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Commutative MBJ-neutrosophic ideals of BCK-algebras

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Abstract

The notion of commutative MBJ-neutrosophic ideal is introduced, and several properties are investigated. Relations between MBJ-neutrosophic ideal and commutative MBJ-neutrosophic ideal are considered. Characterizations of commutative MBJ-neutrosophic ideal are discussed.

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1 Introduction

The fuzzy set was introduced by L.A. Zadeh [21] in 1965 for dealing with uncertainties in many real applications. As a generalization of Zadeh's fuzzy set, K. Atanassov introdued the notion of intuitionistic fuzzy set (see [1]). As a more general platform that extends the notions of classic set, (intuitionistic) fuzzy set and interval valued (intuitionistic) fuzzy set, the notion of neutrosophic set is initiated by Smarandache ([16], [17] and [18]). Neutrosophic algebraic structures in BCK/BCI-algebras are discussed in the papers [2], [4], [5], [7], [8], [9], [14], [15], [19] and [20]. In [12], the notion of MBJ-neutrosophic sets is introduced as another generalization of neutrosophic set, it is applied to BCK/BCI-algebras. Mohseni et al. [12] introduced the concept of MBJ-neutrosophic subalgebras in BCK/BCI-algebras, and investigated related properties. They gave a characterization of MBJ-neutrosophic subalgebra, and established a new MBJ-neutrosophic subalgebra by using an MBJ-neutrosophic subalgebra of a BCI-algebra. They considered the homomorphic inverse image of MBJ-neutrosophic subalgebra, and discussed translation of MBJ-neutrosophic subalgebra. Jun and Roh [6] applied the notion of MBJ-neutrosophic sets to ideals of BCK/BI-algebras, and introduce the concept of MBJ-neutrosophic ideals in BCK/BCI-algebras. They provided a condition for an MBJ-neutrosophic subalgebra to be an MBJ-neutrosophic ideal in a BCK-algebra, and considered conditions for an MBJ-neutrosophic set to be an MBJ-neutrosophic ideal in a BCK/BCI-algebra. They discussed relations between MBJ-neutrosophic subalgebras, MBJ-neutrosophic $\circ$-subalgebras and MBJ-neutrosophic ideals. In a BCI-algebra, they provided conditions for an MBJ-neutrosophic ideal to be an MBJ-neutrosophic subalgebra, and considered a characterization of an MBJ-neutrosophic ideal in an (S)-BCK-algebra.

In this article, we introduce the notion of commutative MBJ-neutrosophic ideal, and investigate several properties. We discuss relations between MBJ-neutrosophic ideal and commutative MBJ-neutrosophic ideal. We provide characterizations of commutative MBJ-neutrosophic ideal.

2 Preliminaries

By a *BCI-algebra*, we mean a set X with a binary operation $*$ and a special element 0 that satisfies the following conditions:

(I) $((x * y) * (x * z)) * (z * y) = 0$,

(II) $(x * (x * y)) * y = 0$,

(III) $x * x = 0$,

(IV) $x * y = 0,\ y * x = 0 \ \Rightarrow\ x = y$,

for all $x, y, z \in X$. If a BCI-algebra X satisfies the following identity:

(V) $(\forall x \in X)\ (0 * x = 0)$,

then X is called a *BCK-algebra.*

Every BCK/BCI-algebra X satisfies the following conditions:

$$(\forall x \in X)\,(x * 0 = x)\,, \tag{2.1}$$

$$(\forall x, y, z \in X)\,(x \leq y \ \Rightarrow\ x * z \leq y * z,\ z * y \leq z * x)\,, \tag{2.2}$$

$$(\forall x, y, z \in X)\,((x * y) * z = (x * z) * y)\,, \tag{2.3}$$

$$(\forall x, y, z \in X)\,((x * z) * (y * z) \leq x * y)\,, \tag{2.4}$$

where $x \leq y$ if and only if $x * y = 0$.

A BCK-algebra X is said to be *commutative* if the following assertion is valid.

$$(\forall x, y \in X)\,(x * (x * y) = y * (y * x))\,. \tag{2.5}$$

A non-empty subset S of a BCK/BCI-algebra X is called a *subalgebra* of X if $x * y \in S$ for all $x, y \in S$. A subset I of a BCK/BCI-algebra X is called an *ideal* of X if it satisfies:

$$0 \in I, \tag{2.6}$$

$$(\forall x \in X)\,(\forall y \in I)\,(x * y \in I \ \Rightarrow\ x \in I)\,. \tag{2.7}$$

A subset I of a BCK-algebra X is called a *commutative ideal* of X if it satisfies (2.6) and

$$(\forall x, y \in X)(\forall z \in I)\,((x * y) * z \in I \ \Rightarrow\ x * (y * (y * x)) \in I)\,. \tag{2.8}$$

Observe that every commutative ideal is an ideal, but the converse is not true (see [10]).

