*Research Article*

# Cartesian Product Of Interval Neutrosophic Automata

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**Abstract**
We introduce Cartesian product of interval neutrosophic automata and prove that Cartesian product of cyclic interval neutrosophic automata is cyclic

**Key words:** Cyclic, Cartesian product.

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## 1 Introduction

The neutrosophic set was introduced by Florentin Smarandache in 1999 [6]. The neutrosophic set is the generalization of classical sets, fuzzy set [11] and so on. The fuzzy set was introduced by Zadeh in 1965[11]. Bipolar fuzzy set, YinYang bipolar fuzzy set, NPN fuzzy set were introduced by W. R. Zhang in [8, 9, 10].

A neutrosophic set N is classified by a Truth membership $T_N$, Indeterminacy membership $I_N$, and Falsity membership $F_N$, where $T_N$, $I_N$, and $F_N$ are real standard and non-standard subsets of] $0^-$, $1^+$[. Interval-valued neutrosophic sets was introduced by Wang etal.,[7]. The concept of interval neutrosophic finite state machine was introduced by Tahir Mahmood [5]. Generalized products of directable fuzzy automata are discussed in [1]. Retrievability, subsystems, and strong subsystems of INA were introduced in the papers [2, 3, 4].

In this paper, we introduce Cartesian product of interval neutrosophic automata and prove that Cartesian product of cyclic interval neutrosophic automata is cyclic.

## 2 Preliminaries

### 2.1 Neutrosophic Set [6]

Let $U$ be the universal set.. A neutrosophic set (NS) $N$ in $U$ is classified by a truth membership $T_N$, an indeterminacy membership $I_N$ and a falsity membership $F_N$, where $T_N$, $I_N$, and $F_N$ are real standard or non-standard subsets of] $0^-$,$1^+$[. That is

$N = \{\langle x, T_N(x), I_N(x), F_N(x) \rangle, x \in U, T_N, I_N, F_N \in ]\, 0^-, 1^+[\, \}$ and $0^- \leq sup\ T_N(x) + sup\ I_N(x) + sup\ F_N(x) \leq 3^+$. We need to take the interval [0,1] for instead of $]\, 0^-, 1^+[$.

### .2.2 Definition [7]

An interval neutrosophic set ($INS$ for short) is $N = \{\langle \alpha_N(x), \beta_N(x), \gamma_N(x)\rangle \mid x \in U\}$ $= \{\langle x, [inf\ \alpha_N(x), sup\ \alpha_N(x)], [inf\ \beta_N(x), sup\ \beta_N(x)], [inf\ \gamma_N(x), sup\ \gamma_N(x)]\rangle\}, x \in U$, where $\alpha_N(x)$, $\beta_N(x)$, and $\gamma_N(x)$ representing the truth-membership, indeterminacy-membership and falsity membership for each $x \in U$. $\alpha_N(x)$, $\beta_N(x)$, $\gamma_N(x) \subseteq [0,1]$ and the condition that $0 \leq sup\ \alpha_N(x) + sup\ \beta_N(x) + sup\ \gamma_N(x) \leq 3$.

### 2.3 Definition [7]

An $INS\ N$ is empty if $inf\ \alpha_N(x) = sup\ \alpha_N(x) = 0, inf\ \beta_N(x) = sup\ \beta_N(x)\ 1, inf\ \gamma_N(x) = sup\ \gamma_N(x) = 1$ for all $x \in U$.

## 3 Interval Neutrosophic Automata

### 3.1 Definition [5]

$M = (Q, \Sigma, N)$ is called interval neutrosophic automaton ($INA$ for short), where $Q$ and $\Sigma$ are non-empty finite sets called the set of states and input symbols respectively, and $N = \{\langle \alpha_N(x), \beta_N(x), \gamma_N(x)\rangle\}$ is an $INS$ in $Q \times$

$\Sigma \times Q$. The set of all words of finite length of $\Sigma$ is denoted by $\Sigma^*$. The empty word is denoted by $\epsilon$ and the length of each $x \in \Sigma^*$ is denoted by $|x|$.

