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Aims and Scope

Neutrosophic theory and its applications have been expanding in all directions at an astonishing rate especially after of the introduction the journal entitled "Neutrosophic Sets and Systems". New theories, techniques, algorithms have been rapidly developed. One of the most striking trends in the neutrosophic theory is the hybridization of neutrosophic set with other potential sets such as rough set, bipolar set, soft set, hesitant fuzzy set, etc. The different hybrid structures such as rough neutrosophic set, single valued neutrosophic rough set, bipolar neutrosophic set, single valued neutrosophic hesitant fuzzy set, etc. are proposed in the literature in a short period of time. Neutrosophic set has been an important tool in the application of various areas such as data mining, decision making, e-learning, engineering, law, medicine, social science, and some more.

(Editors)

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Preface

Neutrosophy, introduced by Dr. Florentin Smarandache in the early 1990s, extends classical logic by incorporating indeterminacy and partial truth. This gives rise to "neutrosophic sets" and "neutrosophic logic," which allow for the modeling of uncertainty and inconsistency—phenomena often encountered in real-world systems. These concepts have led to the development of neutrosophic algebraic structures, which extend traditional algebra to handle issues of ambiguity, indeterminacy, and vagueness.

This book explores the emerging field of Neutrosophic Algebraic Structures, focusing on both their theoretical foundations and practical applications. We apply innovative algorithmic methods to investigate the complex interactions of neutrosophic elements, such as neutrosophic numbers, sets, and functions, within algebraic systems. Our goal is to show how neutrosophic structures challenge and expand traditional algebraic approaches, offering solutions to problems across diverse fields like computer science, engineering, artificial intelligence, and decision-making.

A key theme of this work is the integration of mathematical theory with computational methods. Neutrosophic logic provides more accurate and flexible models for problems involving uncertainty and imprecision, making it a valuable tool in areas where classical approaches fall short. The algorithms introduced here aim to apply neutrosophic theory to solve complex problems that are difficult to address using traditional techniques.

We invite readers to explore the synergy between abstract mathematical theory and cutting-edge computational algorithms. Our aim is to inspire further research in this exciting and rapidly evolving field and demonstrate its relevance in solving real-world problems that involve ambiguity, uncertainty, and indeterminacy.

We hope this work will serve as a valuable resource for students, researchers, and practitioners, contributing to the growing body of knowledge at the intersection of algebraic structures, logic, and modern computational techniques.

http://fs.unm.edu/neutrosophy.htm.

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Chapter 1

Multiple Generalized Set-Valued Neutrosophic Quintuple Sets

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ABSTRACT

In this chapter, the definition and basic properties of multiple generalized set-valued neutrosophic quintuple sets are given. Furthermore, some decision operators (average union, average intersection, optimistic union, optimistic intersection, pessimistic union, pessimistic intersection) for multiple generalized set-valued neutrosophic quintuple sets are defined and examples of these operators are given. Thus, a new structure was obtained by using multiple neutrosophic theory and neutrosophic quintuple theory together.

Keywords: Multi-valued Neutrosophic Sets, Multiple Generalized Set-Valued Neutrosophic Quadruple Sets, Multiple Generalized Set-Valued Neutrosophic Quintuple Sets, Generalized Set-Valued Neutrosophic Quintuple Numbers

1.INTRODUCTION

Classical logic, fuzzy logic [1], intuitionistic logic [2] and neutrosophy [3] are systems of logic that emerged and developed at different stages of this progression. Ancient Greek thinkers, especially Aristotle, developed the basic principles of logic and built classical logic. Aristotle's logic has a mathematical structure based on precise concepts, truth and falsity. Classical logic is known as bivalent logic because propositions are only considered true or false. Developed in 1965 by Lotfi Zadeh [1], fuzzy logic is designed to handle uncertainty and ambiguous information in the real world. Neutrosophy [3] is a theory of logic proposed by Florentin Smarandache in the late 20th century. Neutrosophy is a more complex and versatile logic system that extends fuzzy logic to deal with uncertainty. As a result, neutrosophy is an important step in the development of these logic systems for dealing with complexity and uncertainty, because it directly combines fuzzy and intuitionistic logic systems, offering a more comprehensive perspective. Also, researchers studied this theory [4-19, 28-63]. In 2016, Chatterjee et al. defined quadripartioned neutrosophic sets [20]. Unlike neutrosophic sets, it is defined with a contradiction function and an ambiguity function instead of an uncertainty function. In 2015, Peng and Wang obtained multiple neutrosophic sets [21] and used them in a multi-criteria decision-making application. Also, researchers studied based on this set [22, 23]. Smarandache defined the neutrosophic quadruple set (NQS) [24] in 2015. In general, a neutrosophic quadruple number is of the form (k, lT, mI, nF) (k, l, m, n ∈ R or C). The components T, I and F in this notation are components in neutrosophic logic. However, unlike the NS, NQS has a known part (k) and an unknown part ((lT, mI, nF)). Şahin et al. defined generalized set-valued neutrosophic quadruple numbers (GSVNQN) [25] in 2020. Thanks to this new structure, the NQS theory has become available in the field of applications. Also, in 2023, Şahin et al. defined generalized set-valued neutrosophic quintuple sets (GSVNQS) [26] and some operations on them were defined. Başer and Uluçay [42] defined energy of a neutrosophic soft set and applied it to multi-criteria decision-making problems to show its effectiveness. Başer and Uluçay [46] defined effective q- fuzzy soft expert sets. The above-mentioned theories have been studied in various fields, such as: [64-87]. Recently, Kargın et al. obtained multiple generalized set-valued neutrosophic quintuple sets (MGSVNQS) [27].

The chapter is organized as follows: In The background Section, the basic definitions, and properties to be used in this chapter are indroduced for SVNS, SVQNS, MNS, GSVNQS and MGSVNQS. In the research and findings section, we define the MGSVNQPN and MGSVNQPS and the main features of these new concepts are given. Thus, we have obtained a new structure for SVQNS, GSVNQPN. Optimistic union, pessimistic union, average union, pessimistic intersection, optimistic intersection, average intersection operators were defined for MGSVNQPS. Sample sets from daily life were created and operations were performed on the sets for all the for ementioned properties and definitions. In The Conclusion Section, the results obtained in the study and suggestions for future studies are presented.

2.BACKGROUND

Definition 1 [18] Let X be a universal set. For $\forall \tilde{d}_{\beta} \in X$,

$$0 \leq T^{\widetilde{K}}_{(\tilde{d}_\beta)} + I^{\widetilde{K}}_{(\tilde{d}_\beta)} + F^{\widetilde{K}}_{(\tilde{d}_\beta)} \leq 3$$

$$T^{\widetilde{K}}: X \to [0,1]\ ,\ I^{\widetilde{K}}: X \to [0,1]\ \text{ and } F^{\widetilde{K}}: X \to [0,1]$$

With functions, a $\widetilde{K}$ single-valued NS on X; It is defined as

$$\widetilde{K} = \left\{\langle \tilde{d}_\beta, T^{\widetilde{K}}_{(\tilde{d}_\beta)}, I^{\widetilde{K}}_{(\tilde{d}_\beta)}, F^{\widetilde{K}}_{(\tilde{d}_\beta)} \rangle : \tilde{d}_\beta \in X\right\}.$$

$T^{\widetilde{K}}_{(\tilde{d}_\beta)}, I^{\widetilde{K}}_{(\tilde{d}_\beta)}, F^{\widetilde{K}}_{(\tilde{d}_\beta)}$ are the degree of truth, indeterminancy and falsity of $\tilde{d}_\beta \in X$, respectively.

Definition 2 [21] Let E be an universal set, a MVNS $\widetilde{K}$ over the set E can be defined as follows.

For $\forall x_j \in E$; i=1,...,p and β =1,...,n;

$$\widetilde{K} = \{\langle \tilde{d}_\beta, \left({T^{\widetilde{K}}_{(\tilde{d}_\beta)}}^1, {T^{\widetilde{K}}_{(\tilde{d}_\beta)}}^2, \ldots, {T^{\widetilde{K}}_{(\tilde{d}_\beta)}}^P\right), \left({I^{\widetilde{K}}_{(\tilde{d}_\beta)}}^1, {I^{\widetilde{K}}_{(\tilde{d}_\beta)}}^2, \ldots, {I^{\widetilde{K}}_{(\tilde{d}_\beta)}}^P\right), \left({F^{\widetilde{K}}_{(\tilde{d}_\beta)}}^1, {F^{\widetilde{K}}_{(\tilde{d}_\beta)}}^2, \ldots, {F^{\widetilde{K}}_{(\tilde{d}_\beta)}}^P\right)\rangle : \tilde{d}_\beta \in E\}$$

$${T^{\widetilde{K}}_{(\tilde{d}_\beta)}}^1, {T^{\widetilde{K}}_{(\tilde{d}_\beta)}}^2, \ldots, {T^{\widetilde{K}}_{(\tilde{d}_\beta)}}^P : E \to [0,1]\ ,$$

$${I^{\widetilde{K}}_{(\tilde{d}_\beta)}}^1, {I^{\widetilde{K}}_{(\tilde{d}_\beta)}}^2, \ldots, {I^{\widetilde{K}}_{(\tilde{d}_\beta)}}^P : E \to [0,1],$$

$${F^{\widetilde{K}}_{(\tilde{d}_\beta)}}^1, {F^{\widetilde{K}}_{(\tilde{d}_\beta)}}^2, \ldots, {F^{\widetilde{K}}_{(\tilde{d}_\beta)}}^P : E \to [0,1].$$

It is also,

$$0 \leq {T^{\widetilde{K}}_{(\tilde{d}_\beta)}}^i + {I^{\widetilde{K}}_{(\tilde{d}_\beta)}}^i + {F^{\widetilde{K}}_{(\tilde{d}_\beta)}}^İ \leq 3$$

$$\left({T^{\widetilde{K}}_{(\tilde{d}_\beta)}}^1, {T^{\widetilde{K}}_{(\tilde{d}_\beta)}}^2, \ldots, {T^{\widetilde{K}}_{(\tilde{d}_\beta)}}^P\right), \left({I^{\widetilde{K}}_{(\tilde{d}_\beta)}}^1, {I^{\widetilde{K}}_{(\tilde{d}_\beta)}}^2, \ldots, {I^{\widetilde{K}}_{(\tilde{d}_\beta)}}^P\right) \text{ and } \left({F^{\widetilde{K}}_{(\tilde{d}_\beta)}}^1, {F^{\widetilde{K}}_{(\tilde{d}_\beta)}}^2, \ldots, {F^{\widetilde{K}}_{(\tilde{d}_\beta)}}^P\right)$$

The degree of truth, the degree of indeterminacy, the degree of falsity, respectively.

Definition 3. [20] Let X be a universal set. For $\forall \tilde{d}_\beta \in X$,

$$0 \leq T^{\widetilde{K}}_{(\tilde{d}_\beta)} + U^{\widetilde{K}}_{(\tilde{d}_\beta)} + C^{\widetilde{K}}_{(\tilde{d}_\beta)} + F^{\widetilde{K}}_{(\tilde{d}_\beta)} \leq 4$$

$$T^{\widetilde{K}}: X \to [0,1]\ ,\ U^{\widetilde{K}}: X \to [0,1],\ C^{\widetilde{K}}: X \to [0,1]\ \text{ and } F^{\widetilde{K}}: X \to [0,1]$$

With functions, a $\widetilde{K}$ single-valued quadripartitioned NS on X; It is defined as

$$\widetilde{K} = \left\{\langle \tilde{d}_\beta, T^{\widetilde{K}}_{(\tilde{d}_\beta)}, I^{\widetilde{K}}_{(\tilde{d}_\beta)}, F^{\widetilde{K}}_{(\tilde{d}_\beta)} \rangle : \tilde{d}_\beta \in X\right\}.$$

$T^{\widetilde{K}}_{(\tilde{d}_\beta)}, U^{\widetilde{K}}_{(\tilde{d}_\beta)}, C^{\widetilde{K}}_{(\tilde{d}_\beta)}, F^{\widetilde{K}}_{(\tilde{d}_\beta)}$ degrees of truth, degrees of uncertainty, degrees of contradiction and degrees of falsity, respectively of $\tilde{d}_\beta \in X$, respectively.

Definition 4. [26] Let X be a set and let P(X) be the power set of X. $\widetilde{K}_i$ set of a GSVNQN form

$$\widetilde{K} = \left\{\left(\widetilde{K}_{\beta_{1_i}}, \widetilde{K}_{\beta_{2_i}} T^{\widetilde{K}}_{\beta_{2_i}}, \widetilde{K}_{\beta_{3_i}} U^{\widetilde{K}}_{\beta_{3_i}}, \widetilde{K}_{\beta_{4_i}} U^{\widetilde{K}}_{\beta_{4_i}}, \widetilde{K}_{\beta_{5_i}} F^{\widetilde{K}}_{\beta_{5_i}}\right) : \widetilde{K}_{\beta_{1_i}}, \widetilde{K}_{\beta_{2_i}}, \widetilde{K}_{\beta_{3_i}}, \widetilde{K}_{\beta_{4_i}}, \widetilde{K}_{\beta_{5_i}} \in P(X); i = 1,2,3,\dots,n\right\}$$

$T^{\widetilde{K}}_{\beta_{2_i}}, U^{\widetilde{K}}_{\beta_{3_i}}, C^{\widetilde{K}}_{\beta_{4_i}}$ and $F^{\widetilde{K}}_{\beta_{4_i}}$ are the usual quadripartitioned neutrosophic logic tools. Also, a GSVNQS is defined such that

$$\widetilde{K}^N_i = \left(\widetilde{K}_{\beta_{1_i}}, \widetilde{K}_{\beta_{2_i}} T^{\widetilde{K}}_{\beta_{2_i}}, \widetilde{K}_{\beta_{3_i}} U^{\widetilde{K}}_{\beta_{3_i}}, \widetilde{K}_{\beta_{4_i}} U^{\widetilde{K}}_{\beta_{4_i}}, \widetilde{K}_{\beta_{5_i}} F^{\widetilde{K}}_{\beta_{5_i}}\right).$$

Where, a GSVNQN representing an entity that can be a number, an idea, an object, etc. For a GSVNQN $\left(\widetilde{K}_{\beta_{1_i}}, \widetilde{K}_{\beta_{2_i}} T^{\widetilde{K}}_{\beta_{2_i}}, \widetilde{K}_{\beta_{3_i}} U^{\widetilde{K}}_{\beta_{3_i}}, \widetilde{K}_{\beta_{4_i}} U^{\widetilde{K}}_{\beta_{4_i}}, \widetilde{K}_{\beta_{5_i}} F^{\widetilde{K}}_{\beta_{5_i}}\right)$

$$\widetilde{K}_{\beta_{1_i}}$$

is called the known part and

$$\left(\widetilde{K}_{\beta_{2_i}} T^{\widetilde{K}}_{\beta_{2_i}}, \widetilde{K}_{\beta_{3_i}} U^{\widetilde{K}}_{\beta_{3_i}}, \widetilde{K}_{\beta_{4_i}} U^{\widetilde{K}}_{\beta_{4_i}}, \widetilde{K}_{\beta_{5_i}} F^{\widetilde{K}}_{\beta_{5_i}}\right)$$

is called the unknown part.

We can also show that the GSVNQN consisting of GSVNQS

$$\widetilde{K} = \left\{\widetilde{K}^N_i; i = 1,2,\dots,n\right\}.$$

Definition 5. [27] Let E be universal set and P(E) be power set of E. $\ddot{C}$ MGSVNQS over the set E is defined as

$$\ddot{C} = \{\Big\langle\left(M^1_{1(x_i)}, M^2_{1(x_i)}, M^3_{1(x_i)}, \dots, M^n_{1(x_i)}\right),$$
$$\left(M^1_{2(x_i)}, M^2_{2(x_i)}, M^3_{2(x_i)}, \dots, M^n_{2(x_i)}\right)\left(T^1_{M_{2(x_i)}}, T^2_{M_{2(x_i)}}, T^3_{M_{2(x_i)}}, \dots, T^n_{M_{2(x_i)}}\right),$$

$$\left(M^1_{3(x_i)}, M^2_{3(x_i)}, M^3_{3(x_i)}, \dots, M^n_{3(x_i)}\right)\left(I^1_{M_{3(x_i)}}, I^2_{M_{3(x_i)}}, I^3_{M_{3(x_i)}}, \dots, I^n_{M_{3(x_i)}}\right),$$
$$\left(M^1_{4(x_i)}, M^2_{4(x_i)}, M^3_{4(x_i)}, \dots, M^n_{4(x_i)}\right)$$
$$\left(F^1_{M_{4(x_i)}}, F^2_{M_{4(x_i)}}, F^3_{M_{4(x_i)}}, \dots, F^n_{M_{4(x_i)}}\right)\Big\rangle : M^n_{1(x_i)}, M^n_{2(x_i)}, M^n_{3(x_i)}, M^n_{4(x_i)} \in P(E)\}.$$

Here,

i=1,…,j; n=1,…,p

$$T^1_{M_2}, T^2_{M_2}, T^3_{M_2}, \dots, T^n_{M_2}: E \to [0,1],$$

$$I^1_{M_3}, I^2_{M_3}, I^3_{M_3}, \dots, I^n_{M_3}: E \to [0,1],$$

$$F^1_{M_4}, F^2_{M_4}, F^3_{M_4}, \dots, F^n_{M_4} : E \to [0,1]$$

$$0 \le \dots, T^n_{M_{2(x_i)}} + I^n_{M_{3(x_i)}} + F^n_{M_{4(x_i)}} \le 3$$

and

$$T^1_{M_2}(x_j), T^2_{M_2}(x_j), T^3_{M_2}(x_j), \dots, T^n_{M_2}(x_j),$$

$$I^1_{M_3}(x_j), I^2_{M_3}(x_j), I^3_{M_3}(x_j), \dots, I^n_{M_3}(x_j),$$

$$F^1_{M_4}(x_j), F^2_{M_4}(x_j), F^3_{M_4}(x_j) \dots, F^n_{M_4}(x_j)$$

degrees of truth, degrees of indeterminacy, degrees of falsity, respectively.

3.Research and Findings

Definition 6 Let E be universal set and P(E) be power set of E. $\ddot{C}$ MGSVNQPS over the set E is defined as

$$\ddot{C} = \{\langle (M^1_{1(x_i)}, M^2_{1(x_i)}, M^3_{1(x_i)}, \dots, M^n_{1(x_i)}), (M^1_{2(x_i)}, M^2_{2(x_i)}, M^3_{2(x_i)}, \dots, M^n_{2(x_i)}) \left(T^1_{M_{2(x_i)}}, T^2_{M_{2(x_i)}}, T^3_{M_{2(x_i)}}, \dots, T^n_{M_{2(x_i)}}\right),$$

$$(M^1_{3(x_i)}, M^2_{3(x_i)}, M^3_{3(x_i)}, \dots, M^n_{3(x_i)}) \left(U^1_{M_{3(x_i)}}, U^2_{M_{3(x_i)}}, U^3_{M_{3(x_i)}}, \dots, U^n_{M_{3(x_i)}}\right), (M^1_{4(x_i)}, M^2_{4(x_i)}, M^3_{4(x_i)}, \dots, M^n_{4(x_i)})$$

$$\left(C^1_{M_{4(x_i)}}, C^2_{M_{4(x_i)}}, C^3_{M_{4(x_i)}}, \dots, C^n_{M_{4(x_i)}}\right), (M^1_{5(x_i)}, M^2_{5(x_i)}, M^3_{5(x_i)}, \dots, M^n_{5(x_i)}) \left(F^1_{M_{5(x_i)}}, F^2_{M_{5(x_i)}}, F^3_{M_{5(x_i)}}, \dots, F^n_{M_{5(x_i)}}\right)\rangle :$$

$$M^n_{1(x_i)}, M^n_{2(x_i)}, M^n_{3(x_i)}, M^n_{4(x_i)}, M^n_{5(x_i)} \in P(E)\}.$$

Here,

i=1,…,j; n=1,…,p

$$T^1_{M_2}, T^2_{M_2}, T^3_{M_2}, \dots, T^n_{M_2}: E \to [0,1],$$

$$U^1_{M_3}, U^2_{M_3}, U^3_{M_3}, \dots, U^n_{M_3}: E \to [0,1],$$

$$C^1_{M_3}, C^2_{M_3}, C^3_{M_3}, \dots, C^n_{M_3}: E \to [0,1],$$

$$F_{M_4}^1, F_{M_4}^2, F_{M_4}^3, \dots, F_{M_4}^n : E \to [0,1]$$

$$0 \le \dots, T_{M_{2(x_i)}}^n + U_{M_{3(x_i)}}^n + C_{M_{3(x_i)}}^n + F_{M_{4(x_i)}}^n \le 3$$

and

$$T_{M_2}^1(x_j), T_{M_2}^2(x_j), T_{M_2}^3(x_j), \dots, T_{M_2}^n(x_j),$$

$$U_{M_3}^1(x_j), U_{M_3}^2(x_j), U_{M_3}^3(x_j), \dots, U_{M_3}^n(x_j),$$

$$C_{M_3}^1(x_j), C_{M_3}^2(x_j), C_{M_3}^3(x_j), \dots, C_{M_3}^n(x_j)$$

$$F_{M_4}^1(x_j), F_{M_4}^2(x_j), F_{M_4}^3(x_j) \dots, F_{M_4}^n(x_j)$$

degrees of truth, degrees of uncertainty, degrees of contradiction and degrees of falsity, respectively.

Also, in the MGSVNQPS, p is the number of elements of the set and n is the number of components of each element.

Definition 7 Let

$$\ddot{C} = \{\langle \left(M_{1(x_i)}^1, M_{1(x_i)}^2, M_{1(x_i)}^3, \dots, M_{1(x_i)}^n\right), \left(M_{2(x_i)}^1, M_{2(x_i)}^2, M_{2(x_i)}^3, \dots, M_{2(x_i)}^n\right)\left(T_{M_{2(x_i)}}^1, T_{M_{2(x_i)}}^2, T_{M_{2(x_i)}}^3, \dots, T_{M_{2(x_i)}}^n\right),$$

$$\left(M_{3(x_i)}^1, M_{3(x_i)}^2, M_{3(x_i)}^3, \dots, M_{3(x_i)}^n\right)\left(U_{M_{3(x_i)}}^1, U_{M_{3(x_i)}}^2, U_{M_{3(x_i)}}^3, \dots, U_{M_{3(x_i)}}^n\right), \left(M_{4(x_i)}^1, M_{4(x_i)}^2, M_{4(x_i)}^3, \dots, M_{4(x_i)}^n\right)$$

$$\left(C_{M_{4(x_i)}}^1, C_{M_{4(x_i)}}^2, C_{M_{4(x_i)}}^3, \dots, C_{M_{4(x_i)}}^n\right), \left(M_{5(x_i)}^1, M_{5(x_i)}^2, M_{5(x_i)}^3, \dots, M_{5(x_i)}^n\right)\left(F_{M_{5(x_i)}}^1, F_{M_{5(x_i)}}^2, F_{M_{5(x_i)}}^3, \dots, F_{M_{5(x_i)}}^n\right)\rangle :$$

$$M_{1(x_i)}^n, M_{2(x_i)}^n, M_{3(x_i)}^n, M_{4(x_i)}^n, M_{5(x_i)}^n \in P(E)\}.$$

be a MGSVNQPS and let P(E) be the power set of E. A MGSVNQPN $\ddot{C}_1$ is defined as

$$\ddot{C}_1 = \{(M_{1_{S_{x_1}}{}^1}, M_{1_{S_{x_1}}{}^2}, \dots, M_{1_{S_{x_1}}{}^n}), (M_{2_{S_{x_1}}{}^1}, M_{2_{S_{x_1}}{}^2}, \dots, M_{2_{S_{x_1}}{}^n})(T_{S_{x_1}{}^1}(x_1), T_{S_{x_1}{}^2}(x_1), \dots, T_{S_{x_1}{}^n}(x_1)),$$

$$(M_{3_{S_{x_1}}{}^1}, M_{3_{S_{x_1}}{}^2}, \dots, M_{3_{S_{x_1}}{}^n})(U_{S_{x_1}{}^1}(x_1), U_{S_{x_1}{}^2}(x_1), \dots, U_{S_{x_1}{}^n}(x_1)),$$

$$\left(M_{4_{S_{x_1}}{}^1}, M_{4_{S_{x_1}}{}^2}, \dots, M_{4_{S_{x_1}}{}^n}\right)\left(C_{S_{x_1}{}^1}(x_1), C_{S_{x_1}{}^2}(x_1), \dots, C_{S_{x_1}{}^n}(x_1)\right)$$

$$(M_{5_{S_{x_1}}{}^1}, M_{5_{S_{x_1}}{}^2}, \dots, M_{5_{S_{x_1}}{}^n})(F_{S_{x_1}{}^1}(x_1), F_{S_{x_1}{}^2}(x_1), \dots, F_{S_{x_1}{}^n}(x_1))\}.$$

Where, i=1; n=1,…,p.

As in NQN, an MGSVNQPN representing an entity that can be a number, an idea, an object, etc. For

$$\ddot{C}_1$$

$(M_{1_{S_{x_1}}{}^{1}}, M_{1_{S_{x_1}}{}^{2}}, \dots, M_{1_{S_{x_1}}{}^{n}})$ is called the known part and

$$(M_{2_{S_{x_1}}{}^{1}}, M_{2_{S_{x_1}}{}^{2}}, \dots, M_{2_{S_{x_1}}{}^{n}})(T_{S_{x_1}{}^{1}}(x_1), T_{S_{x_1}{}^{2}}(x_1), \dots, T_{S_{x_1}{}^{n}}(x_1)),$$

$$(M_{3_{S_{x_1}}{}^{1}}, M_{3_{S_{x_1}}{}^{2}}, \dots, M_{3_{S_{x_1}}{}^{n}})(U_{S_{x_1}{}^{1}}(x_1), U_{S_{x_1}{}^{2}}(x_1), \dots, U_{S_{x_1}{}^{n}}(x_1)),$$

$$\left(M_{4_{S_{x_1}}{}^{1}}, M_{4_{S_{x_1}}{}^{2}}, \dots, M_{4_{S_{x_1}}{}^{n}}\right)\left(C_{S_{x_1}{}^{1}}(x_1), C_{S_{x_1}{}^{2}}(x_1), \dots, C_{S_{x_1}{}^{n}}(x_1)\right)$$

$$\left(M_{5_{S_{x_1}}{}^{1}}, M_{5_{S_{x_1}}{}^{2}}, \dots, M_{5_{S_{x_1}}{}^{n}}\right)\left(F_{S_{x_1}{}^{1}}(x_1), F_{S_{x_1}{}^{2}}(x_1), \dots, F_{S_{x_1}{}^{n}}(x_1)\right)$$

is called the unknown part.

MGSVNQPS can also be represented in the form of

$\ddot{C} = \{\ddot{C}_i\ ;\ i = 1,2, \dots, j\}$.

Example 8 Let

$\ddot{C}_{n_1} = \{(\{k, x\}), (\{t, m\})(0.3), (\{s, y\})(0.5), (\{z\})(0.1), (\{y, x\})(0.2)\}$

$\ddot{C}_{n_2} = \{(\{s\}), (\{l, m\})(0.2), (\{k, x\})(0.6), (\{t, z\})(0.3), (\{k, x\})(0.3)\}$

$\ddot{C}_{n_3} = \{(\{m, s\}), (\{y, m\})(0.3), (\{l, z\})(0.5), (\{t\})(0.1), (\{m, x\})(0.2)\}$

$\ddot{C}_{n_4} = \{(\{t\}), (\{l, z\})(0.2), (\{y, x\})(0.6), (\{t, m\})(0.3), (\{k, y\})(0.3)\}$

be four GSVNQPS. Now, if we represent these four sets as a single set, we obtain $\ddot{C}_n$ MGSVNQPS such that

$$\ddot{C}_n =$$

$$\{(\{k, x\}, \{s\}, \{m, s\}, \{t\}), (\{t, m\}, \{l, m\}, \{y, m\}, \{l, z\})(0.3,0.2,0.3,0.2),$$

$$(\{s, y\}, \{k, x\}, \{l, z\}, \{y, x\})(0.5,0.6,0.5,0.6), (\{z\}, \{t, z\}, \{t\}, \{t, m\})(0.1,0.3,0.1,0.3),$$

$$(\{y, x\}, \{k, x\}, \{m, x\}, \{k, y\})(0.2,0.3,0.2,0.3)\}$$

Where, thanks to $\ddot{C}_n$ MGSVNQPS, the sets $\ddot{C}_{n_1}, \ddot{C}_{n_2}, \ddot{C}_{n_3}, \ddot{C}_{n_4}$ were expressed as a single set.

Definition 9 Let

$M^{\ddot{C}} =$
$\{\langle (M^1_{1(x_i)}, M^2_{1(x_i)}, M^3_{1(x_i)}, \dots, M^n_{1(x_i)}),$
$(M^1_{2(x_i)}, M^2_{2(x_i)}, M^3_{2(x_i)}, \dots, M^n_{2(x_i)}) \left(T^1_{M_{2(x_i)}}, T^2_{M_{2(x_i)}}, T^3_{M_{2(x_i)}}, \dots, T^n_{M_{2(x_i)}}\right),$

$(M^1_{3(x_i)}, M^2_{3(x_i)}, M^3_{3(x_i)}, \dots, M^n_{3(x_i)}) \left(U^1_{M_{3(x_i)}}, U^2_{M_{3(x_i)}}, U^3_{M_{3(x_i)}}, \dots, U^n_{M_{3(x_i)}}\right),$
$(M^1_{4(x_i)}, M^2_{4(x_i)}, M^3_{4(x_i)}, \dots, M^n_{4(x_i)})$

$\left(C^1_{M_{4(x_i)}}, C^2_{M_{4(x_i)}}, C^3_{M_{4(x_i)}}, \dots, C^n_{M_{4(x_i)}}\right), (M^1_{5(x_i)}, M^2_{5(x_i)}, M^3_{5(x_i)}, \dots, M^n_{5(x_i)}) \left(F^1_{M_{5(x_i)}}, F^2_{M_{5(x_i)}}, F^3_{M_{5(x_i)}}, \dots, F^n_{M_{5(x_i)}}\right)\rangle:$

$M^n_{1(x_i)}, M^n_{2(x_i)}, M^n_{3(x_i)}, M^n_{4(x_i)}, M^n_{5(x_i)} \in P(E)\}.$

and

$N^{\ddot{C}} =$
$\{\langle (N^1_{1(x_i)}, N^2_{1(x_i)}, N^3_{1(x_i)}, \dots, N^n_{1(x_i)}),$
$(N^1_{2(x_i)}, N^2_{2(x_i)}, N^3_{2(x_i)}, \dots, N^n_{2(x_i)}) \left(T^1_{N_{2(x_i)}}, T^2_{N_{2(x_i)}}, T^3_{N_{2(x_i)}}, \dots, T^n_{N_{2(x_i)}}\right),$

$(N^1_{3(x_i)}, N^2_{3(x_i)}, N^3_{3(x_i)}, \dots, N^n_{3(x_i)}) \left(U^1_{N_{3(x_i)}}, U^2_{N_{3(x_i)}}, U^3_{N_{3(x_i)}}, \dots, U^n_{N_{3(x_i)}}\right),$
$(N^1_{4(x_i)}, N^2_{4(x_i)}, N^3_{4(x_i)}, \dots, N^n_{4(x_i)})$

$\left(C^1_{N_{4(x_i)}}, C^2_{N_{4(x_i)}}, C^3_{N_{4(x_i)}}, \dots, C^n_{N_{4(x_i)}}\right), (N^1_{5(x_i)}, N^2_{5(x_i)}, N^3_{5(x_i)}, \dots, N^n_{5(x_i)}) \left(F^1_{N_{5(x_i)}}, F^2_{N_{5(x_i)}}, F^3_{N_{5(x_i)}}, \dots, F^n_{N_{5(x_i)}}\right)\rangle:$

$N^n_{1(x_i)}, N^n_{2(x_i)}, N^n_{3(x_i)}, N^n_{4(x_i)}, N^n_{5(x_i)} \in P(E)\}.$

be two MGSVNQPSs.

i. If the following conditions are satisfied, we say that $M^{\ddot{C}}$ is a subset of $N^{\ddot{C}}$ and denote $M^{\ddot{C}} \subset N^{\ddot{C}}$.

$$M^n_1(x_i) \subset N^n_1(x_i)$$
$$M^n_2(x_i) \subset N^n_2(x_i)$$
$$M^n_3(x_i) \subset N^n_3(x_i)$$
$$M^n_4(x_i) \subset N^n_4(x_i)$$
$$M^n_5(x_i) \subset N^n_5(x_i)$$

and

$$T^n_{M_{2(x_i)}} \leq T^n_{N_{2(x_i)}}$$

$$U^n_{M_{3(x_i)}} \geq U^n_{N_{3(x_i)}}$$

$$C^n_{M_{4(x_i)}} \geq C^n_{N_{4(x_i)}}$$

$$F^n_{M_{5(x_i)}} \geq F^n_{N_{5(x_i)}}$$

i=1,…,j; n=1,…,p.

ii. If the following conditions are satisfied, $M^{\ddot{C}}$ is equal to $N^{\ddot{C}}$ and is denoted as $M^{\ddot{C}} = N^{\ddot{C}}$.

$$M^n_1(x_i) = N^n_1(x_i)$$

$$M^n_2(x_i) = N^n_2(x_i)$$

$$M^n_3(x_i) = N^n_3(x_i)$$

$$M^n_4(x_i) = N^n_4(x_i)$$

$$M^n_5(x_i) = N^n_5(x_i)$$

and

$$T^n_{M_{2(x_i)}} = T^n_{N_{2(x_i)}}$$

$$U^n_{M_{3(x_i)}} = U^n_{N_{3(x_i)}}$$

$$C^n_{M_{4(x_i)}} = C^n_{N_{4(x_i)}}$$

$$F^n_{M_{5(x_i)}} = F^n_{N_{5(x_i)}}$$

i=1,…,j; n=1,…,p.

Definition 10 Let

$M^{\ddot{C}} =$
$\{\langle \left(M^1_{1(x_i)}, M^2_{1(x_i)}, M^3_{1(x_i)}, \dots, M^n_{1(x_i)}\right),$
$\left(M^1_{2(x_i)}, M^2_{2(x_i)}, M^3_{2(x_i)}, \dots, M^n_{2(x_i)}\right) \left(T^1_{M_{2(x_i)}}, T^2_{M_{2(x_i)}}, T^3_{M_{2(x_i)}}, \dots, T^n_{M_{2(x_i)}}\right),$

$\left(M^1_{3(x_i)}, M^2_{3(x_i)}, M^3_{3(x_i)}, \dots, M^n_{3(x_i)}\right) \left(U^1_{M_{3(x_i)}}, U^2_{M_{3(x_i)}}, U^3_{M_{3(x_i)}}, \dots, U^n_{M_{3(x_i)}}\right),$
$\left(M^1_{4(x_i)}, M^2_{4(x_i)}, M^3_{4(x_i)}, \dots, M^n_{4(x_i)}\right)$

$\left(C^1_{M_{4(x_i)}}, C^2_{M_{4(x_i)}}, C^3_{M_{4(x_i)}}, \ldots, C^n_{M_{4(x_i)}}\right), \left(M^1_{5(x_i)}, M^2_{5(x_i)}, M^3_{5(x_i)}, \ldots, M^n_{5(x_i)}\right) \left(F^1_{M_{5(x_i)}}, F^2_{M_{5(x_i)}}, F^3_{M_{5(x_i)}}, \ldots, F^n_{M_{5(x_i)}}\right)\Big\rangle:$

$M^n_{1(x_i)}, M^n_{2(x_i)}, M^n_{3(x_i)}, M^n_{4(x_i)}, M^n_{5(x_i)} \in P(E)\}.$

and

$N^{\ddot{C}} =$
$\{\Big\langle\left(N^1_{1(x_i)}, N^2_{1(x_i)}, N^3_{1(x_i)}, \ldots, N^n_{1(x_i)}\right),$
$\left(N^1_{2(x_i)}, N^2_{2(x_i)}, N^3_{2(x_i)}, \ldots, N^n_{2(x_i)}\right) \left(T^1_{N_{2(x_i)}}, T^2_{N_{2(x_i)}}, T^3_{N_{2(x_i)}}, \ldots, T^n_{N_{2(x_i)}}\right),$

$\left(N^1_{3(x_i)}, N^2_{3(x_i)}, N^3_{3(x_i)}, \ldots, N^n_{3(x_i)}\right) \left(U^1_{N_{3(x_i)}}, U^2_{N_{3(x_i)}}, U^3_{N_{3(x_i)}}, \ldots, U^n_{N_{3(x_i)}}\right),$
$\left(N^1_{4(x_i)}, N^2_{4(x_i)}, N^3_{4(x_i)}, \ldots, N^n_{4(x_i)}\right)$

$\left(C^1_{N_{4(x_i)}}, C^2_{N_{4(x_i)}}, C^3_{N_{4(x_i)}}, \ldots, C^n_{N_{4(x_i)}}\right), \left(N^1_{5(x_i)}, N^2_{5(x_i)}, N^3_{5(x_i)}, \ldots, N^n_{5(x_i)}\right) \left(F^1_{N_{5(x_i)}}, F^2_{N_{5(x_i)}}, F^3_{N_{5(x_i)}}, \ldots, F^n_{N_{5(x_i)}}\right)\Big\rangle:$

$N^n_{1(x_i)}, N^n_{2(x_i)}, N^n_{3(x_i)}, N^n_{4(x_i)}, N^n_{5(x_i)} \in P(E)\}.$

be two MGSVNQPSs.

i. Average ∪ operation for $M^{\ddot{C}}$ *and* $N^{\ddot{C}}$ is defined as

$M^{\ddot{C}} \ \breve{\cup}_A \ N^{\ddot{C}} = \{\langle(((M,N)^1{}_1(x_i), (M,N)^2{}_1(x_i), \ldots, (M,N)^n{}_1(x_i)),$

$((M,N)^1{}_2(x_i), (M,N)^2{}_2(x_i), \ldots, (M,N)^n{}_2(x_i)) \ (T^1_{(M,N)_2(x_i)}, T^2_{(M,N)_2(x_i)}, \ldots,$
$T^n_{(M,N)_2(x_i)}),$

$((M,N)^1{}_3(x_i), (M,N)^2{}_3(x_i), \ldots, (M,N)^n{}_3(x_i)) \ (U^1_{(M,N)_2(x_i)}, U^2_{(M,N)_2(x_i)}, \ldots,$
$U^n_{(M,N)_2(x_i)}),$

$((M,N)^1{}_4(x_i), (M,N)^2{}_4(x_i), \ldots, (M,N)^n{}_4(x_i)) \ (C^1_{(M,N)_2(x_i)}, C^2_{(M,N)_2(x_i)}, \ldots,$
$C^n_{(M,N)_2(x_i)}),$

$((M,N)^1{}_5(x_i), (M,N)^2{}_5(x_i), \ldots, (M,N)^n{}_5(x_i)) \ (F^1_{(M,N)_2(x_i)}, F^2_{(M,N)_2(x_i)}, \ldots, F^n_{(M,N)_2(x_i)}))\rangle,$

$(M,N)^n{}_1(x_i), (M,N)^n{}_2(x_i), (M,N)^n{}_3(x_i), (M,N)^n{}_4(x_i), (M,N)^n{}_5(x_i) \in$
$P(E)\}.$

Where,

$$(M,N)^n{}_1(x_i) = M^n_1(x_i) \cup N^n_1(x_i)$$

$$(M,N)^n{}_2(x_i) = M^n_2(x_i) \cup N^n_2(x_i)$$

$$(M,N)^{n}{}_{3}(x_i) = M_3^n(x_i) \cup N_3^n(x_i)$$

$$(M,N)^{n}{}_{4}(x_i) = M_4^n(x_i) \cup N_4^n(x_i)$$

$$(M,N)^{n}{}_{5}(x_i) = M_5^n(x_i) \cup N_5^n(x_i)$$

and

$$T^n_{(M,N)_2(x_i)} = \frac{T^n_{(M)_2(x_i)} + T^n_{(N)_2(x_i)}}{2}$$

$$U^n_{(M,N)_3(x_i)} = \frac{U^n_{(M)_3(x_i)} + U^n_{(N)_3(x_i)}}{2}$$

$$C^n_{(M,N)_4(x_i)} = \frac{C^n_{(M)_4(x_i)} + C^n_{(N)_4(x_i)}}{2}$$

$$F^n_{(M,N)_5(x_i)} = \frac{F^n_{(M)_5(x_i)} + F^n_{(N)_5(x_i)}}{2}$$

i=1,…,j; n=1,…,p.

ii. Average ∩ operation for $M^{\ddot{C}}$ *and* $N^{\ddot{C}}$ is obtained by taking the intersection operation (∩) instead of the union operation (∪) in the (i).

iii. Optimistic ∪ operation for $M^{\ddot{C}}$ *and* $N^{\ddot{C}}$ is defined as

$$M^{\ddot{C}} \; \breve{\cup}_O \; N^{\ddot{C}} = \{\langle ((M,N)^{1}{}_{1}(x_i), (M,N)^{2}{}_{1}(x_i), \dots, (M,N)^{n}{}_{1}(x_i)),$$

$$((M,N)^{1}{}_{2}(x_i), (M,N)^{2}{}_{2}(x_i), \dots, (M,N)^{n}{}_{2}(x_i))\ (T^1_{(M,N)_2(x_i)}, T^2_{(M,N)_2(x_i)}, \dots, T^n_{(M,N)_2(x_i)}),$$

$$((M,N)^{1}{}_{3}(x_i), (M,N)^{2}{}_{3}(x_i), \dots, (M,N)^{n}{}_{3}(x_i))\ (U^1_{(M,N)_2(x_i)}, U^2_{(M,N)_2(x_i)}, \dots, U^n_{(M,N)_2(x_i)}),$$

$$((M,N)^{1}{}_{4}(x_i), (M,N)^{2}{}_{4}(x_i), \dots, (M,N)^{n}{}_{4}(x_i))\ (C^1_{(M,N)_2(x_i)}, C^2_{(M,N)_2(x_i)}, \dots, C^n_{(M,N)_2(x_i)}),$$

$$((M,N)^{1}{}_{5}(x_i), (M,N)^{2}{}_{5}(x_i), \dots, (M,N)^{n}{}_{5}(x_i))\ (F^1_{(M,N)_2(x_i)}, F^2_{(M,N)_2(x_i)}, \dots, F^n_{(M,N)_2(x_i)})\rangle,$$

$$(M,N)^{n}{}_{1}(x_i), (M,N)^{n}{}_{2}(x_i), (M,N)^{n}{}_{3}(x_i), (M,N)^{n}{}_{4}(x_i), (M,N)^{n}{}_{5}(x_i) \in P(E)\}.$$

Where,

$$(M,N)^{n}{}_{1}(x_i) = M_1^n(x_i) \cup N_1^n(x_i)$$

$$(M,N)^n{}_2(x_i) = M_2^n(x_i) \cup N_2^n(x_i)$$

$$(M,N)^n{}_3(x_i) = M_3^n(x_i) \cup N_3^n(x_i)$$

$$(M,N)^n{}_4(x_i) = M_4^n(x_i) \cup N_4^n(x_i)$$

$$(M,N)^n{}_5(x_i) = M_5^n(x_i) \cup N_5^n(x_i)$$

and

$$T^n_{(M,N)_2(x_i)=max}\{T^n_{(M)_2(x_i)}, T^n_{(N)_2(x_i)}\}$$

$$U^n_{(M,N)_3(x_i)=min}\{U^n_{(M)_3(x_i)}, U^n_{(N)_3(x_i)}\}$$

$$C^n_{(M,N)_4(x_i)=min}\{C^n_{(M)_4(x_i)}, C^n_{(N)_4(x_i)}\}$$

$$F^n_{(M,N)_5(x_i)=min}\{F^n_{(M)_5(x_i)}, F^n_{(N)_5(x_i)}\}$$

i=1,…,j; n=1,…,p.

iv. Optimistic ∩ operation for $M^{\ddot{C}}$ *ve* $N^{\ddot{C}}$ is obtained by taking the intersection operation (∩) instead of the union operation (∪) in the (iii).

v. Pessimistic ∪ operation for $M^{\ddot{C}}$ *and* $N^{\ddot{C}}$ is defined as

$$M^{\ddot{C}} \ \breve{\cup}_P \ N^{\ddot{C}} = \{\langle((M,N)^1{}_1(x_i), (M,N)^2{}_1(x_i), \dots, (M,N)^n{}_1(x_i)),$$
$$((M,N)^1{}_2(x_i), (M,N)^2{}_2(x_i), \dots, (M,N)^n{}_2(x_i))\ (T^1_{(M,N)_2(x_i)}, T^2_{(M,N)_2(x_i)}, \dots, T^n_{(M,N)_2(x_i)}),$$
$$((M,N)^1{}_3(x_i), (M,N)^2{}_3(x_i), \dots, (M,N)^n{}_3(x_i))\ (U^1_{(M,N)_2(x_i)}, U^2_{(M,N)_2(x_i)}, \dots, U^n_{(M,N)_2(x_i)}),$$
$$((M,N)^1{}_4(x_i), (M,N)^2{}_4(x_i), \dots, (M,N)^n{}_4(x_i))\ (C^1_{(M,N)_2(x_i)}, C^2_{(M,N)_2(x_i)}, \dots, C^n_{(M,N)_2(x_i)}),$$
$$((M,N)^1{}_5(x_i), (M,N)^2{}_5(x_i), \dots, (M,N)^n{}_5(x_i))\ (F^1_{(M,N)_2(x_i)}, F^2_{(M,N)_2(x_i)}, \dots, F^n_{(M,N)_2(x_i)})\rangle,$$
$$(M,N)^n{}_1(x_i), (M,N)^n{}_2(x_i), (M,N)^n{}_3(x_i), (M,N)^n{}_4(x_i), (M,N)^n{}_5(x_i) \in P(E)\}.$$

Where,

$$(M,N)^n{}_1(x_i) = M_1^n(x_i) \cup N_1^n(x_i)$$

$$(M,N)^n{}_2(x_i) = M_2^n(x_i) \cup N_2^n(x_i)$$

$$(M,N)^n{}_3(x_i) = M_3^n(x_i) \cup N_3^n(x_i)$$

$$(M,N)^n{}_4(x_i) = M_4^n(x_i) \cup N_4^n(x_i)$$

$$(M,N)^n{}_5(x_i) = M_5^n(x_i) \cup N_5^n(x_i)$$

and

$$T^n_{(M,N)_2(x_i)}=min\{T^n_{(M)_2(x_i)}, T^n_{(N)_2(x_i)}\}$$

$$U^n_{(M,N)_3(x_i)}=max\{U^n_{(M)_3(x_i)}, U^n_{(N)_3(x_i)}\}$$

$$C^n_{(M,N)_4(x_i)}=max\{C^n_{(M)_4(x_i)}, C^n_{(N)_4(x_i)}\}$$

$$F^n_{(M,N)_5(x_i)}=max\{F^n_{(M)_5(x_i)}, F^n_{(N)_5(x_i)}\}$$

i=1,…,j; n=1,…,p.

vi. Pessimistic ∩ operation for $M^{\ddot{C}}\ ve\ N^{\ddot{C}}$ is is obtained by taking the intersection operation (∩) instead of the union operation (∪) in the (v).

Properties 11 Let

$M^{\ddot{C}} =$

$\{\langle\left(M^1_{1(x_i)}, M^2_{1(x_i)}, M^3_{1(x_i)}, \dots, M^n_{1(x_i)}\right),$

$\left(M^1_{2(x_i)}, M^2_{2(x_i)}, M^3_{2(x_i)}, \dots, M^n_{2(x_i)}\right)\left(T^1_{M_{2(x_i)}}, T^2_{M_{2(x_i)}}, T^3_{M_{2(x_i)}}, \dots, T^n_{M_{2(x_i)}}\right),$

$\left(M^1_{3(x_i)}, M^2_{3(x_i)}, M^3_{3(x_i)}, \dots, M^n_{3(x_i)}\right)\left(U^1_{M_{3(x_i)}}, U^2_{M_{3(x_i)}}, U^3_{M_{3(x_i)}}, \dots, U^n_{M_{3(x_i)}}\right),$

$\left(M^1_{4(x_i)}, M^2_{4(x_i)}, M^3_{4(x_i)}, \dots, M^n_{4(x_i)}\right)$

$\left(C^1_{M_{4(x_i)}}, C^2_{M_{4(x_i)}}, C^3_{M_{4(x_i)}}, \dots, C^n_{M_{4(x_i)}}\right), \left(M^1_{5(x_i)}, M^2_{5(x_i)}, M^3_{5(x_i)}, \dots, M^n_{5(x_i)}\right)\left(F^1_{M_{5(x_i)}}, F^2_{M_{5(x_i)}}, F^3_{M_{5(x_i)}}, \dots, F^n_{M_{5(x_i)}}\right)\rangle:$

$M^n_{1(x_i)}, M^n_{2(x_i)}, M^n_{3(x_i)}, M^n_{4(x_i)}, M^n_{5(x_i)} \in P(E)\}.$

$N^{\ddot{C}} =$

$\{\langle\left(N^1_{1(x_i)}, N^2_{1(x_i)}, N^3_{1(x_i)}, \dots, N^n_{1(x_i)}\right),$

$\left(N^1_{2(x_i)}, N^2_{2(x_i)}, N^3_{2(x_i)}, \dots, N^n_{2(x_i)}\right)\left(T^1_{N_{2(x_i)}}, T^2_{N_{2(x_i)}}, T^3_{N_{2(x_i)}}, \dots, T^n_{N_{2(x_i)}}\right),$

$\left(N^1_{3(x_i)}, N^2_{3(x_i)}, N^3_{3(x_i)}, \dots, N^n_{3(x_i)}\right)\left(U^1_{N_{3(x_i)}}, U^2_{N_{3(x_i)}}, U^3_{N_{3(x_i)}}, \dots, U^n_{N_{3(x_i)}}\right),$

$\left(N^1_{4(x_i)}, N^2_{4(x_i)}, N^3_{4(x_i)}, \dots, N^n_{4(x_i)}\right)$

$\left(C^1_{N_{4(x_i)}}, C^2_{N_{4(x_i)}}, C^3_{N_{4(x_i)}}, \dots, C^n_{N_{4(x_i)}}\right), \left(N^1_{5(x_i)}, N^2_{5(x_i)}, N^3_{5(x_i)}, \dots, N^n_{5(x_i)}\right)\left(F^1_{N_{5(x_i)}}, F^2_{N_{5(x_i)}}, F^3_{N_{5(x_i)}}, \dots, F^n_{N_{5(x_i)}}\right)\rangle:$

$N^n_{1(x_i)}, N^n_{2(x_i)}, N^n_{3(x_i)}, N^n_{4(x_i)}, N^n_{5(x_i)} \in P(E)\}.$

and

$P^{\ddot{C}} =$

$\{\langle (P^1_{1(x_i)}, P^2_{1(x_i)}, P^3_{1(x_i)}, \dots, P^n_{1(x_i)}),$

$(P^1_{2(x_i)}, P^2_{2(x_i)}, P^3_{2(x_i)}, \dots, P^n_{2(x_i)}) \left(T^1_{P_{2(x_i)}}, T^2_{P_{2(x_i)}}, T^3_{P_{2(x_i)}}, \dots, T^n_{P_{2(x_i)}}\right),$

$(P^1_{3(x_i)}, P^2_{3(x_i)}, P^3_{3(x_i)}, \dots, P^n_{3(x_i)}) \left(U^1_{P_{3(x_i)}}, U^2_{P_{3(x_i)}}, U^3_{P_{3(x_i)}}, \dots, U^n_{P_{3(x_i)}}\right),$

$(P^1_{4(x_i)}, P^2_{4(x_i)}, P^3_{4(x_i)}, \dots, P^n_{4(x_i)})$

$\left(C^1_{P_{4(x_i)}}, C^2_{P_{4(x_i)}}, C^3_{P_{4(x_i)}}, \dots, C^n_{P_{4(x_i)}}\right), (P^1_{5(x_i)}, P^2_{5(x_i)}, N^3_{5(x_i)}, \dots, N^n_{5(x_i)}) \left(F^1_{N_{5(x_i)}}, F^2_{N_{5(x_i)}}, F^3_{N_{5(x_i)}}, \dots, F^n_{N_{5(x_i)}}\right)\rangle:$

$P^n_{1(x_i)}, P^n_{2(x_i)}, P^n_{3(x_i)}, P^n_{4(x_i)}, P^n_{5(x_i)} \in P(E)\}.$

be three MGSVNQPSs. Then, the following properties are provided.

i) $M^{\ddot{C}} \breve{\cup}_A N^{\ddot{C}} = N^{\ddot{C}} \breve{\cup}_A M^{\ddot{C}}$

ii) $M^{\ddot{C}} \breve{\cup}_O N^{\ddot{C}} = N^{\ddot{C}} \breve{\cup}_O M^{\ddot{C}}$

iii) $M^{\ddot{C}} \cup'_P N^{\ddot{C}} = N^{\ddot{C}} \cup'_P M^{\ddot{C}}$

iv) $M^{\ddot{C}} \breve{\cap}_A N^{\ddot{C}} = N^{\ddot{C}} \breve{\cap}_A M^{\ddot{C}}$

v) $M^{\ddot{C}} \cap'_O N^{\ddot{C}} = N^{\ddot{C}} \cap'_O M^{\ddot{C}}$

vi) $M^{\ddot{C}} \cap'_P N^{\ddot{C}} = N^{\ddot{C}} \cap'_P M^{\ddot{C}}$

vii) If

$T^n_{M_{2(x_i)}} = T^n_{P_{2(x_i)}}, U^n_{M_{3(x_i)}} = U^n_{P_{3(x_i)}}, C^n_{M_{4(x_i)}} = C^n_{P_{4(x_i)}}, F^n_{M_{5(x_i)}} = F^n_{P_{5(x_i)}}$; i=1,…,j; n=1,…,p.

Then

$$M^{\ddot{C}} \breve{\cup}_A (N^{\ddot{C}} \breve{\cup}_A P^{\ddot{C}}) = (M^{\ddot{C}} \breve{\cup}_A N^{\ddot{C}}) \breve{\cup}_A P^{\ddot{C}}$$

viii) $M^{\ddot{C}} \breve{\cup}_O (N^{\ddot{C}} \breve{\cup}_O P^{\ddot{C}}) = (M^{\ddot{C}} \breve{\cup}_O N^{\ddot{C}}) \breve{\cup}_O P^{\ddot{C}}$

ix) $M^{\ddot{C}} \breve{\cup}_P (N^{\ddot{C}} \breve{\cup}_P P^{\ddot{C}}) = (M^{\ddot{C}} \breve{\cup}_P N^{\ddot{C}}) \breve{\cup}_P P^{\ddot{C}}$

x) If

$T^n_{M_{2(x_i)}} = T^n_{P_{2(x_i)}}, U^n_{M_{3(x_i)}} = U^n_{P_{3(x_i)}}, C^n_{M_{4(x_i)}} = C^n_{P_{4(x_i)}}, F^n_{M_{5(x_i)}} = F^n_{P_{5(x_i)}}$; i=1,…,j; n=1,…,p.

then

$$M^{\ddot{C}} \breve{\cap}_A (N^{\ddot{C}} \breve{\cap}_A P^{\ddot{C}}) = (M^{\ddot{C}} \breve{\cap}_A N^{\ddot{C}}) \breve{\cap}_A P^{\ddot{C}}$$

xi) $M^{\ddot{C}} \cap'_O (N^{\ddot{C}} \cap'_O P^{\ddot{C}}) = (M^{\ddot{C}} \cap'_O N^{\ddot{C}}) \cap'_O P^{\ddot{C}}$

xii) $M^{\ddot{C}} \cap'_P (N^{\ddot{C}} \cap'_P P^{\ddot{C}}) = (M^{\ddot{C}} \cap'_P N^{\ddot{C}}) \cap'_P P^{\ddot{C}}$

xiii) $M^{\ddot{C}} \breve{\cap}_A (N^{\ddot{C}} \breve{\cup}_A P^{\ddot{C}}) = (M^{\ddot{C}} \breve{\cap}_A N^{\ddot{C}}) \breve{\cup}_A (M^{\ddot{C}} \breve{\cap}_A P^{\ddot{C}})$

xiv) $M^{\ddot{C}} \cap'_O (N^{\ddot{C}} \breve{\cup}_A P^{\ddot{C}}) = (M^{\ddot{C}} \cap'_O N^{\ddot{C}}) \breve{\cup}_A (M^{\ddot{C}} \cap'_O P^{\ddot{C}})$

xv) $M^{\ddot{C}} \cap'_P (N^{\ddot{C}} \breve{\cup}_P P^{\ddot{C}}) = (M^{\ddot{C}} \cap'_P N^{\ddot{C}}) \breve{\cup}_P (M^{\ddot{C}} \cap'_P P^{\ddot{C}})$

xvi) $M^{\ddot{C}} \breve{\cup}_A (N^{\ddot{C}} \breve{\cap}_A P^{\ddot{C}}) = (M^{\ddot{C}} \breve{\cup}_A N^{\ddot{C}}) \cap'_O (M^{\ddot{C}} \breve{\cup}_A P^{\ddot{C}})$

xvii) $M^{\ddot{C}} \breve{\cup}_A (N^{\ddot{C}} \cap'_O P^{\ddot{C}}) = (M^{\ddot{C}} \breve{\cup}_A N^{\ddot{C}}) \cap'_O (M^{\ddot{C}} \breve{\cup}_A P^{\ddot{C}})$

xviii) $M^{\ddot{C}} \breve{\cup}_P (M^{\ddot{C}} \cap'_P P^{\ddot{C}}) = (M^{\ddot{C}} \breve{\cup}_P N^{\ddot{C}}) \cap'_P (M^{\ddot{C}} \breve{\cup}_P P^{\ddot{C}})$

xix) İf $M^{\ddot{C}} = N^{\ddot{C}}$ then $M^{\ddot{C}} \breve{\cup}_A N^{\ddot{C}} = M^{\ddot{C}} \breve{\cup}_O N^{\ddot{C}} = M^{\ddot{C}} \cup'_P N^{\ddot{C}}$

xx) İf $M^{\ddot{C}} = N^{Ç}$ then $M^{\ddot{C}} \breve{\cap}_A N^{\ddot{C}} = M^{\ddot{C}} \cap'_O N^{\ddot{C}} = M^{\ddot{C}} \cap'_P N^{\ddot{C}}$

Abbreviations

NS: Neutrosophic Set

NQN: Neutrosophic Quadruple Number

NQS: Neutrosophic Quadruple Set

MVNS: Multi-Valued Neutrosophic Set

SVNQN: Set Valued Neutrosophic Quadruple Number

SVNQS: Set Valued Neutrosophic Quadruple Set

GSVNQS: Generalized Set Valued Neutrosophic Quintuple Set

MGSVNQS: Multiple Generalized Set Valued Neutrosophic Quadruple Set

MGSVNQPS: Multiple Generalized Set Valued Neutrosophic Quintuple Set

MGSVNQPN: Multiple Generalized Set Valued Neutrosophic Quintuple Number

4.Conclusions

In this chapter, multiple generalized set-valued neutrosophic quintuple sets, multiple generalized set-valued neutrosophic quintuple numbers, some operators on them (average union, average intersection, optimistic union, optimistic intersection, pessimistic union, pessimistic intersection) are defined and their basic properties are given. This new set fulfills the properties of both quadripartioned neutrosophic sets, multiple neutrosophic sets and multiple neutrosophic quadruple sets.

5.Future Research Directions

Researchers can use these studies to obtain interval multiple generalized set-valued neutrosophic quintuple sets, bipolar multiple generalized set-valued neutrosophic quintuple sets. In addition, researchers can use the operators obtained in this study in multi-criteria decision making applications to find more objective solutions to real-life problems.

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Hamming Distance Measure for interval generalized set valued neutrosophic quadruple numbers

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ABSTRACT

In this study, we define a Hamming similarity measure for interval generalized set valued neutrosophic quadruple numbers. It is shown that this Hamming similarity measure satisfies the similarity measure conditions. Furthermore, an example application was carried out using the Hamming similarity measure defined for interval generalized set-valued neutrosophic quadruple numbers. In this application, the similarities of 10 fictitious universities to the ideal university were analyzed using the factors affecting university choice. In addition, similarities were also calculated with Hamming measures defined for interval-valued neutrosophic numbers and generalized set-valued neutrosophic quadruple numbers and the results were compared. Although this application was made with fictitious data, researchers can apply this similarity measure with real data to obtain more objective results. Researchers can obtain new results in real-life problems by improving this similarity measure or using it in different decision-making applications.

Keywords: Neutrosophic quadruple sets, Neutrosophic quadruple numbers, Generalized set valued neutrosophic quadruple sets, Generalized set valued quadruple numbers, Hamming similarity measure.

1.INTRODUCTION

In 1998, Florentin Smarandache defined the neutrosophic set theory [1]. In this theory, the elements in the set are expressed by the values of truth T, indeterminacy I and falsity F, making the mathematical expression of ambiguous situations very objective. Also, in 2010, Wang et al. defined SVNSs [2]. In these sets, the truth, indeterminacy and falsity values of each element are in the range of [0,1]. In 2020, Kargın et al. defined neutrosophic triple m-Banach spaces [3]. In 2020, Şahin et al. obtained neutrosophic ternary partial bipolar metric spaces [4]. Moreover, it was shown that neutrosophic ternary partial bipolar metric spaces have different properties from the classical partial metric space, neutrosophic ternary partial metric space and neutrosophic ternary metric space structures. Thus, the theory of neutrosophic metric spaces was shown to be more comprehensive. In 2021, Şahin defined the notion of homomorphism on the neutrosophic multigroup and the homomorphism kerlf, automorphism, homomorphic image and homomorphic preimage of the neutrosophic multigroup, respectively [5]. In 2022, Hawk obtained neutro-sigma algebras and anti-sigma algebras [6]. In 2024, Ali et al. defined some mean and geometric operators based on the q-digit orthopair neutrosophic soft set [7]. In 2024 Kaviyarasu et al. developed circular economy strategies to promote sustainable development using t-neutrosophic fuzzy graphs [8]. In 2024, Futjita aimed to extend the study of neutrosophic graphs by defining general-neutrosophic graphs, anti-neutrosophic graphs, balanced-neutrosophic graphs, and semi-neutrosophic graphs [9].

In 2005, Wang et al. defined IVNSs as an important stage in the progress of neutrosophic theory [10]. This new concept is based on expressing truth, indeterminacy and falsity values as an interval instead of a fixed number. Thus, a wider range of uncertainties can be taken into account in decision-making processes, rather than relying only on a single precise value. Başer and Ulucay [29] defined Effective Q fuzzy soft expert sets its some properties. In 2024, Alqazzazaz and Sallam in Industry 4.0 used a multi-criteria decision making (MCDM) approach for supplier selection process in Industry 4.0, using IVNSs to handle uncertainties, resulting in more flexible and reliable evaluations [11]. Başer and Ulucay [44] defined energy of a neutrosophic soft set. In 2024, Razak et al. proposed a new concept based on Interval Valued Pythagorean Neutrosophic Set to deal with uncertainty and incomplete information [12]. In 2024, Palanikumar et al. presented a new approach to the (δ, ε) interval-valued neutrosophic set, an extension of interval-valued neutrosophic sets [13]. In 2024, Saeed et al. extended the interval-valued neutrosophic fuzzy soft set (IVNFSS) to interval-valued neutrosophic fuzzy set (IVNFSS) and developed a hybrid concept that includes complementary, combinatorial, and integrative operations and a quality assessment distance measure to improve decision-making truth [14].

