Career

25 years at the University of New Mexico, Gallup Campus (1997-2022): Assistant Professor (1997), Associate Professor (2003), Full Professor (2008), Professor Emeritus (2022). Chair,

Mathematical Contributions

In mathematics, introduced: the degree of negation of an axiom or theorem in geometry (Smarandache Geometries, 1969), which can be partially Euclidean and partially non-Euclidean; multi-structures (Smarandache n-structures); multi-spaces; Smarandache Curves, Surfaces, and Sequences; Smarandache Functions, Numbers, and Constants (MathWorld). Co-founder of Dezert-Smarandache Theory (DSmT) in information fusion.

Neutrosophic Science

Founded neutrosophy (1995) as a branch of philosophy studying the origin, nature, and scope of neutralities. From it derived: neutrosophic logic, neutrosophic set, neutrosophic probability, neutrosophic measure and integral, neutrosophic statistics, neutrosophic overset/underset/offset, plithogenic set and logic, refined neutrosophic structures, and the present extension to refined uncertain over/under/off-measure.

Literary & Artistic Work

Author of hundreds of books and research articles in mathematics, philosophy, poetry, prose, and drama. Recipient of numerous international prizes for literature and science. Founder of the Paradoxist Literary Movement and of Neutrosophic Science.

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1 Table of Contents

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2 Part I — Classical Measure Theory

2.1 1.1 Motivation and Historical Background

Classical measure theory arose from the need to rigorously define the notion of size — length, area, volume — for sets more general than simple geometric figures. The Riemann integral fails for many natural functions (e.g. the Dirichlet function). Henri Lebesgue's construction (1901–1904) resolved this by defining measure on σ-algebras. Today measure theory is the foundation of modern probability (Kolmogorov, 1933), functional analysis, ergodic theory, and mathematical physics.

2.2 1.2 Definitions

Definition 1.1 — σ-Algebra. Let X be a non-empty set. A collection $\Sigma \subseteq \mathcal{P}(X)$ is a *σ-algebra* on X if:

(i) $X \in \Sigma$

(ii) $A \in \Sigma \Rightarrow A^c \in \Sigma$ (closed under complement)

(iii) $\{A_n\}_{n\in\mathbb{N}} \subseteq \Sigma \Rightarrow \bigcup_n A_n \in \Sigma$ (closed under countable unions)

The pair (X, Σ) is called a *measurable space*.

Definition 1.2 — Measure. A function $\mu : \Sigma \to [0, +\infty]$ is a *measure* on (X, Σ) if:

(i) $\mu(\emptyset) = 0$

(ii) *Countable additivity (σ-additivity)*: for any pairwise disjoint sequence $\{A_n\} \subseteq \Sigma$:

$$\mu\left(\bigcup_n A_n\right) = \sum_n \mu(A_n)$$

The triple (X, Σ, μ) is called a *measure space*.

Definition 1.3 — Probability Measure. A measure P on (Ω, Σ) is a *probability measure* if $P(\Omega) = 1$. Events $A \in \Sigma$ receive a probability $P(A) \in [0, 1]$.

Definition 1.4 — Lebesgue Measure. On $\mathbb{R}^n$ with Borel σ-algebra $\mathcal{B}(\mathbb{R}^n)$, the *Lebesgue measure* λ satisfies:

$$\lambda([a_1, b_1] \times \cdots \times [a_n, b_n]) = \prod_{i=1}^{n}(b_i - a_i)$$

Definition 1.5 — Measurable Function. $f : X \to Y$ is *measurable* if $f^{-1}(B) \in \Sigma$ for every $B \in \mathcal{T}$.

Definition 1.6 — Lebesgue Integral. For non-negative measurable f:

$$\int_A f\, d\mu = \sup\left\{\int s\, d\mu : 0 \le s \le f,\ s \text{ simple}\right\}$$

For general f, write $f = f^+ - f^-$ and integrate each part.

2.3 1.3 Core Theorems

Theorem 1.1 — Monotone Convergence (Beppo Levi). Let $\{f_n\}$ be non-negative measurable with $f_n \uparrow f$ pointwise. Then:

$$\int f\,d\mu = \lim_{n\to\infty} \int f_n\,d\mu$$

Theorem 1.2 — Dominated Convergence (Lebesgue). Let $f_n \to f$ pointwise, $|f_n| \le g$ with $\int g\,d\mu < \infty$. Then:

$$\int f\,d\mu = \lim_{n\to\infty} \int f_n\,d\mu$$

Theorem 1.3 — Carathéodory Extension. Every σ-finite pre-measure on an algebra $\mathcal{A}$ extends uniquely to a measure on $\sigma(\mathcal{A})$.

Theorem 1.4 — Radon–Nikodym. Let $\nu \ll \mu$ (σ-finite). Then $\exists$ measurable $f \ge 0$ such that:

$$\nu(A) = \int_A f\,d\mu \quad \text{for all } A \in \Sigma$$

The function f is the Radon–Nikodym derivative $\dfrac{d\nu}{d\mu}$.

Theorem 1.5 — Basic Properties.

- *Monotonicity*: $A \subseteq B \Rightarrow \mu(A) \le \mu(B)$
- *Subadditivity*: $\mu\big(\bigcup_n A_n\big) \le \sum_n \mu(A_n)$
- *Continuity from below*: $A_n \uparrow A \Rightarrow \mu(A_n) \uparrow \mu(A)$
- *Continuity from above*: $A_n \downarrow A$, $\mu(A_1) < \infty \Rightarrow \mu(A_n) \downarrow \mu(A)$

2.4 1.4 Examples

Example 1.1 — Counting Measure. On $(\mathbb{N}, \mathcal{P}(\mathbb{N}))$: $\mu(A) = |A|$.

Example 1.2 — Dirac Measure. Fix $x_0 \in X$: $\delta_{x_0}(A) = \mathbf{1}_{x_0 \in A}$.

Example 1.3 — Gaussian Measure. On $(\mathbb{R}, \mathcal{B}(\mathbb{R}))$:

$$P(A) = \frac{1}{\sqrt{2\pi}} \int_A e^{-x^2/2}\,dx$$

2.5 1.5 Applications

- **Probability Theory**: Kolmogorov's axioms (1933); conditional probability via $\dfrac{d\nu}{d\mu}$
- **Functional Analysis**: L^p spaces; L^2 is a Hilbert space
- **Ergodic Theory**: measure-preserving transformations; long-run statistics
- **Finance**: risk-neutral pricing; Girsanov theorem (change of measure)

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3 Part II — Uncertain Measure

3.1 2.1 The Need for Uncertain Measure

Classical measure assumes $\mu : \Sigma \to [0, +\infty]$ is crisp. This fails when set membership is vague, incomplete, or contradictory. By *uncertain* (Smarandache, 2025–2026) we mean all fuzzy-based representations: fuzzy sets, intuitionistic fuzzy sets, neutrosophic sets, picture fuzzy sets, plithogenic sets, q-rung orthopair fuzzy sets, spherical fuzzy sets, and extensions.

3.2 2.2 Fuzzy Measure (Sugeno, 1974)

Definition 2.1 — Fuzzy Measure. $g : \Sigma \to [0, 1]$ is a *fuzzy measure* if:

(i) $g(\emptyset) = 0$; $g(X) = 1$

(ii) *Monotonicity*: $A \subseteq B \Rightarrow g(A) \leq g(B)$

σ-additivity is **not** required; it is replaced by monotonicity.

Definition 2.2 — Choquet Integral. For measurable f and fuzzy measure g:

$$(C) \int f \, dg = \int_0^{\infty} g(\{x : f(x) \geq t\}) \, dt$$

Theorem 2.1 — Fuzzy Measure Properties.

- *Submodularity*: $g(A \cup B) + g(A \cap B) \leq g(A) + g(B)$ (belief function)
- *Supermodularity*: $g(A \cup B) + g(A \cap B) \geq g(A) + g(B)$ (plausibility)

3.3 2.3 Intuitionistic Fuzzy Measure (Atanassov, 1986)

Definition 2.3 — IF Measure. $\mu_{IF} : \Sigma \to [0,1]^2$ assigns $(\mu(A), \nu(A))$ with $\mu(A) + \nu(A) \leq 1$. The *hesitation margin* is:

$$\pi(A) = 1 - \mu(A) - \nu(A) \geq 0$$

3.4 2.4 Neutrosophic Measure (Smarandache, 2013)

Definition 2.4 — Neutrosophic Measure. $\mu_N : \Sigma \to [0,1]^3$ assigns to each $A \in \Sigma$ a triplet $(\mu_T(A), \mu_I(A), \mu_F(A))$ representing:

- $\mu_T(A)$: measure of the *determinate* part of A
- $\mu_I(A)$: measure of the *indeterminate* part of A
- $\mu_F(A)$: measure of the *non-A* (false) part

Axioms:

(i) $\mu_N(\emptyset) = (0, 0, 0)$

(ii) *σ-Neutrosophic additivity*: for pairwise disjoint $\{A_n\}$:

$$\mu_N\left(\bigcup_n A_n\right) = \left(\sum_n \mu_T(A_n),\ \sum_n \mu_I(A_n),\ \sum_n \mu_F(A_n)\right)$$

Definition 2.5 — Normalised Neutrosophic Measure. If:

$$\mu_T(X) + \mu_I(X) + \mu_F(X) = 1$$

the neutrosophic measure is normalised.

Definition 2.6 — Neutrosophic Probability (Smarandache, 1995). A normalised neutrosophic measure $P_N(A) = (T(A), I(A), F(A))$ where:

- $T(A)$ = chance that event A occurs
- $I(A)$ = indeterminate/unknown chance of A's occurrence
- $F(A)$ = chance that A does not occur

Note: $T(A) + I(A) + F(A)$ need not equal 1 in the non-normalised case.

Definition 2.7 — Neutrosophic Integral. For $f_N = (f_T, f_I, f_F)$:

$$\int_N f_N\, d\mu_N = \left(\int f_T\, d\mu_T,\ \int f_I\, d\mu_I,\ \int f_F\, d\mu_F \right)$$

Theorem 2.2 — Neutrosophic Measure Properties.

(i) *Monotonicity*: $A \subseteq B \Rightarrow \mu_T(A) \leq \mu_T(B)$

(ii) μ_N is continuous from below and above

(iii) When $\mu_I \equiv 0$: neutrosophic measure reduces to classical measure

3.5 2.5 Plithogenic Measure (Smarandache, 2017)

Definition 2.8 — Plithogenic Measure. For a plithogenic set with attribute values $\{v_1, \dots, v_n\}$, each with appurtenance degree $d(x, v_i) \in [0, 1]$ and contradiction degree $c(v_1, v_i) \in [0, 1]$ w.r.t. dominant v_1, the plithogenic measure assigns vector $(\mu(A, v_1), \dots, \mu(A, v_n))$ subject to:

$$d(x, v_1 \wedge_P v_2) = d(x, v_1) \cdot_P d(x, v_2)$$

using the plithogenic conjunction operator $\cdot_P$.

3.6 2.6 Theorems

Theorem 2.3 — Generalisation Hierarchy.

$$\text{Classical} \subset \text{Fuzzy} \subset \text{IF} \subset \text{Neutrosophic} \subset \text{Plithogenic}$$

Each inclusion is strict.

Theorem 2.4 — Neutrosophic Bayes Rule. For neutrosophic events A, B with $P_N(B) \neq (0,0,0)$:

$$P_N(A|B) = \frac{P_N(B|A) \cdot P_N(A)}{P_N(B)} \quad \text{(component-wise)}$$

When $I \equiv 0$, this reduces to the classical Bayes theorem $P(A|B) = \dfrac{P(B|A)P(A)}{P(B)}$.

Theorem 2.5 — n-Valued Refined Neutrosophic Probability. With T split into $T_1, \dots, T_p$; I into $I_1, \dots, I_r$; F into $F_1, \dots, F_s$:

$$\sum_{i=1}^{p} T_i(A) + \sum_{j=1}^{r} I_j(A) + \sum_{k=1}^{s} F_k(A) = 1 \quad \text{(normalised)}$$

3.7 2.7 Examples and Applications

Example 2.1 — Neutrosophic Coin. A coin with head (H), tail (T), and indeterminate side (I) (standing on edge):

$$P_N(H) = \left(\tfrac{1}{3}, 0, \tfrac{2}{3}\right), \quad P_N(I) = \left(0, \tfrac{1}{3}, \tfrac{2}{3}\right), \quad P_N(T) = \left(\tfrac{1}{3}, 0, \tfrac{2}{3}\right)$$

Example 2.2 — Medical Diagnosis. Patient X, disease D:

$$\mu_N(\{X \text{ has } D\}) = (0.7,\ 0.2,\ 0.1)$$

$T = 0.7$ (positive evidence), $I = 0.2$ (inconclusive markers), $F = 0.1$ (counter-indicators).

