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## APPLICATIONS OF NEUTROSOPHIC $\mathcal{N}$ -STRUCTURES IN TERNARY SEMIGROUPS

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**Abstract.** In this paper, the notions of neutrosophic  $\mathcal{N}$ -ternary subsemigroups of ternary semigroups are introduced and several properties are investigated.

**Keywords:** ternary semigroups; neutrosophic  $\mathcal{N}$ -structures; neutrosophic  $\mathcal{N}$ -ternary subsemigroups;  $(\alpha, \beta, \gamma)$ -level sets;  $\varepsilon$ -neutrosophic  $\mathcal{N}$ -ternary subsemigroups.

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### 1. INTRODUCTION

The concept of neutrosophic logics was first introduced by Smarandache [11] in 1999. In a neutrosophic set, an element has three associated defining functions characterized by the truth membership function ( $T$ ), the indeterminate membership function ( $I$ ) and the false membership function ( $F$ ) defined on a universe of discourse  $X$ . These three functions are completely independent. Jun et al. [3] introduced a new function, called a negative-valued function, and constructed  $\mathcal{N}$ -structures in 2009. Khan et al. [5] discussed neutrosophic  $\mathcal{N}$ -structures and their

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applications in semigroups in 2017. Jun et al. [3, 4, 12] considered neutrosophic  $\mathcal{N}$ -structures applied to BCK/BCI-algebras and neutrosophic commutative  $\mathcal{N}$ -ideals in BCK-algebras in 2017. Rangasuk et al. [8] discussed neutrosophic  $\mathcal{N}$ -structures and their applications in UP-algebras. Recently, Al-Tahan and Davvaz [1, 2] studied applications of neutrosophic sets and neutrosophic  $\mathcal{N}$ -structures on some algebraic structures and hyperstructures.

The notion of ternary semigroups was known to Banach (cf. Los [6]) who is credited with an example of a ternary semigroup which does not reduce to a semigroup. Los showed that every ternary semigroup can be imbedded in a semigroup [6]. Many results in semigroups were extended to ternary semigroups. Many applications of fuzzy sets and generalized fuzzy sets were studied in ternary semigroups (for example, [7], [10], [13], [14] and [15]).

The aim of this paper is to extend the results from semigroups [5] to ternary semigroups. We introduce some basic notations and definitions in section 2. The third section contains some results on neutrosophic  $\mathcal{N}$ -structures in ternary semigroups. The final section is the conclusion.

## 2. PRELIMINARIES

This section collects some basic notations and definitions needed later.

### 2.1. Ternary Semigroups

In this subsection, we introduce ternary semigroups, ternary subsemigroups, and homomorphisms (cf. [9]).

**Definition 2.1.** Let  $T$  be a nonempty set. The set  $T$  with a ternary operation  $(a, b, c) \mapsto [abc]$  is said to be a *ternary semigroup* if it satisfies the associative law:

$$[[abc]uv] = [a[bcu]v] = [ab[cuv]]$$

for all  $a, b, c, u, v \in T$ .

Any semigroup can be transformed to a ternary semigroup by defining the ternary product to be  $[abc] := abc$ .

**Definition 2.2.** Let  $T$  be a ternary semigroup. A nonempty subset  $S$  of  $T$  is said to be a *ternary subsemigroup* of  $T$  if  $[abc] \in S$  for all  $a, b, c \in S$ .

**Definition 2.3.** Let  $A$  and  $B$  be two ternary semigroups. A mapping  $f : A \rightarrow B$  is said to be a *homomorphism* if

$$f([xyz]) = [f(x)f(y)f(z)]$$

for all  $x, y, z \in A$ .

### 2.2. Neutrosophic $\mathcal{N}$ -structures

The purpose of this subsection is to recall the definitions of neutrosophic  $\mathcal{N}$ -structure, the union and the intersection of two neutrosophic  $\mathcal{N}$ -structures, and  $(\alpha, \beta, \gamma)$ -level set (cf. [5]).

The collection of functions from a set  $X$  to the interval  $[-1, 0]$  is denoted by  $F(X, [-1, 0])$ . An element of  $F(X, [-1, 0])$  is called a *negative-valued function* from  $X$  to the interval  $[-1, 0]$ . An ordered pair  $(X, f)$  of  $X$  and an  $\mathcal{N}$ -function  $f$  on  $X$  is said to be an  $\mathcal{N}$ -structure. Let  $X$  be the nonempty universe of discourse unless otherwise specified.

**Definition 2.4.** A *neutrosophic  $\mathcal{N}$ -structure* over  $X$  is the structure

$$X_N := \frac{X}{(T_N, I_N, F_N)} = \left\{ \frac{x}{(T_N(x), I_N(x), F_N(x))} \mid x \in X \right\}$$

where  $T_N, I_N$  and  $F_N$  are  $\mathcal{N}$ -functions on  $X$  which are called the *negative truth membership function*, the *negative indeterminacy membership function*, and the *negative falsity membership function* on  $X$ , respectively.

**Definition 2.5.** Let  $X_N := \frac{X}{(T_N, I_N, F_N)}$  and  $X_M := \frac{X}{(T_M, I_M, F_M)}$  be neutrosophic  $\mathcal{N}$ -structures over  $X$ . If  $X_M$  satisfies the conditions

$$T_N(x) \geq T_M(x), I_N(x) \leq I_M(x), F_N(x) \geq F_M(x)$$

for all  $x \in X$ , then  $X_N$  is called a *neutrosophic  $\mathcal{N}$ -substructure* of  $X_M$  and denoted by  $X_N \subseteq X_M$ . If  $X_N \subseteq X_M$  and  $X_M \subseteq X_N$ , we say that  $X_N = X_M$ .

**Definition 2.6.** Let  $\{a_i \mid i \in \Lambda\}$  be a family of real numbers. We define

$$\bigvee \{a_i \mid i \in \Lambda\} := \begin{cases} \max\{a_i \mid i \in \Lambda\} & \text{if } \Lambda \text{ is finite,} \\ \sup\{a_i \mid i \in \Lambda\} & \text{otherwise} \end{cases}$$

and

$$\bigwedge \{a_i \mid i \in \Lambda\} := \begin{cases} \min\{a_i \mid i \in \Lambda\} & \text{if } \Lambda \text{ is finite,} \\ \inf\{a_i \mid i \in \Lambda\} & \text{otherwise.} \end{cases}$$

For any two real numbers  $a$  and  $b$ , we use  $a \vee b$  and  $a \wedge b$  instead of  $\vee\{a, b\}$  and  $\wedge\{a, b\}$ , respectively.