In 2015, Smarandache defined neutrosophic quadruple sets and neutrosophic quadruple numbers for the first time, providing a significant expansion in the field of neutrosophic logic [15]. Neutrosophic quadruple sets, like neutrosophic sets, include the parameters truth (T), indeterminacy (I) and falsity (F), but are structured to include both the known and unknown components of these parameters. In 2019, Şahin and Kargın defined set-valued neutrosophic quadruple numbers and neutrosophic triple groups based on neutrosophic quadruple numbers [16]. New mathematical structures were defined by using set-valued neutrosophic quadruple numbers and neutrosophic triple groups together. Thus, new results were obtained in the field of set-valued neutrosophic quadruple numbers. The above-mentioned theories have been studied in various fields [29-90]. In 2020, Şahin et al. generalized neutrosophic quadruple sets and numbers [17]. In 2022, Kargın and Şahin defined IGSVNQSs [18]. A new structure was developed based on GSVNQSs and IVNNs. This structure provides the properties of GSVNQSs and interval neutrosophic sets. In 2024, Kargın et al. defined multivalued generalized set valued neutrosophic quadruples and sets as a generalized form of multivalued neutrosophic numbers and neutrosophic quadruple sets [19]. This new set was constructed using GSVNQSs and multiple neutrosophic sets. The resulting structure contains the properties of both sets. In 2024, Al Tahan et al. defined the concept of neutrosophic quadruple Hv-modules on neutrosophic quadruple Hv-rings [20].

In 2014, Ye defined similarity measures for SVNSs for the first time [21]. In 2014, Ye defined the Hamming similarity measure for IVNSs [22]. In 2021, Kargın et al. developed a Hamming similarity measure for GSVNQNs. They also proved that this new approach provides more accurate results than previous methods for decision-making problems in legal science and other fields [23]. In 2021, Şahin et al. defined Euclidean distance and similarity measures based on GSVNQNs. Moreover, an algorithm based on the generalized Euclidean similarity measure was developed and demonstrated that this algorithm can be used in multi-criteria decision-making processes in the healthcare domain, such as COVID-19 treatment [24]. In 2021, Şahin et al. defined Hausdorff distance and similarity measures based on GSVNQNs and developed a new decision-making algorithm. This algorithm had an application in evaluating the effects of online learning [25]. In 2024, Mustapha et al. developed a hybrid weighted similarity measure based on neutrosophic sets, aiming to improve medical diagnosis processes in analyzing patient symptoms with indeterminacy [26]. In 2024, Rahman et al. developed a new structure and algorithms that provide flexibility and reliability in decision support systems by combining probabilistic neutrosophic set with hypersoft set to effectively deal with uncertainties [27].

In 2016, Odabaş et al. determined the factors affecting university students' university preferences by pairwise comparison method. Data of 863 students studying at Hacettepe, Siirt and Aksaray universities in 2012 were analyzed. The findings show that, in general, the most important factor affecting university preference in all student groups is score ranking. Especially for Hacettepe University students, the quality of education ranked first, while for Siirt and Aksaray University students, being close to family or friends was the most decisive reason for preference. In addition, it was concluded that score ranking was the most important criterion in the groups of students who were satisfied and dissatisfied with their university choices [28].

The chapter is organized as follows: In the background Section, we present the basic definitions and properties to be used in this study. In the Research and Results Section, the Hamming similarity measure for IGSVNQNs numbers is defined and shown to satisfy the similarity conditions. In the Application section, we performed a similarity analysis for the factors affecting university selection using IGSVNQNs. In this application, the similarities of 10 fictitious universities with an ideal university were calculated using the Hamming similarity measure. In the comparison section, the results obtained for IVNNs and GSVNQNs are compared with the IGSVNNs used in the application. This comparison was done in order to evaluate how consistent and reliable the similarity measures are between different sets. In the conclusion, the findings of the study are summarized and suggestions for future research are presented.

2.BACKGROUND

Definition 1: [2] Let $\mathcal{D} \neq \emptyset$. For $\forall \tilde{\mathcal{Y}}_{\tilde{v}_c} \in \mathcal{D}$. A SVNS $\tilde{\mathcal{Y}}$ over $\mathcal{D}$ with functions; is defined as

$$\tilde{\mathcal{Y}} = \left\{ \langle \tilde{\mathcal{Y}}_{\tilde{v}_c}, T_{\tilde{\mathcal{Y}}_{\tilde{v}_c}}, I_{\tilde{\mathcal{Y}}_{\tilde{v}_c}}, F_{\tilde{\mathcal{Y}}_{\tilde{v}_c}} \rangle : \tilde{\mathcal{Y}}_{\tilde{v}_c} \in \mathcal{D}, c = 1,2, \dots, a \right\}$$

such that

$$0 \leq \; T_{\tilde{\mathcal{Y}}_{\tilde{v}_c}} + I_{\tilde{\mathcal{Y}}_{\tilde{v}_c}} + F_{\tilde{\mathcal{Y}}_{\tilde{v}_c}} \; \leq 3,$$

$$T_{\tilde{\mathcal{Y}}}: \mathcal{D} \to [0, 1],\; I_{\tilde{\mathcal{Y}}}: \mathcal{D} \to [0, 1] \text{ and } F_{\tilde{\mathcal{Y}}}: \mathcal{D} \to [0, 1].$$

Where $T_{\tilde{\mathcal{Y}}_{\tilde{v}_c}}, I_{\tilde{\mathcal{Y}}_{\tilde{v}_c}}$ and $F_{\tilde{\mathcal{Y}}_{\tilde{v}_c}}$ are the degree of thuth, degre of indeterminancy and degree of falsity of $\tilde{\mathcal{Y}}_{\tilde{v}_c} \in \mathcal{D}$ respectively.

Definition 2: [10] Let $\mathcal{D} \neq \emptyset$. For $\tilde{\mathcal{Y}}_{\tilde{v}_c} \in \mathcal{D}$. An IVNSs $\tilde{\mathcal{Y}}$ over $\mathcal{D}$ with functions is defined as

$$\tilde{\mathcal{Y}} = \{< \tilde{\mathcal{Y}}_{\tilde{v}_c}, \left[T^L_{\tilde{\mathcal{Y}}_{\tilde{v}_c}}, T^U_{\tilde{\mathcal{Y}}_{\tilde{v}_c}}\right], \left[I^L_{\tilde{\mathcal{Y}}_{\tilde{v}_c}}, I^U_{\tilde{\mathcal{Y}}_{\tilde{v}_c}}\right], \left[F^U_{\tilde{\mathcal{Y}}_{\tilde{v}_c}}, F^U_{\tilde{\mathcal{Y}}_{\tilde{v}_c}}\right] >, \tilde{\mathcal{Y}}_{\tilde{v}_c} \in \mathcal{D}, c = 1,2, \dots, a\}.$$

Here,

$\left[T^L_{\tilde{\mathcal{Y}}_{\tilde{v}_c}}, T^U_{\tilde{\mathcal{Y}}_{\tilde{v}_c}}\right], \left[I^L_{\tilde{\mathcal{Y}}_{\tilde{v}_c}}, I^U_{\tilde{\mathcal{Y}}_{\tilde{v}_c}}\right]$ and $\left[F^U_{\tilde{\mathcal{Y}}_{\tilde{v}_c}}, F^U_{\tilde{\mathcal{Y}}_{\tilde{v}_c}}\right]$ are intervals

$T^L_{\tilde{\mathcal{Y}}}: \mathcal{D} \to [0,1]$, $T^U_{\tilde{\mathcal{Y}}}: \mathcal{D} \to [0,1]$ are truth functions,

$I^L_{\tilde{\mathcal{Y}}}: \mathcal{D} \to [0,1]$, $I^U_{\tilde{\mathcal{Y}}}: \mathcal{D} \to [0,1]$ are indeterminancy functions,

$F^L_{\tilde{\mathcal{Y}}}: \mathcal{D} \to [0,1], F^U_{\tilde{\mathcal{Y}}}: \mathcal{D} \to [0,1]$ are falsity functions.

Definition 3:[16] Let $\mathcal{D} \neq \emptyset$ and P($\mathcal{D}$) be the power set of $\mathcal{D}$. An set valued neutrosophic quadruple numbers

$$(\tilde{\mathcal{Y}}, \tilde{\mathcal{Y}}_T \mathrm{T}, \tilde{\mathcal{Y}}_I I, \tilde{\mathcal{Y}}_F \mathrm{F}).$$

Here, the components T, I and F are respectively the truth, indeterminacy and falsity functions in the SVNSs. In addition,

$$\tilde{\mathcal{Y}}, \tilde{\mathcal{Y}}_T, \tilde{\mathcal{Y}}_I, \tilde{\mathcal{Y}}_F \in \mathrm{P}(\mathcal{D}).$$

Also, a set valued neutrosophic quadruple set

$$\tilde{\mathcal{Y}}_D = \{(\tilde{\mathcal{Y}}, \tilde{\mathcal{Y}}_T \mathrm{T}, \tilde{\mathcal{Y}}_I I, \tilde{\mathcal{Y}}_F \mathrm{F}), \tilde{\mathcal{Y}}\ \tilde{\mathcal{Y}}_T, \tilde{\mathcal{Y}}_I, \tilde{\mathcal{Y}}_F \in \mathrm{P}(\mathcal{D})\}$$

is shown in the form. Here it is

$$\text{“}\tilde{\mathcal{Y}}\text{”}$$

is called known part and

$$\text{“}\tilde{\mathcal{Y}}_T \mathrm{T}, \tilde{\mathcal{Y}}_I I, \tilde{\mathcal{Y}}_F \mathrm{F}\text{”}$$

is called the unknown part.

Definition 4:[18] Let $\mathcal{D}$ be a set and P($\mathcal{D}$) be the power set of $\mathcal{D}$. An IGSVNQS is represented as

$$\tilde{\mathcal{Y}} = \{< (A_{\tilde{\mathcal{Y}}_1}), (A_{\tilde{\mathcal{Y}}_1})_T\left[T^L{}_{\tilde{\mathcal{Y}}_1}, T^U{}_{\tilde{\mathcal{Y}}_1}\right], (A_{\tilde{\mathcal{Y}}_1})_I\left[I^L{}_{\tilde{\mathcal{Y}}_1}, I^U{}_{\tilde{\mathcal{Y}}_1}\right], (A_{\tilde{\mathcal{Y}}_1})_F\left[F^L{}_{\tilde{\mathcal{Y}}_1}, F^U{}_{\tilde{\mathcal{Y}}_1}\right];$$
$$(A_{\tilde{\mathcal{Y}}_2}), (A_{\tilde{\mathcal{Y}}_2})_T\left[T^L{}_{\tilde{\mathcal{Y}}_2}, T^U{}_{\tilde{\mathcal{Y}}_2}\right], (A_{\tilde{\mathcal{Y}}_2})_I\left[I^L{}_{\tilde{\mathcal{Y}}_2}, I^U{}_{\tilde{\mathcal{Y}}_2}\right], (A_{\tilde{\mathcal{Y}}_2})_F\left[F^L{}_{\tilde{\mathcal{Y}}_2}, F^U{}_{\tilde{\mathcal{Y}}_2}\right];$$

$$\begin{aligned}
&\ldots\\
&(A_{\tilde{y}_a}), (A_{\tilde{y}_a})_T[T^L{}_{\tilde{y}_a}, T^U{}_{\tilde{y}_a}], (A_{\tilde{y}_a})_I[I^L{}_{\tilde{y}_a}, I^U{}_{\tilde{y}_a}], (A_{\tilde{y}_a})_F[F^L{}_{\tilde{y}_a}, F^U{}_{\tilde{y}_a}]>;\\
&(A_{\tilde{y}_c}), (A_{\tilde{y}_c})_T, (A_{\tilde{y}_c})_I, (A_{\tilde{y}_c})_F \in P(\mathcal{D}); c = 1,2,3, \ldots, a\}.
\end{aligned}$$

Here, the components $T^L{}_{\tilde{y}_c}, T^U{}_{\tilde{y}_c}, I^L{}_{\tilde{y}_c}, I^U{}_{\tilde{y}_c}, F^L{}_{\tilde{y}_c}, F^U{}_{\tilde{y}_c}$ are truth, indeterminacy and falsity functions in the IVNSs. In addition,

$$(A_{\tilde{y}_c}), (A_{\tilde{y}_c})_T, (A_{\tilde{y}_c})_I, (A_{\tilde{y}_c})_F \in P(\mathcal{D}).$$

Also, an IGSVNQNs

$$(\tilde{\mathcal{Y}}^N)_1 = \{< (A_{\tilde{y}_1}), (A_{\tilde{y}_1})_T[T^L{}_{\tilde{y}_1}, T^U{}_{\tilde{y}_1}], (A_{\tilde{y}_1})_I[I^L{}_{\tilde{y}_1}, I^U{}_{\tilde{y}_1}], (A_{\tilde{y}_1})_F[F^L{}_{\tilde{y}_1}, F^U{}_{\tilde{y}_1}] >\}$$

is shown in the form. Here it is,

$$``(A_{\tilde{y}_1})''$$

is called known part and

$$``(A_{\tilde{y}_1})_T[T^L{}_{\tilde{y}_1}, T^U{}_{\tilde{y}_1}], (A_{\tilde{y}_1})_I[I^L{}_{\tilde{y}_1}, I^U{}_{\tilde{y}_1}], (A_{\tilde{y}_1})_F[F^L{}_{\tilde{y}_1}, F^U{}_{\tilde{y}_1}]''$$

is called the unknown part. Also,

$$Q = \{(\tilde{\mathcal{Y}}^N)_c; c = 1,2, \ldots, a\}$$

is shown in the form.

Definition 5: [18] Let

$$\begin{aligned}
\tilde{\mathcal{Y}} = \{&< (A_{\tilde{y}_1}), (A_{\tilde{y}_1})_T[T^L{}_{\tilde{y}_1}, T^U{}_{\tilde{y}_1}], (A_{\tilde{y}_1})_I[I^L{}_{\tilde{y}_1}, I^U{}_{\tilde{y}_1}], (A_{\tilde{y}_1})_F[F^L{}_{\tilde{y}_1}, F^U{}_{\tilde{y}_1}];\\
&(A_{\tilde{y}_2}), (A_{\tilde{y}_2})_T[T^L{}_{\tilde{y}_2}, T^U{}_{\tilde{y}_2}], (A_{\tilde{y}_2})_I[I^L{}_{\tilde{y}_2}, I^U{}_{\tilde{y}_2}], (A_{\tilde{y}_2})_F[F^L{}_{\tilde{y}_2}, F^U{}_{\tilde{y}_2}];\\
&\ldots\\
&(A_{\tilde{y}_a}), (A_{\tilde{y}_a})_T[T^L{}_{\tilde{y}_a}, T^U{}_{\tilde{y}_a}], (A_{\tilde{y}_a})_I[I^L{}_{\tilde{y}_a}, I^U{}_{\tilde{y}_a}], (A_{\tilde{y}_a})_F[F^L{}_{\tilde{y}_a}, F^U{}_{\tilde{y}_a}]>;\\
&(A_{\tilde{y}_c}), (A_{\tilde{y}_c})_T, (A_{\tilde{y}_c})_I, (A_{\tilde{y}_c})_F \in P(\mathcal{D}); c = 1,2,3, \ldots, a\}
\end{aligned}$$

and

$$\begin{aligned}
\tilde{\mathcal{L}} = \{&< (A_{\tilde{\mathcal{L}}_1}), (A_{\tilde{\mathcal{L}}_1})_T[T^L{}_{\tilde{\mathcal{L}}_1}, T^U{}_{\tilde{\mathcal{L}}_1}], (A_{\tilde{\mathcal{L}}_1})_I[I^L{}_{\tilde{\mathcal{L}}_1}, I^U{}_{\tilde{\mathcal{L}}_1}], (A_{\tilde{\mathcal{L}}_1})_F[F^L{}_{\tilde{\mathcal{L}}_1}, F^U{}_{\tilde{\mathcal{L}}_1}];\\
&(A_{\tilde{\mathcal{L}}_2}), (A_{\tilde{\mathcal{L}}_2})_T[T^L{}_{\tilde{\mathcal{L}}_2}, T^U{}_{\tilde{\mathcal{L}}_2}], (A_{\tilde{\mathcal{L}}_2})_I[I^L{}_{\tilde{\mathcal{L}}_2}, I^U{}_{\tilde{\mathcal{L}}_2}], (A_{\tilde{\mathcal{L}}_2})_F[F^L{}_{\tilde{\mathcal{L}}_2}, F^U{}_{\tilde{\mathcal{L}}_2}];\\
&\ldots\\
&(A_{\tilde{\mathcal{L}}_a}), (A_{\tilde{\mathcal{L}}_a})_T[T^L{}_{\tilde{\mathcal{L}}_a}, T^U{}_{\tilde{y}_a}], (A_{\tilde{\mathcal{L}}_a})_I[I^L{}_{\tilde{\mathcal{L}}_a}, I^U{}_{\tilde{\mathcal{L}}_a}], (A_{\tilde{\mathcal{L}}_a})_F[F^L{}_{\tilde{\mathcal{L}}_a}, F^U{}_{\tilde{\mathcal{L}}_a}]>;
\end{aligned}$$

$$(A_{\tilde{\mathcal{L}}_c}), (A_{\tilde{\mathcal{L}}_c})_T, (A_{\tilde{\mathcal{L}}_c})_I, (A_{\tilde{\mathcal{L}}_c})_F \in P(\mathcal{D}); c = 1,2,3, \dots, a\}$$

be IGSVNQSs.

i. $\tilde{\mathcal{L}}$ is a equal to $\tilde{\mathcal{Y}}$ $(\tilde{\mathcal{Y}} = \tilde{\mathcal{L}})$ if and only if
$\left(A_{\tilde{\mathcal{Y}}_c}\right) = (A_{\tilde{\mathcal{L}}_c})$, $(A_{\tilde{\mathcal{Y}}_c})_T = (A_{\tilde{\mathcal{L}}_c})_T$, $(A_{\tilde{\mathcal{Y}}_c})_I = (A_{\tilde{\mathcal{L}}_c})_I$, $(A_{\tilde{\mathcal{Y}}_c})_F = (A_{\tilde{\mathcal{L}}_c})_F$,
$T^L{}_{\tilde{\mathcal{Y}}_c} = T^L{}_{\tilde{\mathcal{L}}_c}$, $T^U{}_{\tilde{\mathcal{Y}}_c} = T^U{}_{\tilde{\mathcal{L}}_c}$, $I^L{}_{\tilde{\mathcal{Y}}_c} = I^L{}_{\tilde{\mathcal{L}}_c}$, $I^U{}_{\tilde{\mathcal{Y}}_c} = I^U{}_{\tilde{\mathcal{L}}_c}$, $F^L{}_{\tilde{\mathcal{Y}}_c} = F^L{}_{\tilde{\mathcal{L}}_c}$, $F^U{}_{\tilde{\mathcal{Y}}_c} = F^U{}_{\tilde{\mathcal{L}}_c}$.

ii. $\tilde{\mathcal{L}}$ is a subset of $\tilde{\mathcal{Y}}$ $(\tilde{\mathcal{Y}} \subset \tilde{\mathcal{L}})$ if and only if
$(A_{\tilde{\mathcal{Y}}_c}) \subset (A_{\tilde{\mathcal{L}}_c})$, $(A_{\tilde{\mathcal{Y}}_c})_T \subset (A_{\tilde{\mathcal{L}}_c})_T$, $(A_{\tilde{\mathcal{Y}}_c})_I \subset (A_{\tilde{\mathcal{L}}_c})_I$, $(A_{\tilde{\mathcal{Y}}_c})_F \subset (A_{\tilde{\mathcal{L}}_c})_F$,

$T^L{}_{\tilde{\mathcal{Y}}_c} \leq T^L{}_{\tilde{\mathcal{L}}_c}$, $T^U{}_{\tilde{\mathcal{Y}}_c} \leq T^U{}_{\tilde{\mathcal{L}}_c}$, $I^L{}_{\tilde{\mathcal{Y}}_c} \geq I^L{}_{\tilde{\mathcal{L}}_c}$, $I^U{}_{\tilde{\mathcal{Y}}_c} \geq I^U{}_{\tilde{\mathcal{L}}_c}$, $F^L{}_{\tilde{\mathcal{Y}}_c} \geq F^L{}_{\tilde{\mathcal{L}}_c}$, $F^U{}_{\tilde{\mathcal{Y}}_c} \geq F^U{}_{\tilde{\mathcal{L}}_c}$.

Definition 6: [21] Let

$$\tilde{\mathcal{Y}}_1 = \langle \tilde{\mathcal{Y}}_{\tilde{r}_1}, T_{\tilde{\mathcal{Y}}_{\tilde{r}_1}}, I_{\tilde{\mathcal{Y}}_{\tilde{r}_1}}, F_{\tilde{\mathcal{Y}}_{\tilde{r}_1}} \rangle \text{ and } \tilde{\mathcal{Y}}_2 = \langle \tilde{\mathcal{Y}}_{\tilde{r}_2}, T_{\tilde{\mathcal{Y}}_{\tilde{r}_2}}, I_{\tilde{\mathcal{Y}}_{\tilde{r}_2}}, F_{\tilde{\mathcal{Y}}_{\tilde{r}_2}} \rangle$$

be SVNNs, $\widehat{S_H}: \tilde{\mathcal{Y}}_1 \times \tilde{\mathcal{Y}}_2 \to [0,1]$ be a function. The Hamming similarity measure between $\tilde{\mathcal{Y}}_1$ and $\tilde{\mathcal{Y}}_2$ denoted by $\widehat{S_H}(\tilde{\mathcal{Y}}_1, \tilde{\mathcal{Y}}_2)$ such that

$$\widehat{S_H}\left(\tilde{\mathcal{Y}}_1, \tilde{\mathcal{Y}}_2\right) = \frac{1}{3}\left[\left|T_{\tilde{\mathcal{Y}}_{\tilde{r}_1}} - T_{\tilde{\mathcal{Y}}_{\tilde{r}_2}}\right| + \left|I_{\tilde{\mathcal{Y}}_{\tilde{r}_1}} - I_{\tilde{\mathcal{Y}}_{\tilde{r}_2}}\right| + \left|F_{\tilde{\mathcal{Y}}_{\tilde{r}_1}} - F_{\tilde{\mathcal{Y}}_{\tilde{r}_2}}\right|\right].$$

Theorem 1: [21] Let $\tilde{\mathcal{Y}}_1$, $\tilde{\mathcal{Y}}_2$ and $\tilde{\mathcal{Y}}_3$ three SVNNs, $\widehat{S_H}: \tilde{\mathcal{Y}}_1 \times \tilde{\mathcal{Y}}_2 \to [0,1]$ be a Hamming similarity measure. $\widehat{S_H}(\tilde{\mathcal{Y}}_1, \tilde{\mathcal{Y}}_2)$ satisfies below properties.

i. $0 \leq \widehat{S_H}(\tilde{\mathcal{Y}}_1, \tilde{\mathcal{Y}}_2) \leq 1$,
ii. $\widehat{S_H}\left(\tilde{\mathcal{Y}}_1, \tilde{\mathcal{Y}}_2\right) = 1 \Leftrightarrow \tilde{\mathcal{Y}}_1 = \tilde{\mathcal{Y}}_2$
iii. $\widehat{S_H}\left(\tilde{\mathcal{Y}}_1, \tilde{\mathcal{Y}}_2\right) = \widehat{S_H}(\tilde{\mathcal{Y}}_2, \tilde{\mathcal{Y}}_1)$
iv. If $\tilde{\mathcal{Y}}_1 \subseteq \tilde{\mathcal{Y}}_2 \subseteq \tilde{\mathcal{Y}}_3$,then
$\widehat{S_H}(\tilde{\mathcal{Y}}_1, \tilde{\mathcal{Y}}_3) \leq \widehat{S_H}(\tilde{\mathcal{Y}}_1, \tilde{\mathcal{Y}}_2)$ and $\widehat{S_H}(\tilde{\mathcal{Y}}_1, \tilde{\mathcal{Y}}_3) \leq \widehat{S_H}(\tilde{\mathcal{Y}}_2, \tilde{\mathcal{Y}}_3)$.

Definition 7:[22] Let

$$\left(\tilde{\mathcal{Y}}^N\right)_1 = \{< (\tilde{\mathcal{Y}}_1), \left[T^L{}_{\tilde{\mathcal{Y}}_1}, T^U{}_{\tilde{\mathcal{Y}}_1}\right], \left[I^L{}_{\tilde{\mathcal{Y}}_1}, I^U{}_{\tilde{\mathcal{Y}}_1}\right], \left[F^L{}_{\tilde{\mathcal{Y}}_1}, F^U{}_{\tilde{\mathcal{Y}}_1}\right] >\}$$

and

$$\left(\tilde{\mathcal{Y}}^N\right)_2 = \{< (\tilde{\mathcal{Y}}_2), \left[T^L{}_{\tilde{\mathcal{Y}}_2}, T^U{}_{\tilde{\mathcal{Y}}_2}\right], \left[I^L{}_{\tilde{\mathcal{Y}}_2}, I^U{}_{\tilde{\mathcal{Y}}_2}\right], \left[F^L{}_{\tilde{\mathcal{Y}}_2}, F^U{}_{\tilde{\mathcal{Y}}_2}\right] >\}$$

be two IVNNs. $\widehat{S_H}: \left(\tilde{\mathcal{Y}}^N\right)_1 \times \left(\tilde{\mathcal{Y}}^N\right)_2 \to [0,1]$ be a function. The Hamming similarity measure between $\left(\tilde{\mathcal{Y}}^N\right)_1$ and $\left(\tilde{\mathcal{Y}}^N\right)_2$ denoted by $\widehat{S_H}\left(\left(\tilde{\mathcal{Y}}^{\mathrm{N}}\right)_1, \left(\tilde{\mathcal{Y}}^{\mathrm{N}}\right)_2\right)$ such that

$$\widehat{S_H}\left(\left(\tilde{\mathcal{Y}}^{\mathrm{N}}\right)_1, \left(\tilde{\mathcal{Y}}^{\mathrm{N}}\right)_2\right) = \frac{\left[\left|\mathrm{T^L}_{\tilde{\mathcal{Y}}_1} - \mathrm{T^L}_{\tilde{\mathcal{Y}}_2}\right| + \left|\mathrm{I^L}_{\tilde{\mathcal{Y}}_1} - \mathrm{I^L}_{\tilde{\mathcal{Y}}_2}\right| + \left|\mathrm{F^L}_{\tilde{\mathcal{Y}}_1} - \mathrm{F^L}_{\tilde{\mathcal{Y}}_2}\right| + \left|\mathrm{T^U}_{\tilde{\mathcal{Y}}_1} - \mathrm{T^U}_{\tilde{\mathcal{Y}}_2}\right| + \left|\mathrm{I^U}_{\tilde{\mathcal{Y}}_1} - \mathrm{I^U}_{\tilde{\mathcal{Y}}_2}\right| + \left|\mathrm{F^U}_{\tilde{\mathcal{Y}}_1} - \mathrm{F^U}_{\tilde{\mathcal{Y}}_2}\right|\right]}{6}.$$

Definition 8:[23] Let

$$\left(\tilde{\mathcal{Y}}^N\right)_1 = \{< (A_{\tilde{y}_1}),(A_{\tilde{y}_1})_T T_{\tilde{y}_1}, (A_{\tilde{y}_1})_I I_{\tilde{y}_1}, (A_{\tilde{y}_1})_F F_{\tilde{y}_1} >\}$$

and

$$\left(\tilde{\mathcal{Y}}^N\right)_2 = \{< (A_{\tilde{y}_2}),(A_{\tilde{y}_2})_T T_{\tilde{y}_2}, (A_{\tilde{y}_2})_I I_{\tilde{y}_2},(A_{\tilde{y}_2})_F F_{\tilde{y}_2} >\}$$

be two IGSVNQNs. $\widehat{S_H}: \left(\tilde{\mathcal{Y}}^N\right)_1 \times \left(\tilde{\mathcal{Y}}^N\right)_2 \to [0,1]$ be function. The Hamming similarity measure between $\tilde{\mathcal{Y}}_1$ and $\tilde{\mathcal{Y}}_2$ denoted by $\widehat{S_H}(\tilde{\mathcal{Y}}_1, \tilde{\mathcal{Y}}_2)$ such that

$$\widehat{S_H}\left(\left(\tilde{\mathcal{Y}}^N\right)_1, \left(\tilde{\mathcal{Y}}^N\right)_2\right) = 1 - \frac{1}{2}\left[\frac{\left(|T_{\tilde{y}_1} - T_{\tilde{y}_2}| + |I_{\tilde{y}_1} - I_{\tilde{y}_2}| + |F_{\tilde{y}_1} - F_{\tilde{y}_2}|\right)}{3} + \frac{4 - \left(\frac{S\left((A_{\tilde{y}_1})\cap(A_{\tilde{y}_2})\right)}{maks\left\{1, S\left((A_{\tilde{y}_1})\cup(A_{\tilde{y}_2})\right)\right\}} + \frac{S\left((A_{\tilde{y}_1})_T\cap(A_{\tilde{y}_2})_T\right)}{maks\left\{1, S\left((A_{\tilde{y}_1})_T\cup(A_{\tilde{y}_2})_T\right)\right\}} + \frac{S\left((A_{\tilde{y}_1})_I\cap(A_{\tilde{y}_2})_I\right)}{maks\left\{1, S\left((A_{\tilde{y}_1})_I\cup(A_{\tilde{y}_2})_I\right)\right\}} + \frac{S\left((A_{\tilde{y}_1})_F\cap(A_{\tilde{y}_2})_F\right)}{maks\left\{1, S\left((A_{\tilde{y}_1})_F\cup(A_{\tilde{y}_2})_F\right)\right\}}\right)}{4}\right].$$

3.HAMMIING DISTANCE MEASURE FOR INTERVAL GENERALIZED SET-VALUED NEUTROSOPHIC QUADRUPLE NUMBERS

Definition 9 Let

$$\left(\tilde{\mathcal{Y}}^N\right)_1 = \{< (A_{\tilde{y}_1}),(A_{\tilde{y}_1})_T\left[T^L{}_{\tilde{y}_1}, T^U{}_{\tilde{y}_1}\right], (A_{\tilde{y}_1})_I\left[I^L{}_{\tilde{y}_1}, I^U{}_{\tilde{y}_1}\right], (A_{\tilde{y}_1})_F\left[F^L{}_{\tilde{y}_1}, F^U{}_{\tilde{y}_1}\right] >\}$$

$$\left(\tilde{\mathcal{Y}}^N\right)_2 = \{< (A_{\tilde{y}_2}),(A_{\tilde{y}_2})_T\left[T^L{}_{\tilde{y}_2}, T^U{}_{\tilde{y}_2}\right], (A_{\tilde{y}_2})_I\left[I^L{}_{\tilde{y}_2}, I^U{}_{\tilde{y}_2}\right],(A_{\tilde{y}_2})_F\left[F^L{}_{\tilde{y}_2}, F^U{}_{\tilde{y}_2}\right] >\}$$

be two IGSVNQNs. $\widehat{S_H}: \left(\tilde{\mathcal{Y}}^N\right)_1 \times \left(\tilde{\mathcal{Y}}^N\right)_2 \to [0,1]$ be a function. Then,

$$\widehat{S_H}\left(\left(\tilde{\mathcal{Y}}^N\right)_1, \left(\tilde{\mathcal{Y}}^N\right)_2\right) = 1 - \frac{1}{2}\left[\frac{\left|T^L{}_{\tilde{y}_1} - T^L{}_{\tilde{y}_2}\right| + \left|I^L{}_{\tilde{y}_1} - I^L{}_{\tilde{y}_2}\right| + \left|F^L{}_{\tilde{y}_1} - F^L{}_{\tilde{y}_2}\right| + \left|T^U{}_{\tilde{y}_1} - T^U{}_{\tilde{y}_2}\right| + \left|I^U{}_{\tilde{y}_1} - I^U{}_{\tilde{y}_2}\right| + \left|F^U{}_{\tilde{y}_1} - F^U{}_{\tilde{y}_2}\right|}{6} + \frac{4 - \left(\frac{S\left((A_{\tilde{y}_1})\cap(A_{\tilde{y}_2})\right)}{maks\left\{1, S\left((A_{\tilde{y}_1})\cup(A_{\tilde{y}_2})\right)\right\}} + \frac{S\left((A_{\tilde{y}_1})_T\cap(A_{\tilde{y}_2})_T\right)}{maks\left\{1, S\left((A_{\tilde{y}_1})_T\cup(A_{\tilde{y}_2})_T\right)\right\}} + \frac{S\left((A_{\tilde{y}_1})_I\cap(A_{\tilde{y}_2})_I\right)}{maks\left\{1, S\left((A_{\tilde{y}_1})_I\cup(A_{\tilde{y}_2})_I\right)\right\}} + \frac{S\left((A_{\tilde{y}_1})_F\cap(A_{\tilde{y}_2})_F\right)}{maks\left\{1, S\left((A_{\tilde{y}_1})_F\cup(A_{\tilde{y}_2})_F\right)\right\}}\right)}{4}\right].$$

is called generalized Hamming similarity measure for IGSVNQNs.

Example 1: Let

$$\left(\tilde{\mathcal{Y}}^N\right)_1 = \{< \{\hat{\tau}^1, \hat{\tau}^2, \hat{\tau}^4, \hat{\tau}^5, \hat{\tau}^6\}, \{\hat{\tau}^2, \hat{\tau}^4\}[0.32, 0.5], \{\hat{\tau}^1, \hat{\tau}^5\}[0.1, 0.75], \{\hat{\tau}^6, \hat{\tau}^5, \hat{\tau}^4\}[0, 0.2] >\}$$

and

$$\left(\tilde{y}^N\right)_2 = \{< \{\hat{\tau}^1, \hat{\tau}^5, \hat{\tau}^6\}, \{\hat{\tau}^5, \hat{\tau}^1\}[0, 0.25], \{\hat{\tau}^6\}[0.55, 0.7], \{\hat{\tau}^5\}[0.1, 0.8] >\}$$

be two IGSVNQNs and $\widehat{S_H}\left(\left(\tilde{y}^N\right)_1, \left(\tilde{y}^N\right)_2\right)$ be generalized Hamming similarity measure for IGSVNQNs. Then

$$\widehat{S_H}\left(\left(\tilde{y}^N\right)_1, \left(\tilde{y}^N\right)_2\right) = 1 - \frac{1}{2}\left[\frac{\left|T^L{}_{\tilde{y}_1} - T^L{}_{\tilde{y}_2}\right| + \left|I^L{}_{\tilde{y}_1} - I^L{}_{\tilde{y}_2}\right| + \left|F^L{}_{\tilde{y}_1} - F^L{}_{\tilde{y}_2}\right| + \left|T^U{}_{\tilde{y}_1} - T^U{}_{\tilde{y}_2}\right| + \left|I^U{}_{\tilde{y}_1} - I^U{}_{\tilde{y}_2}\right| + \left|F^U{}_{\tilde{y}_1} - F^U{}_{\tilde{y}_2}\right|}{6} + \frac{4 - \left(\frac{S\left((A_{\tilde{y}_1}) \cap (A_{\tilde{y}_2})\right)}{maks\left\{1, S\left((A_{\tilde{y}_1}) \cup (A_{\tilde{y}_2})\right)\right\}} + \frac{S\left((A_{\tilde{y}_1})_T \cap (A_{\tilde{y}_2})_T\right)}{maks\left\{1, S\left((A_{\tilde{y}_1})_T \cup (A_{\tilde{y}_2})_T\right)\right\}} + \frac{S\left((A_{\tilde{y}_1})_I \cap (A_{\tilde{y}_2})_I\right)}{maks\left\{1, S\left((A_{\tilde{y}_1})_I \cup (A_{\tilde{y}_2})_I\right)\right\}} + \frac{S\left((A_{\tilde{y}_1})_F \cap (A_{\tilde{y}_2})_F\right)}{maks\left\{1, S\left((A_{\tilde{y}_1})_F \cup (A_{\tilde{y}_2})_F\right)\right\}}\right)}{4}\right]$$

$$= 1 - \frac{1}{2}\left[\frac{|0.32 - 0| + |0.1 - 0.55| + |0 - 0.1| + |0.5 - 0.25| + |0.75 - 0.7| + |0.2 - 0.8|}{6} + \frac{4 - \left(\frac{3}{maks\{1,6\}} + \frac{0}{maks\{1,4\}} + \frac{0}{maks\{1,3\}} + \frac{1}{maks\{1,3\}}\right)}{4}\right] = 0{,}373.$$

Theorem 2: Let

$\left(\tilde{y}^N\right)_1 = \{< (A_{\tilde{y}_1}), (A_{\tilde{y}_1})_T\left[T^L{}_{\tilde{y}_1}, T^U{}_{\tilde{y}_1}\right], (A_{\tilde{y}_1})_I\left[I^L{}_{\tilde{y}_1}, I^U{}_{\tilde{y}_1}\right], (A_{\tilde{y}_1})_F\left[F^L{}_{\tilde{y}_1}, F^U{}_{\tilde{y}_1}\right] >\}$,

$\left(\tilde{y}^N\right)_2 = \{< (A_{\tilde{y}_2}), (A_{\tilde{y}_2})_T\left[T^L{}_{\tilde{y}_2}, T^U{}_{\tilde{y}_2}\right], (A_{\tilde{y}_2})_I\left[I^L{}_{\tilde{y}_2}, I^U{}_{\tilde{y}_2}\right], (A_{\tilde{y}_2})_F\left[F^L{}_{\tilde{y}_2}, F^U{}_{\tilde{y}_2}\right] >\}$

and

$\left(\tilde{y}^N\right)_3 = \{< (A_{\tilde{y}_3}), (A_{\tilde{y}_3})_T\left[T^L{}_{\tilde{y}_3}, T^U{}_{\tilde{y}_3}\right], (A_{\tilde{y}_3})_I\left[I^L{}_{\tilde{y}_3}, I^U{}_{\tilde{y}_3}\right], (A_{\tilde{y}_3})_F\left[F^L{}_{\tilde{y}_3}, F^U{}_{\tilde{y}_3}\right] >\}$

be three IGSVNQNs and $\widehat{S_H}: \left(\tilde{y}^N\right)_1 \times \left(\tilde{y}^N\right)_2 \rightarrow [0,1]$ be generalized Hamming similarity measure in Definition 9. Then, $\widehat{S_H}$ satisfies the following conditions.

i) $\widehat{S_H}\left(\left(\tilde{y}^N\right)_1, \left(\tilde{y}^N\right)_2\right) \in [0,1]$

ii) $\widehat{S_H}\left(\left(\tilde{y}^N\right)_1, \left(\tilde{y}^N\right)_2\right) = 1 \Leftrightarrow \left(\tilde{y}^N\right)_1 = \left(\tilde{y}^N\right)_2$

iii) $\widehat{S_H}\left(\left(\tilde{y}^N\right)_1, \left(\tilde{y}^N\right)_2\right) = \widehat{S_H}\left(\left(\tilde{y}^N\right)_2, \left(\tilde{y}^N\right)_1\right)$

iv) If $\left(\tilde{y}^N\right)_1 \subset \left(\tilde{y}^N\right)_2 \subset \left(\tilde{y}^N\right)_3$, then

$\widehat{S_H}\left(\left(\tilde{y}^N\right)_1,\left(\tilde{y}^N\right)_3\right) \le \widehat{S_H}\left(\left(\tilde{y}^N\right)_1,\left(\tilde{y}^N\right)_2\right)$ and $\widehat{S_H}\left(\left(\tilde{y}^N\right)_1,\left(\tilde{y}^N\right)_3\right) \le \widehat{S_H}\left(\left(\tilde{y}^N\right)_2,\left(\tilde{y}^N\right)_3\right)$

Proof:

i) Let $\left(\tilde{y}^N\right)_1 = \left(\tilde{y}^N\right)_2$. Then,

$$\widehat{S_H}\left(\left(\tilde{y}^N\right)_1,\left(\tilde{y}^N\right)_2\right) = 1 - \frac{1}{2}\left[\frac{\left|T^L{}_{\tilde{y}_1} - T^L{}_{\tilde{y}_2}\right| + \left|I^L{}_{\tilde{y}_1} - I^L{}_{\tilde{y}_2}\right| + \left|F^L{}_{\tilde{y}_1} - F^L{}_{\tilde{y}_2}\right| + \left|T^U{}_{\tilde{y}_1} - T^U{}_{\tilde{y}_2}\right| + \left|I^U{}_{\tilde{y}_1} - I^U{}_{\tilde{y}_2}\right| + \left|F^U{}_{\tilde{y}_1} - F^U{}_{\tilde{y}_2}\right|}{6} + \frac{4 - \left(\frac{S\left((A_{\tilde{y}_1})\cap(A_{\tilde{y}_2})\right)}{maks\left\{1,S\left((A_{\tilde{y}_1})\cup(A_{\tilde{y}_2})\right)\right\}} + \frac{S\left((A_{\tilde{y}_1})_T\cap(A_{\tilde{y}_2})_T\right)}{maks\left\{1,S\left((A_{\tilde{y}_1})_T\cup(A_{\tilde{y}_2})_T\right)\right\}} + \frac{S\left((A_{\tilde{y}_1})_I\cap(A_{\tilde{y}_2})_I\right)}{maks\left\{1,S\left((A_{\tilde{y}_1})_I\cup(A_{\tilde{y}_2})_I\right)\right\}} + \frac{S\left((A_{\tilde{y}_1})_F\cap(A_{\tilde{y}_2})_F\right)}{maks\left\{1,S\left((A_{\tilde{y}_1})_F\cup(A_{\tilde{y}_2})_F\right)\right\}}\right)}{4}\right]$$

$$=1 - \frac{1}{2}\left[\frac{0+0+0+0+0+0}{6} + \frac{4-[1+1+1+1]}{4}\right]$$

$$=1 \qquad (1)$$

Thus $\max\left\{\widehat{S_H}\left(\left(\tilde{y}^N\right)_1,\left(\tilde{y}^N\right)_2\right)\right\} = 1$.

Now, let

$(A_{\tilde{y}_1}) \cap (A_{\tilde{y}_2}) = \emptyset$, $(A_{\tilde{y}_1})_T \cap (A_{\tilde{y}_2})_T = \emptyset$, $(A_{\tilde{y}_1})_I \cap (A_{\tilde{y}_2})_I = \emptyset$, $(A_{\tilde{y}_1})_F \cap (A_{\tilde{y}_2})_F = \emptyset$

and

$\left|T^L{}_{\tilde{y}_1} - T^L{}_{\tilde{y}_2}\right| = 1$, $\left|T^U{}_{\tilde{y}_1} - T^U{}_{\tilde{y}_2}\right| = 1$,

$\left|I^L{}_{\tilde{y}_1} - I^L{}_{\tilde{y}_2}\right| = 1$, $\left|I^U{}_{\tilde{y}_1} - I^U{}_{\tilde{y}_2}\right| = 1$,

$\left|F^L{}_{\tilde{y}_1} - F^L{}_{\tilde{y}_2}\right| = 1$, $\left|F^U{}_{\tilde{y}_1} - F^U{}_{\tilde{y}_2}\right| = 1$.

Then

$$\widehat{S_H}\left(\left(\tilde{y}^N\right)_1,\left(\tilde{y}^N\right)_2\right) = 1 - \frac{1}{2}\left[\frac{\left|T^L{}_{\tilde{y}_1} - T^L{}_{\tilde{y}_2}\right| + \left|I^L{}_{\tilde{y}_1} - I^L{}_{\tilde{y}_2}\right| + \left|F^L{}_{\tilde{y}_1} - F^L{}_{\tilde{y}_2}\right| + \left|T^U{}_{\tilde{y}_1} - T^U{}_{\tilde{y}_2}\right| + \left|I^U{}_{\tilde{y}_1} - I^U{}_{\tilde{y}_2}\right| + \left|F^U{}_{\tilde{y}_1} - F^U{}_{\tilde{y}_2}\right|}{6} + \frac{4 - \left(\frac{S\left((A_{\tilde{y}_1})\cap(A_{\tilde{y}_2})\right)}{maks\left\{1,S\left((A_{\tilde{y}_1})\cup(A_{\tilde{y}_2})\right)\right\}} + \frac{S\left((A_{\tilde{y}_1})_T\cap(A_{\tilde{y}_2})_T\right)}{maks\left\{1,S\left((A_{\tilde{y}_1})_T\cup(A_{\tilde{y}_2})_T\right)\right\}} + \frac{S\left((A_{\tilde{y}_1})_I\cap(A_{\tilde{y}_2})_I\right)}{maks\left\{1,S\left((A_{\tilde{y}_1})_I\cup(A_{\tilde{y}_2})_I\right)\right\}} + \frac{S\left((A_{\tilde{y}_1})_F\cap(A_{\tilde{y}_2})_F\right)}{maks\left\{1,S\left((A_{\tilde{y}_1})_F\cup(A_{\tilde{y}_2})_F\right)\right\}}\right)}{4}\right]$$

$$=1-\frac{1}{2}\left[\frac{1+1+1+1+1+1}{6}+\frac{4-[0+0+0+0]}{4}\right]$$

$$=0 \qquad (2)$$

Thus, $\min\left\{\widehat{S_H}\left((\tilde{y}^N)_1,(\tilde{y}^N)_2\right)\right\}=0.$

Hence, we obtain

$$\widehat{S_H}\left((\tilde{y}^N)_1,(\tilde{y}^N)_2\right)\in[0,1].$$

ii) Let $(\tilde{y}^N)_1=(\tilde{y}^N)_2$. From (1), we obtain $\widehat{S_H}\left((\tilde{y}^N)_1,(\tilde{y}^N)_2\right)=1$. We assume that

$$\widehat{S_H}\left((\tilde{y}^N)_1,(\tilde{y}^N)_2\right)=1-\frac{1}{2}\left[\frac{|T^L{}_{\tilde{y}_1}-T^L{}_{\tilde{y}_2}|+|I^L{}_{\tilde{y}_1}-I^L{}_{\tilde{y}_2}|+|F^L{}_{\tilde{y}_1}-F^L{}_{\tilde{y}_2}|+|T^U{}_{\tilde{y}_1}-T^U{}_{\tilde{y}_2}|+|I^U{}_{\tilde{y}_1}-I^U{}_{\tilde{y}_2}|+|F^U{}_{\tilde{y}_1}-F^U{}_{\tilde{y}_2}|}{6}+\frac{4-\left(\frac{S\left((A_{\tilde{y}_1})\cap(A_{\tilde{y}_2})\right)}{maks\left\{1,S\left((A_{\tilde{y}_1})\cup(A_{\tilde{y}_2})\right)\right\}}+\frac{S\left((A_{\tilde{y}_1})_T\cap(A_{\tilde{y}_2})_T\right)}{maks\left\{1,S\left((A_{\tilde{y}_1})_T\cup(A_{\tilde{y}_2})_T\right)\right\}}+\frac{S\left((A_{\tilde{y}_1})_I\cap(A_{\tilde{y}_2})_I\right)}{maks\left\{1,S\left((A_{\tilde{y}_1})_I\cup(A_{\tilde{y}_2})_I\right)\right\}}+\frac{S\left((A_{\tilde{y}_1})_F\cap(A_{\tilde{y}_2})_F\right)}{maks\left\{1,S\left((A_{\tilde{y}_1})_F\cup(A_{\tilde{y}_2})_F\right)\right\}}\right)}{4}\right]$$

$$=1$$

Where, it must be

$$\frac{1}{2}\left[\frac{|T^L{}_{\tilde{y}_1}-T^L{}_{\tilde{y}_2}|+|I^L{}_{\tilde{y}_1}-I^L{}_{\tilde{y}_2}|+|F^L{}_{\tilde{y}_1}-F^L{}_{\tilde{y}_2}|+|T^U{}_{\tilde{y}_1}-T^U{}_{\tilde{y}_2}|+|I^U{}_{\tilde{y}_1}-I^U{}_{\tilde{y}_2}|+|F^U{}_{\tilde{y}_1}-F^U{}_{\tilde{y}_2}|}{6}+\frac{4-\left(\frac{S\left((A_{\tilde{y}_1})\cap(A_{\tilde{y}_2})\right)}{maks\left\{1,S\left((A_{\tilde{y}_1})\cup(A_{\tilde{y}_2})\right)\right\}}+\frac{S\left((A_{\tilde{y}_1})_T\cap(A_{\tilde{y}_2})_T\right)}{maks\left\{1,S\left((A_{\tilde{y}_1})_T\cup(A_{\tilde{y}_2})_T\right)\right\}}+\frac{S\left((A_{\tilde{y}_1})_I\cap(A_{\tilde{y}_2})_I\right)}{maks\left\{1,S\left((A_{\tilde{y}_1})_I\cup(A_{\tilde{y}_2})_I\right)\right\}}+\frac{S\left((A_{\tilde{y}_1})_F\cap(A_{\tilde{y}_2})_F\right)}{maks\left\{1,S\left((A_{\tilde{y}_1})_F\cup(A_{\tilde{y}_2})_F\right)\right\}}\right)}{4}\right]$$

$$=0$$

Thus,

$$|T^L{}_{\tilde{y}_1}-T^L{}_{\tilde{y}_2}|+|I^L{}_{\tilde{y}_1}-I^L{}_{\tilde{y}_2}|+|F^L{}_{\tilde{y}_1}-F^L{}_{\tilde{y}_2}|+|T^U{}_{\tilde{y}_1}-T^U{}_{\tilde{y}_2}|+|I^U{}_{\tilde{y}_1}-I^U{}_{\tilde{y}_2}|+|F^U{}_{\tilde{y}_1}-F^U{}_{\tilde{y}_2}|=0$$

and

$$4-\left(\frac{S\left((A_{\tilde{y}_1})\cap(A_{\tilde{y}_2})\right)}{maks\left\{1,S\left((A_{\tilde{y}_1})\cup(A_{\tilde{y}_2})\right)\right\}}+\frac{S\left((A_{\tilde{y}_1})_T\cap(A_{\tilde{y}_2})_T\right)}{maks\left\{1,S\left((A_{\tilde{y}_1})_T\cup(A_{\tilde{y}_2})_T\right)\right\}}+\frac{S\left((A_{\tilde{y}_1})_I\cap(A_{\tilde{y}_2})_I\right)}{maks\left\{1,S\left((A_{\tilde{y}_1})_I\cup(A_{\tilde{y}_2})_I\right)\right\}}+\frac{S\left((A_{\tilde{y}_1})_F\cap(A_{\tilde{y}_2})_F\right)}{maks\left\{1,S\left((A_{\tilde{y}_1})_F\cup(A_{\tilde{y}_2})_F\right)\right\}}\right)=0 \qquad (2)$$

From (2), we obtain that

$\left|T^L{}_{\tilde{y}_1}-T^L{}_{\tilde{y}_2}\right|=0$, $\left|I^L{}_{\tilde{y}_1}-I^L{}_{\tilde{y}_2}\right|=0$, $\left|F^L{}_{\tilde{y}_1}-F^L{}_{\tilde{y}_2}\right|=0$,

$\left|T^U{}_{\tilde{y}_1}-T^U{}_{\tilde{y}_2}\right|=0$, $\left|I^U{}_{\tilde{y}_1}-I^U{}_{\tilde{y}_2}\right|=0$, $\left|F^U{}_{\tilde{y}_1}-F^U{}_{\tilde{y}_2}\right|=0$

and

$$\frac{S\left((A_{\tilde{y}_1})\cap(A_{\tilde{y}_2})\right)}{maks\left\{1,S\left((A_{\tilde{y}_1})\cup(A_{\tilde{y}_2})\right)\right\}}=\frac{S\left((A_{\tilde{y}_1})_T\cap(A_{\tilde{y}_2})_T\right)}{maks\left\{1,S\left((A_{\tilde{y}_1})_T\cup(A_{\tilde{y}_2})_T\right)\right\}}$$
$$=\frac{S\left((A_{\tilde{y}_1})_I\cap(A_{\tilde{y}_2})_I\right)}{maks\left\{1,S\left((A_{\tilde{y}_1})_I\cup(A_{\tilde{y}_2})_I\right)\right\}}=\frac{S\left((A_{\tilde{y}_1})_F\cap(A_{\tilde{y}_2})_F\right)}{maks\left\{1,S\left((A_{\tilde{y}_1})_F\cup(A_{\tilde{y}_2})_F\right)\right\}}=1$$

Thus, we have that

$T^L{}_{\tilde{y}_1}=T^L{}_{\tilde{y}_2}$, $I^L{}_{\tilde{y}_1}=I^L{}_{\tilde{y}_2}$, $F^L{}_{\tilde{y}_1}=F^L{}_{\tilde{y}_2}$, $T^U{}_{\tilde{y}_1}=T^U{}_{\tilde{y}_2}$, $I^U{}_{\tilde{y}_1}=I^U{}_{\tilde{y}_2}$,$F^U{}_{\tilde{y}_1}=F^U{}_{\tilde{y}_2}$,

$\left(A_{\tilde{y}_1}\right)=(A_{\tilde{y}_2})$, $(A_{\tilde{y}_1})_T=(A_{\tilde{y}_2})_T$, $(A_{\tilde{y}_1})_I=(A_{\tilde{y}_2})_I$, $(A_{\tilde{y}_1})_F=(A_{\tilde{y}_2})_F$.

Therefore, from Definition 5; we obtain

$$\left(\tilde{\mathcal{Y}}^N\right)_1=\left(\tilde{\mathcal{Y}}^N\right)_2.$$

iii)

$$\widehat{S_H}\left(\left(\tilde{\mathcal{Y}}^N\right)_1,\left(\tilde{\mathcal{Y}}^N\right)_2\right)$$
$$=1-\frac{1}{2}\left[\frac{\left|T^L{}_{\tilde{y}_1}-T^L{}_{\tilde{y}_2}\right|+\left|I^L{}_{\tilde{y}_1}-I^L{}_{\tilde{y}_2}\right|+\left|F^L{}_{\tilde{y}_1}-F^L{}_{\tilde{y}_2}\right|+\left|T^U{}_{\tilde{y}_1}-T^U{}_{\tilde{y}_2}\right|+\left|I^U{}_{\tilde{y}_1}-I^U{}_{\tilde{y}_2}\right|+\left|F^U{}_{\tilde{y}_1}-F^U{}_{\tilde{y}_2}\right|}{6}\right.$$
$$\left.+\frac{4-\left(\frac{S\left((A_{\tilde{y}_1})\cap(A_{\tilde{y}_2})\right)}{maks\left\{1,S\left((A_{\tilde{y}_1})\cup(A_{\tilde{y}_2})\right)\right\}}+\frac{S\left((A_{\tilde{y}_1})_T\cap(A_{\tilde{y}_2})_T\right)}{maks\left\{1,S\left((A_{\tilde{y}_1})_T\cup(A_{\tilde{y}_2})_T\right)\right\}}+\frac{S\left((A_{\tilde{y}_1})_I\cap(A_{\tilde{y}_2})_I\right)}{maks\left\{1,S\left((A_{\tilde{y}_1})_I\cup(A_{\tilde{y}_2})_I\right)\right\}}+\frac{S\left((A_{\tilde{y}_1})_F\cap(A_{\tilde{y}_2})_F\right)}{maks\left\{1,S\left((A_{\tilde{y}_1})_F\cup(A_{\tilde{y}_2})_F\right)\right\}}\right)}{4}\right]$$

$$=1-\frac{1}{2}\left[\frac{\left|T^L{}_{\tilde{y}_1}-T^L{}_{\tilde{y}_2}\right|+\left|I^L{}_{\tilde{y}_1}-I^L{}_{\tilde{y}_2}\right|+\left|F^L{}_{\tilde{y}_1}-F^L{}_{\tilde{y}_2}\right|+\left|T^U{}_{\tilde{y}_1}-T^U{}_{\tilde{y}_2}\right|+\left|I^U{}_{\tilde{y}_1}-I^U{}_{\tilde{y}_2}\right|+\left|F^U{}_{\tilde{y}_1}-F^U{}_{\tilde{y}_2}\right|}{6}\right.$$

$$+\frac{4-\left(\frac{s\left((A_{\tilde{y}_2})\cap(A_{\tilde{y}_1})\right)}{maks\left\{1,s\left((A_{\tilde{y}_2})\cup(A_{\tilde{y}_1})\right)\right\}}+\frac{s\left((A_{\tilde{y}_2})_T\cap(A_{\tilde{y}_1})_T\right)}{maks\left\{1,s\left((A_{\tilde{y}_2})_T\cup(A_{\tilde{y}_1})_T\right)\right\}}+\frac{s\left((A_{\tilde{y}_2})_I\cap(A_{\tilde{y}_1})_I\right)}{maks\left\{1,s\left((A_{\tilde{y}_2})_I\cup(A_{\tilde{y}_1})_I\right)\right\}}+\frac{s\left((A_{\tilde{y}_2})_F\cap(A_{\tilde{y}_1})_F\right)}{maks\left\{1,s\left((A_{\tilde{y}_2})_F\cup(A_{\tilde{y}_1})_F\right)\right\}}\right)}{4}\Bigg]$$

$$=\widehat{S_H}\left((\tilde{y}^N)_2,(\tilde{y}^N)_1\right).$$

iv) Let $(\tilde{y}^N)_1 \subset (\tilde{y}^N)_2 \subset (\tilde{y}^N)_3$. From Definition 5, we obtain that

$(A_{\tilde{y}_1}) \subset (A_{\tilde{y}_2}) \subset (A_{\tilde{y}_3})$, $(A_{\tilde{y}_1})_T \subset (A_{\tilde{y}_2})_T \subset (A_{\tilde{y}_3})_T$, $(A_{\tilde{y}_1})_I \subset (A_{\tilde{y}_2})_I \subset (A_{\tilde{y}_3})_I$, $(A_{\tilde{y}_1})_F \subset (A_{\tilde{y}_2})_F \subset (A_{\tilde{y}_3})_F$;

$T^L{}_{\tilde{y}_1} \le T^L{}_{\tilde{y}_2} \le T^L{}_{\tilde{y}_3},\ T^U{}_{\tilde{y}_1} \le T^U{}_{\tilde{y}_2} \le T^U{}_{\tilde{y}_3};\ I^L{}_{\tilde{y}_1} \le I^L{}_{\tilde{y}_2} \le I^L{}_{\tilde{y}_3},\ I^U{}_{\tilde{y}_1} \le I^U{}_{\tilde{y}_2} \le I^U{}_{\tilde{y}_3};$

$F^L{}_{\tilde{y}_1} \le F^L{}_{\tilde{y}_2} \le F^L{}_{\tilde{y}_3},\ F^U{}_{\tilde{y}_1} \le F^U{}_{\tilde{y}_2} \le F^U{}_{\tilde{y}_3}$. (3)

From (3), we have that

$$\frac{s\left((A_{\tilde{y}_1})\cap(A_{\tilde{y}_2})\right)}{maks\left\{1,s\left((A_{\tilde{y}_1})\cup(A_{\tilde{y}_2})\right)\right\}}+\frac{s\left((A_{\tilde{y}_1})_T\cap(A_{\tilde{y}_2})_T\right)}{maks\left\{1,s\left((A_{\tilde{y}_1})_T\cup(A_{\tilde{y}_2})_T\right)\right\}}+\frac{s\left((A_{\tilde{y}_1})_I\cap(A_{\tilde{y}_2})_I\right)}{maks\left\{1,s\left((A_{\tilde{y}_1})_I\cup(A_{\tilde{y}_2})_I\right)\right\}}+\frac{s\left((A_{\tilde{y}_1})_F\cap(A_{\tilde{y}_2})_F\right)}{maks\left\{1,s\left((A_{\tilde{y}_1})_F\cup(A_{\tilde{y}_2})_F\right)\right\}}>$$

$$\frac{s\left((A_{\tilde{y}_1})\cap(A_{\tilde{y}_3})\right)}{maks\left\{1,s\left((A_{\tilde{y}_1})\cup(A_{\tilde{y}_3})\right)\right\}}+\frac{s\left((A_{\tilde{y}_1})_T\cap(A_{\tilde{y}_3})_T\right)}{maks\left\{1,s\left((A_{\tilde{y}_1})_T\cup(A_{\tilde{y}_3})_T\right)\right\}}+\frac{s\left((A_{\tilde{y}_1})_I\cap(A_{\tilde{y}_3})_I\right)}{maks\left\{1,s\left((A_{\tilde{y}_1})_I\cup(A_{\tilde{y}_3})_I\right)\right\}}+\frac{s\left((A_{\tilde{y}_1})_F\cap(A_{\tilde{y}_3})_F\right)}{maks\left\{1,s\left((A_{\tilde{y}_1})_F\cup(A_{\tilde{y}_3})_F\right)\right\}} \quad (4)$$

Also, From (4), we have that

$$\left|T^L{}_{\tilde{y}_1}-T^L{}_{\tilde{y}_2}\right|+\left|I^L{}_{\tilde{y}_1}-I^L{}_{\tilde{y}_2}\right|+\left|F^L{}_{\tilde{y}_1}-F^L{}_{\tilde{y}_2}\right|+\left|T^U{}_{\tilde{y}_1}-T^U{}_{\tilde{y}_2}\right|+\left|I^U{}_{\tilde{y}_1}-I^U{}_{\tilde{y}_2}\right|+\left|F^U{}_{\tilde{y}_1}-F^U{}_{\tilde{y}_2}\right|<$$

$$\left|T^L{}_{\tilde{y}_1}-T^L{}_{\tilde{y}_3}\right|+\left|I^L{}_{\tilde{y}_1}-I^L{}_{\tilde{y}_3}\right|+\left|F^L{}_{\tilde{y}_1}-F^L{}_{\tilde{y}_3}\right|+\left|T^U{}_{\tilde{y}_1}-T^U{}_{\tilde{y}_3}\right|+\left|I^U{}_{\tilde{y}_1}-I^U{}_{\tilde{y}_3}\right|+\left|F^U{}_{\tilde{y}_1}-F^U{}_{\tilde{y}_3}\right| \quad (5)$$

Thus, from (4) and (5), we obtain that

$$\frac{1}{2}\left[\frac{\left|T^L{}_{\tilde{y}_1}-T^L{}_{\tilde{y}_2}\right|+\left|I^L{}_{\tilde{y}_1}-I^L{}_{\tilde{y}_2}\right|+\left|F^L{}_{\tilde{y}_1}-F^L{}_{\tilde{y}_2}\right|+\left|T^U{}_{\tilde{y}_1}-T^U{}_{\tilde{y}_2}\right|+\left|I^U{}_{\tilde{y}_1}-I^U{}_{\tilde{y}_2}\right|+\left|F^U{}_{\tilde{y}_1}-F^U{}_{\tilde{y}_2}\right|}{6}\right.$$

$$+\frac{4-\left(\frac{S\left((A_{\tilde{y}_1})\cap(A_{\tilde{y}_2})\right)}{maks\left\{1,S\left((A_{\tilde{y}_1})\cup(A_{\tilde{y}_2})\right)\right\}}+\frac{S\left((A_{\tilde{y}_1})_T\cap(A_{\tilde{y}_2})_T\right)}{maks\left\{1,S\left((A_{\tilde{y}_1})_T\cup(A_{\tilde{y}_2})_T\right)\right\}}+\frac{S\left((A_{\tilde{y}_1})_I\cap(A_{\tilde{y}_2})_I\right)}{maks\left\{1,S\left((A_{\tilde{y}_1})_I\cup(A_{\tilde{y}_2})_I\right)\right\}}+\frac{S\left((A_{\tilde{y}_1})_F\cap(A_{\tilde{y}_2})_F\right)}{maks\left\{1,S\left((A_{\tilde{y}_1})_F\cup(A_{\tilde{y}_2})_F\right)\right\}}\right)}{4}\Bigg]$$

$$<$$

$$\frac{1}{2}\Bigg[\frac{\left|T^L{}_{\tilde{y}_1}-T^L{}_{\tilde{y}_3}\right|+\left|I^L{}_{\tilde{y}_1}-I^L{}_{\tilde{y}_3}\right|+\left|F^L{}_{\tilde{y}_1}-F^L{}_{\tilde{y}_3}\right|+\left|T^U{}_{\tilde{y}_1}-T^U{}_{\tilde{y}_3}\right|+\left|I^U{}_{\tilde{y}_1}-I^U{}_{\tilde{y}_3}\right|+\left|F^U{}_{\tilde{y}_1}-F^U{}_{\tilde{y}_3}\right|}{6}$$

$$+\frac{4-\left(\frac{S\left((A_{\tilde{y}_1})\cap(A_{\tilde{y}_3})\right)}{maks\left\{1,S\left((A_{\tilde{y}_1})\cup(A_{\tilde{y}_3})\right)\right\}}+\frac{S\left((A_{\tilde{y}_1})_T\cap(A_{\tilde{y}_3})_T\right)}{maks\left\{1,S\left((A_{\tilde{y}_1})_T\cup(A_{\tilde{y}_3})_T\right)\right\}}+\frac{S\left((A_{\tilde{y}_1})_I\cap(A_{\tilde{y}_3})_I\right)}{maks\left\{1,S\left((A_{\tilde{y}_1})_I\cup(A_{\tilde{y}_3})_I\right)\right\}}+\frac{S\left((A_{\tilde{y}_1})_F\cap(A_{\tilde{y}_3})_F\right)}{maks\left\{1,S\left((A_{\tilde{y}_1})_F\cup(A_{\tilde{y}_3})_F\right)\right\}}\right)}{4}\Bigg]$$

(6)

Hence, from (6), we have that

$$1-\frac{1}{2}\Bigg[\frac{\left|T^L{}_{\tilde{y}_1}-T^L{}_{\tilde{y}_3}\right|+\left|I^L{}_{\tilde{y}_1}-I^L{}_{\tilde{y}_3}\right|+\left|F^L{}_{\tilde{y}_1}-F^L{}_{\tilde{y}_3}\right|+\left|T^U{}_{\tilde{y}_1}-T^U{}_{\tilde{y}_3}\right|+\left|I^U{}_{\tilde{y}_1}-I^U{}_{\tilde{y}_3}\right|+\left|F^U{}_{\tilde{y}_1}-F^U{}_{\tilde{y}_3}\right|}{6}$$

$$+\frac{4-\left(\frac{S\left((A_{\tilde{y}_1})\cap(A_{\tilde{y}_3})\right)}{maks\left\{1,S\left((A_{\tilde{y}_1})\cup(A_{\tilde{y}_3})\right)\right\}}+\frac{S\left((A_{\tilde{y}_1})_T\cap(A_{\tilde{y}_3})_T\right)}{maks\left\{1,S\left((A_{\tilde{y}_1})_T\cup(A_{\tilde{y}_3})_T\right)\right\}}+\frac{S\left((A_{\tilde{y}_1})_I\cap(A_{\tilde{y}_3})_I\right)}{maks\left\{1,S\left((A_{\tilde{y}_1})_I\cup(A_{\tilde{y}_3})_I\right)\right\}}+\frac{S\left((A_{\tilde{y}_1})_F\cap(A_{\tilde{y}_3})_F\right)}{maks\left\{1,S\left((A_{\tilde{y}_1})_F\cup(A_{\tilde{y}_3})_F\right)\right\}}\right)}{4}\Bigg]$$

$$>$$

$$1-\frac{1}{2}\Bigg[\frac{\left|T^L{}_{\tilde{y}_1}-T^L{}_{\tilde{y}_2}\right|+\left|I^L{}_{\tilde{y}_1}-I^L{}_{\tilde{y}_2}\right|+\left|F^L{}_{\tilde{y}_1}-F^L{}_{\tilde{y}_2}\right|+\left|T^U{}_{\tilde{y}_1}-T^U{}_{\tilde{y}_2}\right|+\left|I^U{}_{\tilde{y}_1}-I^U{}_{\tilde{y}_2}\right|+\left|F^U{}_{\tilde{y}_1}-F^U{}_{\tilde{y}_2}\right|}{6}$$

$$+\frac{4-\left(\frac{S\left((A_{\tilde{y}_1})\cap(A_{\tilde{y}_2})\right)}{maks\left\{1,S\left((A_{\tilde{y}_1})\cup(A_{\tilde{y}_2})\right)\right\}}+\frac{S\left((A_{\tilde{y}_1})_T\cap(A_{\tilde{y}_2})_T\right)}{maks\left\{1,S\left((A_{\tilde{y}_1})_T\cup(A_{\tilde{y}_2})_T\right)\right\}}+\frac{S\left((A_{\tilde{y}_1})_I\cap(A_{\tilde{y}_2})_I\right)}{maks\left\{1,S\left((A_{\tilde{y}_1})_I\cup(A_{\tilde{y}_2})_I\right)\right\}}+\frac{S\left((A_{\tilde{y}_1})_F\cap(A_{\tilde{y}_2})_F\right)}{maks\left\{1,S\left((A_{\tilde{y}_1})_F\cup(A_{\tilde{y}_2})_F\right)\right\}}\right)}{4}\Bigg]$$

Therefore, we obtain $\widehat{S_H}\left(\left(\tilde{y}^N\right)_1,\left(\tilde{y}^N\right)_3\right)\leq\widehat{S_H}\left(\left(\tilde{y}^N\right)_1,\left(\tilde{y}^N\right)_2\right)$

Also, $\widehat{S_H}\left(\left(\tilde{y}^N\right)_1,\left(\tilde{y}^N\right)_3\right)\leq\widehat{S_H}\left(\left(\tilde{y}^N\right)_2,\left(\tilde{y}^N\right)_3\right)$ can be proved similar to $\widehat{S_H}\left(\left(\tilde{y}^N\right)_1,\left(\tilde{y}^N\right)_3\right)\leq\widehat{S_H}\left(\left(\tilde{y}^N\right)_1,\left(\tilde{y}^N\right)_2\right)$.