Example 2.3 — Soccer. $P_N(\text{A wins}) = (0.6, 0.1, 0.3)$; $P_N(\text{draw}) = (0.2, 0.15, 0.65)$.

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4 Part III — Refined Uncertain Measure

4.1 3.1 Motivation

In many problems, T, I, F are not monolithic. Indeterminacy may arise from measurement noise, semantic ambiguity, or conflicting sources — each deserving separate tracking.

4.2 3.2 Refined Uncertain Set / Measure (Smarandache, 2026)

Definition 3.1 — Refined Uncertain Set. An uncertainty representation with n sub-components:

$$T_1, T_2, \dots, T_p; \quad I_1, I_2, \dots, I_r; \quad F_1, F_2, \dots, F_s$$

where $p, r, s \geq 1$, $p + r + s = n$, and $T_i, I_j, F_k \in [0, 1]$.

Definition 3.2 — Refined Uncertain Measure. $\mu_{RU} : \Sigma \to [0, 1]^n$ assigns:

$$(T_1(A), \dots, T_p(A),\ I_1(A), \dots, I_r(A),\ F_1(A), \dots, F_s(A)) \in [0, 1]^n$$

with component-wise σ-additivity: for disjoint $\{A_n\}$:

$$C_k\left(\bigcup_n A_n\right) = \sum_n C_k(A_n), \quad C_k \in \{T_1, \dots, T_p, I_1, \dots, I_r, F_1, \dots, F_s\}$$

4.3 3.3 Refined Nonstandard Uncertain Measure

Definition 3.3. Sub-components take values in the nonstandard interval:

$$T_i, I_j, F_k \in {}^{-}0, 1^{+}[$$

where ${}^{-}0$ is infinitesimally below 0 and 1^{+} infinitesimally above 1. Components may be hyperreal numbers, monads, or binads.

Theorem 3.1 — Reduction Hierarchy.

$$\text{Refined Nonstandard} \supseteq \text{Refined Uncertain} \supseteq \text{Neutrosophic} \supseteq \text{IF} \supseteq \text{Fuzzy} \supseteq \text{Classical}$$

Theorem 3.2 — Expressiveness. A Refined Uncertain Measure with $n = p+r+s$ sub-components represents:

- Classical measure: $n = 1$, T_1 only
- IF measure: $n = 2$, (T, F) with $\pi = 1 - T - F$

- Neutrosophic measure: $n = 3$, (T, I, F)
- Picture fuzzy measure: $n = 4$
- Any q-rung orthopair fuzzy measure as special case

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5 Part IV — Uncertain Over-/Under-/Off-Measure

5.1 4.1 Motivation: Beyond $[0, 1]$

The restriction to $[0, 1]$ is violated in many practical situations:

- **Over-membership** (> 1): an overtime employee deserves performance score > 1
- **Under-membership** (< 0): an employee causing net damage deserves score < 0
- **Off**: both effects coexist in the same system

5.2 4.2 Neutrosophic Overset, Underset, Offset (Smarandache, 2007)

Definition 4.1 — Neutrosophic Overset. A neutrosophic set where at least one component exceeds 1:

$$T(x) > 1 \quad \text{or} \quad I(x) > 1 \quad \text{or} \quad F(x) > 1 \text{ for some } x$$

Components take values in $[0, \psi]$, $\psi > 1$.

Definition 4.2 — Neutrosophic Underset. At least one component is < 0; components in $[\phi, 1]$, $\phi < 0$.

Definition 4.3 — Neutrosophic Offset. Both over- and under-values present; components in $[\phi, \psi]$ where:

$$\phi < 0 < 1 < \psi$$

5.3 4.3 Uncertain Over-/Under-/Off-Measure

Definition 4.4 — Uncertain Off-Measure. $\mu_{off} : \Sigma \to [\phi, \psi]^3$ with $\phi < 0 < 1 < \psi$, satisfying σ-additivity over the extended domain:

$$\mu_{off}\left(\bigcup_n A_n\right) = \sum_n \mu_{off}(A_n) \quad \text{(component-wise)}$$

Definition 4.5 — Uncertain Off-Probability. $P_{off} : \Sigma \to [\phi, \psi]$ satisfies:

$$\phi \leq P_{off}(A) \leq \psi \quad \forall A \in \Sigma$$

5.4 4.4 Operators: OverNorm / UnderNorm / OffNorm (Smarandache, 2025)

Definition 4.6 — OverNorm. $T_{over} : [0, \psi]^2 \to [0, \psi]$ satisfying:

$$T_{over}(x, y) = T_{over}(y, x), \quad T_{over}(x, \psi) = x$$

Example — OverMinimum: $T_{over}(x, y) = \min(x, y)$ for $x, y \in [0, \psi]$.

Definition 4.7 — OffNorm. $T_{off} : [\phi, \psi]^2 \to [\phi, \psi]$ with neutral element ψ:

$$T_{off}(x, y) = T_{off}(y, x), \quad T_{off}(x, \psi) = x$$

Theorem 4.1 — Extension. Every classical t-norm T on $[0, 1]$ extends to an OffNorm on $[\phi, \psi]$ preserving commutativity, associativity, and monotonicity. When restricted to $[0, 1]$: reduces to T.

5.5 4.5 Examples

Example 4.1 — Employee Performance.

Employee	T	I	F	Domain
Full-time (baseline)	1.0	0.0	0.0	$[0, 1]$
Overtime worker	**1.3**	0.0	0.0	$[0, 1.5]$
Saboteur	**–0.2**	0.1	0.8	$[-0.5, 1]$

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6 Part V — Refined Uncertain Over-/Under-/Off-Measure

6.1 5.1 Definitions (Smarandache, 2026)

Definition 5.1 — Refined Uncertain Off-Measure. Sub-components satisfy:

$$T_i, I_j, F_k \in [\Omega, \Psi] \quad \forall\, i, j, k$$

where $\Omega < 0 < 1 < \Psi$.

Definition 5.2 — Refined Nonstandard Uncertain Off-Measure. The most general form:

$$T_i, I_j, F_k \in {}^{-}\Omega, \Psi^{+}[$$

where components may be real numbers, hyperreal numbers, monads, or binads.

6.2 5.2 Theorems

Theorem 5.1 — Maximal Generalisation. The Refined Nonstandard Uncertain Off-Measure encompasses as special cases:

Structure	n	Domain
Classical Measure	1	$[0, \infty]$
Fuzzy Measure	1	$[0, 1]$
IF Measure	2	$[0, 1]^2$
Neutrosophic Measure	3	$[0, 1]^3$
Refined Uncertain Measure	≥ 4	$[0, 1]^n$
Uncertain Off-Measure	3	$[\phi, \psi]^3$
Refined Uncertain Off-Measure	≥ 4	$[\Omega, \Psi]^n$
Refined Nonstandard Off-Measure	≥ 4	$]^{-}\Omega, \Psi^{+}[^n$

Theorem 5.2 — Refined Off-Probability Bounds. For P_{RO} with $n = p + r + s$ sub-components:

$$n \cdot \Omega \leq \sum_{i=1}^{p} T_i(A) + \sum_{j=1}^{r} I_j(A) + \sum_{k=1}^{s} F_k(A) \leq n \cdot \Psi$$

7 Part VI

Connections with Probability Theory and Statistics

7.1 6.1 Classical Probability as a Special Case

Classical (Kolmogorov) probability is the best-studied special case of uncertain measure. A probability space (Omega, Sigma, P) assigns to each event A in Sigma a real number P(A) in [0,1] with P(Omega) = 1 and sigma-additivity. This is precisely a normalised classical measure. In the neutrosophic framework, it corresponds to the degenerate case where I(A) = 0 and F(A) = 1 - T(A) for every event A, giving P_N(A) = (P(A), 0, 1-P(A)).

Connection 6.1 — Classical Probability

Formal Reduction. Let P_N be a neutrosophic probability measure. Setting I(A) = 0 for all A in Sigma and F(A) = 1 - T(A) yields: P_N(A) = (P(A), 0, 1-P(A)) which is isomorphic to classical probability P(A). Conversely, any classical probability space embeds into neutrosophic probability by the canonical injection P -> (P, 0, 1-P). Thus: Classical Probability = Neutrosophic Probability restricted to I = 0. Neutrosophic statistics (Smarandache, 2014) extends classical statistics to datasets containing indeterminate values. Classical estimators (mean, variance, regression coefficients) are computed in the T-component; indeterminate observations contribute to the I-component rather than being discarded or imputed.

7.2 6.2 Bayesian Probability vs. Neutrosophic Probability

Bayesian probability interprets P(A) as a degree of belief, updated via Bayes' theorem: P(A|B) = P(B|A) P(A) / P(B). It is crisp: the posterior is a single real number, even when prior knowledge is vague. This creates tension when the evidence is genuinely indeterminate rather than merely uncertain.

Comparison 6.2 — Bayesian vs. Neutrosophic

Key Difference. In Bayesian probability, vague prior knowledge is encoded as a prior distribution (e.g. a diffuse or Jeffreys prior) — but the result is still a crisp number in [0,1]. In neutrosophic probability, genuine indeterminacy yields I(A) > 0 as a first-class component: P_N(A) = (T, I, F) where I represents 'irreducibly unknown chance.' Bayesian updating of P_N(A|B) proceeds component-wise, but the I-component may remain nonzero even after conditioning — a situation impossible in classical Bayes. Neutrosophic probability thus models a fundamentally different epistemic state: not 'spread-out certainty' (Bayesian prior) but 'structured indecision' (neutrosophic I).

7.3 6.3 Imprecise Probability and Credal Sets

Imprecise probability (Walley, 1991; Weichselberger, 2000) replaces a single probability P(A) with an interval [P_lower(A), P_upper(A)] or a set of probability measures (a credal set). This captures the state of an agent who cannot assign a single coherent probability but can bound it. The gap P_upper - P_lower is often interpreted as imprecision.

Comparison 6.3 — Imprecise Probability

Imprecise Probability vs. Neutrosophic Probability. An imprecise probability interval [l, u] with l + u <= 1 can be embedded in the neutrosophic framework as T(A) = l (confirmed lower bound), F(A) = 1 - u (confirmed upper bound on non-occurrence), I(A) = u - l (the imprecision gap). However, neutrosophic probability is more general: T + I + F need not lie in [0,1], and the three components are semantically distinct (they are not simply the lower bound, gap, and upper complement). Imprecise probability is a special calibrated case of neutrosophic probability where I precisely equals the imprecision gap.

7.4 6.4 Dempster-Shafer Theory and Neutrosophic Measure

Dempster-Shafer (DS) theory (Shafer, 1976) assigns a basic probability assignment (BPA) m: 2^Omega -> [0,1] with Sum_{A subseteq Omega} m(A) = 1 and m(empty) = 0. From the BPA, belief Bel(A) and plausibility Pl(A) are defined: Bel(A) = Sum_{B subseteq A} m(B) (evidence fully supporting A), Pl(A) = Sum_{B n A != empty} m(B) (evidence not contradicting A). The interval [Bel(A), Pl(A)] captures the range of certainty about A.

Comparison 6.4 — Dempster-Shafer vs. Neutrosophic

DS Theory vs. Neutrosophic Measure. The mapping $T(A) = Bel(A)$, $F(A) = Bel(\text{not-}A)$, $I(A) = Pl(A) - Bel(A)$ embeds DS theory into the neutrosophic framework. Key difference: DS theory requires m to be normalised over a fixed frame of discernment and uses Dempster's rule of combination, which can produce counterintuitive results on highly conflicting sources (Zadeh's paradox). The Dezert-Smarandanche Theory (DSmT), developed by Dezert and Smarandache, generalises DS theory by allowing non-exclusive and non-exhaustive hypotheses (the DSm free model) and using PCR5/PCR6 instead of Dempster's rule, which handles total conflict more robustly. Neutrosophic measure further generalises by not requiring the BPA structure at all.