**Definition 2.7.** Let  $X_N := \frac{X}{(T_N, I_N, F_N)}$  and  $X_M := \frac{X}{(T_M, I_M, F_M)}$  be neutrosophic  $\mathcal{N}$ -structures on  $X$ . Then the *union* of  $X_N$  and  $X_M$  is a neutrosophic  $\mathcal{N}$ -structure

$$X_{NUM} := \frac{X}{(T_{NUM}, I_{NUM}, F_{NUM})}$$

where

$$T_{NUM}(x) = T_N(x) \wedge T_M(x), I_{NUM}(x) = I_N(x) \vee I_M(x), F_{NUM}(x) = F_N(x) \wedge F_M(x)$$

for all  $x \in X$ .

**Definition 2.8.** Let  $X_N := \frac{X}{(T_N, I_N, F_N)}$  and  $X_M := \frac{X}{(T_M, I_M, F_M)}$  be neutrosophic  $\mathcal{N}$ -structures on  $X$ . Then the *intersection* of  $X_N$  and  $X_M$  is a neutrosophic  $\mathcal{N}$ -structure

$$X_{N \cap M} := \frac{X}{(T_{N \cap M}, I_{N \cap M}, F_{N \cap M})}$$

where

$$T_{N \cap M}(x) = T_N(x) \vee T_M(x), I_{N \cap M}(x) = I_N(x) \wedge I_M(x), F_{N \cap M}(x) = F_N(x) \vee F_M(x)$$

for all  $x \in X$ .

**Definition 2.9.** Let  $X_N := \frac{X}{(T_N, I_N, F_N)}$  be a neutrosophic  $\mathcal{N}$ -structures over  $X$ . Then the *complement* of  $X_N$  is defined to a neutrosophic  $\mathcal{N}$ -structure

$$X_{N^c} := \frac{X}{(T_{N^c}, I_{N^c}, F_{N^c})}$$

where

$$T_{N^c}(x) = -1 - T_N(x), I_{N^c}(x) = -1 - I_N(x), F_{N^c}(x) = -1 - F_N(x)$$

for all  $x \in X$ .

**Definition 2.10.** Let  $X_N$  be a neutrosophic  $\mathcal{N}$ -structure over  $X$  and let  $\alpha, \beta, \gamma$  be real numbers on the interval  $[-1, 0]$ . Consider the following sets:

$$T_N^\alpha := \{x \in X \mid T_N(x) \leq \alpha\},$$

$$I_N^\beta := \{x \in X \mid I_N(x) \geq \beta\},$$

$$F_N^\gamma := \{x \in X \mid F_N(x) \leq \gamma\}.$$

The set

$$X_N(\alpha, \beta, \gamma) := \{x \in X \mid T_N(x) \leq \alpha, I_N(x) \geq \beta, F_N(x) \leq \gamma\}$$

is called an  $(\alpha, \beta, \gamma)$ -level set of  $X_N$ . We note that

$$X_N(\alpha, \beta, \gamma) = T_N^\alpha \cap I_N^\beta \cap F_N^\gamma.$$

### 3. MAIN RESULTS

In this section, we discuss on neutrosophic  $\mathcal{N}$ -ternary subsemigroups, the  $(\alpha, \beta, \gamma)$ -level set, the intersection of neutrosophic  $\mathcal{N}$ -ternary subsemigroups, neutrosophic  $\mathcal{N}$ -products,  $\varepsilon$ -neutrosophic  $\mathcal{N}$ -ternary subsemigroups, homomorphic preimage of the neutrosophic  $\mathcal{N}$ -ternary subsemigroup and onto homomorphic image of the neutrosophic  $\mathcal{N}$ -ternary subsemigroup. Throughout this section, we denote a ternary semigroup  $X$  as the universe of discourse unless otherwise specified.

**Definition 3.1.** Let  $X_N$  be a neutrosophic  $\mathcal{N}$ -structure over  $X$ . Then  $X_N$  is said to be a neutrosophic  $\mathcal{N}$ -ternary subsemigroup of  $X$  if it satisfies the following conditions:

$$T_N([xyz]) \leq \bigvee \{T_N(x), T_N(y), T_N(z)\},$$

$$I_N([xyz]) \geq \bigwedge \{I_N(x), I_N(y), I_N(z)\},$$

$$F_N([xyz]) \leq \bigvee \{F_N(x), F_N(y), F_N(z)\},$$

for all  $x, y, z \in X$ .

**Theorem 3.1.** Let  $X_N$  be a neutrosophic  $\mathcal{N}$ -structure over  $X$  and let  $\alpha, \beta, \gamma$  be real numbers on the interval  $[-1, 0]$ . If  $X_N$  is a neutrosophic  $\mathcal{N}$ -ternary subsemigroup of  $X$ , then the nonempty  $(\alpha, \beta, \gamma)$ -level set of  $X_N$  is a ternary subsemigroup of  $X$ .

*Proof.* Suppose that  $X_N(\alpha, \beta, \gamma) \neq \emptyset$  for  $\alpha, \beta, \gamma \in [-1, 0]$ . Let  $x, y, z \in X_N(\alpha, \beta, \gamma)$ . Thus

$$T_N(x) \leq \alpha, I_N(x) \geq \beta, F_N(x) \leq \gamma,$$

$$T_N(y) \leq \alpha, I_N(y) \geq \beta, F_N(y) \leq \gamma,$$

$$T_N(z) \leq \alpha, I_N(z) \geq \beta, F_N(z) \leq \gamma.$$

It follows that

$$T_N([xyz]) = \bigvee \{T_N(x), T_N(y), T_N(z)\} \leq \alpha,$$

$$I_N([xyz]) = \bigwedge \{I_N(x), I_N(y), I_N(z)\} \geq \beta,$$

$$F_N([xyz]) = \bigvee \{F_N(x), F_N(y), F_N(z)\} \leq \gamma.$$

Hence  $[xyz] \in X_N(\alpha, \beta, \gamma)$ . It implies that  $X_N(\alpha, \beta, \gamma)$  is a ternary subsemigroup of  $X$ .  $\square$

**Theorem 3.2.** *Let  $X_N$  be a neutrosophic  $\mathcal{N}$ -structure over  $X$ . If  $T_N^\alpha, I_N^\beta$  and  $F_N^\gamma$  are ternary subsemigroups of  $X$  for all  $\alpha, \beta, \gamma \in [-1, 0]$ , then  $X_N$  is a neutrosophic  $\mathcal{N}$ -ternary subsemigroup of  $X$ .*

*Proof.* We prove this theorem by contradiction. Assume first that there exist  $a, b, c \in X$  such that  $T_N([abc]) > \bigvee \{T_N(a), T_N(b), T_N(c)\}$ . Then  $T_N([abc]) > t_\alpha \geq \bigvee \{T_N(a), T_N(b), T_N(c)\}$  for some  $t_\alpha \in [-1, 0)$ . Hence  $a, b, c \in T_N^{t_\alpha}$ , but  $[abc] \notin T_N^{t_\alpha}$ , which is a contradiction. Therefore

$$T_N([xyz]) \leq \bigvee \{T_N(x), T_N(y), T_N(z)\}$$

for all  $x, y, z \in X$ .