**3.2 Definition [5]**

$M = (Q, \Sigma, N)$ be an $INA$. Define an $INS\ N^* = \{\langle \alpha_{N^*}(x),\ \beta_{N^*}(x), \gamma_{N^*}(x)\rangle\}$ in $Q \times \Sigma^* \times Q$ by

$$\alpha_{N^*}(q_i, \epsilon, q_j) = \begin{cases} [1,1]\ if\ q_i = q_j \\ [0,0]\ if\ q_i \neq q_j \end{cases}$$

$$\beta_{N^*}(q_i, \epsilon, q_j) = \begin{cases} [0,0]\ if\ q_i = q_j \\ [1,1]\ if\ q_i \neq q_j \end{cases}$$

$$\gamma_{N^*}(q_i, \epsilon, q_j) = \begin{cases} [0,0]\ if\ q_i = q_j \\ [1,1]\ if\ q_i \neq q_j \end{cases}$$

$$\alpha_{N^*}(q_i, w, q_j) = \alpha_{N^*}(q_i,\ xy,\ q_j) = \vee_{q_r \in Q} \left[ \alpha_{N^*}(q_i,\ x,\ q_r) \wedge \alpha_{N^*}(q_r,\ y,\ q_j)\right]$$

$$\beta_{N^*}(q_i, w, q_j) = \beta_{N^*}(q_i,\ xy,\ q_j) = \wedge_{q_r \in Q} \left[ \beta_{N^*}(q_i,\ x,\ q_r) \vee \beta_{N^*}(q_r,\ y,\ q_j)\right]$$

$\gamma_{N^*}(q_i, w, q_j) = \gamma_{N^*}(q_i,\ xy,\ q_j) = \wedge_{q_r \in Q} \left[ \gamma_{N^*}(q_i,\ x,\ q_r) \vee \gamma_{N^*}(q_r,\ y,\ q_j)\right]\ \forall\, q_i,\ q_j \in Q, w = xy, x \in \Sigma^*$ and $y \in \Sigma$.

**4 Cartesian Composition of Interval Neutrosophic Automata**

**4.1 Definition**

Let $M_i = (Q_i, \Sigma_i, N_i), i = 1, 2$ be interval neutrosophic automata and let $\Sigma_1 \cap \Sigma_2 = \emptyset$. Let $M_1 \times M_2 = (Q_1 \times Q_2, \Sigma_1 \cup \Sigma_2, N_1 \times N_2)$, where

$$(\alpha_1 \times \alpha_2)\left((q_i, q_j), a, (q_k, q_l)\right) = \begin{cases} \alpha_1(q_i, a, q_k) > [0,0]\ if\ a \in \Sigma_1\ and\ q_j = q_l \\ \alpha_2(q_i, a, q_k) > [0,0]\ if\ a \in \Sigma_2\ and\ q_i = q_k \\ 0 \quad otherwise \end{cases}$$

$$(\beta_1 \times \beta_2)\left((q_i, q_j), a, (q_k, q_l)\right) = \begin{cases} \beta_1(q_i, a, q_k) < [1,1]\ if\ a \in \Sigma_1\ and\ q_j = q_l \\ \beta_2(q_i, a, q_k) < [1,1]\ if\ a \in \Sigma_2\ and\ q_i = q_k \\ 0 \quad otherwise \end{cases}$$

$$(\gamma_1 \times \gamma_2)\left((q_i, q_j), a, (q_k, q_l)\right) = \begin{cases} \gamma_1(q_i, a, q_k) < [1,1]\ if\ a \in \Sigma_1\ and\ q_j = q_l \\ \gamma_2(q_i, a, q_k) < [1,1]\ if\ a \in \Sigma_2\ and\ q_i = q_k \\ 0 \quad otherwise \end{cases}$$

$\forall (q_i, q_j),\ (q_k, q_l) \in Q_1 \times Q_2,\ a \in \Sigma_1 \cup \Sigma_2$. Then $M_1 \times M_2$ is called the Cartesian product of interval neutrosophic automata.

**4.2 Definition**

Let $M = (Q, \Sigma, N)$ be an INA. $M$ is cyclic if $\exists\ q_i \in$ Q such that Q = S($q_i$).