7.5 6.5 Comparison Table: Probability Frameworks

Framework	Domain	Additivity	Indeterminacy	Key Feature
Classical Prob.	[0,1]	Strict	None	Kolmogorov axioms
Bayesian Prob.	[0,1]	Strict	Via prior	Bayes updating
Imprecise Prob.	[l,u]	Interval	Gap u-l	

Credal sets

DS Theory

[0,1]

Via BPA

Bel-Pl gap

Belief functions

Fuzzy Measure

[0,1]

Monotone

None

Choquet integral

IF Measure

[0,1]^2

Monotone

Hesitation pi

Atanassov 1986

Neutrosophic P.

[0,1]^3

Component

Independent I

T+I+F free

Plithogenic P.

[0,1]^n

Attrib.

Multi-attrib.

Contradiction deg.

Off-Probability

[phi,psi]

Extended

Independent I

Values outside [0,1]

8 Part VII

Connections with Possibility Theory and Evidence Theory

8.1 7.1 Possibility Theory (Zadeh / Dubois-Prade)

Possibility theory (Zadeh, 1978; Dubois and Prade, 1988) is built on a possibility distribution pi: Omega -> [0,1] with max_{omega} pi(omega) = 1. It defines two dual measures: Pi(A) = max_{omega in A} pi(omega) (possibility: is A possible at all?), N(A) = 1 - Pi(not-A) = min_{omega not in A} (1 pi(omega)) (necessity: is A certain?). Possibility theory captures qualitative, ordinal uncertainty — it asks 'how possible?' rather than 'how probable?'

Definition 7.1 — Possibility and Necessity Measures

Possibility Measure. A function Pi: 2^Omega -> [0,1] is a possibility measure if: (i) Pi(empty) = 0; Pi(Omega) = 1; (ii) Pi(A U B) = max(Pi(A), Pi(B)) for all A, B (max-decomposability). Its dual: N(A) = 1 - Pi(not-A) is a necessity measure satisfying N(A n B) = min(N(A), N(B)).

Comparison 7.1 — Possibility vs. Neutrosophic

Possibility vs. Neutrosophic Measure. The duality (Pi, N) maps naturally to (T, F) in neutrosophic measure: T(A) ~ N(A) (degree of necessity/certainty), F(A) ~ N(not-A), I(A) ~ Pi(A) - N(A) (the possibility-necessity gap, representing indeterminacy). Neutrosophic measure is strictly more expressive: T, I, F are independently assignable, while possibility theory constrains them by Pi(A) = max over all omega, enforcing the max-decomposability axiom that neutrosophic measure does not require. Possibility theory is the ordinal, qualitative cousin; neutrosophic measure is the quantitative, independent-component generalisation.

8.2 7.2 Evidence Theory: Belief and Plausibility Functions

A belief function (Shafer, 1976) Bel: 2^Omega -> [0,1] is a special case of a fuzzy measure satisfying the stronger condition of n-monotonicity for all n >= 2: Bel(A_1 U … U A_n) >= Sum over non-empty I subseteq {1..n} of (-1)^(|I|+1) Bel(n_{i in I} A_i). Belief functions arise from the mass assignment m and satisfy Bel(A) + Bel(not-A) <= 1 (the interval [Bel, Pl] always contains 1/2 or is asymmetric).

Theorem 7.1 — Belief Functions as Neutrosophic

Belief Functions as Neutrosophic Measures. Every belief function Bel on (Omega, 2^Omega) defines a neutrosophic measure by: T(A) = Bel(A), I(A) = Pl(A) - Bel(A), F(A) = 1 - Pl(A). The axioms of neutrosophic measure are satisfied: T(empty)=I(empty)=F(empty)=0. The reverse does not hold: a general neutrosophic measure need not be n-monotone. Thus: Belief functions are a strict subclass of neutrosophic measures.

8.3 7.3 Dezert-Smarandache Theory (DSmT)

DSmT (Dezert and Smarandache, 2004-2015) was developed specifically to handle fusion of highly uncertain, conflicting, and imprecise sources of information. It replaces the Shafer frame (mutually exclusive hypotheses) with the DSm free model where hypotheses may overlap, and introduces the PCR5/PCR6 combination rules that redistribute conflict proportionally to the reliability of the conflicting sources.

Comparison 7.2 — DSmT vs. DS vs. Neutrosophic

DSmT vs. DS Theory vs. Neutrosophic Measure. DS Theory: mutually exclusive frame; Dempster rule; fails at total conflict. DSmT: overlapping hypotheses allowed; PCR5/6 rules; handles total conflict correctly. Neutrosophic Measure: no frame of discernment required; indeterminacy I is a first-class semantic component, not just redistribution of conflict. DSmT is the most powerful information-fusion framework and connects naturally to neutrosophic probability: DSmT fusion output can be expressed as a neutrosophic triple (T, I, F) where I captures the residual unresolved conflict after PCR redistribution.

Example 7.1 — Sensor Fusion

Sensor Fusion under Total Conflict. Two sensors assess whether a target exists in sector A: Sensor 1: m_1(A) = 0.9, m_1(not-A) = 0.1. Sensor 2: m_2(A) = 0.05, m_2(not-A) = 0.95. DS rule: total conflict K = 0.9 x 0.95 + 0.1 x 0.05 = 0.86; normalised result is dominated by the normalisation artefact. PCR5 (DSmT): redistributes the conflicting mass 0.86 proportionally, giving a more reliable combined assessment. Neutrosophic: P_N(A) = (0.9, 0.86, 0.1) from Sensor 1's view with I = conflict, directly encoding the indeterminate situation without normalisation.

8.4 7.4 Unified View: When Each Theory Applies

Classical probability: well-defined random experiment; no vagueness; all probabilities knowable in principle. Use when: dice, coins, cards, sampling. Bayesian probability: subjective beliefs that can be updated; prior information available. Use when: sequential inference, parameter estimation.

Possibility theory: qualitative, ordinal uncertainty; linguistic variables; expert rules. Use when: fuzzy control, linguistic modelling. DS Theory: evidence from independent sources, mutually exclusive frame. Use when: sensor fusion with non-conflicting sources. DSmT: highly conflicting, overlapping hypotheses, unreliable sources. Use when: multi-sensor fusion, intelligence analysis. Neutrosophic measure: genuine indeterminacy coexists with truth and falsity; independent I-component required. Use when: medical diagnosis, quantum systems, ambiguous legal evidence, philosophical propositions. Plithogenic / Refined measures: multiple attribute dimensions; subcomponents of uncertainty semantically distinct. Use when: multi-attribute evaluation, AI uncertainty decomposition.

9 Part VIII

Connections with Non-Additive and Capacity Measures

9.1 8.1 Capacities and 2-Monotone Measures

A capacity (Choquet, 1953) is a monotone set function v: 2^X -> [0,1] with v(empty) = 0, v(X) = 1, and A subseteq B => v(A) <= v(B). Unlike probability measures, capacities do not require additivity. They model ‘importance’ or ‘weight’ in the absence of a frequency interpretation. A capacity is k-monotone if it satisfies the k-th order inclusion-exclusion inequality. 2-monotone capacities (also called convex capacities or supermodular games) satisfy v(A U B) + v(A n B) >= v(A) + v(B) for all A, B.

Definition 8.1 — Capacity and Choquet Integral

Capacity (Choquet, 1953). A set function v: 2^X -> [0,+inf) with v(empty) = 0 and A subseteq B => v(A) <= v(B). The Choquet integral of f w.r.t. v is: (C) Integral f dv = Integral_0^+inf v({x: f(x) >= t}) dt + Integral_{-inf}^0 [v({x:f(x)>=t}) - v(X)] dt. For non-negative f: (C) Integral f dv = Integral_0^+inf v({x: f(x) >= t}) dt.

Theorem 8.1 — Choquet Integral Properties

Choquet Integral Properties. For a 2-monotone capacity v and bounded measurable f, g: (i) Positive homogeneity: (C) Integral (lambda f) dv = lambda (C) Integral f dv for lambda >= 0. (ii) Monotonicity: f <= g => (C) Integral f dv <= (C) Integral g dv. (iii) Comonotone additivity: if f and g are comonotone, (C) Integral (f+g) dv = (C) Integral f dv + (C) Integral g dv. (iv) When v is additive (a probability measure), the Choquet integral = Lebesgue integral.

9.2 8.2 Belief Functions as Special Capacities

Every belief function Bel is a completely monotone capacity (infinite-order monotone), and every plausibility function Pl is a completely alternating capacity. The pair (Bel, Pl) brackets the true probability: Bel(A) <= P(A) <= Pl(A) for any probability consistent with the mass assignment m.

Comparison 8.1 — Capacity Hierarchy

Capacity Hierarchy. Additive measure (probability) SUBSET 2-monotone capacity SUBSET completely monotone capacity (belief function) SUBSET monotone capacity (fuzzy measure). Neutrosophic measure sits outside this hierarchy: it is a vector-valued capacity (T, I, F) where each component is monotone, but the triple together is not a scalar capacity. It captures a richer structure than any scalar capacity.

9.3 8.3 Interaction Indices: Shapley and Banzhaf

For a capacity v on a finite set N = {1,…,n}, the Shapley interaction index quantifies the contribution of each element or subset. The Shapley value phi_i(v) = Sum_{S subseteq N{i}} [v(S U {i}) - v(S)] * w(|S|,n) measures average marginal contribution of element i. These indices are widely used in cooperative game theory, explainable AI (SHAP values), and multi-criteria decision analysis.

Comparison 8.2 — Neutrosophic Shapley

Shapley Index and Neutrosophic Measure. When v = μ_T (the truth-component of a neutrosophic measure), the Shapley value measures the average truth-marginal contribution of criterion i. Analogously, phi_i(μ_I) measures indeterminacy contribution and phi_i(μ_F) measures falsity contribution. Together, (phi_i(μ_T), phi_i(μ_I), phi_i(μ_F)) is a neutrosophic Shapley triple — a richer attribution than a single real number. This extends explainable AI (SHAP) from classical to neutrosophic settings.

Example 8.1 — Neutrosophic Shapley

MCDM with Neutrosophic Shapley. Five criteria {C_1,…,C_5} evaluate a policy proposal. Neutrosophic Shapley triples: phi_1 = (0.35, 0.05, 0.12), phi_2 = (0.28, 0.18, 0.09), phi_3 = (0.20, 0.30, 0.15), phi_4 = (0.10, 0.08, 0.40), phi_5 = (0.07, 0.39, 0.24). Criterion C_3 has the highest indeterminacy contribution (phi_I = 0.30): it is the most ambiguous criterion and should be prioritised for expert clarification. C_4 has the highest falsity contribution (phi_F = 0.40): it actively contradicts the proposal and deserves special scrutiny.

10 Part IX

Connections with Quantum Probability and Physics

10.1 9.1 Quantum Probability: Born Rule and Measurement

Quantum probability (von Neumann, 1932) assigns probabilities to measurement outcomes via the Born rule: P(outcome A | state psi) = ||P_A psi||^2, where P_A is the orthogonal projector onto the eigenspace of A. Unlike classical probability, quantum probability is defined on a non-commutative lattice of projectors (a quantum logic), not a Boolean sigma-algebra. Joint probabilities P(A and B) may not exist when A and B are incompatible observables (non-commuting operators).

Comparison 9.1 — Quantum vs. Neutrosophic

Quantum vs. Classical Probability. Classical: Boolean sigma-algebra; all events simultaneously measurable; Kolmogorov axioms. Quantum: orthomodular lattice; incompatible events not jointly measurable; Born rule replaces Kolmogorov. Neutrosophic: Boolean sigma-algebra retained; indeterminacy I explicitly modelled as a first-class component rather than arising from non-commutativity. Quantum indeterminacy is structural (Heisenberg uncertainty); neutrosophic indeterminacy is informational (unknown / undecidable).

10.2 9.2 Contextuality and Non-Kolmogorovian Probability

Contextuality (Kochen-Specker, 1967; Bell, 1964) is the property that quantum measurement outcomes cannot be assigned definite pre-existing values independent of the measurement context. This rules out hidden-variable models and implies that classical (Kolmogorovian) probability cannot adequately describe quantum systems. Khrennikov (1999-2020) has developed p-adic and non-

Kolmogorovian probability models to accommodate contextual phenomena.

Comparison 9.2 — Contextuality

Contextuality and Neutrosophic Indeterminacy. In contextual quantum systems, the value of observable A may be (T, I, F) = (defined probability in context C_1, indeterminate across contexts, contradicted in context C_2). The I-component of neutrosophic measure can absorb contextual indeterminacy: I(A) encodes 'the outcome of A depends on context in a way that cannot be resolved without specifying the context.' This is formally distinct from the standard Hilbert-space Born rule but provides a classical proxy for contextual ambiguity in approximate models.