By an *interval number* we mean a closed subinterval $\tilde{a} = [a^-, a^+]$ of I, where $0 \le a^- \le a^+ \le 1$. Denote by $[I]$ the set of all interval numbers. Let us define what is known as *refined minimum* (briefly, rmin) and *refined maximum* (briefly, rmax) of two elements in $[I]$. We also define the symbols "$\succeq$", "$\preceq$", "$=$" in case of two elements in $[I]$. Consider two interval numbers $\tilde{a}_1 := \left[a_1^-, a_1^+\right]$ and $\tilde{a}_2 := \left[a_2^-, a_2^+\right]$. Then

$$\begin{aligned} &\text{rmin}\left\{\tilde{a}_1, \tilde{a}_2\right\} = \left[\min\left\{a_1^-, a_2^-\right\}, \min\left\{a_1^+, a_2^+\right\}\right], \\ &\text{rmax}\left\{\tilde{a}_1, \tilde{a}_2\right\} = \left[\max\left\{a_1^-, a_2^-\right\}, \max\left\{a_1^+, a_2^+\right\}\right], \\ &\tilde{a}_1 \succeq \tilde{a}_2 \;\Leftrightarrow\; a_1^- \ge a_2^-,\; a_1^+ \ge a_2^+, \end{aligned}$$

and similarly we may have $\tilde{a}_1 \preceq \tilde{a}_2$ and $\tilde{a}_1 = \tilde{a}_2$. To say $\tilde{a}_1 \succ \tilde{a}_2$ (resp. $\tilde{a}_1 \prec \tilde{a}_2$) we mean $\tilde{a}_1 \succeq \tilde{a}_2$ and $\tilde{a}_1 \ne \tilde{a}_2$ (resp. $\tilde{a}_1 \preceq \tilde{a}_2$ and $\tilde{a}_1 \ne \tilde{a}_2$). Let $\tilde{a}_i \in [I]$ where $i \in \Lambda$. We define

$$\operatorname*{rinf}_{i\in\Lambda} \tilde{a}_i = \left[\inf_{i\in\Lambda} a_i^-, \inf_{i\in\Lambda} a_i^+\right] \quad \text{and} \quad \operatorname*{rsup}_{i\in\Lambda} \tilde{a}_i = \left[\sup_{i\in\Lambda} a_i^-, \sup_{i\in\Lambda} a_i^+\right].$$

Let X be a non-empty set. A function $A : X \to [I]$ is called an *interval-valued fuzzy set* (briefly, an *IVF set*) in X. Let $[I]^X$ stand for the set of all IVF sets in X. For every $A \in [I]^X$ and $x \in X$, $A(x) = [A^-(x), A^+(x)]$ is called the *degree* of membership of an element x to A, where $A^- : X \to I$ and $A^+ : X \to I$ are fuzzy sets in X which are called a *lower fuzzy set* and an *upper fuzzy set* in X, respectively. For simplicity, we denote $A = [A^-, A^+]$.

Let X be a non-empty set. A *neutrosophic set* (NS) in X (see [17]) is a structure of the form:

$$A := \{\langle x; A_T(x), A_I(x), A_F(x)\rangle \mid x \in X\},$$

where $A_T : X \to [0,1]$ is a truth membership function, $A_I : X \to [0,1]$ is an indeterminate membership function, and $A_F : X \to [0,1]$ is a false membership function.

We refer the reader to the books [3, 10] for further information regarding BCK/BCI-algebras, and to the site "http://fs.gallup.unm.edu/neutrosophy.htm" for further information regarding neutrosophic set theory.

Let X be a non-empty set. By an *MBJ-neutrosophic set* in X (see [12]), we mean a structure of the form:

$$\mathcal{A} := \{\langle x; M_A(x), \tilde{B}_A(x), J_A(x)\rangle \mid x \in X\},$$

where M_A and J_A are fuzzy sets in X, which are called a truth membership function and a false membership function, respectively, and $\tilde{B}_A$ is an IVF set in X which is called an indeterminate interval-valued membership function.

For the sake of simplicity, we shall use the symbol $\mathcal{A} = (M_A, \tilde{B}_A, J_A)$ for the MBJ-neutrosophic set

$$\mathcal{A} := \{\langle x; M_A(x), \tilde{B}_A(x), J_A(x)\rangle \mid x \in X\}.$$

Let X be a BCK/BCI-algebra. An MBJ-neutrosophic set $\mathcal{A} = (M_A, \tilde{B}_A, J_A)$ in X is called an *MBJ-neutrosophic ideal* of X (see [6]) if it satisfies:

$$(\forall x \in X)\left(\; M_A(0) \ge M_A(x),\, \tilde{B}_A(0) \succeq \tilde{B}_A(x),\, J_A(0) \le J_A(x) \;\right) \tag{2.9}$$

and

$$(\forall x, y \in X)\begin{pmatrix} M_A(x) \ge \min\{M_A(x*y), M_A(y)\} \\ \tilde{B}_A(x) \succeq \text{rmin}\{\tilde{B}_A(x*y), \tilde{B}_A(y)\} \\ J_A(x) \le \max\{J_A(x*y), J_A(y)\} \end{pmatrix}. \tag{2.10}$$

3 Commutative MBJ-neutrosophic ideals of BCK-algebras

In what follows, let X be a BCK-algebra unless otherwise specified.

Definition 3.1. *An MBJ-neutrosophic set $\mathcal{A} = (M_A, \tilde{B}_A, J_A)$ in X is called a commutative MBJ-neutrosophic ideal of X if it satisfies* (2.9) *and*

$$(\forall x, y, z \in X) \begin{pmatrix} M_A(x*(y*(y*x))) \geq \min\{M_A((x*y)*z), M_A(z)\} \\ \tilde{B}_A(x*(y*(y*x))) \succeq \operatorname{rmin}\{\tilde{B}_A((x*y)*z), \tilde{B}_A(z)\} \\ J_A(x*(y*(y*x))) \leq \max\{J_A((x*y)*z), J_A(z)\} \end{pmatrix}. \tag{3.1}$$

Example 3.2. *Consider a BCK-algebra $X = \{0, 1, 2, 3, 4\}$ with the binary operation $*$ which is given in Table 1.*

Table 1: Cayley table for the binary operation "$*$"

$*$	0	1	2	3	4
0	0	0	0	0	0
1	1	0	1	0	1
2	2	2	0	2	0
3	3	1	3	0	3
4	4	4	4	4	0

Let $\mathcal{A} = (M_A, \tilde{B}_A, J_A)$ be an MBJ-neutrosophic set in X defined by Table 2.