4.Application

In this section, we aim to identify the optimal university by considering various factors affecting university selection. For this purpose, the Hamming similarity measure defined for IGSVNQNs in Definition 9 is used. In this application, fictitious data is used to demonstrate the usability of the Hamming similarity measure. Researchers can use this application to obtain solutions to their own problems with real data. The factors used in the application were taken from Odabaş, Yakar and Gündeğer's study titled "Scaling the reasons of university students for choosing university by pairwise comparison method" [28].

Example 2: Using the Hamming similarity measure defined for interval generalized set-valued neutrosophic quadruple numbers, the factors affecting university selection and the most ideal university will be determined.

Step 1: The set of factors influencing university choice is defined as

$\partial = \{\partial_1, \partial_2, \partial_3, \partial_4, \partial_5, \partial_6, \partial_7\}$.

Here,

∂_1 = Quality of education at the university

∂_2 = Social and physical facilities of the university

∂_3 = Social and physical facilities of the city where the university is located

∂_4 = Place in the standings

∂_5 = Location in the same/close province with family or friends

∂_6 = Family request

∂_7 = Guidance from the guidance specialist

7 factors were identified.

Step 2: The set of universities is defined as

$\mathcal{C} = \{\mathcal{C}_1, \mathcal{C}_2, \mathcal{C}_3, \mathcal{C}_4, \mathcal{C}_5, \mathcal{C}_6, \mathcal{C}_7, \mathcal{C}_8, \mathcal{C}_9, \mathcal{C}_{10}\}$.

Step 3: Using the factors influencing university choice, each university is determined as an IGSVNQNs.

$\mathcal{C}_1 = \{< \{\partial_1, \partial_2, \partial_4, \partial_5, \partial_7\}, \{\partial_1, \partial_4, \partial_5\}[0.54, 0.6], \{\partial_1, \partial_2\}[0.2, 0.7], \{\partial_5, \partial_7\}[0.45, 0.65] >\}$

$\mathcal{C}_2 = \{< \{\partial_1, \partial_3, \partial_4, \partial_6, \partial_7\}, \{\partial_3, \partial_6, \partial_7\}[0.7, 0.85], \{\partial_1, \partial_4, \partial_6\}[0.32, 0.5], \{\partial_4\}[0.2, 0.45] >\}$

$\mathcal{C}_3 = \{< \{\partial_2, \partial_3, \partial_5, \partial_6, \partial_7\}, \{\partial_2 \partial_5\}[0.9, 1], \{\partial_3, \partial_7\}[0.6, 0.7], \{\partial_3, \partial_5, \partial_6\}[0.15, 0.3] >\}$

$\mathcal{C}_4 = \{< \{\partial_1, \partial_2, \partial_3, \partial_4, \partial_5, \partial_6\}, \{\partial_1, \partial_3, \partial_4, \partial_6\}[0.75, 0.9], \{\partial_2\}[0.2, 0.3], \{\partial_2, \partial_4, \partial_5\}[0.5, 0.65] >\}$

$\mathcal{C}_5 = \{< \{\partial_1, \partial_2, \partial_3, \partial_6, \partial_7\}, \{\partial_1\}[0.6, 0.85], \{\partial_3, \partial_6\}[0.4, 0.52], \{\partial_2, \partial_3, \partial_7\}[0.1, 0.3] >\}$

$\mathcal{C}_6 = \{< \{\partial_2, \partial_3, \partial_4, \partial_5\}, \{\partial_4, \partial_5\}[0.5, 0.6], \{\partial_3\}[0.2, 0.35], \{\partial_2, \partial_5\}[0, 0.2] >\}$

$\mathcal{C}_7 = \{< \{\partial_1, \partial_2, \partial_3, \partial_4, \partial_5\}, \{\partial_3, \partial_4, \partial_5\}[0.2, 0.65], \{\partial_1, \partial_2, \partial_3\}[0.55, 0.7], \{\partial_3, \partial_4\}[0.3, 0.5] >\}$

$\mathcal{C}_8 = \{< \{\partial_2, \partial_3, \partial_4, \partial_5, \partial_6, \partial_7\}, \{\partial_2, \partial_3, \partial_4, \partial_5\}[0.8, 0.95], \{\partial_5, \partial_6, \partial_7\}[0.5, 0.6], \{\partial_4, \partial_5\}[0.4, 0.7] >\}$

$\mathcal{C}_9 = \{< \{\partial_1, \partial_4, \partial_5, \partial_6, \partial_7\}, \{\partial_4, \partial_5, \partial_6\}[0.35, 0.4], \{\partial_1, \partial_7\}[0.8, 0.85], \{\partial_5, \partial_6, \partial_7\}[0.5, 0.7] >\}$

$\mathcal{C}_{10} = \{< \{\partial_1, \partial_2, \partial_3, \partial_4, \partial_5, \partial_6, \partial_7\}, \{\partial_3, \partial_4, \partial_5\}[0.65, 0.8], \{\partial_1, \partial_5, \partial_6, \partial_7\}[0.4, 0.6], \{\partial_1, \partial_5\}[0.1, 0.3] >\}$

Step 4: To compare universities, the ideal university is determined as an IGSVNQN

$\mathcal{C}_{\mathrm{i}} = \{< \{\partial_1, \partial_2, \partial_3, \partial_4, \partial_5, \partial_6, \partial_7\}, \{\partial_1, \partial_2, \partial_3, \partial_4, \partial_5, \partial_6, \partial_7\}[1, 1], \emptyset[0, 0], \emptyset[0, 0] >\}.$

Step 5: The similarity of the universities with the ideal university is calculated using the Hamming similarity measure in Definition 3.1, which is defined for IGSVNQNs.

$$\widehat{S_H}(\mathcal{C}_{\mathrm{i}}, \mathcal{C}_1) = 1 - \frac{1}{2}\left[\frac{|1-0.54|+|0-0.2|+|0-0.45|+|1-0.6|+|0-0.7|+|0-0.65|}{6} + \frac{4-\left(4-\frac{5}{maks\{1,7\}}+\frac{3}{maks\{1,7\}}+\frac{0}{maks\{1,S2\}}+\frac{0}{maks\{1,2\}}\right)}{4}\right] = 0.2973.$$

$$\widehat{S_H}(\mathcal{C}_{\mathrm{i}}, \mathcal{C}_2) = 1 - \frac{1}{2}\left[\frac{|1-0.7|+|0-0.32|+|0-0.2|+|1-0.85|+|0-0.5|+|0-0.45|}{6} + \frac{4-\left(\frac{5}{maks\{1,7\}}+\frac{3}{maks\{1,7\}}+\frac{0}{maks\{1,3\}}+\frac{0}{maks\{1,1\}}\right)}{4}\right] = 0.3757.$$

$$\widehat{S_H}(\mathcal{C}_{\mathrm{i}}, \mathcal{C}_3) = 1 - \frac{1}{2}\left[\frac{|1-0.9|+|0-0.6|+|0-0.15|+|1-1|+|0-0.7|+|0-0.3|}{6} + \frac{4-\left(\frac{5}{maks\{1,7\}}+\frac{2}{maks\{1,7\}}+\frac{0}{maks\{1,2\}}+\frac{0}{maks\{1,3\}}\right)}{4}\right] = 0.3994.$$

$$\widehat{S_H}(\mathcal{C}_{\mathrm{i}}, \mathcal{C}_4) = 1 - \frac{1}{2}\left[\frac{|1-0.75|+|0-0.2|+|0-0.5|+|1-0.9|+|0-0.3|+|0-0.65|}{6} + \frac{4-\left(\frac{6}{maks\{1,7\}}+\frac{4}{maks\{1,7\}}+\frac{0}{maks\{1,1\}}+\frac{0}{maks\{1,3\}}\right)}{4}\right] = 0.3690.$$

$$\widehat{S_H}(\mathcal{C}_{\mathrm{i}}, \mathcal{C}_5) = 1 - \frac{1}{2}\left[\frac{|1-0.6|+|0-0.4|+|0-0.1|+|1-0.85|+|0-0.52|+|0-0.3|}{6} + \frac{4-\left(\frac{5}{maks\{1,7\}}+\frac{1}{maks\{1,7\}}+\frac{0}{maks\{1,2\}}+\frac{0}{maks\{1,3\}}\right)}{4}\right] = 0.4155.$$

$$\widehat{S_H}(\mathcal{C}_{\mathrm{i}}, \mathcal{C}_6) = 1 - \frac{1}{2}\left[\frac{|1-0.5|+|0-0.2|+|0-0|+|1-0.6|+|0-0.35|+|0-0.2|}{6} + \frac{4-\left(\frac{4}{maks\{1,7\}}+\frac{2}{maks\{1,7\}}+\frac{0}{maks\{1,1\}}+\frac{0}{maks\{1,2\}}\right)}{4}\right] = 0.3982.$$

$$\widehat{S_H}(\mathcal{C}_{\mathrm{i}}, \mathcal{C}_7) = 1 - \frac{1}{2}\left[\frac{|1-0.2|+|0-0.55|+|0-0.3|+|1-0.65|+|0-0.7|+|0-0.5|}{6} + \frac{4-\left(\frac{5}{maks\{1,7\}}+\frac{3}{maks\{1,7\}}+\frac{0}{maks\{1,3\}}+\frac{0}{maks\{1,2\}}\right)}{4}\right] = 0.2960.$$

$$\widehat{S_H}(\mathcal{C}_{\mathrm{i}}, \mathcal{C}_8) = 1 - \frac{1}{2}\left[\frac{|1-0.8|+|0-0.5|+|0-0.4|+|1-0.95|+|0-0.6|+|0-0.7|}{6} + \frac{4-\left(\frac{6}{maks\{1,7\}}+\frac{4}{maks\{1,7\}}+\frac{0}{maks\{1,3\}}+\frac{0}{maks\{1,2\}}\right)}{4}\right] = 0.3315.$$

$$\widehat{S_H}(\mathcal{C}_{\mathrm{i}}, \mathcal{C}_9) = 1 - \frac{1}{2}\left[\frac{|1-0.35|+|0-0.8|+|0-0.5|+|1-0.4|+|0-0.85|+|0-0.7|}{6} + \frac{4-\left(\frac{5}{maks\{1,7\}}+\frac{3}{maks\{1,7\}}+\frac{0}{maks\{1,2\}}+\frac{0}{maks\{1,3\}}\right)}{4}\right] = 0.1940.$$

$$\widehat{S_H}(\mathcal{C}_{\mathrm{i}}, \mathcal{C}_{10}) = 1 - \frac{1}{2}\left[\frac{|1-0.65|+|0-0.4|+|0-0.1|+|1-0.8|+|0-0.6|+|0-0.3|}{6} + \frac{4-\left(\frac{7}{maks\{1,7\}}+\frac{3}{maks\{1,7\}}+\frac{0}{maks\{1,4\}}+\frac{0}{maks\{1,2\}}\right)}{4}\right] = 0.4089.$$

Step 6: The similarities obtained are given in Table 1.

Table 1: Similarities of universities

Universities	Similarity
$\mathcal{C}_1$	0.2973
$\mathcal{C}_2$	0.3757
$\mathcal{C}_3$	0.3994

C_4	0.3690
C_5	0.4155
C_6	0.3982
C_7	0.2960
C_8	0.3315
C_9	0.1940
C_{10}	0.4089

5.Comparison

In this section, we compare the methods in Step 3 of Example 2, in which different numbers of IGSVNQNs are obtained and their similarities are calculated using the Hamming similarity measures defined for the obtained numbers.

i. method: Calculated in Step 5 of Example 2 and given in Table 2. The ranking of the similarities obtained is given in Table 3.

ii. method: One IVNNs was obtained by subtracting the sets from the IGSVNQNs in Step 3 of Example 2. Using these IVNNs and the Hamming similarity measure in Definition 6, similarities are calculated and given in Table 2. The ranking of the similarities obtained is given in Table 3.

iii. method: One GSVNQNs was obtained by taking the upper bounds of the intervals of the IGSVNQNs in Step 3 of Example 2. Using these numbers and the Hamming similarity measure in Definition 7, similarities are calculated and given in Table 2. The ranking of the similarities obtained is given in Table 3.

iv. method: One GSVNQNs was obtained by taking the lower bounds of the intervals of the IGSVNQNs in Step 3 of Example 2. Using these numbers and the Hamming similarity measure in Definition 7, similarities are calculated and given in Table 2. The ranking of the similarities obtained is given in Table 3.

v. method: One GSVNQNs was obtained by averaging the lower and upper bounds of the intervals of the IGSVNQNs in Step 3 of Example 2. Using these numbers and the Hamming similarity measure in Definition 7, similarities are calculated and given in Table 2. The ranking of the similarities obtained is given in Table 3.

Table 2: Comparison of similarities of universities

	i. method	ii. method	iii. method	iv. method	v. method
C_1	0.2973	0.476	0.3507	0.2440	0.2973
C_2	0.3757	0.320	0.3990	0.3523	0.3757
C_3	0.3994	0.4583	0.4119	0.3869	0.3994

C_4	0.3690	0.333	0.3773	0.3607	0.3648
C_5	0.4155	0.3116	0.4214	0.4097	0.4155
C_6	0.3982	0.275	0.4190	0.3773	0.3982
C_7	0.2960	0.533	0.2607	0.2773	0.2690
C_8	0.3315	0.4083	0.3523	0.3107	0.3357
C_9	0.1940	0.683	0.2107	0.1773	0.1774
C_{10}	0.4089	0.325	0.4297	0.3880	0.4089

When the data in Table 2 are analyzed, it is observed that the similarity of C_1 university to the structure defined as "ideal university" varies according to the methods used. Accordingly, the similarity is calculated as 0.2973 according to i. method, 0.467 according to method ii, 0.3507 according to method iii, 0.2440 according to iv. method and 0.2973 according to v. method. These values reveal that the method with the highest fit of C_1 university to the ideal structure is ii. method and the method with the lowest fit is method iv. It is also seen that the results obtained from i. and iv. methods are equal. In this context, the selection of the method used in similarity analysis in accordance with the structure of the problem is of great importance in terms of the accuracy and reliability of the results to be obtained. The fact that each method offers a different perspective enables a more comprehensive analysis by considering various dimensions in the evaluation of the results.

Table 3: Similarity rankings

Method	Similarity Rankings of Universities
i.	$C_5 > C_{10} > C_3 > C_6 > C_2 > C_4 > C_8 > C_1 > C_7 > C_9$
ii.	$C_9 > C_7 > C_1 > C_3 > C_8 > C_4 > C_{10} > C_2 > C_5 > C_6$
iii.	$C_{10} > C_5 > C_6 > C_3 > C_2 > C_4 > C_8 > C_1 > C_7 > C_9$
iv.	$C_5 > C_{10} > C_3 > C_6 > C_4 > C_2 > C_8 > C_7 > C_1 > C_9$
v.	$C_5 > C_{10} > C_3 > C_6 > C_2 > C_4 > C_8 > C_1 > C_7 > C_9$

As can be seen in Table 3, while the same rankings were obtained in methods i and v, different results were obtained in other methods. C_9 university, which was determined as the least similar university according to i, iii, iv. and v. methods, was determined as the most similar university according to ii. method. Similarly, C_5 university, which was one of the first two most similar universities according to i., iii., iv. and v. methods, was determined as the second least similar university in ii.

method. This shows that the methods used can lead to quite different results in the similarity assessment between universities.

6.Conclusions

In this paper, we define a Hamming similarity measure for IGSVNQNs. It is proved mathematically that this similarity measure satisfies the similarity measure criteria. Furthermore, an example application is realized using this Hamming similarity measure. In this application, the similarity of 10 fictitious universities to an ideal university is analyzed by considering the factors that affect university choice. Moreover, similarity calculations were computed using Hamming similarity measures defined for both IVNNs and GSVNQNs and the results were compared. It is concretely demonstrated that similar or different results can be obtained in decision-making processes when different numbers are used. Therefore, the numbers to be used in decision-making applications should be selected by considering the characteristics and objectives of the problem domain.

7.Future Research Directions

Researchers can use the similarity measure in this study with real data in different applications. They can also define Euclidean and Hausorff similarity measures for IGSVNQNs.

Abbreviations

SVNN: Single valued neutrosophic number
SVNS: Single valued neutrosophic set
IVNN: Interval valued neutrosophic number
IVNS: Interval valued neutrosophic set
GSVNQN: Generalized set valued neutrosophic quadruple number
GSVNQS: Generalized set valued neutrosophic quadruple set
IGSVNQN: Interval Generalized set valued neutrosophic quadruple number
IGSVNQS: Interval Generalized set valued neutrosophic quadruple set

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Chapter 3

On The Symbolic 4-plithogenic and 5-Plithogenic Vector Spaces

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ABSTRACT

Symbolic n-plithogenic algebraic structures are considered as symmetric generalizations of classical algebraic structures by using $n+1$ symmetric components. This paper is dedicated to generalize symbolic 3-plithogenic Vector Spaces by defining symbolic 4-plithogenic Vector Spaces and 5-plithogenic Vector Spaces, where these new classes of n-symbolic plithogenic algebraic structures will be defined for the first time, and it will be studied through its algebraic substructures.

Keywords: Neutrosophic sets, Symbolic 4-plithogenic Vector Spaces, , Symbolic 4-plithogenic Vector Spaces homomorphism, Symbolic 5-plithogenic Vector Spaces, 5-plithogenic Vector Spaces homomorphism.

1.INTRODUCTION

The vagueness or uncertainty representation of imperfect knowledge becomes a crucial issue in the areas of computer science and artificial intelligence. To deal with the uncertainty, the fuzzy set proposed by Zadeh [29] allows the uncertainty of a set with a membership degree between 0 and 1. Then, Atanassov [30] introduced an intuitionistic Fuzzy set (IFS) as a generalization of the Fuzzy set.

The IFS represents the uncertainty with respect to both membership and non-membership. However, it can only handle incomplete information but not the indeterminate and inconsistent information which exists commonly in real situations. Therefore, Smarandache [31] proposed a neutrosophic set. It can independently express truth-membership degree, indeterminacy-membership degree, and false membership degree and deal with incomplete, indeterminate, and inconsistent information.

During the last two years, the research steps interested in studying symbolic n-plithogenic algebraic structures. These structures were supposed by Smarandache in [4]. For $n = 2$, we get the symbolic 2-plithogenic algebraic structures such as symbolic 2-plithogenic spaces/rings/modules, and integers [1-3,5-8]. For $n = 3$, we get the symbolic 3- plithogenic algebraic structures such as symbolic 3-plithogenic spaces/rings/modules, and integers [20-23].

All of the mentioned algebraic structures are characterized by having very similar algebraic properties to the refined neutrosophic structures [9-18, 24-30,33-45]. Başer and Uluçay [39] defined effective q- fuzzy soft expert sets. Then, Başer and Ulucay [68] defined energy of a neutrosophic soft set. Recently, studies on extensions of fuzzy sets have been continuing very rapidly on applications and algebraic structures [46-89]

This is what prompted us to follow up the previous scientific efforts and to study 4-plithogenic Vector Spaces for the first time, by providing basic definitions and proofs that describe the algebraic behavior of the elements of these Vector Spaces. It is noteworthy that these new rings will be very useful in more extensive classes of algebraic modules, and cryptographic algorithms.

2.BACKGROUND

Definition 1. [31]Let $\mathcal{U}$ be a universe. $\mathcal{A}$ neutrosophic sets $\mathcal{A}$ over $\mathcal{U}$ is defined by

$$\mathcal{A} = \{\prec u, (\mu_{\mathcal{A}}(u), v_{\mathcal{A}}(u), w_{\mathcal{A}}(u)) \succ : u \in \mathcal{U}\}$$

where, $\mu_{\mathcal{A}}(u)$, $v_{\mathcal{A}}(u)$ and $w_{\mathcal{A}}(u)$ are called truth-membership function, indeterminacy-membership function and falsity- membership function, respectively. They are respectively defined by

$$\mu_{\mathcal{A}}: \mathcal{U} \to]^{-}0, 1^{+}[\,, \qquad v_{\mathcal{A}}: \mathcal{U} \to]^{-}0, 1^{+}[\,, \qquad w_{\mathcal{A}}: \mathcal{U} \to]^{-}0, 1^{+}[$$

such that $0^{-} \leq \mu_{\mathcal{A}}(u)+v_{\mathcal{A}}(u)+w_{\mathcal{A}}(u) \leq 3^{+}$.

Definition 2 [32] Let $\mathcal{U}$ be a universe. An single valued neutrosophic set (SVN-set) over $\mathcal{U}$ is a neutrosophic set over $\mathcal{U}$, but the truth-membership function, indeterminacy-membership function and falsity- membership function are respectively defined by

$$\mu_{\mathcal{A}}: \mathcal{U} \to [0,1], \qquad v_{\mathcal{A}}: \mathcal{U} \to [0,1]\,, \qquad w_{\mathcal{A}}: \mathcal{U} \to [0,1]$$

Such that $0 \leq \mu_{\mathcal{A}}(u)+v_{\mathcal{A}}(u)+w_{\mathcal{A}}(u) \leq 3$.

In the next section, we will define a new Hybrid Distance-Based Similarity Measures for Refined Neutrosophic Sets (RNSs).

Definition 3. [3] Let's consider two plithogenic numbers:

$$PN_1 = a_0 + a_1P_1 + a_2P_2 + \cdots + a_nP_n$$

$$PN_2 = b_0 + b_1P_1 + b_2P_2 + \cdots + b_nP_n$$

1. Addition of Plithogenic Numbers: $PN_1 + PN_2 = (a_0 + b_0) + \sum_{i=1}^{n}(a_i + b_i)P_i$
2. Subtraction of Plithogenic Numbers: $PN_1 - PN_2 = (a_0 - b_0) + \sum_{i=1}^{n}(a_i - b_i)P_i$
3. Scalar Multiplication of Plithogenic Numbers: $rPN_1 = ra_0 + ra_1P_1 + ra_2P_2 + \cdots + ra_nP_n$
4. Multiplication of Plithogenic Numbers

$$PN_1 \cdot PN_2 = (a_0 + a_1P_1 + a_2P_2 + \cdots + a_nP_n) \cdot (b_0 + b_1P_1 + b_2P_2 + \cdots + b_nP_n)$$

And then one multiplies them, term by term $a_iP_i \cdot a_jP_j = a_i \cdot a_jP_{\max(i,j)}$ where $\cdot$ is the classical multiplication as in classical algebra, using the above multiplication of symbolic plithogenic components.

As particular case:

1. $0.P_i = 0$
2. $1 = 1 + 0.P_1 + 0.P_2 + \cdots + 0.P_n$

3.Symbolic 4-plithogenic Vector Spaces

Definition 4. Let V be a Vector Spaces, the symbolic 4-plithogenic Vector Spaces is:

$4-SP_R = \{a_0 + a_1P_1 + a_2P_2 + a_3P_3 + a_4P_4;\ a_i \in R, {P_j}^2 = P_j, P_i \times P_j = P_{max(i,j)}\}$.

Operations on $4-SP_R$:

Addition:

$[a_0 + a_1P_1 + a_2P_2 + a_3P_3 + a_4P_4] + [b_0 + b_1P_1 + b_2P_2 + b_3P_3 + b_4P_4] =$

$(a_0 + b_0) + (a_1 + b_1)P_1 + (a_2 + b_2)P_2 + (a_3 + b_3)P_3 + (a_4 + b_4)P_4$.

Multiplication:

$$[a_0 + a_1P_1 + a_2P_2 + a_3P_3 + a_4P_4].[b_0 + b_1P_1 + b_2P_2 + b_3P_3 + b_4P_4] =$$

$$a_0b_0 + (a_0b_1 + a_1b_0 + a_1b_1)P_1 + (a_0b_2 + a_1b_2 + a_2b_0 + a_2b_1 + a_2b_2)P_2 +$$

$$(a_0b_3 + a_1b_3 + a_2b_3 + a_3b_3 + a_3b_0 + a_3b_1 + a_3b_2)P_3 +$$

$(a_0b_4 + a_1b_4 + a_2b_4 + a_3b_4 + a_4b_0 + a_4b_1 + a_4b_2 + a_4b_3 + a_4b_4)P_4$.

Scalar Multiplication:

$$r(a_0 + a_1P_1 + a_2P_2 + a_3P_3 + a_4P_4) = ra_0 + ra_1P_1 + ra_2P_2 + ra_3P_3 + ra_4P_4$$

As particular case:

3. $0.P_i = 0$
4. $1 = 1 + 0.P_1 + 0.P_2 + 0.P_3 + 0.P_4$

It is clear that $(4-SP_R)$ is a Vector Spaces.

Example 5. Consider the Vector Spaces $R = Z_3 = \{0,1,2\}$, the corresponding $4-SP_R$ is:

$4-SP_R = \{a + bP_1 + cP_2 + dP_3 + eP_4; a, b, c, d, e \in Z_3\}$.

If $X = 2 + 2P_1 + 1P_2 + P_3 + P_4, Y = P_2 + 2P_3$, then:

$$X + Y = 2 + 2P_1 + 2P_2 + P_4,$$

$$X - Y = 2 + 2P_1 - P_3 + P_4,$$

$X.Y = 2P_2 + 4P_3 + 2P_2 + 4P_3 + P_2 + 2P_3 + P_3 + 2P_3 + P_4 + 2P_4 = 5P_2 + 13P_3 + 3P_4$.

Theorem 6. Let $4 - SP_R$ be a 4-plithogenic symbolic Vector Spaces, Let $X = e_0 + e_1P_1 + e_2P_2 + e_3P_3 + e_4P_4$ be an arbitrary element, then:

1. X is invertible if and only if $e_0, e_0 + e_1, e_0 + e_1 + e_2$, $e_0 + e_1 + e_2 + e_3$, $e_0 + e_1 + e_2 + e_3 + e_4$ are invertible.
2. $X^{-1} = e_0{}^{-1} + [(e_0 + e_1)^{-1} - e_0{}^{-1}]P_1 + [(e_0 + e_1 + e_2)^{-1} - (e_0 + e_1)^{-1}]P_2$+$[(e_0 + e_1 + e_2 + e_3)^{-1} - (e_0 + e_1 + e_2)^{-1}]P_3 + [(e_0 + e_1 + e_2 + e_3 + e_4)^{-1} - (e_0 + e_1 + e_2 + e_3)^{-1}]P_4$.

Proof.

1. Assume that X is invertible, than there exists $Y = n_0 + n_1P_1 + n_2P_2 + n_3P_3 + n_4P_4$ such that $X.Y = 1$, hence:

$$\begin{cases} e_0n_3 + e_1n_3 + e_2n_3 + e_3n_3 + e_3n_1 + e_3n_2 + e_3n_0 = 0 \ (1) \\ e_0n_0 = 1 \dots \ (2) \\ e_0n_1 + e_1n_0 + e_1n_1 = 0 \dots \ (3) \\ e_0n_2 + e_2n_0 + e_2n_2 + e_1n_2 + e_2n_1 = 0 \dots \ (4), \\ e_0n_4 + e_1n_4 + e_2n_4 + e_3n_4 + e_4n_0 + e_4n_1 + e_4n_3 + e_4n_4 (5) \end{cases}$$

From (2), e_0 is invertible.

By adding (3) to (2), we get $(e_0 + e_1)(n_0 + n_1) = 1$, thus $e_0 + e_1$ is invertible.

By adding (4) to (3)to (2), $(e_0 + e_1 + e_2)(n_0 + n_1 + n_2) = 1$, hence $e_0 + e_1 + e_2$ is invertible.

By adding (1) to (2) to (3) to(4),

$(e_0 + e_1 + e_2 + e_3)(n_0 + n_1 + n_2 + n_3) = 1$, hence $e_0 + e_1 + e_2 + e_3$ is invertible.

Adding all equations gives:

$(e_0 + e_1 + e_2 + e_3 + e_4)(n_0 + n_1 + n_2 + n_3 + n_4) = 1$, hence $e_0 + e_1 + e_2 + e_3 + e_4$ is invertible.

2. From the first part. We have:

$n_0 = e_0{}^{-1}, n_0 + n_1 = (e_0 + e_1)^{-1}, n_0 + n_1 + n_2 = (e_0 + e_1 + e_2)^{-1}, (e_0 + e_1 + e_2 + e_3)^{-1} = n_0 + n_1 + n_2 + n_3, (e_0 + e_1 + e_2 + e_3 + e_4)^{-1} = n_0 + n_1 + n_2 + n_3 + n_4$, then:

$Y = {e_0}^{-1} + [(e_0 + e_1)^{-1} - {e_0}^{-1}]P_1 + [(e_0 + e_1 + e_2)^{-1} - (e_0 + e_1)^{-1}]P_2 +$

$[(e_0 + e_1 + e_2 + e_3)^{-1} - (e_0 + e_1 + e_2)^{-1}]P_3 +$

$[(e_0 + e_1 + e_2 + e_3 + e_4)^{-1} - (e_0 + e_1 + e_2 + e_3)^{-1}]P_4 = X^{-1}.$

Example 7. Take $R = Z_3 = \{0,1,2\}$, $4 - SP_{Z_3}$ is the corresponding symbolic 4-plithogenic Vector Spaces, consider $X = 2 + 2P_2 + P_4 \in 4 - SP_{Z_3}$, then:

$e_0 = 2$ is invertible with ${e_0}^{-1} = 2$, $e_0 + e_1 = 2$ is invertible with $(e_0 + e_1)^{-1} = 2$,

$e_0 + e_1 + e_2 = 1$ is invertible with $(e_0 + e_1 + e_2)^{-1} = 1$,

$$e_0 + e_1 + e_2 + e_3 = 1, (e_0 + e_1 + e_2 + e_3)^{-1} = 1\,,$$

$e_0 + e_1 + e_2 + e_3 + e_4 = 2, (e_0 + e_1 + e_2 + e_3 + e_4)^{-1} = 2$ hence:

$X^{-1} = 2 + (2 - 2)P_1 + (1 - 2)P_2 + (1 - 1)P_3 + (2 - 1)P_4 = 2 + 2P_2 + P_4.$

Theorem 8. Let $4 - SP_R$ be a symbolic 4-plithogenic Vector Spaces, hence if $X = m + nP_1 + cP_2 + qP_3 + lP_4$, then:

$X^n = m^n + [(m + n)^n - m^n]P_1 + [(m + n + c)^n - (m + n)^n]P_2 + [(m + n + c + q)^n - (m + n + c)^n]P_3 + [(m + n + c + q + l)^n - (m + n + c + q)^n]P_4$ for every $n \in Z^+$.

Proof.

For $n = 1$, it holds easily. Assume that it is true for $n = k$, we prove it for $n = k + 1$.

$$X^{k+1} = X.X^k =$$

$(m + nP_1 + cP_2 + qP_3 + lP_4)(m^k + [(m + n)^k - m^k]P_1 + [(m + n + c)^k - (m + n)^k]P_2 + [(m + n + c + q)^k - (m + n + c)^k]P_3 + [(m + n + c + q + l)^k - (m + n + c + q)^k]P_4) = m^{k+1} + [(m + n)^{k+1} - m^{k+1}]P_1 + [(m + n + c)^{k+1} - (m + n)^{k+1}]P_2 +$

$[(m + n + c + q)^{k+1} - (m + n + c)^{k+1}]P_3 +$

$[(m + n + c + q + l)^{k+1} - (m + n + c + q)^{k+1}]P_4.$

Therefore, that proof is complete by induction.

Definition 9. Let R, T be two Vector Spaces, $4-SP_R, 4-SP_T$ are the corresponding symbolic 4-plithogenic Vector Spaces, let $f_0, f_1, f_2, f_3, f_4: R \to T$ be Vector Spaces homomorphisms, we define the symbolic 4-plithogenic Vector Spaces homomorphism:

$f: 4-SP_R \to 4-SP_T$ such that:

$f(m + nP_1 + cP_2 + qP_3 + lP_4) = f_0(m) + f_1(n)P_1 + f_2(c)P_2 + f_3(q)P_3 + f_4(l)P_4$

If $f_0 = f_1 = f_2 = f_3 = f_4$, then f is called symbolic 4-plithogenic Vector Spaces homomorphism.

Remark 10. If f_0, f_1, f_2, f_3, f_4 is isomorphisms, then f is called symbolic 4-plithogenic Vector Spaces isomorphism.

Example 11. Take $R = Z,\ T = Z_4,\ f_0, f_1: R \to T$ such that:

$f_0(x) = x(mod\ 4), f_1(2) = 2x(mod\ 4)$. It is clear that f_0, f_1 are homomorphisms.

We define $f: 4-SP_R \to 4-SP_T$, where:
$f(m + nP_1 + cP_2 + qP_3 + lP_4) = f_0(m) + f_1(n)P_1 + f_0(c)P_2 + f_1(q)P_3 + f_1(l)P_4 = m(mod\ 4) + 2n(mod\ 4)P_1 + (c\ mod\ 4)P_2 + (2q\ mod\ 4)P_3 + (2l\ mod\ 4)P_4$

Which is a symbolic 4-plithogenic Vector Spaces homomorphism.

Theorem 12. Let $f = f_0 + f_1P_1 + f_2P_2 + f_3P_3 + f_4P_4: 4-SP_R \to 4-SP_T$ be a mapping, then:

1. If f is symbolic 4-plithogenic Vector Spaces homomorphism, then f is a Vector Spaces homomorphism.
2. If f is an symbolic 4-plithogenic Vector Spaces homomorphism, then it is an isomorphism.

Proof.

1. Assume that f is an symbolic 4-plithogenic Vector Spaces homomorphism, then:

   $f_0 = f_1 = f_2 = f_3 = f_4$ are homomorphisms.

Let $X = d_0 + d_1P_1 + d_2P_2 + d_3P_3 + d_4P_4, Y = c_0 + c_1P_1 + c_2P_2 + c_3P_3 + c_4P_4 \in 4-SP_R$, we have:

$f(X+Y) =$

$f_0(d_0+c_0)+f_0(d_1+c_1)P_1+f_0(d_2+c_2)P_2+f_0(d_3+c_3)P_3+f_0(d_4+c_4)P_4$
$= f(X)+f(Y)$

$$f(X.Y) = f_0(d_0c_0)+f_0(d_0c_1+d_1c_0+d_1c_1)P_1 + f_0(d_0c_2+d_2c_0+d_2c_2+d_2c_1+d_1c_2)P_2 + f_0(d_0c_3+d_1c_3+d_2c_3+d_3c_3+d_3c_1+d_3c_0+d_3c_2)P_3 +$$

$$(d_0c_4+d_1c_4+d_2c_4+d_3c_4+d_4c_0+d_4c_1+d_4c_2+d_4c_3+d_4c_4)P_4$$
$$= f_0(d_0)f_0(c_0)+\big(f_0(d_0)f_0(c_1)+f_0(d_1)f_0(c_0)+f_0(d_1)f_0(c_1)\big)P_1 + \big(f_0(d_0)f_0(c_2)+f_0(d_2)f_0(y_0)+f_0(d_2)f_0(c_2)+f_0(d_2)f_0(c_1) + f_0(d_1)f_0(c_2)\big)P_2 + \big[\big(f_0(d_0)f_0(c_3)+f_0(d_1)f_0(c_3)+f_0(d_2)f_0(c_3)+f_0(d_3)f_0(c_3) + f_0(d_3)f_0(c_1)+f_0(d_3)f_0(c_2)+f_0(d_3)f_0(c_0)\big)\big]P_3 + \big[\big(f_0(d_0)f_0(c_4)+f_0(d_1)f_0(c_4)+f_0(d_2)f_0(c_4)+f_0(d_3)f_0(c_4) + f_0(d_4)f_0(c_1)+f_0(d_4)f_0(c_2)+f_0(d_4)f_0(c_0)+f_0(d_4)f_0(c_4) + f_0(d_4)f_0(c_3)\big)\big]P_4 =$$

$[f_0(d_0)+f_0(d_1)P_1+f_0(d_2)P_2+f_0(d_3)P_3+f_0(d_4)P_4][f_0(c_0)+f_0(c_1)P_1+f_0(c_2)P_2+f_0(c_3)P_3+f_0(c_4)P_4] = f(X).f(Y).$

This implies the proof.

4. Symbolic 5-plithogenic Vector Spaces

Definition 13. Let R be a Vector Spaces, the symbolic 5-plithogenic Vector Spaces is:
$5-SP_R = \{a_0+a_1P_1+a_2P_2+a_3P_3+a_4P_4+a_5P_5;\ a_i \in R, {P_j}^2 = P_j, P_i \times P_j = P_{max(i,j)}\}$.
Operations on $5-SP_R$:
Addition:
$$[a_0+a_1P_1+a_2P_2+a_3P_3+a_4P_4+a_5P_5]+[b_0+b_1P_1+b_2P_2+b_3P_3+b_4P_4+b_5P_5] = (a_0+b_0)+(a_1+b_1)P_1+(a_2+b_2)P_2+(a_3+b_3)P_3+(a_4+b_4)P_4+(a_5+b_5)P_5.$$
Multiplication:
$$[a_0+a_1P_1+a_2P_2+a_3P_3+a_4P_4+a_5P_5].[b_0+b_1P_1+b_2P_2+b_3P_3+b_4P_4+b_5P_5] = a_0b_0+(a_0b_1+a_1b_0+a_1b_1)P_1+(a_0b_2+a_1b_2+a_2b_0+a_2b_1+a_2b_2)P_2 + (a_0b_3+a_1b_3+a_2b_3+a_3b_3+a_3b_0+a_3b_1+a_3b_2)P_3 +$$

$(a_0b_4 + a_1b_4 + a_2b_4 + a_3b_4 + a_4b_0 + a_4b_1 + a_4b_2 + a_4b_3 + a_4b_4)P_4$ +($a_0b_5 + a_1b_5 + a_2b_5 + a_3b_5 + a_4b_5 + a_5b_0 + a_5b_1 + a_5b_2 + a_5b_3 + a_5b_4 + a_5b_5)P_5$.

Scalar Multiplication:

$$r(a_0 + a_1P_1 + a_2P_2 + a_3P_3 + a_4P_4 + a_5b_5)$$
$$= ra_0 + ra_1P_1 + ra_2P_2 + ra_3P_3 + ra_4P_4 + ra_5b_5$$

As particular case:

1. $0.P_i = 0$
2. $1 = 1 + 0.P_1 + 0.P_2 + 0.P_3 + 0.P_4 + 0.b_5$

It is clear that $(4 - SP_R)$ is a Vector Spaces.

Example 14. Consider the Vector Spaces $R = Z_3 = \{0,1,2\}$, the corresponding $5 - SP_R$ is:

$4 - SP_R = \{a + bP_1 + cP_2 + dP_3 + eP_4 + wP_5; a, b, c, d, e, w \in Z_3\}$.

If $X = 2 + 2P_1 + P_4, Y = P_2 + 2P_5$, then:

$$X + Y = 2 + 2P_1 + P_2 + P_4 + 2P_5,$$
$$X - Y = 2 + 2P_1 - P_2 + P_4 - 2P_5,$$

$X.Y = 2P_2 + 4P_3 + 2P_2 + 4P_5 + P_4 + 2P_5 = 4P_2 + 4P_3 + P_4 + 2P_5$.

Theorem 15. Let $5 - SP_R$ be a 5-plithogenic symbolic Vector Spaces, with unity (1). Let $X = e_0 + e_1P_1 + e_2P_2 + e_3P_3 + e_4P_4 + e_5P_5$ be an arbitrary element, then:

3. X is invertible if and only if $e_0, e_0 + e_1, e_0 + e_1 + e_2$, $e_0 + e_1 + e_2 + e_3$, $e_0 + e_1 + e_2 + e_3 + e_4, e_0 + e_1 + e_2 + e_3 + e_4 + e_5$ are invertible.
4. $X^{-1} = e_0{}^{-1} + [(e_0 + e_1)^{-1} - e_0{}^{-1}]P_1 + [(e_0 + e_1 + e_2)^{-1} - (e_0 + e_1)^{-1}]P_2$ +[$(e_0 + e_1 + e_2 + e_3)^{-1} - (e_0 + e_1 + e_2)^{-1}]P_3 + [(e_0 + e_1 + e_2 + e_3 + e_4)^{-1} - (e_0 + e_1 + e_2 + e_3)^{-1}]P_4 + [(e_0 + e_1 + e_2 + e_3 + e_4 + e_5)^{-1} - (e_0 + e_1 + e_2 + e_3 + e_4)^{-1}]P_4$.

Proof.

3. Assume that X is invertible, than there exists $Y = n_0 + n_1P_1 + n_2P_2 + n_3P_3 + n_4P_4 + n_5P_5$ such that $X.Y = 1$, hence:

$$\begin{cases} e_0n_3 + e_1n_3 + e_2n_3 + e_3n_3 + e_3n_1 + e_3n_2 + e_3n_0 = 0 \ (1) \\ e_0n_0 = 1 \ldots \ (2) \\ e_0n_1 + e_1n_0 + e_1n_1 = 0 \ldots \ (3) \\ e_0n_2 + e_2n_0 + e_2n_2 + e_1n_2 + e_2n_1 = 0 \ldots \ (4), \\ e_0n_4 + e_1n_4 + e_2n_4 + e_3n_4 + e_4n_0 + e_4n_1 + e_4n_3 + e_4n_4 (5) \\ e_0n_5 + e_1n_5 + e_2n_5 + e_3n_5 + e_4n_5 + e_5n_0 + e_5n_1 + e_5n_2 + e_5n_3 + e_5n_4 + e_5n_5 = 0 \ (6) \end{cases}$$

From (2), e_0 is invertible.

By adding (3) to (2), we get $(e_0 + e_1)(n_0 + n_1) = 1$, thus $e_0 + e_1$ is invertible.

By adding (4) to (3)to (2), $(e_0 + e_1 + e_2)(n_0 + n_1 + n_2) = 1$, hence $e_0 + e_1 + e_2$ is invertible.

By adding (1) to (2) to (3) to(4),

$(e_0+e_1+e_2+e_3)(n_0+n_1+n_2+n_3)=1$, hence $e_0+e_1+e_2+e_3$ is invertible.
Adding all equations (1),(2),(3),(4),(5) gives:
$(e_0+e_1+e_2+e_3+e_4)(n_0+n_1+n_2+n_3+n_4)=1,$ hence $e_0+e_1+e_2+e_3+e_4$ is invertible.
Adding all equations (1) to (6) gives:
$(e_0+e_1+e_2+e_3+e_4+e_5)(n_0+n_1+n_2+n_3+n_4+n_5)=1,$
hence $e_0+e_1+e_2+e_3+e_4+e_5$ is invertible.

4. From the first part. We have:

$n_0={e_0}^{-1}, n_0+n_1=(e_0+e_1)^{-1}, n_0+n_1+n_2=(e_0+e_1+e_2)^{-1}, (e_0+e_1+e_2+e_3)^{-1}=n_0+n_1+n_2+n_3, (e_0+e_1+e_2+e_3+e_4)^{-1}=n_0+n_1+n_2+n_3+n_4, (e_0+e_1+e_2+e_3+e_4+e_5)^{-1}=n_0+n_1+n_2+n_3+n_4+n_5$, then:
$Y={e_0}^{-1}+[(e_0+e_1)^{-1}-{e_0}^{-1}]P_1+[(e_0+e_1+e_2)^{-1}-(e_0+e_1)^{-1}]P_2+$
$[(e_0+e_1+e_2+e_3)^{-1}-(e_0+e_1+e_2)^{-1}]P_3+$
$[(e_0+e_1+e_2+e_3+e_4)^{-1}-(e_0+e_1+e_2+e_3)^{-1}]P_4+$
$[(e_0+e_1+e_2+e_3+e_4+e_5)^{-1}-(e_0+e_1+e_2+e_3+e_4)^{-1}]P_5=X^{-1}.$

Example 16. Take $R=Z_3=\{0,1,2\}$, $5-SP_{Z_3}$ is the corresponding symbolic 5-plithogenic Vector Spaces, consider $X=2+2P_2+P_5\in 5-SP_{Z_3}$, then:
$e_0=2$ is invertible with ${e_0}^{-1}=2$, $e_0+e_1=2$ is invertible with $(e_0+e_1)^{-1}=2$,
$e_0+e_1+e_2=1$ is invertible with $(e_0+e_1+e_2)^{-1}=1$,
$e_0+e_1+e_2+e_3=1, (e_0+e_1+e_2+e_3)^{-1}=1$,
$e_0+e_1+e_2+e_3+e_4=1, (e_0+e_1+e_2+e_3+e_4)^{-1}=1, e_0+e_1+e_2+e_3+e_4+e_5=2, (e_0+e_1+e_2+e_3+e_4+e_5)^{-1}=2$ hence:
$X^{-1}=2+(2-2)P_1+(1-2)P_2+(1-1)P_3+(1-1)P_4+(2-1)P_5=2+2P_2+P_5$.

Definition 17. Let R,T be two Vector Spaces, $5-SP_R, 5-SP_T$ are the corresponding symbolic 5-plithogenic Vector Spaces, let $f_0,f_1,f_2,f_3,f_4,f_5: R\to T$ be symbolic 5-plithogenic Vector Spaces homomorphisms $f: 5-SP_R\to 5-SP_T$ such that:
$f(m+nP_1+cP_2+qP_3+lP_4+kP_5)=f_0(m)+f_1(n)P_1+f_2(c)P_2+f_3(q)P_3+f_4(l)P_4+f_5(k)P_5$
If $f_0=f_1=f_2=f_3=f_4=f_5$, then f is called symbolic 5-plithogenic Vector Spaces homomorphism.

Remark 18. If f_0,f_1,f_2,f_3,f_4,f_5 are isomorphisms, then f is called symbolic 5-plithogenic Vector Spaces isomorphism.

Example 19. Take $R=Z, T=Z_4$, $f_0,f_1: R\to T$ such that:
$f_0(x)=x(mod\ 4), f_1(2)=2x(mod\ 4)$. It is clear that f_0,f_1 are homomorphisms.

We define $f: 5-SP_R \to 5-SP_T$, where: $f(m+nP_1+cP_2+qP_3+lP_4+kP_5) = f_0(m)+f_1(n)P_1+f_0(c)P_2+f_1(q)P_3+f_1(l)P_4+f_1(k)P_5 = m(mod\ 4)+2n(mod\ 4)P_1+(c\ mod\ 4)P_2 + (2q\ mod\ 4)P_3+(2l\ mod\ 4)P_4+(2k\ mod\ 4)P_5$

Which is an symbolic 5-plithogenic Vector Spaces homomorphism.

Theorem 20. Let $f = f_0+f_1P_1+f_2P_2+f_3P_3+f_4P_4+f_5P_5: 5-SP_R \to 5-SP_T$ be a mapping, then:

1. If f is an symbolic 5-plithogenic Vector Spaces homomorphism, then f is a Vector Spaces homomorphism.
2. If f is an symbolic 5-plithogenic Vector Spaces homomorphism, then it is an isomorphism.

Proof.

1. Assume that f is an symbolic 5-plithogenic Vector Spaces homomorphism, then $f_0 = f_1 = f_2 = f_3 = f_4 = f_5$ are homomorphisms.

Let

$X = d_0+d_1P_1+d_2P_2+d_3P_3+d_4P_4+d_5P_5, Y = c_0+c_1P_1+c_2P_2+c_3P_3+c_4P_4+c_5P_5 \in 5-SP_R$, we have:

$$f(X+Y) = f_0(d_0+c_0)+f_0(d_1+c_1)P_1+f_0(d_2+c_2)P_2+f_0(d_3+c_3)P_3+f_0(d_4+c_4)P_4+f_0(d_5+c_5)P_5 = f(X)+f(Y)$$

$$\begin{aligned}
f(X.Y) = {} & f_0(d_0c_0)+f_0(d_0c_1+d_1c_0+d_1c_1)P_1 \\
& + f_0(d_0c_2+d_2c_0+d_2c_2+d_2c_1+d_1c_2)P_2 \\
& + f_0(d_0c_3+d_1c_3+d_2c_3+d_3c_3+d_3c_1+d_3c_0+d_3c_2)P_3 + \\
& (d_0c_4+d_1c_4+d_2c_4+d_3c_4+d_4c_0+d_4c_1+d_4c_2+d_4c_3+d_4c_4)P_4 \\
& + (d_0c_5+d_1c_5+d_2c_5+d_3c_5+d_4c_5+d_5c_0+d_5c_1+d_5c_2+d_5c_3 \\
& + d_5c_4+d_5c_5)P_5 \\
& = f_0(d_0)f_0(c_0)+\big(f_0(d_0)f_0(c_1)+f_0(d_1)f_0(c_0)+f_0(d_1)f_0(c_1)\big)P_1 \\
& + \big(f_0(d_0)f_0(c_2)+f_0(d_2)f_0(y_0)+f_0(d_2)f_0(c_2)+f_0(d_2)f_0(c_1) \\
& + f_0(d_1)f_0(c_2)\big)P_2 \\
& + \big[\big(f_0(d_0)f_0(c_3)+f_0(d_1)f_0(c_3)+f_0(d_2)f_0(c_3)+f_0(d_3)f_0(c_3) \\
& + f_0(d_3)f_0(c_1)+f_0(d_3)f_0(c_2)+f_0(d_3)f_0(c_0)\big)\big]P_3 \\
& + \big[\big(f_0(d_0)f_0(c_4)+f_0(d_1)f_0(c_4)+f_0(d_2)f_0(c_4)+f_0(d_3)f_0(c_4) \\
& + f_0(d_4)f_0(c_1)+f_0(d_4)f_0(c_2)+f_0(d_4)f_0(c_0)+f_0(d_4)f_0(c_4) \\
& + f_0(d_4)f_0(c_3)\big)\big]P_4 + [f_0(d_0)f_0(c_5)+f_0(d_1)f_0(c_5)+f_0(d_2)f_0(c_5) \\
& + f_0(d_3)f_0(c_5)+f_0(d_4)f_0(c_5)+f_0(d_5)f_0(c_0)+f_0(d_5)f_0(c_1) \\
& + f_0(d_5)f_0(c_2)+f_0(d_5)f_0(c_3)+f_0(d_5)f_0(c_4)+f_0(d_5)f_0(c_5)]P_5 =
\end{aligned}$$

$[f_0(d_0) + f_0(d_1)P_1 + f_0(d_2)P_2 + f_0(d_3)P_3 + f_0(d_4)P_4 + f_0(d_5)P_5][f_0(c_0) + f_0(c_1)P_1 + f_0(c_2)P_2 + f_0(c_3)P_3 + f_0(c_4)P_4 + f_0(c_5)P_5] = f(X).f(Y)$.

This implies the proof.

5.Conclusions

In this paper, we have defined the symbolic 4-plithogenic Vector Spaces and 5-plithogenic Vector Spaces, with many algebraic properties. Also, we have shown some related substructures such as symbolic 4-plithogenic Vector Spaces homomorphisms and symbolic 5-plithogenic Vector Spaces homomorphisms and many illustrative examples were presented.

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Chapter 4

A New Approach for Application of Cyber Wars with Q-Fuzzy Sets

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Abstract

In today's world, where almost all information is stored and processed in the cyber environment, devices are managed in the cyber environment, and the operational superiority of the armies is based on tactics such as electronic communication, image acquisition and remote sensing, attack and defense strategies and tactics are developed according to the fifth battlefield, the cyber environment. In international relations, a time has come when cyber armies and attacks are included to resolve conflicts between countries. Therefore, to facilitate the decision-making process in complex decision problems, many multi-criteria decision-making methods have been proposed and developed from past to present. These methods continue to develop by developing new approaches and methods and use in wide application areas. Therefore, a new mathematical tool has been proposed to achieve consensus among the countries here. Our aim of this study was to develop the VIKOR method on Q-fuzzy sets, which is a new method for multi-criteria decision-making methods. By applying this method to the real-life problem of cyber warfare, he demonstrated the flexibility, effectiveness, and feasibility of the proposed VIKOR method.

Keywords: Cyber War, Cyber Law, fuzzy set, Q-fuzzy set, Decision-making.

1. Introduction

With the revolutionary developments in the last quarter century in the field of technology, mankind has reached a standard of living and style that he could not even imagine. The enlargement of the possibilities in the information environment of the newly developing technology is the biggest problem for the lawyers in defining the terms. Computers have shrunk, mobile phones have become almost computers, and the living space where these two devices cannot be taken or used has almost disappeared. These developments have also caused some important problems in the field of law, new concepts have emerged, the definitions of these concepts have begun to be discussed and to have legal consequences. However, following the developments in this field and putting forth appropriate definitions, reconciliation in the international arena and regulation in the national field is the difficulties faced by the lawyers. So, solving fuzzy phenomena and uncertain events in real life is necessary as science and technology advance. Zadeh [1] first explicitly and systematically proposed fuzzy sets to solve these problems effectively.

Decision making is the act of identifying and choosing alternatives to find out the best solution from a pool of options based on different factors and considering the decision maker’s expectations. Every decision is made within an environment, which is defined as the collection of information, alternatives, values and preferences available at the time when the decision must be made. Başer and Ulucay [35] defined energy of a neutrosophic soft set and its applications to multi-criteria decision-making problems. Till date, several MCDM methods [17-53] have been proposed and successfully deployed to solve complex decision-making problems arising from different corners of engineering and management. Amongst those techniques, VIKOR (the Serbian name is ‘Vlse Kriterijumska Optimizacija Kompromisno Resenje’ which means multi-criteria optimization and compromise solution) method has gained much popularity among the decision-making community due to its simple and easily comprehensible computational steps [2]. Opricovic used the VIKOR model to investigate some MCGDM problems with conflicting criteria [3,4]. Also, we solve an MCDM problem with cyber wars related example based on VIKOR strategy in Q- fuzzy sets. Multi Q -fuzzy soft expert set [9]. Effective Q- fuzzy soft expert sets [31], Q -fuzzy soft sets [10-12], and multi-Q fuzzy soft sets [13-15] were proposed by Adam and Hassan.

Recently, it has been used by other researchers [5-8, 54-75]. To fill up this research gap, we propose a new VIKOR method to deal with MCDM problems in Q- fuzzy set environment. Since Q-fuzzy sets are developing and novel sets, there is a scarcity in the number of studies in the literature, which is a contribution of this study.

2. Preliminaries

Definition 2.1[1] The fuzzy sets defined on a non-empty V as objects having the form $A = \{< v: \mu_A(v) >, v \in V\}$ where the functions $\mu: V \rightarrow [0, 1]$ for $v \in V$.

Definition 2.2 [16] Let I be unit interval and k be a positive integer. A multi Q - fuzzy set $\tilde{A}_Q$ in V and a non-empty set Q is a set of ordered sequences $\tilde{A}_Q = \{(v.q), \mu_i(v,q): v \in V, q \in Q\}$ where

$$\mu_i: V \times Q \rightarrow I^k, \quad i = 1,2, \dots, k.$$

The function $(\mu_1(v,q), \mu_2(v,q), \dots, \mu_k(v,q))$ is called the membership function of multi Q- fuzzy set $\tilde{A}_Q$: and $\mu_1(v,q) + \mu_2(v,q) + \cdots + \mu_k(v,q) \leq 1, k$ is called the dimension of $\tilde{A}_Q$. The set of all multi Q- fuzzy sets of dimension k in V and Q is denoted by $M^k QF(V)$.

3. A New Method on Q-Fuzzy Sets

To address the uncertainties and conflicts that arise in real-world decision-making, the VIKOR method is often employed. This method is particularly effective in solving MCDM problems by incorporating more practical and flexible preference information provided by decision-makers. In this context, the VIKOR method is adapted to operate within Q-fuzzy sets, enhancing its ability to manage uncertainty and imprecision. The method offers a structured, systematic approach to decision-making, enabling the identification of both the best and a compromise solution, particularly in settings involving fuzzy logic.

Figure 1 shows the steps followed when the VIKOR method is applied to MCDM problems using fuzzy sets. It shows how this approach leads to systematic selection of the best solutions in situations involving uncertainty, using criteria characterized by the uncertainty.

Step 1: Criteria are determined and measurement scales are created.

Step 2: A direct relationship matrix is created in Q fuzzy sets.

Step 3: The direct relationship matrix is replaced with grammar in Q fuzzy sets.

Step 4: The weight of the decision makers is determined.

Step 5: A batch Q fuzzy sets martial is created.

Step 6: The optimal weights of the criteria are obtained by considering the subjective weight and objective weight.

Step 7: PIS and NIS values are calculated.

Step 8: Benefit (S) and Cost (R) values are calculated.

Step 9: The priority value (Q) is calculated.

Step 10: S, R and Q values are sorted according to max criteria.

Step 11: Alternatives are listed and a compromise solution is obtained.

Figure 1: VIKOR Method

According to Figure 1, VIKOR consists of 11 steps. There are two types of optimal criterion weights, which distinguish VIKOR from other methods, these are subjective and objective weights.

Let $D^{(k)}$ be a set of decision makers where k=1,2,…,p. A_i represents alternatives where i=1,2,…,m and C_i represent criteria j=1,2,…,n. The criteria are classified as cost criteria and benefit criteria.

Step 1: Criteria are defined and measurement scales are created.