10.3 9.3 Neutrosophic Measure and Quantum Indeterminacy

Heisenberg's uncertainty principle states that Delta_x * Delta_p >= hbar/2: position and momentum cannot both be precisely known simultaneously. This is a hard physical constraint, not a limitation of knowledge. In the neutrosophic framework, a physical observable whose value is uncertain due to the Heisenberg principle would carry I(A) > 0 representing the irreducible quantum indeterminacy. Falsity F(A) would represent the probability of the complementary outcome under the Born rule.

Example 9.1 — Quantum Spin

Spin Measurement. A spin-1/2 particle prepared in state |+x> is measured in the z-direction. Born rule: P(spin up) = 0.5, P(spin down) = 0.5. Before measurement, the spin is genuinely indeterminate (not merely unknown). Neutrosophic encoding: P_N(spin-up) = (0.5, delta, 0.5) where delta > 0 represents the irreducible pre-measurement indeterminacy. After measurement, delta -> 0 (state collapses). This provides a classical bookkeeping of quantum indeterminacy without claiming to replace the full Hilbert-space formalism.

10.4 9.4 Over/Under-Measure and Quantum Negative Probability

Negative probability was introduced by Dirac (1942) and Feynman (1987) as a computational tool in quantum field theory (e.g. Wigner quasi-probability distribution). The Wigner function W(x, p) of a quantum state can be negative in certain regions of phase space — a signature of non-classicality. Similarly, Kirkwood-Dirac quasi-probabilities can exceed 1 or be negative.

Comparison 9.3 — Negative Quasi-Probability

Negative Probability and Uncertain Under-Measure. The neutrosophic Under-Measure (components in [phi, 1] with phi < 0) is the natural mathematical home for negative quasi-probabilities.

P_under(A) = phi < 0 represents 'anti-probability': the event A actively destructively interferes with the rest of the probability space. Similarly, P_over(A) > 1 (Over-Measure) corresponds to Wigner function values that exceed 1, representing quantum coherence peaks. The Off-Measure domain [phi, psi] with phi < 0 < 1 < psi thus accommodates the full range of quantum quasi-probabilities within a unified uncertain measure framework.

11 Part X

Connections with AI, Machine Learning, and Decision Theory

11.1 10.1 Uncertainty in Machine Learning: Epistemic vs. Aleatoric

Modern machine learning distinguishes two types of uncertainty: Aleatoric uncertainty is irreducible noise inherent in the data — the label noise in a training set, the natural randomness of outcomes. It cannot be reduced by collecting more data. Epistemic uncertainty arises from limited knowledge of the true model — it decreases with more data or better models. Standard neural networks produce neither type correctly: their softmax output is overconfident and does not distinguish the two.

Comparison 10.1 — ML Uncertainty

ML Uncertainty and Neutrosophic Measure. The neutrosophic triple (T, I, F) maps naturally: T(A) ~ P(A | model is correct) [aleatoric-dominated], I(A) ~ epistemic uncertainty (model uncertainty, out-of-distribution), F(A) ~ 1 - T(A) - I(A) (residual). Bayesian Neural Networks (BNNs) estimate epistemic uncertainty via posterior variance over weights. Ensemble methods (deep ensembles) approximate it via disagreement. Both can be mapped to the I-component of a neutrosophic measure, giving T = mean prediction, I = ensemble disagreement, F = 1 - T - I. This provides a neutrosophic interpretation of BNN output.

11.2 10.2 Conformal Prediction and Uncertain Measure

Conformal prediction (Vovk, Gammerman, Shafer, 1999-2005) produces prediction sets C(x) with guaranteed coverage: P(y in C(x)) >= 1 - alpha for any alpha in (0,1), without distributional assumptions. The set C(x) is large when the model is uncertain and small when confident. It provides a valid, distribution-free uncertainty quantification.

Comparison 10.2 — Conformal Prediction

Conformal Prediction and Refined Uncertain Measure. For a conformal predictor with significance level alpha, the prediction set C_alpha(x) can be mapped to a neutrosophic measure: T(y) = 1 - p-value(y, x) for y in C_alpha(x), I(y) = width of C_alpha(x) / |Y| (set-size-relative indeterminacy), F(y) = p-value(y, x) for y not in C_alpha(x). For a refined conformal predictor that distinguishes multiple non-conformity scores, the Refined Uncertain Measure (T_1,…,T_p, I_1,…,I_r, F_1,…,F_s) provides a natural framework with each component corresponding to a different non-conformity measure or calibration source.

11.3 10.3 Multi-Criteria Decision Making (MCDM)

MCDM problems require choosing among alternatives A_1,…,A_m evaluated on criteria C_1,…,C_n under uncertainty. Classical approaches (AHP, TOPSIS, VIKOR) use crisp weights and scores. Fuzzy MCDM (Bellman-Zadeh, 1970) uses fuzzy sets to capture vague evaluations. Intuitionistic fuzzy MCDM (Xu, 2007) adds non-membership. Neutrosophic MCDM further adds an independent indeterminacy component.

Definition 10.1 — Neutrosophic TOPSIS

Neutrosophic TOPSIS. Step 1: Build the neutrosophic decision matrix D_ij = (T_ij, I_ij, F_ij). Step 2: Determine the neutrosophic positive ideal solution (NPIS) and negative ideal solution (NNIS) component-wise. Step 3: Compute neutrosophic distances d+(A_i) = distance from A_i to NPIS, d-(A_i) = distance from A_i to NNIS using a neutrosophic distance metric. Step 4: Relative closeness CC_i = d-(A_i) / (d+(A_i) + d-(A_i)). Step 5: Rank alternatives by CC_i. This extends classical TOPSIS by preserving indeterminacy throughout the ranking process.

Example 10.1 — Neutrosophic TOPSIS

Supplier Selection. Three suppliers {S_1, S_2, S_3} evaluated on cost (C_1), quality (C_2), delivery reliability (C_3). Expert assessments in neutrosophic form: S_1: ((0.7,0.1,0.2), (0.8,0.1,0.1), (0.6,0.2,0.2)). S_2: ((0.5,0.3,0.2), (0.7,0.2,0.1), (0.8,0.1,0.1)). S_3: ((0.6,0.2,0.2), (0.6,0.3,0.1), (0.5,0.3,0.2)). Neutrosophic TOPSIS with equal weights ranks S_2 highest due to superior delivery reliability with low indeterminacy. The I-components identify C_3 of S_3 (I=0.3) as the most uncertain criterion, flagging it for further investigation.

11.4 10.4 Neutrosophic and Plithogenic MCDM Operators

Several aggregation operators extend classical weighted averaging to neutrosophic settings: Single-Valued Neutrosophic Weighted Average (SVNWA): SVNWA(a_1,…,a_n) = (1 Prod(1-T_i)^w_i, Prod(I_i)^w_i, Prod(F_i)^w_i) where w = (w_1,…,w_n) are normalised weights.

Single-Valued Neutrosophic Geometric (SVNWG): SVNWG(a_1,…,a_n) = (Prod(T_i)^w_i, 1-Prod(1-I_i)^w_i, 1-Prod(1-F_i)^w_i). Plithogenic Weighted Average: extends SVNWA by incorporating contradiction degrees c(v_1, v_k) to weight the aggregation of each attribute value. Refined Neutrosophic OWA: applies ordered weighting to the n-component refined uncertain vector, preserving sub-component structure in the output.

11.5 10.5 Fuzzy Rule-Based Systems and Uncertain Measure

Fuzzy rule-based systems (Mamdani, 1974; Takagi-Sugeno, 1985) model systems with linguistic rules: IF x is A THEN y is B, where A, B are fuzzy sets. The inference uses t-norms (AND) and t-conorms (OR) from the standard [0,1] domain. Neutrosophic rule-based systems extend this to rules of the form: IF x is (T_A, I_A, F_A) THEN y is (T_B, I_B, F_B), using OffNorms when the degree of applicability of a rule may exceed classical bounds.

Comparison 10.3 — Fuzzy vs. Neutrosophic Rules

Mamdani Fuzzy vs. Neutrosophic Rule System. Mamdani: IF temperature is hot (mu=0.8) THEN cooling is high. Neutrosophic: IF temperature is (hot T=0.8, uncertain sensor I=0.15, not-hot F=0.05) THEN cooling is (high T=0.7, uncertain output I=0.15, not-high F=0.15). The I-component propagates through the rule, preserving the sensor's uncertainty into the output — impossible in classical Mamdani fuzzy control. OffNorm extension: if the temperature reading is 1.2 (over-range), the OffNorm allows the rule to fire with T > 1, triggering an emergency cooling protocol.

12 Part XI

Grand Synthesis: A Unified Map of Uncertainty Theories

12.1 11.1 The Lattice of Uncertainty Theories

The following diagram represents the generalisation lattice of uncertainty theories covered in this monograph. Each arrow denotes a strict generalisation: the target theory models strictly more uncertainty structures than the source. Classical Measure | v Fuzzy Measure (Sugeno) <– Possibility Measure (Zadeh) | v IF Measure (Atanassov) <– Belief Functions (Shafer) <– DSmT (Dezert-Smarandache) | v Neutrosophic Measure (Smarandache 2013) | v Plithogenic Measure (Smarandache 2017) | v Refined Uncertain Measure (Smarandache 2026) | v Uncertain Off-Measure [phi, psi], phi<0<1 | v Refined Uncertain Off-Measure [Omega,Psi]^n (Smarandache 2026) | v Refined Nonstandard Uncertain Off-Measure]Omega-, Psi+[^n (Smarandache 2026)

12.2 11.2 Selection Guide: Which Theory for Which Problem?

Situation / Need

Recommended Theory

Key Reason

Events with known frequencies

Classical Probability

Kolmogorov axioms exact

Sequential belief updating

Bayesian Probability

Bayes theorem applies

Qualitative / linguistic uncertainty

Possibility Theory

Ordinal, not quantitative

Non-additive criterion weights

Fuzzy Measure + Choquet

Super/sub-additivity

Membership + non-membership known

IF Measure (Atanassov)

Hesitation margin pi

Conflicting sensor data

DSmT (Dezert-Smarandache)

PCR5/6 handles conflict

Genuine undecidable indeterminacy

Neutrosophic Measure

Independent I component

Multi-attribute with contradiction

Plithogenic Measure

Contradiction degrees

Sub-types of T, I, F needed

Refined Uncertain Measure

Component decomposition

Scores exceed [0,1] naturally

Uncertain Off-Measure

phi<=0, psi>=1 domain

Quantum quasi-probabilities

Uncertain Under-Measure

Negative values phi<0

All of the above combined

Refined Nonstandard Off-MeasureMaximal generalisation

12.3 11.3 Open Problems and Research Directions

P1. Choquet Integral for Neutrosophic Measures. Define a fully rigorous Choquet-type integral (C) Integral f d μ_N for neutrosophic measures μ_N = (μ_T, μ_I, μ_F), prove convergence theorems analogous to the Monotone and Dominated Convergence Theorems. P2. Radon-Nikodym for Neutrosophic Off-Measures. Develop a neutrosophic Radon-Nikodym theorem: given two neutrosophic measures μ_N, nu_N with nu_N « μ_N (absolute continuity in each component), characterise the neutrosophic derivative d nu_N / d μ_N. This would enable a neutrosophic change-of-measure formula (analogue of Girsanov). P3. Neutrosophic Ergodic Theory. Extend Birkhoff's Ergodic Theorem to neutrosophic measure-preserving transformations: T: (X, μ_N) -> (X, μ_N). Define neutrosophic time averages and characterise ergodicity, mixing, and entropy in the neutrosophic framework. P4. Refined Off-Measure and Quantum Field Theory. Formalise the connection between Wigner quasi-probability distributions and Uncertain Under-Measures. Determine whether the PCTs (PCT theorem, CPT invariance) can be reformulated using neutrosophic or off-measures. P5. DSmT and Neutrosophic Measure Unification. Develop a formal embedding of DSmT fusion operators (PCR5, PCR6) into the neutrosophic measure framework, and characterise which neutrosophic update rules are equivalent to which DSm combination rules. P6. Neutrosophic Martingales and Finance. Define neutrosophic martingales: stochastic processes (M_t) = (T_t, I_t, F_t) with neutrosophic conditional expectation E_N[M_t | F_s] = M_s for s <= t. Develop neutrosophic stochastic calculus and a neutrosophic Black-Scholes formula. P7. Topological and Algebraic Structure. Study the nonstandard monad]Omega-, Psi+[as a topological ring or field. Characterise its Galois group over standard reals. Determine whether

Refined Nonstandard Uncertain Off-Measures form a complete lattice. P8. Computational Complexity. Determine the computational complexity of key operations on Refined Uncertain Off-Measures: normalisation, operator application, integration. Develop efficient algorithms for Neutrosophic TOPSIS, AHP, and MCDM at scale.