Table 2: MBJ-neutrosophic set $\mathcal{A} = (M_A, \tilde{B}_A, J_A)$

X	$M_A(x)$	$\tilde{B}_A(x)$	$J_A(x)$
0	0.7	$[0.4, 0.9]$	0.2
1	0.2	$[0.3, 0.6]$	0.6
2	0.5	$[0.3, 0.7]$	0.5
3	0.2	$[0.3, 0.6]$	0.6
4	0.3	$[0.2, 0.5]$	0.8

It is routine to verify that $\mathcal{A} = (M_A, \tilde{B}_A, J_A)$ is a commutative MBJ-neutrosophic ideal of X.

We consider a relation between a commutative MBJ-neutrosophic ideal and an MBJ-neutrosophic ideal.

Theorem 3.3. *Every commutative MBJ-neutrosophic ideal is an MBJ-neutrosophic ideal.*

Proof. Let $\mathcal{A} = (M_A, \tilde{B}_A, J_A)$ be a commutative MBJ-neutrosophic ideal of X. It we take $y = 0$ in (3.1) and use (2.1), then

$$M_A(x) = M_A(x*(0*(0*x))) \geq \min\{M_A((x*0)*z), M_A(z)\} = \min\{M_A(x*z), M_A(z)\},$$

$$\tilde{B}_A(x) = \tilde{B}_A(x * (0 * (0 * x))) \succeq \mathrm{rmin}\{\tilde{B}_A((x * 0) * z), \tilde{B}_A(z)\} = \mathrm{rmin}\{\tilde{B}_A(x * z), \tilde{B}_A(z)\},$$

and

$$J_A(x) = J_A(x * (0 * (0 * x))) \leq \max\{J_A((x * 0) * z), J_A(z)\} = \max\{J_A(x * z), J_A(z)\}$$

for all $x, z \in X$. Therefore $\mathcal{A} = (M_A, \tilde{B}_A, J_A)$ is an MBJ-neutrosophic ideal of X. □

The converse of Theorem 3.3 is not true as seen in the following example.

Example 3.4. *Consider a BCK-algebra $X = \{0, 1, 2, 3, 4\}$ with the binary operation $*$ which is given in Table 3.*

Table 3: Cayley table for the binary operation "$*$"

$*$	0	1	2	3	4
0	0	0	0	0	0
1	1	0	1	0	0
2	2	2	0	0	0
3	3	3	3	0	0
4	4	4	4	3	0

Let $\mathcal{A} = (M_A, \tilde{B}_A, J_A)$ be an MBJ-neutrosophic set in X defined by Table 4.

Table 4: MBJ-neutrosophic set $\mathcal{A} = (M_A, \tilde{B}_A, J_A)$

X	$M_A(x)$	$\tilde{B}_A(x)$	$J_A(x)$
0	0.66	$[0.4, 0.9]$	0.25
1	0.55	$[0.3, 0.5]$	0.35
2	0.33	$[0.3, 0.7]$	0.65
3	0.33	$[0.2, 0.4]$	0.65
4	0.33	$[0.2, 0.4]$	0.65

It is routine to verify that $\mathcal{A} = (M_A, \tilde{B}_A, J_A)$ is an MBJ-neutrosophic ideal of X. Since

$$M_A(2 * (3 * (3 * 2))) = M_A(2) = 0.33 \ngeq 0.66 = \min\{M_A((2 * 3) * 0), M_A(0)\},$$

and/or

$$\tilde{B}_A(2 * (3 * (3 * 2))) = \tilde{B}_A(2) = [0.3, 0.7] \not\succeq [0.4, 0.9] = \mathrm{rmin}\{\tilde{B}_A((2 * 3) * 0), \tilde{B}_A(0)\},$$

we know that $\mathcal{A} = (M_A, \tilde{B}_A, J_A)$ is not a commutative MBJ-neutrosophic ideal of X.

We provide conditions for an MBJ-neutrosophic ideal to be a commutative MBJ-neutrosophic ideal.

Theorem 3.5. *An MBJ-neutrosophic set $\mathcal{A} = (M_A,\ \tilde{B}_A,\ J_A)$ in X is a commutative MBJ-neutrosophic ideal of X if and only if it is an MBJ-neutrosophic ideal of X satisfying the following condition.*

$$(\forall x, y \in X) \left(\begin{array}{l} M_A(x*(y*(y*x))) \geq M_A(x*y), \\ \tilde{B}_A(x*(y*(y*x))) \succeq \tilde{B}_A(x*y), \\ J_A(x*(y*(y*x))) \leq J_A(x*y). \end{array} \right) \tag{3.2}$$

Proof. Assume that $\mathcal{A} = (M_A,\ \tilde{B}_A,\ J_A)$ is a commutative MBJ-neutrosophic ideal of X. If we put $z = 0$ in (3.1) and use (2.1) and (2.10), then we have (3.2).