Now assume that there are  $a, b, c \in X$  such that  $I_N([abc]) < \bigwedge \{I_N(a), I_N(b), I_N(c)\}$ . Then  $a, b, c \in I_N^{t_\beta}$  and  $[abc] \notin I_N^{t_\beta}$  for  $t_\beta := \bigwedge \{I_N(a), I_N(b), I_N(c)\}$ . This is a contradiction. Hence

$$I_N([xyz]) \geq \bigwedge \{I_N(x), I_N(y), I_N(z)\}$$

for all  $x, y, z \in X$ .

Finally, suppose that there are  $a, b, c \in X$  such that  $F_N([abc]) > \bigvee \{F_N(a), F_N(b), F_N(c)\}$ . Then  $F_N([abc]) > t_\gamma \geq \bigvee \{F_N(a), F_N(b), F_N(c)\}$  for some  $t_\gamma \in [-1, 0)$ . This implies that  $a, b, c \in T_N^{t_\gamma}$ ,

but  $[abc] \notin T_N^{ty}$ , which is a contradiction. Therefore

$$F_N([xyz]) \leq \bigvee \{F_N(x), F_N(y), F_N(z)\}$$

for all  $x, y, z \in X$ .

Therefore  $X_N$  is a neutrosophic  $\mathcal{N}$ -ternary subsemigroup of  $X$ . □

**Theorem 3.3.** Let  $X_N := \frac{X}{(T_N, I_N, F_N)}$  and  $X_M := \frac{X}{(T_M, I_M, F_M)}$  be neutrosophic  $\mathcal{N}$ -ternary subsemigroups of  $X$ . The intersection of  $X_N$  and  $X_M$ ,  $X_{N \cap M}$ , is also a neutrosophic  $\mathcal{N}$ -ternary subsemigroup of  $X$ .

*Proof.* Let  $x, y, z \in X$ . Then

$$\begin{aligned} T_{N \cap M}([xyz]) &= \bigvee \{T_N([xyz]), T_M([xyz])\} \\ &\leq \bigvee \{ \bigvee \{T_N(x), T_N(y), T_N(z)\}, \bigvee \{T_M(x), T_M(y), T_M(z)\} \} \\ &= \bigvee \{ \bigvee \{T_N(x), T_M(x)\}, \bigvee \{T_N(y), T_M(y)\}, \bigvee \{T_N(z), T_M(z)\} \} \\ &= \bigvee \{T_{N \cap M}(x), T_{N \cap M}(y), T_{N \cap M}(z)\}, \end{aligned}$$

$$\begin{aligned} I_{N \cap M}([xyz]) &= \bigwedge \{I_N([xyz]), I_M([xyz])\} \\ &\geq \bigwedge \{ \bigwedge \{I_N(x), I_N(y), I_N(z)\}, \bigwedge \{I_M(x), I_M(y), I_M(z)\} \} \\ &= \bigwedge \{ \bigwedge \{I_N(x), I_M(x)\}, \bigwedge \{I_N(y), I_M(y)\}, \bigwedge \{I_N(z), I_M(z)\} \} \\ &= \bigwedge \{I_{N \cap M}(x), I_{N \cap M}(y), I_{N \cap M}(z)\} \end{aligned}$$

and

$$\begin{aligned} F_{N \cap M}([xyz]) &= \bigvee \{F_N([xyz]), F_M([xyz])\} \\ &\leq \bigvee \{ \bigvee \{F_N(x), F_N(y), F_N(z)\}, \bigvee \{F_M(x), F_M(y), F_M(z)\} \} \\ &= \bigvee \{ \bigvee \{F_N(x), F_M(x)\}, \bigvee \{F_N(y), F_M(y)\}, \bigvee \{F_N(z), F_M(z)\} \} \\ &= \bigvee \{F_{N \cap M}(x), F_{N \cap M}(y), F_{N \cap M}(z)\} \end{aligned}$$

for all  $x, y, z \in X$ . Thus  $X_{N \cap M}$  is a neutrosophic  $\mathcal{N}$ -ternary subsemigroup of  $X$ . □



**Corollary 3.4.** *Let  $\{X_{N_i} \mid i \in \mathbb{N}\}$  be a family of neutrosophic  $\mathcal{N}$ -ternary subsemigroup of  $X$ . Then the intersection of  $X_{N_i}$ , denoted by  $X_{\cap N_i}$ , is also a neutrosophic  $\mathcal{N}$ -ternary subsemigroup of  $X$ .*

Let  $X_N := \frac{X}{(T_N, I_N, F_N)}$ ,  $X_M := \frac{X}{(T_M, I_M, F_M)}$  and  $X_Q := \frac{X}{(T_Q, I_Q, F_Q)}$  be neutrosophic  $\mathcal{N}$ -structures over  $X$ . The neutrosophic  $\mathcal{N}$ -product of  $X_N, X_M$  and  $X_Q$  is defined to be a neutrosophic  $\mathcal{N}$ -structure over  $X$

$$\begin{aligned} X_N \odot X_M \odot X_Q &:= \frac{X}{(T_{N \circ M \circ Q}, I_{N \circ M \circ Q}, F_{N \circ M \circ Q})} \\ &= \left\{ \frac{x}{(T_{N \circ M \circ Q}(x), I_{N \circ M \circ Q}(x), F_{N \circ M \circ Q}(x))} \mid x \in X \right\} \end{aligned}$$

where

$$\begin{aligned} T_{N \circ M \circ Q} &= \begin{cases} \bigwedge_{x=[abc]} \{T_N(a) \vee T_M(b) \vee T_Q(c)\} & \text{if } a, b, c \in X \text{ such that } x = [abc], \\ 0 & \text{otherwise,} \end{cases} \\ I_{N \circ M \circ Q} &= \begin{cases} \bigvee_{x=[abc]} \{I_N(a) \wedge I_M(b) \wedge I_Q(c)\} & \text{if } a, b, c \in X \text{ such that } x = [abc], \\ -1 & \text{otherwise,} \end{cases} \\ F_{N \circ M \circ Q} &= \begin{cases} \bigwedge_{x=[abc]} \{F_N(a) \vee F_M(b) \vee F_Q(c)\} & \text{if } a, b, c \in X \text{ such that } x = [abc], \\ 0 & \text{otherwise.} \end{cases} \end{aligned}$$

For any  $x \in X$ , the element  $\frac{x}{(T_{N \circ M \circ Q}(x), I_{N \circ M \circ Q}(x), F_{N \circ M \circ Q}(x))}$  is simply denoted by

$$(X_N \odot X_M \odot X_Q)(x) := (T_{N \circ M \circ Q}(x), I_{N \circ M \circ Q}(x), F_{N \circ M \circ Q}(x)).$$