**4.3 Definition [2]**

Let $M = (Q, \Sigma, N)$ be INA. M is connected if $\forall\ q_j, q_i$ and $\exists$ a $\in \Sigma$ such that either $\alpha_N\left(q_i, a, q_j\right) > [0,0]$, $\beta_N\left(q_i, a, q_j\right) < [1,1]$, $\gamma_N\left(q_i, a, q_j\right) < [1,1]$ or

$\alpha_N\left(q_j, a, q_i\right) > [0,0]$, $\beta_N\left(q_j, a, q_i\right) < [1,1]$, $\gamma_N\left(q_j, a, q_i\right) < [1,1]$.

**4.4 Definition [2]**

Let $M = (Q, \Sigma, N)$ be INA. M is strongly connected if for every $q_i, q_j \in Q$, there exists u $\in \Sigma^*$ such that $\alpha_{N^*}(q_i, u, q_j) > [0,0], \beta_{N^*}(q_i, u, q_j) < [1,1], \gamma_{N^*}(q_i, u, q_j) < [1,1]$.

**Theorem 4.1** Let $M_i = (Q_i, \Sigma_i, N_i), i = 1,2$ be interval neutrosophic automata and let $\Sigma_1 \cap \Sigma_2 = \emptyset$. Let $M_1 \times M_2 = (Q_1 \times Q_2, \Sigma_1 \cup \Sigma_2, N_1 \times N_2)$ be the Cartesian product of $M_1$ and $M_2$. Then $\forall\, x \in {\Sigma_1}^* \cup {\Sigma_2}^*, x \neq \epsilon$

$$(\alpha_1 \times \alpha_2)^*\left((q_i, q_j), x, (q_k, q_l)\right) = \begin{cases} \alpha_1(q_i, x, q_k) > [0,0] \; if \; x \in {\Sigma_1}^* \; and \; q_j = q_l \\ \alpha_2(q_i, x, q_k) > [0,0] \; if \; x \in {\Sigma_2}^* and \; q_i = q_k \\ 0 \qquad otherwise \end{cases}$$

$$(\beta_1 \times \beta_2)^*\left((q_i, q_j), x, (q_k, q_l)\right) = \begin{cases} \beta_1(q_i, x, q_k) < [1,1] \; if \; x \in {\Sigma_1}^* \; and \; q_j = q_l \\ \beta_2(q_i, x, q_k) < [1,1] \; if \; x \in {\Sigma_2}^* \; and \; q_i = q_k \\ 0 \qquad otherwise \end{cases}$$

$$(\gamma_1 \times \gamma_2)^*\left((q_i, q_j), x, (q_k, q_l)\right) = \begin{cases} \gamma_1(q_i, x, q_k) < [1,1] \; if \; x \in {\Sigma_1}^* and \; q_j = q_l \\ \gamma_2(q_i, x, q_k) < [1,1] \; if \; x \in {\Sigma_1}^* and \; q_i = q_k \\ 0 \qquad otherwise \end{cases}$$

$\forall (q_i, q_j), (q_k, q_l) \in Q_1 \times Q_2$.

**Proof .** Let $x \in {\Sigma_1}^* \cup {\Sigma_2}^*, x \neq \epsilon$ and let $|x| = m$. Let $x \in {\Sigma_1}^*$. The result is trivial if $m = 1$. Let the result is true $\forall\, y \in {\Sigma_1}^*, |y| = m - 1, m > 1$. Let $x = ay$ where $a \in \Sigma_1, y \in {\Sigma_1}^*$. Now,

$$(\alpha_{N_1} \times \alpha_{N_2})^*\left((q_i, q_j), x, (q_k, q_l)\right) = (\alpha_{N_1} \times \alpha_{N_2})^*\left((q_i, q_j), ay, (q_k, q_l)\right)$$
$$= \vee_{(q_r, q_s) \in Q_1 \times Q_2} \left\{(\alpha_{N_1} \times \alpha_{N_2})\left((q_i, q_j), a, (q_r, q_s)\right) \wedge (\alpha_1 \times \alpha_2)^*\left((q_r, q_s), y, (q_k, q_l)\right)\right\}$$
$$= \vee_{q_r \in Q_1} \left\{\alpha_{N_1}(q_i, a, q_r) \wedge (\alpha_{N_1} \times \alpha_{N_2})^*\left((q_r, q_s), y, (q_k, q_l)\right)\right\}$$
$$= \begin{cases} \vee_{q_r \in Q_1} \left\{\alpha_{N_1}(q_i, a, q_r) \wedge {\alpha_{N_1}}^*(q_r, y, q_k)\right\} > [0,0] \; if \; q_j = q_l \\ 0 \qquad otherwise \end{cases}$$
$$= \begin{cases} {\alpha_{N_1}}^*(q_i, ay, q_k) > [0,0] \; if \; q_j = q_l \\ 0 \qquad otherwise \end{cases}$$