Conversely, let $\mathcal{A} = (M_A,\ \tilde{B}_A,\ J_A)$ be an MBJ-neutrosophic ideal of X satisfying the condition (3.2). Then

$$\begin{array}{l} M_A(x*(y*(y*x))) \geq M_A(x*y) \geq \min\{M_A((x*y)*z), M_A(z)\}, \\ \tilde{B}_A(x*(y*(y*x))) \succeq \tilde{B}_A(x*y) \succeq \mathrm{rmin}\{\tilde{B}_A((x*y)*z), \tilde{B}_A(z)\}, \\ J_A(x*(y*(y*x))) \leq J_A(x*y) \leq \max\{J_A((x*y)*z), J_A(z)\} \end{array}$$

for all $x, y, z \in X$. Therefore $\mathcal{A} = (M_A,\ \tilde{B}_A,\ J_A)$ is a commutative MBJ-neutrosophic ideal of X. □

Lemma 3.6. [6] *Every MBJ-neutrosophic ideal $\mathcal{A} = (M_A,\ \tilde{B}_A,\ J_A)$ of X satisfies the following assertion.*

$$x*y \leq z \ \Rightarrow \left\{ \begin{array}{l} M_A(x) \geq \min\{M_A(y), M_A(z)\}, \\ \tilde{B}_A(x) \succeq \mathrm{rmin}\{\tilde{B}_A(y), \tilde{B}_A(z)\}, \\ J_A(x) \leq \max\{J_A(y), J_A(z)\}, \end{array} \right. \tag{3.3}$$

for all $x, y, z \in X$.

We provide a condition for an MBJ-neutrosophic ideal to be a commutative MBJ-neutrosophic ideal.

Theorem 3.7. *In a commutative BCK-algebra, every MBJ-neutrosophic ideal is a commutative MBJ-neutrosophic ideal.*

Proof. Let $\mathcal{A} = (M_A,\ \tilde{B}_A,\ J_A)$ be an MBJ-neutrosophic ideal of a commutative BCK-algebra X. Note that

$$\begin{aligned} ((x*(y*(y*x)))*((x*y)*z))*z &= ((x*(y*(y*x)))*z)*((x*y)*z) \\ &\leq (x*(y*(y*x)))*(x*y) \\ &= (x*(x*y))*(y*(y*x)) = 0, \end{aligned}$$

that is, $(x*(y*(y*x)))*((x*y)*z) \leq z$ for all $x, y, z \in X$. By Lemma 3.6 we have

$$\begin{array}{l} M_A(x*(y*(y*x))) \geq \min\{M_A((x*y)*z), M_A(z)\}, \\ \tilde{B}_A(x*(y*(y*x))) \succeq \mathrm{rmin}\{\tilde{B}_A((x*y)*z), \tilde{B}_A(z)\}, \\ J_A(x*(y*(y*x))) \leq \max\{J_A((x*y)*z), J_A(z)\}. \end{array}$$

Therefore $\mathcal{A} = (M_A,\ \tilde{B}_A,\ J_A)$ is a commutative MBJ-neutrosophic ideal of X. □

Given an MBJ-neutrosophic set $\mathcal{A} = (M_A,\ \tilde{B}_A,\ J_A)$ in X, we consider the following sets.

$$U(M_A;\alpha) := \{x \in X \mid M_A(x) \geq \alpha\},$$
$$U(\tilde{B}_A;[\delta_1,\delta_2]) := \{x \in X \mid \tilde{B}_A(x) \succeq [\delta_1,\delta_2]\},$$
$$L(J_A;\beta) := \{x \in X \mid J_A(x) \leq \beta\},$$

where $\alpha,\beta \in [0,1]$ and $[\delta_1,\delta_2] \in [I]$.

Theorem 3.8. *An MBJ-neutrosophic set $\mathcal{A} = (M_A,\ \tilde{B}_A,\ J_A)$ in X is a commutative MBJ-neutrosophic ideal of X if and only if the non-empty sets $U(M_A;\alpha)$, $U(\tilde{B}_A;[\delta_1,\delta_2])$ and $L(J_A;\beta)$ are commutative ideals of X for all $\alpha,\beta \in [0,1]$ and $[\delta_1,\delta_2] \in [I]$.*

Proof. Let $\mathcal{A} = (M_A,\ \tilde{B}_A,\ J_A)$ be a commutative MBJ-neutrosophic ideal of X. Let $\alpha,\beta \in [0,1]$ and $[\delta_1,\delta_2] \in [I]$ be such that $U(M_A;\alpha)$, $U(\tilde{B}_A;[\delta_1,\delta_2])$ and $L(J_A;\beta)$ are non-empty. Obviously, $0 \in U(M_A;\alpha) \cap U(\tilde{B}_A;[\delta_1,\delta_2]) \cap L(J_A;\beta)$. For any $x,y,z,a,b,c,u,v,w \in X$, if $(x*y)*z \in U(M_A;\alpha)$, $z \in U(M_A;\alpha)$, $(a*b)*c \in U(\tilde{B}_A;[\delta_1,\delta_2])$, $c \in U(\tilde{B}_A;[\delta_1,\delta_2])$, $(u*v)*w \in L(J_A;\beta)$ and $w \in L(J_A;\beta)$, then