12.4 11.4 Concluding Remarks

This monograph has traced the complete generalisation pathway of measure theory from its classical Lebesgue-Kolmogorov foundation to the most general uncertain measure structures available today. Along the way it has situated each extension within the landscape of uncertainty theories — probability, possibility, evidence, capacity, quantum, and AI uncertainty — demonstrating that each classical framework arises as a special case, a special calibration, or a structural constraint of the neutrosophic and refined uncertain measure hierarchy. Several deep themes emerge from this synthesis. First: indeterminacy is not noise. Classical theories treat everything that is not probability as residual uncertainty to be absorbed, redistributed, or discarded. Neutrosophic measure refuses this: I is a first-class component with its own axioms, its own carriers, its own contribution to every theorem and every application. Second: the boundary [0,1] is a convention, not a law. Real systems — employees, quantum states, financial hedges, sensor readings — produce values outside this interval. The Off-Measure framework gives them a rigorous home without ad hoc clipping or normalisation. Third: refinement is information. Collapsing T into a single number discards the semantic distinction between 'evidence from source 1' and 'evidence from source 2.' Refined Uncertain Measure preserves it, making auditable, interpretable uncertainty quantification possible. The open problems listed in Section 11.3 are not decorative. A neutrosophic Choquet integral would give a new non-additive expectation theory. A neutrosophic Radon-Nikodym theorem would give a new change-of-measure calculus. A neutrosophic Girsanov theorem would give a new financial mathematics. These are not small extensions: they are invitations to reconstruct, from the ground up, the entire edifice of measure-theoretic probability on foundations that are honest about indeterminacy from the first axiom.

Selected References [1] Atanassov, K. (1986). Intuitionistic fuzzy sets. Fuzzy Sets and Systems, 20(1), 87-96. [2] Bell, J. S. (1964). On the Einstein-Podolsky-Rosen paradox. Physics, 1(3), 195-200. [3] Choquet, G. (1953). Theory of capacities. Annales de l'Institut Fourier, 5, 131-295. [4] Dezert, J. and Smarandache, F. (2004-2015). Advances and Applications of DSmT. American Research Press. https://fs.unm.edu/DSmT.htm

[5] Dubois, D. and Prade, H. (1988). Possibility Theory. Plenum Press. [6] Feynman, R. P. (1987). Negative probability. In Quantum Implications, Routledge. [7] Lebesgue, H. (1904). Lecons sur l'integration. Gauthier-Villars. [8] Shafer, G. (1976). A Mathematical Theory of Evidence. Princeton University Press. [9] Smarandache, F. (1995). Neutrosophy: Neutrosophic Probability, Set, and Logic. American Research Press. [10] Smarandache, F. (2013). Introduction to Neutrosophic Measure, Neutrosophic Integral, and Neutrosophic Probability. Sitech/Education Publishing. [11] Smarandache, F. (2016). Neutrosophic Overset, Neutrosophic Underset, and Neutrosophic Offset. Pons Editions. https://fs.unm.edu/NeutrosophicOversetUndersetOffset.pdf [12] Smarandache, F. (2025). Operators for Uncertain Over/Under/Off-Sets/-Logics/-Probabilities. Neutrosophic Sets and Systems, 79. [13] Smarandache, F. (2026). Refined Uncertain Set, Refined Nonstandard Uncertain Set, Refined Uncertain Over/Under/Off-Set. Neutrosophic Knowledge, 11. [14] Sugeno, M. (1974). Theory of Fuzzy Integrals and its Applications. PhD Thesis, Tokyo Institute of Technology. [15] Walley, P. (1991). Statistical Reasoning with Imprecise Probabilities. Chapman and Hall. [16] Zadeh, L. A. (1965). Fuzzy sets. Information and Control, 8(3), 338-353. [17] Zadeh, L. A. (1978). Fuzzy sets as a basis for a theory of possibility. Fuzzy Sets and Systems, 1(1), 3-28.

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13 Part XII

Extended Worked Case Studies This Part provides extended, end-to-end worked case studies demonstrating the practical application of the uncertainty measure hierarchy. Each study begins with a real-world problem, identifies the appropriate theory, applies the formal framework, and interprets the results. The studies are designed to be self-contained teaching examples.

13.1 12.1 Case Study I: Clinical Trial with Indeterminate Outcomes

A Phase III clinical trial tests a new drug D for condition C. The trial has three patient cohorts, and outcomes are evaluated by an independent committee. Due to borderline test results and conflicting biomarker readings, the committee cannot assign crisp success/failure judgments to all patients.

Comparison 12.1 — Trial Data Encoding

Classical vs. Neutrosophic Approach. Classical statistics: each patient is coded as ‘success’ (1) or ‘failure’ (0), with borderline cases resolved by committee vote or excluded. This discards information and inflates variance. Neutrosophic approach: each patient outcome is coded (T, I, F) where T = degree of therapeutic success, I = degree of clinical indeterminacy (e.g. partial response, borderline biomarker), F = degree of failure. No information is discarded.

Data Table (Cohort A, n=5 representative patients): Patient

T (success)

I (indeterminate)

F (failure)

Classical code

P01

0.85

0.10

0.05

Success (1)

P02

0.40

0.45

0.15

Excluded

P03

0.20

0.15

0.65

Failure (0)

P04

0.70

0.25

0.05

Success (1)*

P05

0.55

0.30

0.15

Disputed

Neutrosophic trial success rate for Cohort A: NP_success = mean(T_i) = (0.85+0.40+0.20+0.70+0.55)/5 = 0.54 NP_indeterminate = mean(I_i) = (0.10+0.45+0.15+0.25+0.30)/5 = 0.25 NP_failure = mean(F_i) = (0.05+0.15+0.65+0.05+0.15)/5 = 0.21

The neutrosophic summary (0.54, 0.25, 0.21) tells regulators: 54% clear success, 25% indeterminate (requiring follow-up), 21% failure. Classical coding would report 40% success (2/5), 20% failure (1/5), with 40% excluded — a very different, and less informative, picture. The I-component of 0.25 triggers a protocol for extended monitoring of borderline patients rather than forcing a binary decision.

13.2 12.2 Case Study II: Sensor Fusion Under Off-Domain Readings

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An autonomous vehicle uses four sensors (radar, lidar, camera, ultrasonic) to estimate obstacle proximity. Each sensor outputs a confidence score. In adverse conditions (glare, rain, sensor malfunction), readings may be outside normal [0,1] bounds.

Example 12.2 — Autonomous Vehicle Sensor Fusion

Off-Measure Sensor Fusion. Normal operating range: confidence in [0,1]. Adverse readings: lidar returns 1.3 (super-confident due to calibration overshoot), camera returns -0.1 (active counter-confidence: detects own lens obstruction and flags that its reading should reduce overall confidence). Uncertain Off-Measure encoding: Obstacle = (T=1.3, I=0.1, F=-0.1) in [phi,psi]=[-0.2, 1.5]. OffNorm fusion of four sensors: T_fused = T_off(1.3, 0.9, 0.7, -0.1) = min(1.3, 0.9, 0.7, -0.1)

= -0.1 using OffMinimum (conservative fusion). This correctly triggers a 'sensor failure' alert since one sensor actively contradicts the reading. Classical fusion (clamped to [0,1]): T_fused = (1.0 +

13.3 0.9 + 0.7 + 0.0)/4 = 0.65 — a dangerously misleading 'moderate confidence' that hides the sensor

failure.

13.4 12.3 Case Study III: Environmental Risk — Refined Uncertain Measure

An environmental agency assesses the contamination risk of a coastal site. Three independent labs provide measurements on different biomarkers. Truth-membership in 'contaminated' is refined into T_1 (chemical contamination) and T_2 (biological contamination). Indeterminacy is refined into I_1 (equipment calibration uncertainty) and I_2 (seasonal variability). Falsity F_1 (non-contaminated).

Definition 12.1 — Refined Site Risk Encoding

Refined Uncertain Risk Vector for the site: (T_1, T_2, I_1, I_2, F_1) = (0.75, 0.60, 0.18, 0.12, 0.20). Interpretation: chemical contamination strongly supported (T_1=0.75); biological contamination moderately supported (T_2=0.60); equipment calibration contributes I_1=0.18 uncertainty; seasonal variation contributes I_2=0.12 uncertainty; non-contamination evidence F_1=0.20. A classical single-valued risk score would collapse this to 0.675 (average of T_1, T_2), losing the distinction between chemical and biological risk — critical information for remediation strategy.

13.5 12.4 Case Study IV: Financial Portfolio — Neutrosophic Off-Probability

A portfolio manager assigns neutrosophic probabilities to four asset categories under extreme market conditions. In stress testing, certain assets can exhibit 'super-hedging' (T > 1: provides more protection than expected) or 'anti-performance' (F < 0: the asset actively amplifies loss beyond its stated risk). Asset

T (performs)

I (uncertain)

F (underperforms)

Domain

Treasury bonds

1.15

0.05

0.00

[0, 1.5]

Technology stocks

0.60

0.25

0.15

[0, 1]

Commodities hedge

0.80

0.10

-0.10

[-0.2, 1]

Crypto derivative

0.30

0.50

0.20

[0, 1]

Treasury bonds with T=1.15 indicate a super-hedge: in the stress scenario, they outperform even the most optimistic projection (flight-to-safety effect). The commodities hedge with F=-0.10 signals that in this scenario it not only fails to hedge but reduces overall portfolio stability — a negative performance beyond zero. Only the Uncertain Off-Measure framework captures both effects within a single coherent mathematical structure.

14 Part XIII

Neutrosophic Statistics and Estimation Theory Classical statistics assumes data are precise real numbers. Neutrosophic statistics, introduced by Smarandache (2014, extended 2022), generalises classical statistics to datasets where observations are neutrosophic numbers — values carrying inherent indeterminacy. This is the statistical companion to neutrosophic measure.

14.1 13.1 Neutrosophic Numbers and Data

Definition 13.1 — Neutrosophic Number (Smarandache, 2014)

Neutrosophic Number. A neutrosophic number has the form N = a + bI where a is the determinate part, b is the indeterminate coefficient, and I in [I_L, I_U] is an indeterminacy interval. Example: N = 5 + 2I with I in [0.1, 0.4] represents a value in [5.2, 5.8]. When I = 0, N reduces to the classical number a. Neutrosophic arithmetic: (a + bI) + (c + dI) = (a+c) + (b+d)I; (a + bI)(c + dI) = ac + (ad + bc + bd)I (since I.I = I in neutrosophic arithmetic).

Definition 13.2 — Neutrosophic Dataset

Neutrosophic Dataset. A dataset {x_1, ..., x_n} is neutrosophic if each x_i = a_i + b_i.I_i for intervals I_i in [I_L_i, I_U_i]. This models: measurement imprecision (I = measurement error interval), linguistic uncertainty (I = range of interpretations), sensor indeterminacy (I = calibration uncertainty band).

14.2 13.2 Neutrosophic Descriptive Statistics

Definition 13.3 — Neutrosophic Mean

Neutrosophic Mean. For dataset {x_i = a_i + b_i.I}, i=1..n: x_N_bar = (1/n) Sum(a_i) + (1/n) Sum(b_i).I = A_bar + B_bar.I where A_bar = (1/n) Sum(a_i) is the classical mean of determinate parts, B_bar.I is the mean indeterminate component. The neutrosophic mean lies in the interval [A_bar + B_bar.I_L, A_bar + B_bar.I_U].

Definition 13.4 — Neutrosophic Variance and Standard Deviation

Neutrosophic Variance. S_N^2 = S_a^2 + S_b^2.I where S_a^2 = (1/(n-1)) Sum((a_i - A_bar)^2) is the classical variance of determinate parts and S_b^2.I captures the variance contributed by indeterminacy. The neutrosophic standard deviation S_N = sqrt(S_a^2 + S_b^2.I) is an interval.

14.3 13.3 Neutrosophic Hypothesis Testing

Classical hypothesis testing (Student t-test, chi-squared test) produces crisp p-values. Neutrosophic hypothesis testing produces neutrosophic p-values — intervals that reflect genuine uncertainty about the test conclusion.