**Theorem 3.5.** *A neutrosophic  $\mathcal{N}$ -structure  $X_N$  over  $X$  is a neutrosophic ternary  $\mathcal{N}$ -ternary subsemigroup of  $X$  if and only if  $X_N \odot X_N \odot X_N \subseteq X_N$ .*

*Proof.* To show the necessity condition, we assume that  $X_N$  is a neutrosophic  $\mathcal{N}$ -ternary subsemigroup of  $X$ . Let  $x$  be an element of  $X$ . If  $x \neq [abc]$  for all  $a, b, c \in X$ , then it is clear that

$X_N \odot X_N \odot X_N \subseteq X_N$ . Suppose that there are  $a, b, c \in X$  such that  $x = [abc]$ .

$$T_{N \circ N \circ N}(x) = \bigwedge_{x=[abc]} \{T_N(a) \vee T_N(b) \vee T_N(c)\} \geq \bigwedge_{x=[abc]} T_N([abc]) = T_N(x).$$

$$I_{N \circ N \circ N}(x) = \bigvee_{x=[abc]} \{I_N(a) \wedge I_N(b) \wedge I_N(c)\} \leq \bigvee_{x=[abc]} I_N([abc]) = I_N(x),$$

$$F_{N \circ N \circ N}(x) = \bigwedge_{x=[abc]} \{F_N(a) \vee F_N(b) \vee F_N(c)\} \geq \bigwedge_{x=[abc]} F_N([abc]) = F_N(x).$$

Hence  $X_N \odot X_N \odot X_N \subseteq X_N$ .

Conversely, let  $X_N$  be any neutrosophic  $\mathcal{N}$ -structure over  $X$  such that  $X_N \odot X_N \odot X_N \subseteq X_N$ .

Let  $x, y, z$  be any elements of  $X$  and let  $d = [xyz]$ . Then

$$T_N([xyz]) = T_N(d) \leq T_{N \circ N \circ N}(d) = \bigwedge_{d=[abc]} \{T_N(a) \vee T_N(b) \vee T_N(c)\} \leq T_N(x) \vee T_N(y) \vee T_N(z),$$

$$I_N([xyz]) = I_N(d) \geq I_{N \circ N \circ N}(d) = \bigvee_{d=[abc]} \{I_N(a) \wedge I_N(b) \wedge I_N(c)\} \geq I_N(x) \wedge I_N(y) \wedge I_N(z),$$

$$F_N([xyz]) = F_N(d) \leq F_{N \circ N \circ N}(d) = \bigwedge_{d=[abc]} \{F_N(a) \vee F_N(b) \vee F_N(c)\} \leq F_N(x) \vee F_N(y) \vee F_N(z).$$

Therefore  $X_N$  is a neutrosophic  $\mathcal{N}$ -ternary subsemigroup of  $X$ . □

**Theorem 3.6.** *Let  $X$  be a ternary semigroup with identity  $e$ . Let  $X_N := \frac{X}{(T_N, I_N, F_N)}$  be a neutrosophic  $\mathcal{N}$ -structure over  $X$  such that  $X_N(e) \geq X_N(x)$  for all  $x \in X$ , that is,  $T_N(e) \leq T_N(x), I_N(e) \geq I_N(x)$  and  $F_N(e) \leq F_N(x)$  for all  $x \in X$ . If  $X_N$  is a neutrosophic  $\mathcal{N}$ -ternary subsemigroup of  $X$ , then  $X_N \odot X_N \odot X_N = X_N$ .*

*Proof.* For any  $x \in X$ , we have

$$T_{N \circ N \circ N}(x) = \bigwedge_{x=[abc]} \{T_N(a) \vee T_N(b) \vee T_N(c)\} \leq T_N(x) \vee T_N(e) = T_N(x),$$

$$I_{N \circ N \circ N}(x) = \bigvee_{x=[abc]} \{I_N(a) \wedge I_N(b) \wedge I_N(c)\} \geq I_N(x) \wedge I_N(e) = I_N(x),$$

$$F_{N \circ N \circ N}(x) = \bigwedge_{x=[abc]} \{F_N(a) \vee F_N(b) \vee F_N(c)\} \leq F_N(x) \vee F_N(e) = F_N(x).$$

It implies that  $X_N \subseteq X_N \odot X_N \odot X_N$ . By Theorem 3.5, we know that  $X_N \odot X_N \odot X_N \subseteq X_N$ .

Therefore  $X_N \odot X_N \odot X_N = X_N$ . □

**Definition 3.2.** Let  $X_N$  be a neutrosophic  $\mathcal{N}$ -structure over  $X$ . Then  $X_N$  is said to be an  $\varepsilon$ -neutrosophic  $\mathcal{N}$ -ternary subsemigroup of  $X$  if it satisfies the conditions

$$\begin{aligned} T_N([xyz]) &\leq \bigvee \{T_N(x), T_N(y), T_N(z), \varepsilon_T\}, \\ I_N([xyz]) &\geq \bigwedge \{I_N(x), I_N(y), I_N(z), \varepsilon_I\}, \\ F_N([xyz]) &\leq \bigvee \{F_N(x), F_N(y), F_N(z), \varepsilon_F\}, \end{aligned}$$

for all  $x, y, z \in X$  where  $\varepsilon_T, \varepsilon_I, \varepsilon_F \in [-1, 0]$ .

**Proposition 3.7.** Let  $X_N$  be an  $\varepsilon$ -neutrosophic  $\mathcal{N}$ -ternary subsemigroup of  $X$ . Then  $X_N$  is a neutrosophic  $\mathcal{N}$ -ternary subsemigroup of  $X$  if it satisfies the condition  $X_N(x) \leq (\varepsilon_T, \varepsilon_I, \varepsilon_F)$  for all  $x \in X$ , that is,  $T_N(x) \geq \varepsilon_T, I_N(x) \leq \varepsilon_I$  and  $F_N(x) \geq \varepsilon_F$ .

**Theorem 3.8.** Let  $X_N$  be a neutrosophic  $\mathcal{N}$ -structure over  $X$  and let  $\alpha, \beta, \gamma$  be real numbers on the interval  $[-1, 0]$ . If  $X_N$  is an  $\varepsilon$ -neutrosophic  $\mathcal{N}$ -ternary subsemigroup of  $X$ , then the  $(\alpha, \beta, \gamma)$ -level set of  $X_N$  is a ternary subsemigroup of  $X$  whenever  $(\alpha, \beta, \gamma) \leq (\varepsilon_T, \varepsilon_I, \varepsilon_F)$ , that is,  $\alpha \geq \varepsilon_T, \beta \leq \varepsilon_I$  and  $\gamma \geq \varepsilon_F$ .