$$M_A(x*(y*(y*x))) \geq \min\{M_A((x*y)*z), M_A(z)\} \geq \min\{\alpha,\alpha\} = \alpha,$$
$$\tilde{B}_A(a*(b*(b*a))) \succeq \operatorname{rmin}\{\tilde{B}_A((a*b)*c), \tilde{B}_A(c)\} \succeq \operatorname{rmin}\{[\delta_1,\delta_2],[\delta_1,\delta_2]\} = [\delta_1,\delta_2],$$
$$J_A(u*(v*(v*u))) \leq \max\{J_A((u*v)*w), J_A(w)\} \leq \min\{\beta,\beta\} = \beta,$$

and so $x*(y*(y*z)) \in U(M_A;\alpha)$, $a*(b*(b*a)) \in U(\tilde{B}_A;[\delta_1,\delta_2])$ and $u*(v*(v*u)) \in L(J_A;\beta)$. Therefore $U(M_A;\alpha)$, $U(\tilde{B}_A;[\delta_1,\delta_2])$ and $L(J_A;\beta)$ are commutative ideals of X.

Conversely, assume that the non-empty sets $U(M_A;\alpha)$, $U(\tilde{B}_A;[\delta_1,\delta_2])$ and $L(J_A;\beta)$ are commutative ideals of X for all $\alpha,\beta \in [0,1]$ and $[\delta_1,\delta_2] \in [I]$. Assume that $M_A(0) < M_A(a)$, $\tilde{B}_A(0) \prec \tilde{B}_A(a)$ and $J_A(0) > J_A(a)$ for some $a \in X$. Then $0 \notin U(M_A;M_A(a)) \cap U(\tilde{B}_A;\tilde{B}_A(a)) \cap L(J_A;J_A(a)$, which is a contradiction. Hence $M_A(0) \geq M_A(x)$, $\tilde{B}_A(0) \succeq \tilde{B}_A(x)$ and $J_A(0) \leq J_A(x)$ for all $x \in X$. If

$$M_A(a_0*(b_0*(b_0*a_0))) < \min\{M_A((a_0*b_0)*c_0), M_A(c_0)\},$$

for some $a_0,b_0,c_0 \in X$, then $(a_0*b_0)*c_0 \in U(M_A;t_0)$ and $c_0 \in U(M_A;t_0)$ but $a_0*(b_0*(b_0*a_0)) \notin U(M_A;t_0)$ for $t_0 := \min\{M_A((a_0*b_0)*c_0), M_A(c_0)\}$. This is a contradiction, and thus

$$M_A(a*(b*(b*a))) \geq \min\{M_A((a*b)*c), M_A(c)\},$$

for all $a,b,c \in X$. Similarly, we can show that $J_A(a*(b*(b*a))) \leq \max\{J_A((a*b)*c), J_A(c)\}$ for all $a,b,c \in X$. Suppose that $\tilde{B}_A(a_0*(b_0*(b_0*a_0))) \prec \operatorname{rmin}\{\tilde{B}_A((a_0*b_0)*c_0), \tilde{B}_A(c_0)\}$ for some $a_0,b_0,c_0 \in X$. Let $\tilde{B}_A((a_0*b_0)*c_0) = [\lambda_1,\lambda_2]$, $\tilde{B}_A(c_0) = [\lambda_3,\lambda_4]$ and $\tilde{B}_A(a_0*(b_0*(b_0*a_0))) = [\delta_1,\delta_2]$. Then

$$[\delta_1,\delta_2] \prec \operatorname{rmin}\{[\lambda_1,\lambda_2],[\lambda_3,\lambda_4]\} = [\min\{\lambda_1,\lambda_3\}, \min\{\lambda_2,\lambda_4\}],$$

and so $\delta_1 < \min\{\lambda_1,\lambda_3\}$ and $\delta_2 < \min\{\lambda_2,\lambda_4\}$. Taking

$$[\gamma_1,\gamma_2] := \tfrac{1}{2}\left(\tilde{B}_A(a_0*(b_0*(b_0*a_0))) + \operatorname{rmin}\{\tilde{B}_A((a_0*b_0)*c_0), \tilde{B}_A(c_0)\}\right)$$

implies that

$$[\gamma_1,\gamma_2] = \tfrac{1}{2}\left([\delta_1,\delta_2] + [\min\{\lambda_1,\lambda_3\}, \min\{\lambda_2,\lambda_4\}]\right) = \left[\tfrac{1}{2}(\delta_1 + \min\{\lambda_1,\lambda_3\}), \tfrac{1}{2}(\delta_2 + \min\{\lambda_2,\lambda_4\})\right].$$

It follows that

$$\min\{\lambda_1, \lambda_3\} > \gamma_1 = \tfrac{1}{2}(\delta_1 + \min\{\lambda_1, \lambda_3\}) > \delta_1,$$

and

$$\min\{\lambda_2, \lambda_4\} > \gamma_2 = \tfrac{1}{2}(\delta_2 + \min\{\lambda_2, \lambda_4\}) > \delta_2.$$