$$(\beta_{N_1} \times \beta_{N_2})^*\left((q_i, q_j), x, (q_k, q_l)\right) = (\beta_{N_1} \times \beta_{N_2})^*\left((q_i, q_j), ay, (q_k, q_l)\right)$$
$$= \wedge_{(q_r, q_s) \in Q_1 \times Q_2} \left\{(\beta_{N_1} \times \beta_{N_2})\left((q_i, q_j), a, (q_r, q_s)\right) \vee (\beta_{N_1} \times \beta_{N_2})^*\left((q_r, q_s), y, (q_k, q_l)\right)\right\}$$
$$= \wedge_{q_r \in Q_1} \left\{\beta_{N_1}(q_i, a, q_r) \vee (\beta_{N_1} \times \beta_{N_2})^*\left((q_r, q_s), y, (q_k, q_l)\right)\right\}$$
$$= \begin{cases} \wedge_{q_r \in Q_1} \left\{\beta_{N_1}(q_i, a, q_r) \vee {\beta_{N_1}}^*(q_r, y, q_k)\right\} < [1,1] \; if \; q_j = q_l \\ 0 \qquad otherwise \end{cases}$$
$$= \begin{cases} {\beta_{N_1}}^*(q_i, ay, q_k) < [1,1] \; if \; q_j = q_l \\ 0 \qquad otherwise \end{cases}$$

$$(\gamma_{N_1} \times \gamma_{N_2})^*\left((q_i, q_j), x, (q_k, q_l)\right) = (\gamma_{N_1} \times \gamma_{N_2})^*\left((q_i, q_j), ay, (q_k, q_l)\right)$$
$$= \wedge_{(q_r, q_s) \in Q_1 \times Q_2} \left\{(\gamma_{N_1} \times \gamma_{N_2})\left((q_i, q_j), a, (q_r, q_s)\right) \vee (\gamma_{N_1} \times \gamma_{N_2})^*\left((q_r, q_s), y, (q_k, q_l)\right)\right\}$$
$$= \wedge_{q_r \in Q_1} \left\{\gamma_{N_1}(q_i, a, q_r) \vee (\gamma_{N_1} \times \gamma_{N_2})^*\left((q_r, q_s), y, (q_k, q_l)\right)\right\}$$
$$= \begin{cases} \wedge_{q_r \in Q_1} \left\{\gamma_{N_1}(q_i, a, q_r) \vee {\gamma_{N_1}}^*(q_r, y, q_k)\right\} < [1,1] \; if \; q_j = q_l \\ 0 \qquad otherwise \end{cases}$$
$$= \begin{cases} {\gamma_{N_1}}^*(q_i, ay, q_k) < [1,1] \; if \; q_j = q_l \\ 0 \qquad otherwise \end{cases}$$

The result is follows by induction. The Proof is similar if $y \in {\Sigma_2}^*$.

**Theorem 4.2** Let $M_i = (Q_i, \Sigma_i, N_i), i = 1,2$ be INA and let $\Sigma_1 \cap \Sigma_2 = \emptyset$. Then $\forall\, x \in {\Sigma_1}^*, y \in {\Sigma_2}^*$,