Definition 13.5 — Neutrosophic t-test

Neutrosophic t-test. For neutrosophic sample {x_i = a_i + b_i.I}: Compute neutrosophic t-statistic: t_N = (x_N_bar - mu_0) / (S_N / sqrt(n)) = (A_bar - mu_0 + B_bar.I) / ((S_a + S_b.I) / sqrt(n)). This yields an interval t_N in [t_L, t_U]. Decision: if entire interval [t_L, t_U] exceeds critical value t_alpha,n-1 -> reject H_0; if entire interval falls below -> accept H_0; if t_alpha falls inside [t_L, t_U] -> neutrosophically indeterminate (neither reject nor accept).

Theorem 13.1 — Classical Reduction

Reduction to Classical. When I_L = I_U = 0 (no indeterminacy), the neutrosophic t-test reduces exactly to the classical Student t-test. The indeterminate case [t_L, t_U] with t_alpha in the interior corresponds to Walley's (1991) notion of 'prior-data conflict' in imprecise probability, and to the 'inconclusive' region in sequential probability ratio tests.

Example 13.1 — Neutrosophic Blood Pressure Test

Blood Pressure Study. n=30 patients, neutrosophic blood pressure measurements: x_bar_N = 128 + 4I mmHg, I in [0.05, 0.2] (reflects uncertainty in home vs. clinic readings). Neutrosophic standard deviation: S_N = 12 + 3I. Test H_0: mu = 120 mmHg at alpha=0.05 (t_crit = 2.045 for df=29). t_N = (128 + 4I - 120) / ((12 + 3I)/sqrt(30)) = (8 + 4I) / (2.19 + 0.55I). t_L = 8/2.74 = 2.92 (I=0.05 extreme), t_U = 8.8/2.30 = 3.83 (I=0.2 extreme). Since [2.92, 3.83] > 2.045 entirely, we reject H_0 neutrosophically: the indeterminacy does not change the conclusion. Had t_L < 2.045 < t_U, the conclusion would be neutrosophically indeterminate.

14.4 13.4 Neutrosophic Confidence Intervals

Definition 13.6 — Neutrosophic Confidence Interval

Neutrosophic Confidence Interval. A 100(1-alpha)% neutrosophic confidence interval for mu is: CI_N = [x_N_bar - t_(alpha/2) . S_N/sqrt(n), x_N_bar + t_(alpha/2) . S_N/sqrt(n)]. Since x_N_bar and S_N are intervals, CI_N is an interval of intervals — a neutrosophic interval. This represents not just sampling uncertainty (classical width) but also measurement indeterminacy (neutrosophic component).

14.5 13.5 Comparison: Classical vs. Neutrosophic Statistics

Feature

Classical Statistics

Neutrosophic Statistics

Data type

Crisp numbers

Neutrosophic numbers a + bI

Mean

Single value x_bar

Interval [x_L, x_U]

Variance

Single value S^2

Neutrosophic S_N^2

p-value

Single value in (0,1)

Interval [p_L, p_U]

Test decision

Reject / Accept

Reject / Accept / Indeterminate

Confidence interval

Fixed-width interval

Neutrosophic interval

Indeterminacy

Absorbed as error

Explicitly modeled as I

Data exclusion

Outliers removed

Kept as high-I observations

Reduces to classical

—

When I_L = I_U = 0

Part XIV

Historical Notes and Philosophical Foundations

14.6 14.1 The Historical Development of Measure Theory

The trajectory from classical to uncertain measure spans more than a century of intellectual development. Understanding this trajectory situates the neutrosophic and refined uncertain frameworks within the broader history of mathematical thought about size, probability, and indeterminacy. Year

Author(s)

Contribution

1854

Riemann

Riemann integral; integration via partition of domain

1875

Cantor

Set theory; foundation for measure-theoretic thinking

1898

Borel

Borel sets; first countably additive measure on R

1901

Lebesgue

Lebesgue integral; measure via partition of range

1906

Vitali

Non-measurable set; limits of Lebesgue measure

1914

Hausdorff

Hausdorff measure; fractional dimensions

1933

Kolmogorov

Axiomatic probability as measure; sigma-algebra foundation

1953

Choquet

Theory of capacities; first non-additive measures

1965

Zadeh

Fuzzy sets; membership in [0,1]; vagueness formalised

1974

Sugeno

Fuzzy measure and fuzzy integral; non-additive aggregation

1976

Shafer

Mathematical Theory of Evidence; belief functions

1978

Zadeh

Possibility theory; fuzzy measures for ordinal uncertainty

1983

Walley

Imprecise probabilities; credal sets; robust Bayesianism

1986

Atanassov

Intuitionistic fuzzy sets; membership + non-membership

1995

Smarandache

Neutrosophy; neutrosophic logic, set, and probability

1996

Dubois/Prade

Possibility measures as special fuzzy measures

2004

Dezert/Smarandache

DSmT; fusion of highly conflicting sources

2013

Smarandache

Neutrosophic measure, integral, and probability

2016

Smarandache

Neutrosophic Overset/Underset/Offset; beyond [0,1]

2017

Smarandache

Plithogenic set; multi-attribute uncertainty

2025

Smarandache

Uncertain Over/Under/Off operators; OffNorm

2026

Smarandache

Refined Uncertain Set; maximal generalisation

14.7 14.2 Philosophical Foundations: The Status of Indeterminacy

The central philosophical question this monograph addresses is: what is the ontological status of indeterminacy? Three positions have historically competed: Epistemic view (Laplace, Kolmogorov): Indeterminacy is ignorance. A sufficiently informed agent has no uncertainty. Probability measures our ignorance, not nature's indeterminacy. This view underpins classical Bayesian statistics. Aleatory view (frequentists, quantum mechanicists): Indeterminacy is irreducible. Even a fully informed agent cannot predict the outcome of a quantum measurement or a fair coin toss. Probability measures objective chance. Neutrosophic view (Smarandache): Indeterminacy is a third, independent category — neither ignorance nor irreducible chance, but structural non-resolution. A proposition may be simultaneously partially true, partially indeterminate, and partially false, and all three components carry independent information. I cannot be reduced to 1-T-F. The neutrosophic view has empirical consequences: it predicts that some real systems cannot be characterised by any classical probability distribution without loss of information (the soccer example: T+I+F > 1 is possible). This is analogous to the way that quantum mechanics predicted phenomena (entanglement, Bell inequality violations) that no classical probability model can accommodate.

14.8 14.3 The [0,1] Convention: A Historical Accident?

The restriction of probability and membership to [0,1] is so deeply embedded in the mathematical tradition that it appears necessary. But its origins are largely conventional: - Kolmogorov (1933) chose P(Omega)=1 as a normalisation convention, not a logical necessity. The axiom could equally read P(Omega) = c for any positive c. - Zadeh (1965) chose [0,1] for fuzzy membership because it is the natural range for a 'degree', and because it makes arithmetic convenient. - In quantum mechanics, Feynman (1987) explicitly argued for 'negative probability' as a legitimate computational tool in path integrals — an early precursor to the Under-Measure concept. - In economics, Arrow-Debreu state prices can be negative (arbitrage conditions violated), corresponding formally to an Under-Measure on states of the world. The Uncertain Off-Measure framework gives these observations a rigorous mathematical home. Rather than treating values outside [0,1] as errors to be corrected, it treats them as carriers of genuine structural information about the system being modelled.

14.9 14.4 Neutrosophy as a Philosophy

Neutrosophy, founded by Florentin Smarandache in 1995, is a branch of philosophy studying the origin, nature, and scope of neutralities — the third category between and beyond opposites. Its central thesis:

every idea has an opposite and a neutral that is neither nor but related to both. In the context of measure theory: = the event has a determinate measure (T component). = the event definitely does not have that measure (F component). = the measure of the event is indeterminate, neither T nor F (I component). This philosophical tripartition maps directly onto the neutrosophic measure triple (T, I, F), giving the mathematical framework a coherent philosophical foundation. The independence of I from T and F is not a mathematical convenience: it reflects the philosophical claim that genuine indeterminacy is irreducible to any combination of truth and falsity.

Part XV

The Neutrosophic Integral: A Deeper Treatment The Lebesgue integral is the backbone of classical measure theory. Part II introduced the neutrosophic integral briefly. This Part develops it more thoroughly, with convergence theorems, the connection to Choquet integration, and worked examples.

14.10 15.1 Definition and Basic Properties

Definition 15.1 — Neutrosophic Integral

Neutrosophic Integral (Extended Definition). Let (X, Sigma, mu_N) be a neutrosophic measure space with mu_N = (mu_T, mu_I, mu_F). For a neutrosophic measurable function f_N = (f_T, f_I, f_F) on X: Integral_N f_N d mu_N = (Integral f_T d mu_T, Integral f_I d mu_I, Integral f_F d mu_F) where each component integral is a standard Lebesgue integral on the corresponding sub-space. For a scalar function f: X -> R and neutrosophic measure mu_N: Integral_N f d mu_N = (Integral_D f d mu_T, Integral_Ind f d mu_I, Integral_F f d mu_F) where D, Ind, F are the determinate, indeterminate, and false sub-spaces of X.

Theorem 15.1 — Neutrosophic Monotone Convergence

Neutrosophic Monotone Convergence. Let {f_n} be a sequence of non-negative neutrosophic measurable functions with f_n up f pointwise (component-wise). Then: Integral_N f d mu_N = lim_n Integral_N f_n d mu_N where the limit is taken component-wise: (Integral f_T d mu_T, Integral f_I d mu_I, Integral f_F d mu_F) = lim_n (Integral f_n_T d mu_T, Integral f_n_I d mu_I, Integral f_n_F d mu_F).

Theorem 15.2 — Neutrosophic Dominated Convergence

Neutrosophic Dominated Convergence. Let f_n_N -> f_N pointwise (component-wise) and |f_n_N|

<= g_N component-wise with Integral_N g_N d mu_N < (inf, inf, inf). Then: Integral_N f_N d mu_N = lim_n Integral_N f_n_N d mu_N.

14.11 15.2 Indeterminacy in the Integrand vs. Indeterminacy in the Measure

A crucial distinction, first identified by Smarandache (2013), concerns where the indeterminacy originates:

Comparison 15.1 — Two Types of Integral Indeterminacy

Type 1 — Indeterminacy in the Function. f_N: X -> R is itself indeterminate on a subset Y subset X. For example, f may be the temperature function on a building, and on the corridor Y its value is unknown. Then Integral_N f d mu = (Integral_{XY} f d mu, Integral_Y c d mu, 0) where c is an indeterminate constant. The I-component of the integral captures the unknown contribution of Y. Type 2 — Indeterminacy in the Measure. The measure mu_N itself has an indeterminate component mu_I on a subset Z. The function f is fully known. Then Integral_N f d mu_N has an I-component = Integral_Z f d mu_I, reflecting that the 'size' of Z is indeterminate even though f is known there.

Example 15.1 — Neutrosophic Temperature Integral

Example: Temperature Integral. X = [0,10] metres (a corridor). Temperature f(x) = 20 + 2x degrees (known). Neutrosophic measure: mu_N([0,7]) = (7, 0, 0) (determinate); mu_N([7,10]) = (0, 3, 0) (indeterminate — this zone has uncertain occupancy). Integral_N f d mu_N: T-component = Integral_0^7 (20+2x) dx = [20x + x^2]_0^7 = 140 + 49 = 189. I-component = Integral_7^10 (20+2x) dx = [20x + x^2]_7^10 = (200+100)-(140+49) = 111. F-component = 0. Result: (189, 111, 0). Interpretation: the determinate heat load is 189 degree.metres; 111 degree.metres are indeterminate due to uncertain occupancy of [7,10].

14.12 15.3 Connection to the Choquet Integral

The Choquet integral (Part II) integrates a function with respect to a non-additive (fuzzy) measure. The neutrosophic integral relates to it as follows:

Proposition 15.1 — Relation to Choquet Integral

Choquet-Neutrosophic Connection. If mu_N is a neutrosophic measure with mu_I = 0 (no indeterminacy) and mu_F = 1 - mu_T (complementary), then the T-component of the neutrosophic integral reduces to a Lebesgue integral, and if additionally mu_T is non-additive (monotone only), the appropriate extension is the Choquet integral: Integral_N f d mu_N|_{I=0} = (C) Integral f d mu_T. Thus: Lebesgue integral (additive) subset Choquet integral (monotone) subset Neutrosophic integral (triplet-valued, sigma-additive component-wise).

Part XVI

Axiom Systems: A Comparative Overview Each uncertainty theory rests on a set of axioms. This Part places the axiom systems of the major theories side by side, identifying which axioms are shared, which are relaxed, and which are added at each step of the generalisation hierarchy.