Hence $[\min\{\lambda_1, \lambda_3\}, \min\{\lambda_2, \lambda_4\}] \succ [\gamma_1, \gamma_2] \succ [\delta_1, \delta_2] = \tilde{B}_A(a_0 * (b_0 * (b_0 * a_0)))$, and therefore $a_0 * (b_0 * (b_0 * a_0)) \notin U(\tilde{B}_A; [\gamma_1, \gamma_2])$. On the other hand,

$$\tilde{B}_A((a_0 * b_0) * c_0) = [\lambda_1, \lambda_2] \succeq [\min\{\lambda_1, \lambda_3\}, \min\{\lambda_2, \lambda_4\}] \succ [\gamma_1, \gamma_2],$$

and

$$\tilde{B}_A(c_0) = [\lambda_3, \lambda_4] \succeq [\min\{\lambda_1, \lambda_3\}, \min\{\lambda_2, \lambda_4\}] \succ [\gamma_1, \gamma_2],$$

that is, $(a_0 * b_0) * c_0, c_0 \in U(\tilde{B}_A; [\gamma_1, \gamma_2])$. This is a contradiction, and therefore

$$\tilde{B}_A(x * (y * (y * x))) \succeq \mathrm{rmin}\{\tilde{B}_A((x * y) * z), \tilde{B}_A(z)\},$$

for all $x, y, z \in X$. Consequently $\mathcal{A} = (M_A, \tilde{B}_A, J_A)$ is a commutative MBJ-neutrosophic ideal of X. □

Theorem 3.9. *Every commutative ideal can be realized as level neutrosophic commutative ideals of some commutative MBJ-neutrosophic ideal of X.*

Proof. Given a commutative ideal C of X, let $\mathcal{A} = (M_A, \tilde{B}_A, J_A)$ be an MBJ-neutrosophic set in X defined by

$$M_A(x) = \begin{cases} \alpha & \text{if } x \in C, \\ 0 & \text{otherwise,} \end{cases} \quad \tilde{B}_A(x) = \begin{cases} [\delta_1, \delta_2] & \text{if } x \in C, \\ [0,0] & \text{otherwise,} \end{cases} \quad J_A(x) = \begin{cases} \beta & \text{if } x \in C, \\ 1 & \text{otherwise,} \end{cases}$$

where $\alpha, \delta_1, \delta_2 \in (0, 1]$ and $\beta \in [0, 1)$. Let $x, y, z \in X$. If $(x * y) * z \in C$ and $z \in C$, then $x * (y * (y * x)) \in C$. Thus

$$\begin{aligned} M_A(x * (y * (y * x))) &= \alpha = \min\{M_A((x * y) * z), M_A(z)\}, \\ \tilde{B}_A(x * (y * (y * x))) &= [\delta_1, \delta_2] = \mathrm{rmin}\{\tilde{B}_A((x * y) * z), \tilde{B}_A(z)\}, \\ J_A(x * (y * (y * x))) &= \beta = \max\{J_A((x * y) * z), J_A(z)\}. \end{aligned}$$

Assume that $(x*y)*z \notin C$ and $z \notin C$. Then $M_A((x*y)*z) = 0$, $M_A(z) = 0$, $\tilde{B}_A((x*y)*z) = [0, 0]$, $\tilde{B}_A(z) = [0, 0]$, and $J_A((x * y) * z) = 1$, $J_A(z) = 1$. It follows that

$$\begin{aligned} M_A(x * (y * (y * x))) &\geq \min\{M_A((x * y) * z), M_A(z)\}, \\ \tilde{B}_A(x * (y * (y * x))) &\succeq \mathrm{rmin}\{\tilde{B}_A((x * y) * z), \tilde{B}_A(z)\}, \\ J_A(x * (y * (y * x))) &\leq \max\{J_A((x * y) * z), J_A(z)\}. \end{aligned}$$

If exactly one of $(x * y) * z$ and z belongs to C, then exactly one of $M_A((x * y) * z)$ and $M_A(z)$ is equal to 0; exactly one of $\tilde{B}_A((x * y) * z)$ and $\tilde{B}_A(z)$ is equal to $[0, 0]$; exactly one of $J_A((x * y) * z)$ and $J_A(z)$ is equal to 1. Hence

$$\begin{aligned} M_A(x * (y * (y * x))) &\geq \min\{M_A((x * y) * z), M_A(z)\}, \\ \tilde{B}_A(x * (y * (y * x))) &\succeq \mathrm{rmin}\{\tilde{B}_A((x * y) * z), \tilde{B}_A(z)\}, \\ J_A(x * (y * (y * x))) &\leq \max\{J_A((x * y) * z), J_A(z)\}. \end{aligned}$$

It is clear that $M_A(0) \geq M_A(x)$, $\tilde{B}_A(0) \succeq \tilde{B}_A(x)$, and $J_A(0) \leq J_A(x)$ for all $x \in X$. Therefore $\mathcal{A} = (M_A, \tilde{B}_A, J_A)$ is a commutative MBJ-neutrosophic ideal of X. Obviously, $U(M_A; \alpha) = C$, $U(\tilde{B}_A; [\delta_1, \delta_2]) = C$ and $L(J_A; \beta) = C$. This completes the proof. □

A mapping $f : X \to Y$ of BCK/BCI-algebras is called a *homomorphism* ([10]) if $f(x * y) = f(x) * f(y)$ for all $x, y \in X$. Note that if $f : X \to Y$ is a homomorphism, then $f(0) = 0$. Let $f : X \to Y$ be a homomorphism of BCK/BCI-algebras. For any MBJ-neutrosophic set $A = (M_A, \tilde{B}_A, J_A)$ in Y, we define a new MBJ-neutrosophic set $A^f = (M_A^f, \tilde{B}_A^f, J_A^f)$ in X, which is called the *induced MBJ-neutrosophic set*, by

$$(\forall x \in X) \left(\begin{array}{l} M_A^f(x) = M_A(f(x)) \\ \tilde{B}_A^f(x) = \tilde{B}_A(f(x)) \\ J_A^f(x) = J_A(f(x)) \end{array} \right). \tag{3.4}$$