*Proof.* Assume that  $X_N(\alpha, \beta, \gamma) \neq \emptyset$  for  $\alpha, \beta, \gamma \in [-1, 0]$ . Let  $x, y, z \in X_N(\alpha, \beta, \gamma)$ . Then

$$\begin{aligned} T_N(x) &\leq \alpha, I_N(x) \geq \beta, F_N(x) \leq \gamma, \\ T_N(y) &\leq \alpha, I_N(y) \geq \beta, F_N(y) \leq \gamma, \\ T_N(z) &\leq \alpha, I_N(z) \geq \beta, F_N(z) \leq \gamma. \end{aligned}$$

It implies that

$$\begin{aligned} T_N([xyz]) &\leq \bigvee \{T_N(x), T_N(y), T_N(z), \varepsilon_T\} \leq \bigvee \{\alpha, \varepsilon_T\} = \alpha, \\ I_N([xyz]) &\geq \bigwedge \{I_N(x), I_N(y), I_N(z), \varepsilon_I\} \geq \bigwedge \{\beta, \varepsilon_I\} = \beta, \\ F_N([xyz]) &\leq \bigvee \{F_N(x), F_N(y), F_N(z), \varepsilon_F\} \leq \bigvee \{\gamma, \varepsilon_F\} = \gamma. \end{aligned}$$

Hence  $[xyz] \in X_N(\alpha, \beta, \gamma)$ . It implies that  $X_N(\alpha, \beta, \gamma)$  is a ternary subsemigroup of  $X$ .  $\square$

**Theorem 3.9.** *Let  $X_N$  be a neutrosophic  $\mathcal{N}$ -structure over  $X$  and let  $\alpha, \beta, \gamma$  be real numbers on the interval  $[-1, 0]$ . If  $T_N^\alpha, I_N^\beta$  and  $F_N^\gamma$  are ternary subsemigroups of  $X$  for all  $\varepsilon_T, \varepsilon_I, \varepsilon_F \in [-1, 0]$  and  $(\alpha, \beta, \gamma) \leq (\varepsilon_T, \varepsilon_I, \varepsilon_F)$ , then  $X_N$  is an  $\varepsilon$ -neutrosophic  $\mathcal{N}$ -ternary subsemigroup of  $X$ .*

*Proof.* Assume first that there exist  $a, b, c \in X$  such that

$$T_N([abc]) > \bigvee \{T_N(a), T_N(b), T_N(c), \varepsilon_T\}.$$

Then

$$T_N([abc]) > t_\alpha \geq \bigvee \{T_N(a), T_N(b), T_N(c), \varepsilon_T\}$$

for some  $t_\alpha \in [-1, 0)$ . It implies that  $a, b, c \in T_N^{t_\alpha}, [abc] \notin T_N^{t_\alpha}$  and  $t_\alpha \geq \varepsilon_T$ . By the hypothesis,  $T_N^{t_\alpha}$  is a ternary subsemigroup of  $X$ , this is a contradiction. Thus

$$T_N([xyz]) \leq \bigvee \{T_N(x), T_N(y), T_N(z), \varepsilon_T\}$$

for all  $x, y, z \in X$ .

Suppose now that there are  $a, b, c \in X$  such that

$$I_N([abc]) < \bigwedge \{I_N(a), I_N(b), I_N(c), \varepsilon_I\}.$$

We define  $t_\beta := \bigwedge \{I_N(x), I_N(y), I_N(z), \varepsilon_I\}$ . Then  $a, b, c \in I_N^{t_\beta}, [abc] \notin I_N^{t_\beta}$ , and  $t_\beta \leq \varepsilon_I$ , a contradiction. Hence

$$I_N([xyz]) \geq \bigwedge \{I_N(x), I_N(y), I_N(z), \varepsilon_I\}$$

for all  $x, y, z \in X$ .

Finally, suppose that there exist  $a, b, c \in X$  and  $t_\gamma \in [-1, 0)$  such that

$$F_N([abc]) > t_\gamma \geq \bigvee \{F_N(a), F_N(b), F_N(c), \varepsilon_F\}.$$

Hence  $a, b, c \in F_N^{t_\gamma}, [abc] \notin F_N^{t_\gamma}$  and  $t_\gamma \geq \varepsilon_F$ , which is a contradiction. Then

$$F_N([xyz]) \leq \bigvee \{F_N(x), F_N(y), F_N(z), \varepsilon_F\}$$

for all  $x, y, z \in X$ .

Therefore  $X_N$  is an  $\varepsilon$ -neutrosophic  $\mathcal{N}$ -ternary subsemigroup of  $X$ . □

**Theorem 3.10.** *Let  $\varepsilon_T, \varepsilon_I, \varepsilon_F, \delta_T, \delta_I, \delta_F \in [-1, 0]$ . If  $X_N$  and  $X_M$  are  $\varepsilon$ -neutrosophic  $\mathcal{N}$ -ternary subsemigroup and a  $\delta$ -neutrosophic  $\mathcal{N}$ -ternary subsemigroup of  $X$ , respectively, then  $X_{N \cap M}$  is a  $\xi$ -neutrosophic  $\mathcal{N}$ -ternary subsemigroup of  $X$  for  $\xi := \varepsilon \wedge \delta$ , that is,*

$$(\xi_T, \xi_I, \xi_F) = (\varepsilon_T \vee \delta_T, \varepsilon_I \wedge \delta_I, \varepsilon_F \vee \delta_F).$$

*Proof.* Let  $x, y, z \in X$ . Then

$$\begin{aligned} T_{N \cap M}([xyz]) &= \bigvee \{T_N([xyz]), T_M([xyz])\} \\ &\leq \bigvee \{ \bigvee \{T_N(x), T_N(y), T_N(z), \varepsilon_T\}, \bigvee \{T_M(x), T_M(y), T_M(z), \delta_T\} \} \\ &\leq \bigvee \{ \bigvee \{T_N(x), T_N(y), T_N(z), \xi_T\}, \bigvee \{T_M(x), T_M(y), T_M(z), \xi_T\} \} \\ &= \bigvee \{ \bigvee \{T_N(x), T_M(x), \xi_T\}, \bigvee \{T_N(y), T_M(y), \xi_T\}, \bigvee \{T_N(z), T_M(z), \xi_T\} \} \\ &= \bigvee \{ \bigvee \{T_N(x), T_M(x)\}, \bigvee \{T_N(y), T_M(y)\}, \bigvee \{T_N(z), T_M(z)\}, \xi_T \} \\ &= \bigvee \{T_{N \cap M}(x), T_{N \cap M}(y), T_{N \cap M}(z), \xi_T\}, \end{aligned}$$