14.13 16.1 The Kolmogorov Axioms (Classical Probability)

K1 (Non-negativity): P(A) >= 0 for all A in Sigma. K2 (Normalisation): P(Omega) = 1. K3 (Sigma-additivity): P(Union_n A_n) = Sum_n P(A_n) for pairwise disjoint {A_n}.

14.14 16.2 Fuzzy Measure Axioms (Sugeno)

Retains K1 (non-negativity), replaces K3 with monotonicity, relaxes K2. FM1 (Boundary): g(empty) = 0; g(X) = 1. FM2 (Monotonicity): A subseteq B ==> g(A) <= g(B). FM3 (Continuity) [optional]: g is continuous from below and above. K3 (sigma-additivity) is dropped. This is the key departure from classical measure.

14.15 16.3 Neutrosophic Measure Axioms

Extends classical axioms to three independent components: NM1 (Triplet boundary): mu_N(empty) = (0,0,0). NM2 (Component sigma-additivity): For disjoint {A_n}: mu_N(Union A_n) = (Sum mu_T(A_n), Sum mu_I(A_n), Sum mu_F(A_n)). NM3 (Independence of I): mu_I is not determined by mu_T and mu_F: mu_I != 1 - mu_T - mu_F in general. NM4 (Reduction): When mu_I = 0 everywhere, NM1-NM3 reduce to classical measure axioms.

14.16 16.4 Off-Measure Axioms

OM1 (Extended domain): mu_off: Sigma -> [Omega, Psi]^3 with Omega < 0 < 1 < Psi.

OM2 (Off-additivity): sigma-additivity holds over [Omega, Psi]: mu_off(Union A_n) = Sum mu_off(A_n) component-wise (sums may exceed [0,1]). OM3 (Boundary relaxation): mu_off(empty) = (0,0,0) or (Omega, Omega, Omega) by convention. OM4 (Reduction): When Omega=0, Psi=1, reduces to neutrosophic measure axioms.

14.17 16.5 Axiom Comparison Table

Axiom

Classical

Fuzzy

IF

Neutrosophic

Domain

[0,inf]

[0,1]

[0,1]^2

[0,1]^3

Non-negativity

Yes

Yes

Yes

Per component

Relaxed

Relaxed

Normalisation

Yes (=1)

Yes (=1)

Sum<=1

Optional

Optional

Optional

Sigma-additivity

Yes

No

Monotone

Component-wise

Extended

Extended

Monotonicity

Yes

Yes

Yes

Yes (T-comp)

Partial

Partial

mu+nu<=1

T+I+F free

Free

Free

Complement rule

P(A^c)=1-P(A) g(A^c) free

Off-Measure

Refined Off

[Omega,Psi]^3 [Omega,Psi]^n

Indeterminacy

None

None

pi=1-mu-nu

Independent I

Independent I

I_1..I_r

Over-values (>1)

Possible (mass)

No

No

No

Yes

Yes

Under-values (<0)

No

No

No

No

Yes

Yes

Theorem 16.1 — Monotone Axiom Relaxation

Monotone Generalisation Theorem. Each row in the axiom table is either preserved or relaxed (never strengthened) as we move left to right. This guarantees that every theorem valid in Classical Measure remains valid in all extensions, under appropriate specialisation of the parameters. Conversely, no extension introduces an axiom incompatible with the previous level.

Supplementary References [18] Birkhoff, G. D. (1931). Proof of the ergodic theorem. Proc. Natl. Acad. Sci. USA, 17(12), 656-660. [19] Borel, E. (1898). Lecons sur la theorie des fonctions. Gauthier-Villars. [20] Carathéodory, C. (1914). Über das lineare Mass von Punktmengen. Göttinger Nachrichten, 404-426. [21] Dezert, J. and Smarandache, F. (2004). On the generation of hyper-power sets for the DSmT. In Proc. Fusion 2004, Stockholm. [22] Hausdorff, F. (1914). Grundzüge der Mengenlehre. Veit und Co. [23] Kantorovitch, L. V. and Akilov, G. P. (1982).

Functional Analysis. Pergamon. [24] Kolmogorov, A. N. (1933). Grundbegriffe der Wahrscheinlichkeitsrechnung. Springer. (English trans. 1956, Chelsea.) [25] Lebesgue, H. (1904). Lecons sur l'intégration. Gauthier-Villars. [26] Neutrosophic Statistics: Smarandache, F. (2014). Introduction to Neutrosophic Statistics. Sitech and Education Publishing. [27] Radon, J. (1913). Theorie und Anwendungen der absolut additiven Mengenfunktionen. Sitzungsberichte Akad. Wiss. Wien, 122, 1295-1438. [28] Robinson, A. (1966). Non-Standard Analysis. North-Holland. [29] Smarandache, F. (1995). Neutrosophy: Neutrosophic Probability, Set, and Logic. American Research Press. [30] Smarandache, F. (1998). Unifying Field in Logics: Neutrosophic Logic. American Research Press. [31] Smarandache, F. (2017). Plithogeny, Plithogenic Set, Logic, Probability, and Statistics. Pons Publishing. [32] Smarandache, F. (2022). Neutrosophic Statistics is an Extension of Interval Statistics. Neutrosophic Sets and Systems, 45. [33] Stroock, D. W. (1994). A Concise Introduction to the Theory of Integration. Birkhauser. [34] Walley, P. (1991). Statistical Reasoning with Imprecise Probabilities. Chapman and Hall. [35] Wigner, E. P. (1932). On the quantum correction for thermodynamic equilibrium. Phys. Rev., 40, 749-759.

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15 Part XII — Extended Worked Case Studies

This Part provides end-to-end worked case studies demonstrating the practical application of the uncertainty measure hierarchy. Each study begins with a real-world problem, identifies the appropriate theory, applies the formal framework, and interprets the results.

15.1 12.1 Case Study I: Clinical Trial with Indeterminate Outcomes

A Phase III clinical trial tests a new drug for a condition. Due to borderline test results and conflicting biomarker readings, the committee cannot assign crisp success/failure judgments to all patients.

Comparison 12.1 — Classical vs. Neutrosophic Trial Encoding

> Classical statistics codes each patient as success (1) or failure (0), with borderline cases excluded. This discards information. The neutrosophic approach codes each patient as (T, I, F) where T = degree of therapeutic success, I = clinical indeterminacy, F = degree of failure. For $n = 5$ patients: $\bar{T} = 0.54$, $\bar{I} = 0.25$, $\bar{F} = 0.21$. The I-component of 0.25 triggers a protocol for extended monitoring of borderline patients.

Classical coding would report 40% success, 20% failure, 40% excluded — far less informative.

15.2 12.2 Case Study II: Sensor Fusion Under Off-Domain Readings

An autonomous vehicle uses four sensors. In adverse conditions (glare, rain, malfunction), readings may be outside normal $[0, 1]$ bounds. Lidar returns $T = 1.3$ (calibration overshoot); camera returns $T = -0.1$ (lens obstruction flag — active counter-confidence).

Example 12.2 — Uncertain Off-Measure Sensor Fusion

> OffMinimum fusion: $T_{fused} = \min(1.3, 0.9, 0.7, -0.1) = -0.1$. This correctly triggers a sensor-failure alert since one sensor actively contradicts the reading. Classical fusion (clamped to $[0, 1]$): $T_{fused} = (1.0 + 0.9 + 0.7 + 0.0)/4 = 0.65$ — dangerously misleading, hiding the sensor failure entirely.

15.3 12.3 Case Study III: Environmental Risk — Refined Uncertain Measure

An environmental agency assesses coastal contamination. Truth-membership is refined into T_1 (chemical contamination) and T_2 (biological contamination). Indeterminacy is refined into I_1 (equipment calibration uncertainty) and I_2 (seasonal variability).

Definition 12.1 — Refined Site Risk Encoding

> Refined risk vector: $(T_1, T_2, I_1, I_2, F_1) = (0.75, 0.60, 0.18, 0.12, 0.20)$. A classical single score would collapse this to 0.675, losing the distinction between chemical and biological risk — critical information for remediation strategy.

15.4 12.4 Case Study IV: Financial Portfolio — Off-Probability

Under extreme market stress, certain assets exhibit super-hedging ($T > 1$) or anti-performance ($F < 0$): Treasury bonds $T = 1.15$ (super-hedge, flight-to-safety); commodities hedge $F = -0.10$ (actively amplifies loss). Only the Uncertain Off-Measure captures both within a single structure.

16 Part XIII — Neutrosophic Statistics and Estimation Theory

Classical statistics assumes data are precise real numbers. Neutrosophic statistics (Smarandache, 2014) generalises this to datasets where observations are neutrosophic numbers $N = a + bI$, where $I \in [I_L, I_U]$ is an indeterminacy interval.

16.1 13.1 Neutrosophic Numbers and Descriptive Statistics

Definition 13.1 — Neutrosophic Mean

For dataset $\{x_i = a_i + b_i I\}$: $\bar{x}_N = \bar{A} + \bar{B} \cdot I$ where $\bar{A} = \frac{1}{n}\sum a_i$ and $\bar{B} = \frac{1}{n}\sum b_i$. The neutrosophic mean lies in $[\bar{A} + \bar{B}I_L,\ \bar{A} + \bar{B}I_U]$.

Definition 13.2 — Neutrosophic Variance

$S_N^2 = S_a^2 + S_b^2 \cdot I$ where $S_a^2 = \frac{1}{n-1}\sum(a_i - \bar{A})^2$.

16.2 13.2 Neutrosophic Hypothesis Testing

Definition 13.3 — Neutrosophic t-test

Neutrosophic t-statistic: $t_N = \frac{\bar{x}_N - \mu_0}{S_N/\sqrt{n}} \in [t_L, t_U]$. Decision rule: if $[t_L, t_U] > t_{\alpha,n-1}$ — reject H_0; if $t_{\alpha,n-1} \in (t_L, t_U)$ — **neutrosophically indeterminate** (neither reject nor accept).

Example 13.1 — Blood Pressure Study

$n = 30$ patients, $\bar{x}_N = 128 + 4I$ mmHg, $I \in [0.05, 0.2]$, $S_N = 12 + 3I$. Test $H_0 : \mu = 120$: $t_L = 3.69$, $t_U = 3.83$, $t_{crit} = 2.045$. Since $[3.69, 3.83] > 2.045$ entirely: reject H_0 neutrosophically.

17 Part XIV — Historical Notes and Philosophical Foundations

17.1 14.1 Development Timeline

The trajectory from classical to uncertain measure spans more than a century:

Year	Author	Contribution
1901	Lebesgue	Lebesgue integral and σ-algebra foundation
1933	Kolmogorov	Axiomatic probability as measure
1953	Choquet	Non-additive capacities
1965	Zadeh	Fuzzy sets; $[0, 1]$ membership
1974	Sugeno	Fuzzy measure and Choquet integral
1976	Shafer	Mathematical Theory of Evidence
1986	Atanassov	Intuitionistic fuzzy sets
1995	Smarandache	Neutrosophy; neutrosophic probability
2013	Smarandache	Neutrosophic measure and integral
2016	Smarandache	Neutrosophic Overset/Underset/Offset
2026	Smarandache	Refined Uncertain Set; maximal generalisation

17.2 14.2 Three Philosophical Views of Indeterminacy

1. **Epistemic (Laplace, Kolmogorov):** Indeterminacy is ignorance. P measures lack of knowledge. With complete information, indeterminacy vanishes.
2. **Aleatory (frequentist, quantum):** Indeterminacy is irreducible. Even a fully informed agent cannot predict individual quantum events. P measures objective chance.
3. **Neutrosophic (Smarandache, 1995):** Indeterminacy is a *third, independent category* — neither ignorance nor chance, but structural non-resolution. I is not $1 - T - F$. The three components carry genuinely independent information.

17.3 14.3 The $[0, 1]$ Convention

The restriction $P \in [0, 1]$ is conventional, not logically necessary:

- Kolmogorov chose $P(\Omega) = 1$ as normalisation; any positive constant works

- Feynman (1987) argued for negative probability in path integral calculations
- Arrow-Debreu state prices can be negative (arbitrage violations)

The Uncertain Off-Measure domain $[\phi, \psi]$ with $\phi < 0 < 1 < \psi$ gives these phenomena a rigorous and unified mathematical home.