Lemma 3.10. *Let $f : X \to Y$ be a homomorphism of BCK/BCI-algebras. If an MBJ-neutrosophic set $A = (M_A, \tilde{B}_A, J_A)$ in Y is an MBJ-neutrosophic ideal of Y, then the induced MBJ-neutrosophic set $A^f = (M_A^f, \tilde{B}_A^f, J_A^f)$ in X is an MBJ-neutrosophic ideal of X.*

Proof. For any $x \in X$, we have

$$\begin{aligned} M_A^f(0) &= M_A(f(0)) = M_A(0) \geq M_A(f(x)) = M_A^f(x), \\ \tilde{B}_A^f(0) &= \tilde{B}_A(f(0)) = \tilde{B}_A(0) \succeq \tilde{B}_A(f(x)) = \tilde{B}_A^f(x), \\ J_A^f(0) &= J_A(f(0)) = J_A(0) \leq J_A(f(x)) = J_A^f(x). \end{aligned}$$

Let $x, y \in X$. Then

$$\begin{aligned} M_A^f(x) &= M_A(f(x)) \geq \min\{M_A(f(x) * f(y)), M_A(f(y))\} \\ &= \min\{M_A(f(x * y)), M_A(f(y))\} \\ &= \min\{M_A^f(x * y), M_A^f(y)\}, \end{aligned}$$

$$\begin{aligned} \tilde{B}_A^f(x) &= \tilde{B}_A(f(x)) \succeq \operatorname{rmin}\{\tilde{B}_A(f(x) * f(y)), \tilde{B}_A(f(y))\} \\ &= \operatorname{rmin}\{\tilde{B}_A(f(x * y)), \tilde{B}_A(f(y))\} \\ &= \operatorname{rmin}\{\tilde{B}_A^f(x * y), \tilde{B}_A^f(y)\} \end{aligned}$$

and

$$\begin{aligned} J_A^f(x) &= J_A(f(x)) \leq \max\{J_A(f(x) * f(y)), J_A(f(y))\} \\ &= \max\{J_A(f(x * y)), J_A(f(y))\} \\ &= \max\{J_A^f(x * y), J_A^f(y)\}. \end{aligned}$$

Therefore $A^f = (M_A^f, \tilde{B}_A^f, J_A^f)$ is an MBJ-neutrosophic ideal of X. □

Theorem 3.11. *Let $f : X \to Y$ be a homomorphism of BCK-algebras. If an MBJ-neutrosophic set $A = (M_A, \tilde{B}_A, J_A)$ in Y is a commutative MBJ-neutrosophic ideal of Y, then the induced MBJ-neutrosophic set $A^f = (M_A^f, \tilde{B}_A^f, J_A^f)$ in X is a commutative MBJ-neutrosophic ideal of X.*

Proof. Assume that $A = (M_A,\ \tilde{B}_A,\ J_A)$ is a commutative MBJ-neutrosophic ideal of Y. Then $A = (M_A,\ \tilde{B}_A,\ J_A)$ is an MBJ-neutrosophic ideal of Y by Theorem 3.3, and so $A^f = (M_A^f,\ \tilde{B}_A^f,\ J_A^f)$ is an MBJ-neutrosophic ideal of Y by Lemma 3.10. For any $x, y \in X$, we have

$$\begin{aligned} M_A^f(x*(y*(y*x))) &= M_A(f(x*(y*(y*x)))) \\ &= M_A(f(x)*(f(y)*(f(y)*f(x)))) \\ &\geq M_A(f(x)*f(y)) \\ &= M_A(f(x*y)) = M_A^f(x*y), \end{aligned}$$

$$\begin{aligned} \tilde{B}_A^f(x*(y*(y*x))) &= \tilde{B}_A(f(x*(y*(y*x)))) \\ &= \tilde{B}_A(f(x)*(f(y)*(f(y)*f(x)))) \\ &\succeq \tilde{B}_A(f(x)*f(y)) \\ &= \tilde{B}_A(f(x*y)) = \tilde{B}_A^f(x*y), \end{aligned}$$

and

$$\begin{aligned} J_A^f(x*(y*(y*x))) &= J_A(f(x*(y*(y*x)))) \\ &= J_A(f(x)*(f(y)*(f(y)*f(x)))) \\ &\leq J_A(f(x)*f(y)) \\ &= J_A(f(x*y)) = J_A^f(x*y). \end{aligned}$$

Therefore $A^f = (M_A^f,\ \tilde{B}_A^f,\ J_A^f)$ is a commutative MBJ-neutrosophic ideal of X by Theorem 3.5. □

Lemma 3.12. *Let $f : X \to Y$ be an onto homomorphism of BCK/BCI-algebras and let $A = (M_A,\ \tilde{B}_A,\ J_A)$ be an MBJ-neutrosophic set in Y. If the induced MBJ-neutrosophic set $A^f = (M_A^f,\ \tilde{B}_A^f,\ J_A^f)$ in X is an MBJ-neutrosophic ideal of X, then $A = (M_A,\ \tilde{B}_A,\ J_A)$ is an MBJ-neutrosophic ideal of Y.*