Step 2: A Direct Relationship Matrix is established with Q fuzzy sets. We will obtain the fuzzy Decision matrix $[R_{ij}]_{mxn}$, $(i = 1,2, \dots, k \ ve \ j = 1,2, \dots, p)$ as follows:

$$C_1 \quad C_2 \quad \dots \quad C_n$$

$$[R_{ij}]_{mxn} = \begin{matrix} A_1 \\ A_2 \\ \vdots \\ A_m \end{matrix} \begin{bmatrix} R_{11} & R_{12} \dots & R_{1n} \\ R_{21} & R_{22} \dots & R_{2n} \\ \vdots & \vdots & \vdots \\ R_{m1} & R_{m2} \dots & R_{mn} \end{bmatrix}$$

where $R_{ij} = \langle (x_{ij} q_{ij}) : (\mu_{ij}) \rangle$

Step 3: The Direct Relationship Matrix is replaced with grammatical information.

Step 4: The weights of the decision makers are determined. The weights of the decision makers can be obtained with the following formula:

$$\varpi_k = \frac{\mu_k}{\sum_{k=1}^{p} \mu_k}, \quad \varpi_k \geq 0, \sum_{k=1}^{p} \varpi_k = 1$$

$\mu: accuracy \ - \ represents\ the\ membership\ function$

Step 5: Consolidated Q fuzzy sets decision matrix is created $\mathcal{N}^{(k)} = (\mathcal{N}_{ij}^{k})_{mxn}$ k. Let the decision maker have a single value decision matrix Q fuzzy sets weight operator is used to sum all the individual decision matrices $\mathcal{N}$.

$\mathcal{N}^{(k)} = (\mathcal{N}_{ij}^{k})_{mxn}, k = 1,2, \dots, p, i = 1,2, \dots, m\ ve\ j = 1,2, \dots, n$.

$$d_{ij} = SNWA\left(d_{ij}^{(1)}, d_{ij}^{(2)}, \dots, d_{ij}^{(p)}\right) = (1 - \prod_{k=1}^{p} \left(1 - \mu_{ij}^{(k)}\right)^{\frac{1}{p}})$$

The aggregate decision matrix D is defined as follows:

$$\mathcal{N} = \begin{bmatrix} d_{11} & d_{12} \dots & d_{1n} \\ d_{21} & d_{22} \dots & d_{2n} \\ \vdots & \vdots & \vdots \\ d_{m1} & d_{m2} \dots & d_{mn} \end{bmatrix}, d_{ij} = \langle (x_{ij} q_{ij}) : (\mu_{ij}) \rangle$$

Step 6: The optimal weights of the criterion are obtained.

There are two types of weights in this section that are subjective and need to be considered.

3.1. Subjective Weight

The rating of the alternatives according to each criterion is collected by the decision makers. The importance weights of the criteria corresponding to the alternatives are determined using the linguistic rating scale as follows.

LANGUAGE TERMS	*IMPACT SCORE*	***Fuzzy Value***
TOO LOW	1	(0)
LOW	2	(0)
MEDIUM LOW	3	(0.1)
MEDIUM	4	(0.3)
MEDIUM HIGH	5	(0.5)
HIGH	6	(0.7)
TOO HIGH	7	(0.9)

Considering that the criterion weight is obtained using the equation:

$$w_j = QFSWA(w_j^1, w_j^2, \dots, w_j^l) = \left(1 - \prod_{k=1}^{l}\left(1 - \mu_{ij}^{(k)}\right)^{\psi_k}\right)$$

$j = 1,2,\dots,n \ ve \ w_j = (\mu_j)$

The subjective weight of each criterion is obtained using the formula below.

$$w_j^s = \mu_k + \left[-\left(\frac{1}{\ln m}\right)\sum_{i=1}^{m}\mu_k\right]^{-1}$$

$j = 1,2,\dots,n \ ve \ \sum_{j=1}^{n} w_j^s = 1.$

3.2. Objective Weight

The evaluation criteria are normalized using the following equation.

$$P_{ij} = \frac{R_{ij}}{\sum_{i=1}^{m} R_{ij}}$$

where P_{ij} is the predicted result of criterion j. Next, the predicted results of the j criterion entropy set E_j are calculated.

$$E_j = -\left(\frac{1}{\ln m}\right)\sum_{i=1}^{m} P_{ij} lnP_{ij}$$

where m is the number of criteria and $0 \le E_j \le 1$.

After that, to obtain the objective weights of the criteria, the divergence div_j, which is the degree of deviation of the information of the j criterion, must be defined.

$$div_j = 1 - E_j$$

The greater the degree of deviation of this criterion, the more important the criterion is in the decision-making process. Finally, objective weights can be obtained using the following equation.

$$w_j^o = \frac{div_j}{\sum_{j=1}^{n} div_j}$$

Step 7: Fuzzy values of positive ideal solution (PIS) are calculated $f_j^+ = \langle (x_j q_j):(\mu_j^+) \rangle$ and fuzzy values of negative ideal solution (NIS) are calculated $f_j^- = \langle (x_j q_j):(\mu_j^-) \rangle$, <, $j = 1,2, \dots, n$.

$$f_j^- = \begin{Bmatrix} \min R_{ij}\ , benefit\ criterion \\ \max R_{ij}\ , cost\ criterion \end{Bmatrix}$$

$$f_j^+ = \begin{Bmatrix} \max R_{ij}\ , benefit\ criterion \\ \min R_{ij}\ , cost\ criterion \end{Bmatrix}$$

$j = 1,2, \dots, n$.

Step 8: Calculate the utility measure (S_i) and regret measure (R_i) for the alternative as follows:

$$S_i = \sum_{j=1}^{n} \frac{w_j \| f_j^+ - f_{1j} \|}{\| f_j^+ - f_j^- \|}\ , i = 1,2, \dots, m$$

$$R_i = max \left\{ \frac{w_j \| f_j^+ - f_{1j} \|}{\| f_j^+ - f_j^- \|} \right\}\ i = 1,2, \dots, m$$

Here w_j, represents the weight of the combination for each criterion.

$$w_j = v w_j^s + (1 - v) \frac{(R_i - R^+)}{(R^- - R^+)}$$

Where v, denotes the relative importance between subjective weights and objective weights. It can be any value from 0 to 1, but is usually set to 0.5.

Step 9: The priority value $Q_i, i = 1,2, \dots, m$ is applied using the following formulas

$$Q_i = \nu \frac{(S_i - S^+)}{(S^- - S^+)} + (1-\nu)\frac{(R_i - R^+)}{(R^- - R^+)}$$

$$S^+ = minS_i \text{ , } S^- = maxS_j$$

$$R^+ = minR_i \text{ , } R^- = maxR_j$$

ν, indicates the weight of the strategy of the constraints of the criteria and is usually assumed to be 0.5.

Step 10: Sort S_i, R_i and Q_i values by maximum criterion. Sort the sort results by descending alternatives.

Step 11: List the alternatives and find the compromise solution.

At the minimum value of Q, the best alternative $A^{(1)}$ (top alternative) must satisfy the following 2 conditions.

Condition 1: Acceptable benefits

$$Q(A^{(2)}) - Q(A^{(1)}) \geq \frac{1}{m-1}$$

Here $A^{(1)}$ and $A^{(2)}$, Q_i is also the two best alternatives.

Condition 2: Acceptable Status

The best alternatives should be ranked best by S_i and R_i. If one of the above conditions is not met, a number of compromise solutions have been proposed:

1. $A^{(1)}$ and $A^{(2)}$ alternatives are also acceptable if only stability requirement is not achieved;
2. Alternatives $A^{(1)}, A^{(2)}, \dots, A^{(u)}$ are accepted if the advantage condition is not met. $A^{(u)}$ for max u $Q(A^{(u)}) - Q(A^{(1)}) \geq \frac{1}{m-1}$ It is determined by the formula. (the positions of these alternatives are close to each other.)

4. Application of Cyber Wars of VIKOR Method for Decision-Making Problems

The increase in cyber war threats in the world obliges states to take precautions in this regard. Although developed countries have come a long way in this regard, there are still many countries that do not take adequate steps in this regard. Especially underdeveloped and developing countries, as they are insufficient in cyber warfare, can be vulnerable and suffer victimization in case of any cyber-attack. To prevent this situation, a few developing countries that decided to act have taken the models of the countries that have achieved success in this subject to examination and have decided to take the model they found suitable for them as an example. As a result of the examination, the policies that developed countries have already implemented and are considering implementing soon have been taken into consideration. Especially developing countries wants to use proposed method when choosing a model. The models he can take to are Türkiye model (A1), England model (A2), USA model (A3) and Q=$\{q_1, q_2, q_3\}$. It considers four criteria for three alternatives: models' price (C1), usability (C2), speed (C3) and security after buy. While making this choice, he consults 3 different cyber warfare experts (DM1, DM2, DM3).

Then we can take a model based on their parameters using Q-fuzzy sets, by applying the following algorithm.

Step 1: Criteria are defined, and measurement scales are created. Compiling the perspectives of 3 different decision makers (DM1, DM2, DM3), the student, who met with his family, carefully selected the decision makers by considering the cost and benefit. Tables 2 and 3 describe the decision makers' perspectives on the weight of the criteria and the evaluation of alternatives according to the criteria.

Table 2 Impact score of the three decision makers on the importance of the criteria

	C1	**C2**	**C3**	**C4**
DM1	5	6	5	4
DM2	6	6	6	4
DM3	6	5	6	4

Table 3 Rating of evaluation of alternatives according to criteria

		C1	**C2**	**C3**	**C4**
A1(x_1, q_1)	DM1	6	6	1	3
	DM2	3	4	2	2
	DM3	5	6	1	3
A2(x_1, q_2)	DM1	6	6	7	7
	DM2	6	6	5	7
	DM3	6	6	6	7
A3(x_1, q_3)	DM1	3	4	5	4
	DM2	3	3	4	6
	DM3	3	5	4	4

Step 2 and Step 3: A Direct Relationship Matrix is Constructed in the Q fuzzy set. The Language Expressions of Q fuzzy Sets are modified in table 3 to create the Q fuzzy set Direct Relationship Matrix (see Table 4).

Table 4 Q-FS Direct Relationship Matrix

		C1	**C2**	**C3**	**C4**
A1$(x_1 q_1)$	DM1	0.7	0.7	0	0.1
	DM2	0.1	0.3	0	0
	DM3	0.5	0.7	0	0.1
A2$(x_1 q_2)$	DM1	0.7	0.7	0.9	0.9
	DM2	0.7	0.7	0.5	0.9
	DM3	0.7	0.7	0.7	0.9
A3$(x_1 q_3)$	DM1	0.1	0.3	0.5	0.3
	DM2	0.1	0.1	0.3	0.7
	DM3	0.1	0.5	0.3	0.3

Step 4: Determine the weights of the decision makers. The weights of the decision makers are obtained by the following equation:

$$DM1 = 0.5\,, DM2 = 0.2\,, DM3 = 0.3$$

Step 5: An aggregated QFS decision matrix is generated. The importance weights of the decision makers are shown in Table 5.

Table 5 Importance weight of decision makers

	Linguistic Data	**Impact Score**	**Weight**
DM1	Medium High	(0.5)	0.5
DM2	Medium High	(0.2)	0.2
DM3	Medium	(0.3)	0.3

To aggregate the weights of all decision makers, the QFSWA operator is applied.

$\mu_{11} = 1 - ((1 - 0.7)^{0.5} \times (1{-}0.1)^{0.2} \times (1 - 0.5)^{0.3}) = 0.564$

The remaining calculations are calculated in a similar way. The detailed calculation of the aggregated QFS matrix is shown in Table 6.

Table 6 Aggregate Q-FS matrix

	C1	**C2**	**C3**	**C4**
A1 (x_1q_1)	0.564	0.645	0	0.081
A2(x_1q_2)	0.7	0.7	0.81	0.9
A3(x_1q_3)	0.1	0.335	0.408	0.409

Step 6: The optimal weight of the criterion is obtained. Linguistic variables are shown in Table 2. The overall subjective weight of the criterion is calculated as follows:

$\mu_1 = 1 - ((1 - 0.5)^{0.5} \times (1{-}0.7)^{0.2} \times (1 - 0.7)^{0.3}) = 0.613$

The result of the collective subjective weight calculation is shown in Table 7.

Table 7 Total subjective weight

	$\boldsymbol{\mu}$
C1	0.613
C2	0.650
C3	0.613

C4	0.3

The subjective weight of the criterion is calculated and presented as follows:

$$s(x_{ij}) = \frac{\mu_{ij}}{4}$$

$$s(x_{11}) = \frac{0.613}{4} = 0.153$$

The aggregated net matrix is given in Table 8.

Table 8 Aggregate net matrix

	C1	C2	C3	C4
A1($x_1 q_1$)	0.153	0.163	0.153	0.075
A2($x_1 q_2$)	0.2666	0.2666	0.3178	0.3
A3($x_1 q_3$)	0.4333	0.3983	0.3476	0.5147

The criteria evaluation is then normalized as follows.

$$P_{11} = \frac{0.3959}{0.3959 + 0.2666 + 0.4333} = 0.3612$$

$$P_{21} = \frac{0.2666}{0.3959 + 0.2666 + 0.4333} = 0.2432$$

$$P_{31} = \frac{0.4333}{0.3959 + 0.2666 + 0.4333} = 0.3954$$

$$P_{12} = 0.4528$$

$$P_{22} = 0.2193$$

$$P_{23} = 0.3277$$

$$P_{13} = 0.4803$$

$$P_{23} = 0.2481$$

$$P_{33} = 0.2714$$

$$P_{14} = 0.4034$$

$$P_{24} = 0.2196$$

$$P_{34} = 0.3768$$

Next, entropy Ej is calculated.

lnP_{ij}:

$$lnP_{11} = -1.01832$$

$$lnP_{21} = -1.4114$$

$$lnP_{31} = -0.9278$$

$$lnP_{12} = -0.7923$$

$$lnP_{22} = -1.5173$$

$$lnP_{23} = -1.1156$$

$$lnP_{13} = -0.7333$$

$$lnP_{23} = -1.3939$$

$$lnP_{33} = -1.3041$$

$$lnP_{14} = -0.9078$$

$$lnP_{24} = -1.5159$$

$$lnP_{34} = -0.9769$$

$$\sum_{i=1}^{m} P_{x_i q_j} lnP_{x_i q_j} = (0.3612 \times -1.01832) + (0.2432 \times -1.4114) + (0.3954 \times -0.9278)$$

$$= -1.07792$$

$$\therefore E_1 = -\left(\frac{1}{ln3}\right)(-1.07792) = 0.9811$$

$$\therefore E_2 = 0.96218$$

$$\therefore E_3 = 0.95753$$

$$\therefore E_4 = 0.97108$$

After that, the distance value is calculated.

$$div_1 = 1 - 0.9811 = 0.0189$$

$$div_2 = 0.0378$$

$$div_3 = 0.0424$$

$$div_4 = 0.0289$$

$$\sum_{j=1}^{n} div_j = 0.0189 + 0.0378 + 0.0424 + 0.0289 = 0.128$$

Objective weights are calculated.

$$w_1^o = \frac{div1}{\sum div_j} = \frac{0.0189}{0.128} = 0.1476$$

$$w_2^o = 0.2953$$

$$w_3^o = 0.3312$$

$$w_4^o = 0.2257$$

The remaining objective weights are calculated similarly. The calculated objective weight and subjective weight results are shown in Table 9.

Table 9 Objective weight and subjective weight of the criteria

	Subjective weight	**Objective weight**
C1	0.2571	0.1476
C2	0.3644	0.2953
C3	0.3299	0.3312
C4	0.0483	0.2257

According to the result of the subjective weight for each criterion, it is seen that education (C3) is the most important criterion, and employment opportunity (C4) is the least important criterion according to the decision makers. According to the analysis of the objective weights of the criteria, we have seen that education (C3) is again the most important criterion and transportation (C1) is the least important criterion.

Step 7: The Q fuzzy value of the Positive Ideal Solution (PIS) and the Negative Ideal Solution (NIS) is determined. Benefit and Cost criteria are determined.

Benefit criteria: It is determined as Placement (C2), Education (C3) and Employment opportunity (C4), on the other hand Cost criteria; Available as Transport (C1).

The PIS and NIS in Table 8 are obtained, respectively. PIS and NIS values of all criteria are given in Table 10.

Table 10 PIS and NIS of all criteria

	PIS	**NIS**
C1	0.2666	0.4333
C2	0.5504	0.2666
C3	0.6152	0.3178
C4	0.5510	0.3

Step 8: The utility measure (S_i) and regret measure (R_i) are calculated for the alternative.

$\nu = 0.5$

$$w_{c1} = (0.5) \times (0.2571) + (1 - 0.5) \times 0.1476 = 0.2023$$

$$w_{c2} = 0.3298$$

$$w_{c3} = 0.3305$$

$$w_{c4} = 0.137$$

Then,

$$S_{11} = \frac{w_{c1}\|f_1^+ - f_{11}\|}{\|f_1^+ - f_1^-\|} = \frac{\|0.2023(0.2666 - 0.3959)\|}{\|0.2666 - 04333\|} = 0.1569$$

$S_{21} = 0$

$S_{31} = 0.2023$

$S_{12} = 0$

$S_{13} = 0$

$S_{14} = 0$

$S_{22} = 0.3298$

$S_{24} = 0.137$

$S_{32} = 0.1767$

$S_{34} = 0.0198$

$S_{23} = 0.3305$

$S_{33} = 0.2973$

From here,

$S_{A1} = 0.1569 + 0 + 0 + 0 = 0.1519$

$S_{A2} = 0.7973$

$S_{A3} = 0.6961$

And

$$R_i = max\left\{\frac{w_{c1}\|f_1^+ - f_{11}\|}{\|f_1^+ - f_1^-\|}\right\}$$

Table 11 S_i *and* R_i values

	$\boldsymbol{R_i}$	$\boldsymbol{R_i}$
C1	0.1569	0.1569
C2	0.7973	0.3305
C3	0.6961	0.2973
C4		

Step 9: Calculate the priority value.

$$S_i^+ min = 0.1569$$

$$R_i^+ min = 0.1569$$

$$S_i^- max = 0.7973$$

$$R_i^- max = 0.3305$$

$$Q_i = v\frac{(S_i - S^+)}{(S^- - S^+)} + (1 - v)\frac{(R_i - R^+)}{(R^- - R^+)}$$

$$Q_{A1} = 0$$

$$Q_{A2} = 1$$

$$Q_{A3} = 0.8253$$

Step 10: S_i, R_i and Q_i are sorted by maximum criteria.

Table 12: Calculated of $\boldsymbol{S_i}$, $\boldsymbol{R_i}$ and $\boldsymbol{Q_i}$

	S	Rank	R	Rank	Q	Rank
A1($\boldsymbol{x_1 q_1}$)	0.1569	1	0.1569	1	0	1
A2($\boldsymbol{x_1 q_2}$)	0.7973	3	0.3305	3	1	3
A3($\boldsymbol{x_1 q_3}$)	0.6961	2	0.2973	2	0.8253	2

Step 11: Sort alternatives by rank.

$$A2_{(x_1 q_1)} > A3_{(x_1 q_2)} > A1_{(x_1 q_3)}$$

5 Conclusions

The primary focus of the paper is on selecting the optimal model for combating cyber warfare, demonstrating the effectiveness of our proposed method in this context. The unique advantage of these methods lies in their ability to handle three aggregated arguments, as opposed to the conventional two or one, making them more sensitive and adaptable to complex situations. This enhanced sensitivity is particularly valuable in the dynamic and unpredictable environment of cyber warfare, where multiple conflicting factors must be considered simultaneously. This approach offers a more scientific and robust framework for addressing conflicting attributes, providing a clearer and more reliable decision-making process.

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Chapter 5

Effective *Q* - Neutrosophic Soft Sets and Architucture Application in Decision-Making

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Abstract

It becomes even more complex with complex architectural problems, and decision-making methods are needed, and it is understood how important decision-making methods are. While the use of decision-making methods in the field of engineering is dominant, their use in the field of architecture is becoming more and more widespread. It can be listed as reaching an optimum solution with the targeted and designed alternatives with these methods, evolving the design process, allowing recycling, controlling these processes, and creating data for architecture in the future Clustering plays an important role in data mining, pattern recognition, and machine learning. In this section, a new algorithm is proposed based on Effective Q - Neutrosophic Soft Sets. Finally, we illustrate the feasibility of the new method by an example in architecture.

Keywords: Neutrosophic sets; Effective Q - Neutrosophic Soft Sets, Architecture.

1. Introduction

Housing has always been more than just a shelter; it serves as a reflection of the lifestyle, culture, and preferences of the family, group, or community to which it belongs. Moreover, housing is a mirror of the user's personality and worldview, evident in its materials, shape, and design. The types of housing, construction methods, the number of rooms per person, and the functional spaces within a home all contribute to the complexity and uncertainties faced by its inhabitants. In this context, decision-making in architecture becomes a critical process, influencing the design and usage of living spaces. To properly describe objects in an environment characterized by uncertainty and ambiguity, it is necessary to handle indeterminate and incomplete information effectively. In this regard, the concept of intuitionistic fuzzy sets, introduced by Atanassov [1], followed by Molodtsov's work on soft sets [2], and neutrosophic logic [3], as well as neutrosophic sets [4,5] by Smarandache, has gained significant attention. The term "neutrosophy" refers to the study of neutral thought, which distinguishes neutrosophic logic from traditional fuzzy and intuitionistic fuzzy logic by introducing the notion of neutrality. Baser and Ulucay [56] defined concept of neutrosophic soft energy. Currently, research on soft set theory is progressing rapidly, with extensive literature on Q-fuzzy sets. For example, Q-fuzzy soft sets [6–8] and multi Q-fuzzy sets [9–11] have opened up many applications [12-17], including the development of multi Q-fuzzy soft expert sets [18]. Şahin et al. [19] introduced neutrosophic soft expert sets, and Hassan et al. [20] expanded this further with the Q-neutrosophic soft expert set. In 2022, Alkhazaleh [21] introduced the concept of the Effective Fuzzy Soft Set (EFSS), which was designed to extend the notion of external effectiveness within the framework of soft sets. Later, the concept of Effective Fuzzy Soft Expert Sets [22] was proposed, incorporating expert opinions into a unified model. Başer and Uluçay [23] further defined Effective Q-Fuzzy Soft Expert Sets. In 2023, the Effective Neutrosophic Soft Set [24] was introduced and later extended to the generalized Effective Neutrosophic Soft Set (ENSS) [25], incorporating the concept of effectiveness across three independent membership functions: truth (T), indeterminacy (I), and falsity (F). Furthermore, the concept of the Effective Neutrosophic Soft Expert Set [26] was introduced, along with associated operations and practical examples. Building on this, we develop a new mathematical tool by combining the concept of the Effective Neutrosophic Soft Set with the supply set Q—a mathematical framework designed to capture the nuances of uncertain information through three distinct membership functions representing degrees of truth (T), uncertainty (I), and falsity (F). We introduce the Effective Q-Neutrosophic Soft Set, a new concept that enhances the framework's ability to model complex real-world scenarios. This innovative approach combines the strengths of both the Effective Neutrosophic Soft Set and the Effective Neutrosophic Soft Expert Set, offering a versatile and comprehensive representation of uncertainty.

These advanced set theories have been successfully applied to various decision-making problems across multiple domains [27–67]. The aim of this chapter, besides the objective evaluation, a decision-making model that can be effective in expressing the subjective evaluations within the structure of architecture (mass, spatial, semantic, form and experience) has been developed.

2. Preliminaries

Definition 1 [4] Let $\mathcal{U}$ be a universe of discourse, with a generic element in $\mathcal{U}$ denoted by u then a neutrosophic set (NS) N is an object having the form

$$N = \{< \mathrm{u} : \theta_N(\mathrm{u}), \zeta_N(\mathrm{u}), \eta_N(\mathrm{u}), \mathrm{u} \in \mathcal{U}\}$$

where the functions θ , ζ, η : $\mathcal{U} \rightarrow$ $]^-0,\ 1^+[$ define respectively the degree of membership (or Truth) , the degree of indeterminacy, and the degree of non-membership (or Falsehood) of the element $\mathrm{u} \in \mathcal{U}$ to the set N with the condition.

$$^-0 \leq \theta_N(\mathrm{u}) + \zeta_N(\mathrm{u}) + \eta_N(\mathrm{u}) \leq 3^+$$

Definition 2 [5] A neutrosophic set N is contained in another neutrosophic set N_1 i.e. $N \subseteq N_1$ if $\forall \mathrm{u} \in \mathcal{U},\ \theta_N(\mathrm{u}) \leq \theta_{N_1}(\mathrm{u}),\ \zeta_N(\mathrm{u}) \leq \zeta_{N_1}(\mathrm{u}),\ \eta_N(\mathrm{u}) \geq \eta_{N_1}(\mathrm{u})$.

Definition 3 [14] Let $\mathcal{U}$ be an initial universe set and ϵ be a set of parameters. Consider $N \subseteq \epsilon$. Let $P(\mathcal{U})$ denotes the set of all neutrosophic sets of $\mathcal{U}$. The collection (F, N) is termed to be the neutrosophic soft set over $\mathcal{U}$, where F is a mapping given by $F: N \rightarrow P(\mathcal{U})$.

Proposition 1 [14]

For two NSS over the universe $\mathcal{U}$ and ϵ be a set of parameters.

1. $(N, \epsilon) \tilde{\subset} (N_1, \epsilon)$ if and only if

$$\theta_{N(e)}(\mathrm{u}) \tilde{\leq} \theta_{N_1(e_1)}(\mathrm{u}),\ \zeta_{N(e)} \tilde{\leq} \zeta_{N_1(e_1)}(\mathrm{u}),\ \eta_{N(e)}(\mathrm{u}) \tilde{\geq} \eta_{N_1(e_1)}(\mathrm{u})$$

$$\forall e, e_1 \in \epsilon, \mathrm{u} \in \mathcal{U}.$$

2. $(N, \epsilon) = (N_1, \epsilon)$ if and only if

$$\theta_{N(e)}(\mathrm{u}) = \theta_{N_1(e_1)}(\mathrm{u}),\ \zeta_{N(e)} = \zeta_{N_1(e_1)}(\mathrm{u}),\ \eta_{N(e)}(\mathrm{u}) = \eta_{N_1(e_1)}(\mathrm{u})$$

$$\forall e, e_1 \in \epsilon, \mathrm{u} \in \mathcal{U}.$$

3. $(N, \epsilon)^c = \{\mathrm{u},\ \theta_{N^c(\mathrm{u})} = \eta_{N(\mathrm{u})},\ \zeta_{N^c(\mathrm{u})} = \zeta_{N(\mathrm{u})},\ \eta_{N^c(\mathrm{u})} = \theta_{N(\mathrm{u})} : \forall e \in \epsilon, \mathrm{u} \in \mathcal{U}\ \}$.

4. $(N,\epsilon)\tilde{\cup}(N_1,\epsilon) = \begin{cases} max\{\theta_{N(e)}(u), \theta_{N_1(e_1)}(u)\} \\ max\{\zeta_{N(e)}(u), \zeta_{N_1(e_1)}(u)\} \\ min\{\eta_{N(e)}(u), \eta_{N_1(e_1)}(u)\} \end{cases}$

5. $(N,\epsilon)\tilde{\cap}(N_1,\epsilon) = \begin{cases} min\{\theta_{N(e)}(u), \theta_{N_1(e_1)}(u)\} \\ min\{\zeta_{N(e)}(u), \zeta_{N_1(e_1)}(u)\} \\ max\{\eta_{N(e)}(u), \eta_{N_1(e_1)}(u)\} \end{cases}$

Definition 4 [11] Let I be unit interval and k be a positive integer. A multi Q -fuzzy set $\tilde{N}_Q$ in $\mathcal{U}$ and a non-empty set Q is a set of ordered sequences $\tilde{N}_Q = \{(\mathrm{u}, q), \theta_i(\mathrm{u}, q) : \mathrm{u} \in \mathcal{U}, q \in Q\}$ where

$$\theta_i : \mathcal{U} \times Q \to \mathrm{I}^k, \quad i = 1,2, \dots, k.$$

The function $(\theta_1(\mathrm{u},q), \theta_2(\mathrm{u},q), \dots, \theta_k(\mathrm{u},q))$ is called the membership function of multi Q- fuzzy set $\tilde{N}_Q$ and $\theta_1(\mathrm{u},q) + \theta_2(\mathrm{u},q) + \cdots + \theta_k(\mathrm{u},q) \leq 1, k$ is called the dimension of $\tilde{N}_Q$. The set of all multi Q- fuzzy sets of dimension k in $\mathcal{U}$ and Q is denoted by $M^k QF(\mathcal{U})$.

Definition 5 [20] (F_Q, N) is a *QNSES* over $\mathcal{U}$, where F_Q is the mapping $F_Q : N \to$ *QNSES* such that *QNSES* is the set of all *QNSES* over $\mathcal{U}$.

Definition 6 [24] A neutrosophic set Λ in a universe of discourse $\mathcal{U}_1$, where $\Lambda : \mathcal{U}_1 \to [0,1]$ a function, is an effective set. $\mathcal{U}_1$ is a set of effective parameters that can change membership and it's written in the following way;

$\Lambda = \{< \alpha, (\theta_\Lambda(\alpha), \zeta_\Lambda(\alpha), \eta_\Lambda(\alpha)) > : \alpha \in \mathcal{U}_1\}$

Definition 7 [24] Let $\mathcal{U}$ be an initial universe, ϵ be a set of all parameters, $\mathcal{U}_1$ be a set of effective parameters, Λ be a effective set over $\mathcal{U}_1$ and $P(\mathcal{U})$ represent the power set of $\mathcal{U}$. In this case $(\tilde{N}, \mathcal{U}_1)_\Lambda$ is called on effective neutrosophic soft set over $\mathcal{U}$, where $\tilde{N}$ is mapping represented by $N : \mathcal{U}_1 \to P(\mathcal{U})$ and it may be expressed as a collection of ordered pairs;

$(\tilde{N}, \mathcal{U}_1)_\Lambda = \{((u_j, < u_j, \theta_{\tilde{N}(e_j)}(u_j)_\Lambda, \zeta_{\tilde{N}(e_j)}(u_j)_\Lambda, \eta_{\tilde{N}(e_j)}(u_j)_\Lambda >) : u_j \in \mathcal{U}, e_j \in \epsilon\}$

and $\theta_{\tilde{N}(e_j)}(u_j)_\Lambda, \zeta_{\tilde{N}(e_j)}(u_j)_\Lambda, \eta_{\tilde{N}(e_j)}(u_j)_\Lambda$ membership values for $\forall\, \alpha \in \mathcal{U}_1$ is calculated as

$$\theta_{\tilde{N}(e_j)}(u_j)_\Lambda = \begin{cases} \theta_{\tilde{N}(e_j)}(u_j) + \left(\dfrac{\left(1-\theta_{\tilde{N}(e_j)}(u_j)\right)\sum \theta_{\Lambda\, U_j}(\alpha)}{|\Lambda|}\right), & if\ \theta_{\tilde{N}(e_j)}(u_j) \in (0,1) \\ \theta_{\tilde{N}(e_j)}(u_j), & O.W \end{cases}$$

$$\zeta_{\tilde{N}(e_j)}(u_j)_\Lambda = \begin{cases} \zeta_{\tilde{N}(e_j)}(u_j) + \left(\dfrac{\left(1-\zeta_{\tilde{N}(e_j)}(u_j)\right)\sum \zeta_{\Lambda\, U_j}(\alpha)}{|\Lambda|}\right), & if\ \zeta_{\tilde{N}(e_j)}(u_j) \in (0,1) \\ \zeta_{\tilde{N}(e_j)}(u_j), & O.W \end{cases}$$

$$\eta_{\tilde{N}(e_j)}(u_j)_\Lambda = \begin{cases} (1-\eta_{\tilde{N}(e_j)}(u_j)_\Lambda) + \left(\dfrac{\eta_{\tilde{N}(e_j)}(u_j)_\Lambda \sum \eta_{\Lambda\, U_j}(\alpha)}{|\Lambda|}\right), & if\ \eta_{\tilde{N}(e_j)}(u_j) \in (0,1) \\ \eta_{\tilde{N}(e_j)}(u_j), & O.W \end{cases}$$

3. Effective Q- Neutrosophic Soft Sets

We will now propose the definition of Effective *Q*-Neutrosophic Soft Sets (EQNSS) and some of their properties. Throughout this discussion, let $\mathcal{U}$ be the initial universe, $A \subseteq \epsilon$ be the set of parameters, Q be a set of supplies, and Λ be the set of effective parameters. Let $S \subseteq \epsilon \times Q \times \Lambda$.

Definition 8 $(\tilde{N}_Q, A)$ is a *QNSS* over $\mathcal{U}$, where $\tilde{N}_Q$ is the mapping $\tilde{N}_Q : A \to$ *QNSS*($\mathcal{U}$) such that *QNSS*($\mathcal{U}$) is the set of all *QNSS* over $\mathcal{U}$.

$$\tilde{N}_{Q_\Lambda}(S) = \left\{ \frac{(u_j, k_r)}{< \theta_{\tilde{N}(e_j)}(u_j, k_r)_\Lambda, \zeta_{\tilde{N}(e_j)}(u_j, k_r)_\Lambda, \eta_{\tilde{N}(e_j)}(u_j, k_r)_\Lambda >} : u_j \in \mathcal{U}, k_r \in Q \right\}$$

For all $s \in S$ and for all $\alpha \in \Lambda$, we have:

$$\theta_{\tilde{N}(e_j)}(u_j, k_r)_\Lambda = \begin{cases} \theta_{\tilde{N}(e_j)}(u_j, k_r) + \dfrac{\left(1-\theta_{\tilde{N}(e_j)}(u_j, k_r)\right)\sum_k \theta_{\Lambda_{x_j}}(a_k)}{|\Lambda|}, & if\ \theta_{\tilde{N}(e_j)}(u_j, k_r) \in (0,1) \\ \theta_{\tilde{N}(e_j)}(u_j, k_r), & O.W \end{cases}$$

$$\zeta_{\tilde{N}(e_j)}(u_j, k_r)_{\Lambda} = \begin{cases} \zeta_{\tilde{N}(e_j)}(u_j, k_r) - \dfrac{\zeta_{\tilde{N}(e_j)}(u_j, k_r) \sum_k \zeta_{\Lambda_{x_j}}(a_k)}{|\Lambda|}, & if \ \zeta_{\tilde{N}(e_j)}(u_j, k_r) \in (0,1) \\ \zeta_{\tilde{N}(e_j)}(u_j, k_r) \quad , & O.W \end{cases}$$

$$\eta_{\tilde{N}(e_j)}(u_j, k_r)_{\Lambda} = \begin{cases} \eta_{\tilde{N}(e_j)}(u_j, k_r) - \dfrac{\eta_{\tilde{N}(e_j)}(u_j, k_r) \sum_k \eta_{\Lambda_{x_j}}(a_k)}{|\Lambda|}, & if \ \eta_{\tilde{N}(e_j)}(u_j, k_r) \in (0,1) \\ \eta_{\tilde{N}(e_j)}(u_j, k_r) \quad , & O.W \end{cases}$$

Example 1 Suppose a customer who wants to build a new house wants to get feedback from several experts. Let $\mathcal{U} = \{u_1, u_2\}$ be the set of houses, $Q = \{k_1, k_2\}$ be the set of construction companies, $\epsilon = \{e_1, e_2\}$ be the set of decision parameters and the set of effective parameters is represented by $\Lambda = \{l_1, l_2\}$. Assume that;

$$\Lambda^1(u_1, k_1) = \left\{\frac{l_1}{(0.5, 0.3, 0.6)}, \frac{l_2}{(0.4, 0.6, 0.3)}\right\}, \quad \Lambda^2(u_1, k_2) = \left\{\frac{l_1}{(0.3, 0.5, 0.4)}, \frac{l_2}{(0.8, 0.3, 0.6)}\right\}$$

$$\Lambda^3(u_2, k_1) = \left\{\frac{l_1}{(0.1, 0.7, 0.3)}, \frac{l_2}{(0.4, 0.5, 0.6)}\right\}, \quad \Lambda^4(u_2, k_2) = \left\{\frac{l_1}{(0.1, 0.5, 0.4)}, \frac{l_2}{(0.2, 0.3, 0.7)}\right\}$$

Let N be the Q - neutrosophic soft set (QNSS) defined as follows:

$$N_Q(e_1) = \left\{\left(\frac{(u_1, k_1)}{0.6, 0.8, 0.5}, \frac{(u_1, k_2)}{0.5, 0.4, 0.7}, \frac{(u_2, k_1)}{0.4, 0.3, 0.1}, \frac{(u_2, k_2)}{0.2, 0.1, 0.4}\right)\right\}$$

$$N_Q(e_2) = \left\{\left(\frac{(u_1, k_1)}{0.6, 0.4, 0.5}, \frac{(u_1, k_2)}{0.4, 0.2, 0.3}, \frac{(u_2, k_1)}{0.7, 0.5, 0.4}, \frac{(u_2, k_2)}{0.5, 0.3, 0.4}\right)\right\}$$

Then by applying Definition 8 we get,

$$N_Q(e_1)_\Lambda = \left\{ \begin{array}{c} \dfrac{(u_1,k_1)}{<0.6+\left[(1-0.6)\dfrac{0.5+0.4}{2}\right], 0.8-\left[0.8\dfrac{0.3+0.6}{2}\right], 0.5-\left[0.5\dfrac{0.3+0.6}{2}\right]>}, \\ \dfrac{(u_1,k_2)}{<0.5+\left[(1-0.5)\dfrac{0.3+0.8}{2}\right], 0.4-\left[0.4\dfrac{0.5+0.3}{2}\right], 0.7-\left[0.7\dfrac{0.4+0.6}{2}\right]>}, \\ \dfrac{(u_2,k_1)}{<0.4+\left[(1-0.4)\dfrac{0.1+0.4}{2}\right],\ 0.3-\left[0.3\dfrac{0.7+0.5}{2}\right], 0.1-\left[0.1\dfrac{0.3+0.6}{2}\right]>}, \\ \dfrac{(u_2,k_2)}{<0.2+\left[(1-0.2)\dfrac{0.1+0.2}{2}\right], 0.1-\left[0.1\dfrac{0.5+0.3}{2}\right], 0.4-\left[0.4\dfrac{0.4+0.7}{2}\right]>} \end{array} \right\}$$

$$= \frac{(u_1,k_1)}{<0.8,0.4,0.3>}, \frac{(u_1,k_2)}{<0.8,0.2,0.4>}, \frac{(u_2,k_1)}{<0.6,0.1,0.1>}, \frac{(u_2,k_2)}{<0.3,0.1,0.2>}$$

Similarly, when the calculations are continued, $N_Q(e_2)_\Lambda$ effective Q-neutrosophic soft set is found as follows;

$$N_Q(e_2)_\Lambda = \left\{ \begin{array}{c} \dfrac{(u_1,k_1)}{<0.6+\left[(1-0.6)\dfrac{0.5+0.4}{2}\right], 0.4-\left[0.8\dfrac{0.3+0.6}{2}\right], 0.5-\left[0.5\dfrac{0.3+0.6}{2}\right]>}, \\ \dfrac{(u_1,k_2)}{<0.4+\left[(1-0.4)\dfrac{0.3+0.8}{2}\right], 0.2-\left[0.2\dfrac{0.5+0.3}{2}\right], 0.3-\left[0.3\dfrac{0.4+0.6}{2}\right]>}, \\ \dfrac{(u_2,k_1)}{<0.7+\left[(1-0.7)\dfrac{0.1+0.4}{2}\right],\ 0.5-\left[0.5\dfrac{0.7+0.5}{2}\right], 0.4-\left[0.4\dfrac{0.3+0.6}{2}\right]>}, \\ \dfrac{(u_2,k_2)}{<0.5+\left[(1-0.5)\dfrac{0.1+0.2}{2}\right], 0.3-\left[0.3\dfrac{0.5+0.3}{2}\right], 0.4-\left[0.4\dfrac{0.4+0.7}{2}\right]>} \end{array} \right\}$$

$$= \frac{(u_1,k_1)}{<0.8,0.2,0.3>}, \frac{(u_1,k_2)}{<0.7,0.1,0.2>}, \frac{(u_2,k_1)}{<0.8,0.2,0.2>}, \frac{(u_2,k_2)}{<0.6,0.2,0.2>}$$

$$(N_Q,A)_\Lambda = \left\{ \left[(e_1), \frac{(u_1,k_1)}{<0.8,0.4,0.3>}, \frac{(u_1,k_2)}{<0.8,0.2,0.4>}, \frac{(u_2,k_1)}{<0.6,0.1,0.1>}, \frac{(u_2,k_2)}{<0.3,0.1,0.2>}\right] \right.$$
$$\left. \left[(e_2), \frac{(u_1,k_1)}{<0.8,0.2,0.3>}, \frac{(u_1,k_2)}{<0.7,0.1,0.2>}, \frac{(u_2,k_1)}{<0.8,0.2,0.2>}, \frac{(u_2,k_2)}{<0.6,0.2,0.2>}\right] \right\}$$

Definition 9 For two *EQNSS* $(N_Q,A)_\Lambda$ and $(N'_Q,B)_\Lambda$ over $\mathcal{U}$, $(N_Q,A)_\Lambda$ is called an effective Q-neutrosophic soft subset of $(N'_Q,B)_\Lambda$ if

i. $B \subseteq A$,

ii. $N_Q(T)_\Lambda$ is effective Q-neutrosophic soft subset $N'_Q(T)_\Lambda$ for all $T \in B$.

Definition 10 Two *EQNSS* $(N_Q,A)_\Lambda$ and $(N'_Q,A)_\Lambda$ over $\mathcal{U}$ are equal if $(N_Q,A)_\Lambda$ is a *EQNSS* subset of $(N'_Q,A)_\Lambda$ and $(N'_Q,A)_\Lambda$ is a *EQNSS* subset of $(N_Q,A)_\Lambda$.

Definition 11 The complement of a *EQNSS* $(N_Q,Z)_\Lambda$ is

$$(N_Q,Z)_\Lambda^{\;c} = \left(N_Q^{\;(c)},\neg A\right)_\Lambda$$

such that $N_Q^{\;(c)}{}_\Lambda : \neg A \to EQNSS(\mathcal{U})$ a mapping

$$N_Q^{\;(c)}(\propto)_\Lambda = \left\{\theta_{N_Q^{\;(c)}(\propto)_\Lambda} = \eta_{N_Q^{\;(c)}(\propto)_\Lambda}, \zeta_{N_Q^{\;(c)}(\propto)_\Lambda} = \bar{1} - \zeta_{N_Q^{\;(c)}(\propto)_\Lambda}, \eta_{N_Q^{\;(c)}(\propto)_\Lambda} = \theta_{N_Q^{\;(c)}(\propto)_\Lambda}\right\}$$

for each $\propto \in \epsilon$. It is *clear that* $\left((N_Q,A)_\Lambda^{\;c}\right)^c = (N_Q,A)_\Lambda$.

Example 2 Using our previous Example 1, the complement of the *EQNSS* $N_Q(e_1)_\Lambda$ denoted by $N_Q(e_1)_\Lambda^{\;c}$ is given as follows:

$$N_Q(e_1)_\Lambda^{\;c} = \left\{\frac{(u_1,k_1)}{<0.3, 0.6, 0.8>}, \frac{(u_1,k_2)}{<0.2, 0.9, 0.7>}, \frac{(u_2,k_1)}{<0.2, 0.8, 0.8>}, \frac{(u_2,k_2)}{<0.2, 0.8, 0.6>}\right\}$$

Definition 12 The union of two *EQNSS* $(N_Q,A)_\Lambda$ and $(N'_Q,A)_\Lambda$ over $\mathcal{U}$, denoted by

$$(N_Q,A)_\Lambda \,\tilde{\cup}\, (N'_Q,A)_\Lambda$$

is the *EQNSS* $(H_Q,C)_\Lambda$ such that $C = A \cup B$ and the memberships of truth, indeterminacy, and falsity of $(H_Q,C)_\Lambda$ are respectively as follows:

$$\theta_{H_Q(e)} = \begin{cases} \theta_{N_Q(e)}(u_j,k_r) & if \quad e \in A - B \\ \theta_{N'_Q(e)}\ (u_j,k_r) & if \quad e \in B - A \\ max\left(\theta_{N_Q(e)}(mu_j,k_r), \theta_{N'_Q(e)}(u_j,k_r)\right) & if \quad e \in A \cap B \end{cases}$$

$$\zeta_{H_Q(e)} = \begin{cases} \zeta_{N_Q(e)}(u_j, k_r) & if \quad e \in A - B \\ \zeta_{N'_Q(e)}\ (u_j, k_r) & if \quad e \in B - A \\ min\left(\zeta_{N_Q(e)}(u_j, k_r), \zeta_{N'_Q(e)}(u_j, k_r)\right) & if \quad e \in A \cap B \end{cases}$$

$$\eta_{H_Q(e)} = \begin{cases} \eta_{N_Q(e)}(u_j, k_r) & if \quad e \in A - B \\ \eta_{N'_Q(e)}\ (u_j, k_r) & if \quad e \in B - A \\ min\left(\eta_{N_Q(e)}(u_j, k_r), \eta_{N'_Q(e)}(u_j, k_r)\right) & if \quad e \in A \cap B \end{cases}$$

Example 3 Suppose that $(N_Q, A)_\Lambda$ and $(N'_Q, A)_\Lambda$ are two *EQNSS* over V, such that

$$(N_Q, A)_\Lambda = \left\{\left[(e_1)\left(\frac{(u_1,k_1)}{0.8,0.2,0.4}, \frac{(u_1,k_2)}{0.7,0.3,0.5}, \frac{(u_2,k_1)}{0.3,0.6,0.2}, \frac{(u_2,k_2)}{0.3,0.2,0.4}\right)\right],\right.$$

$$\left.\left[(e_2),\left(\frac{(u_1,k_1)}{0.4,0.2,0.8}, \frac{(u_1,k_2)}{0.3,0.6,0.2}, \frac{(u_2,k_1)}{0.8,0.2,0.5}, \frac{(u_2,k_2)}{0.1,0.2,0.8}\right)\right]\right\}$$

$$(N'_Q, A)_\Lambda = \left\{\left[(e_1)\left(\frac{(u_1,k_1)}{0.4,0.6,0.6}, \frac{(u_1,k_2)}{0.7,0.3,0.8}, \frac{(u_2,k_1)}{0.5,0.3,0.4}, \frac{(u_2,k_2)}{0.2,0.6,0.8}\right)\right]\right\},$$

Then $(N_Q, A)_\Lambda \tilde{\cup} (N'_Q, A)_\Lambda = (H_Q, C)_\Lambda$ where

$$(H_Q, C)_\Lambda = \left\{\left[(e_1),\left(\frac{(u_1,k_1)}{0.8,0.2,0.4}, \frac{(u_1,k_2)}{0.7,0.3,0.5}, \frac{(u_2,k_1)}{0.5,0.3,0.2}, \frac{(u_2,k_2)}{0.3,0.2,0.4}\right)\right]\right.$$

$$\left.\left[(e_2),\left(\frac{(u_1,k_1)}{0.4,0.2,0.8}, \frac{(u_1,k_2)}{0.3,0.6,0.2}, \frac{(u_2,k_1)}{0.8,0.2,0.5}, \frac{(u_2,k_2)}{0.1,0.2,0.8}\right)\right]\right\}$$

Proposition 2 If $(N_Q, A)_\Lambda, (N'_Q, A)_\Lambda$ and $(H_Q, C)_\Lambda$ are three *EQNSS* over $\mathcal{U}$, then

i. $\left((N_Q, A)_\Lambda \tilde{\cup} (N'_Q, A)_\Lambda\right) \tilde{\cup} (H_Q, C)_\Lambda = (N_Q, A)_\Lambda \tilde{\cup} \left((N'_Q, A)_\Lambda \tilde{\cup} (H_Q, C)_\Lambda\right)$.

ii. $(N_Q, A)_\Lambda \tilde{\cup} (N_Q, A)_\Lambda \tilde{\subseteq} (N_Q, A)_\Lambda$.

Definition 13 Suppose $(N_Q, A)_\Lambda$ and $(N'_Q, A)_\Lambda$ are two *EQNSS* over the common universe $\mathcal{U}$. The intersection of $(N_Q, A)_\Lambda$ and $(N'_Q, A)_\Lambda$ is $(N_Q, A)_\Lambda \tilde{\cap}$ $(N'_Q, A)_\Lambda = (K_Q, C)_\Lambda$ such that $C = A \cap B$ and the memberships of truth, indeterminacy and falsity of $(K_Q, C)_\Lambda$ are:

$$\theta_{K_Q(e)} = \begin{cases} \theta_{N_Q(e)}(u_j, k_r) & if \quad e \in A - B \\ \theta_{N'_Q(e)}\ (u_j, k_r) & if \quad e \in B - A \\ min\left(\theta_{N_Q(e)}(u_j, k_r), \theta_{N'_Q(e)}(u_j, k_r)\right) & if \quad e \in A \cap B \end{cases}$$

$$\zeta_{K_Q(e)} = \begin{cases} \zeta_{N_Q(e)}(u_j, k_r) & if \quad e \in A - B \\ \zeta_{N'_Q(e)}\ (u_j, k_r) & if \quad e \in B - A \\ max\left(\zeta_{N_Q(e)}(u_j, k_r), \zeta_{N'_Q(e)}(u_j, k_r)\right) & if \quad e \in A \cap B \end{cases}$$

$$\eta_{K_Q(e)} = \begin{cases} \eta_{N_Q(e)}(u_j, k_r) & if \quad e \in A - B \\ \eta_{N'_Q(e)}(u_j, k_r) & if \quad e \in B - A \\ max\left(\eta_{N_Q(e)}(u_j, k_r), \eta_{N'_Q(e)}(u_j, k_r)\right) & if \quad e \in A \cap B \end{cases}$$

Example 4 Suppose that $(N_Q, A)_\Lambda$ and $(N'_Q, A)_\Lambda$ are two *EQNSS* over $\mathcal{U}$, such that

$$(N_Q, A)_\Lambda = \left\{\left[(e_1), \left(\frac{(u_1, k_1)}{0.3,0.4,0.3}, \frac{(u_2, k_1)}{0.5,0.2,0.9}\right)\right],\right.$$

$$\left[(e_2), \left(\frac{(u_1, k_1)}{0.8,0.2,0.5}, \frac{(u_2, k_1)}{0.4,0.2,0.3}\right)\right],$$

$$\left.\left[(e_2), \left(\frac{(u_1, k_1)}{0.5,0.4,0.7}, \frac{(u_2, k_1)}{0.8,0.4,0.4}\right)\right]\right\}.$$

$$(N'_Q, A)_\Lambda = \left\{\left[(e_1), \left(\frac{(u_1, k_1)}{0.5,0.4,0.2}, \frac{(u_2, k_1)}{0.7,0.3,0.2}\right)\right]\right.$$

Then $(N_Q, A)_\Lambda \tilde{\cap} (N'_Q, A)_\Lambda = (K_Q, C)_\Lambda$ where

$$(K_Q, C)_\Lambda = \left\{\left[(e_1), \left(\frac{(u_1, k_1)}{0.3,0.4,0.3}, \frac{(u_2, k_1)}{0.5,0.3,0.9}\right)\right]\right\}.$$

Proposition 3 If $(N_Q, A)_\Lambda, (N'_Q, A)_\Lambda$ and $(K_Q, C)_\Lambda$ are three *EQNSS* over $\mathcal{U}$, the following properties hold true.

i. $\left((N_Q, A)_\Lambda \tilde{\cap} (N'_Q, A)_\Lambda\right) \tilde{\cap} (K_Q, C)_\Lambda = (N_Q, A)_\Lambda \tilde{\cap} \left((N'_Q, A)_\Lambda \tilde{\cap} (K_Q, C)_\Lambda\right)$

ii. $(N_Q, A)_\Lambda \tilde{\cap} (N_Q, A)_\Lambda \subseteq (N_Q, A)_\Lambda$.

Proposition 4 If $(N_Q,A)_\Lambda, (N'_Q,A)_\Lambda$ and $(K_Q,C)_\Lambda$ are three *EQNSS* over $\mathcal{U}$, then

i.
$$\left((N_Q,A)_\Lambda \tilde{\cup} (N'_Q,A)_\Lambda\right) \tilde{\cap} (K_Q,C)_\Lambda = \left((N_Q,A)_\Lambda \tilde{\cap} (K_Q,C)_\Lambda\right) \tilde{\cup} \left((N'_Q,A)_\Lambda \tilde{\cap} (K_Q,C)_\Lambda\right)$$

ii.
$$\left((N_Q,A)_\Lambda \tilde{\cap} (N'_Q,A)_\Lambda\right) \tilde{\cup} (K_Q,C)_\Lambda = \left((N_Q,A)_\Lambda \tilde{\cap} (K_Q,C)_\Lambda\right) \tilde{\cap} \left((N'_Q,A)_\Lambda \tilde{\cup} (K_Q,C)_\Lambda\right)$$

Definition 14 If $(N_Q,A)_\Lambda$ and $(N'_Q,A)_\Lambda$ are two *EQNSS* over $\mathcal{U}$, then "$(N_Q,A)_\Lambda$ AND $(N'_Q,A)_\Lambda$ " is

$$(N_Q,A)_\Lambda \wedge (N'_Q,A)_\Lambda = (H_Q,A \times B)_\Lambda$$

such that $H_Q(\alpha,\beta) = N_Q(\alpha) \cap N'_Q(\beta)$ and memberships of truth, indeterminacy, and falsity of of $(H_Q,A \times B)_\Lambda$ are as follows:

$$\theta_{H_Q(\alpha,\beta)}(u_j,k_r) = min\left(\theta_{N_Q(\alpha)}(u_j,k_r), \theta_{N'_Q(\beta)}(u_j,k_r)\right),$$

$$\zeta_{H_Q(\alpha,\beta)}(u_j,k_r) = max\left(\zeta_{N_Q(\alpha)}(u_j,k_r), \zeta_{N'_Q(\beta)}(u_j,k_r)\right),$$

$$\eta_{H_Q(\alpha,\beta)}(mu_j,k_r) = max\left(\eta_{N_Q(\alpha)}(u_j,k_r), \eta_{N'_Q(\beta)}(u_j,k_r)\right)$$

where $\forall\alpha \in A, \forall\beta \in B$.

Example 5 Suppose that $(N_Q,A)_\Lambda$ and $(N'_Q,B)_\Lambda$ are two *EQNSS* over $\mathcal{U}$, such that

$$(N_Q,A)_\Lambda = \left\{\left[(e_1),\left(\frac{(u_1,k_1)}{0.2,0.5,0.6},\frac{(u_1,k_2)}{0.4,0.1,0.3},\frac{(u_2,k_1)}{0.6,0.4,0.7},\frac{(u_2,k_2)}{0.3,0.1,0.8}\right)\right]\right.,$$

$$(N'_Q,B)_\Lambda = \left\{\left[(e_1),\left(\frac{(u_1,k_1)}{0.5,0.3,0.6},\frac{(u_1,k_2)}{0.1,0.9,0.8},\frac{(u_2,k_1)}{0.1,0.4,0.3},\frac{(u_2,k_2)}{0.1,0.2,0.6}\right)\right]\right.,$$

$$\left.\left[(e_2),\left(\frac{(u_1,k_1)}{0.4,0.1,0.9},\frac{(u_1,k_2)}{0.3,0.1,0.7},\frac{(u_2,k_1)}{0.8,0.4,0.3},\frac{(u_2,k_2)}{0.7,0.3,0.2}\right)\right]\right\}.$$

Then $(N_Q,A)_\Lambda \wedge (N'_Q,A)_\Lambda = (H_Q,A \times B)_\Lambda$ where

$$(H_Q,A \times B)_\Lambda = \left\{\left[(e_1,e_1),\left(\frac{(u_1,k_1)}{0.2,0.5,0.6},\frac{(u_1,k_2)}{0.1,0.9,0.8},\frac{(u_2,k_1)}{0.1,0.4,0.7},\frac{(u_2,k_2)}{0.1,0.2,0.8}\right)\right]\right.,$$

$$\left[(e_1,e_2),\left(\frac{(u_1,k_1)}{0.2,0.5,0.9},\frac{(u_1,k_2)}{0.3,0.1,0.7},\frac{(u_2,k_1)}{0.6,0.4,0.7},\frac{(u_2,k_2)}{0.3,0.3,0.8}\right)\right]$$

Definition 15 If $(N_Q,A)_\Lambda$ and $(N'_Q,A)_\Lambda$ are two *EQNSS* over $\mathcal{U}$, then " $(N_Q,A)_\Lambda$ OR $(N'_Q,A)_\Lambda$ " is

$$(N_Q,A)_\Lambda \vee (N'_Q,A)_\Lambda = (K_Q,A\times B)_\Lambda$$

such that $K_Q(\alpha,\beta) = N_Q(\alpha)\cup N'_Q(\beta)$ and the memberships of truth, indeterminacy, and falsity of $(K_Q,A\times B)_\Lambda$ are as follows:

$$\theta_{K_Q(\alpha,\beta)}(u_j,k_r) = max\left(\theta_{N_Q(\alpha)}(u_j,k_r),\theta_{N'_Q(\beta)}(u_j,k_r)\right),$$

$$\zeta_{K_Q(\alpha,\beta)}(u_j,k_r) = min\left(\zeta_{N_Q(\alpha)}(u_j,k_r),\zeta_{N'_Q(\beta)}(u_j,k_r)\right),$$

$$\eta_{K_Q(\alpha,\beta)}(u_j,k_r) = min\left(\eta_{N_Q(\alpha)}(u_j,k_r),\eta_{N'_Q(\beta)}(u_j,k_r)\right)$$

where $\forall\alpha\in A,\forall\beta\in B$.

Example 6 Suppose that $(N_Q,A)_\Lambda$ and $(N'_Q,A)_\Lambda$ are two *EQNSS* over $\mathcal{U}$, such that

$$(N_Q,A)_\Lambda = \left\{\left[(e_1),\left(\frac{(u_1,k_1)}{0.4,0.2,0.6},\frac{(u_1,k_2)}{0.3,0.2,0.7},\frac{(u_2,k_1)}{0.5,0.8,0.4},\frac{(u_2,k_2)}{0.3,0.1,0.1}\right)\right],\right.$$

$$(N'_Q,A)_\Lambda = \left\{\left[(e_1),\left(\frac{(u_1,k_1)}{0.7,0.4,0.8},\frac{(u_1,k_2)}{0.5,0.4,0.6},\frac{(u_2,k_1)}{0.6,0.7,0.1},\frac{(u_2,k_2)}{0.2,0.2,0.8}\right)\right],\right.$$

$$\left.\left[(e_2),\left(\frac{(u_1,k_1)}{0.6,0.4,0.3},\frac{(u_1,k_2)}{0.2,0.5,0.4},\frac{(u_2,k_1)}{0.3,0.7,0.8},\frac{(u_2,k_2)}{0.9,0.1,0.2}\right)\right]\right\}.$$

Then $(N_Q,A)_\Lambda \vee (N'_Q,A)_\Lambda = (K_Q,A\times B)_\Lambda$ where

$$(K_Q,A\times B)_\Lambda = \left\{\left[(e_1,e_1)\left(\frac{(u_1,k_1)}{0.7,0.2,0.6},\frac{(u_1,k_2)}{0.5,0.2,0.6},\frac{(u_2,k_1)}{0.6,0.7,0.1},\frac{(u_2,k_2)}{0.3,0.1,0.1}\right)\right],\right.$$

$$\left[(e_1,e_2),\left(\frac{(u_1,k_1)}{0.6,0.2,0.3},\frac{(u_1,k_2)}{0.3,0.2,0.4},\frac{(u_2,k_1)}{0.5,0.7,0.4},\frac{(u_2,k_2)}{0.9,0.1,0.1}\right)\right],$$

Proposition 5 If $(N_Q,A)_\Lambda$ nd $(N'_Q,A)_\Lambda$ are *EQNSS* over $\mathcal{U}$, then

i.$\left((N_Q,A)_\Lambda \wedge (N'_Q,A)_\Lambda\right)^c = (N_Q,A)_\Lambda{}^c \vee (N'_Q,A)_\Lambda{}^c$

ii.$\left((N_Q,A)_\Lambda \vee (N'_Q,A)_\Lambda\right)^c = (N_Q,A)_\Lambda{}^c \wedge (N'_Q,A)_\Lambda{}^c$

4. An Application of *EQNSS*

We will now present an application of *EQNSS* theory to illustrate that this concept can be successfully applied to decision-making problems with uncertain information. The following algorithm is suggested to solve an effective *Q*-neutrosophic soft expert-based decision-making problem below. For comparison purposes in this section, an example is used in [28].

People both in line with their own actions and needs and the conditions of the living space and the environment; It meets the main act of housing in different types of housing. The planning and design of the immediate surroundings of the house depends on what is expected of it, how it will be used and how it will serve the house in it. Answers to these questions should be sought before starting design. As everywhere in the world's climate is the main factor that determines the types of housing conditions in Turkey. In addition, natural natural conditions such as geological structure and vegetation determine the housing types. Economic and cultural development in our country reduces the impact of the natural environment on housing types. Therefore Ezgi Construction Company wants to build housing in four different regions in the site planning. There are three alternatives $\mathcal{U} = \{u_1, u_2, u_3\}$, with two types of qualifications $Q = \{k_1, k_2\}$ and there are two parameters $\epsilon = \{e_1, e_2\}$ with e_i $(i = 1,2)$ standing for "transportation" and "location" respectively and the set of effective parameters is represented by Λ ={ $l_1 =$ "*environment*", $l_2 =$ "*price*"}.

$$\Lambda^1(u_1,k_1) = \left\{\frac{l_1}{(0.6,0.3,0.7)}, \frac{l_2}{(0.3,0.4,0.7)}\right\}, \qquad \Lambda^2(u_1,k_2) = \left\{\frac{l_1}{(0.4,0.2,0,7)}, \frac{l_2}{(0.9,0.5,0.4)}\right\}$$

$$\Lambda^5(u_2,k_1) = \left\{\frac{l_1}{(0.4,0.3,0.5)}, \frac{l_2}{(0.8,0.1,0.6)}\right\}, \qquad \Lambda^7(u_2,k_2) = \left\{\frac{l_1}{(0.2,0.5,0.4)}, \frac{l_2}{(0.5,0.6,0.7)}\right\}$$

$$\Lambda^{10}(u_3,k_1) = \left\{\frac{l_1}{(0.7,0.2,0.6)}, \frac{l_2}{(0.8,0.4,0.3)}\right\}, \qquad \Lambda^{12}(u_3,k_2) = \left\{\frac{l_1}{(0.6,0.3,0.5)}, \frac{l_2}{(0.8,0.4,0.3)}\right\}$$

$$(N_Q,Z) = \left\{ \left[(e_1),\left(\frac{(u_1,k_1)}{0.3,0.4,0.5},\frac{(u_1,k_2)}{0.7,0.3,0.7},\frac{(u_2,k_1)}{0.5,0.4,0.6},\frac{(u_2,k_2)}{0.5,0.3,0.2},\frac{(u_3,k_1)}{0.5,0.8,0.2},\frac{(u_3,k_2)}{0.6,0.3,0.7}\right)\right],\right.$$

$$\left[(e_2),\left(\frac{(u_1,k_1)}{0.6,0.7,0.2},\frac{(u_1,k_2)}{0.9,0.3,0.5},\frac{(u_2,k_1)}{0.7,0.4,0.3},\frac{(u_2,k_2)}{0.3,0.3,0.6},\frac{(u_3,k_1)}{0.7,0.1,0.5},\frac{(u_3,k_2)}{0.3,0.5,0.8}\right)\right],$$

Tables 1 presents the *EQNSS*.