18 Part XV — The Neutrosophic Integral: Deeper Treatment

18.1 15.1 Two Types of Integral Indeterminacy

Type 1 — Indeterminacy in the function: f_N is indeterminate on $Y \subseteq X$:

$$\int_N f_N \, d\mu = \left(\int_{X \setminus Y} f \, d\mu_T, \; \int_Y c \, d\mu_I, \; 0 \right)$$

Type 2 — Indeterminacy in the measure: μ_N has indeterminate component μ_I on Z:

$$\int_N f \, d\mu_N = \left(\int_D f \, d\mu_T, \; \int_Z f \, d\mu_I, \; \int_F f \, d\mu_F \right)$$

18.2 15.2 Convergence Theorems

Theorem 15.1 — Neutrosophic Monotone Convergence. $f_n \uparrow f$ component-wise:

$$\int_N f \, d\mu_N = \lim_{n\to\infty} \int_N f_n \, d\mu_N \quad \text{(component-wise)}$$

Theorem 15.2 — Neutrosophic Dominated Convergence. If $f_n \to f$ component-wise and $|f_n| \leq g_N$ with $\int_N g_N \, d\mu_N < (\infty, \infty, \infty)$:

$$\int_N f_N \, d\mu_N = \lim_{n\to\infty} \int_N f_{n,N} \, d\mu_N$$

18.3 15.3 Example: Temperature Corridor

$X = [0,10]$ m, $f(x) = 20 + 2x$ (fully known temperature). $\mu_N([0,7]) = (7,0,0)$; $\mu_N([7,10]) = (0,3,0)$ (indeterminate occupancy).

$$T\text{-component} = \int_0^7 (20+2x)\,dx = \left[20x + x^2\right]_0^7 = 189$$

$$I\text{-component} = \int_7^{10} (20+2x)\,dx = 111$$

Result: $(189, 111, 0)$ — determinate heat load 189 degree·m; 111 degree·m indeterminate.

19 Part XVI — Axiom Systems: A Comparative Overview

19.1 16.1 The Six Axiom Systems Side by Side

The following table places the axiom systems of the six major uncertainty theories side by side, showing exactly which axioms are shared, relaxed, or added at each level of generalisation.

Axiom	Classical	Fuzzy	IF	Neutrosophic	Off-Measure	Refined Off
Domain	$[0,\infty]$	$[0,1]$	$[0,1]^2$	$[0,1]^3$	$[\phi,\psi]^3$	$[\Omega,\Psi]^n$
Non-negativity	Yes	Yes	Yes	Per component	Relaxed	Relaxed
Normalisation	$=1$	$=1$	≤ 1	Optional	Optional	Optional
σ-additivity	Strict	None	Monotone	Component-wise	Extended	Extended
Monotonicity	Yes	Yes	Yes	T-component	Partial	Partial
Indeterminacy	None	None	$\pi = 1{-}\mu{-}\nu$	Independent I	Independent I	$I_1, \dots, I_r$

Axiom	Classical	Fuzzy	IF	Neutrosophic	Off-Measure	Refined Off
Over-values > 1	No	No	No	No	**Yes**	**Yes**
Under-values < 0	No	No	No	No	**Yes**	**Yes**

Theorem 16.1 — Monotone Axiom Relaxation. Each row is preserved or relaxed (never strengthened) moving left to right. Every theorem valid in Classical Measure remains valid in all extensions under appropriate parameter specialisation.

19.2 16.2 Selection Guide

Situation	Recommended Theory	Key Reason
Known frequencies	Classical Probability	Kolmogorov axioms exact
Linguistic uncertainty	Possibility Theory	Ordinal, not quantitative
Non-additive weights	Fuzzy Measure + Choquet	Super/sub-additivity
Membership + non-membership	IF Measure	Hesitation margin π
Conflicting sources	DSmT	PCR5/6 handles conflict
Genuine indeterminacy	Neutrosophic Measure	Independent I
Multi-attribute + contradiction	Plithogenic Measure	Contradiction degrees
Sub-types of T, I, F	Refined Uncertain Measure	Component decomposition
Scores exceed $[0, 1]$	Uncertain Off-Measure	$\phi \leq 0,\ \psi \geq 1$
All of the above	Refined Nonstandard Off	Maximal generalisation

20 Open Problems and Future Research Directions

The following open problems identify the frontiers of the theory developed in this monograph. Each represents a direction where rigorous mathematical work remains to be done.

P1. Neutrosophic Choquet Integral. Define a fully rigorous Choquet-type integral $(C) \int f \, d\mu_N$ for neutrosophic measures $\mu_N = (\mu_T, \mu_I, \mu_F)$. Prove convergence theorems analogous to the Monotone and Dominated Convergence Theorems. Characterise when the neutrosophic Choquet integral reduces to the classical Choquet integral.

P2. Radon–Nikodym for Neutrosophic Off-Measures. Develop a neutrosophic Radon–Nikodym theorem: given two neutrosophic measures μ_N, ν_N with $\nu_N \ll \mu_N$ (absolute continuity in each component), characterise the neutrosophic derivative $\frac{d\nu_N}{d\mu_N}$. This would enable a neutrosophic change-of-measure formula, the analogue of the Girsanov theorem in stochastic calculus.

P3. Neutrosophic Ergodic Theory. Extend Birkhoff's Ergodic Theorem to neutrosophic measure-preserving transformations $T : (X, \mu_N) \to (X, \mu_N)$. Define neutrosophic time averages and characterise ergodicity, mixing, and entropy in the neutrosophic framework.

P4. Wigner Quasi-Probability and Off-Measure. Formalise the connection between Wigner quasi-probability distributions $W(x, p)$ (which can be negative) and Uncertain Under-Measures ($\phi < 0$). Determine whether PCT and CPT invariance in quantum field theory can be reformulated using neutrosophic or off-measures.

P5. DSmT–Neutrosophic Unification. Develop a formal embedding of DSmT fusion operators (PCR5, PCR6) into the neutrosophic measure framework. Characterise which neutrosophic update rules are equivalent to which DSm combination rules.

P6. Neutrosophic Martingales and Finance. Define neutrosophic martingales: stochastic processes $(M_t) = (T_t, I_t, F_t)$ with neutrosophic conditional expectation $E_N[M_t|\mathcal{F}_s] = M_s$ for $s \leq t$. Develop neutrosophic stochastic calculus and a neutrosophic Black–Scholes formula.

P7. Topological and Algebraic Structure. Study the nonstandard interval $]^{-}\Omega, \Psi^{+}[$ as a topological ring. Characterise its Galois group over standard reals. Determine whether Refined Nonstandard Uncertain Off-Measures form a complete lattice.

P8. Computational Complexity. Determine the complexity of key operations on Refined Uncertain Off-Measures: normalisation, operator application, integration. Develop efficient algorithms for Neutrosophic TOPSIS, AHP, and MCDM at scale.

21 Concluding Remarks

This monograph has traced the complete generalisation pathway of measure theory from its classical Lebesgue–Kolmogorov foundation to the most general uncertain measure structures available today. Along the way it has situated each extension within the landscape of uncertainty theories — probability, possibility, evidence, capacity, quantum, and AI uncertainty — demonstrating that each classical framework arises as a special case, a special calibration, or a structural constraint of the neutrosophic and refined uncertain measure hierarchy.

Three deep themes emerge from this synthesis.

First: indeterminacy is not noise. Classical theories treat everything that is not probability as residual uncertainty to be absorbed, redistributed, or discarded. Neutrosophic measure refuses this: I is a first-class component with its own axioms, its own carriers, its own contribution to every theorem and every application.

Second: the boundary $[0, 1]$ is a convention, not a law. Real systems — employees, quantum states, financial hedges, sensor readings — produce values outside this interval. The Off-Measure framework gives them a rigorous home without ad hoc clipping or normalisation.

Third: refinement is information. Collapsing T into a single number discards the semantic distinction between evidence from source 1 and evidence from source 2. Refined Uncertain Measure preserves it, making auditable, interpretable uncertainty quantification possible.

The open problems listed above are not decorative. A neutrosophic Choquet integral would give a new non-additive expectation theory. A neutrosophic Radon–Nikodym theorem would give a new change-of-measure calculus. A neutrosophic Girsanov theorem would give a new financial mathematics. These are invitations to reconstruct, from the ground up, the entire edifice of measure-theoretic probability on foundations that are honest about indeterminacy from the first axiom.

22 Selected References

[1] Atanassov, K. (1986). Intuitionistic fuzzy sets. *Fuzzy Sets and Systems*, 20(1), 87–96. [2] Bell, J. S. (1964). On the Einstein-Podolsky-Rosen paradox. *Physics*, 1(3), 195–200. [3] Birkhoff, G. D. (1931). Proof of the ergodic theorem. *PNAS*, 17(12), 656–660. [4] Choquet, G. (1953). Theory of capacities. *Annales de l'Institut Fourier*, 5, 131–295. [5] Dezert, J. and Smarandache, F.

(2004). *Advances and Applications of DSmT*. American Research Press. [6] Dubois, D. and Prade, H. (1988). *Possibility Theory*. Plenum Press. [7] Feynman, R. P. (1987). Negative probability. In *Quantum Implications*, Routledge. [8] Kolmogorov, A. N. (1933). *Grundbegriffe der Wahrscheinlichkeitsrechnung*. Springer. [9] Lebesgue, H. (1904). *Lecons sur l'integration*. Gauthier-Villars. [10] Robinson, A. (1966). *Non-Standard Analysis*. North-Holland. [11] Shafer, G. (1976). *A Mathematical Theory of Evidence*. Princeton University Press. [12] Smarandache, F. (1995). *Neutrosophy*. American Research Press. [13] Smarandache, F. (2013). *Introduction to Neutrosophic Measure, Integral, and Probability*. Sitech. [14] Smarandache, F. (2014). *Introduction to Neutrosophic Statistics*. Sitech. [15] Smarandache, F. (2016). *Neutrosophic Overset, Underset, and Offset*. Pons Editions. [16] Smarandache, F. (2017). *Plithogeny, Plithogenic Set, Logic, Probability, and Statistics*. Pons. [17] Smarandache, F. (2025). Operators for Uncertain Over/Under/Off-Sets. *NSS*, 79. [18] Smarandache, F. (2026). Refined Uncertain Set. *Neutrosophic Knowledge*, 11. [19] Sugeno, M. (1974). *Theory of Fuzzy Integrals*. PhD Thesis, Tokyo Institute of Technology. [20] Walley, P. (1991). *Statistical Reasoning with Imprecise Probabilities*. Chapman and Hall. [21] Wigner, E. P. (1932). Quantum correction for thermodynamic equilibrium. *Phys. Rev.*, 40, 749. [22] Zadeh, L. A. (1965). Fuzzy sets. *Information and Control*, 8(3), 338–353. [23] Zadeh, L. A. (1978). Fuzzy sets as a basis for a theory of possibility. *FSS*, 1(1), 3–28.

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ABOUT THIS BOOK

This monograph traces a complete, rigorous progression from classical measure theory to the most general uncertain measure structures known today. Beginning with Lebesgue's sigma-additive framework, it extends successively through fuzzy measures, intuitionistic fuzzy measures, neutrosophic measures, plithogenic measures, and their refined and nonstandard variants.

The central contribution is the unified treatment of uncertain Over-/Under-/Off-Measures: measure functions whose values exceed the classical interval [0, 1], reaching into the extended domain [Omega, Psi] with Omega < 0 and Psi > 1. The final Part V synthesises both refinement (decomposition of truth, indeterminacy, and falsity into sub-components) and over/under/off extension, yielding the Refined Nonstandard Uncertain Off-Measure as the maximal generalisation.

Each Part supplies formal definitions, theorems with proof sketches, worked examples, and concrete applications in medicine, AI, finance, environmental science, and quantum systems. The monograph is self-contained and accessible to graduate students in mathematics, computer science, and engineering.

KEY CONCEPTS

Classical Measure:	sigma-additive function mu: Sigma -> [0, +inf]
Fuzzy Measure:	Monotone, non-additive; Choquet integral (Sugeno, 1974)
Neutrosophic Measure:	Triple (T, I, F) with independent indeterminacy component
Plithogenic Measure:	Multi-attribute appurtenance with contradiction degrees
Refined Uncertain Measure:	Sub-components (T1..Tp, I1..Ir, F1..Fs) in [0,1]^n
Over-/Under-/Off-Measure:	Components in [Omega, Psi] with Omega < 0 < 1 < Psi
OffNorm / OffConorm:	Extended t-norm / t-conorm over [Omega, Psi]
Refined Off-Measure:	Maximal synthesis: refinement + over/under/off + nonstandard

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