$$(\alpha_{N_1} \times \alpha_{N_2})^*\left((p_i, p_j), xy, (q_i, q_j)\right) = {\alpha_{N_1}}^*(p_i, x, q_i) \wedge {\alpha_{N_2}}^*(p_j, y, q_j)$$
$$= (\alpha_{N_1} \times \alpha_{N_2})^*\left((p_i, p_j), yx, (q_i, q_j)\right)$$
$$(\beta_{N_1} \times \beta_{N_2})^*\left((p_i, p_j), xy, (q_i, q_j)\right) = {\beta_{N_1}}^*(p_i, x, q_i) \vee {\beta_{N_2}}^*(p_j, y, q_j)$$
$$= (\beta_{N_1} \times \beta_{N_2})^*\left((p_i, p_j), yx, (q_i, q_j)\right)$$
$$(\gamma_{N_1} \times \gamma_{N_2})^*\left((p_i, p_j), xy, (q_i, q_j)\right) = {\gamma_{N_1}}^*(p_i, x, q_i) \vee {\gamma_{N_2}}^*(p_j, y, q_j)$$
$$= (\gamma_{N_1} \times \gamma_{N_2})^*\left((p_i, p_j), yx, (q_i, q_j)\right),$$

$(p_i, p_j), (q_i, q_j) \in Q_1 \times Q_2$.

**Proof .**

Let $\in \Sigma_1{}^*, y \in \Sigma_2{}^*$, $(p_i, p_j), (q_i, q_j) \in Q_1 \times Q_2$. If $x = \epsilon = y$, then $xy = \epsilon$. Suppose $(p_i, p_j) = (q_i, q_j)$. Then $p_i = q_i$ and $p_j = q_j$. Hence

$$(\alpha_{N_1} \times \alpha_{N_2})^*\left((p_i,p_j), xy, (q_i,q_j)\right) = [1,1] = [1,1] \wedge [1,1] = \alpha_{N_1}{}^*(p_i,x,q_i) \wedge \alpha_{N_2}{}^*(p_j,y,q_j)$$

$$(\beta_{N_1} \times \beta_{N_2})^*\left((p_i,p_j), xy, (q_i,q_j)\right) = [0,0] = [0,0] \vee [0,0] = \beta_{N_1}{}^*(p_i,x,q_i) \vee \beta_{N_1}{}^*(p_i,x,q_i)$$

$$(\gamma_{N_1} \times \gamma_{N_2})^*\left((p_i,p_j), xy, (q_i,q_j)\right) = [0,0] = [0,0] \vee [0,0] = \gamma_{N_1}{}^*(p_i,x,q_i) \vee \gamma_{N_2}{}^*(p_j,y,q_j)$$

If $(p_i,p_j) \neq (q_i,q_j)$, then either $p_i \neq q_i$ or $p_j \neq q_j$.

Thus, $\alpha_{N_1}{}^*(p_i,x,q_i) \wedge \alpha_{N_2}{}^*(p_j,y,q_j) = [0,0]$, $\beta_{N_1}{}^*(p_i,x,q_i) \vee \beta_{N_1}{}^*(p_i,x,q_i) = [1,1]$, $\gamma_{N_1}{}^*(p_i,x,q_i) \vee \gamma_{N_2}{}^*(p_j,y,q_j) = [1,1]$.

Hence $(\alpha_{N_1} \times \alpha_{N_2})^*\left((p_i,p_j), xy, (q_i,q_j)\right) = [0,0] = \alpha_{N_1}{}^*(p_i,x,q_i) \wedge \alpha_{N_2}{}^*(p_j,y,q_j)$

$$(\beta_{N_1} \times \beta_{N_2})^*\left((p_i,p_j), xy, (q_i,q_j)\right) = [1,1] = \beta_{N_1}{}^*(p_i,x,q_i) \vee \beta_{N_1}{}^*(p_i,x,q_i)$$

$$(\gamma_{N_1} \times \gamma_{N_2})^*\left((p_i,p_j), xy, (q_i,q_j)\right) = [1,1] = \gamma_{N_1}{}^*(p_i,x,q_i) \vee \gamma_{N_2}{}^*(p_j,y,q_j)$$

If $x = \epsilon$ and $y \neq \epsilon$ or $x \neq \epsilon$ and $y = \epsilon$, then the result follows by Theorem 4.1. Suppose $x \neq \epsilon$ and $y \neq \epsilon$. Now,

$$(\alpha_{N_1} \times \alpha_{N_2})^*\left((p_i,p_j), xy, (q_i,q_j)\right) = \vee_{(r_i,r_j) \in Q_1 \times Q_2}\left\{(\alpha_{N_1} \times \alpha_{N_2})^*\left((p_i,p_j), x, (r_i,r_j)\right) \wedge (\alpha_{N_1} \times \alpha_{N_2})^*\left((r_i,r_j), y, (q_i,q_j)\right)\right\}$$