The following algorithm may be used to choose the most qualified candidate to fill the vacancy.

1. Input the *QNSS* (N_Q,Z).
2. Compute the EQNSS $(N_Q,Z)_\Lambda$.
3. Compute the EQNSS $s_j(u,k) = \frac{|\theta - \zeta - \eta|}{\theta + \eta}$.
4. Determine $\max s_j$.

Table 1: *EQNSS*

$\mathcal{U} \times Q$	(u_1,k_1)	(u_1,k_2)	(u_2,k_1)	(u_2,k_2)	(u_3,k_1)	(u_3,k_2)
$N(e_1)_\Lambda$	(0.6, 0.3, 0.2)	(0.9, 0.2, 0.3)	(0.8, 0.3, 0.3)	(0.7, 0.1, 0.1)	(0.9, 0.6, 0.1)	(0.9, 0.2, 0.4)
$N(e_2)_\Lambda$	(0.8, 0.5, 0.1)	(1.0, 0.2, 0.2)	(0.9, 0.3, 0.1)	(0.5, 0.1, 0.3)	(0.9, 0.1, 0.3)	(0.9, 0.3, 0.5)
$s_j(u,k) = \frac{\lvert\theta - \zeta - \eta\rvert}{\theta + \eta}$	0.3	0.5	0.4	0.6	0.5	0.2

As can be seen, the maximum score is $\max s_j = 0.6$. So, the best alternative is u_2 and best qualification is k_2.

5. Conclusion

We have introduced the concept of a effective Q -neutrosophic soft set along with its operations of equality, union, intersection, OR, and AND. The application of this novel concept to a decision-making process is illustrated and compared to those in existing literature. It is shown that this proposed concept is more inclusive by considering the membership of falsity and indeterminacy, expert, neutrosophy and Q -fuzzy. Thus, the proposed approach is shown to be useful in handling realistic uncertain problems.

6. Future Research Directions

This study can be extended by using other type of neutrosophic decision making approaches, including interval valued neutrosophic soft sets, bipolar neutrosophic soft sets.

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Chapter 6

On Symbolic n- Pilthogenic $R - module$

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ABSTRACT

Modules are one of fundamental and rich algebraic structure with respect to some binary operations in the study of algebra. In this paper definition of Symbolic n- Pilthogenic R-modules and Symbolic n- Pilthogenic submodules in algebra are introduced. Some properties of Symbolic n- Pilthogenic R-modules and Symbolic n- Pilthogenic submodules are presented. More precisely, classical modules, ring and Symbolic n- Pilthogenic rings are utilized. Consequently, Symbolic n- Pilthogenic R- modules which are completely different from the classical modular in the structural properties are introduced. Also, Symbolic n- Pilthogenic R-module homomorphism is explained and some definitions and theorems are presented.

Keywords: Neutrosophic sets, Symbolic n- Pilthogenic ring, Symbolic n- Pilthogenic R-module, weak Symbolic n- Pilthogenic R-module, strong Symbolic n- Pilthogenic R-module, Symbolic n- Pilthogenic R-module homomorphism.

1.INTRODUCTION

Neutrosophy is a new branch of philosophy which studies the nature, origin and scope of neutralities as well as their interaction with ideational spectra. Neutrosophy is the base of neutrosophic logic, which is an extension of fuzzy logic in which indeterminacy is included.

Florentin Smarandanche introduced the notion of neutrosophy as a new branch of philosophy in 1980. After that, he introduced the concept of neutrosophic logic and neutrosophic set where each proposition in neutrosophic logic is approximated to have the percentage of truth in a subset T the percentage of indeterminacy in a subset I and the percentage of falsity in a subset F so that this neutrosophic logic is called an extension of fuzzy logic as well as an extension of intuitionistic fuzzy logic.

In fact, neutrosophic set is the generalization of classical set, neutrosophic group and neutrosophic ring the generalization of classical group and ring etc. The same way neutrosophic R-module is the generalization of classical R-module. By utilizing the idea of neutrosophic theory, Vasantha Kanadasamy and Florentin Samarandanche [1-2] studied neutrosophic algebraic structures in by inserting an indeterminate element I in the algebraic structure and then combining I with corresponding binary operation.

One of the most attractive concepts for mathematicians is algebraic structures due to their analog properties and close relationship with other branches of mathematics, such as geometry and matrix theory [3,4]. During the last two years, researchers have become interested in studying symbolic n-plithogenic algebraic structures.

These structures were supposed by Smarandache in [5] as novel generalizations of classical algebraic structures that have symmetric logical elements combined with algebraic elements.

Başer and Uluçay [30] defined effective q- fuzzy soft expert sets. Then, Başer and Ulucay [59] defined energy of a neutrosophic soft set and its applications to multi-criteria decision-making problems. These algebraic structures [71-80] have been constructed in a manner similar to their analogues using neutrosophic logic, where it is possible to clearly see that the method that was used to construct the neutrosophic structures [6,7], the split-complex numbers [8,9], and the weak fuzzy numbers [10] was used in the extension of algebraic rings by plithogenic sets.

For the case of $n = 2$, we find many studies that deal with corresponding plithogenic structures. In [11], Merkeci et al. defined the symbolic 2-plithogenic ring and studied its elementary properties and substructures, such as AH-ideals, AH-homomorphisms, and kernels. Laterally, their results were used by Taffach and other authors to define and study symbolic 2-plithogenic vector spaces [12], symbolic 2-plithogenic modules [13], and symbolic 2-plithogenic number theory [14]. A wide review of symbolic 2-plithogenic algebraic structures is provided in [15,16]. This is what prompted other researchers to generalize the previous results to the symbolic n-plithogenic case. In [17,18], symbolic 3-plithogenic, 4-plithogenic, 5-plithogenic rings were handled for the first time by Albasheer and A.Hatip ; then, symbolic 3-plithogenic vector spaces, modules, and number theoretical concepts were defined and studied (see [19,20,21,22]). Keskin and Başer [29] presented an investigation of the Baer–Kaplansky property. These advanced set theories have been successfully applied to various extension of fuzzy sets across multiple domains [24–70].

This is what prompted us to follow up the previous scientific efforts and to study 4-plithogenic rings for the first time by providing basic definitions and proofs that describe the algebraic behavior of the elements of these rings. It is noteworthy that these new rings will be very useful in more extensive classes of algebraic modules and vector spaces, and also cryptographic algorithms.

The present paper is devoted to the study of Symbolic n- Pilthogenic R-module. More properties of Symbolic n- Pilthogenic R-module will be presented.

2.BACKGROUND

In this section, we will give some definitions, examples and results that will be useful in other sections of the research.

Definition 1. [2] Let $\mathcal{U}$ be a universe. $\mathcal{A}$ neutrosophic sets $\mathcal{A}$ over $\mathcal{U}$ is defined by

$$\mathcal{A} = \{< u, (T_{\mathcal{A}}(u), I_{\mathcal{A}}(u), F_{\mathcal{A}}(u)) >: u \in \mathcal{U}\}$$

where, $T_{\mathcal{A}}(u)$, $I_{\mathcal{A}}(u)$ and $F_{\mathcal{A}}(u)$ are called truth-membership function, indeterminacy-membership function and falsity- membership function, respectively. They are respectively defined by

$$T_{\mathcal{A}}: \mathcal{U} \to]^{-}0, 1^{+}[, \quad I_{\mathcal{A}}: \mathcal{U} \to]^{-}0, 1^{+}[, \quad F_{\mathcal{A}}: \mathcal{U} \to]^{-}0, 1^{+}[$$

such that $0^{-} \le T_{\mathcal{A}}(u) + I_{\mathcal{A}}(u) + F_{\mathcal{A}}(u) \le 3^{+}$.

Definition 2. [23] Let $\mathcal{U}$ be a universe. A single valued neutrosophic set (SVN-set) over $\mathcal{U}$ is a neutrosophic set over $\mathcal{U}$, but the truth-membership function T, indeterminacy-membership function I and falsity- membership function F are respectively defined by

$$T_{\mathcal{A}}: \mathcal{U} \to]^{-}0, 1^{+}[, \quad I_{\mathcal{A}}: \mathcal{U} \to]^{-}0, 1^{+}[, \quad F_{\mathcal{A}}: \mathcal{U} \to]^{-}0, 1^{+}[$$

Such that $0 \le T_{\mathcal{A}}(u) + I_{\mathcal{A}}(u) + F_{\mathcal{A}}(u) \le 3$.

Definition 3. [23] The Plithogenic Numbers (PN) of the form

$PN = a_0 + a_1 P_1 + a_2 P_2 + \cdots + a_n P_n$

defined as above are called Plithogenic Numbers.

Definition 4. [23] Let's consider two plithogenic numbers:

$$PN_1 = a_0 + a_1 P_1 + a_2 P_2 + \cdots + a_n P_n$$

$$PN_2 = b_0 + b_1 P_1 + b_2 P_2 + \cdots + b_n P_n$$

1. Addition of Plithogenic Numbers: $PN_1 + PN_2 = (a_0 + b_0) + \sum_{i=1}^{n} (a_i + b_i) P_i$

2. Subtraction of Plithogenic Numbers:
   $PN_1 - PN_2 = (a_0 - b_0) + \sum_{i=1}^{n}(a_i - b_i)P_i$
3. Scalar Multiplication of Plithogenic Numbers:
   $rPN_1 = ra_0 + ra_1P_1 + ra_2P_2 + \cdots + ra_nP_n$
4. Multiplication of Plithogenic Numbers

$PN_1 \cdot PN_2 = (a_0 + a_1P_1 + a_2P_2 + \cdots + a_nP_n) \cdot (b_0 + b_1P_1 + b_2P_2 + \cdots + b_nP_n)$

and then one multiplies them, term by term $a_iP_i \cdot a_jP_j = a_i \cdot a_jP_{\max(i,j)}$ where $\cdot$ is the classical multiplication as in classical algebra, using the above multiplication of symbolic plithogenic components.

As particular case:

1. $0.P_i = 0$
2. $1 = 1 + 0.P_1 + 0.P_2 + \cdots + 0.P_n$

3.On Symbolic n- Pilthogenic $R - Module$ and Their Properties

In this section, we define the Symbolic n- Pilthogenic $R - Module$ and Symbolic n- Pilthogenic $R - SupModule$. Then, we point out that Symbolic n- Pilthogenic $R - Module$ has more properties than the classical $R - Module$.

Definition 5. Let $(M, +, \cdot)$ be any R-module over a commutative ring R. The triple $(n - SPM, +, \cdot)$ is called a weak Symbolic n- Pilthogenic R-module over a ring R generated by $M. P_1, P_2, \dots, P_n$.

If $(n - SPM, +, \cdot)$ is a Symbolic n- Pilthogenic R-module over a refined neutrosophic ring $(n - SPR, +, \cdot)$ then $(n - SPM, +, \cdot)$is called a strong Symbolic n- Pilthogenic R-module.

Theorem 6. Every strong Symbolic n- Pilthogenic neutrosophic R-module is a weak Symbolic n- Pilthogenic R-module.

Proof: Suppose that $(n - SPM, +, \cdot)$is a strong Symbolic n- Pilthogenic R-module over a Symbolic n- Pilthogenic ring $(n - SPR, +, \cdot)$. Since $R \subset (n - SPR, +, \cdot)$ for every ring R, it follows that $(n - SPM, +, \cdot)$is a weak Symbolic n- Pilthogenic R-module.

Theorem 7. Every weak Symbolic n- Pilthogenic R-module is an R-module.

Proof: If we have $PN_1, PN_1 \in (n - SPM, +, \cdot)$ and $r, s \in R$ then:

1. $r(PN_1 + PN_1) = r[(a_0 + b_0) + \sum_{i=1}^{n}(a_i + b_i)P_i] = r(a_0 + b_0) + r\sum_{i=1}^{n}(a_i + b_i)P_i$

$$= ra_0 + r\sum_{i=1}^{n}(a_i)P_i + rb_0 + r\sum_{i=1}^{n}(b_i)P_i = rPN_1 + rPN_1$$

2. $(r+s)PN_1 = (r+s)a_0 + (r+s)a_1P_1 + (r+s)a_2P_2 + \cdots + (r+s)a_nP_n$

   $ra_0 + ra_1P_1 + ra_2P_2 + \cdots + ra_nP_n + sa_0 + sa_1P_1 + sa_2P_2 + \cdots + sa_nP_n$

   $= rPN_1 + sPN_1$

3. $(r \cdot s)PN_1 = (r \cdot s)a_0 + (r \cdot s)a_1P_1 + (r \cdot s)a_2P_2 + \cdots + (r \cdot s)a_nP_n = r(s \cdot PN_1)$

4. $(1 + 0.P_1 + 0.P_2 + \cdots + 0.P_n)PN_1 = PN_1$

Therefore, that $(n - SPM, +,\cdot)$ is an R-module.

Definition 8. Let $(n - SPM, +,\cdot)$ be a strong Symbolic n- Pilthogenic R- module over a Symbolic n- Pilthogenic ring $(n - SPR, +,\cdot)$ and let $(n - SPN, +,\cdot)$be a nonempty subset of $(n - SPM, +,\cdot)$. $(n - SPN, +,\cdot)$ is called a strong Symbolic n- Pilthogenic submodule of $(n - SPM, +,\cdot)$ if $(n - SPN, +,\cdot)$ is itself a strong Symbolic n- Pilthogenic R- module over $(n - SPR, +,\cdot)$. It is essential that $(n - SPN, +,\cdot)$ contains a proper subset which is an R-module.

Definition 9. Let $(n - SPM, +,\cdot)$ be a weak Symbolic n- Pilthogenic R- module over a Symbolic n- Pilthogenic ring $(n - SPR, +,\cdot)$ and let $(n - SPN, +,\cdot)$ be a nonempty subset of $(n - SPM, +,\cdot)$. $(n - SPN, +,\cdot)$ is called a weak Symbolic n- Pilthogenic submodule of $(n - SPM, +,\cdot)$ if $(n - SPN, +,\cdot)$ $N\left(I_1, I_2\right)$ is itself a weak Symbolic n- Pilthogenic R- module over $(n - SPR, +,\cdot)$. It is essential that $(n - SPN, +,\cdot)$ contains a proper subset which is an R-module.

Theorem 10. If we have $(n - SPM, +,\cdot)$ as a Symbolic n- Pilthogenic R-module over a ring R and if we take $(n - SPN, +,\cdot)$ as a subset of $(n - SPM, +,\cdot)$, $(n - SPN, +,\cdot)$ is a weak Symbolic n- Pilthogenic submodule of $(n - SPM, +,\cdot)$ if and only if the following conditions hold:

1. $PN_1, PN_2 \in (n - SPN, +,\cdot) \Rightarrow PN_1 + PN_2 \in (n - SPN, +,\cdot)$
2. For all $r \in R, PN_1 \in (n - SPN, +,\cdot) \Rightarrow rPN_1 \in (n - SPN, +,\cdot)$

3. $(n - SPN, +,\cdot)$ must has a proper subset which is a R- module.

Corollary 11. If we have $(n - SPM, +,\cdot)$ as a Symbolic n- Pilthogenic R-module over a ring R and if we take $(n - SPN, +,\cdot)$ as a subset of $(n - SPM, +,\cdot)$, $(n - SPN, +,\cdot)$ is a weak Symbolic n- Pilthogenic submodule of $(n - SPM, +,\cdot)$ if and only if the following conditions hold:

1. For all $r, s \in R$ *and* $PN_1, PN_2 \in (n - SPN, +,\cdot) \Rightarrow rPN_1 + sPN_2 \in (n - SPN, +,\cdot)$
2. $(n - SPN, +,\cdot)$ must has a proper subset which is a R- module.

Example 12. Let $(n - SPM, +,\cdot)$ be a weak Symbolic n- Pilthogenic R-module. $(n - SPM, +,\cdot)$is a weak Symbolic n- Pilthogenic submodule called a trivial weak Symbolic n- Pilthogenic submodule.

Theorem 13. Let $(n - SPM, +,\cdot)$ be a Symbolic n- Pilthogenic R-module over a ring R and let $\{n - SP_t\}_{t\in\lambda}$ be a family of Symbolic n- Pilthogenic submodules of $(n - SPM, +,\cdot)$. Then $\cap\{n - SP_t\}_{t\in\lambda}$ is a Symbolic n- Pilthogenic submodule of $(n - SPM, +,\cdot)$.

Remark 14. Let $(n - SPM, +,\cdot)$ be a Symbolic n- Pilthogenic R-module over a ring R and let $(n - SPN_1, +,\cdot)$ and $(n - SPN_2, +,\cdot)$ be two distinct Symbolic n- Pilthogenic submodule of $(n - SPM, +,\cdot)$. Generally, is not a $(n - SPN_1, +,\cdot) \cap (n - SPN_2, +,\cdot)$ Symbolic n- Pilthogenic submodule of $(n - SPM, +,\cdot)$.

However, if $(n - SPN_1, +,\cdot) \subset (n - SPN_2, +,\cdot)$ or $(n - SPN_1, +,\cdot) \supset (n - SPN_2, +,\cdot)$ then

$(n - SPN_1, +,\cdot) \cup (n - SPN_2, +,\cdot)$ is a Symbolic n- Pilthogenic submodule of $(n - SPM, +,\cdot)$.

4.Symbolic n- Pilthogenic R- Modules homomorphism

Definition 15. If we have $(n-SPM,+,\cdot)$ and $(n-SPN,+,\cdot)$ as two Symbolic n-Pilthogenic $R-$ modules over a ring R, a mapping $\varphi:(n-SPM,+,\cdot)\longrightarrow(n-SPN,+,\cdot)$ is said to be a Symbolic n- Pilthogenic homomorphism $R-$module, precisely when:

1. $\varphi(rPN_1+sPN_2)=r\varphi(PN_1)+s\varphi(PN_2)$ for all $r,s\in R\ \ and\ \ PN_1,PN_2\in(n-SPM,+,\cdot)$
2. $\varphi(P_i)=P_i\quad :i=1,2,\dots,n$

Endomorphism, epimorphism, monomorphism, automorphism, and isomorphism of φ have the same definitions as those of the classical cases.

Definition 16. Let $(n-SPM,+,\cdot)$and $(n-SPN,+,\cdot)$ be Symbolic n- Pilthogenic R-modules over a ring R and let $\psi:(n-SPM,+,\cdot)\longrightarrow(n-SPN,+,\cdot)$ be a Symbolic n-Pilthogenic R-module homomorphism then:

(1) The kernel of ψ denoted by $\ker\psi$ is defined by the set

$$\ker\psi=\{PN\in(n-SPM,+,\cdot):\psi(PN)=0+0.P_1+0.P_2+\cdots+0.P_n\}$$

(2) The image of ψ denoted by $\mathrm{Im}\psi$ is defined by the set

$$Im\psi=\{PN_1\in(n-SPN,+,\cdot),\exists PN_2\in(n-SPM,+,\cdot):\psi(PN_2)=PN_1\}$$

Example 17. Let $(n-SPM,+,\cdot)$be a Symbolic n- Pilthogenic R-module over a ring R. The mapping $\psi:(n-SPM,+,\cdot)\longrightarrow(n-SPM,+,\cdot)$ defined by $\psi(PN)=PN$ for all $PN\in(n-SPM,+,\cdot)$ is Symbolic n- Pilthogenic R-module homomorphism and

1. $\ker\psi=0+0.P_1+0.P_2+\cdots+0.P_n$
2. $Im\psi=(n-SPM,+,\cdot)$

Example 18. The mapping $\psi:(n-SPM,+,\cdot)\longrightarrow(n-SPM,+,\cdot)$ defined by

$$\psi(PN) = 0 + 0.P_1 + 0.P_2 + \cdots + 0.P_n$$

for all $PN \in (n - SPM, +,\cdot)$ is Symbolic n- Pilthogenic R-module homomorphism.

Definition 19. Let $(n - SPM, +,\cdot)$and $(n - SPN, +,\cdot)$ be Symbolic n- Pilthogenic R-modules over a ring R and let $\psi: (n - SPM, +,\cdot) \longrightarrow (n - SPN, +,\cdot)$ be a Symbolic n-Pilthogenic R-module homomorphism then: :

1. $\ker\psi$ is not a Symbolic n- Pilthogenic submodule of $(n - SPM, +,\cdot)$but a submodule of M.
2. $\operatorname{Im}\psi$ is a Symbolic n- Pilthogenic submodule of $(n - SPN, +,\cdot)$.

Proof:

1. Obviously $P_1 = 0 + 1.P_1 + 0.P_2 + \cdots + 0.P_n \in (n - SPM, +,\cdot)$ but

$\psi(P_1) = P_1 \neq 0 + 0.P_1 + 0.P_2 + \cdots + 0.P_n$. That $\ker\psi$ is a submodule of M is clear.

2. Clear.

5. Conclusion

In this paper, we defined the Symbolic n- Pilthogenic R-modules and Symbolic n-Pilthogenic submodules which are completely different from the classical modules and submodules in the structural properties. It was shown that every Symbolic n-Pilthogenic R-module is an R-module. Finally, Symbolic n- Pilthogenic R-module homomorphism were explained and some definitions and theorems were given and many illustrative examples were presented.

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Chapter 7

A Neutrosophic Approach to Regression Problems: Handling Uncertainty, Indeterminacy, and Inconsistency

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ABSTRACT

In regression problems, neutrosophic sets provide a structured way to address the uncertainty, indeterminacy, and inconsistency often present in complex or incomplete data. Unlike traditional binary frameworks limited to true/false values, neutrosophic logic expands the scope of data interpretation by incorporating three distinct degrees: truth, indeterminacy, and falsity. This approach allows for a more nuanced representation of information, enabling the model to better handle ambiguous or conflicting data. By assigning varying degrees to these three factors, neutrosophic sets enhance regression analysis, making it more robust in scenarios where traditional regression might struggle with data imperfections or variability.

Keywords: Neutrosophic sets, Linear regression, Non-linear regression

1 INTRODUCTION

Supervised learning is a machine learning technique that involves mapping inputs to outputs using sample input-output pairs as a guide. Regression problems are considered supervised learning problems. They try to predict outcomes within a continuous output, that is, they try to map variables to some continuous function.

Given that the world is filled with uncertainties, neutrosophic concepts have been increasingly adopted in up-to-date investigations in the fields of machine learning. Smarandache introduced the foundational principles of neutrosophic sets in his research [1]. Başer and Uluçay [49] defined effective q- fuzzy soft expert sets. Then, Başer and Uluçay [45] defined the energy of a neutrosophic soft set and applied it to multi-criteria decision-making problems to show its applicability. As research increasingly seeks to address the complexities and uncertainties in various fields, the integration of neutrosophic logic into statistical analyses, particularly regression, offers a promising approach for enhancing predictive modeling [17-50]. Regression analysis focuses on examining how a dependent variable is influenced by one or more independent variables to predict its future values. Regression is commonly applied to continuous numerical data, assessing how independent variables impact variations in the dependent variable. Regression problems are frequently used across various fields to make predictions or to understand relationships between variables. There are two primary types such as Linear Regression and Non-linear Regression. Linear Regression used when the dependent variable has a linear relationship with independent variables. Non-linear regression is used when the connection between dependent and independent variables does not follow a linear pattern [2]. Regression analysis serves as an effective method for generating forecasts, creating decision support systems, and testing hypotheses in scientific research. Neutrosophic statistics, grounded in neutrosophic logic, focus on quantifying uncertainty [51-67]. They extend intuitionistic fuzzy sets, making them suitable for handling uncertain environments. In this framework, neutrosophic statistics facilitate variance and significance testing, even when observations fall outside traditional fuzzy boundaries, effectively broadening the scope of both classical and fuzzy statistical methods [3]. In comparison, neutrosophic regression analysis builds on the same principles of neutrosophic logic but focuses on modeling the connection between the target variable and predictor variables under conditions of uncertainty and indeterminacy. Unlike traditional regression, neutrosophic regression can handle inconsistent or incomplete data by assigning truth, indeterminacy, and falsity values to each observation. This allows for a more flexible and robust model that accommodates ambiguous data, enhancing the predictive accuracy and applicability of regression in uncertain contexts.

Within machine learning, especially when dealing with regression models, the precision of data characteristics is crucial. The effectiveness of a regression model depends on its capacity to detect relationships and trends in the input data [4]. Therefore, ensuring the accuracy of these characteristics is critical for the overall prediction process. Additionally, flawed or unreliable data features can severely hinder the model's ability to make generalizations, leading to noise that compromises the integrity of the predictions. Such inaccuracies can result in unreliable forecasts and diminished model performance. Even minor discrepancies in the input data can propagate through the modeling process, ultimately distorting the outcomes and undermining the decision-making process. Thus, meticulous attention to the quality of data features is essential for developing robust regression models that can produce reliable and valid predictions in various applications. To effectively evaluate the performance of these models, it is crucial to rely on performance measures, or error metrics, which play a vital role in evaluation frameworks. These metrics are defined as logical and mathematical constructs that assess the proximity between actual results and predicted outcomes. Among the most widely used error metrics in regression analysis are mean square error (MSE), root mean square error (RMSE), mean absolute percentage error (MAPE), and mean absolute error (MAE) [5]. It has been applied in areas such as neutrosophic simple linear regression [6], neutrosophic multiple regression [3], neutrosophic non-linear regression [7], neutrosophic fuzzy regression [8], and interval prediction regression [9] promising results have been obtained.

In the rest of the article, the application of single-valued neutrosophic (SVN) sets and interval-valued neutrosophic sets to regression analysis is examined, and each approaches is examined by dividing them into subheadings.

2 SINGLE-VALUED NEUTROSOPHY FOR REGRESSION MODEL

A regression model is a statistical method employed to analyze the connection between one or more predictor variables and a dependent variable. The main goal of regression analysis is to estimate the dependent variable's value using the independent variables and to evaluate the strength and type of their interrelationships. Regression models are widely used in various fields, including economics, biology, engineering, social sciences, and business, for tasks such as trend analysis, forecasting, and hypothesis testing. Neutrosophic regression analysis is an effective approach for modeling connections between variables and for assessing the uncertainty present in the observed data [3]. While determining the correlation coefficients of correlation and neutrosophic data in probability spaces, the correlation coefficient plays an important role in measuring the strength of the linear relationship between two variables.

A SVN set S each element x in the universe U is denoted by SVN as follows [10].

$$S(x) = < T(x), I(x), F(x) >$$

$$T(x), I(x), F(x) \in [0,1]$$

$$^{-1}0 \leq T(x) + I(x) + F(x) \leq {}^{+}3$$

if we have one input independet variable Y→ output, X→input, a→ intercept, b→ slope, and e→ residual then regression equation like at the below [11].

$$Y = a + bX + \in$$

if we have more than one input independent variable equation like at the below.

$$Y = a + b_1X_1 + b_2X_2 + \cdots + b_nX_n + \in$$

For a SVN linear regression model, each observation of the dependent variable Y, and the independent variables X_i is represented by three components:

$$Y = < Y_T, Y_I, Y_F >$$

$$X_i = < X_{i,T}, X_{i,I}, X_{i,F} >$$

$$Y = < a_{0,T}, a_{0,I}, a_{0,F} > + \sum_{i=1}^{n} < b_{iT}, b_{iI}, b_{iF} > . < X_{iT}, X_{iI}, X_{iF} > + < \in_T, \in_I, \in_F >$$

To estimate the parameters b_{iT}, b_{iI}, b_{iF} , we can use modified optimization techniques (neutrosophic least squares) to minimize the neutrosophic error terms across truth, indeterminacy, and falsity dimensions.

Error metrics such as MSE evaluate for neutrosophic real value can be defined like $Y_{i,T}, Y_{i,I}, Y_{i,F}$ and predicted value can be defined like $Y'_{i,T}, Y'_{i,I}, Y'_{i,F}$ for input value $X_{i,T}, X_{i,I}, X_{i,F}$

$$MSE = \frac{1}{n}\sum_{i=1}^{n}[(Y_{i,T} - Y'_{i,T})^2, ((Y_{i,I} - Y'_{i,I})^2), ((Y_{i,F} - Y'_{i,F})^2)]$$

3 The Interval valued Neutrosophy for RegressIon Model

In the realm of neutrosophic statistics, the interval prediction model enhances the effectiveness of regression analysis by incorporating the inherent uncertainties associated with data and model estimations. Unlike traditional regression methods that yield point estimates, the interval prediction model acknowledges that real-world phenomena often exhibit variability and imprecision. By generating prediction intervals that encompass a range of possible outcomes, this model provides a probabilistic framework for understanding the uncertainty surrounding predictions [12]. Specifically, the prediction interval is constructed by assessing both the uncertainty in the regression estimates and the variability present in the observed data. This dual consideration allows practitioners to express their predictions not just as a single value but as a range within which the true value of the dependent variable is expected to lie with a specified level of confidence [9]. Consequently, the use of neutrosophic interval predictions empowers decision-makers with a more nuanced understanding of potential outcomes, enabling them to account for uncertainties and make more informed choices in various applications of machine learning.

Interval-Valued Neutrosophic Number (IVNN): IVNN is a structure in which the truth, uncertainty, and falsity components of an element are each expressed as intervals [13].

For these three components, each is defined in a specific interval. For an input variable x_i in the dataset, IVNN representation is given by:

$$x_i = < (T_i^L, T_i^U), (I_i^L, I_i^U), (F_i^L, F_i^U) >$$

$T_i^L, T_i^U, I_i^L, I_i^U, F_i^L, F_i^U$ lower and upper bounds for the truth, indeterminacy, and falsity intervals, respectively.

Assume we are building a linear regression model for each components $T, I, and\ F$. The regression model aims to predict intervals for each component of the output based on the input intervals. The regression model can be formulated as [14]:

$$\hat{Y}_i = < (\hat{T}_i^L, \hat{T}_i^U), (\hat{I}_i^L, \hat{I}_i^U), (\hat{F}_i^L, \hat{F}_i^U) >$$

$$Y = a + bX + \in$$

$$a_T^L \ , a_T^U \ , b_T^U, \text{ and } b_T^L$$

$$\hat{T}_i^L = a_T^L + \sum_{J=1}^{m} b_{T,j}^L X_{i,j} \ , \hat{T}_i^U = a_T^U + \sum_{J=1}^{m} b_{T,j}^U X_{i,j}$$

$$\hat{I}_i^L = a_I^L + \sum_{J=1}^{m} b_{I,j}^L X_{i,j} \ , \hat{I}_i^U = a_I^U + \sum_{J=1}^{m} b_{I,j}^U X_{i,j}$$

$$\hat{F}_i^L = a_F^L + \sum_{J=1}^{m} b_{F,j}^L X_{i,j} \ , \hat{F}_i^U = a_F^U + \sum_{J=1}^{m} b_{F,j}^U X_{i,j}$$

To evaluate the performance of this interval-valued neutrosophic regression model, we define an interval-valued Mean Squared Error (MSE) for each component:

$$MSE_T = \frac{1}{n} \sum_{i=1}^{n} ((\hat{T}_i^L - T_i^L)^2 + (\hat{T}_i^U - T_i^U)^2)$$

$$MSE_I = \frac{1}{n}\sum_{i=1}^{n}((\hat{I}_i^L - I_i^L)^2 + (\hat{I}_i^U - I_i^U)^2)$$

$$MSE_F = \frac{1}{n}\sum_{i=1}^{n}((\hat{F}_i^L - F_i^L)^2 + (\hat{F}_i^U - F_i^U)^2)$$

$$MSE = MSE_T + MSE_I + MSE_F$$

For prediction interval, it is essential to account for the uncertainty in the regression model's predictions and the inherent variability of the data; the interval is typically symmetrical, extending a specified amount above and below the predicted value, and is determined based on the standard error of the prediction and the residual standard deviation that reflects the spread of the model's errors. The use of this method is to obtain the result by obtaining a range instead of a single floating number obtained from the regression analysis. In many real-life examples, the prediction results often actually contain a distribution of probabilities within a certain range rather than a single number. Such as the probability of recovery from diseases.

4. Conclusions

Neutrosophic regression models, both single-valued and interval-valued, offer a robust framework for handling uncertainty, indeterminacy, and inconsistency in data. These models are particularly valuable in fields where data imperfections are prevalent, and more traditional methods struggle to produce accurate predictions. As machine learning continues to evolve, incorporating neutrosophic logic into regression analysis will play an essential role in enhancing the reliability and robustness of predictive models across a variety of industries, from healthcare to finance and beyond. In summary, neutrosophic regression provides a flexible and powerful framework for modeling relationships between variables under uncertainty. By incorporating the three key components of truth, indeterminacy, and falsity, this approach offers significant advantages over traditional regression methods, especially when dealing with incomplete, conflicting, or imprecise interval-valued data. The use of interval-valued neutrosophic sets further enhances prediction accuracy by providing a range of possible outcomes rather than a single estimate, which is critical for decision-making in uncertain environments. Future research could focus on integrating neutrosophic regression with modern machine learning techniques, further improving its applicability and performance in complex, real-world problems.

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Effective Q - Neutrosophic Soft Expert Sets and Its Application in Decision Making

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Abstract

We introduce the concept of an effective neutrosophic soft set, which aims to capture the influence on three independent membership functions representing degrees of truth (T), indeterminacy (I) and falsity (F). We go further by presenting a generalization of the effective Q- neutrosophic soft expert set, which includes the incorporation of a degree to signify the potential for an approximate value-set. This enhancement contributes to improved efficiency and realism in the concept. Notably, this innovative approach leverages the strengths of both the neutrosophic soft set and the effective neutrosophic soft expert set. The subsequent sections delve into fundamental operations on the generalized effective Q-neutrosophic soft expert set, providing clarity through illustrative examples and propositions.

Keywords: Neutrosophic sets; Effective Q - Neutrosophic Soft expert Sets, Decision-Making.

1.Introduction

In this regard, the concept of intuitionistic fuzzy sets, introduced by Atanassov [1], followed by Molodtsov's work on soft sets [2], and neutrosophic logic [3], as well as neutrosophic sets [4,5] by Smarandache, has gained significant attention. The term "neutrosophy" refers to the study of neutral thought, which distinguishes neutrosophic logic from traditional fuzzy and intuitionistic fuzzy logic by introducing the notion of neutrality. Currently, research on soft set theory is progressing rapidly, with extensive literature on Q-fuzzy sets. For example, Q-fuzzy soft sets [6–8] and multi-Q-fuzzy sets [9–11] have opened up many applications [12-17], including the development of multi Q-fuzzy soft expert sets [18]. Şahin et al. [19] introduced neutrosophic soft expert sets, and Hassan et al. [20] expanded this further with the Q-neutrosophic soft expert set. In 2022, Alkhazaleh [21] introduced the concept of the Effective Fuzzy Soft Set (EFSS), which was designed to extend the notion of external effectiveness within the framework of soft sets. Later, the concept of Effective Fuzzy Soft Expert Sets [22] was proposed, incorporating expert opinions into a unified model. Başer and Uluçay [23] further defined effective Q-fuzzy soft expert sets. In 2023, the Effective Neutrosophic Soft Set [24] was introduced and later extended to the generalized Effective Neutrosophic Soft Set (ENSS) [25], incorporating the concept of effectiveness across three independent membership functions: truth (T), indeterminacy (I), and falsity (F). Furthermore, the concept of the Effective Neutrosophic Soft Expert Set [26] was introduced, along with associated operations and practical examples. Building on this, we develop a new mathematical tool by combining the concept of the Effective Neutrosophic Soft Set with the supply set Q—a mathematical framework designed to capture the nuances of uncertain information through three distinct membership functions representing degrees of truth (T), uncertainty (I), and falsity (F). We introduce the Effective Q-Neutrosophic Soft Expert Set, a new concept that enhances the framework's ability to model complex real-world scenarios. This innovative approach combines the strengths of both the Effective Neutrosophic Soft Set and the Effective Neutrosophic Soft Expert Set, offering a versatile and comprehensive representation of uncertainty.

Başer and Uluçay [27] defined the energy of a neutrosophic soft set and applied it to multi-criteria decision-making problems to show its applicability. Then, these advanced set theories have been successfully applied to various decision-making problems across multiple domains [28–63]. The aim of this chapter, this research advances mathematical models for managing uncertainty, bridging the gap between theoretical foundations and practical applications. By introducing a comprehensive framework that integrates elements of neutrosophic sets and soft expert sets, the study offers a novel approach to addressing uncertainty and ambiguity in decision-making processes, providing valuable insights and practical solutions.

2. Preliminaries

Definition 1 [4] Let $\mathcal{E}$ be a universe of discourse, with a generic element in $\mathcal{E}$ denoted by e then a neutrosophic set (NS) N is an object having the form

$$N = \{< \mathrm{e} : \mathcal{T}_N(\mathrm{e}), \mathcal{I}_N(\mathrm{e}), \mathcal{F}_N(\mathrm{e}), \mathrm{e} \in \mathcal{E}\}$$

where the functions $\mathcal{T}, \mathcal{I}, \mathcal{F} : \mathcal{E} \to]^-0, 1^+[$ define respectively the degree of membership (or Truth) , the degree of indeterminacy, and the degree of non-membership (or Falsehood) of the element $\mathrm{e} \in \mathcal{E}$ to the set N with the condition.

$$^-0 \leq \mathcal{T}_N(\mathrm{e}) + \mathcal{I}_N(\mathrm{e}) + \mathcal{F}_N(\mathrm{e}) \leq 3^+$$

Definition 2 [14] Let $\mathcal{E}$ be an initial universe set and ϵ be a set of parameters. Consider $N \subseteq \epsilon$. Let $P(\mathcal{E})$ denotes the set of all neutrosophic sets of $\mathcal{E}$. The collection (φ, N) is termed to be the soft neutrosophic set over $\mathcal{E}$, where F is a mapping given by $\varphi : N \to P(\mathcal{E})$.

Definition 3 [5] A neutrosophic set N is contained in another neutrosophic set N_1 i.e. $N \subseteq N_1$ if $\forall \mathrm{e} \in \mathcal{E}$; $\mathcal{T}_N(\mathrm{e}) \leq \mathcal{T}_{N_1}(\mathrm{e})$, $\mathcal{I}_N(\mathrm{e}) \leq \mathcal{I}_{N_1}(\mathrm{e})$, $\mathcal{F}_N(\mathrm{e}) \geq \mathcal{F}_{N_1}(\mathrm{e})$.

Definition 4 [22] $\mathcal{E}$ is an initial universe, ϵ is a set of parameters X is a set of experts (agents), and O ={agree=1, disagree=0} a set of opinions. Let $\hat{Z} = \epsilon \times X \times O$ and $N \subseteq \hat{Z}$. A pair (φ, N) is called a soft expert set over $\mathcal{U}$, where F is mapping given by

$$\varphi : N \to P(\mathcal{E})$$

where $P(\mathcal{E})$ denotes the power set of $\mathcal{E}$.

Definition 5 [19] A pair (φ, N) is called a neutrosophic soft expert set (NSES) over $\mathcal{E}$, where φ is a mapping given by

$$\varphi : N \to P(\mathcal{E})$$

where $P(\mathcal{E})$ denotes the power neutrosophic set of $\mathcal{E}$.

Proposition 1[19]

For two NSES

6. $(\varphi, N)\tilde{\subseteq}(\psi, N_1)$ if and only if

$$\mathcal{T}_{\varphi(n)}(e)\tilde{\leq}\mathcal{T}_{\psi(n_1)}(e),\ \mathcal{I}_{\varphi(n)}(e)\tilde{\leq}\mathcal{I}_{\psi(n_1)}(e), \mathcal{F}_{\varphi(n)}(e)\tilde{\geq}\mathcal{F}_{\psi(n_1)}(e)$$

$\forall n \in N, n_1 \in N_1, e \in \mathcal{E}$.

7. $(\varphi, N) = (\psi, N_1)$ if and only if

$$\mathcal{T}_{\varphi(n)}(e) = \mathcal{T}_{\psi(n_1)}(e),\ \mathcal{I}_{\varphi(n)}(e) = \mathcal{I}_{\psi(n_1)}(e), \mathcal{F}_{\varphi(n)}(e) = \mathcal{F}_{\psi(n_1)}(e)$$

$\forall n \in N, n_1 \in N_1, e \in \mathcal{E}$.

8. $(\varphi, N)^c =$ $\{e,\ \mathcal{T}_{\varphi^c(n)} = \mathcal{F}_{\varphi(n)},\ \mathcal{I}_{\varphi^c(n)}(\mathrm{e}) = \mathcal{I}_{\varphi(n)}(\mathrm{e}),\ \mathcal{F}_{\varphi^c(n)} = \mathcal{T}_{\varphi(n)} : \forall n \in N, e \in \mathcal{E}\}$.
9. An agree-NSES,

$$(\varphi, N)^1 = \{\varphi^1(a) : a \in \epsilon \times X \times \{1\}\}.$$

10. A disagree-NSES,

$$(\varphi, N)^0 = \{\varphi^0(a) : a \in \epsilon \times X \times \{0\}\}.$$

11. $$(\underset{\cdot}{\mathrm{H}}, N)\tilde{\cup}(\bar{\mathrm{G}}, N_1) = \begin{cases} \mathcal{T}_{\underset{\cdot}{\mathrm{H}}(n)}(a), & if\ n \in N - N_1, \\ \mathcal{T}_{\bar{\mathrm{G}}(n)}(a), & if\ n \in N_1 - N, \\ max\left(\mathcal{T}_{\underset{\cdot}{\mathrm{H}}(n)}(a), \mathcal{T}_{\bar{\mathrm{G}}(n,p)}(a)\right), & if\ n \in N \cap N_1. \\ \mathcal{I}_{\underset{\cdot}{\mathrm{H}}(n)}(a), & if\ n \in N - N_1, \\ \mathcal{I}_{\bar{\mathrm{G}}(n)}(a), & if\ n \in N_1 - N, \\ \frac{\mathcal{I}_{\underset{\cdot}{\mathrm{H}}(n)}(a) + \mathcal{I}_{\bar{\mathrm{G}}(n)}(a)}{2}, & if\ n \in N \cap N_1. \\ \mathcal{F}_{\underset{\cdot}{\mathrm{H}}(n)}(a), & if\ n \in N - N_1, \\ \mathcal{F}_{\bar{\mathrm{G}}(n)}(a), & if\ n \in N_1 - N, \\ min\left(\mathcal{F}_{\underset{\cdot}{\mathrm{H}}(n)}(a), \mathcal{F}_{\bar{\mathrm{G}}(n)}(a)\right), & if\ n \in N \cap N_1. \end{cases}$$

12. $$(\underset{\cdot}{\mathrm{H}}, N)\tilde{\cap}(\bar{\mathrm{G}}, N_1) = \begin{cases} min\left(\mathcal{T}_{\underset{\cdot}{\mathrm{H}}(n)}(a), \mathcal{T}_{\bar{\mathrm{G}}(n)}(a)\right), \\ \frac{\mathcal{I}_{\underset{\cdot}{\mathrm{H}}(n)}(a) + \mathcal{I}_{\bar{\mathrm{G}}(n)}(a)}{2}, \\ max\left(\mathcal{F}_{\underset{\cdot}{\mathrm{H}}(n)}(a), \mathcal{F}_{\bar{\mathrm{G}}(n)}(a)\right), if\ n \in N \cap N_1. \end{cases}$$

Definition 6 [11] Let I be unit interval and k be a positive integer. A multi Q-fuzzy set $\tilde{\mathrm{N}}_Q$ in $\mathcal{E}$ and a non-empty set Q is a set of ordered sequences $\tilde{\mathrm{N}}_Q = \{(e,q),\mathcal{T}_i(e,q): e \in \mathcal{E}, q \in Q\}$ where

$$\mathcal{T}_i: \mathcal{E} \times Q \rightarrow I^k, \quad i = 1,2,\dots,k.$$

The function $(\mathcal{T}_1(e,q),\mathcal{T}_2(e,q),\dots,\mathcal{T}_k(e,q))$ is called the membership function of multi Q- fuzzy set $\tilde{\mathrm{N}}_Q$: and $\mathcal{T}_1(e,q)+\mathcal{T}_2(e,q)+\cdots+\mathcal{T}_k(e,q) \leq 1, k$ is called the dimension of $\tilde{\mathrm{N}}_Q$. The set of all multi Q- fuzzy sets of dimension k in $\mathcal{E}$ and Q is denoted by $M^k Q\varphi(\mathcal{E})$.

Definition 7 [20] (φ_Q, N) is a *QNSES* over $\mathcal{E}$, where φ_Q is the mapping $\varphi_Q: N \rightarrow$ *QNSES* such that *QNSES* is the set of all *QNSES* over $\mathcal{E}$.

Definition 8 [24] A neutrosophic set δ in a universe of discourse $\mathcal{E}_1$, where $\delta: \mathcal{E}_1 \rightarrow [0,1]$ a function, is an effective set. $\mathcal{E}_1$ is a set of effective parameters that can change membership and it's written in the following way;

$$\delta = \{< \alpha, (\mathcal{T}_\delta(\alpha), \mathcal{I}_\delta(\alpha), \mathcal{F}_\delta(\alpha)) > : \alpha \in \mathcal{E}_1\}$$

Definition 9 [26] Let $\mathcal{E}$ be an initial universe, ϵ be a set of all parameters, $\mathcal{E}_1$ be a set of effective parameters, δ be a effective set over $\mathcal{E}_1$ and $P(\mathcal{E})$ represent the power set of $\mathcal{E}$. In this case $(\tilde{\mathrm{N}},\mathcal{E}_1)_\delta$ is called on effective neutrosophic soft expert set over $\mathcal{E}$, where $\tilde{\mathrm{N}}$ is mapping represented by $\tilde{\mathrm{N}}: \mathcal{E}_1 \rightarrow P(\mathcal{E})$ and it may be expressed as a collection of ordered pairs;

$$\left(\tilde{\mathrm{N}},\mathcal{E}_1\right)_\delta = \left\{\left((e_j, < e_j, \mathcal{T}_{\tilde{\mathrm{N}}(x_j)}(e_j)_\delta, \mathcal{I}_{\tilde{\mathrm{N}}(x_j)}(e_j)_\delta, \mathcal{F}_{\tilde{\mathrm{N}}(x_j)}(e_j)_\delta >\right): e_j \in \mathcal{E}, x_j \in \epsilon\right\}$$

and $\mathcal{T}_{\tilde{\mathrm{N}}(x_j)}(e_j)_\delta$, $\mathcal{I}_{\tilde{\mathrm{N}}(x_j)}(e_j)_\delta$, $\mathcal{F}_{\tilde{\mathrm{N}}(x_j)}(e_j)_\delta$ membership values for $\forall\, \alpha \in \mathcal{E}$ is calculated as

$$\mathcal{T}_{\tilde{\mathrm{N}}(x_j)}(e_j)_\delta = \begin{cases} \mathcal{T}_{\tilde{\mathrm{N}}(x_j)}(e_j) + \dfrac{\left(1-\mathcal{T}_{\tilde{\mathrm{N}}(x_j)}(e_j)\right)\sum_k \mathcal{T}_{\delta_{x_j}}(a_k)}{|\delta|}, & if\ \mathcal{T}_{\tilde{\mathrm{N}}(x_j)}(e_j) \in (0,1) \\ \mathcal{T}_{\tilde{\mathrm{N}}(x_j)}(e_j), & O.W \end{cases}$$

$$\mathcal{I}_{\tilde{\mathrm{N}}(x_j)}(e_j)_\delta = \begin{cases} \mathcal{I}_{\tilde{\mathrm{N}}(x_j)}(e_j) - \dfrac{\mathcal{I}_{\tilde{\mathrm{N}}(x_j)}(e_j) \sum_k \mathcal{I}_{\delta_{x_j}}(a_k)}{|\delta|}, & if\ \mathcal{I}_{\tilde{\mathrm{N}}(x_j)}(e_j) \in (0,1) \\ \mathcal{I}_{\tilde{\mathrm{N}}(x_j)}(e_j), & O.W \end{cases}$$

$$\mathcal{F}_{\tilde{\mathrm{N}}(x_j)}(e_j)_\delta = \begin{cases} \mathcal{F}_{\tilde{\mathrm{N}}(x_j)}(e_j) - \dfrac{\mathcal{F}_{\tilde{\mathrm{N}}(x_j)}(e_j) \sum_k \mathcal{F}_{\delta_{x_j}}(a_k)}{|\delta|}, & if\ \mathcal{F}_{\tilde{\mathrm{N}}(x_j)}(e_j) \in (0,1) \\ \mathcal{F}_{\tilde{\mathrm{N}}(x_j)}(e_j), & O.W \end{cases}$$

3. Effective Q- Neutrosophic Soft Expert Sets

We will now propose the definition of effective *Q*-neutrosophic soft expert sets *EQNSES*, and propose some of its properties. Throughout the discussion, $\mathcal{E}$ is the initial universe, ϵ is the set of parameters, *Q be* a set of supply, δ is the set of effective parameters, X is the set of experts, and $O = \{agree = 1, disagree = 0\}$ a set of suggestions. Let $\breve{\mathrm{A}} \subseteq \hat{\mathrm{Z}}$ where $\hat{Z} = \epsilon \times X \times O$.

Definition 10 $\left(\tilde{\mathrm{N}}_Q, \breve{\mathrm{A}}\right)$ is a *QNSES* over $\mathcal{E}$, where $\tilde{\mathrm{N}}_Q$ is the mapping $\tilde{\mathrm{N}}_Q : \breve{\mathrm{A}} \to$ *QNSES* such that *QNSES* is the set of all *QNSES* over $\mathcal{E}$.

Definition 11 Let $\mathcal{E}$ be an initial universe, $\left(\tilde{\mathrm{N}}_Q, \breve{\mathrm{A}}\right)$ is a *QNSES* over $\mathcal{E}$. ϵ be a set of all parameters ϵ be a set of all parameters, $\mathcal{E}_1$ be a set of effective parameters, δ be a effective set over $\mathcal{E}_1$ and $P(\mathcal{E})$ represent the power set of $\mathcal{E}$, X is the set of experts. In this case $(\tilde{\mathrm{N}}, \mathcal{E}_1)_\delta$ is called on effective q-neutrosophic soft expert set over $\mathcal{E}$, where $\tilde{\mathrm{N}}$ is mapping represented by $\tilde{\mathrm{N}} : \mathcal{E}_1 \to P(\mathcal{E})$ and it

$$\left(\tilde{\mathrm{N}}, \mathcal{E}_1\right)_\delta = \Big\{\Big((x_j, p_j, 1), < (e_i, k_j), \mathcal{T}_{\tilde{\mathrm{N}}(x_j)}(e_i, k_j)_\delta, \mathcal{I}_{\tilde{\mathrm{N}}(x_j)}(e_i, k_j)_\delta, \mathcal{F}_{\tilde{\mathrm{N}}(x_j)}(e_i, k_j)_\delta > \Big) : x_j \in \mathcal{E}, e_j \in \epsilon, p_j \in X$$

$$\Big((x_j, p_j, 0), < (e_i, k_j), \mathcal{T}_{\tilde{\mathrm{N}}(x_j)}(e_i, k_j)_\delta, \mathcal{I}_{\tilde{\mathrm{N}}(x_j)}(e_i, k_j)_\delta, \mathcal{F}_{\tilde{\mathrm{N}}(x_j)}(e_i, k_j)_\delta > \Big) : x_j \in \mathcal{E}, e_j \in \epsilon, p_j \in X\Big\}$$

and $\mathcal{T}_{\tilde{\mathrm{N}}(x_j)}(e_i, k_j)_\delta$, $\mathcal{I}_{\tilde{\mathrm{N}}(x_j)}(e_i, k_j)_\delta$, $\mathcal{F}_{\tilde{\mathrm{N}}(x_j)}(e_i, k_j)_\delta$ m embership values for $\forall\, \alpha \in \mathcal{E}$ is calculated as ;

$$\mathcal{T}_{\tilde{\mathrm{N}}_Q(x_j)}(e_i,k_j)_\delta = \begin{cases} \mathcal{T}_{\tilde{\mathrm{N}}_Q(x_j)}(e_i,k_j) + \dfrac{\left(1-\mathcal{T}_{\tilde{\mathrm{N}}_Q(x_j)}(e_i,k_j)\right)\sum_k \mathcal{T}_{\delta_{x_j}}(a_k)}{|\delta|}, & if\ \mathcal{T}_{\tilde{\mathrm{N}}_Q(x_j)}(e_i,k_j) \in (0,1) \\ \mathcal{T}_{\tilde{\mathrm{N}}_Q(x_j)}(e_i,k_j), & O.W \end{cases}$$

$$\mathcal{I}_{\tilde{\mathrm{N}}_Q(x_j)}(e_i,k_j)_\delta = \begin{cases} \mathcal{I}_{\tilde{\mathrm{N}}_Q(x_j)}(e_i,k_j) - \dfrac{\mathcal{I}_{\tilde{\mathrm{N}}_Q(x_j)}(e_i,k_j)\sum_k \mathcal{I}_{\delta_{x_j}}(a_k)}{|\delta|}, & if\ \mathcal{I}_{\tilde{\mathrm{N}}_Q(x_j)}(e_i,k_j) \in (0,1) \\ \mathcal{I}_{\tilde{\mathrm{N}}_Q(x_j)}(e_i,k_j), & O.W \end{cases}$$

$$\mathcal{F}_{\tilde{\mathrm{N}}_Q(x_j)}(e_i,k_j)_\delta = \begin{cases} \mathcal{F}_{\tilde{\mathrm{N}}_Q(x_j)}(e_i,k_j) - \dfrac{\mathcal{F}_{\tilde{\mathrm{N}}_Q(x_j)}(e_i,k_j)\sum_k \mathcal{F}_{\delta_{x_j}}(a_k)}{|\delta|}, & if\ \mathcal{F}_{\tilde{\mathrm{N}}_Q(x_j)}(e_i,k_j) \in (0,1) \\ \mathcal{F}_{\tilde{\mathrm{N}}_Q(x_j)}(e_i,k_j), & O.W \end{cases}$$

Example 1 Suppose a customer who wants to build a new house wants to get feedback from several experts. Let $\mathcal{E} = \{e_1, e_2\}$ be the set of houses, $Q = \{k_1, k_2\}$ be the set of construction companies, $\epsilon = \{x_1, x_2\}$ be the set of decision parameters and the set of effective parameters is represented by $\delta = \{l_1, l_2\}$. Let $X = \{p_1, p_2\}$ be the set of experts. Assume that;

$$\delta^1(e_1,k_1,\mathrm{p}_1) = \left\{\frac{l_1}{(0.3,0.6,0.2)}, \frac{l_2}{(0.6,0.8,0.1)}\right\},\ \delta^2(e_1,k_2,\mathrm{p}_1) = \left\{\frac{l_1}{(0.1,0.6,0.4)}, \frac{l_2}{(0.5,0.8,0.4)}\right\},$$

$$\delta^3(e_1,k_1,\mathrm{p}_2) = \left\{\frac{l_1}{(0.7,0.1,0.5)}, \frac{l_2}{(0.1,0.4,0.7)}\right\},\ \delta^4(e_1,k_2,\mathrm{p}_2) = \left\{\frac{l_1}{(0.2,0.9,0.3)}, \frac{l_2}{(0.6,0.3,0.8)}\right\},$$

$$\delta^5(e_2,k_1,\mathrm{p}_1) = \left\{\frac{l_1}{(0.4,0.6,0.1)}, \frac{l_2}{(0.4,0.5,0.3)}\right\},\ \delta^6(e_2,k_1,\mathrm{p}_2) = \left\{\frac{l_1}{(0.5,0.9,1.0)}, \frac{l_2}{(0.4,0.5,0.2)}\right\}$$

$$\delta^7(e_2,k_2,\mathrm{p}_1) = \left\{\frac{l_1}{(0.3,0.6,0.1)}, \frac{l_2}{(0.2,0.7,0.8)}\right\},\ \delta^8(e_2,k_2,\mathrm{p}_2) = \left\{\frac{l_1}{(0.6,0.1,0.9)}, \frac{l_2}{(0.7,0.8,0.1)}\right\}$$

Let $\tilde{\mathrm{N}}$ be the neutrosophic expert set (QNSES) defined as follows:

$$\tilde{\mathrm{N}}_Q(x_1,p_1,1) = \left\{\left(\frac{(e_1,k_1)}{0.3,0.6,0.3}, \frac{(e_1,k_2)}{0.6,0.1,0.9}, \frac{(e_2,k_1)}{0.7,0.2,0.8}, \frac{(e_2,k_2)}{0.6,0.5,0.7}\right)\right\}$$

$$\tilde{\mathrm{N}}_Q(x_1,p_2,1) = \left\{\left(\frac{(e_1,k_1)}{0.8,0.5,0.3}, \frac{(e_1,k_2)}{0.1,0.4,0.6}, \frac{(e_2,k_1)}{0.6,0.3,0.7}, \frac{(e_2,k_2)}{0.3,0.3,0.3}\right)\right\}$$

$$\tilde{N}_Q(x_2,p_1,1) = \left\{\left(\frac{(e_1,k_1)}{0.7,0.5,0.8},\frac{(e_1,k_2)}{0.3,0.1,0.4},\frac{(e_2,k_1)}{0.6,0.6,0.3},\frac{(e_2,k_2)}{0.6,0.4,0.5}\right)\right\}$$

$$\tilde{N}_Q(x_2,p_2,1) = \left\{\left(\frac{(e_1,k_1)}{0.6,0.7,0.3},\frac{(e_1,k_2)}{0.7,0.5,0.6},\frac{(e_2,k_1)}{0.4,0.2,0.8},\frac{(e_2,k_2)}{0.6,0.4,0.4}\right)\right\}$$

$$\tilde{N}_Q(x_1,p_1,0) = \left\{\left(\frac{(e_1,k_1)}{0.7,0.1,0.4},\frac{(e_1,k_2)}{0.5,0.3,0.6},\frac{(e_2,k_1)}{0.4,0.3,0.3},\frac{(e_2,k_2)}{0.1,0.6,0.8}\right)\right\}$$

$$\tilde{N}_Q(x_1,p_2,0) = \left\{\left(\frac{(e_1,k_1)}{0.9,0.1,0.5},\frac{(e_1,k_2)}{0.4,0.5,0.7},\frac{(e_2,k_1)}{0.7,0.5,0.5},\frac{(e_2,k_2)}{0.6,0.7,0.8}\right)\right\}$$

$$\tilde{N}_Q(x_2,p_1,0) = \left\{\left(\frac{(e_1,k_1)}{0.1,0.2,0.6},\frac{(e_1,k_2)}{0.3,0.6,0.7},\frac{(e_2,k_1)}{0.3,0.2,0.6},\frac{(e_2,k_2)}{0.2,0.2,0.8}\right)\right\}$$

$$\tilde{N}_Q(x_2,p_2,0) = \left\{\left(\frac{(e_1,k_1)}{0.7,0.3,0.1},\frac{(e_1,k_2)}{0.7,0.1,0.4},\frac{(e_2,k_1)}{0.8,0.4,0.9},\frac{(e_2,k_2)}{1.0,0.1,0.2}\right)\right\}$$

Then by applying **Definition 11** we get,

$$\tilde{N}_Q(x_1,p_1,1)_\delta = \left\{\begin{array}{c}\dfrac{(e_1,k_1)}{< 0.3+\left[(1-0.3)\frac{0.3+0.6}{2}\right], 0.6-\left[0.6\frac{0.6+0.8}{2}\right], 0.3-\left[0.3\frac{0.2+0.1}{2}\right] >}, \\ \dfrac{(e_1,k_2)}{< 0.6+\left[(1-0.6)\frac{0.1+0.5}{2}\right], 0.1-\left[0.1\frac{0.6+0.8}{2}\right], 0.9-\left[0.9\frac{0.4+0.4}{2}\right] >}, \\ \dfrac{(e_2,k_1)}{< 0.7+\left[(1-0.7)\frac{0.4+0.4}{2}\right], 0.2-\left[0.2\frac{0.6+0.5}{2}\right], 0.8-\left[0.8\frac{0.1+03}{2}\right] >}, \\ \dfrac{(e_2,k_2)}{< 0.6+\left[(1-0.6)\frac{0.3+0.2}{2}\right], 0.5-\left[0.5\frac{0.6+0.7}{2}\right], 0.7-\left[0.7\frac{0.1+0.8}{2}\right] >}\end{array}\right\}$$

$$= \frac{(e_1,k_1)}{< 0.6, 0.2, 0.3 >} \frac{(e_1,k_2)}{< 0.8,0.0,0.8 >} \frac{(e_2,k_1)}{< 0.8,0.1,0.7 >} \frac{(e_2,k_2)}{< 0.8,0.4,0.6 >}$$

Similarly, when the calculations are continued, the effective Q-neutrosophic soft exper set is found as follows;

$$\tilde{N}_Q(x_1,p_2,1)_\delta = \frac{(e_1,k_1)}{<0.9,0.2,0.2>},\frac{(e_1,k_2)}{<0.4,0.1,0.4>},\frac{(e_2,k_1)}{<0.7,0.1,0.4>},\frac{(e_2,k_2)}{<0.5,0.2,0.2>}$$

$$\tilde{N}_Q(x_2,p_1,1)_\delta = \frac{(e_1,k_1)}{<0.8,0.2,0.6>},\frac{(e_1,k_2)}{<0.6,0.0,0.3>},\frac{(e_2,k_1)}{<0.8,0.3,0.2>},\frac{(e_2,k_2)}{<0.8,0.3,0.4>}$$

$$\tilde{N}_Q(x_2,p_2,1)_\delta = \frac{(e_1,k_1)}{<0.7,0.2,0.2>},\frac{(e_1,k_2)}{<0.8,0.2,0.3>},\frac{(e_2,k_1)}{<0.6,0.1,0.4>},\frac{(e_2,k_2)}{<0.7,0.2,0.2>}$$

$$\tilde{N}_Q(x_1,p_1,0)_\delta = \frac{(e_1,k_1)}{<0.8,0.1,0.2>},\frac{(e_1,k_2)}{<0.7,0.2,0.2>},\frac{(e_2,k_1)}{<0.6,0.2,0.1>},\frac{(e_2,k_2)}{<0.5,0.2,0.3>}$$

$$\tilde{N}_Q(x_1,p_2,0)_\delta = \frac{(e_1,k_1)}{<0.9,0.0,0.2>},\frac{(e_1,k_2)}{<0.6,0.2,0.3>},\frac{(e_2,k_1)}{<0.8,0.2,0.2>},\frac{(e_2,k_2)}{<0.8,0.3,0.4>}$$

$$\tilde{N}_Q(x_2,p_1,0)_\delta = \frac{(e_1,k_1)}{<0.5,0.1,0.2>},\frac{(e_1,k_2)}{<0.6,0.2,0.3>},\frac{(e_2,k_1)}{<0.6,0.1,0.2>},\frac{(e_2,k_2)}{<0.6,0.1,0.3>}$$

$$\tilde{N}_Q(x_2,p_2,0)_\delta = \frac{(e_1,k_1)}{<0.9,0.2,0.1>},\frac{(e_1,k_2)}{<0.9,0.1,0.2>},\frac{(e_2,k_1)}{<0.9,0.2,0.5>},\frac{(e_2,k_2)}{<1.0,0.1,0.1>}$$

Definition 12 For two *EQNSES* $(N_Q,A)_\delta$ and $(N_{1_Q},B)_\delta$ over $\mathcal{E}$, $(N_Q,A)_\delta$ is called a effective Q-neutrosophic soft expert subset of $(N_{1_Q},B)_\delta$ if

i. $B \subseteq A$,

ii. $N_{1_Q}(b)_\delta$ is neutrosophic soft expert subset $N_Q(\varepsilon)_\delta$ for all $b \in B$.