$$\begin{aligned} I_{N \cap M}([xyz]) &= \bigwedge \{I_N([xyz]), I_M([xyz])\} \\ &\geq \bigwedge \{ \bigwedge \{I_N(x), I_N(y), I_N(z), \varepsilon_I\}, \bigwedge \{I_M(x), I_M(y), I_M(z), \delta_I\} \} \\ &\geq \bigwedge \{ \bigwedge \{I_N(x), I_N(y), I_N(z), \xi_I\}, \bigwedge \{I_M(x), I_M(y), I_M(z), \xi_I\} \} \\ &= \bigwedge \{ \bigwedge \{I_N(x), I_M(x), \xi_I\}, \bigwedge \{I_N(y), I_M(y), \xi_I\}, \bigwedge \{I_N(z), I_M(z), \xi_I\} \} \\ &= \bigwedge \{ \bigwedge \{I_N(x), I_M(x)\}, \bigwedge \{I_N(y), I_M(y)\}, \bigwedge \{I_N(z), I_M(z)\}, \xi_I \} \\ &= \bigwedge \{I_{N \cap M}(x), I_{N \cap M}(y), I_{N \cap M}(z), \xi_I\} \end{aligned}$$

and

$$\begin{aligned} F_{N \cap M}([xyz]) &= \bigvee \{F_N([xyz]), F_M([xyz])\} \\ &\leq \bigvee \{ \bigvee \{F_N(x), F_N(y), F_N(z), \varepsilon_F\}, \bigvee \{F_M(x), F_M(y), F_M(z), \delta_F\} \} \\ &\leq \bigvee \{ \bigvee \{F_N(x), F_N(y), F_N(z), \xi_F\}, \bigvee \{F_M(x), F_M(y), F_M(z), \xi_F\} \} \end{aligned}$$

$$\begin{aligned} &= \bigvee \{ \bigvee \{ F_N(x), F_M(x), \xi_F \}, \bigvee \{ F_N(y), F_M(y), \xi_F \}, \bigvee \{ F_N(z), F_M(z), \xi_F \} \} \\ &= \bigvee \{ \bigvee \{ F_N(x), F_M(x) \}, \bigvee \{ F_N(y), F_M(y) \}, \bigvee \{ F_N(z), F_M(z) \}, \xi_F \} \\ &= \bigvee \{ F_{N \cap M}(x), F_{N \cap M}(y), F_{N \cap M}(z), \xi_F \}. \end{aligned}$$

Therefore  $X_{N \cap M}$  is a  $\xi$ -neutrosophic  $\mathcal{N}$ -ternary subsemigroup of  $X$ . □

**Theorem 3.11.** *Let  $X_N$  be an  $\varepsilon$ -neutrosophic  $\mathcal{N}$ -ternary subsemigroup of  $X$ . If*

$$\kappa := (\kappa_T, \kappa_I, \kappa_F) = \left( \bigvee_{x \in X} \{ T_N(x) \}, \bigwedge_{x \in X} \{ I_N(x) \}, \bigvee_{x \in X} \{ F_N(x) \} \right),$$

then the set

$$\Omega := \{ x \in X \mid T_N(x) \leq \kappa_T \vee \varepsilon_T, I_N(x) \geq \kappa_I \wedge \varepsilon_I, F_N(x) \leq \kappa_F \vee \varepsilon_F \}$$

is a ternary subsemigroup of  $X$ .

*Proof.* Let  $x, y, z \in \Omega$ . Then

$$T_N(x) \leq \kappa_T \vee \varepsilon_T = \bigvee_{x \in X} \{ T_N(x) \} \vee \varepsilon_T, I_N(x) \geq \kappa_I \wedge \varepsilon_I = \bigwedge_{x \in X} \{ I_N(x) \} \wedge \varepsilon_I,$$

$$F_N(x) \leq \kappa_F \vee \varepsilon_F = \bigvee_{x \in X} \{ F_N(x) \} \vee \varepsilon_F,$$

$$T_N(y) \leq \kappa_T \vee \varepsilon_T = \bigvee_{y \in X} \{ T_N(y) \} \vee \varepsilon_T, I_N(y) \geq \kappa_I \wedge \varepsilon_I = \bigwedge_{y \in X} \{ I_N(y) \} \wedge \varepsilon_I,$$

$$F_N(y) \leq \kappa_F \vee \varepsilon_F = \bigvee_{y \in X} \{ F_N(y) \} \vee \varepsilon_F,$$

$$T_N(z) \leq \kappa_T \vee \varepsilon_T = \bigvee_{z \in X} \{ T_N(z) \} \vee \varepsilon_T, I_N(z) \geq \kappa_I \wedge \varepsilon_I = \bigwedge_{z \in X} \{ I_N(z) \} \wedge \varepsilon_I,$$

$$F_N(z) \leq \kappa_F \vee \varepsilon_F = \bigvee_{z \in X} \{ F_N(z) \} \vee \varepsilon_F.$$

It follows that

$$\begin{aligned} T_N([xyz]) &\leq \bigvee \{ T_N(x), T_N(y), T_N(z), \varepsilon_T \} \\ &\leq \bigvee \{ \kappa_T \vee \varepsilon_T, \kappa_T \vee \varepsilon_T, \kappa_T \vee \varepsilon_T, \varepsilon_T \} \\ &= \kappa_T \vee \varepsilon_T, \end{aligned}$$

$$\begin{aligned}
I_N([xyz]) &\geq \bigwedge \{I_N(x), I_N(y), I_N(z), \varepsilon_I\} \\
&\geq \bigwedge \{\kappa_I \wedge \varepsilon_I, \kappa_I \wedge \varepsilon_I, \kappa_I \wedge \varepsilon_I, \varepsilon_I\} \\
&= \kappa_I \wedge \varepsilon_I
\end{aligned}$$

and

$$\begin{aligned}
F_N([xyz]) &\leq \bigvee \{F_N(x), F_N(y), F_N(z), \varepsilon_F\} \\
&\leq \bigvee \{\kappa_F \vee \varepsilon_F, \kappa_F \vee \varepsilon_F, \kappa_F \vee \varepsilon_F, \varepsilon_F\} \\
&= \kappa_F \vee \varepsilon_F.
\end{aligned}$$

Hence  $[xyz] \in \Omega$ . It implies that  $\Omega$  is a ternary subsemigroup of  $X$ .  $\square$

Let  $f : X \rightarrow Y$  be a mapping of ternary semigroups and  $Y_N := \frac{Y}{(T_N, I_N, F_N)}$  be a neutrosophic  $\mathcal{N}$ -structure over  $Y$  with  $\varepsilon = (\varepsilon_T, \varepsilon_I, \varepsilon_F)$ . Then  $X_N^\varepsilon := \frac{X}{(T_N^\varepsilon, I_N^\varepsilon, F_N^\varepsilon)}$  is a neutrosophic  $\mathcal{N}$ -structure over  $X$  where

$$\begin{aligned}
T_N^\varepsilon : X &\rightarrow [-1, 0], \quad x \mapsto \bigvee \{T_N(f(x)), \varepsilon_T\}, \\
I_N^\varepsilon : X &\rightarrow [-1, 0], \quad x \mapsto \bigwedge \{I_N(f(x)), \varepsilon_I\}, \\
F_N^\varepsilon : X &\rightarrow [-1, 0], \quad x \mapsto \bigvee \{F_N(f(x)), \varepsilon_F\}.
\end{aligned}$$