$$= \vee_{r_i \in Q_1}\left\{(\alpha_{N_1} \times \alpha_{N_2})^*\left((p_i,p_j), x, (r_i,p_j)\right) \wedge (\alpha_{N_1} \times \alpha_{N_2})^*\left((r_i,p_j), y, (q_i,q_j)\right)\right\}$$

$$= \alpha_{N_1}{}^*(p_i,x,q_i) \wedge \alpha_{N_2}{}^*(p_j,y,q_j)$$

$$(\beta_{N_1} \times \beta_{N_2})^*\left((p_i,p_j), xy, (q_i,q_j)\right) = \wedge_{(r_i,r_j) \in Q_1 \times Q_2}\left\{(\beta_{N_1} \times \beta_{N_2})^*\left((p_i,p_j), x, (r_i,r_j)\right) \vee (\beta_{N_1} \times \beta_{N_2})^*\left((r_i,r_j), y, (q_i,q_j)\right)\right\}$$

$$= \wedge_{r_i \in Q_1}\left\{(\beta_{N_1} \times \beta_{N_2})^*\left((p_i,p_j), x, (r_i,p_j)\right) \vee (\beta_{N_1} \times \beta_{N_2})^*\left((r_i,p_j), y, (q_i,q_j)\right)\right\}$$

$$= \beta_{N_1}{}^*(p_i,x,q_i) \vee \beta_{N_2}{}^*(p_j,y,q_j)$$

$$(\gamma_{N_1} \times \gamma_{N_2})^*\left((p_i,p_j), xy, (q_i,q_j)\right) = \wedge_{(r_i,r_j) \in Q_1 \times Q_2}\left\{(\gamma_{N_1} \times \gamma_{N_2})^*\left((p_i,p_j), x, (r_i,r_j)\right) \vee (\gamma_{N_1} \times \gamma_{N_2})^*\left((r_i,r_j), y, (q_i,q_j)\right)\right\}$$

$$= \wedge_{r_i \in Q_1}\left\{(\gamma_{N_1} \times \gamma_{N_2})^*\left((p_i,p_j), x, (r_i,p_j)\right) \vee (\gamma_{N_1} \times \gamma_{N_2})^*\left((r_i,p_j), y, (q_i,q_j)\right)\right\}$$

$$= \gamma_{N_1}{}^*(p_i,x,q_i) \vee \gamma_{N_2}{}^*(p_j,y,q_j)$$

Similarly

$$(\alpha_{N_1} \times \alpha_{N_2})^*\left((p_i,p_j), yx, (q_i,q_j)\right) = \alpha_{N_2}{}^*(p_j,y,q_j) \wedge \alpha_{N_1}{}^*(p_i,x,q_i)$$

$$(\beta_{N_1} \times \beta_{N_2})^*\left((p_i,p_j), yx, (q_i,q_j)\right) = \beta_{N_2}{}^*(p_j,y,q_j) \vee \beta_{N_1}{}^*(p_i,x,q_i)$$

$$(\gamma_{N_1} \times \gamma_{N_2})^*\left((p_i,p_j), yx, (q_i,q_j)\right) = \gamma_{N_2}{}^*(p_j,y,q_j) \vee \gamma_{N_1}{}^*(p_i,x,q_i).$$

**Theorem 4.3** Let $M_i = (Q_i, \Sigma_i, N_i), i = 1, 2$ be INA and let $\Sigma_1 \cap \Sigma_2 = \emptyset$. Cartesian product of $M_1 \times M_2$ is cyclic iff $M_1$ and $M_2$ are cyclic.