Definition 13 Two *EQNSES* $(N_Q,A)_\delta$ and $(N_{1_Q},B)_\delta$ over $\mathcal{E}$ are equal if $(N_Q,A)_\delta$ is a *EQNSES* subset of $(N_{1_Q},B)_\delta$ and $(N_{1_Q},B)_\delta$ is a *EQNSES* subset of $(N_Q,A)_\delta$.

Definition 14 Agree-*EQNSES* $(\tilde{N}_Q,A)_\delta{}^1$ over $\mathcal{E}$ is a *EQNSES* subset of $(\tilde{N}_Q,A)_\delta$ defined as

$$(\tilde{N}_Q,A)_\delta{}^1 = \{\tilde{N}_Q{}^1(\propto)_\delta : \propto \in \epsilon \times X \times \{1\}\}.$$

Example 3 Using our previous Example 1, the agree-*EQNSES* $(\tilde{N}_Q,A)_\delta{}^1$ over $\mathcal{E}$ is

$(\tilde{N}_Q,A)_\delta{}^1 =$

$$\left\{(x_1,p_1,1)_{\delta},\frac{(e_1,k_1)}{<0.6,0.2,0.3>}\ \frac{(e_1,k_2)}{<0.8,0.0,0.8>}\ \frac{(e_2,k_1)}{<0.8,0.1,0.7>}\ \frac{(e_2,k_2)}{<0.8,0.4,0.6>},\right.$$

$$(x_1,p_2,1)_{\delta},\frac{(e_1,k_1)}{<0.9,0.2,0.2>},\frac{(e_1,k_2)}{<0.4,0.1,0.4>},\frac{(e_2,k_1)}{<0.7,0.1,0.4>},\frac{(e_2,k_2)}{<0.5,0.2,0.2>},$$

$$(x_2,p_1,1)_{\delta},\frac{(e_1,k_1)}{<0.8,0.2,0.6>},\frac{(e_1,k_2)}{<0.6,0.0,0.3>},\frac{(e_2,k_1)}{<0.8,0.3,0.2>},\frac{(e_2,k_2)}{<0.8,0.3,0.4>},$$

$$\left.(x_2,p_2,1)_{\delta},\frac{(e_1,k_1)}{<0.7,0.2,0.2>},\frac{(e_1,k_2)}{<0.8,0.2,0.3>},\frac{(e_2,k_1)}{<0.6,0.1,0.4>},\frac{(e_2,k_2)}{<0.7,0.2,0.2>}\right\}$$

Definition 15 A disagree-*EQNSES* $(N_Q,A)_{\delta}{}^{0}$ over $\mathcal{E}$ is a *EQNSES* subset of $(N_Q,A)_{\delta}$ defined as

$$(N_Q,A)_{\delta}{}^{0}=\{N_Q{}^{0}(\propto)_{\Lambda}:\propto\in\epsilon\times X\times\{0\}\}.$$

Example 4 Using our previous Example 1, the disagree- *EQNSES* $(\tilde{\dot{N}}_Q,A)_{\delta}{}^{0}$ over $\mathcal{E}$ is

$$(\tilde{\dot{N}}_Q,A)_{\delta}{}^{0}=$$

$$\left\{(x_1,p_1,0)_{\delta},\frac{(e_1,k_1)}{<0.8,0.1,0.2>},\frac{(e_1,k_2)}{<0.7,0.2,0.2>},\frac{(e_2,k_1)}{<0.6,0.2,0.1>},\frac{(e_2,k_2)}{<0.5,0.2,0.3>},\right.$$

$$(x_1,p_2,0)_{\delta},\frac{(e_1,k_1)}{<0.9,0.0,0.2>},\frac{(e_1,k_2)}{<0.6,0.2,0.3>},\frac{(e_2,k_1)}{<0.8,0.2,0.2>},\frac{(e_2,k_2)}{<0.8,0.3,0.4>},$$

$$(x_2,p_1,0)_{\delta},\frac{(e_1,k_1)}{<0.5,0.1,0.2>},\frac{(e_1,k_2)}{<0.6,0.2,0.3>},\frac{(e_2,k_1)}{<0.6,0.1,0.2>},\frac{(e_2,k_2)}{<0.6,0.1,0.3>},$$

$$\left.(x_2,p_2,0)_{\delta},\frac{(e_1,k_1)}{<0.9,0.2,0.1>},\frac{(e_1,k_2)}{<0.9,0.1,0.2>},\frac{(e_2,k_1)}{<0.9,0.2,0.5>},\frac{(e_2,k_2)}{<1.0,0.1,0.1>}\right\}$$

Definition 16 The complement of a *EQNSES* $(\tilde{\dot{N}}_Q,A)_{\delta}$ is

$$(\tilde{\dot{N}}_Q,A)_{\delta}{}^{c}=(\tilde{\dot{N}}_Q{}^{(c)},\neg A)_{\delta}$$

such that ${\tilde{\mathrm{N}}_Q}^{(c)}{}_{\delta}: \neg A \rightarrow EQNSE(\mathcal{E})$ a mapping

$$
{\tilde{\mathrm{N}}_Q}^{(c)}(\propto)_\delta = \left\{\mathcal{T}_{\tilde{\mathrm{N}}_Q(\propto)^{(c)}{}_\delta} = \mathcal{F}_{\tilde{\mathrm{N}}_Q(\propto)_\delta}, \mathcal{I}_{\tilde{\mathrm{N}}_Q(\propto)^{(c)}{}_\delta} = \bar{1} - \mathcal{I}_{\tilde{\mathrm{N}}_Q(\propto)_\delta}, \mathcal{F}_{\tilde{\mathrm{N}}_Q(\propto)^{(c)}{}_\delta} = \mathcal{T}_{\tilde{\mathrm{N}}_Q(\propto)_\delta}\right\}
$$

for each $\propto \in \epsilon$. It is clear that it is $\left(\left(\tilde{\mathrm{N}}_Q, A\right)_\delta{}^c\right)^c = \left(\tilde{\mathrm{N}}_Q, A\right)_\delta$

Example 5 Using our previous Example 1, the complement of the *EQNSES* $\tilde{\mathrm{N}}_Q$ denoted by ${\tilde{\mathrm{N}}_Q}^{(c)}{}_\delta$ is given as follows:

$$
\left(\tilde{\mathrm{N}}_Q, Z\right)_\delta{}^c =
$$

$$
\left\{\neg(x_1, p_1, 1)_\delta, \frac{(e_1,k_1)}{<0.3, 0.8, 0.6>} \frac{(e_1,k_2)}{<0.8,1.0,0.8>} \frac{(e_2,k_1)}{<0.7,0.9,0.8>} \frac{(e_2,k_2)}{<0.6,0.6,0.8>},\right.
$$

$$
\neg(x_1, p_2, 1)_\delta, \frac{(e_1,k_1)}{<0.2, 0.8, 0.9>}, \frac{(e_1,k_2)}{<0.4, 0.9, 0.4>}, \frac{(e_2,k_1)}{<0.4, 0.9, 0.7>}, \frac{(e_2,k_2)}{<0.2, 0.8, 0.5>},
$$

$$
\neg(x_2, p_1, 1)_\delta, \frac{(e_1,k_1)}{<0.6, 0.8, 0.8>}, \frac{(e_1,k_2)}{<0.3, 1.0, 0.6>}, \frac{(e_2,k_1)}{<0.2, 0.7, 0.8>}, \frac{(e_2,k_2)}{<0.4, 0.7, 0.8>},
$$

$$
\left.\neg(x_2, p_2, 1)_\delta, \frac{(e_1,k_1)}{<0.2, 0.8, 0.7>}, \frac{(e_1,k_2)}{<0.3, 0.8, 0.8>}, \frac{(e_2,k_1)}{<0.4, 0.9, 0.6>}, \frac{(e_2,k_2)}{<0.2, 0.8, 0.7>}\right\}
$$

Definition 17 The union of two *EQNSES* $(N_Q, A)_\delta$ and $\left(N_{1_Q}, B\right)_\delta$ over $\mathcal{E}$, denoted by

$$
(N_Q, A)_\delta \,\breve{\cup}\, \left(N_{1_Q}, B\right)_\delta
$$

is the *EQNSES* $(H_Q, C)_\delta$ such that $C = A \cup B$ and the memberships of truth, indeterminacy and falsity of $(H_Q, C)_\delta$ are respectively as follows: $\mathcal{T}_{\tilde{\mathrm{N}}(x_j)}(e_i, p_j)_\delta$

$$
\mathcal{T}_{H_Q(n)}(e_i, k_j)_\delta = \begin{cases} \mathcal{T}_{N_Q(n)}(e_i, k_j)_\delta & if \;\; n \in N - N_1 \\ \mathcal{T}_{N_{1_Q}(n)}(e_i, k_j)_\delta & if \;\; n \in N_1 - N \\ max\left(\mathcal{T}_{N_Q(n)}(e_i, k_j)_\delta, \mathcal{T}_{N_{1_Q}(n)}(e_i, k_j)_\delta\right) & if \;\; n \in N \cap N_1 \end{cases}
$$

$$\mathcal{I}_{H_Q(n)}(e_i,k_j)_\delta = \begin{cases} \mathcal{I}_{N_Q(n)}(e_i,k_j)_\delta & if \;\; n \in N - N_1 \\ \mathcal{I}_{N_{1Q}(n)}(e_i,k_j)_\delta & if \;\; n \in N_1 - N \\ min\left(\mathcal{I}_{N_Q(n)}(e_i,k_j)_\delta, \mathcal{I}_{N_{1Q}(n)}(e_i,k_j)_\delta\right) & if \;\; n \in N \cap N_1 \end{cases}$$

$$\mathcal{F}_{H_Q(n)}(e_i,k_j)_\delta = \begin{cases} \mathcal{F}_{N_Q(n)}(e_i,k_j)_\delta & if \;\; n \in N - N_1 \\ \mathcal{F}_{N_{1Q}(n)}(e_i,k_j)_\delta & if \;\; n \in N_1 - N \\ min\left(\mathcal{F}_{N_Q(n)}(e_i,k_j)_\delta, \mathcal{F}_{N_{1Q}(n)}(e_i,k_j)_\delta\right) & if \;\; n \in N \cap N_1. \end{cases}$$

Example 6 Suppose that $(N_Q,A)_\delta$ and $(N_{1_Q},B)_\delta$ are two *EQNSES* over $\mathcal{E}$, such that

$$(N_Q,A)_\delta = \left\{\left[(x_1,p,0)\left(\frac{(e_1,k_1)}{0.8,0.2,0.4},\frac{(e_1,k_2)}{0.7,0.3,0.5},\frac{(e_2,k_1)}{0.3,0.6,0.2},\frac{(e_2,k_2)}{0.3,0.2,0.4}\right)\right]\right.$$

$$\left.\left[(x_2,p,0),\left(\frac{(e_1,k_1)}{0.4,0.2,0.8},\frac{(e_1,k_2)}{0.3,0.6,0.2},\frac{(e_2,k_1)}{0.8,0.2,0.5},\frac{(e_2,k_2)}{0.1,0.2,0.8}\right)\right]\right\}$$

$$(N_{1_Q},B)_\delta = \left\{\left[(x_1,p,0)\left(\frac{(e_1,k_1)}{0.4,0.6,0.6},\frac{(e_1,k_2)}{0.7,0.3,0.8},\frac{(e_2,k_1)}{0.5,0.3,0.4},\frac{(e_2,k_2)}{0.2,0.6,0.8}\right)\right],\right.$$

$$\left.\left[(x_2,p,1),\left(\frac{(e_1,k_1)}{0.6,0.8,0.4},\frac{(e_1,k_2)}{0.7,0.4,0.9},\frac{(e_2,k_1)}{0.3,0.2,0.9},\frac{(e_2,k_2)}{0.5,0.6,0.7}\right)\right]\right\}$$

Then $(N_Q,A)_\delta \tilde{\cup} (N_{1_Q},B)_\delta = (H_Q,C)_\delta$ where

$$(H_Q,C)_\delta = \left\{\left[(x_1,p,0),\left(\frac{(e_1,k_1)}{0.8,0.2,0.4},\frac{(e_1,k_2)}{0.7,0.3,0.5},\frac{(e_2,k_1)}{0.5,0.3,0.2},\frac{(e_2,k_2)}{0.3,0.2,0.4}\right)\right]\right.$$

$$\left[(x_2,p,0),\left(\frac{(e_1,k_1)}{0.4,0.2,0.8},\frac{(e_1,k_2)}{0.3,0.6,0.2},\frac{(e_2,k_1)}{0.8,0.2,0.5},\frac{(e_2,k_2)}{0.1,0.2,0.8}\right)\right]$$

$$\left.\left[(x_2,p,1),\left(\frac{(e_1,k_1)}{0.6,0.8,0.4},\frac{(e_1,k_2)}{0.7,0.4,0.9},\frac{(e_2,k_1)}{0.3,0.2,0.9},\frac{(e_2,k_2)}{0.5,0.6,0.7}\right)\right]\right\}$$

.

Proposition 2 If $(N_Q,A)_\delta, (N_{1_Q},B)_\delta$ and $(H_Q,C)_\Lambda$ are three *EQNSES* over $\mathcal{E}$, then *i.* $\left(\left(N_Q,A\right)_\delta \tilde{\cup} \left(N_{1_Q},B\right)_\delta\right) \tilde{\cup} \left(H_Q,C\right)_\Lambda = \left(N_Q,A\right)_\delta \tilde{\cup} \left(\left(N_{1_Q},B\right)_\delta \tilde{\cup} \left(H_Q,C\right)_\Lambda\right)$.

ii. $\left(N_Q,A\right)_\delta \left(N_Q,A\right)_\delta \tilde{\subseteq} \left(N_Q,A\right)_\delta$.

Definition 18 Suppose $\left(N_Q,A\right)_\delta$ and $\left(N_{1_Q},B\right)_\delta$ are two *EQNSES* over the common universe $\mathcal{E}$. The intersection of $\left(N_Q,A\right)_\delta$ and $\left(N_{1_Q},B\right)_\delta$ is $\left(N_Q,A\right)_\delta \tilde{\cap} \left(N_{1_Q},B\right)_\delta = \left(K_Q,C\right)_\Lambda$ such that $C = A \cap B$ and the memberships of truth, indeterminacy and falsity of $\left(K_Q,C\right)_\Lambda$ are:

$$\mathcal{T}_{H_Q(n)}\left(e_i,k_j\right)_\delta = \begin{cases} \mathcal{T}_{N_Q(n,p)}\left(e_i,k_j\right)_\delta & if\ \ n \in N - N_1 \\ \mathcal{T}_{N_{1_Q}(n,p)}\left(e_i,k_j\right)_\delta & if\ \ n \in N_1 - N \\ min\left(\mathcal{T}_{N_Q(n,p)}\left(e_i,k_j\right)_\delta ,\mathcal{T}_{N_{1_Q}(n,p)}\left(e_i,k_j\right)_\delta\right) & if\ \ n \in N \cap N_1 \end{cases}$$

$$\mathcal{I}_{H_Q(n)}\left(e_i,k_j\right)_\delta = \begin{cases} \mathcal{I}_{N_Q(n)}\left(e_i,k_j\right)_\delta & if\ \ n \in N - N_1 \\ \mathcal{I}_{N_{1_Q}(n)}\left(e_i,k_j\right)_\delta & if\ \ n \in N_1 - N \\ max\left(\mathcal{I}_{N_Q(n)}\left(e_i,k_j\right)_\delta ,\mathcal{I}_{N_{1_Q}(n)}\left(e_i,k_j\right)_\delta\right) & if\ \ n \in N \cap N_1 \end{cases}$$

$$\mathcal{F}_{H_Q(n)}\left(e_i,k_j\right)_\delta = \begin{cases} \mathcal{F}_{N_Q(n)}\left(e_i,k_j\right)_\delta & if\ \ n \in N - N_1 \\ \mathcal{F}_{N_{1_Q}(n)}\left(e_i,k_j\right)_\delta & if\ \ n \in N_1 - N \\ max\left(\mathcal{F}_{N_Q(n)}\left(e_i,k_j\right)_\delta ,\mathcal{F}_{N_{1_Q}(n)}\left(e_i,k_j\right)_\delta\right) & if\ \ n \in N \cap N_1. \end{cases}$$

Example 7 Suppose that $\left(N_Q,A\right)_\delta$ and $\left(N_{1_Q},B\right)_\delta$ are two *EQNSES* over $\mathcal{E}$, such that

$$\left(N_Q,A\right)_\delta = \left\{\left[(x_1,p,1),\left(\frac{(e_1,k_1)}{0.3,0.4,0.3},\frac{(e_2,k_1)}{0.5,0.2,0.9}\right)\right],\right.$$

$$\left[(x_2,p,1),\left(\frac{(e_1,k_1)}{0.8,0.2,0.5},\frac{(e_2,k_1)}{0.4,0.2,0.3}\right)\right],$$

$$\left.\left[(x_2,p,0),\left(\frac{(e_1,k_1)}{0.5,0.4,0.7},\frac{(e_2,k_1)}{0.8,0.4,0.4}\right)\right]\right\}.$$

$$\left(N_{1_Q},B\right)_\delta = \left\{\left[(x_1,p,1),\left(\frac{(e_1,k_1)}{0.5,0.4,0.2},\frac{(e_2,k_1)}{0.7,0.3,0.2}\right)\right]\right.$$

Then $\left(N_Q,A\right)_\delta \tilde{\cap} \left(N_{1_Q},B\right)_\delta = \left(K_Q,C\right)_\Lambda$ where

$(K_Q, C)_\Lambda = \left\{\left[(x_1, p, 1), \left(\frac{(e_1, k_1)}{0.3,0.4,0.3}, \frac{(e_2, k_1)}{0.5,0.3,0.9}\right)\right]\right\}.$

Proposition 3 If $(N_Q, A)_\delta$,$(N_{1_Q}, B)_\delta$ and $(K_Q, C)_\Lambda$ are three *EQNSES* over $\mathcal{E}$, the following properties hold true.

i. $\left((N_Q, A)_\delta \tilde{\cap} (N_{1_Q}, B)_\delta\right) \tilde{\cap} (K_Q, C)_\Lambda = (N_Q, A)_\delta \tilde{\cap} \left((N_{1_Q}, B)_\delta \tilde{\cap} (K_Q, C)_\Lambda\right)$

ii. $(N_Q, A)_\delta \tilde{\cap} (N_Q, A)_\delta \subseteq (N_Q, A)_\delta$.

Proposition 4 If If $(N_Q, A)_\delta$,$(N_{1_Q}, B)_\delta$ and $(K_Q, C)_\Lambda$ are three *EQNSES* over $\mathcal{E}$, then

i.
$\left((N_Q, A)_\delta \tilde{\cup} (N_{1_Q}, B)_\delta\right) \tilde{\cap} (K_Q, C)_\Lambda = \left((N_Q, A)_\delta \tilde{\cap} (K_Q, C)_\Lambda\right) \tilde{\cup} \left((N_{1_Q}, B)_\delta \tilde{\cap} (K_Q, C)_\Lambda\right)$

ii.
$\left((N_Q, A)_\delta \tilde{\cap} (N_{1_Q}, B)_\delta\right) \tilde{\cup} (K_Q, C)_\Lambda = \left((N_Q, A)_\delta \tilde{\cup} (K_Q, C)_\Lambda\right) \tilde{\cap} \left((N_{1_Q}, B)_\delta \tilde{\cup} (K_Q, C)_\Lambda\right)$

Definition 19 If $(N_Q, A)_\delta$ and $(N_{1_Q}, B)_\delta$ are two *EQNSES* over $\mathcal{E}$, then "$(N_Q, A)_\delta$ AND $(N_{1_Q}, B)_\delta$" is

$(N_Q, A)_\delta \wedge (N_{1_Q}, B)_\delta = (H_Q, A \times B)_\delta$

such that $H_Q(\alpha, \beta) = N_Q(\alpha) \cap N_{1_Q}(\beta)$ and memberships of truth, indeterminacy, and falsity of of $(H_Q, A \times B)_\Lambda$ are as follows:

$\mathcal{T}_{H_Q(\alpha,\beta)}(e_i, p_j)_\delta = min\left(\mathcal{T}_{N_Q(\alpha)}(e_i, k_j)_\delta, \mathcal{T}_{N_{1Q}(\beta)}(e_i, k_j)_\delta\right),$

$\mathcal{I}_{H_Q(\alpha,\beta)}(e_i, p_j)_\delta = max\left(\mathcal{I}_{N_Q(\alpha)}(e_i, k_j)_\delta, \mathcal{I}_{N_{1Q}(\beta)}(e_i, k_j)_\delta\right),$

$\mathcal{F}_{H_Q(\alpha,\beta)}(e_i, p_j)_\delta = max\left(\mathcal{F}_{N_Q(\alpha)}(e_i, k_j)_\delta, \mathcal{F}_{N_{1Q}(\beta)}(e_i, k_j)_\delta\right)$

where $\forall \alpha \in A, \forall \beta \in B$.

Example 8 Suppose that $(N_Q,A)_\delta$ and $(N_{1_Q},B)_\delta$ are two *EQNSES* over $\mathcal{E}$, such that

$$(N_Q,A)_\delta = \left\{\left[(x_1,p,1),\left(\frac{(e_1,k_1)}{0.2,0.5,0.6},\frac{(e_1,k_2)}{0.4,0.1,0.3},\frac{(e_2,k_1)}{0.6,0.4,0.7},\frac{(e_2,k_2)}{0.3,0.1,0.8}\right)\right]\right\},$$

$$(G_Q,B)_\Lambda = \left\{\left[(x_1,p,1),\left(\frac{(e_1,k_1)}{0.5,0.3,0.6},\frac{(e_1,k_2)}{0.1,0.9,0.8},\frac{(e_2,k_1)}{0.1,0.4,0.3},\frac{(e_2,k_2)}{0.1,0.2,0.6}\right)\right],\right.$$

$$\left.\left[(x_2,q,0),\left(\frac{(e_1,k_1)}{0.4,0.1,0.9},\frac{(e_1,k_2)}{0.3,0.1,0.7},\frac{(e_2,k_1)}{0.8,0.4,0.3},\frac{(e_2,k_2)}{0.7,0.3,0.2}\right)\right]\right\}.$$

Then $(N_Q,A)_\delta \wedge (N_{1_Q},B)_\delta = (H_Q,A\times B)_\delta$ where

$$(H_Q,A\times B)_\delta = \left\{\left[(x_1,p,1),(x_1,p,1),\left(\frac{(e_1,k_1)}{0.2,0.5,0.6},\frac{(e_1,k_2)}{0.1,0.9,0.8},\frac{(e_2,k_1)}{0.1,0.4,0.7},\frac{(e_2,k_2)}{0.1,0.2,0.8}\right)\right],\right.$$

$$\left.\left[(x_1,p,1),(x_2,q,0),\left(\frac{(e_1,k_1)}{0.2,0.5,0.9},\frac{(e_1,k_2)}{0.3,0.1,0.7},\frac{(e_2,k_1)}{0.6,0.4,0.7},\frac{(e_2,k_2)}{0.3,0.3,0.8}\right)\right]\right\}$$

Definition 20 If $(N_Q,A)_\delta$ and $(N_{1_Q},B)_\delta$ are two *EQNSES* over $\mathcal{E}$, then "$(N_Q,A)_\delta$ OR $(N_{1_Q},B)_\delta$" is

$$(N_Q,A)_\delta \vee (N_{1_Q},B)_\delta = (K_Q,A\times B)_\delta$$

such that $K_Q(\alpha,\beta) = N_Q(\alpha)\cup N_{1_Q}(\beta)$ and the memberships of truth, indeterminacy, and falsity of $(K_Q,A\times B)_\delta$ are as follows:

$$\mathcal{T}_{H_Q(\alpha,\beta)}(e_i,k_j)_\delta = max\left(\mathcal{T}_{N_Q(\alpha)}(e_i,k_j)_\delta, \mathcal{T}_{N_{1}Q(\beta)}(e_i,k_j)_\delta\right),$$

$$\mathcal{I}_{H_Q(\alpha,\beta)}(e_i,k_j)_\delta = min\left(\mathcal{I}_{N_Q(\alpha)}(e_i,k_j)_\delta, \mathcal{I}_{N_{1}Q(\beta)}(e_i,k_j)_\delta\right),$$

$$\mathcal{F}_{H_Q(\alpha,\beta)}(e_i,k_j)_\delta = min\left(\mathcal{F}_{N_Q(\alpha)}(e_i,k_j)_\delta, \mathcal{F}_{N_{1}Q(\beta)}(e_i,k_j)_\delta\right)$$

where $\forall\alpha\in A, \forall\beta\in B$.

Example 9 Suppose that $(N_Q,A)_\delta$ and $(N_{1_Q},B)_\delta$ are two *EQNSES* over $\mathcal{E}$, such that

$$(N_Q,A)_\delta = \left\{\left[(x_1,p,1),\left(\frac{(e_1,k_1)}{0.4,0.2,0.6},\frac{(e_1,k_2)}{0.3,0.2,0.7},\frac{(e_2,k_1)}{0.5,0.8,0.4},\frac{(e_2,k_2)}{0.3,0.1,0.1}\right)\right]\right\}$$

$$(N_{1_Q},B)_\delta = \left\{\left[(x_1,p,1),\left(\frac{(e_1,k_1)}{0.7,0.4,0.8},\frac{(e_1,k_2)}{0.5,0.4,0.6},\frac{(e_2,k_1)}{0.6,0.7,0.1},\frac{(e_2,k_2)}{0.2,0.2,0.8}\right)\right],\right.$$

$$\left.\left[(x_2,p,0),\left(\frac{(e_1,k_1)}{0.6,0.4,0.3},\frac{(e_1,k_2)}{0.2,0.5,0.4},\frac{(e_2,k_1)}{0.3,0.7,0.8},\frac{(e_2,k_2)}{0.9,0.1,0.2}\right)\right]\right\}.$$

Then $(N_Q,A)_\delta \vee (N_{1_Q},B)_\delta = (K_Q,A\times B)_\Lambda$ where

$$(K_Q,A\times B)_\delta = \left\{\left[(x_1,p,1),(x_1,p,1)\left(\frac{(e_1,k_1)}{0.7,0.2,0.6},\frac{(e_1,k_2)}{0.5,0.2,0.6},\frac{(e_2,k_1)}{0.6,0.7,0.1},\frac{(e_2,k_2)}{0.3,0.1,0.1}\right)\right],\right.$$

$$\left.\left[(x_1,p,1),(x_2,p,0),\left(\frac{(e_1,k_1)}{0.6,0.2,0.3},\frac{(e_1,k_2)}{0.3,0.2,0.4},\frac{(e_2,k_1)}{0.5,0.7,0.4},\frac{(e_2,k_2)}{0.9,0.1,0.1}\right)\right]\right\}$$

Proposition 5 If $(N_Q,A)_\delta$ and $(N_{1_Q},B)_\delta$ are *EQNSES* over $\mathcal{E}$, then

i. $\left((N_Q,A)_\delta \wedge (N_{1_Q},B)_\delta\right)^c = (N_Q,A)_\delta{}^c \vee (N_{1_Q},B)_\delta{}^c$

ii. $\left((N_Q,A)_\delta \vee (N_{1_Q},B)_\delta\right)^c = (N_Q,A)_\delta{}^c \wedge (N_{1_Q},B)_\delta{}^c$

4. An Application of *EQNSES*

We will now present an application of *EQNSES* theory to illustrate that this concept can be successfully applied to decision-making problems with uncertain information. The following algorithm is suggested to solve a effective *Q*-neutrosophic soft expert based decision making problem below.

Assume that a book selection will be made for mathematics students. There are three alternatives $\mathcal{E} = \{e_1,e_2,e_3\}$, with two types of qualifications *Q*= $\{k_1,k_2\}$ and there are two parameters $\epsilon = \{x_1,x_2\}$ with x_i $(i = 1,2)$ standing for "price range" and "reviews" respectively and the set of effective parameters is represented by δ ={l_1 = *"popularity"*, l_2 = *"writing style"*}. Suppose $X = \{p_1, p_2\}$ is the set of two expert faculty members will decide which book to choose. After long discussions, the experts construct the *QNSES* below.

$$\delta^1(e_1,k_1,p_1)=\left\{\frac{l_1}{(0.6,0.7,0.4)},\frac{l_2}{(0.3,0.7,0.5)}\right\},\ \delta^2(e_1,k_2,p_1)=\left\{\frac{l_1}{(0.4,0.6,0.4)},\frac{l_2}{(0.9,0.3,0.6)}\right\},$$

$$\delta^3(e_1,k_1,p_2)=\left\{\frac{l_1}{(0.8,0.3,0.1)},\frac{l_2}{(0.2,0.8,0.5)}\right\},\ \delta^4(e_1,k_2,p_2)=\left\{\frac{l_1}{(0.6,0.5,0.1)},\frac{l_2}{(0.7,0.2,0.4)}\right\},$$

$$\delta^5(e_2,k_1,p_1)=\left\{\frac{l_1}{(0.4,0.1,0.7)},\frac{l_2}{(0.8,0.5,0.3)}\right\},\ \delta^6(e_2,k_1,p_2)=\left\{\frac{l_1}{(0.8,0.4,0.2)},\frac{l_2}{(0.9,0.2,0.6)}\right\},$$

$$\delta^7(e_2,k_2,p_1)=\left\{\frac{l_1}{(0.2,0.3,0.4)},\frac{l_2}{(0.5,0.3,0.9)}\right\},\ \delta^8(e_2,k_2,p_2)=\left\{\frac{l_1}{(0.3,0.4,0.7)},\frac{l_2}{(0.8,0.7,0.4)}\right\},$$

$$\delta^9(e_3,k_2,p_2)=\left\{\frac{l_1}{(0.3,0.4,0.7)},\frac{l_2}{(0.2,0.6,0.1)}\right\},\ \delta^{10}(e_3,k_1,p_1)=\left\{\frac{l_1}{(0.7,0.4,0.6)},\frac{l_2}{(0.8,0.4,0.6)}\right\},$$

$$\delta^{11}(e_3,k_1,p_2)=\left\{\frac{l_1}{(0.3,0.5,0.9)},\frac{l_2}{(0.1,0.5,0.7)}\right\},\ \delta^{12}(e_3,k_2,p_1)=\left\{\frac{l_1}{(0.6,0.4,0.9)},\frac{l_2}{(0.8,0.5,0.2)}\right\}$$

$$(N_Q,Z)$$

$$=\left\{\ \left[(x_1,p_1,1),\left(\frac{(e_1,k_1)}{0.3,0.4,0.5},\frac{(e_1,k_2)}{0.7,0.3,0.7},\frac{(e_2,k_1)}{0.5,0.4,0.6},\frac{(e_2,k_2)}{0.5,0.3,0.2},\frac{(e_3,k_1)}{0.5,0.2,0.2},\frac{(e_3,k_2)}{0.6,0.3,0.7}\right)\right],\right.$$

$$\left[(x_1,p_2,1),\left(\frac{(e_1,k_1)}{0.3,0.5,0.1},\frac{(e_1,k_2)}{0.4,0.3,0.7},\frac{(e_2,k_1)}{0.5,0.4,0.4},\frac{(e_2,k_2)}{0.3,0.1,0.4},\frac{(e_3,k_1)}{0.5,0.4,0.2},\frac{(e_3,k_2)}{0.2,0.1,0.8}\right)\right],$$

$$\left[(x_2,p_1,1),\left(\frac{(e_1,k_1)}{0.6,0.7,0.2},\frac{(e_1,k_2)}{0.9,0.3,0.5},\frac{(e_2,k_1)}{0.7,0.4,0.3},\frac{(e_2,k_2)}{0.3,0.3,0.6},\frac{(e_3,k_1)}{0.7,0.1,0.5},\frac{(e_3,k_2)}{0.3,0.5,0.8}\right)\right],$$

$$\left[(x_2,p_2,1),\left(\frac{(e_1,k_1)}{0.3,0.4,0.5},\frac{(e_1,k_2)}{0.9,0.3,0.5},\frac{(e_2,k_1)}{0.7,0.2,0.6},\frac{(e_2,k_2)}{0.3,0.1,0.9},\frac{(e_3,k_1)}{0.5,0.4,0.3},\frac{(e_3,k_2)}{0.2,0.6,0.6}\right)\right],$$

$$\left[(x_1,p_1,0),\left(\frac{(e_1,k_1)}{0.3,0.5,0.8},\frac{(e_1,k_2)}{0.1,0.8,0.4},\frac{(e_2,k_1)}{0.4,0.1,0.7},\frac{(e_2,k_2)}{0.3,0.5,0.8},\frac{(e_3,k_1)}{0.4,0.5,0.6},\frac{(e_3,k_2)}{0.7,0.4,0.3}\right)\right],$$

$$\left[(x_1,p_2,0),\left(\frac{(e_1,k_1)}{0.1,0.5,0.5},\frac{(e_1,k_2)}{0.7,0.3,0.5},\frac{(e_2,k_1)}{0.7,0.4,0.4},\frac{(e_2,k_2)}{0.3,0.5,0.4},\frac{(e_3,k_1)}{0.5,0.4,0.2},\frac{(e_3,k_2)}{0.2,0.1,0.5}\right)\right],$$

$$\left[(x_2,p_1,0),\left(\frac{(e_1,k_1)}{0.3,0.5,0.6},\frac{(e_1,k_2)}{0.7,0.5,0.4},\frac{(e_2,k_1)}{0.5,0.4,0.1},\frac{(e_2,k_2)}{0.5,0.3,0.4},\frac{(e_3,k_1)}{0.5,0.3,0.2},\frac{(e_3,k_2)}{0.7,0.3,0.3}\right)\right],$$

$$\left[(x_2,p_2,0),\left(\frac{(e_1,k_1)}{0.3,0.5,0.6},\frac{(e_1,k_2)}{0.7,0.3,0.7},\frac{(e_2,k_1)}{0.5,0.2,0.6},\frac{(e_2,k_2)}{0.5,0.3,0.2},\frac{(e_3,k_1)}{0.5,0.4,0.2},\frac{(e_3,k_2)}{0.2,0.3,0.5}\right)\right],$$

Tables 1 presents the agree-*EQNSES* while Table 2 presents the disagree-*EQNSES* by using the mean of each *EQNSES*.

The following algorithm may be used to choose the most qualified candidate to fill the vacancy.

5. Input the $QNSES\ (N_Q, Z)$.
6. Compute the $EQNSES\ (N_Q, Z)_\delta$.
7. Find the agree-*EQNSES* and disagree-*EQNSES*.
8. Calculate $C_j = \sum_i (x,p)_{ij} = \frac{|\mathcal{T}-\mathcal{J}-\mathcal{F}|}{\mathcal{T}+\mathcal{F}}$ for agree-*EQNSES*.
9. Calculate $K_j = \sum_i (x,p)_{ij} = \frac{|\mathcal{T}-\mathcal{J}-\mathcal{F}|}{\mathcal{T}+\mathcal{F}}$ for disagree-*EQNSES*.
10. Determine $s_j = \frac{c_j - k_j}{2}$.
11. Determine r, for which $s_r = max s_j$. If there is has more than a one value of r, then the college can have alternative choices.

Table 1: Agree- *EQNSES*

$\mathcal{E} \times Q$	(e_1,k_1)	(e_1,k_2)	(e_2,k_1)	(e_2,k_2)	(e_3,k_1)	(e_3,k_2)
$(x_1,p_1,1)$	(0.6,0.1,0.3)	(0.8,0.1,0.4)	(0.7,0.1,0.3)	(0.7,0.2,0.1)	(0.7,0.1,0.1)	(0.8,0.2,0.4)
$(x_1,p_2,1)$	(0.8,0.3,0.1)	(0.8,0.2,0.4)	(0.8,0.2,0.2)	(0.8,0.1,0.2)	(0.8,0.2,0.1)	(0.7,0.1,0.4)
$(x_2,p_1,1)$	(0.8,0.5,0.1)	(1.0,0.2,0.3)	(0.9,0.3,0.2)	(0.7,0.2,0.3)	(0.9,0.1,0.3)	(0.7,0.3,0.4)
$(x_2,p_2,1)$	(0.5,0.3,0.2)	(0.9,0.2,0.2)	(0.8,0.1,0.2)	(0.5,0.0,0.3)	(0.7,0.1,0.1)	(0.5,0.2,0.2)
$C_j = \sum_i (x,p)_{ij}$	$c_1 = 1.2$	$c_2 = 1.4$	$c_3 = 1.5$	$c_4 = 1.6$	$c_5 = 2.2$	$c_6 = 0.6$

***Table 2**: Disagree-EQNSES*

$\mathcal{E} \times Q$	(e_1,k_1)	(e_1,k_2)	(e_2,k_1)	(e_2,k_2)	(e_3,k_1)	(e_3,k_2)
$(x_1,p_1,1)$	(0.7,0.2,0.6)	(0.6,0.4,0.3)	(0.7,0.0,0.5)	(0.7,0.4,0.6)	(0.7,0.4,0.4)	(0.9,0.3,0.2)
$(x_1,p_2,1)$	(0.7,0.3,0.4)	(0.9,0.2,0.4)	(0.9,0.3,0.3)	(0.8,0.4,0.3)	(0.8,0.3,0.2)	(0.7,0.1,0.4)
$(x_2,p_1,1)$	(0.9,0.4,0.4)	(1.0,0.4,0.2)	(0.9,0.3,0.1)	(0.9,0.2,0.2)	(0.9,0.2,0.1)	(1.0,0.2,0.2)
$(x_2,p_2,1)$	(0.7,0.2,0.3)	(0.9,0.1,0.3)	(0.8,0.1,0.3)	(0.8,0.1,0.1)	(0.8,0.2,0.1)	(0.6,0.1,0.2)
$K_j = \sum_i (x,p)_{ij}$	$k_1 = 0.5$	$k_2 = 1.0$	$k_3 = 1.4$	$k_4 = 1.4$	$k_5 = 1.6$	$k_6 = 1.4$

Table 3: $s_j = \frac{c_j - k_j}{2}$ for x_1

j	$\mathcal{E} \times Q$	c_j	k_j	$c_j - k_j$	$s_j = \frac{c_j - k_j}{2}$
1	(e_1, k_1)	1.2	0.5	0.7	0.4
2	(e_1, k_2)	1.4	1.0	0.4	0.2
3	(e_2, k_1)	1.5	1.4	0.1	0.05
4	(e_2, k_2)	1.6	1.4	0.2	0.1
5	(e_3, k_1)	2.2	1.6	0.6	0.3
6	(e_3, k_2)	0.6	1.4	-0.8	-0.4

From Tables 1 and 2 we are able to calculate the values of $s_j = \frac{c_j - k_j}{2}$ as in Table 3.

As can be seen, the maximum score is $s_r = max s_j$=0.4 for (e_1, p_1) .

5. Conclusion

We have introduced the concept of an effective Q -neutrosophic soft expert set along with its operations of equality, union, intersection, subset, OR, and AND. The application of this novel concept to a decision-making process is illustrated and compared to those in existing literature. It is shown that this proposed concept is more inclusive by considering the membership of falsity and indeterminacy, expert, neutrosophy and Q -fuzzy. Thus, the proposed approach is shown to be useful in handling realistic uncertain problems.

6. Future Research Directions

This study can be extended by using other type of neutrosophic decision-making approaches, including bipolar neutrosophic soft sets, interval valued neutrosophic soft sets.

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Generalized Bonferroni mean operators of trapezoidal fuzzy multi numbers and their application to decision making problem

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Abstract

In cases where conventional sets are insufficient in use in daily life, we use trapezoidal fuzzy multi sets to solve decision-making problems in many areas. Hence, The aim of this paper is to investigate how to solve a model selection problem with multi-criteria against poverty and hunger by using trapezoidal fuzzy multi numbers. For this, we introduce two aggregation methods called trapezoidal fuzzy multi generalized weighted Bonferroni arithmetic mean operator and trapezoidal fuzzy multi generalized weighted Bonferroni geometric mean operator. Later, we investigate their properties and some special cases. Moreover, we introduce a process to solve multi-criteria decision making problems with trapezoidal fuzzy multi numbers. Then, we apply introduced methods to a model selection problem against poverty and hunger.
Keywords: *fuzzy multi sets, trapezoidal fuzzy number, trapezoidal fuzzy multi numbers, generalized Bonferroni arithmetic mean, poverty, hunger*

1 Introduction

In 1965, fuzzy set theory was proposed by Zadeh [40] as an extension of a classical concept of a set for ambiguous information. With the introduction of the theory, he offered a new way which makes decision-making process using fuzzy concepts of some information more useful. This methodology basically, a fuzzy set is a set that has no clearly known boundaries and can only contain elements only in some degree i.e. elements can have a degree of membership determined by appropriate functions namely membership functions. These functions are used to determine the membership degree of each element in a fuzzy set. In time, a kind of fuzzy sets were introduced by Yager [38] which is called multi-fuzzy sets (fuzzy bags). The notion presents a new generalization of fuzzy sets. In addition, it gives complete information for some problems including situations in which each element has different membership values. Miyamoto [17] and Sebastian and Ramakrishnan [24, 25] expanded and studied detailedly the Yager's multi-sets and multi-fuzzy sets. Since some situations have multi possibility of same or different membership values, Ulucay et al. [29] developed trapezoidal fuzzy multi-numbers on real number set $\mathbb{R}$. They are expansion of both multi-fuzzy sets and fuzzy numbers enabling the recurrent occurrences of any element. Later, many studies have been conducted by many scientists. Readers may find the studies about trapezoidal fuzzy multi-numbers application of fuzzy logic in different areas and in [1, 4, 8, 10, 11, 15, 16, 18, 19, 20, 21, 22, 23, 28, 30, 42].

The Bonferroni mean (BM), firstly created by Bonferroni [3] is an aggregation technique which is useful to aggregate the crisp data. It can see the interrelationships among arguments, which plays an important role in multi-criteria decision-making problems.This is why Yager [39] presented a elaborated paper of BM and proposed some generalizations that extend it's capability. Beliakov et al.[2] got the BM more enhanced by handling the interrelation of any three aggregated elements instead of any two. However, Xu and Yager [37] introduced that the elements are suitable to be aggregated by the BM and generalized BM can only take the forms of crisp numbers rather than any other kind of numbers, which limits the potential applications of the BM to more enhanced areas and applied the BM to intuitionistic fuzzy environment and proposed the intuitionistic fuzzy Bonferroni mean (IFBM) and the intuitionistic fuzzy weighted Bonferroni mean (IFWBM). Then, they applied the IFWBM to multi criteria decision making. Moreover, Xia et al. [34, 35] submitted a generalized weighted Bonferroni mean (GWBM), generalized intuitionistic fuzzy weighted Bonferroni mean (GIFWBM) and geometric Bonferroni means and discussed their applications in multi-criteria decision making. In 2021, Deli [6] extended Bonferroni mean operators to generalized trapezoidal hesitant fuzzy numbers and gave their application to decision-making problems. Kesen and Deli [15] introduced weighted Bonferroni harmonic mean operator on trapezoidal fuzzy multi-numbers in 2022.

This article have six chapters. In second chapter, we give definition of some basic concepts such as fuzzy multi sets, trapezoidal fuzzy multi numbers, hamming distance and CRITIC (Criteria Importance Through Intercriteria Correlation) method etc. Then, we propose some properties and operations of fuzzy multi sets and trapezoidal fuzzy multi numbers. In third chapter, we introduce two aggregation operators which are called trapezoidal fuzzy multi generalized weighted Bonferroni arithmetic mean and trapezoidal fuzzy multi generalized weighted Bonferroni geometric mean and we give their some properties. In the fourth chapter, we give an approach to multi-attribute group decision making problem and to see application to multi-attribute group decision making problems. At the end of the chapter a numerical example is given. In fifth chapter, we propose a comparison table to compare given methods with existing methods.

1.1 Novelty

This paper proposes two main novelties as follows:

1. Mathematically, we developed generalized Bonferroni mean operators. In normal, Bonferroni mean operators only tackle with the conditions having correlations between any two aggregated elements, but not the conditions having connections among any three aggregated elements. This is a drawback of these operators. In order to solve this issue, generalized Bonferroni mean operators have been developed. Moreover, generalized Bonferroni mean operators have been extended to TFM-numbers.
2. Politically, we developed two methods to select best model against poverty and hunger by considering all policies in the problem. In this aspect, we present a new perspective to select and build models for fighting against poverty and hunger. Therefore, these methods can be extended and utilised for more complex humanitarian problems.

2 Preliminary

In this section, we give some basic concepts of fuzzy set [40], fuzzy number [13] and fuzzy multi set [24], fuzzy Bonferroni aggregation operators, trapezoidal fuzzy multi numbers etc. For more, readers may look at Deli and Karaaslan [7], Deli [5], Torra [26], Torra and Narukawa [27], Wang et al.[31], Wei [32], Xia and Xu [33], Xu [36].

Definition 2.1 *[40] Let X be a non-empty set. A fuzzy set F on X is defined as:*

$$F = \{\langle x, \mu_F(x)\rangle : x \in X\}$$

where $\mu_F : X \to [0,1]$ for $x \in X$.

Definition 2.2 *[41] t-norms are monotonic, commutative and associative functions t with two valued mapping from $[0,1]\times[0,1]$ into $[0,1]$ and satisfying following conditions:*

1. $t(0,0)=0$, $t(\mu_{F_1}(x),1)=t(1,\mu_{F_1}(x))=\mu_{F_1}(x)$
2. If $\mu_{F_1}(x)\le\mu_{F_3}(x)$ and $\mu_{F_2}(x)\le\mu_{F_4}(x)$, then $t(\mu_{F_1}(x),\mu_{F_2}(x))\le t(\mu_{F_3}x),\mu_{F_4}(x))$
3. $t(\mu_{F_1}(x),\mu_{F_2}(x))=t(\mu_{F_2}(x),\mu_{F_1}(x))$
4. $t(\mu_{F_1}(x),t(\mu_{F_2}(x),\mu_{F_3}(x)))=t(t(\mu_{F_1}(x),\mu_{F_2})(x),\mu_{F_3}(x))$

Definition 2.3 *[41] s-norm are monotonic, commutative and associative functions t with two placed mapping from $[0,1]\times[0,1]$ into $[0,1]$ and satisfying following conditions:*

1. $s(1,1)=1, s(\mu_{F_1}(x),0)=s(0,\mu_{F_1}(x))=\mu_{F_1}(x)$
2. if $\mu_{F_1}(x)\le\mu_{F_3}(x)$ and $\mu_{F_2}(x)\le\mu_{F_4}(x)$, then $s(\mu_{F_1}(x),\mu_{F_2}(x))\le s(\mu_{F_3}(x),\mu_{F_4}(x))$
3. $s(\mu_{F_1}(x),\mu_{F_2}(x))=s(\mu_{F_2}(x),\mu_{F_1}(x))$
4. $s(\mu_{F_1}(x),s(\mu_{F_2}(x),\mu_{F_3}(x)))=s(s(\mu_{F_1}(x),\mu_{F_2})(x),\mu_{F_3}(x))$

t-norm and s-norm are related in a sense of logical duality. Typical dual pairs of non-parameterized t-norm and s-norm are complied below:

1. Drastic product:

$$t_w(\mu_{F_1}(x),\mu_{F_2}(x))=\begin{cases}\min\{\mu_{F_1}(x),\mu_{F_2}(x)\}, & \max\{\mu_{F_1}(x),\mu_{F_2}(x)\}=1\\ 0, & otherwise\end{cases}$$

2. Drastic sum:

$$s_w(\mu_{F_1}(x),\mu_{F_2}(x))=\begin{cases}\max\{\mu_{F_1}(x),\mu_{F_2}(x)\}, & \min\{\mu_{F_1}(x),\mu_{F_2}(x)\}=0\\ 1, & otherwise\end{cases}$$

3. Bounded product:

$$t_1(\mu_{F_1}(x),\mu_{F_2}(x))=\max\{0,\mu_{F_1}(x)+\mu_{F_2}(x)-1\}$$

4. Bounded sum:

$$s_1(\mu_{F_1}(x),\mu_{F_2}(x))=\min\{1,\mu_{F_1}(x)+\mu_{F_2}(x)\}$$

5. Einstein product:

$$t_{1.5}(\mu_{F_1}(x),\mu_{F_2}(x))=\frac{\mu_{F_1}(x).\mu_{F_2}(x)}{2-[\mu_{F_1}(x)+\mu_{F_2}(x)-\mu_{F_1}(x).\mu_{F_2}(x)]}$$

6. Einstein sum:

$$s_{1.5}(\mu_{F_1}(x),\mu_{F_2}(x))=\frac{\mu_{F_1}(x)+\mu_{F_2}(x)}{1+\mu_{F_1}(x).\mu_{F_2}(x)}$$

7. Algebraic product:

$$t_2(\mu_{F_1}(x),\mu_{F_2}(x))=\mu_{F_1}(x).\mu_{F_2}(x)$$

8. Algebraic sum:

$$s_2(\mu_{F_1}(x),\mu_{F_2}(x))=\mu_{F_1}(x)+\mu_{F_2}(x)-\mu_{F_1}(x).\mu_{F_2}(x)$$

9. Hamacher product:

$$t_{2.5}(\mu_{F_1}(x),\mu_{F_2}(x))=\frac{\mu_{F_1}(x).\mu_{F_2}(x)}{\mu_{F_1}(x)+\mu_{F_2}(x)-\mu_{F_1}(x).\mu_{F_2}(x)}$$

10. Hamacher sum:

$$s_{2.5}(\mu_{F_1}(x),\mu_{F_2}(x))=\frac{\mu_{F_1}(x)+\mu_{F_2}(x)-2.\mu_{F_1}(x).\mu_{F_2}(x)}{1-\mu_{F_1}(x).\mu_{F_2}(x)}$$

11. Minumum:

$$t_3(\mu_{F_1}(x),\mu_{F_2}(x))=\min\{\mu_{F_1}(x),\mu_{F_2}(x)\}$$

12. Maximum:

$$s_3(\mu_{F_1}(x),\mu_{F_2}(x))=\max\{\mu_{F_1}(x),\mu_{F_2}(x)\}$$

Definition 2.4 *[24] Let X be a non-empty set. A multi-fuzzy set G on X is defined as;*

$$G=\{\langle x,\mu_G^1(x),\mu_G^2(x),...,\mu_G^i(x),...\rangle : x\in X\}$$

where $\mu_G^i : X\to[0,1]$ for all $i\in\{1,2,...,p\}$ and $x\in X$.

In the paper, I_m , I_n and I_P will be used instead of $\{1,2,...,m\}$, $\{1,2,...,n\}$ and $\{1,2,...,P\}$ respectively.

Definition 2.5 *[29] Let $\eta_T\in[0,1]$ and $\epsilon_1,\epsilon_2,\epsilon_3,\epsilon_4\in R$ such that $\epsilon_1\le\epsilon_2\le\epsilon_3\le\epsilon_4$. Then, a generalized trapezoidal fuzzy number (GTF-number) $T=\langle(\epsilon_1,\epsilon_2,\epsilon_3,\epsilon_4);\eta_T\rangle$ is a special* *fuzzy set on the real number set R. Its membership functions are defined as follows:*

$$\mu_T(x)=\begin{cases}(x-\epsilon_1)\eta_T/(\epsilon_2-\epsilon_1) & \epsilon_1\le x<\epsilon_2\\ \eta_T & \epsilon_2\le x\le\epsilon_3\\ (\epsilon_4-x)\eta_T/(\epsilon_4-\epsilon_3) & \epsilon_3<x\le\epsilon_4\\ 0 & otherwise\end{cases}$$

Definition 2.6 *[29] Let* $\eta_T^i\in[0,1]$ $(i\in I_P)$ *and* $\epsilon_1,\epsilon_2,\epsilon_3,\epsilon_4\in R$ *such that* $\epsilon_1\le\epsilon_2\le\epsilon_3\le\epsilon_4$. *Then, a trapezoidal fuzzy multi-number (TFM-number)* $T=\langle(\epsilon_1,\epsilon_2,\epsilon_3,\epsilon_4);\eta_T^1,\eta_T^2,...,\eta_T^P\rangle$ *is a special fuzzy multi-set on the real number set* $\mathbb{R}$. *Its membership functions are defined as follows:*

$$\mu_T^i(x)=\begin{cases}(x-\epsilon_1)\eta_T^i/(\epsilon_2-\epsilon_1) & \epsilon_1\le x<\epsilon_2\\ \eta_T^i & \epsilon_2\le x\le\epsilon_3\\ (\epsilon_4-x)\eta_T^i/(\epsilon_4-\epsilon_3) & \epsilon_3<x\le\epsilon_4\\ 0 & otherwise\end{cases}$$

Note that the set of all TFM-number on $\mathbb{R}^+$ will be denoted by $\mho(\mathbb{R}^+)$.

Definition 2.7 *[29] Let* $T_1=\langle(\epsilon_1,\epsilon_2,\epsilon_3,\epsilon_4);\eta_{T_1}^1,\eta_{T_1}^2,...,\eta_{T_1}^P\rangle$ *and* $T_2=\langle(\rho_1,\rho_2,\rho_3,\rho_4);\eta_{T_2}^1,\eta_{T_2}^2,...,\eta_{T_2}^P\rangle$ *be two TFM-numbers and* $\gamma\ne0$ *be any real number. Then,*

1. $T_1\oplus T_2=(s(\epsilon_1,\rho_1),s(\epsilon_2,\rho_2),s(\epsilon_3,\rho_3),s(\epsilon_4,\rho_4));s(\eta_{T_1}^1,\eta_{T_2}^1),s(\eta_{T_1}^2,\eta_{T_2}^2),...,s(\eta_{T_1}^p,\eta_{T_2}^p))\rangle$

2. $$T_1\otimes T_2=\begin{cases}\langle(t(\epsilon_1\rho_1),t(\epsilon_2\rho_2),t(\epsilon_3\rho_3),t(\epsilon_4\rho_4));t(\eta_{T_1}^1\eta_{T_2}^1),t(\eta_{T_1}^2,\eta_{T_2}^2),...,t(\eta_{T_1}^p,\eta_{T_2}^p))\rangle & (\epsilon_4>0,\rho_4>0)\\ \langle(t(\epsilon_1\rho_4),t(\epsilon_2\rho_3),t(\epsilon_3\rho_2),t(\epsilon_4\rho_1));t(\eta_{T_1}^1,\eta_{T_2}^1),t(\eta_{T_1}^2,\eta_{T_2}^2),...,t(\eta_{T_1}^p,\eta_{T_2}^p))\rangle & (\epsilon_4<0,\rho_4>0)\\ \langle(t(\epsilon_4\rho_4),t(\epsilon_3\rho_3),t(\epsilon_2\rho_2),t(\epsilon_1\rho_1));t(\eta_{T_1}^1,\eta_{T_2}^1),t(\eta_{T_1}^2,\eta_{T_2}^2),...,t(\eta_{T_1}^p,\eta_{T_2}^p))\rangle & (\epsilon_4<0,\rho_4<0)\end{cases}$$

3. $\gamma T_1=\langle(\gamma\epsilon_1,\gamma\epsilon_2,\gamma\epsilon_3,\gamma\epsilon_4);1-(1-\eta_{T_1}^1)^\gamma,1-(1-\eta_{T_1}^2)^\gamma,...,1-(1-\eta_{T_1}^p)^\gamma\rangle(\gamma\ge0)$
4. $T_1^\gamma=\langle(\epsilon_1^\gamma,\epsilon_2^\gamma,\epsilon_3^\gamma,\epsilon_4^\gamma);(\eta_{T_1}^1)^\gamma,(\eta_{T_1}^2)^\gamma,...,(\eta_{T_1}^P)^\gamma\rangle(\gamma\ge0)$

Definition 2.8 *[15] Let* $T_1 = \langle(\epsilon_1, \epsilon_2, \epsilon_3, \epsilon_4); \eta_{T_1}^1, \eta_{T_1}^2, ..., \eta_{T_1}^P\rangle$, $T_2 = \langle(\rho_1, \rho_2, \rho_3, \rho_4); \eta_{T_2}^1, \eta_{T_2}^2, ..., \eta_{T_2}^P\rangle$ *be two TFM-numbers. The followings hold:*

1. If $\epsilon_1 < \rho_1$, $\epsilon_2 < \rho_2$, $\epsilon_3 < \rho_3$, $\epsilon_4 < \rho_4$, and $\eta_{T_1}^1 < \eta_{T_2}^1$, $\eta_{T_1}^2 < \eta_{T_2}^2$, ..., $\eta_{T_1}^P < \eta_{T_2}^P$, then $T_1 < T_2$.
2. If $\epsilon_1 > \rho_1$, $\epsilon_2 > \rho_2$, $\epsilon_3 > \rho_3$, $\epsilon_4 > \rho_4$, and $\eta_{T_1}^1 > \eta_{T_2}^1$, $\eta_{T_1}^2 > \eta_{T_2}^2$, ..., $\eta_{T_1}^P > \eta_{T_2}^P$, then $T_1 > T_2$.
3. If $\epsilon_1 = \rho_1$, $\epsilon_2 = \rho_2$, $\epsilon_3 = \rho_3$, $\epsilon_4 = \rho_4$, and $\eta_{T_1}^1 = \eta_{T_2}^1$, $\eta_{T_1}^2 = \eta_{T_2}^2$, ..., $\eta_{T_1}^P = \eta_{T_2}^P$, then $T_1 = T_2$.

Definition 2.9 *[29] Let* $T_1 = \langle(\epsilon_1, \epsilon_2, \epsilon_3, \epsilon_4); \eta_{T_1}^1, \eta_{T_1}^2, ..., \eta_{T_1}^P\rangle$, $T_2 = \langle(\rho_1, \rho_2, \rho_3, \rho_4); \eta_{T_2}^1, \eta_{T_2}^2, ..., \eta_{T_2}^P\rangle$ *be two TFM-numbers. Then, the Hamming distance between* T_1 *and* T_2 *is defined as follows:*

$$d(T_1,T_2) = \frac{1}{8p}\sum_{i=1}^{p}(|(1+\eta_{T_1}^i)\epsilon_1 - (1+\eta_{T_2}^i)\rho_1| + |(1+\eta_{T_1}^i)\epsilon_2 - (1+\eta_{T_2}^i)\rho_2| + \\ |(1+\eta_{T_1}^i)\epsilon_3 - (1+\eta_{T_2}^i)\rho_3| + |(1+\eta_{T_1}^i)\epsilon_4 - (1+\eta_{T_2}^i)\rho_4|)$$

Definition 2.10 *[9] Let* $T = \langle(\epsilon_1, \epsilon_2, \epsilon_3, \epsilon_4); \eta_T^1, \eta_T^2, ..., \eta_T^P\rangle$ *be a TFM-number. Value of T based on centroid point denoted by* $Val(T)$ *is computed as:*

$$Val(T) = \frac{\sum_{i=1}^{P} deff(T_i)}{P}$$

where

$$deff(T_i) = \frac{\int_{\epsilon_1}^{\epsilon_2} x\frac{(x-\epsilon_1)\eta_T^i}{(\epsilon_2-\epsilon_1)}dx + \int_{\epsilon_2}^{\epsilon_3} x\eta_T^i dx + \int_{\epsilon_3}^{\epsilon_4} x\frac{(\epsilon_4-x)\eta_T^i}{(\epsilon_4-\epsilon_3)}dx}{\int_{\epsilon_1}^{\epsilon_2} \frac{(x-\epsilon_1)\eta_T^i}{(\epsilon_2-\epsilon_1)}dx + \int_{\epsilon_2}^{\epsilon_3} \eta_T^i dx + \int_{\epsilon_3}^{\epsilon_4} \frac{(\epsilon_4-x)\eta_T^i}{(\epsilon_4-\epsilon_3)}dx}, (i \in I_p)$$

2.1 TFM Bonferroni Means

Here, we give two definitions called trapezoidal fuzzy multi Bonferroni arithmetic mean and trapezoidal fuzzy multi Bonferroni geometric mean which will be base of rest of the paper.

Definition 2.11 *[14] Let* $T_i = \langle(\epsilon_i, \rho_i, \delta_i, \gamma_i); \eta_{T_i}^1, \eta_{T_i}^2, ..., \eta_{T_i}^P\rangle$ $(i \in I_n)$ *be a collection of TFM-numbers. For any* $p, q > 0$*, if*

$$TFMBAM^{(p,q)}(T_1,T_2,...,T_n)=\left(\frac{1}{n.(n-1)}\bigoplus_{\substack{i,j=1\\ i\neq j}}^{n}(T_i^{p}\otimes T_j^{q})\right)^{\frac{1}{p+q}}$$

then $TFMBAM^{(p,q)}$ is called trapezoidal fuzzy multi Bonferroni arithmetic mean.

Definition 2.12 *[14] Let $T_i=\langle(\epsilon_i,\rho_i,\delta_i,\gamma_i);\eta_{T_i}^1,\eta_{T_i}^2,...,\eta_{T_i}^P\rangle$ $(i\in I_n)$ be a collection of TFM-numbers, $\upsilon=(\upsilon_1,\upsilon_2,...,\upsilon_n)^T$ their weight vector, where υ_i indicates the importance degree of T_i, satisfying $\upsilon_i>0$ ($i\in I_n$), and $\sum_{i=1}^{n}\upsilon_i=1$. For any $p,q>0$, If*

$$TFMBAM_{\upsilon}^{(p,q)}(T_1,T_2,...,T_n)=\left(\frac{1}{n.(n-1)}\bigoplus_{i,j=1 i\neq j}^{n}((\upsilon_i.T_i^{p})\otimes(\upsilon_j.T_j^{q}))^{\frac{1}{p+q}}\right)$$

then $TFMBAM_{\upsilon}$ is called trapezoidal fuzzy multi weighted Bonferroni mean.

Definition 2.13 *[14] Let $T_i=\langle(\epsilon_i,\rho_i,\delta_i,\gamma_i);\eta_{T_i}^1,\eta_{T_i}^2,...,\eta_{T_i}^P\rangle$ $(i\in I_n)$ be a collection of TFM-numbers. For any $p,q>0$. If*

$$TFMBGM^{(p,q)}(T_1,T_2,...,T_n)=\frac{1}{p+q}\bigotimes_{\substack{i,j=1\\ i\neq j}}^{n}((p.T_i\oplus q.T_j))^{\frac{1}{n.(n-1)}} \quad (1)$$

then, $TFMBGM^{(p,q)}$ is called trapezoidal fuzzy multi Bonferroni geometric mean operator.

Considering the weight vector of the aggregated arguments, the weighted form is also proposed:

Definition 2.14 *[14] Let Let $T_i=\langle(\epsilon_i,\rho_i,\delta_i,\gamma_i);\eta_{T_i}^1,\eta_{T_i}^2,...,\eta_{T_i}^P\rangle$ $(i\in I_n)$ be a collection of TFM-numbers, $\upsilon=(\upsilon_1,\upsilon_2,...,\upsilon_n)^T$ their weight vector, where υ_i indicates the importance degree of T_i, satisfying $\upsilon_i>0$ ($i\in I_n$), and $\sum_{i=1}^{n}\upsilon_i=1$. For any $p,q>0$, If*

$$TFMBGM_{\upsilon}^{(p,q)}(T_1,T_2,...,T_n)=\frac{1}{p+q}\left(\bigotimes_{i,j=1,i\neq j}^{n}((p.F_i^{w_i}\oplus q.F_j^{\upsilon_j}))^{\frac{2}{n.(n-1)}}\right.$$

then, $TFMBGM_{\upsilon}^{(p,q)}$ is called trapezoidal fuzzy weighted Bonferroni geometric mean.