**Theorem 3.12.** *Let  $f : X \rightarrow Y$  be a homomorphism of ternary semigroups. If a neutrosophic  $\mathcal{N}$ -structure  $Y_N := \frac{Y}{(T_N, I_N, F_N)}$  over  $Y$  is an  $\varepsilon$ -neutrosophic  $\mathcal{N}$ -ternary subsemigroup of  $Y$ , then  $X_N^\varepsilon := \frac{X}{(T_N^\varepsilon, I_N^\varepsilon, F_N^\varepsilon)}$  is an  $\varepsilon$ -neutrosophic  $\mathcal{N}$ -ternary subsemigroup of  $X$ .*

*Proof.* Let  $x, y, z \in X$ . Then

$$\begin{aligned}
T_N^\varepsilon([xyz]) &= \bigvee \{T_N(f([xyz])), \varepsilon_T\} \\
&= \bigvee \{T_N([f(x)f(y)f(z)]), \varepsilon_T\} \\
&\leq \bigvee \{\bigvee \{T_N(f(x)), T_N(f(y)), T_N(f(z)), \varepsilon_T\}, \varepsilon_T\} \\
&= \bigvee \{\bigvee \{T_N(f(x)), \varepsilon_T\}, \bigvee \{T_N(f(y)), \varepsilon_T\}, \bigvee \{T_N(f(z)), \varepsilon_T\}, \varepsilon_T\} \\
&= \bigvee \{T_N^\varepsilon(x), T_N^\varepsilon(y), T_N^\varepsilon(z), \varepsilon_T\},
\end{aligned}$$

$$\begin{aligned}
 I_N^\varepsilon([xyz]) &= \bigwedge \{I_N(f([xyz])), \varepsilon_I\} \\
 &= \bigwedge \{I_N([f(x)f(y)f(z)]), \varepsilon_I\} \\
 &\geq \bigwedge \{\bigwedge \{I_N(f(x)), I_N(f(y)), I_N(f(z)), \varepsilon_I\}, \varepsilon_I\} \\
 &= \bigwedge \{\bigwedge \{I_N(f(x)), \varepsilon_I\}, \bigwedge \{I_N(f(y)), \varepsilon_I\}, \bigwedge \{I_N(f(z)), \varepsilon_I\}, \varepsilon_I\} \\
 &= \bigwedge \{I_N^\varepsilon(x), I_N^\varepsilon(y), I_N^\varepsilon(z), \varepsilon_I\}
 \end{aligned}$$

and

$$\begin{aligned}
 F_N^\varepsilon([xyz]) &= \bigvee \{F_N(f([xyz])), \varepsilon_F\} \\
 &= \bigvee \{F_N([f(x)f(y)f(z)]), \varepsilon_F\} \\
 &\leq \bigvee \{\bigvee \{F_N(f(x)), F_N(f(y)), F_N(f(z)), \varepsilon_F\}, \varepsilon_F\} \\
 &= \bigvee \{\bigvee \{F_N(f(x)), \varepsilon_F\}, \bigvee \{F_N(f(y)), \varepsilon_F\}, \bigvee \{F_N(f(z)), \varepsilon_F\}, \varepsilon_F\} \\
 &= \bigvee \{F_N^\varepsilon(x), F_N^\varepsilon(y), F_N^\varepsilon(z), \varepsilon_F\}
 \end{aligned}$$

Hence  $X_N^\varepsilon := \frac{X}{(T_N^\varepsilon, I_N^\varepsilon, F_N^\varepsilon)}$  is an  $\varepsilon$ -neutrosophic  $\mathcal{N}$ -ternary subsemigroup of  $X$ . □

Let  $f : X \rightarrow Y$  be a function of sets. If  $Y_M := \frac{Y}{(T_M, I_M, F_M)}$  is a neutrosophic  $\mathcal{N}$ -structure over  $Y$ , then the preimage of  $Y_M$  under  $f$

$$f^{-1}(Y_M) := \frac{X}{(f^{-1}(T_M), f^{-1}(I_M), f^{-1}(F_M))}$$

is defined to be a neutrosophic  $\mathcal{N}$ -structure over  $X$  where

$$f^{-1}(T_M)(x) = T_M(f(x)), f^{-1}(I_M)(x) = I_M(f(x)) \text{ and } f^{-1}(F_M)(x) = F_M(f(x))$$

for all  $x \in X$ .

**Theorem 3.13.** *Let  $X, Y$  be ternary semigroups and  $f : X \rightarrow Y$  a homomorphism. If  $Y_M := \frac{Y}{(T_M, I_M, F_M)}$  is a neutrosophic  $\mathcal{N}$ -ternary subsemigroup of  $Y$ , then the preimage of  $Y_M$  under  $f$ ,*

$$f^{-1}(Y_M) := \frac{X}{(f^{-1}(T_M), f^{-1}(I_M), f^{-1}(F_M))}$$

*is a neutrosophic  $\mathcal{N}$ -ternary subsemigroup of  $X$ .*



*Proof.* Let  $x, y, z \in X$ . Then

$$\begin{aligned} f^{-1}(T_M)([xyz]) &= T_M(f([xyz])) = T_M([f(x)f(y)f(z)]) \\ &\leq \bigvee \{T_M(f(x)), T_M(f(y)), T_M(f(z))\} \\ &= \bigvee \{f^{-1}(T_M)(x), f^{-1}(T_M)(y), f^{-1}(T_M)(z)\}, \end{aligned}$$

$$\begin{aligned} f^{-1}(I_M)([xyz]) &= I_M(f([xyz])) = I_M([f(x)f(y)f(z)]) \\ &\geq \bigwedge \{I_M(f(x)), I_M(f(y)), I_M(f(z))\} \\ &= \bigwedge \{f^{-1}(I_M)(x), f^{-1}(I_M)(y), f^{-1}(I_M)(z)\} \end{aligned}$$

and

$$\begin{aligned} f^{-1}(F_M)([xyz]) &= F_M(f([xyz])) = F_M([f(x)f(y)f(z)]) \\ &\leq \bigvee \{F_M(f(x)), F_M(f(y)), F_M(f(z))\} \\ &= \bigvee \{f^{-1}(F_M)(x), f^{-1}(F_M)(y), f^{-1}(F_M)(z)\}. \end{aligned}$$