**Proof.** Let $M_1$ and $M_2$ are cyclic. Then $Q_1 = S(q_i)$ and $Q_2 = S(p_j)$ for some $q_i \in Q_1, p_j \in Q_2$. Let $(q_k, p_l) \in Q_1 \times Q_2$. Then $\exists\, x \in \Sigma_1{}^*$ and $y \in \Sigma_2{}^*$ such that

$\alpha_{N_1}{}^*(q_i,x,q_k) > [0,0]$, $\beta_{N_1}{}^*(q_i,x,q_k) < [1,1]$, $\gamma_{N_1}{}^*(q_i,x,q_k) < [1,1]$ and

$\alpha_{N_2}{}^*(p_j,y,p_l) > [0,0]$, $\beta_{N_2}{}^*(p_j,y,p_l) < [1,1]$, $\gamma_{N_2}{}^*(p_j,y,p_l) < [1,1]$. Thus

$$(\alpha_{N_1} \times \alpha_{N_2})^*\left((q_i,p_j), xy, (q_k,p_l)\right) = \alpha_{N_1}{}^*(q_i,x,q_k) \wedge \alpha_{N_2}{}^*(p_j,y,p_l) > [0,0]$$

$$(\beta_{N_1} \times \beta_{N_2})^*\left((q_i,p_j), xy, (q_k,p_l)\right) = \beta_{N_1}{}^*(q_i,x,q_k) \vee \beta_{N_2}{}^*(p_j,y,p_l) < [1,1]$$

$$(\gamma_{N_1} \times \gamma_{N_2})^*\left((q_i,p_j), xy, (q_k,p_l)\right) = \gamma_{N_1}{}^*(q_i,x,q_k) \vee \gamma_{N_2}{}^*(p_j,y,p_l) < [1,1].$$

Hence $(q_k,p_l) \in S\left((q_i,p_j)\right)$. $Q_1 \times Q_2 = S\left((q_i,p_j)\right)$. Hence $M_1 \times M_2$ is cyclic.

Conversely, let $M_1 \times M_2$ is cyclic. Then $Q_1 \times Q_2 = S\left((q_i,p_j)\right)$ for some $(q_i,p_j) \in Q_1 \times Q_2$.

Let $q_k \in Q_1$ and $p_l \in Q_2$. Then $\exists\, w \in (\Sigma_1 \cup \Sigma_2)^*$ such that

$(\alpha_{N_1} \times \alpha_{N_2})^*\left((q_i,p_j), w, (q_k,p_l)\right) > [0,0]$, $(\beta_{N_1} \times \beta_{N_2})^*\left((q_i,p_j), w, (q_k,p_l)\right) < [1,1]$ and

$(\gamma_{N_1} \times \gamma_{N_2})^*\left((q_i,p_j), w, (q_k,p_l)\right) < [1,1]$. Then by the theorem 4.2 $\exists\, u \in \Sigma_1{}^*$ and $v \in \Sigma_2{}^*$ such that

$\alpha_{N_1}{}^*(q_i,u,q_k) \wedge \alpha_{N_2}{}^*(p_j,v,p_l) = (\alpha_{N_1} \times \alpha_{N_2})^*\left((q_i,p_j),w,(q_k,p_l)\right) > [0,0]$
$\beta_{N_1}{}^*(q_i,u,q_k) \vee \beta_{N_2}{}^*(p_j,v,p_l) = (\beta_{N_1} \times \beta_{N_2})^*\left((q_i,p_j),w,(q_k,p_l)\right) < [1,1]$
$\gamma_{N_1}{}^*(q_i,u,q_k) \vee \gamma_{N_2}{}^*(p_j,v,p_l) = (\gamma_{N_1} \times \gamma_{N_2})^*\left((q_i,p_j),w,(q_k,p_l)\right) < [1,1]$.
Hence $\exists\, u \in \Sigma_1{}^*$ and $v \in \Sigma_2{}^*$ such that $\alpha_{N_1}{}^*(q_i,u,q_k) > [0,0]$, $\beta_{N_1}{}^*(q_i,u,q_k) < [1,1]$, $\gamma_{N_1}{}^*(q_i,u,q_k) < [1,1]$ and $\alpha_{N_2}{}^*(p_j,v,p_l) > [0,0]$, $\beta_{N_2}{}^*(p_j,v,p_l) < [1,1]$, $\gamma_{N_2}{}^*(p_j,v,p_l) < [1,1]$. Thus $q_k \in S(q_i)$ and $p_l \in S(p_j)$. Hence $Q_1 \in S(q_i)$ and $Q_2 \in S(p_j)$. Therefore $M_1 \times M_2$ is cyclic.

**5 Conclusion**

The purpose of this paper is to study the Cartesian product of INA. We prove that Cartesian product of cyclic of interval neutrosophic automata is cyclic.

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