2.2 CRITIC method for determining of weight of criteria

CRITIC method which was firstly introduced by Diakoulaki et al. [12] helps to decision makers to determine the weight of each criteria by means of values in the decision matrix.

Let $M = \{M_1, M_2, ..., M_m\}$ be set of alternatives, $C = \{c_1, c_2, ..., c_n\}$ be set of attributes and x_{ij} be a values of M_i alternative based on c_j attribute. Then, algorithm of CRITIC method is given as;

Algorithm:

Step 1 Construct the decision matrix $(x_{ij})_{mxn}$ according to decision makers' preferences as;

$$(x_{ij})_{mxn} = \begin{pmatrix} x_{11} & x_{12} & \cdots & x_{1x} \\ x_{21} & x_{22} & \cdots & x_{2x} \\ \vdots & \vdots & \ddots & \vdots \\ x_{m1} & x_{m2} & \cdots & x_{mn} \end{pmatrix}$$

where x_{ij} $(i \in I_m$ and $j \in I_n)$ represents the real numbers.

Step 2 Find normalized decision matrix $(\overline{x}_{ij})_{mxn}$ of the decision matrix $(x_{ij})_{mxn}$ as;

$$(\overline{x}_{ij})_{mxn} = \begin{pmatrix} \overline{x}_{11} & \overline{x}_{12} & \cdots & \overline{x}_{1\overline{x}} \\ \overline{x}_{21} & \overline{x}_{22} & \cdots & \overline{x}_{2\overline{x}} \\ \vdots & \vdots & \ddots & \vdots \\ \overline{x}_{m1} & \overline{x}_{m2} & \cdots & \overline{x}_{mn} \end{pmatrix}$$

where

$$\overline{x}_{ij} = \begin{cases} \dfrac{x_{ij} - \min\limits_{k \in I_n}\{x_{ik}\}}{\max\limits_{k \in I_n}\{x_{ik}\} - \min\limits_{k \in I_n}\{x_{ik}\}}, & \textit{for benefit attribute} \\[2ex] \dfrac{\max\limits_{k \in I_n}\{x_{ik}\} - x_{ij}}{\max\limits_{k \in I_n}\{x_{ik}\} - \min\limits_{k \in I_n}\{x_{ik}\}}, & \textit{for cost attribute} \end{cases}$$

such that and $j \in I_n$.

Step 3. Construct the relation-coefficient matrix $(r_{ij})_{n \times n}$ of the normalized decision matrix $(\overline{x}_{ij})_{mxn}$ as;

$$(r_{ij})_{n\times n} = \begin{pmatrix} r_{11} & r_{12} & \cdots & r_{1n} \\ r_{21} & r_{22} & \cdots & r_{2n} \\ \vdots & \vdots & \ddots & \vdots \\ r_{n1} & r_{n2} & \cdots & r_{nn} \end{pmatrix}$$

where

$$r_{jk} = \frac{\sum_{i=1}^{m}(\overline{x}_{ij} - \tilde{x}_j).(\overline{x}_{ik} - \tilde{x}_k)}{\sqrt{\sum_{i=1}^{m}(\overline{x}_{ij} - \tilde{x}_j)^2.\sum_{i=1}^{m}(\overline{x}_{ik} - \tilde{x}_k)^2}}$$

such that $j,k \in I_n$. Here, $\tilde{x}_j$ and $\tilde{x}_k$ are arithmetic means of $\overline{x}_{ij}$ and $\overline{x}_{ik}$, respectively.

Step 4. Find c_j to get information from contrast and conflicts in *j*th criterion as;

$$c_j = \sigma_j \sum_{k=1}^{n}(1 - r_{jk})$$

where

$$\sigma_j = \sqrt{\frac{\sum_{i=1}^{m}(\overline{x}_{ij} - \tilde{x}_j)^2}{m-1}}$$

and $\tilde{x}_j$ is arithmetic mean of $\overline{x}_{ij}$ for $j \in I_n$.

Step 5. Compute weights, denoted by w_j $(j \in I_n)$, of *j*th criterion c_j as;

$$w_j = \frac{c_j}{\sum_{j=1}^{n} c_j}, \quad (j \in I_n)$$

2.3 Generalized Bonferroni Means

By taking the correlations of any three aggregated elements instead of any two, Beliakov et al. [2] defined the Bonferroni means.

Definition 2.15 *[2] Let $p,q,r \geq 0$ and F_i $(i \in I_n)$ be a collection of nonnegative numbers. If*

$$GBM^{(p,q,r)}(F_1, F_2, ..., F_n) = \left(\frac{1}{n.(n-1).(n-2)} \sum_{\substack{i,j,k=1 \\ i\neq j\neq k}}^{n} F_i^{\,p} F_j^{\,q} F_k^{\,r}\right)^{\frac{1}{p+q+r}} \qquad (2)$$

then $GBM^{(p,q,r)}$ is called a generalized Bonferroni mean (GBM).

Definition 2.16 *[35] Let* $p,q,r \geq 0$, $F_i \quad (i \in I_n)$ *be a collection of nonnegative numbers,* $\upsilon = (\upsilon_1, \upsilon_2, ..., \upsilon_n)^T$ *be the weight vector of* F_i *such that* $\upsilon_i > 0 \ (i \in I_n)$ *and* $\sum_{i=1}^{n} \upsilon_i = 1$. *If*

$$GWBM^{(p,q,r)}(F_1, F_2, ..., F_n) = (\sum_{i,j,k=1}^{n} \upsilon_i \upsilon_j \upsilon_k (F_i^p F_j^q F_k^r))^{\frac{1}{p+q+r}} \tag{3}$$

then $GWBM^{(p,q,r)}$ is called a generalized weighed Bonferroni mean (GWBM).

Note 2.17 *[35] if* $\upsilon = (1/n, 1/n, ..., 1/n)^T$ *, then the GWBM converted into the following:*

$$RBM^{(p,q,r)}(F_1, F_2, ..., F_n) = (\frac{1}{n^3} \sum_{i,j,k=1}^{n} F_i^p F_j^q F_k^r)^{\frac{1}{p+q+r}} \tag{4}$$

which is called the revised Bonferroni mean (RBM).

Theorem 2.18 *[35] Let* $p,q,r \geq 0$ *and* F_i *and* $G_i \ (i \in I_n)$ *be two collections of nonnegative numbers. GWBM has some properties as follows:*

1. $GWBM^{(p,q,r)}(0, 0, ..., 0) = 0$
2. $GWBM^{(p,q,r)}(F, F, ..., F) = F \quad if \quad F_i = F \ for all i$.
3. $GWBM^{(p,q,r)}(F_1, F_2, ..., F_n) \geq GWBM^{(p,q,r)}(G_1, G_2, ..., G_n)$ i.e., $GWBM^{(p,q,r)}$ is monotonic if
$F_i \geq G_i$ for all i.
4. $\min\{F_i\} \leq GWBM^{(p,q,r)}(F_1, F_2, ..., F_n) \leq \max\{F_i\}$.

Remark 2.19 *[35] We can get some special cases as the change of the parameters:*

1. If r = 0, then the $GWBM^{(p,q,r)}$ converted into the following:

$$\begin{aligned} GWBM^{(p,q,0)}(F_1, F_2, ..., F_n) \quad &= (\sum_{i,j,k=1}^{n} \upsilon_i \upsilon_j \upsilon_k (F_i^p F_j^q))^{\frac{1}{p+q}} \\ &= (\sum_{i,j}^{n} \upsilon_i \upsilon_j (F_i^p F_j^q) \sum_{k=1}^{n} \upsilon_k)^{\frac{1}{p+q}} \\ &= (\sum_{i,j}^{n} \upsilon_i \upsilon_j (F_i^p F_j^q))^{\frac{1}{p+q}} \end{aligned} \tag{5}$$

which we call a weighted Bonferroni mean (WBM).

2. If q = 0 and r = 0, then

$$\begin{aligned} GWBM^{(p,0,0)}(F_1,F_2,...,F_n) &= (\sum_{i,j,k=1}^{n} \upsilon_i \upsilon_j \upsilon_k F_i^{\,p})^{\frac{1}{p}} \\ &= (\sum_{i=1}^{n} \upsilon_i F_i^{\,p} \sum_{j=1}^{n} \upsilon_j \sum_{k=1}^{n} \upsilon_k)^{\frac{1}{p}} \\ &= (\sum_{i}^{n} \upsilon_i F_i^{\,p})^{\frac{1}{p}} \end{aligned} \tag{6}$$

which is the generalized weighted averaging operator ([39]).

Definition 2.20 *[35] Let* $p,q,r \ge 0$, F_i $(i \in I_n)$ *be a collection of nonnegative numbers,* $\upsilon = (\upsilon_1, \upsilon_2, ..., \upsilon_n)^T$ *be the weight vector of* F_i *such that* $\upsilon_i > 0$ $(i \in I_n)$ *and* $\sum_{i=1}^{n} \upsilon_i = 1$. *If*

$$GWBGM^{(p,q,r)}(F_1,F_2,...,F_n) = \frac{1}{p+q+r} \bigotimes_{i,j,k=1}^{n} (pF_i \oplus qF_j \oplus rF_k)^{\upsilon_i \upsilon_j \upsilon_k} \tag{7}$$

then $GWBGM^{(p,q,r)}$ is called a generalized weighted Bonferroni Geometric mean (GWBGM).

Theorem 2.21 *[35] Let* $p,q,r \ge 0$ *and* F_i *and* G_i $(i \in I_n)$ *be two collections of nonnegative numbers.* $GWBGM^{(p,q,r)}$ *has some properties as follows:*

1. $GWBGM^{(p,q,r)}(0,0,...,0) = 0$
2. $GWBGM^{(p,q,r)}(F,F,...,F) = F$ *if* $F_i = F$ *for all i*.
3. $GWBGM^{(p,q,r)}(F_1,F_2,...,F_n) \ge GWBGM^{(p,q,r)}(G_1,G_2,...,G_n)$ i.e., $GWBGM^{(p,q,r)}$ is monotonic if $F_i \ge G_i$ for all i.
4. $\min\{F_i\} \le GWBGM^{(p,q,r)}(F_1,F_2,...,F_n) \le \max\{F_i\}$.

In addition, we can get some special cases as the change of the parameters:

1. If r=0 then $GWBGM^{(p,q,r)}$ converted into:

$$\begin{aligned} GWBGM^{(p,q,0)}(F_1,F_2,...,F_n) &= \frac{1}{p+q} \bigotimes_{i,j,k=1}^{n} (pF_i \oplus qF_j)^{\upsilon_i \upsilon_j \upsilon_k} \\ &= \frac{1}{p+q} \bigotimes_{i,j,k=1}^{n} (pF_i \oplus qF_j)^{\upsilon_i \upsilon_j \sum_{k=1}^{n} \upsilon_k} \\ &= \frac{1}{p+q} \bigotimes_{i,j=1}^{n} (pF_i \oplus qF_j)^{\upsilon_i \upsilon_j} \end{aligned} \tag{8}$$

which is called a weighted Bonferroni geometric mean (WBGM).

2. If q=r=0, then

$$\begin{aligned} GWBGM^{(p,0,0)}(F_1,F_2,...,F_n) &= \frac{1}{p}\bigotimes_{i,j,k=1}^{n}(pF_i)^{\upsilon_i\upsilon_j\upsilon_k} \\ &= \frac{1}{p}\bigotimes_{i,j,k=1}^{n}(pF_i)^{\upsilon_i\sum_{j=1}^{n}\upsilon_j\sum_{k=1}^{n}\upsilon_k} \\ &= \bigotimes_{i=1}^{n}F_i^{\upsilon_i} \end{aligned} \tag{9}$$

which is the usual geometric mean.

The aggregation methods given above can only cope with the situations in which the elements are given as nonnegative numbers. If the arguments given in other forms such as the trapezoidal fuzzy multi numbers, the methods will be formed as follows:

3 Generalized Weighted Bonferroni Mean of TFM-Numbers

Both the $TFMBAM_{\upsilon}^{(p,q)}$ and the $TFMBGM_{\upsilon}^{(p,q)}$ which are given in Definitions 2.12 and 2.14 only tackle with the conditions in which there are correlations between any two aggregated elements, but not the conditions where there are connections among any three aggregated elements. In order to get rid of this drawback, motivated by Definition 2.16, we extended the generalized Bonferroni mean to trapezoidal fuzzy multi environment and proposed the following definition:

Definition 3.1 *Let* $F_i = \langle(\epsilon_i,\rho_i,\delta_i,\gamma_i);\eta_{F_i}^1,\eta_{F_i}^2,...,\eta_{F_i}^P\rangle$ *be a collection of TFM-numbers,* $\upsilon=(\upsilon_1,\upsilon_2,...,\upsilon_n)^T$ *their weight vector, where* υ_i *indicates the importance degree of* F_i*, satisfying* $\upsilon_i>0$, *(*$i\in I_n$*) and* $\sum_{i=1}^{n}\upsilon_i=1$. *For any* $p,q,r>0$*, If*

$$TFMGBAM_{\upsilon}^{(p,q,r)}(F_1,F_2,...,F_n)=\Big(\bigoplus_{i,j,k=1}^{n}\upsilon_i\upsilon_j\upsilon_k(F_i^{\,p}\otimes F_j^{\,q}\otimes F_k^{\,r})\Big)^{\frac{1}{p+q+r}} \tag{10}$$

then $TFMGBAM_{\upsilon}^{(p,q,r)}$ is called trapezoidal fuzzy multi generalized weighted Bonferroni arithmetic mean.

Remark 3.2 *Let* $F_i = \langle(\epsilon_i,\rho_i,\delta_i,\gamma_i);\eta_{F_i}^1,\eta_{F_i}^2,...,\eta_{F_i}^P\rangle$ $(i\in I_n)$ *be a collection of TFM-numbers,* $\upsilon=(\upsilon_1,\upsilon_2,...,\upsilon_n)^T$ *their weight vector, where* υ_i *indicates the importance degree of* F_i*, satisfying* $\upsilon_i>0$, *(*$i\in I_n$*) and* $\sum_{i=1}^{n}\upsilon_i=1$. *By considering different values of* p*,* q *and* r*, several specific cases of the* $TFMGBAM_{\upsilon}^{(p,q,r)}$ *are obtained as follows:*

1. Especially, if $r \to 0$, then the $TFMGBAM_{\upsilon}^{(p,q,r)}$ converted into:

$$\begin{aligned}\lim_{r\to 0} TFMGBAM_{\upsilon}^{(p,q,r)}(F_1,F_2,...,F_n) &= (\bigoplus_{i,j,k=1}^{n} \upsilon_i\upsilon_j\upsilon_k(F_i^{\,p} \otimes F_j^{\,q}))^{\frac{1}{p+q}} \\ &= ((\sum_{k=1}^{n}\upsilon_k)\bigoplus_{i,j=1}^{n}\upsilon_i\upsilon_j(F_i^{\,p} \otimes F_j^{\,q}))^{\frac{1}{p+q}} \\ &= (\bigoplus_{i,j=1}^{n}\upsilon_i\upsilon_j(F_i^{\,p} \otimes F_j^{\,q}))^{\frac{1}{p+q}}\end{aligned} \tag{11}$$

which is called an trapezoidal fuzzy multi weighted Bonferroni mean ($TFMBAM_{\upsilon}$) which is given in Definition 2.12.

2. Especially, if $q \to 0$ and $r \to 0$ then the $TFMGBAM_{\upsilon}^{(p,q,r)}$ converted into:

$$\begin{aligned}\lim_{q\to 0}(\lim_{r\to 0} TFMGBAM_{\upsilon}^{(p,q,r)}(F_1,F_2,...,F_n)) &= \lim_{q\to 0}((\bigoplus_{i,j,k=1}^{n} \upsilon_i\upsilon_j\upsilon_k(F_i^{\,p} \otimes F_j^{\,q}))^{\frac{1}{p+q}}) \\ &= (\bigoplus_{i,j,k=1}^{n} \upsilon_i\upsilon_j\upsilon_k F_i^{\,p})^{\frac{1}{p}} \\ &= (\sum_{j=1}^{n}\upsilon_j\sum_{k=1}^{n}\upsilon_k\bigoplus_{i=1}^{n}\upsilon_i F_i^{\,p})^{\frac{1}{p}} \\ &= (\bigoplus_{i=1}^{n}\upsilon_i F_i^{\,p})^{\frac{1}{p}}\end{aligned} \tag{12}$$

which is the generalized trapezoidal fuzzy weighted mean.

Based on the operational laws of TFM-numbers given in Definition 2.7, we can give the following theorem:
In the following theorem, algebraic product and algebraic sum are used in computations.

Theorem 3.3 *Let* $F_i = \langle(\epsilon_i,\rho_i,\delta_i,\gamma_i);\eta_{F_i}^{1},\eta_{F_i}^{2},...,\eta_{F_i}^{P}\rangle$ $(i \in I_n)$ *be a collection of TFM-numbers and* $p,q,r > 0$, *then the aggregated value by using the* $TFMGBAM_{\upsilon}^{(p,q,r)}$ *is also a TFM-number and computed as follows:*

$$\begin{aligned}
TFMGBAM_{\upsilon}^{(p,q,r)}(F_1,F_2,...,F_n) &= (\bigoplus_{i,j,k=1}^{n}\upsilon_i\upsilon_j\upsilon_k(F_i^{\,p}\otimes F_j^{\,q}\otimes F_k^{\,r}))^{\frac{1}{p+q+r}} \\
&= \langle((1-\prod_{i,j,k=1}^{n}(1-(\epsilon_i^{\,p}\epsilon_j^{\,q}\epsilon_k^{\,r}))^{\upsilon_i\upsilon_j\upsilon_k})^{\frac{1}{p+q+r}}, \\
&(1-\prod_{i,j,k=1}^{n}(1-(\rho_i^{\,p}\rho_j^{\,q}\rho_k^{\,r}))^{\upsilon_i\upsilon_j\upsilon_k})^{\frac{1}{p+q+r}}, \\
&(1-\prod_{i,j,k=1}^{n}(1-(\delta_i^{\,p}\delta_j^{\,q}\delta_k^{\,r}))^{\upsilon_i\upsilon_j\upsilon_k})^{\frac{1}{p+q+r}}, \\
&(1-\prod_{i,j,k=1}^{n}(1-(\gamma_i^{\,p}\gamma_j^{\,q}\gamma_k^{\,r}))^{\upsilon_i\upsilon_j\upsilon_k})^{\frac{1}{p+q+r}}; \\
&(1-\prod_{i,j,k=1}^{n}(1-(\eta_{F_i}^{1})^{p}(\eta_{F_j}^{1})^{q}(\eta_{F_k}^{1})^{r})^{\upsilon_i\upsilon_j\upsilon_k})^{\frac{1}{p+q+r}}, \\
&(1-\prod_{i,j,k=1}^{n}(1-(\eta_{F_i}^{2})^{p}(\eta_{F_j}^{2})^{q}(\eta_{F_k}^{2})^{r})^{\upsilon_i\upsilon_j\upsilon_k})^{\frac{1}{p+q+r}},..., \\
&(1-\prod_{i,j,k=1}^{n}(1-(\eta_{F_i}^{P})^{p}(\eta_{F_j}^{P})^{q}(\eta_{F_k}^{P})^{r})^{\upsilon_i\upsilon_j\upsilon_k})^{\frac{1}{p+q+r}}\rangle
\end{aligned} \tag{13}$$

Proof: To proof the Equation (13), we need to use mathematical induction on n.

Firstly, by the operational rules given in Definition 2.7, we get that:

$$\begin{aligned}
\upsilon_i\upsilon_j\upsilon_k(F_i^{\,p}\otimes F_j^{\,q}\otimes F_k^{\,r}) = \langle(1-(1-\epsilon_i^{\,p}\epsilon_j^{\,q}\epsilon_k^{\,r})^{\upsilon_i\upsilon_j\upsilon_k},(1-(1-\rho_i^{\,p}\rho_j^{\,q}\rho_k^{\,r})^{\upsilon_i\upsilon_j\upsilon_k}, \\
(1-(1-\delta_i^{\,p}\delta_j^{\,q}\delta_k^{\,r})^{\upsilon_i\upsilon_j\upsilon_k},(1-(1-\gamma_i^{\,p}\gamma_j^{\,q}\gamma_k^{\,r})^{\upsilon_i\upsilon_j\upsilon_k}); \\
1-(1-(\eta_{F_i}^{1})^{p}(\eta_{F_j}^{1})^{q}(\eta_{F_k}^{1})^{r})^{\upsilon_i\upsilon_j\upsilon_k}, \\
1-(1-(\eta_{F_i}^{2})^{p}(\eta_{F_j}^{2})^{q}(\eta_{F_k}^{2})^{r})^{\upsilon_i\upsilon_j\upsilon_k},\ldots, \\
1-(1-(\eta_{F_i}^{P})^{p}(\eta_{F_j}^{P})^{q}(\eta_{F_k}^{P})^{r})^{\upsilon_i\upsilon_j\upsilon_k}\rangle
\end{aligned}$$

1. When $n=2$, we can get:

$$\begin{aligned}
\bigoplus_{i,j,k=1}^{2}\upsilon_i\upsilon_j\upsilon_k(F_i^{\,p}\otimes F_j^{\,q}\otimes F_k^{\,r}) = \upsilon_1\upsilon_1\upsilon_1(F_1^{\,p}\otimes F_1^{\,q}\otimes F_1^{\,r}) \\
\oplus\upsilon_1\upsilon_1\upsilon_2(F_1^{\,p}\otimes F_1^{\,q}\otimes F_2^{\,r}) \\
\oplus\upsilon_1\upsilon_2\upsilon_1(F_1^{\,p}\otimes F_2^{\,q}\otimes F_1^{\,r})
\end{aligned}$$

$$
\begin{aligned}
&\oplus\upsilon_2\upsilon_1\upsilon_1(F_2^{\,p}\otimes F_1^{\,q}\otimes F_1^{\,r})\\
&\oplus\upsilon_1\upsilon_2\upsilon_2(F_1^{\,p}\otimes F_2^{\,q}\otimes F_2^{\,r})\\
&\oplus\upsilon_2\upsilon_1\upsilon_2(F_2^{\,p}\otimes F_1^{\,q}\otimes F_2^{\,r})\\
&\oplus\upsilon_2\upsilon_2\upsilon_1(F_2^{\,p}\otimes F_2^{\,q}\otimes F_1^{\,r})\\
&\oplus\upsilon_2\upsilon_2\upsilon_2(F_2^{\,p}\otimes F_2^{\,q}\otimes F_2^{\,r})\\
&=\langle(1-(1-\epsilon_1^{p}\epsilon_1^{q}\epsilon_1^{r})^{\upsilon_1\upsilon_1\upsilon_1},(1-(1-\rho_1^{p}\rho_1^{q}\rho_1^{r})^{\upsilon_1\upsilon_1\upsilon_1},\\
&\quad(1-(1-\delta_1^{p}\delta_1^{q}\delta_1^{r})^{\upsilon_1\upsilon_1\upsilon_1},(1-(1-\gamma_1^{p}\gamma_1^{q}\gamma_1^{r})^{\upsilon_1\upsilon_1\upsilon_1});\\
&\quad1-(1-(\eta_{F_1}^{1})^{p}(\eta_{F_1}^{1})^{q}(\eta_{F_1}^{1})^{r})^{\upsilon_1\upsilon_1\upsilon_1},\\
&\quad1-(1-(\eta_{F_1}^{2})^{p}(\eta_{F_1}^{2})^{q}(\eta_{F_1}^{2})^{r})^{\upsilon_1\upsilon_1\upsilon_1},\ldots,\\
&\quad1-(1-(\eta_{F_1}^{P})^{p}(\eta_{F_1}^{P})^{q}(\eta_{F_1}^{P})^{r})^{\upsilon_1\upsilon_1\upsilon_1}\rangle\\
&\quad\oplus\langle(1-(1-\epsilon_1^{p}\epsilon_1^{q}\epsilon_2^{r})^{\upsilon_1\upsilon_1\upsilon_2},(1-(1-\rho_1^{p}\rho_1^{q}\rho_2^{r})^{\upsilon_1\upsilon_1\upsilon_2},\\
&\quad(1-(1-\delta_1^{p}\delta_1^{q}\delta_2^{r})^{\upsilon_1\upsilon_1\upsilon_2},(1-(1-\gamma_1^{p}\gamma_1^{q}\gamma_2^{r})^{\upsilon_1\upsilon_1\upsilon_2});\\
&\quad1-(1-(\eta_{F_1}^{1})^{p}(\eta_{F_1}^{1})^{q}(\eta_{F_2}^{1})^{r})^{\upsilon_1\upsilon_1\upsilon_2},\\
&\quad1-(1-(\eta_{F_1}^{2})^{p}(\eta_{F_1}^{2})^{q}(\eta_{F_2}^{2})^{r})^{\upsilon_1\upsilon_1\upsilon_2},\ldots,\\
&\quad1-(1-(\eta_{F_1}^{P})^{p}(\eta_{F_1}^{P})^{q}(\eta_{F_2}^{P})^{r})^{\upsilon_1\upsilon_1\upsilon_2}\rangle\\
&\quad\oplus\langle(1-(1-\epsilon_1^{p}\epsilon_2^{q}\epsilon_1^{r})^{\upsilon_1\upsilon_2\upsilon_1},(1-(1-\rho_1^{p}\rho_2^{q}\rho_1^{r})^{\upsilon_1\upsilon_2\upsilon_1},\\
&\quad(1-(1-\delta_1^{p}\delta_2^{q}\delta_1^{r})^{\upsilon_1\upsilon_2\upsilon_1},(1-(1-\gamma_1^{p}\gamma_2^{q}\gamma_1^{r})^{\upsilon_1\upsilon_2\upsilon_1});\\
&\quad1-(1-(\eta_{F_1}^{1})^{p}(\eta_{F_2}^{1})^{q}(\eta_{F_1}^{1})^{r})^{\upsilon_1\upsilon_2\upsilon_1},\\
&\quad1-(1-(\eta_{F_1}^{2})^{p}(\eta_{F_2}^{2})^{q}(\eta_{F_1}^{2})^{r})^{\upsilon_1\upsilon_2\upsilon_1},\ldots,\\
&\quad1-(1-(\eta_{F_1}^{P})^{p}(\eta_{F_2}^{P})^{q}(\eta_{F_1}^{P})^{r})^{\upsilon_1\upsilon_2\upsilon_1}\rangle\\
&\quad\oplus\langle(1-(1-\epsilon_2^{p}\epsilon_1^{q}\epsilon_1^{r})^{\upsilon_2\upsilon_1\upsilon_1},(1-(1-\rho_2^{p}\rho_1^{q}\rho_1^{r})^{\upsilon_2\upsilon_1\upsilon_1},\\
&\quad(1-(1-\delta_2^{p}\delta_1^{q}\delta_1^{r})^{\upsilon_2\upsilon_1\upsilon_1},(1-(1-\gamma_2^{p}\gamma_1^{q}\gamma_1^{r})^{\upsilon_2\upsilon_1\upsilon_1});\\
&\quad1-(1-(\eta_{F_2}^{1})^{p}(\eta_{F_1}^{1})^{q}(\eta_{F_1}^{1})^{r})^{\upsilon_2\upsilon_1\upsilon_1},\\
&\quad1-(1-(\eta_{F_2}^{2})^{p}(\eta_{F_1}^{2})^{q}(\eta_{F_1}^{2})^{r})^{\upsilon_2\upsilon_1\upsilon_1},\ldots,\\
&\quad1-(1-(\eta_{F_2}^{P})^{p}(\eta_{F_1}^{P})^{q}(\eta_{F_1}^{P})^{r})^{\upsilon_2\upsilon_1\upsilon_1}\rangle\\
&\quad\oplus\langle(1-(1-\epsilon_1^{p}\epsilon_2^{q}\epsilon_2^{r})^{\upsilon_1\upsilon_2\upsilon_2},(1-(1-\rho_1^{p}\rho_2^{q}\rho_2^{r})^{\upsilon_1\upsilon_2\upsilon_2},\\
&\quad(1-(1-\delta_1^{p}\delta_2^{q}\delta_2^{r})^{\upsilon_1\upsilon_2\upsilon_2},(1-(1-\gamma_1^{p}\gamma_2^{q}\gamma_2^{r})^{\upsilon_1\upsilon_2\upsilon_2});\\
&\quad1-(1-(\eta_{F_1}^{1})^{p}(\eta_{F_2}^{1})^{q}(\eta_{F_2}^{1})^{r})^{\upsilon_1\upsilon_2\upsilon_2},\\
&\quad1-(1-(\eta_{F_1}^{2})^{p}(\eta_{F_2}^{2})^{q}(\eta_{F_2}^{2})^{r})^{\upsilon_1\upsilon_2\upsilon_2},\ldots,
\end{aligned}
$$

$$
\begin{aligned}
&1-(1-(\eta_{F_1}^{P})^{p}(\eta_{F_2}^{P})^{q}(\eta_{F_2}^{P})^{r})^{\upsilon_1\upsilon_2\upsilon_2}\rangle\\
&\oplus\langle(1-(1-\epsilon_2^{p}\epsilon_1^{q}\epsilon_2^{r})^{\upsilon_2\upsilon_1\upsilon_2},(1-(1-\rho_2^{p}\rho_1^{q}\rho_2^{r})^{\upsilon_2\upsilon_1\upsilon_2},\\
&(1-(1-\delta_2^{p}\delta_1^{q}\delta_2^{r})^{\upsilon_2\upsilon_1\upsilon_2},(1-(1-\gamma_2^{p}\gamma_1^{q}\gamma_2^{r})^{\upsilon_2\upsilon_1\upsilon_2});\\
&1-(1-(\eta_{F_2}^{1})^{p}(\eta_{F_1}^{1})^{q}(\eta_{F_2}^{1})^{r})^{\upsilon_2\upsilon_1\upsilon_2},\\
&1-(1-(\eta_{F_2}^{2})^{p}(\eta_{F_1}^{2})^{q}(\eta_{F_2}^{2})^{r})^{\upsilon_2\upsilon_1\upsilon_2},\ldots,\\
&1-(1-(\eta_{F_2}^{P})^{p}(\eta_{F_1}^{P})^{q}(\eta_{F_2}^{P})^{r})^{\upsilon_2\upsilon_1\upsilon_2}\rangle\\
&\oplus\langle(1-(1-\epsilon_2^{p}\epsilon_2^{q}\epsilon_1^{r})^{\upsilon_2\upsilon_2\upsilon_1},(1-(1-\rho_2^{p}\rho_2^{q}\rho_1^{r})^{\upsilon_2\upsilon_2\upsilon_1},\\
&(1-(1-\delta_2^{p}\delta_2^{q}\delta_1^{r})^{\upsilon_2\upsilon_2\upsilon_1},(1-(1-\gamma_2^{p}\gamma_2^{q}\gamma_1^{r})^{\upsilon_2\upsilon_2\upsilon_1});\\
&1-(1-(\eta_{F_2}^{1})^{p}(\eta_{F_2}^{1})^{q}(\eta_{F_1}^{1})^{r})^{\upsilon_2\upsilon_2\upsilon_1},\\
&1-(1-(\eta_{F_2}^{2})^{p}(\eta_{F_2}^{2})^{q}(\eta_{F_1}^{2})^{r})^{\upsilon_2\upsilon_2\upsilon_1},\ldots,\\
&\oplus\langle(1-(1-\epsilon_2^{p}\epsilon_2^{q}\epsilon_2^{r})^{\upsilon_2\upsilon_2\upsilon_2},(1-(1-\rho_2^{p}\rho_2^{q}\rho_2^{r})^{\upsilon_2\upsilon_2\upsilon_2},\\
&(1-(1-\delta_2^{p}\delta_2^{q}\delta_2^{r})^{\upsilon_2\upsilon_2\upsilon_2},(1-(1-\gamma_2^{p}\gamma_2^{q}\gamma_2^{r})^{\upsilon_2\upsilon_2\upsilon_2});\\
&1-(1-(\eta_{F_2}^{1})^{p}(\eta_{F_2}^{1})^{q}(\eta_{F_2}^{1})^{r})^{\upsilon_2\upsilon_2\upsilon_2},\\
&1-(1-(\eta_{F_2}^{2})^{p}(\eta_{F_2}^{2})^{q}(\eta_{F_2}^{2})^{r})^{\upsilon_2\upsilon_2\upsilon_2},\ldots,\\
&1-(1-(\eta_{F_2}^{P})^{p}(\eta_{F_2}^{P})^{q}(\eta_{F_2}^{P})^{r})^{\upsilon_2\upsilon_2\upsilon_2}\rangle\\
&=\langle(1-\prod_{i,j,k=1}^{2}(1-(\epsilon_i^{p}\epsilon_j^{q}\epsilon_k^{r})^{\upsilon_i\upsilon_j\upsilon_k}))^{\frac{1}{p+q+r}},\\
&(1-\prod_{i,j,k=1}^{2}(1-(\rho_i^{p}\rho_j^{q}\rho_k^{r})^{\upsilon_i\upsilon_j\upsilon_k}))^{\frac{1}{p+q+r}},\\
&(1-\prod_{i,j,k=1}^{2}(1-(\delta_i^{p}\delta_j^{q}\delta_k^{r})^{\upsilon_i\upsilon_j\upsilon_k}))^{\frac{1}{p+q+r}},\\
&(1-\prod_{i,j,k=1}^{2}(1-(\gamma_i^{p}\gamma_j^{q}\gamma_k^{r})^{\upsilon_i\upsilon_j\upsilon_k}))^{\frac{1}{p+q+r}};\\
&1-\prod_{i,j,k=1}^{2}(1-(\eta_{F_i}^{1})^{p}(\eta_{F_j}^{1})^{q}(\eta_{F_k}^{1})^{r})^{\upsilon_i\upsilon_j\upsilon_k},\\
&1-\prod_{i,j,k=1}^{2}(1-(\eta_{F_i}^{2})^{p}(\eta_{F_j}^{2})^{q}(\eta_{F_k}^{2})^{r})^{\upsilon_i\upsilon_j\upsilon_k},\ldots,\\
&1-\prod_{i,j,k=1}^{2}(1-(\eta_{F_i}^{P})^{p}(\eta_{F_j}^{P})^{q}(\eta_{F_k}^{P})^{r})^{\upsilon_i\upsilon_j\upsilon_k}\rangle
\end{aligned}
$$

So, when $n=2$, the Equation (13) is right.

2. Suppose when $n = t$, the Equation (13) is right, i.e

$$TFMGBAM_{\upsilon}^{(p,q,r)}(F_1, F_2, ..., F_n) = \left(\bigoplus_{i,j,k=1}^{t} \upsilon_i \upsilon_j \upsilon_k (F_i^{\ p} \otimes F_j^{\ q} \otimes F_k^{\ r}) \right)^{\frac{1}{p+q+r}}$$

$$= \langle (1 - \prod_{i,j,k=1}^{t} (1 - (\epsilon_i^{\ p} \epsilon_j^{\ q} \epsilon_k^{\ r}))^{\upsilon_i \upsilon_j \upsilon_k})^{\frac{1}{p+q+r}},$$

$$(1 - \prod_{i,j,k=1}^{t} (1 - (\rho_i^{\ p} \rho_j^{\ q} \rho_k^{\ r}))^{\upsilon_i \upsilon_j \upsilon_k})^{\frac{1}{p+q+r}},$$

$$(1 - \prod_{i,j,k=1}^{t} (1 - (\delta_i^{\ p} \delta_j^{\ q} \delta_k^{\ r}))^{\upsilon_i \upsilon_j \upsilon_k})^{\frac{1}{p+q+r}},$$

$$(1 - \prod_{i,j,k=1}^{t} (1 - (\gamma_i^{\ p} \gamma_j^{\ q} \gamma_k^{\ r}))^{\upsilon_i \upsilon_j \upsilon_k})^{\frac{1}{p+q+r}}; \tag{14}$$

$$(1 - \prod_{i,j,k=1}^{t} (1 - (\eta_{F_i}^{1})^p (\eta_{F_j}^{1})^q (\eta_{F_k}^{1})^r)^{\upsilon_i \upsilon_j \upsilon_k})^{\frac{1}{p+q+r}}, \ldots,$$

$$(1 - \prod_{i,j,k=1}^{t} (1 - (\eta_{F_i}^{2})^p (\eta_{F_j}^{2})^q (\eta_{F_k}^{2})^r)^{\upsilon_i \upsilon_j \upsilon_k})^{\frac{1}{p+q+r}},$$

$$(1 - \prod_{i,j,k=1}^{t} (1 - (\eta_{F_i}^{P})^p (\eta_{F_j}^{P})^q (\eta_{F_k}^{P})^r)^{\upsilon_i \upsilon_j \upsilon_k})^{\frac{1}{p+q+r}} \rangle$$

then, when $n = t+1$, we have

$$\left(\bigoplus_{i,j,k=1}^{t+1} \upsilon_i \upsilon_j \upsilon_k (F_i^{\ p} \otimes F_j^{\ q} \otimes F_k^{\ r}) \right)^{\frac{1}{p+q+r}} = \left(\bigoplus_{i,j,k=1}^{t} \upsilon_i \upsilon_j \upsilon_k (F_i^{\ p} \otimes F_j^{\ q} \otimes F_k^{\ r}) \right)^{\frac{1}{p+q+r}}$$

$$\oplus \left(\bigoplus_{i=1}^{t} \upsilon_i \upsilon_{t+1} \upsilon_{t+1} (F_i^{\ p} \otimes F_{t+1}^{\ q} \otimes F_{t+1}^{\ r}) \right)^{\frac{1}{p+q+r}}$$

$$\oplus \left(\bigoplus_{j=1}^{t} \upsilon_{t+1} \upsilon_j \upsilon_{t+1} (F_{t+1}^{\ p} \otimes F_j^{\ q} \otimes F_{t+1}^{\ r}) \right)^{\frac{1}{p+q+r}}$$

$$\oplus \left(\bigoplus_{k=1}^{t} \upsilon_{t+1} \upsilon_{t+1} \upsilon_k (F_{t+1}^{\ p} \otimes F_{t+1}^{\ q} \otimes F_k^{\ r}) \right)^{\frac{1}{p+q+r}} \tag{15}$$

$$\oplus \left(\bigoplus_{i,j=1}^{t} \upsilon_i \upsilon_j \upsilon_{t+1} (F_i^{\ p} \otimes F_j^{\ q} \otimes F_{t+1}^{\ r}) \right)^{\frac{1}{p+q+r}}$$

$$\oplus \left(\bigoplus_{i,k=1}^{t} \upsilon_i \upsilon_{t+1} \upsilon_k (F_i^{\ p} \otimes F_{t+1}^{\ q} \otimes F_k^{\ r}) \right)^{\frac{1}{p+q+r}}$$

$$\oplus\left(\bigoplus_{j,k=1}^{t}\upsilon_{t+1}\upsilon_j\upsilon_k(F_{t+1}^p\otimes F_j^q\otimes F_k^r)\right)^{\frac{1}{p+q+r}}$$

$$\oplus\left(\bigoplus_{i,j,k=1}^{t}\upsilon_{t+1}\upsilon_{t+1}\upsilon_{t+1}(F_{t+1}^p\otimes F_{t+1}^q\otimes F_{t+1}^r)\right)^{\frac{1}{p+q+r}}$$

by Definition 3.1 and operational laws in Definition 2.7 we get following equalities;

$$\left(\bigoplus_{i=1}^{t}\upsilon_i\upsilon_{t+1}\upsilon_{t+1}(F_i^p\otimes F_{t+1}^q\otimes F_{t+1}^r)\right)^{\frac{1}{p+q+r}}=\langle((1-\prod_{i=1}^{t}(1-(\epsilon_i^p\epsilon_{t+1}^q\epsilon_{t+1}^r))^{\upsilon_i\upsilon_{t+1}\upsilon_{t+1}})^{\frac{1}{p+q+r}},$$

$$(1-\prod_{i=1}^{t}(1-(\rho_i^p\rho_{t+1}^q\rho_{t+1}^r))^{\upsilon_i\upsilon_{t+1}\upsilon_{t+1}})^{\frac{1}{p+q+r}},$$

$$(1-\prod_{i=1}^{t}(1-(\delta_i^p\delta_{t+1}^q\delta_{t+1}^r))^{\upsilon_i\upsilon_{t+1}\upsilon_{t+1}})^{\frac{1}{p+q+r}},$$

$$(1-\prod_{i=1}^{t}(1-(\gamma_i^p\gamma_{t+1}^q\gamma_{t+1}^r))^{\upsilon_i\upsilon_{t+1}\upsilon_{t+1}})^{\frac{1}{p+q+r}}); \tag{16}$$

$$(1-\prod_{i=1}^{t}\left(1-(\eta_{F_i}^1)^p(\eta_{F_{t+1}}^1)^q(\eta_{F_{t+1}}^1)^r\right)^{\upsilon_i\upsilon_{t+1}\upsilon_{t+1}})^{\frac{1}{p+q+r}},$$

$$(1-\prod_{i=1}^{t}\left(1-(\eta_{F_i}^2)^p(\eta_{F_{t+1}}^2)^q(\eta_{F_{t+1}}^2)^r\right)^{\upsilon_i\upsilon_{t+1}\upsilon_{t+1}})^{\frac{1}{p+q+r}},\ldots,$$

$$(1-\prod_{i=1}^{t}\left(1-(\eta_{F_i}^P)^p(\eta_{F_{t+1}}^P)^q(\eta_{F_{t+1}}^P)^r\right)^{\upsilon_i\upsilon_{t+1}\upsilon_{t+1}})^{\frac{1}{p+q+r}}\rangle$$

,

$$\left(\bigoplus_{j=1}^{t}\upsilon_{t+1}\upsilon_j\upsilon_{t+1}(F_{t+1}^p\otimes F_j^q\otimes F_{t+1}^r)\right)^{\frac{1}{p+q+r}}=\langle((1-\prod_{j=1}^{t}(1-(\epsilon_{t+1}^p\epsilon_j^q\epsilon_{t+1}^r))^{\upsilon_{t+1}\upsilon_j\upsilon_{t+1}})^{\frac{1}{p+q+r}},$$

$$(1-\prod_{j=1}^{t}(1-(\rho_{t+1}^p\rho_j^q\rho_{t+1}^r))^{\upsilon_{t+1}\upsilon_j\upsilon_{t+1}})^{\frac{1}{p+q+r}},$$

$$(1-\prod_{j=1}^{t}(1-(\delta_{t+1}^p\delta_j^q\delta_{t+1}^r))^{\upsilon_{t+1}\upsilon_j\upsilon_{t+1}})^{\frac{1}{p+q+r}},$$

$$(1-\prod_{j=1}^{t}(1-(\gamma_{t+1}^p\gamma_j^q\gamma_{t+1}^r))^{\upsilon_{t+1}\upsilon_j\upsilon_{t+1}})^{\frac{1}{p+q+r}}); \tag{17}$$

$$(1-\prod_{j=1}^{t}\left(1-(\eta_{F_{t+1}}^{1})^{p}(\eta_{F_{j}}^{1})^{q}(\eta_{F_{t+1}}^{1})^{r}\right)^{\upsilon_{t+1}\upsilon_{j}\upsilon_{t+1}})^{\frac{1}{p+q+r}},$$

$$(1-\prod_{j=1}^{t}\left(1-(\eta_{F_{t+1}}^{2})^{p}(\eta_{F_{j}}^{2})^{q}(\eta_{F_{t+1}}^{2})^{r}\right)^{\upsilon_{t+1}\upsilon_{j}\upsilon_{t+1}})^{\frac{1}{p+q+r}},\ldots,$$

$$(1-\prod_{j=1}^{t}\left(1-(\eta_{F_{t+1}}^{P})^{p}(\eta_{F_{j}}^{P})^{q}(\eta_{F_{t+1}}^{P})^{r}\right)^{\upsilon_{t+1}\upsilon_{j}\upsilon_{t+1}})^{\frac{1}{p+q+r}}\rangle$$

,

$$\left(\bigoplus_{k=1}^{t}\upsilon_{t+1}\upsilon_{t+1}\upsilon_{k}(F_{t+1}^{p}\otimes F_{t+1}^{q}\otimes F_{k}^{r})\right)^{\frac{1}{p+q+r}}=\langle((1-\prod_{k=1}^{t}(1-(\epsilon_{t+1}^{p}\epsilon_{t+1}^{q}\epsilon_{k}^{r}))^{\upsilon_{t+1}\upsilon_{t+1}\upsilon_{k}})^{\frac{1}{p+q+r}},$$

$$(1-\prod_{k=1}^{t}(1-(\rho_{t+1}^{p}\rho_{t+1}^{q}\rho_{k}^{r}))^{\upsilon_{t+1}\upsilon_{t+1}\upsilon_{k}})^{\frac{1}{p+q+r}},$$

$$(1-\prod_{k=1}^{t}(1-(\delta_{t+1}^{p}\delta_{t+1}^{q}\delta_{k}^{r}))^{\upsilon_{t+1}\upsilon_{t+1}\upsilon_{k}})^{\frac{1}{p+q+r}},$$

$$(1-\prod_{k=1}^{t}(1-(\gamma_{t+1}^{p}\gamma_{t+1}^{q}\gamma_{k}^{r}))^{\upsilon_{t+1}\upsilon_{t+1}\upsilon_{k}})^{\frac{1}{p+q+r}}); \tag{18}$$

$$(1-\prod_{k=1}^{t}\left(1-(\eta_{F_{t+1}}^{1})^{p}(\eta_{F_{t+1}}^{1})^{q}(\eta_{F_{k}}^{1})^{r}\right)^{\upsilon_{t+1}\upsilon_{t+1}\upsilon_{k}})^{\frac{1}{p+q+r}},$$

$$(1-\prod_{k=1}^{t}\left(1-(\eta_{F_{t+1}}^{2})^{p}(\eta_{F_{t+1}}^{2})^{q}(\eta_{F_{k}}^{2})^{r}\right)^{\upsilon_{t+1}\upsilon_{t+1}\upsilon_{k}})^{\frac{1}{p+q+r}},\ldots,$$

$$(1-\prod_{k=1}^{t}\left(1-(\eta_{F_{t+1}}^{P})^{p}(\eta_{F_{t+1}}^{P})^{q}(\eta_{F_{k}}^{P})^{r}\right)^{\upsilon_{t+1}\upsilon_{t+1}\upsilon_{k}})^{\frac{1}{p+q+r}}\rangle$$

,

$$\left(\bigoplus_{i,j=1}^{t}\upsilon_{i}\upsilon_{j}\upsilon_{t+1}(F_{i}^{p}\otimes F_{j}^{q}\otimes F_{t+1}^{r})\right)^{\frac{1}{p+q+r}}=\langle((1-\prod_{i,j=1}^{t}(1-(\epsilon_{i}^{p}\epsilon_{j}^{q}\epsilon_{t+1}^{r}))^{\upsilon_{i}\upsilon_{j}\upsilon_{t+1}})^{\frac{1}{p+q+r}},$$

$$(1-\prod_{i,j=1}^{t}(1-(\rho_{i}^{p}\rho_{j}^{q}\rho_{t+1}^{r}))^{\upsilon_{i}\upsilon_{j}\upsilon_{t+1}})^{\frac{1}{p+q+r}},$$

$$(1-\prod_{i,j=1}^{t}(1-(\delta_{i}^{p}\delta_{j}^{q}\delta_{t+1}^{r}))^{\upsilon_{i}\upsilon_{j}\upsilon_{t+1}})^{\frac{1}{p+q+r}}, \tag{19}$$

$$(1-\prod_{i,j=1}^{t}\left(1-(\eta_{F_{i}}^{1})^{p}(\eta_{F_{j}}^{1})^{q}(\eta_{F_{t+1}}^{1})^{r}\right)^{\upsilon_{i}\upsilon_{j}\upsilon_{t+1}})^{\frac{1}{p+q+r}},$$

$$(1-\prod_{i,j=1}^{t}\left(1-(\eta_{F_{i}}^{2})^{p}(\eta_{F_{j}}^{2})^{q}(\eta_{F_{t+1}}^{2})^{r}\right)^{\upsilon_{i}\upsilon_{j}\upsilon_{t+1}})^{\frac{1}{p+q+r}},\ldots,$$

$$(1-\prod_{i,j=1}^{t}\left(1-(\eta_{F_i}^{P})^{p}(\eta_{F_j}^{P})^{q}(\eta_{F_{t+1}}^{P})^{r}\right)^{\upsilon_i\upsilon_j\upsilon_{t+1}})^{\frac{1}{p+q+r}}\rangle$$

,

$$\left(\bigoplus_{i,k=1}^{t}\upsilon_i\upsilon_{t+1}\upsilon_k(F_i^{\,p}\otimes F_{t+1}^{q}\otimes F_k^{r})\right)^{\frac{1}{p+q+r}}=\langle(((1-\prod_{i,k=1}^{t}(1-(\epsilon_i^{\,p}\epsilon_{t+1}^{q}\epsilon_k^{r}))^{\upsilon_i\upsilon_{t+1}\upsilon_k})^{\frac{1}{p+q+r}},$$

$$(1-\prod_{i,k=1}^{t}(1-(\rho_i^{\,p}\rho_{t+1}^{q}\rho_k^{r}))^{\upsilon_i\upsilon_{t+1}\upsilon_k})^{\frac{1}{p+q+r}},$$

$$(1-\prod_{i,k=1}^{t}(1-(\delta_i^{\,p}\delta_{t+1}^{q}\delta_k^{r}))^{\upsilon_i\upsilon_{t+1}\upsilon_k})^{\frac{1}{p+q+r}},$$

$$(1-\prod_{i,k=1}^{t}(1-(\gamma_i^{\,p}\gamma_{t+1}^{q}\gamma_k^{r}))^{\upsilon_i\upsilon_{t+1}\upsilon_k})^{\frac{1}{p+q+r}}); \tag{20}$$

$$(1-\prod_{i,k=1}^{t}\left(1-(\eta_{F_i}^{1})^{p}(\eta_{F_{t+1}}^{1})^{q}(\eta_{F_k}^{1})^{r}\right)^{\upsilon_i\upsilon_{t+1}\upsilon_k})^{\frac{1}{p+q+r}},$$

$$(1-\prod_{i,k=1}^{t}\left(1-(\eta_{F_i}^{2})^{p}(\eta_{F_{t+1}}^{2})^{q}(\eta_{F_k}^{2})^{r}\right)^{\upsilon_i\upsilon_{t+1}\upsilon_k})^{\frac{1}{p+q+r}},\ldots,$$

$$(1-\prod_{i,k=1}^{t}\left(1-(\eta_{F_i}^{P})^{p}(\eta_{F_{t+1}}^{P})^{q}(\eta_{F_k}^{P})^{r}\right)^{\upsilon_i\upsilon_{t+1}\upsilon_k})^{\frac{1}{p+q+r}}\rangle$$

,

$$\left(\bigoplus_{j,k=1}^{t}\upsilon_{t+1}\upsilon_j\upsilon_k(F_{t+1}^{p}\otimes F_j^{q}\otimes F_k^{r})\right)^{\frac{1}{p+q+r}}=\langle(((1-\prod_{j,k=1}^{t}(1-(\epsilon_{t+1}^{p}\epsilon_j^{q}\epsilon_k^{r}))^{\upsilon_{t+1}\upsilon_j\upsilon_k})^{\frac{1}{p+q+r}},$$

$$(1-\prod_{j,k=1}^{t}(1-(\rho_{t+1}^{p}\rho_j^{q}\rho_k^{r}))^{\upsilon_{t+1}\upsilon_j\upsilon_k})^{\frac{1}{p+q+r}},$$

$$(1-\prod_{j,k=1}^{t}(1-(\delta_{t+1}^{p}\delta_j^{q}\delta_k^{r}))^{\upsilon_{t+1}\upsilon_j\upsilon_k})^{\frac{1}{p+q+r}},$$

$$(1-\prod_{j,k=1}^{t}(1-(\gamma_{t+1}^{p}\gamma_j^{q}\gamma_k^{r}))^{\upsilon_{t+1}\upsilon_j\upsilon_k})^{\frac{1}{p+q+r}}); \tag{21}$$

$$(1-\prod_{j,k=1}^{t}\left(1-(\eta_{F_{t+1}}^{1})^{p}(\eta_{F_j}^{1})^{q}(\eta_{F_k}^{1})^{r}\right)^{\upsilon_{t+1}\upsilon_j\upsilon_k})^{\frac{1}{p+q+r}},$$

$$(1-\prod_{j,k=1}^{t}\left(1-(\eta_{F_{t+1}}^{2})^{p}(\eta_{F_j}^{2})^{q}(\eta_{F_k}^{2})^{r}\right)^{\upsilon_{t+1}\upsilon_j\upsilon_k})^{\frac{1}{p+q+r}},\ldots,$$

$$(1-\prod_{j,k=1}^{t}\left(1-(\eta_{F_{t+1}}^{P})^{p}(\eta_{F_j}^{P})^{q}(\eta_{F_k}^{P})^{r}\right)^{\upsilon_{t+1}\upsilon_j\upsilon_k})^{\frac{1}{p+q+r}}\rangle$$

and

$$\left(\bigoplus_{i,j,k=1}^{t} \upsilon_{t+1}\upsilon_{t+1}\upsilon_{t+1}(F_{t+1}^{p}\otimes F_{t+1}^{q}\otimes F_{t+1}^{r})\right)^{\frac{1}{p+q+r}} = \langle((1-\prod_{i,j,k=1}^{t}(1-(\epsilon_{t+1}^{p}\epsilon_{t+1}^{q}\epsilon_{t+1}^{r}))^{\upsilon_{t+1}\upsilon_{t+1}\upsilon_{t+1}})^{\frac{1}{p+q+r}},$$

$$(1-\prod_{i,j,k=1}^{t}(1-(\rho_{t+1}^{p}\rho_{t+1}^{q}\rho_{t+1}^{r}))^{\upsilon_{t+1}\upsilon_{t+1}\upsilon_{t+1}})^{\frac{1}{p+q+r}},$$

$$(1-\prod_{i,j,k=1}^{t}(1-(\delta_{t+1}^{p}\delta_{t+1}^{q}\delta_{t+1}^{r}))^{\upsilon_{t+1}\upsilon_{t+1}\upsilon_{t+1}})^{\frac{1}{p+q+r}},$$

$$(1-\prod_{i,j,k=1}^{t}(1-(\gamma_{t+1}^{p}\gamma_{t+1}^{q}\gamma_{t+1}^{r}))^{\upsilon_{t+1}\upsilon_{t+1}\upsilon_{t+1}})^{\frac{1}{p+q+r}}); \quad (22)$$

$$(1-\prod_{i,j,k=1}^{t}\left(1-(\eta_{F_{t+1}}^{1})^{p}(\eta_{F_{t+1}}^{1})^{q}(\eta_{F_{t+1}}^{1})^{r}\right)^{\upsilon_{t+1}\upsilon_{t+1}\upsilon_{t+1}})^{\frac{1}{p+q+r}},$$

$$(1-\prod_{i,j,k=1}^{t}\left(1-(\eta_{F_{t+1}}^{2})^{p}(\eta_{F_{t+1}}^{2})^{q}(\eta_{F_{t+1}}^{2})^{r}\right)^{\upsilon_{t+1}\upsilon_{t+1}\upsilon_{t+1}})^{\frac{1}{p+q+r}},\ldots,$$

$$(1-\prod_{i,j,k=1}^{t}\left(1-(\eta_{F_{t+1}}^{P})^{p}(\eta_{F_{t+1}}^{P})^{q}(\eta_{F_{t+1}}^{P})^{r}\right)^{\upsilon_{t+1}\upsilon_{t+1}\upsilon_{t+1}})^{\frac{1}{p+q+r}}\rangle$$

by using Equations (14), (15), (16), (17), (18), (19), (20), (21) and (22), we have

$$\left(\bigoplus_{i,j,k=1}^{t+1} \upsilon_{i}\upsilon_{j}\upsilon_{k}(F_{i}^{p}\otimes F_{j}^{q}\otimes F_{k}^{r})\right)^{\frac{1}{p+q+r}} = \langle((1-\prod_{i,j,k=1}^{t+1}(1-(\epsilon_{i}^{p}\epsilon_{j}^{q}\epsilon_{k}^{r}))^{\upsilon_{i}\upsilon_{j}\upsilon_{k}})^{\frac{1}{p+q+r}},$$

$$(1-\prod_{i,j,k=1}^{t+1}(1-(\rho_{i}^{p}\rho_{j}^{q}\rho_{k}^{r}))^{\upsilon_{i}\upsilon_{j}\upsilon_{k}})^{\frac{1}{p+q+r}},$$

$$(1-\prod_{i,j,k=1}^{t+1}(1-(\delta_{i}^{p}\delta_{j}^{q}\delta_{k}^{r}))^{\upsilon_{i}\upsilon_{j}\upsilon_{k}})^{\frac{1}{p+q+r}},$$

$$(1-\prod_{i,j,k=1}^{t+1}(1-(\gamma_{i}^{p}\gamma_{j}^{q}\gamma_{k}^{r}))^{\upsilon_{i}\upsilon_{j}\upsilon_{k}})^{\frac{1}{p+q+r}}); \quad (23)$$

$$(1-\prod_{i,j,k=1}^{t+1}\left(1-(\eta_{F_i}^{1})^{p}(\eta_{F_j}^{1})^{q}(\eta_{F_k}^{1})^{r}\right)^{\upsilon_{i}\upsilon_{j}\upsilon_{k}})^{\frac{1}{p+q+r}},$$

$$(1-\prod_{i,j,k=1}^{t+1}\left(1-(\eta_{F_i}^{2})^{p}(\eta_{F_j}^{2})^{q}(\eta_{F_k}^{2})^{r}\right)^{\upsilon_{i}\upsilon_{j}\upsilon_{k}})^{\frac{1}{p+q+r}},\ldots,$$

$$(1-\prod_{i,j,k=1}^{t+1}\left(1-(\eta_{F_i}^{P})^{p}(\eta_{F_j}^{P})^{q}(\eta_{F_k}^{P})^{r}\right)^{\upsilon_{i}\upsilon_{j}\upsilon_{k}})^{\frac{1}{p+q+r}}\rangle$$

Thus, when $n=t+1$, Equation (23) is right. So, the Equation (13) is right for all n.

By operational laws in Definition 2.7 and the Equations (14), (15), (16), (17), (18), (19), (20), (21), (22) and (23), we get finally that:

$$TFMGBAM_{\upsilon}^{(p,q,r)}(F_1,F_2,...,F_n)=\left(\bigoplus_{i,j,k=1}^{n}\upsilon_i\upsilon_j\upsilon_k(F_i^{\,p}\otimes F_j^{\,q}\otimes F_k^{\,r})\right)^{\frac{1}{p+q+r}}$$

$$=\langle((1-\prod_{i,j,k=1}^{n}(1-(\epsilon_i^{p}\epsilon_j^{q}\epsilon_k^{r}))^{\upsilon_i\upsilon_j\upsilon_k})^{\frac{1}{p+q+r}},$$

$$(1-\prod_{i,j,k=1}^{n}(1-(\rho_i^{p}\rho_j^{q}\rho_k^{r}))^{\upsilon_i\upsilon_j\upsilon_k})^{\frac{1}{p+q+r}},$$

$$(1-\prod_{i,j,k=1}^{n}(1-(\delta_i^{p}\delta_j^{q}\delta_k^{r}))^{\upsilon_i\upsilon_j\upsilon_k})^{\frac{1}{p+q+r}},$$

$$(1-\prod_{i,j,k=1}^{n}(1-(\gamma_i^{p}\gamma_j^{q}\gamma_k^{r}))^{\upsilon_i\upsilon_j\upsilon_k})^{\frac{1}{p+q+r}});$$

$$(1-\prod_{i,j,k=1}^{n}\left(1-(\eta_{F_i}^{1})^{p}(\eta_{F_j}^{1})^{q}(\eta_{F_k}^{1})^{r}\right)^{\upsilon_i\upsilon_j\upsilon_k})^{\frac{1}{p+q+r}},$$

$$(1-\prod_{i,j,k=1}^{n}\left(1-(\eta_{F_i}^{2})^{p}(\eta_{F_j}^{2})^{q}(\eta_{F_k}^{2})^{r}\right)^{\upsilon_i\upsilon_j\upsilon_k})^{\frac{1}{p+q+r}},\ldots,$$

$$(1-\prod_{i,j,k=1}^{n}\left(1-(\eta_{F_i}^{P})^{p}(\eta_{F_j}^{P})^{q}(\eta_{F_k}^{P})^{r}\right)^{\upsilon_i\upsilon_j\upsilon_k})^{\frac{1}{p+q+r}}\rangle$$

and proof is completed.

In the following proposition, algebraic product and algebraic sum are used in computations.