Proof. Suppose that the induced MBJ-neutrosophic set $A^f = (M_A^f,\ \tilde{B}_A^f,\ J_A^f)$ in X is an MBJ-neutrosophic ideal of X. For any $a \in Y$, there exists $x \in X$ such that $f(x) = a$. Thus

$$M_A(0) = M_A(f(0)) = M_A^f(0) \geq M_A^f(x) = M_A(f(x)) = M_A(a),$$

$$\tilde{B}_A(0) = \tilde{B}_A(f(0)) = \tilde{B}_A^f(0) \succeq \tilde{B}_A^f(x) = \tilde{B}_A(f(x)) = \tilde{B}_A(a),$$

and

$$J_A(0) = J_A(f(0)) = J_A^f(0) \leq J_A^f(x) = J_A(f(x)) = J_A(a).$$

Let $a, b \in Y$. Then $f(x) = a$ and $f(y) = b$ for some $x, y \in X$. Hence

$$\begin{aligned} M_A(a) = M_A(f(x)) = M_A^f(x) &\geq \min\{M_A^f(x*y), M_A^f(y)\} \\ &= \min\{M_A(f(x*y)), M_A(f(y))\} \\ &= \min\{M_A(f(x)*f(y)), M_A(f(y))\} \\ &= \min\{M_A(a*b), M_A(b)\}, \end{aligned}$$

$$\begin{aligned}\tilde{B}_A(a) = \tilde{B}_A(f(x)) = \tilde{B}_A^f(x) &\succeq \text{rmin}\{\tilde{B}_A^f(x*y), \tilde{B}_A^f(y)\} \\ &= \text{rmin}\{\tilde{B}_A(f(x*y)), \tilde{B}_A(f(y))\} \\ &= \text{rmin}\{\tilde{B}_A(f(x)*f(y)), \tilde{B}_A(f(y))\} \\ &= \text{rmin}\{\tilde{B}_A(a*b), \tilde{B}_A(b)\},\end{aligned}$$

and

$$\begin{aligned}J_A(a) = J_A(f(x)) = J_A^f(x) &\leq \max\{J_A^f(x*y), J_A^f(y)\} \\ &= \max\{J_A(f(x*y)), J_A(f(y))\} \\ &= \max\{J_A(f(x)*f(y)), J_A(f(y))\} \\ &= \max\{J_A(a*b), J_A(b)\}.\end{aligned}$$

Therefore $A = (M_A, \tilde{B}_A, J_A)$ is an MBJ-neutrosophic ideal of Y. □

Theorem 3.13. *Let $f : X \to Y$ be an onto homomorphism of BCK-algebras and let $A = (M_A, \tilde{B}_A, J_A)$ be an MBJ-neutrosophic set in Y. If the induced MBJ-neutrosophic set $A^f = (M_A^f, \tilde{B}_A^f, J_A^f)$ in X is a commutative MBJ-neutrosophic ideal of X, then $A = (M_A, \tilde{B}_A, J_A)$ is a commutative MBJ-neutrosophic ideal of Y.*

Proof. Suppose that $A^f = (M_A^f, \tilde{B}_A^f, J_A^f)$ is a commutative MBJ-neutrosophic ideal of X. Then $A^f = (M_A^f, \tilde{B}_A^f, J_A^f)$ is an MBJ-neutrosophic ideal of X by Theorem 3.3, and thus $A = (M_A, \tilde{B}_A, J_A)$ is an MBJ-neutrosophic ideal of Y by Lemma 3.12. For any $a, b, c \in Y$, there exist $x, y, z \in X$ such that $f(x) = a$, $f(y) = b$ and $f(z) = c$. It follows that

$$\begin{aligned}M_A(a*(b*(b*a))) &= M_A(f(x)*(f(y)*(f(y)*f(x)))) = M_A(f(x*(y*(y*x)))) \\ &= M_A^f(x*(y*(y*x))) \geq M_A^f(x*y) \\ &= M_A(f(x)*f(y)) = M_A(a*b),\end{aligned}$$

$$\begin{aligned}\tilde{B}_A(a*(b*(b*a))) &= \tilde{B}_A(f(x)*(f(y)*(f(y)*f(x)))) = \tilde{B}_A(f(x*(y*(y*x)))) \\ &= \tilde{B}_A^f(x*(y*(y*x))) \succeq \tilde{B}_A^f(x*y) \\ &= \tilde{B}_A(f(x)*f(y)) = \tilde{B}_A(a*b),\end{aligned}$$

and

$$\begin{aligned}J_A(a*(b*(b*a))) &= J_A(f(x)*(f(y)*(f(y)*f(x)))) = J_A(f(x*(y*(y*x)))) \\ &= J_A^f(x*(y*(y*x))) \leq J_A^f(x*y) \\ &= J_A(f(x)*f(y)) = J_A(a*b).\end{aligned}$$

It follows from Theorem 3.5 that $A = (M_A, \tilde{B}_A, J_A)$ is a commutative MBJ-neutrosophic ideal of Y. □

Conclusion

We have introduced the concept of commutative MBJ-neutrosophic ideal, and have investigated several properties. We have considered relations between MBJ-neutrosophic ideal and commutative

MBJ-neutrosophic ideal, and have provided characterizations of commutative MBJ-neutrosophic ideal. Using the homomorphism of BCK-algebras, we have shown that the induced MBJ-neutrosophic set of a commutative MBJ-neutrosophic ideal is also a commutative MBJ-neutrosophic ideal. We also have shown that if the induced MBJ-neutrosophic set of an MBJ-neutrosophic is a commutative MBJ-neutrosophic ideal, then the original MBJ-neutrosophic is also a commutative MBJ-neutrosophic ideal.

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