Therefore  $f^{-1}(Y_M)$  is a neutrosophic  $\mathcal{N}$ -ternary subsemigroup of  $X$ , which completes the proof.  $\square$

Let  $X, Y$  be sets and  $f : X \rightarrow Y$  be an onto function. If  $X_N := \frac{X}{(T_N, I_N, F_N)}$  is a neutrosophic  $\mathcal{N}$ -structure over  $X$ , then the image of  $X_N$  under  $f$

$$f(X_N) := \frac{Y}{(f(T_N), f(I_N), f(F_N))}$$

is defined to be a neutrosophic  $\mathcal{N}$ -structure over  $Y$  where

$$f(T_N)(y) = \bigwedge_{x \in f^{-1}(y)} T_N(x),$$

$$f(I_N)(y) = \bigvee_{x \in f^{-1}(y)} I_N(x),$$

$$f(F_N)(y) = \bigwedge_{x \in f^{-1}(y)} F_N(x).$$

**Theorem 3.14.** *Let  $X, Y$  be ternary semigroups and  $f : X \rightarrow Y$  be an onto homomorphism. Let  $X_N := \frac{X}{(T_N, I_N, F_N)}$  be a neutrosophic  $\mathcal{N}$ -structure of  $X$  such that*

$$T_N(x_0) = \bigwedge_{z \in A} T_N(z), \quad I_N(x_0) = \bigvee_{z \in A} I_N(z), \quad F_N(x_0) = \bigwedge_{z \in A} F_N(z).$$

*for all  $A \subseteq X$  and some  $x_0 \in A$ . If  $X_N$  is a neutrosophic  $\mathcal{N}$ -structure of  $X$ , then the image of  $X_N$  under  $f$*

$$f(X_N) := \frac{Y}{(f(T_N), f(I_N), f(F_N))}$$

*is a neutrosophic  $\mathcal{N}$ -ternary subsemigroup of  $Y$ .*

*Proof.* Let  $a, b, c \in Y$ . Then  $f^{-1}(a) \neq \emptyset, f^{-1}(b) \neq \emptyset$  and  $f^{-1}(c) \neq \emptyset$  in  $X$ . It follows that there exist  $x_a \in f^{-1}(a), x_b \in f^{-1}(b)$  and  $x_c \in f^{-1}(c)$  such that

$$\begin{aligned} T_N(x_a) &= \bigwedge_{u \in f^{-1}(a)} T_N(u), & I_N(x_a) &= \bigvee_{u \in f^{-1}(a)} I_N(u), & F_N(x_a) &= \bigwedge_{u \in f^{-1}(a)} F_N(u), \\ T_N(x_b) &= \bigwedge_{v \in f^{-1}(b)} T_N(v), & I_N(x_b) &= \bigvee_{v \in f^{-1}(b)} I_N(v), & F_N(x_b) &= \bigwedge_{v \in f^{-1}(b)} F_N(v), \\ T_N(x_c) &= \bigwedge_{w \in f^{-1}(c)} T_N(w), & I_N(x_c) &= \bigvee_{w \in f^{-1}(c)} I_N(w), & F_N(x_c) &= \bigwedge_{w \in f^{-1}(c)} F_N(w). \end{aligned}$$

Hence

$$\begin{aligned} f(T_N)([abc]) &= \bigwedge_{x \in f^{-1}([abc])} T_N(x) \leq T_N([x_a x_b x_c]) \\ &\leq \bigvee \{T_N(x_a), T_N(x_b), T_N(x_c)\} \\ &= \bigvee \left\{ \bigwedge_{u \in f^{-1}(a)} T_N(u), \bigwedge_{v \in f^{-1}(b)} T_N(v), \bigwedge_{w \in f^{-1}(c)} T_N(w) \right\} \\ &= \bigvee \{f(T_N)(a), f(T_N)(b), f(T_N)(c)\}, \end{aligned}$$

$$\begin{aligned} f(I_N)([abc]) &= \bigvee_{x \in f^{-1}([abc])} I_N(x) \geq I_N([x_a x_b x_c]) \\ &\geq \bigwedge \{I_N(x_a), I_N(x_b), I_N(x_c)\} \\ &= \bigwedge \left\{ \bigvee_{u \in f^{-1}(a)} I_N(u), \bigvee_{v \in f^{-1}(b)} I_N(v), \bigvee_{w \in f^{-1}(c)} I_N(w) \right\} \\ &= \bigwedge \{f(I_N)(a), f(I_N)(b), f(I_N)(c)\}, \end{aligned}$$

and

$$\begin{aligned}
 f(F_N)([abc]) &= \bigwedge_{x \in f^{-1}([abc])} F_N(x) \leq F_N([x_a x_b x_c]) \\
 &\leq \bigvee \{F_N(x_a), F_N(x_b), F_N(x_c)\} \\
 &= \bigvee \left\{ \bigwedge_{u \in f^{-1}(a)} F_N(u), \bigwedge_{v \in f^{-1}(b)} F_N(v), \bigwedge_{w \in f^{-1}(c)} F_N(w) \right\} \\
 &= \bigvee \{f(F_N)(a), f(F_N)(b), f(F_N)(c)\}.
 \end{aligned}$$

Then  $f(X_N)$  is a neutrosophic  $\mathcal{N}$ -ternary subsemigroup of  $Y$ . □

#### 4. CONCLUSIONS

In this paper, we applied neutrosophic  $\mathcal{N}$ -structure to ternary semigroups. We also investigated the notion of neutrosophic  $\mathcal{N}$ -ternary subsemigroups and showed some properties. Moreover, the conditions for neutrosophic  $\mathcal{N}$ -structure to be neutrosophic  $\mathcal{N}$ -ternary subsemigroup have been investigated. We also defined neutrosophic  $\mathcal{N}$ -products and discuss about the characterization of neutrosophic  $\mathcal{N}$ -ternary subsemigroups. In addition, we have introduced  $\varepsilon$ -neutrosophic  $\mathcal{N}$ -ternary subsemigroups and shown the relation between neutrosophic ternary subsemigroups and  $\varepsilon$ -neutrosophic  $\mathcal{N}$ -ternary subsemigroups. Finally, we showed that the homomorphic preimage of the neutrosophic  $\mathcal{N}$ -ternary subsemigroup is a neutrosophic  $\mathcal{N}$ -ternary subsemigroup and the onto homomorphic image of the neutrosophic  $\mathcal{N}$ -ternary subsemigroup is a neutrosophic  $\mathcal{N}$ -ternary subsemigroup.

In our future study, we will apply these notion/results to other types of neutrosophic  $\mathcal{N}$ -structures in ternary semigroups. We will also study the soft set theory/cubic set theory of such neutrosophic  $\mathcal{N}$ -structures.

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#### CONFLICT OF INTERESTS

The authors declare that there is no conflict of interests.

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