Proposition 3.4 *Let* $F_i=\langle(\epsilon_i,\rho_i,\delta_i,\gamma_i);\eta_{F_i}^{1},\eta_{F_i}^{2},...,\eta_{F_i}^{P}\rangle$ *and* $G_i=\langle(\varepsilon_i,\zeta_i,\theta_i,\varsigma_i);\eta_{G_i}^{1},\eta_{G_i}^{2},...,\eta_{G_i}^{P}\rangle$ $(i\in I_n)$ *be two collections of TFM-numbers.* $TFMGBAM_{\upsilon}^{(p,q,r)}$ *also has the following properties:*

1. If all $F_i=F\ \ (i\in I_n)$ for all i, then

$$TFMGBAM_{\upsilon}^{(p,q,r)}(F_1,F_2,...,F_n)=TFMGBAM_{\upsilon}^{(p,q,r)}(F,F,...,F)=F$$

2. if $F_i\geq G_i$ for all i, then $TFMGBAM_{\upsilon}^{(p,q,r)}$ is monotonic that is,

$$TFMGBAM_{\upsilon}^{(p,q,r)}(F_1,F_2,...,F_n)\geq TFMGBAM_{\upsilon}^{(p,q,r)}(G_1,G_2,...,G_n)$$

3. Let $(\dot{F}_1, \dot{F}_2, ..., \dot{F}_n)$ be any permutation of $(F_1, F_2, ..., F_n)$ and $(\dot{\upsilon}_1, \dot{\upsilon}_2, ..., \dot{\upsilon}_n)$ be weight vector of $(\dot{F}_1, \dot{F}_2, ..., \dot{F}_n)$. Then,

$$TFMGBAM_{\upsilon}^{(p,q,r)}(F_1, F_2, ..., F_n) = TFMGBAM_{\dot{\upsilon}}^{(p,q,r)}(\dot{F}_1, \dot{F}_2, ..., \dot{F}_n)$$

4.

$$F^{-} \le TFMGBAM_{\upsilon}^{(p,q,r)}(F_1, F_2, ..., F_n) \le F^{+}$$

where

$$F^{+} = \langle (\max_{\{i\in I_n\}}\{\epsilon_i\}, \max_{\{i\in I_n\}}\{\rho_i\}, \max_{\{i\in I_n\}}\{\delta_i\}, \max_{\{i\in I_n\}}\{\gamma_i\}); \max_{\{i\in I_n\}}\{\eta_{F_i}^{1}\}, \max_{\{i\in I_n\}}\{\eta_{F_i}^{2}\}, ..., \max_{\{i\in I_n\}}\{\eta_{F_i}^{P}\}\rangle$$

and

$$F^{-} = \langle (\min_{\{i\in I_n\}}\{\epsilon_i\}, \min_{\{i\in I_n\}}\{\rho_i\}, \min_{\{i\in I_n\}}\{\delta_i\}, \min_{\{i\in I_n\}}\{\gamma_i\}); \min_{\{i\in I_n\}}\{\eta_{F_i}^{1}\}, \min_{\{i\in I_n\}}\{\eta_{F_i}^{2}\}, ..., \min_{\{i\in I_n\}}\{\eta_{F_i}^{P}\}\rangle$$

Proof

1. Let $F_i = F \ \ (i \in I_n)$ for all i and let $F = \langle(\epsilon, \rho, \delta, \gamma); \eta_F^1, \eta_F^2, \ldots, \eta_F^P\rangle$. That is, , $\rho_i = \rho$, $\delta_i = \delta$, $\gamma_i = \gamma$ and $\eta_{F_i}^1 = \eta_F^1$, $\eta_{F_i}^2 = \eta_F^2$,..., $\eta_{F_i}^P = \eta_F^P$.

$$TFMGBAM_{\upsilon}^{(p,q,r)}(F_1, F_2, ..., F_n) = TFMGBAM_{\upsilon}^{(p,q,r)}(F, F, ..., F)$$

$$= \left(\bigoplus_{i,j,k=1}^{n} \upsilon_i \upsilon_j \upsilon_k (F^{p} \otimes F^{q} \otimes F^{r}) \right)^{\frac{1}{p+q+r}}$$

$$= \langle(((1-\prod_{i,j,k=1}^{n}(1-(\epsilon_i^{p}\epsilon_j^{q}\epsilon_k^{r}))^{\upsilon_i\upsilon_j\upsilon_k})^{\frac{1}{p+q+r}},$$

$$(1-\prod_{i,j,k=1}^{n}(1-(\rho_i^{p}\rho_j^{q}\rho_k^{r}))^{\upsilon_i\upsilon_j\upsilon_k})^{\frac{1}{p+q+r}},$$

$$(1-\prod_{i,j,k=1}^{n}(1-(\delta_i^{p}\delta_j^{q}\delta_k^{r}))^{\upsilon_i\upsilon_j\upsilon_k})^{\frac{1}{p+q+r}},$$

$$(1-\prod_{i,j,k=1}^{n}(1-(\gamma_i^{p}\gamma_j^{q}\gamma_k^{r}))^{\upsilon_i\upsilon_j\upsilon_k})^{\frac{1}{p+q+r}});$$

$$(1-\prod_{i,j,k=1}^{n}\left(1-(\eta_F^1)^{p}(\eta_F^1)^{q}(\eta_F^1)^{r}\right)^{\upsilon_i\upsilon_j\upsilon_k})^{\frac{1}{p+q+r}},$$

$$(1-\prod_{i,j,k=1}^{n}\left(1-(\eta_F^2)^{p}(\eta_F^2)^{q}(\eta_F^2)^{r}\right)^{\upsilon_i\upsilon_j\upsilon_k})^{\frac{1}{p+q+r}}, \ldots,$$

$$(1-\prod_{i,j,k=1}^{n}\left(1-(\eta_F^P)^{p}(\eta_F^P)^{q}(\eta_F^P)^{r}\right)^{\upsilon_i\upsilon_j\upsilon_k})^{\frac{1}{p+q+r}}\rangle$$

$$= \langle((1-\prod_{i,j,k=1}^{n}(1-\epsilon^{p+q+r})^{\upsilon_i\upsilon_j\upsilon_k})^{\frac{1}{p+q+r}},$$

$$(1-\prod_{i,j,k=1}^{n}(1-\rho^{p+q+r})^{\upsilon_i\upsilon_j\upsilon_k})^{\frac{1}{p+q+r}},$$

$$(1-\prod_{i,j,k=1}^{n}(1-\delta^{p+q+r})^{\upsilon_i\upsilon_j\upsilon_k})^{\frac{1}{p+q+r}},$$

$$(1-\prod_{i,j,k=1}^{n}(1-\gamma^{p+q+r})^{\upsilon_i\upsilon_j\upsilon_k})^{\frac{1}{p+q+r}});$$

$$(1-\prod_{i,j,k=1}^{n}\left(1-(\eta_F^1)^{p+q+r}\right)^{\upsilon_i\upsilon_j\upsilon_k})^{\frac{1}{p+q+r}},$$

$$(1-\prod_{i,j,k=1}^{n}\left(1-(\eta_F^2)^{p+q+r}\right)^{\upsilon_i\upsilon_j\upsilon_k})^{\frac{1}{p+q+r}},\dots,$$

$$(1-\prod_{i,j,k=1}^{n}\left(1-(\eta_F^P)^{p+q+r}\right)^{\upsilon_i\upsilon_j\upsilon_k})^{\frac{1}{p+q+r}}\rangle$$

$$= \langle((1-\left(1-\epsilon^{p+q+r}\right)_{\sum_{i,j,k=1}^{n}\upsilon_i\upsilon_j\upsilon_k})^{\frac{1}{p+q+r}},$$

$$(1-\left(1-\rho^{p+q+r}\right)_{\sum_{i,j,k=1}^{n}\upsilon_i\upsilon_j\upsilon_k})^{\frac{1}{p+q+r}},$$

$$(1-\left(1-\delta^{p+q+r}\right)_{\sum_{i,j,k=1}^{n}\upsilon_i\upsilon_j\upsilon_k})^{\frac{1}{p+q+r}},$$

$$(1-\left(1-\gamma^{p+q+r}\right)_{\sum_{i,j,k=1}^{n}\upsilon_i\upsilon_j\upsilon_k})^{\frac{1}{p+q+r}});$$

$$(1-\left(1-(\eta_F^1)^{p+q+r}\right)_{\sum_{i,j,k=1}^{n}\upsilon_i\upsilon_j\upsilon_k})^{\frac{1}{p+q+r}},$$

$$(1-\left(1-(\eta_F^2)^{p+q+r}\right)_{\sum_{i,j,k=1}^{n}\upsilon_i\upsilon_j\upsilon_k})^{\frac{1}{p+q+r}},\dots,$$

$$(1-\left(1-(\eta_F^P)^{p+q+r}\right)_{\sum_{i,j,k=1}^{n}\upsilon_i\upsilon_j\upsilon_k})^{\frac{1}{p+q+r}}\rangle$$

$$= \langle((1-\left(1-\epsilon^{p+q+r}\right))^{\frac{1}{p+q+r}},\ (1-\left(1-\rho^{p+q+r}\right))^{\frac{1}{p+q+r}},$$

$$(1-\left(1-\delta^{p+q+r}\right))^{\frac{1}{p+q+r}},\ (1-\left(1-\epsilon^{p+q+r}\right))^{\frac{1}{p+q+r}});$$

$$(1-\left(1-(\eta_F^1)^{p+q+r}\right))^{\frac{1}{p+q+r}},$$

$$(1-\left(1-(\eta_F^2)^{p+q+r}\right))^{\frac{1}{p+q+r}},\dots,$$

$$(1-\left(1-(\eta_F^P)^{p+q+r}\right))^{\frac{1}{p+q+r}}\rangle$$
$$=\langle(\epsilon,\rho,\delta,\gamma);\eta_F^1,\eta_F^2,\ldots,\eta_F^P\rangle$$
$$=F$$

2. Since $F_i \geq G_i$, it is obvious that

$$\epsilon_i \geq \varepsilon_i, \rho_i \geq \zeta_i, \delta_i \geq \theta_i, \gamma_i \geq \varsigma_i \tag{24}$$

$$\Rightarrow \epsilon_i^p\epsilon_j^q\epsilon_k^r \geq \varepsilon_i^p\varepsilon_j^q\varepsilon_k^r, \rho_i^p\rho_j^q\rho_k^r \geq \zeta_i^p\zeta_j^q\zeta_k^r, \delta_i^p\delta_j^q\delta_k^r \geq \theta_i^p\theta_j^q\theta_k^r, \gamma_i^p\gamma_j^q\gamma_k^r \geq \varsigma_i^p\varsigma_j^q\varsigma_k^r$$

$$\Rightarrow 1-(1-\epsilon_i^p\epsilon_j^q\epsilon_k^r)^{\upsilon_i\upsilon_j\upsilon_k} \leq 1-(1-\varepsilon_i^p\varepsilon_j^q\varepsilon_k^r)^{\upsilon_i\upsilon_j\upsilon_k}, 1-(1-\rho_i^p\rho_j^q\rho_k^r)^{\upsilon_i\upsilon_j\upsilon_k} \leq 1-(1-\zeta_i^p\zeta_j^q\zeta_k^r)^{\upsilon_i\upsilon_j\upsilon_k},$$

$$1-(1-\delta_i^p\delta_j^q\delta_k^r)^{\upsilon_i\upsilon_j\upsilon_k} \leq 1-(1-\theta_i^p\theta_j^q\theta_k^r)^{\upsilon_i\upsilon_j\upsilon_k}, 1-(1-\gamma_i^p\gamma_j^q\gamma_k^r)^{\upsilon_i\upsilon_j\upsilon_k} \leq 1-(1-\varsigma_i^p\varsigma_j^q\varsigma_k^r)^{\upsilon_i\upsilon_j\upsilon_k}$$

$$\Rightarrow (1-\prod_{i,j,k=1}^{n}(1-\epsilon_i^p\epsilon_j^q\epsilon_k^r)^{\upsilon_i\upsilon_j\upsilon_k})^{\frac{1}{p+q+r}} \geq (1-\prod_{i,j,k=1}^{n}(1-\varepsilon_i^p\varepsilon_j^q\varepsilon_k^r)^{\upsilon_i\upsilon_j\upsilon_k})^{\frac{1}{p+q+r}},$$

$$(1-\prod_{i,j,k=1}^{n}(1-\rho_i^p\rho_j^q\rho_k^r)^{\upsilon_i\upsilon_j\upsilon_k})^{\frac{1}{p+q+r}} \geq (1-\prod_{i,j,k=1}^{n}(1-\zeta_i^p\zeta_j^q\zeta_k^r)^{\upsilon_i\upsilon_j\upsilon_k})^{\frac{1}{p+q+r}},$$

$$(1-\prod_{i,j,k=1}^{n}(1-\delta_i^p\delta_j^q\delta_k^r)^{\upsilon_i\upsilon_j\upsilon_k})^{\frac{1}{p+q+r}} \geq (1-\prod_{i,j,k=1}^{n}(1-\theta_i^p\theta_j^q\theta_k^r)^{\upsilon_i\upsilon_j\upsilon_k})^{\frac{1}{p+q+r}},$$

$$(1-\prod_{i,j,k=1}^{n}(1-\gamma_i^p\gamma_j^q\gamma_k^r)^{\upsilon_i\upsilon_j\upsilon_k})^{\frac{1}{p+q+r}} \geq (1-\prod_{i,j,k=1}^{n}(1-\varsigma_i^p\varsigma_j^q\varsigma_k^r)^{\upsilon_i\upsilon_j\upsilon_k})^{\frac{1}{p+q+r}}$$

On the other hand similarly we can write that

$$\eta_{F_i}^P \geq \eta_{G_i}^P, \eta_{F_j}^P \geq \eta_{G_j}^P, \eta_{F_k}^P \geq \eta_{G_k}^P$$

By using operational laws of TFM-numbers:

$$(\eta_{F_i}^P)^p(\eta_{F_j}^P)^q(\eta_{F_k}^P)^r \geq (\eta_{G_i}^P)^p(\eta_{G_j}^P)^q(\eta_{G_k}^P)^r$$

$$\Rightarrow (1-(\eta_{F_i}^P)^p(\eta_{F_j}^P)^q(\eta_{F_k}^P)^r)^{\upsilon_i\upsilon_j\upsilon_k} \leq (1-(\eta_{G_i}^P)^p(\eta_{G_j}^P)^q(\eta_{G_k}^P)^r)^{\upsilon_i\upsilon_j\upsilon_k}$$

$$\Rightarrow (1-\prod_{i,j,k=1}^{n}(1-(\eta_{F_i}^{P})^{p}(\eta_{F_j}^{P})^{q}(\eta_{F_k}^{P})^{r})^{\upsilon_i\upsilon_j\upsilon_k})^{\frac{1}{p+q+r}} \geq (1-\prod_{i,j,k=1}^{n}(1-(\eta_{G_i}^{P})^{p}(\eta_{G_j}^{P})^{q}(\eta_{G_k}^{P})^{r})^{\upsilon_i\upsilon_j\upsilon_k})^{\frac{1}{p+q+r}} \quad (25)$$

If we consider Equations (24) and (25), we get that and complete the proof:

$$TFMGBAM_{\upsilon}^{(p,q,r)}(F_1, F_2, ..., F_n) \geq TFMGBAM_{\upsilon}^{(p,q,r)}(G_1, G_2, ..., G_n)$$

3. Let $(\dot{F}_1, \dot{F}_2, ..., \dot{F}_n)$ be any permutation of $(F_1, F_2, ..., F_n)$ and $(\dot{\upsilon}_1, \dot{\upsilon}_2, ..., \dot{\upsilon}_n)$ be weight vector of $(\dot{F}_1, \dot{F}_2, ..., \dot{F}_n)$. Then,

$$TFMGBAM_{\upsilon}^{(p,q,r)}(F_1, F_2, ..., F_n) = \left(\bigoplus_{i,j,k=1}^{n} \upsilon_i\upsilon_j\upsilon_k \left(F_i^{p} \otimes F_j^{q} \otimes F_k^{r}\right)\right)^{\frac{1}{p+q+r}}$$

$$= \left(\bigoplus_{i,j,k=1}^{n} \dot{\upsilon}_i\dot{\upsilon}_j\dot{\upsilon}_k \left(\dot{F}_i^{p} \otimes \dot{F}_j^{q} \otimes \dot{F}_k^{r}\right)\right)^{\frac{1}{p+q+r}}$$

$$= TFMGBAM_{\upsilon}^{(p,q,r)}(\dot{F}_1, \dot{F}_2, ..., \dot{F}_n)$$

4. From Proposition (3.4),

$$TFMGBAM_{\upsilon}^{(p,q,r)}(F^-, F^-, ..., F^-) = F^-$$

and

and from Proposition (3.4) and Definition 2.8,

$$TFMGBAM_{\upsilon}^{(p,q,r)}(F^-, F^-, ..., F^-) \leq TFMGBAM_{\upsilon}^{(p,q,r)}(F_1, F_2, ..., F_n)$$

and

$$TFMGBAM_{\upsilon}^{(p,q,r)}(F_1, F_2, ..., F_n) \leq TFMGBAM_{\upsilon}^{(p,q,r)}(F^+, F^+, ..., F^+)$$

Therefore,

$$F^- \leq TFMGBAM_{\upsilon}^{(p,q,r)}(F_1, F_2, ..., F_n) \leq F^+$$

3.1 Generalized Weighted Bonferroni Geometric Mean of Trapezoidal Fuzzy Multi Numbers

In this section, we introduce the generalized weighted Bonferroni geometric mean and based on that, we give the trapezoidal fuzzy multi generalized weighted

Bonferroni geometric mean.

Definition 3.5 *Let* $F_i = \langle(\epsilon_i, \rho_i, \delta_i, \gamma_i); \eta^1_{F_i}, \eta^2_{F_i}, ..., \eta^P_{F_i}\rangle$ $(i \in I_n)$ *be a collection of TFM-numbers,* $\upsilon = (\upsilon_1, \upsilon_2, ..., \upsilon_n)^T$ *their weight vector, where* υ_i *indicates the importance degree of* F_i*, satisfying* $\upsilon_i > 0$ *(* $i \in I_n$*), and* $\sum_{i=1}^{n} \upsilon_i = 1$. *For any* $p, q, r > 0$*, If*

$$TFMGBGM_{\upsilon}^{(p,q,r)}(F_1, F_2, ..., F_n) = \frac{1}{p+q+r}\left(\bigotimes_{i,j,k=1}^{n} (pF_i \oplus qF_j \oplus rF_k)^{\upsilon_i \upsilon_j \upsilon_k}\right) \quad (26)$$

then $TFMGBGM_{\upsilon}^{(p,q,r)}$ is called a trapezoidal fuzzy multi weighted Bonferroni geometric mean operator.

In the following theorem, algebraic product and algebraic sum are used in computations.

Theorem 3.6 *Let* $F_i = \langle(\epsilon_i, \rho_i, \delta_i, \gamma_i); \eta^1_{F_i}, \eta^2_{F_i}, ..., \eta^P_{F_i}\rangle$ $(i \in I_n)$ *be a collection of TFM-numbers and* $p, q, r > 0$*, then aggregated value by using the* $TFMGBGM_{\upsilon}^{(p,q,r)}$ *is also an TFM-number and computed as follows:*

$$\begin{aligned}
TFMGBGM_{\upsilon}^{(p,q,r)}(F_1, F_2, ..., F_n) &= \frac{1}{p+q+r}\left(\bigotimes_{i,j,k=1}^{n} (pF_i \oplus qF_j \oplus rF_k)^{\upsilon_i \upsilon_j \upsilon_k}\right) \\
&= \langle(1-(1-\prod_{i,j,k=1}^{n}[1-(1-\epsilon_i)^p(1-\epsilon_j)^q(1-\epsilon_k)^r]^{\upsilon_i \upsilon_j \upsilon_k})^{\frac{1}{p+q+r}}, \\
&\quad 1-(1-\prod_{i,j,k=1}^{n}[1-(1-\rho_i)^p(1-\rho_j)^q(1-\rho_k)^r]^{\upsilon_i \upsilon_j \upsilon_k})^{\frac{1}{p+q+r}}, \\
&\quad 1-(1-\prod_{i,j,k=1}^{n}[1-(1-\delta_i)^p(1-\delta_j)^q(1-\delta_k)^r]^{\upsilon_i \upsilon_j \upsilon_k})^{\frac{1}{p+q+r}}, \\
&\quad 1-(1-\prod_{i,j,k=1}^{n}[1-(1-\gamma_i)^p(1-\gamma_j)^q(1-\gamma_k)^r]^{\upsilon_i \upsilon_j \upsilon_k})^{\frac{1}{p+q+r}}); \\
&\quad 1-(1-\prod_{i,j,k=1}^{n}[1-(1-\eta^1_{F_i})^p(1-\eta^1_{F_j})^q(1-\eta^1_{F_k})^r]^{\upsilon_i \upsilon_j \upsilon_k})^{\frac{1}{p+q+r}}, \\
&\quad 1-(1-\prod_{i,j,k=1}^{n}[1-(1-\eta^2_{F_i})^p(1-\eta^2_{F_j})^q(1-\eta^2_{F_k})^r]^{\upsilon_i \upsilon_j \upsilon_k})^{\frac{1}{p+q+r}}, ..., \\
&\quad 1-(1-\prod_{i,j,k=1}^{n}[1-(1-\eta^P_{F_i})^p(1-\eta^P_{F_j})^q(1-\eta^P_{F_k})^r]^{\upsilon_i \upsilon_j \upsilon_k})^{\frac{1}{p+q+r}}\rangle
\end{aligned}$$

(27)

Proof: The theorem can be easily done similar to Theorem 3.3.

Proposition 3.7 *Let* $F_i = \langle(\epsilon_i, \rho_i, \delta_i, \gamma_i); \eta^1_{F_i}, \eta^2_{F_i}, ..., \eta^P_{F_i}\rangle$ *and* $G_i = \langle(\varepsilon_i, \zeta_i, \theta_i, \varsigma_i); \eta^1_{G_i}, \eta^2_{G_i}, ..., \eta^P_{G_i}\rangle$ $(i \in I_n)$ *be two collections of TFM-numbers.* $TFMGBGM_{\upsilon}^{(p,q,r)}$ *also has the following properties:*

1. If all $F_i = F \quad (i \in I_n)$ for all i, then

$$TFMGBGM_{\upsilon}^{(p,q,r)}(F_1, F_2, ..., F_n) = F$$

2. if $F_i \geq G_i$ for all i, then

$$TFMGBGM_{\upsilon}^{(p,q,r)}(F_1, F_2, ..., F_n) \geq TFMGBGM_{\upsilon}^{(p,q,r)}(G_1, G_2, ..., G_n)$$

3. Let $(\dot{F}_1, \dot{F}_2, ..., \dot{F}_n)$ be any permutation of $(F_1, F_2, ..., F_n)$ and $(\dot{\upsilon}_1, \dot{\upsilon}_2, ..., \dot{\upsilon}_n)$ be weight vector of $(\dot{F}_1, \dot{F}_2, ..., \dot{F}_n)$. Then,

$$TFMGBGM_{\upsilon}^{(p,q,r)}(F_1, F_2, ..., F_n) = TFMGBGM_{\dot{\upsilon}}^{(p,q,r)}(\dot{F}_1, \dot{F}_2, ..., \dot{F}_n)$$

4.

$$F^- \leq TFMGBGM_{\upsilon}^{(p,q,r)}(F_1, F_2, ..., F_n) \leq F^+$$

where

$$F^+ = \langle(\max_{\{i\in I_n\}}\{\epsilon_i\}, \max_{\{i\in I_n\}}\{\rho_i\}, \max_{\{i\in I_n\}}\{\delta_i\}, \max_{\{i\in I_n\}}\{\gamma_i\}); \max_{\{i\in I_n\}}\{\eta^1_{F_i}\}, \max_{\{i\in I_n\}}\{\eta^2_{F_i}\}, ..., \max_{\{i\in I_n\}}\{\eta^P_{F_i}\}\rangle$$

and

$$F^- = \langle(\min_{\{i\in I_n\}}\{\epsilon_i\}, \min_{\{i\in I_n\}}\{\rho_i\}, \min_{\{i\in I_n\}}\{\delta_i\}, \min_{\{i\in I_n\}}\{\gamma_i\}); \min_{\{i\in I_n\}}\{\eta^1_{F_i}\}, \min_{\{i\in I_n\}}\{\eta^2_{F_i}\}, ..., \min_{\{i\in I_n\}}\{\eta^P_{F_i}\}\rangle$$

Proof Items can be proven similar to Proposition 3.4.

4 Application

In this section, we give process to solve decision-making problems given under trapezoidal fuzzy-multi environment. Then, we give an application to show running of the process.

Decision Making Process

Step 1 Present a TFM decision matrix $((L_{ij})_{mxn})$ for each group of the criteria, showing results of evaluation based upon the characteristic of the alternative $M = \{M_1, M_2, ..., M_m\}$ satisfying the attribute $C = \{c_1, c_2, ..., c_n\}$ based on linguistic terms Table 2 as follows:

$$(L_{ij})_{mxn} = \begin{pmatrix} L_{11} & L_{12} & \cdots & L_{1n} \\ L_{21} & L_{22} & \cdots & L_{2n} \\ \vdots & \vdots & \ddots & \vdots \\ L_{m1} & L_{m2} & \cdots & L_{mn} \end{pmatrix}$$

Step 2 Find a weight vector for each group of the criteria as follows:

Substep 1 For each TFM decision matrix, construct a matrix $((x_{ij}))$ consisting of real numbers by value of TFM-numbers obtain from defuzzification of each element of the decision matrix $(L_{ij})_{mxn}$ by using Definition 2.10 as follows:

$$(x_{ij})_{mxn} = \begin{pmatrix} x_{11} & x_{12} & \cdots & x_{1x} \\ x_{21} & x_{22} & \cdots & x_{2x} \\ \vdots & \vdots & \ddots & \vdots \\ x_{m1} & x_{m2} & \cdots & x_{mn} \end{pmatrix}$$

Substep 2 Find the weights of criteria according to criteria in the decision-making problem and values in $(D_{ij})_{mxn}$ matrix by using CRITIC method given in Subsection 2.2:

$$\upsilon = (\upsilon_1, \upsilon_2, ..., \upsilon_n)$$

where $\sum_{i=1}^{n} \upsilon_i = 1$.

Step 3 For all i $(i \in I_m)$, find the aggregation values by using Equation (13) (or Equation (27)) according to first group of criteria, to get the performance value corresponding to the alternative M_i, denoted by ${}_1L_i$, $(i \in I_m)$ as follows:

$${}_1L_i = TFMGBAM_{\upsilon}^{(p,q,r)}(L_{i1}, L_{i2}, ..., L_{in})(i \in I_m)$$

Step 4 For all i $(i \in I_m)$, find the aggregation values by using Equation (13) (or Equation (27)) according to second group of criteria, to get the performance value corresponding to the alternative M_i, denoted by ${}_2L_i$, $(i \in I_m)$ as follows:

$${}_2L_i = TFMGBAM_{\upsilon}^{(p,q,r)}(L_{i1}, L_{i2}, ..., L_{in})(i \in I_m)$$

Step 5 Compute the Hamming distance given in Definition 2.9 between ${}_1L_i$ and

$(i \in I_m)$

Step 6 Rank the results. The smaller result, the better alternative.

4.1 Numerical Example

The thread of increasing in poverty and hunger in the world obliges states to take precautions in this regard. Although developed countries have come a long way in this regard, even this is no longer a problem for them, there are still many countries that do not take adequate steps in this issue. Especially underdeveloped countries, as they are insufficient economically, are vulnerable and suffer victimization in poverty and hunger. In order to prevent this situation, some countries that decided to take action have taken the models of the countries that have achieved success in this subject to examination and have decided to take the model they found suitable for them as an example. As a result of the examination, the policies that developed countries have already implemented and are considering to implement in the near future have been taken into consideration. Countries will decide on the choice of model by looking at how well the currently implemented policies lead to the policies that are planned to be implemented as next step of precautions against poverty and hunger. Models which can be choosen are as follows:

1. Model of Country 1 (M_1)
2. Model of Country 2 (M_2)
3. Model of Country 3 (M_3)
4. Model of Country 4 (M_4)
5. Model of Country 5 (M_5)
6. Model of Country 6 (M_6)

Group of policies, according to United Nations, given as follows:

Table 1: Groups of Policies

	Already Implemented Policies (First Group of Criteria)(c_1,c_2,...,c_7)		Targeted Policies (Second Group of Criteria)($\dot{c}_1$,$\dot{c}_2$,...,$\dot{c}_9$)
c_1	To eradicate extreme poverty for all people everywhere, defined as living on less than 1.25 per day	$\dot{c}_1$	Ensuring universal access to reliable and nutritious food
c_2	To reduce at least by half the proportion of men, women, and children of all ages living in poverty in all its dimensions according to national definitions	$\dot{c}_2$	Eliminating all forms of malnutrition for everyone
c_3	To implement nationally appropriate social protection systems and measures, including minimum safeguards, to ensure coverage for all and achieve substantial coverage of the poor and vulnerable	$\dot{c}_3$	Increasing the productivity and incomes of small-scale food producers
c_4	To ensure that all men and women, particularly the poor and vulnerable, have equal rights to economic resources, as well as access to basic services, ownership, and control over land and other forms of property, inheritance, natural resources, appropriate new technology, and financial services, including microfinance	$\dot{c}_4$	Ensuring sustainable food production systems and implementing resilient agricultural practices
c_5	To build the resilience of the poor and those in vulnerable situations and reduce their exposure and vulnerability to climate-related extreme events and other economic, social, and environmental shocks and disasters	$\dot{c}_5$	Preserving genetic diversity in food production
c_6	To ensure significant mobilization of resources from a variety of sources, including through enhanced development cooperation, in order to provide adequate and predictable means for developing countries, in particular the least developed countries, to implement programs and policies to end poverty in all its dimensions	$\dot{c}_6$	Enhancing investments in rural infrastructure, agricultural research and extension services, technology development, and plant and animal gene banks in developing countries through improved international cooperation to increase agricultural production capacity
c_7	To create sound policy frameworks, at the national, regional, and international levels, based on pro-poor and gender-sensitive development strategies, to support accelerated investment in poverty eradication actions	$\dot{c}_7$	Preventing agricultural trade restrictions, market distortions, and export subsidies
		$\dot{c}_8$	Ensuring stability and timely access to information in food commodity markets
		$\dot{c}_9$	Enhancing investments in rural infrastructure, agricultural research and extension services, technology development, and plant and animal gene banks in developing countries, particularly the least developed countries, through improved international cooperation to increase agricultural production capacity

Table 2: TFM-numbers of linguistic terms

Linguistic terms	TFM-numbers
Definitely-low(DL)	$\langle(0.01, 0.05, 0.10, 0.15); 0.1, 0.2, 0.3, 0.4\rangle$
Too-Low(TL)	$\langle(0.05, 0.10, 0.15, 0.20); 0.2, 0.3, 0.4, 0.1\rangle$
Very-Low(VL)	$\langle(0.10, 0.15, 0.15, 0.20); 0.2, 0.4, 0.5, 0.3\rangle$
Low(L)	$\langle(0.10, 0.20, 0.20, 0.30); 0.3, 0.4, 0.8, 0.1\rangle$
Fairly-low(FL)	$\langle(0.15, 0.20, 0.25, 0.30); 0.4, 0.6, 0.2, 0.5\rangle$
Medium(M)	$\langle(0.25, 0.30, 0.35, 0.40); 0.4, 0.5, 0.6, 0.8\rangle$
Fairly-high(FH)	$\langle(0.30, 0.35, 0.40, 0.45); 0.6, 0.1, 0.8, 0.4\rangle$
High(H)	$\langle(0.40, 0.45, 0.50, 0.55); 0.8, 0.9, 0.3, 0.6\rangle$
Very-High(VH)	$\langle(0.45, 0.55, 0.65, 0.75); 0.7, 0.8, 0.6, 0.3\rangle$

Too-High(TH)	$\langle(0.50, 0.60, 0.70, 0.80); 0.1, 0.7, 0.8, 0.9\rangle$
Definitely-high(DH)	$\langle(0.70, 0.80, 0.90, 1.00); 0.7, 0.8, 0.9, 0.2\rangle$

Step 1 Present two TFM decision matrices since there are two groups of the criteria:

Decision Matrix-1 according to first group of the criteria ($c_1, c_2, ..., c_7$):

M_1	$\langle(0.70, 0.80, 0.90, 1.00); 0.7, 0.8, 0.9, 0.2\rangle$	$\langle(0.15, 0.20, 0.25, 0.30); 0.4, 0.6, 0.2, 0.5\rangle$	$\langle(0.05, 0.10, 0.15, 0.20); 0.2, 0.3, 0.4, 0.1\rangle$	$\langle(0.10, 0.20, 0.20, 0.30); 0.3, 0.4, 0.8, 0.1\rangle$	$\langle(0.05, 0.10, 0.15, 0.20); 0.2, 0.3, 0.4, 0.1\rangle$	$\langle(0.40, 0.45, 0.50, 0.55); 0.8, 0.9, 0.3, 0.6\rangle$	$\langle(0.25, 0.30, 0.35, 0.40); 0.4, 0.5, 0.6, 0.8\rangle$
M_2	$\langle(0.50, 0.60, 0.70, 0.80); 0.1, 0.7, 0.8, 0.9\rangle$	$\langle(0.25, 0.30, 0.35, 0.40); 0.4, 0.5, 0.6, 0.8\rangle$	$\langle(0.40, 0.45, 0.50, 0.55); 0.8, 0.9, 0.3, 0.6\rangle$	$\langle(0.50, 0.60, 0.70, 0.80); 0.1, 0.7, 0.8, 0.9\rangle$	$\langle(0.10, 0.15, 0.15, 0.20); 0.2, 0.4, 0.5, 0.3\rangle$	$\langle(0.01, 0.05, 0.10, 0.15); 0.1, 0.2, 0.3, 0.4\rangle$	$\langle(0.30, 0.35, 0.40, 0.45); 0.6, 0.1, 0.8, 0.4\rangle$
M_3	$\langle(0.70, 0.80, 0.90, 1.00); 0.7, 0.8, 0.9, 0.2\rangle$	$\langle(0.01, 0.05, 0.10, 0.15); 0.1, 0.2, 0.3, 0.4\rangle$	$\langle(0.05, 0.10, 0.15, 0.20); 0.2, 0.3, 0.4, 0.1\rangle$	$\langle(0.10, 0.20, 0.20, 0.30); 0.3, 0.4, 0.8, 0.1\rangle$	$\langle(0.30, 0.35, 0.40, 0.45); 0.6, 0.1, 0.8, 0.4\rangle$	$\langle(0.70, 0.80, 0.90, 1.00); 0.7, 0.8, 0.9, 0.2\rangle$	$\langle(0.45, 0.55, 0.65, 0.75); 0.7, 0.8, 0.6, 0.3\rangle$
M_4	$\langle(0.50, 0.60, 0.70, 0.80); 0.1, 0.7, 0.8, 0.9\rangle$	$\langle(0.40, 0.45, 0.50, 0.55); 0.8, 0.9, 0.3, 0.6\rangle$	$\langle(0.25, 0.30, 0.35, 0.40); 0.4, 0.5, 0.6, 0.8\rangle$	$\langle(0.10, 0.20, 0.20, 0.30); 0.3, 0.4, 0.8, 0.1\rangle$	$\langle(0.05, 0.10, 0.15, 0.20); 0.2, 0.3, 0.4, 0.1\rangle$	$\langle(0.70, 0.80, 0.90, 1.00); 0.7, 0.8, 0.9, 0.2\rangle$	$\langle(0.30, 0.35, 0.40, 0.45); 0.6, 0.1, 0.8, 0.4\rangle$
M_5	$\langle(0.01, 0.05, 0.10, 0.15); 0.1, 0.2, 0.3, 0.4\rangle$	$\langle(0.45, 0.55, 0.65, 0.75); 0.7, 0.8, 0.6, 0.3\rangle$	$\langle(0.01, 0.05, 0.10, 0.15); 0.1, 0.2, 0.3, 0.4\rangle$	$\langle(0.45, 0.55, 0.65, 0.75); 0.7, 0.8, 0.6, 0.3\rangle$	$\langle(0.15, 0.20, 0.25, 0.30); 0.4, 0.6, 0.2, 0.5\rangle$	$\langle(0.45, 0.55, 0.65, 0.75); 0.7, 0.8, 0.6, 0.3\rangle$	$\langle(0.01, 0.05, 0.10, 0.15); 0.1, 0.2, 0.3, 0.4\rangle$
M_6	$\langle(0.10, 0.20, 0.20, 0.30); 0.3, 0.4, 0.8, 0.1\rangle$	$\langle(0.15, 0.20, 0.25, 0.30); 0.4, 0.6, 0.2, 0.5\rangle$	$\langle(0.70, 0.80, 0.90, 1.00); 0.7, 0.8, 0.9, 0.2\rangle$	$\langle(0.25, 0.30, 0.35, 0.40); 0.4, 0.5, 0.6, 0.8\rangle$	$\langle(0.05, 0.10, 0.15, 0.20); 0.2, 0.3, 0.4, 0.1\rangle$	$\langle(0.05, 0.10, 0.15, 0.20); 0.2, 0.3, 0.4, 0.1\rangle$	$\langle(0.30, 0.35, 0.40, 0.45); 0.6, 0.1, 0.8, 0.4\rangle$

Decision Matrix-2 according to second group of the criteria ($\dot{c}_1, \dot{c}_2, ..., \dot{c}_9$):

M_1	$\langle(0.45, 0.55, 0.65, 0.75); 0.7, 0.8, 0.6, 0.3\rangle$	$\langle(0.30, 0.35, 0.40, 0.45); 0.6, 0.1, 0.8, 0.4\rangle$	$\langle(0.01, 0.05, 0.10, 0.15); 0.1, 0.2, 0.3, 0.4\rangle$	$\langle(0.05, 0.10, 0.15, 0.20); 0.2, 0.3, 0.4, 0.1\rangle$	$\langle(0.10, 0.15, 0.15, 0.20); 0.2, 0.4, 0.5, 0.3\rangle$	$\langle(0.30, 0.35, 0.40, 0.45); 0.6, 0.1, 0.8, 0.4\rangle$	$\langle(0.05, 0.10, 0.15, 0.20); 0.2, 0.3, 0.4, 0.1\rangle$	$\langle(0.50, 0.60, 0.70, 0.80); 0.1, 0.7, 0.8, 0.9\rangle$	$\langle(0.40, 0.45, 0.50, 0.55); 0.8, 0.9, 0.3, 0.6\rangle$
M_2	$\langle(0.01, 0.05, 0.10, 0.15); 0.1, 0.2, 0.3, 0.4\rangle$	$\langle(0.50, 0.60, 0.70, 0.80); 0.1, 0.7, 0.8, 0.9\rangle$	$\langle(0.15, 0.20, 0.25, 0.30); 0.4, 0.6, 0.2, 0.5\rangle$	$\langle(0.50, 0.60, 0.70, 0.80); 0.1, 0.7, 0.8, 0.9\rangle$	$\langle(0.30, 0.35, 0.40, 0.45); 0.6, 0.1, 0.8, 0.4\rangle$	$\langle(0.05, 0.10, 0.15, 0.20); 0.2, 0.3, 0.4, 0.1\rangle$	$\langle(0.70, 0.80, 0.90, 1.00); 0.7, 0.8, 0.9, 0.2\rangle$	$\langle(0.10, 0.20, 0.20, 0.30); 0.3, 0.4, 0.8, 0.1\rangle$	$\langle(0.30, 0.35, 0.40, 0.45); 0.6, 0.1, 0.8, 0.4\rangle$
M_3	$\langle(0.05, 0.10, 0.15, 0.20); 0.2, 0.3, 0.4, 0.1\rangle$	$\langle(0.05, 0.10, 0.15, 0.20); 0.2, 0.3, 0.4, 0.1\rangle$	$\langle(0.70, 0.80, 0.90, 1.00); 0.7, 0.8, 0.9, 0.2\rangle$	$\langle(0.30, 0.35, 0.40, 0.45); 0.6, 0.1, 0.8, 0.4\rangle$	$\langle(0.05, 0.10, 0.15, 0.20); 0.2, 0.3, 0.4, 0.1\rangle$	$\langle(0.25, 0.30, 0.35, 0.40); 0.4, 0.5, 0.6, 0.8\rangle$	$\langle(0.45, 0.55, 0.65, 0.75); 0.7, 0.8, 0.6, 0.3\rangle$	$\langle(0.01, 0.05, 0.10, 0.15); 0.1, 0.2, 0.3, 0.4\rangle$	$\langle(0.05, 0.10, 0.15, 0.20); 0.2, 0.3, 0.4, 0.1\rangle$
M_4	$\langle(0.15, 0.20, 0.25, 0.30); 0.4, 0.6, 0.2, 0.5\rangle$	$\langle(0.40, 0.45, 0.50, 0.55); 0.8, 0.9, 0.3, 0.6\rangle$	$\langle(0.10, 0.20, 0.20, 0.30); 0.3, 0.4, 0.8, 0.1\rangle$	$\langle(0.50, 0.60, 0.70, 0.80); 0.1, 0.7, 0.8, 0.9\rangle$	$\langle(0.05, 0.10, 0.15, 0.20); 0.2, 0.3, 0.4, 0.1\rangle$	$\langle(0.05, 0.10, 0.15, 0.20); 0.2, 0.3, 0.4, 0.1\rangle$	$\langle(0.30, 0.35, 0.40, 0.45); 0.6, 0.1, 0.8, 0.4\rangle$	$\langle(0.25, 0.30, 0.35, 0.40); 0.4, 0.5, 0.6, 0.8\rangle$	$\langle(0.50, 0.60, 0.70, 0.80); 0.1, 0.7, 0.8, 0.9\rangle$
M_5	$\langle(0.70, 0.80, 0.90, 1.00); 0.7, 0.8, 0.9, 0.2\rangle$	$\langle(0.25, 0.30, 0.35, 0.40); 0.4, 0.5, 0.6, 0.8\rangle$	$\langle(0.01, 0.05, 0.10, 0.15); 0.1, 0.2, 0.3, 0.4\rangle$	$\langle(0.25, 0.30, 0.35, 0.40); 0.4, 0.5, 0.6, 0.8\rangle$	$\langle(0.15, 0.20, 0.25, 0.30); 0.4, 0.6, 0.2, 0.5\rangle$	$\langle(0.25, 0.30, 0.35, 0.40); 0.4, 0.5, 0.6, 0.8\rangle$	$\langle(0.01, 0.05, 0.10, 0.15); 0.1, 0.2, 0.3, 0.4\rangle$	$\langle(0.15, 0.20, 0.25, 0.30); 0.4, 0.6, 0.2, 0.5\rangle$	$\langle(0.45, 0.55, 0.65, 0.75); 0.7, 0.8, 0.6, 0.3\rangle$
M_6	$\langle(0.01, 0.05, 0.10, 0.15); 0.1, 0.2, 0.3, 0.4\rangle$	$\langle(0.15, 0.20, 0.25, 0.30); 0.4, 0.6, 0.2, 0.5\rangle$	$\langle(0.40, 0.45, 0.50, 0.55); 0.8, 0.9, 0.3, 0.6\rangle$	$\langle(0.25, 0.30, 0.35, 0.40); 0.4, 0.5, 0.6, 0.8\rangle$	$\langle(0.01, 0.05, 0.10, 0.15); 0.1, 0.2, 0.3, 0.4\rangle$	$\langle(0.30, 0.35, 0.40, 0.45); 0.6, 0.1, 0.8, 0.4\rangle$	$\langle(0.25, 0.30, 0.35, 0.40); 0.4, 0.5, 0.6, 0.8\rangle$	$\langle(0.15, 0.20, 0.25, 0.30); 0.4, 0.6, 0.2, 0.5\rangle$	$\langle(0.05, 0.10, 0.15, 0.20); 0.2, 0.3, 0.4, 0.1\rangle$

Step 2 Find two weight vectors since we have two decision matrices:
Substep 1 For each decision matrix, construct a matrix consisting of real numbers by value of TFM-numbers obtain from defuzzification of each element of the decision matrices given above by using Definition 2.10 as follows:
Decision matrix for first group of criteria is built as follows:

$$(x_{ij})^{(1)}_{mxn} = \begin{pmatrix} 0.2000 & 0.2250 & 0.1250 & 0.2000 & 0.1250 & 0.4750 & 0.2350 \\ 0.6500 & 0.3250 & 0.4750 & 0.6500 & 0.1500 & 0.0779 & 0.3750 \\ 0.8500 & 0.0779 & 0.1250 & 0.2000 & 0.1250 & 0.8500 & 0.3750 \\ 0.6500 & 0.4750 & 0.3250 & 0.2000 & 0.1250 & 0.8500 & 0.3750 \\ 0.0779 & 0.6000 & 0.0779 & 0.6000 & 0.2250 & 0.6000 & 0.0779 \\ 0.2000 & 0.2250 & 0.8500 & 0.3250 & 0.1250 & 0.1250 & 0.3750 \end{pmatrix}$$

(columns $c_1, c_2, c_3, c_4, c_5, c_6, c_7$)

and decision matrix for second group of criteria is built as follows:

$$(x_{ij})^{(2)}_{mxn} = \begin{pmatrix} 0.6000 & 0.3750 & 0.2000 & 0.1230 & 0.1500 & 0.3750 & 0.1230 & 0.6500 & 0.4750 \\ 0.2000 & 0.6500 & 0.2250 & 0.6500 & 0.3750 & 0.1230 & 0.8500 & 0.2000 & 0.3750 \\ 0.1230 & 0.1230 & 0.8500 & 0.3750 & 0.1230 & 0.3250 & 0.6000 & 0.2000 & 0.1230 \\ 0.2250 & 0.4750 & 0.2000 & 0.6500 & 0.1230 & 0.1230 & 0.3750 & 0.3250 & 0.6500 \\ 0.8500 & 0.3250 & 0.2000 & 0.3250 & 0.2250 & 0.3250 & 0.2000 & 0.2250 & 0.6000 \\ 0.2000 & 0.2250 & 0.4750 & 0.3250 & 0.2000 & 0.3750 & 0.3250 & 0.2250 & 0.1230 \end{pmatrix}$$

(columns $\dot{c}_1, \dot{c}_2, \dot{c}_3, \dot{c}_4, \dot{c}_5, \dot{c}_6, \dot{c}_7, \dot{c}_8, \dot{c}_9$)

Substep 2 Find two weight vectors of criteria according to matrices given above:
Weight vector of first group of criteria is taken as follows:

$$w_1 = (0.1524, 0.1467, 0.0966, 0.1513, 0.1380, 0.1153, 0.1993)$$

Weight vector of second group of criteria is taken as follows:

$$w_2 = (0.1294, 0.09084, 0.1249, 0.0924, 0.0896, 0.1162, 0.1343, 0.0923, 0.1296)$$

Step 3 Find the aggregation values according to first group of criteria:

Aggregation results of $TFMGBAM_{\upsilon}^{(p,q,r)}(L_{i1}, L_{i2}, ..., L_{in})(i \in I_m)$

$$_1L_1 = TFMGBAM_{\upsilon}^{(1,1,1)}(L_{11}, L_{12}, ..., L_{17})$$
$$= \langle 0.1210, 0.1350, 0.1462, 0.1784; 0.1960, 0.2560, 0.2450, 0.1510 \rangle$$

Similarly,

$$_1L_2 = \langle 0.1135, 0.1350, 0.1680, 0.2080; 0.1370, 0.2190, 0.2930, 0.2950 \rangle$$
$$_1L_3 = \langle 0.1290, 0.1450, 0.1560, 0.1870; 0.2210, 0.2260, 0.3320, 0.1120 \rangle$$
$$_1L_4 = \langle 0.1380, 0.1660, 0.1870, 0.2040; 0.1990, 0.2330, 0.3230, 0.1920 \rangle$$
$$_1L_5 = \langle 0.1350, 0.1550, 0.1550, 0.1910; 0.1990, 0.2330, 0.3230, 0.1920 \rangle$$
$$_1L_6 = \langle 0.1020, 0.1144, 0.1450, 0.1468; 0.1830, 0.1840, 0.2780, 0.1380 \rangle$$

Step 4 Find the aggregation values according to second group of criteria:
Aggregation results of $TFMGBAM_{\upsilon}^{(p,q,r)}(L_{i1}, L_{i2}, ..., L_{in})$

$$_{2}L_{1} = TFMGBAM_{\upsilon}^{(1,1,1)}(L_{11}, L_{12}, ..., L_{19})$$
$$= \langle 0.1240, 0.1370, 0.1570, 0.1740); 0.1880, 0.2020, 0.2620, 0.1780 \rangle$$

Similarly,

$$_{2}L_{2} = \langle 0.1370, 0.1540, 0.1740, 0.1870; 0.1660, 0.2040, 0.3220, 0.1990 \rangle$$
$$_{2}L_{3} = \langle 0.0910, 0.1340, 0.1470, 0.1740; 0.1790, 0.2000, 0.2670, 0.1280 \rangle$$
$$_{2}L_{4} = \langle 0.1240, 0.1450, 0.1680, 0.2740; 0.1600, 0.2370, 0.2860, 0.2320 \rangle$$
$$_{2}L_{5} = \langle 0.1170, 0.1470, 0.1570, 0.1600; 0.1880, 0.2530, 0.2360, 0.2470 \rangle$$
$$_{2}L_{6} = \langle 0.0680, 0.1240, 0.1290, 0.1380; 0.1780, 0.2050, 0.2020, 0.2410 \rangle$$

Step 5 Compute the Hamming distance between $_{1}L_{i}$ and $_{2}L_{i}$ $(i \in I_{6})$ as follows:
For $TFMGBAM_{\upsilon}^{(p,q,r)}(L_{i1}, L_{i2}, ..., L_{in})$:

$$d(_{1}L_{1}, _{2}L_{1}) = 0.0034$$
$$d(_{1}L_{2}, _{2}L_{2}) = 0.0108$$
$$d(_{1}L_{3}, _{2}L_{3}) = 0.0128$$
$$d(_{1}L_{4}, _{2}L_{4}) = 0.0193$$
$$d(_{1}L_{5}, _{2}L_{5}) = 0.0100$$
$$d(_{1}L_{6}, _{2}L_{6}) = 0.0102$$

Step 6 Rank the results and determine the best alternative:

For $TFMGBAM_{\upsilon}^{(p,q,r)}$:

$$M_{4} < M_{3} < M_{2} < M_{6} < M_{5} < M_{1}$$

As seen, the best alternative is M_{4}.

5 Comparison Table

Table 3: Some rankings in terms of other methods and proposed methods

Methods	Operator	Ranking
Proposed Method 1	$TFMGBAM_{\upsilon}^{(1,1,1)}$	$M_{1} > M_{5} > M_{6} > M_{2} > M_{3} > M_{4}$
Proposed Method 2	$TFMGBGM_{\upsilon}^{(1,1,1)}$	$M_{3} > M_{5} > M_{1} > M_{6} > M_{4} > M_{2}$
Method of Kesen and Deli [15]	$TFMBHM_{v}^{(1,1)}$	$M_{1} > M_{3} > M_{5} > M_{2} > M_{4}$
Method of Deli and Keles [8]	$S^{i}(M_{i})$	$M_{5} > M_{3} > M_{4} > M_{1} > M_{2}$
Method of Ulucay et al. [29]	$TFMG_{v}$	$M_{5} > M_{3} > M_{4} > M_{1} > M_{2}$
Method of Sahin et al. [21]	D_{v}	$M_{3} > M_{5} > M_{1} > M_{4} > M_{2}$
Method of Ulucay [28]	S_{v}	$M_{4} > M_{3} > M_{1} > M_{5} > M_{2}$

In Table 3, we gave a brief comparison of introduced operators with some existing operators such as weighted Bonferroni harmonic mean operator given by Kesen and Deli [15], distance measure operator proposed by Deli and Keles [8], TFM weighted geometric operator introduced by Ulucay et al. [29], weighted dice vector similarity operator submitted by Sahin et al. [21] and vector similarity operator given by Ulucay [28]. Poverty and hunger has been still drawing attention all around the world. This is the main reason why we chose the selection of a model against poverty and hunger. As for operators we introduced, we used them efficiently on the problem. If the comparison table is analyzed, results of the proposed aggregation methods presents a new perspective to decision making process and generally compatible with the existing methods. Therefore, decision makers can easily use proposed methods to solve decision-making problems with multiple criteria.

6 Conclusion

In the paper we introduced two new aggregation methods in TFM-numbers which are called trapezoidal fuzzy multi generalized weighted Bonferroni arithmetic mean operator and trapezoidal fuzzy multi generalized weighted Bonferroni geometric mean operator. Then, we analyze their properties and special cases by changing p,q and r values. At the end, firstly we gave a solution process. Secondly, we gave numerical example to see application of the operators.

The paper mainly deals the methods to select the best model for fighting against poverty and hunger. These methods can efficiently used for such situations. The specific characteristic of these methods is that they deal with the three aggregated arguments instead of two or one. This makes these methods more sensitive. This is why the application of these methods in fighting against poverty and hunger performs well. In the future, new mathematical modelling will be proposed for selection problems in many areas which draw attention such as zero waste, artificial intelligence, machine learning, deep learning, big data etc.

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Chapter 10

Application areas of the Neutrosophic K-Means

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Abstract

With the increasing collection of data in areas such as healthcare, finance, social media, and scientific research, significant challenges arise in machine learning applications due to data issues such as noise, missing labels, and inconsistencies. This chapter examines neutrosophic theory, which includes truth, uncertainty, and falsity, as a framework to effectively manage these challenges. It presents studies that demonstrate improved robustness to clustering and classification tasks involving uncertain or missing data by integrating neutrosophic principles into machine learning models such as Neutrosophic K-Means. The study highlights the potential of neutrosophic to significantly advance data preprocessing and predictive performance in complex, uncertain environments by demonstrating how neutrosophic distance metrics can increase model accuracy compared to traditional approaches.

Keywords: Neutrosophic sets, Dataset, Prediction, Neutrosophic K-Means,

1 Introduction

With the rapid advances in technology, an enormous amount of data is being collected and accumulated across various fields such as healthcare, finance, social media, and scientific research. This real-world data, while highly valuable, often comes with a variety of challenges when it comes to utilization in machine learning or artificial intelligence models. Common issues include incomplete or missing labels, noisy data, inconsistencies, and gaps in the dataset. These challenges make it difficult to derive accurate insights or predictions without appropriate preprocessing or handling techniques. In particular, deep learning algorithms—despite their powerful capacity to model complex patterns in data—can face significant issues when learning from noisy or imperfect datasets. These algorithms are known to have a strong tendency to memorize the data they are trained on, which can lead to overfitting. Overfitting occurs when a model becomes so tailored to the training data, including noise and irrelevant patterns, that it fails to generalize well to unseen data. Studies exploring deep learning's response to noisy data have found that, even when using advanced techniques such as normalization (which scales the data to a consistent range) and dropout (which randomly disables neurons during training to prevent co-adaptation), there can still be a significant degradation in performance. Despite these regularization techniques, the presence of noise in the data continues to hinder the model's ability to effectively generalize, leading to substantial loss in performance when applied to real-world tasks. These findings highlight the ongoing challenge of developing robust deep learning models capable of learning from noisy and imperfect data without overfitting.

Neutrosophy, a theory developed by Florentin Smarandache to handle uncertainty, inconsistency, and incomplete information, introduces three essential components: truth (T), indeterminacy (I), and falsehood (F) [1]. Unlike binary logic, which assigns values of either true or false, or fuzzy logic, which assigns degrees of truth, neutrosophic logic allows for the representation of all three components simultaneously, offering a more detailed and flexible attempt to managing real-world data, which often contains noise, missing values, and contradictions [17-40].

The ability to work with uncertain, imprecise, and incomplete data makes neutrosophy particularly valuable in various domains. [2] For instance, in datasets with high levels of uncertainty, such as those in medical diagnostics, financial market analysis, and climate studies, traditional logic systems often fall short due to their rigid structure. Neutrosophic logic [41-50], however, provides a more versatile framework, which is especially useful when dealing with datasets where uncertainty and inconsistency cannot be eliminated but must instead be accounted for and managed. Recently, many researchers continued to work rapidly in this field. [51-67].

In data preprocessing for machine learning applications, the entity of noisy, missing, or contradictory information in datasets can reduce the error performance of models. To address this, before applying machine learning algorithms, it is essential to either cleanse the dataset of errors, inconsistencies, and missing information, or mark the uncertain data points, thus preserving the integrity of the dataset If deleting uncertain data points leads to the loss of valuable information, neutrosophic approaches allow for the retention of these data points by assigning them a degree of indeterminacy, rather than excluding them outright.

There are several machine learning algorithms that have been adapted to work with neutrosophic data, enhancing their ability to handle uncertainty. Algorithms such as Neutrosophic K-Means, Neutrosophic K-Nearest Neighbors (N-KNN), Neutrosophic Decision Trees (N-DT), and Neutrosophic Support Vector Machines (N-SVM) extend their traditional counterparts by integrating neutrosophic principles. These adaptations allow the algorithms to better classify, cluster, and predict outcomes from data with incomplete or contradictory information.

In addition, deep learning models, like Neutrosophic Neural Networks (N-NN) and Neutrosophic Deep Learning (N-DL), are gaining traction. These models can process data where the truth, falsehood, and indeterminacy of information are all simultaneously considered, making them more robust for tasks like image recognition, signal processing, and natural language processing, especially in environments where data is noisy or incomplete. For instance, in healthcare, where data can be imprecise or contain contradictory diagnoses, neutrosophic models can improve the accuracy of predictions and diagnoses by better handling uncertainty.

The main goal of applying neutrosophic approaches in machine learning and data mining is to extract meaningful patterns, clusters, and relationships from large datasets. In this context, clustering algorithms, which fall under the umbrella of unsupervised learning, are highly effective. Unlike supervised learning, where data is labeled, unsupervised learning algorithms, like neutrosophic clustering, must automatically detect the underlying structure in the data. This makes neutrosophic clustering especially useful in fields like social media analysis, where user behaviors and interactions are dynamic, or in earthquake prediction [3], where sensor data may be incomplete or contain noise. For instance, they transformed the error and missing data in the occupancy status information for various sectors, including data from cameras and other sensors, into neutrosphic sets (NS) in two ways. The first method is T(i), F(i), and I(i) for the data representing the i. sample in the dataset, while the second method is based on the principle that the i. and j. samples affect each other, expressed as T(i,j), F(i,j), and I(i,j). They concluded that using NS is more efficient than the other methods they examined [4]. Başer and Uluçay [31] defined the energy of a neutrosophic soft set and applied it to multi-criteria decision-making problems to show its applicability and effectiveness. And then, Başer and Uluçay [35] defined effective q- fuzzy soft expert sets.

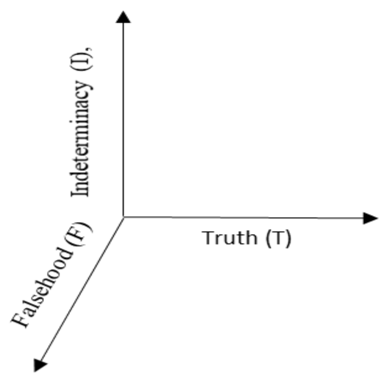
Although neutrosophic approaches offer significant theoretical potential, their application remains relatively limited, primarily due to the complexity of integrating neutrosophic principles into existing machine learning frameworks. However, as interest in managing uncertain and imprecise data grows, particularly in domains where traditional methods struggle, it is likely that neutrosophic methods will see broader adoption. Future research could focus on developing more accessible tools and frameworks for applying neutrosophy in a variety of practical applications, and on creating benchmark datasets that highlight the advantages of neutrosophic approaches over more traditional methods. This would include healthcare, finance, social media, climate studies, earthquake prediction, image and signal processing, and uncertainties for many scenarios, as more datasets emerge where neutrosophic approaches could be effective in practice. Single-valued neutrosophic numbers are used to deal with noisy data in the data preprocessing step as they provide a powerful capacity for modeling complex information [5].

2 Neutrosophic K-Means

The K-Means technique can be considered as a data mining and machine learning algorithm used to process large amounts of data by clustering or grouping. It lies at the intersection between these two fields because it serves the purposes of both fields. Neutrosophic K-Nearest Neighbor (KNN) is a version of the K-Nearest Neighbor (KNN) algorithm adapted to Neutrosophic logic. While KNN classifies by looking at the neighbors of each data point, Neutrosophic KNN makes decisions by taking into account the accuracy, uncertainty and error components of each data point.

Truth (T): Represents the degree of truth or certainty about an object.
Indeterminacy (I): Represents the degree of uncertainty or indeterminacy.
Falsehood (F): Represents the degree of falsehood or contradiction

Figure 1: K-Means Algorithm for Neutrosophic Explanation of Components



It is used in classification problems and provides more reliable results especially in noisy, missing or contradictory data. K-means clustering algorithm is one of the widely used algorithms considering its easy-to-understand structure, fast convergence and ease of application to ML algorithms [6]. However, since the number K must be given as input to the algorithm and the correct choice of K directly affects the result, the choice of the K value is very important [7].

Neutrosophic Distance: Neutrosophic distance is used instead of the classical Euclidean distance in KNN. This metric is an effective way to handle and interpret uncertain, imprecise, or contradictory information in various machine learning and data analysis applications, especially in fields like medical diagnosis, environmental studies, and social sciences, where data uncertainty is common

Neighbor Selection: The distance between each data point and its neighbors is determined according to the accuracy, uncertainty and error components.

Neutrosophic distance is calculated with a formula based on accuracy, uncertainty and error weights. The weights of the components W_T truth, W_I uncertainty and W_F inaccuracy. These weights are usually chosen so that their sum is 1.

$$W_T + W_I + W_F = 1$$

Data points are positioned in three dimensions as accuracy, uncertainty and inaccuracy axes. Using Euclidean distance, the weights of the contribution of each axis to the distance are calculated and adjusted. This distance determines the similarity or difference between two Neutrosophic data points.

Neutrosophic Distance Formula; Let there be two data points A and B.

$\boldsymbol{T_A}, I_A, F_A \rightarrow$ are the neutrosophic components of data point A

$\boldsymbol{T_B}, I_B, F_B \rightarrow$ are the components of the cluster center B

$$d_N(A,B) = \sqrt{W_T\,(T_A - T_B)^2 + W_I(I_A - I_B)^2 + W_F(F_A - F_B)^2}$$

The weights of W_T, W_I, W_F can be adjusted according to the importance of the application, for example, in scenarios where uncertainty is critical, such as spam filters [8], the uncertainty coefficient can be increased. Thus helping machine learning algorithms work more effectively with uncertain and incomplete data.

The K-Means algorithm is one of the unsupervised learning methods and is usually used to separate similar data in a dataset into groups (clustering) [9].

Table 1: K-Means algorithm

Here is the pseudo code of the K-Means algorithm [10].

```
1. K = number_of_clusters (Determine the number of clusters K)
2. centroids = randomly select K data_points (Randomly select K cluster centers)
3. for each data_point: (Assign each data point to the nearest cluster center)
        find the nearest centroid
        assign data_point to this cluster
4. for each cluster: (Calculate new cluster centers for each cluster)
        calculate the mean of all data_points in the cluster
        update the centroid
5. new_assignment = False (Reassign data points to the new cluster centers)
   for each data_point:
        find the nearest new centroid
        if the cluster changed:
             assign data_point to the new cluster
             new_assignment = True
6. if new_assignment is True: (Repeat iterations until cluster centers do not change)
        go back to step 4
7. results = final cluster assignment for each data_point (Once clustering is complete, obtain the results)
```

3 Conclusions

Deep learning algorithms are prone to memorization and suffer from overfitting, especially when learning from noisy data. Research has shown a significant loss in performance, despite the use of normalization and dropout techniques [11]. Another study [12] examined the problem of noisy label data negatively impacting classification performance, concluding that traditional machine learning algorithms also struggle with label noise, and dropout alone is insufficient to prevent overfitting. Detecting and cleaning erroneous data with various heuristics has been proposed as an alternative solution. However, as these existing methods have limitations, this remains an open area for further research. Interestingly, many studies analyzing machine learning algorithms do not consider neutrosophy as a potential solution to these challenges. Nonetheless, there are examples of leveraging uncertainty in classification problems. For instance, a study that applied Rough Neutrosophic Sets (RNS) in the tourism sector with k-means clustering found that identifying essential and non-essential attributes using RNS (though computationally intensive) led to improved classification performance [13]. Monitoring seismic data and denoising the obtained data is a challenging task. In the study where they combined the K-Means method with neutrosophic, they applied it to earthquake datasets in Ecuador. They concluded that it is more effective in determining patterns in the data in the presence of data that partially belong to more than one cluster and the results are improved [14]. Image data consists of pixels, if we think of pixels as a two-dimensional array, we add the values of accuracy, inaccuracy and uncertainty to this array for each pixel, thus converting the image into a neutrosophic cluster. Clusters are performed iteratively for segmentation of the image, and the image is segmented at the point where the number of clusters stops increasing. With this method, they achieved better results than the traditional K-means [15]. They proposed an optimized NS K-mean to increase the performance of the Automatic Vehicle License Plate Recognition System. When there are distortions in the acquired image data, traditional license plate recognition methods achieved 79% accuracy, while their proposed method achieved 92.5% [16]. These examples illustrate how neutrosophy can achieve remarkable success in clustering, capturing attention as a valuable addition to traditional methods.

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A *Neutrosophic Algebraic Structures and Their Applications*ؘ ١Ƙ6ı167ؗ

Algebraic Structures In the Universe of Neutrosophic: Analysis with Innovative Algorithmic Approaches

Neutrosophic Logic is a general framework for unification of many existing logics, such as fuzzy logic (especially intuitionistic fuzzy logic), paraconsistent logic, intuitionistic logic, etc. University of New Mexico (UNM)'s website on neutrosophic theories and their applications is: http://fs.unm.edu/neutrosophy.htm

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