

FURTHER RESULTS OF NEUTROSOPHIC SUBALGEBRAS IN *BCK/BCI-ALGEBRAS* BASED ON NEUTROSOPHIC POINTS

G. MUHIUDDIN¹, YOUNG BAE JUN², §

ABSTRACT. In this paper, we investigate several properties of $(\in, \in \vee q_{(k_T, k_I, k_F)})$ -neutrosophic subalgebra, $(\in, q_{(k_T, k_I, k_F)})$ -neutrosophic subalgebra, $(q_{(k_T, k_I, k_F)}, \in \vee q_{(k_T, k_I, k_F)})$ -neutrosophic subalgebra, and $(q_{(k_T, k_I, k_F)}, \in)$ -neutrosophic subalgebra.

Keywords: $(\in, \in \vee q_{(k_T, k_I, k_F)})$ -neutrosophic subalgebra; $(\in, q_{(k_T, k_I, k_F)})$ -neutrosophic subalgebra; $(q_{(k_T, k_I, k_F)}, \in \vee q_{(k_T, k_I, k_F)})$ -neutrosophic subalgebra.

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1. INTRODUCTION

Smarandache [10, 11] introduced the concept of neutrosophic sets which is more general platform to extend the notions of the classical set and (intuitionistic, interval valued) fuzzy set. Neutrosophic set theory is applied to several parts which is referred to the site <http://fs.gallup.unm.edu/neutrosophy.htm>. Jun [3] introduced the notion of neutrosophic subalgebras in *BCK/BCI*-algebras based on neutrosophic points. Borumand and Jun [1] studied several properties of $(\in, \in \vee q)$ -neutrosophic subalgebras and $(q, \in \vee q)$ -neutrosophic subalgebras in *BCK/BCI*-algebras. Muhiuddin et al. [9] studied further results on (\in, \in) -neutrosophic subalgebras and ideals in *BCK/BCI*-algebras. Also, Kim et al. [4] considered a general form of neutrosophic points, and then they discussed generalizations of the papers [3] and [1]. As a generalization of $(\in, \in \vee q)$ -neutrosophic subalgebras, they introduced the notions of $(\in, \in \vee q_{(k_T, k_I, k_F)})$ -neutrosophic subalgebra, and $(\in, q_{(k_T, k_I, k_F)})$ -neutrosophic subalgebra in *BCK/BCI*-algebras, and investigated several properties. They discussed characterizations of $(\in, \in \vee q_{(k_T, k_I, k_F)})$ -neutrosophic subalgebra, and considered relations between (\in, \in) -neutrosophic subalgebra, $(\in, q_{(k_T, k_I, k_F)})$ -neutrosophic subalgebra and $(\in, \in \vee q_{(k_T, k_I, k_F)})$ -neutrosophic subalgebra. Recently, Muhiuddin et al. applied the neutrosophic set theory to the *BCK/BCI*-algebras on various aspects (see for e.g., [6], [7], [8], [9]).

¹ Department of Mathematics, University of Tabuk, Tabuk 71491, Saudi Arabia.
e-mail: chishtygm@gmail.com; ORCID: <http://orcid.org/0000-0002-5596-5841>.

² Department of Mathematics Education, Gyeongsang National University, Jinju 52828, Korea.
e-mail: skywine@gmail.com; ORCID: <https://orcid.org/0000-0002-0181-8969>.

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In this paper, we investigate further properties of $(\in, \in \vee q_{(k_T, k_I, k_F)})$ -neutrosophic subalgebra, $(q_{(k_T, k_I, k_F)}, \in \vee q_{(k_T, k_I, k_F)})$ -neutrosophic subalgebra, $(\in, q_{(k_T, k_I, k_F)})$ -neutrosophic subalgebra and $(q_{(k_T, k_I, k_F)}, \in)$ -neutrosophic subalgebra in *BCK/BCI*-algebras.

2. PRELIMINARIES

By a *BCI-algebra* we mean a set X with a binary operation $*$ and the special element 0 satisfying the axioms:

- (a1) $((x * y) * (x * z)) * (z * y) = 0,$
- (a2) $(x * (x * y)) * y = 0,$
- (a3) $x * x = 0,$
- (a4) $x * y = y * x = 0 \Rightarrow x = y,$

for all $x, y, z \in X$. If a *BCI-algebra* X satisfies the axiom

- (a5) $0 * x = 0$ for all $x \in X,$

then we say that X is a *BCK-algebra*. A nonempty subset S of a *BCK/BCI-algebra* X is called a *subalgebra* of X if $x * y \in S$ for all $x, y \in S$.

The collection of all *BCK-algebras* and all *BCI-algebras* are denoted by $\mathcal{B}_K(X)$ and $\mathcal{B}_I(X)$, respectively. Also $\mathcal{B}(X) := \mathcal{B}_K(X) \cup \mathcal{B}_I(X)$.

We refer the reader to the books [2] and [5] for further information regarding *BCK/BCI-algebras*.

Let X be a non-empty set. A neutrosophic set (NS) in X (see [10]) is a structure of the form:

$$A := \{ \langle x; A_T(x), A_I(x), A_F(x) \rangle \mid x \in X \}$$

where $A_T : X \rightarrow [0, 1]$ is a truth membership function, $A_I : X \rightarrow [0, 1]$ is an indeterminate membership function, and $A_F : X \rightarrow [0, 1]$ is a false membership function. For the sake of simplicity, we shall use the symbol $A = (A_T, A_I, A_F)$ for the neutrosophic set

$$A := \{ \langle x; A_T(x), A_I(x), A_F(x) \rangle \mid x \in X \}.$$

Given a neutrosophic set $A = (A_T, A_I, A_F)$ in a set X , $\alpha, \beta \in (0, 1]$ and $\gamma \in [0, 1)$, we consider the following sets (see [3]):

- $T_{\in}(A; \alpha) := \{x \in X \mid A_T(x) \geq \alpha\},$
- $I_{\in}(A; \beta) := \{x \in X \mid A_I(x) \geq \beta\},$
- $F_{\in}(A; \gamma) := \{x \in X \mid A_F(x) \leq \gamma\},$
- $T_q(A; \alpha) := \{x \in X \mid A_T(x) + \alpha > 1\},$
- $I_q(A; \beta) := \{x \in X \mid A_I(x) + \beta > 1\},$
- $F_q(A; \gamma) := \{x \in X \mid A_F(x) + \gamma < 1\},$
- $T_{\in \vee q}(A; \alpha) := \{x \in X \mid A_T(x) \geq \alpha \text{ or } A_T(x) + \alpha > 1\},$
- $I_{\in \vee q}(A; \beta) := \{x \in X \mid A_I(x) \geq \beta \text{ or } A_I(x) + \beta > 1\},$
- $F_{\in \vee q}(A; \gamma) := \{x \in X \mid A_F(x) \leq \gamma \text{ or } A_F(x) + \gamma < 1\}.$

We say $T_{\in}(A; \alpha)$, $I_{\in}(A; \beta)$ and $F_{\in}(A; \gamma)$ are *neutrosophic \in -subsets*; $T_q(A; \alpha)$, $I_q(A; \beta)$ and $F_q(A; \gamma)$ are *neutrosophic q -subsets*; and $T_{\in \vee q}(A; \alpha)$, $I_{\in \vee q}(A; \beta)$ and $F_{\in \vee q}(A; \gamma)$ are

neutrosophic $\in \vee q$ -subsets. It is clear that

$$T_{\in \vee q}(A; \alpha) = T_{\in}(A; \alpha) \cup T_q(A; \alpha), \quad (1)$$

$$I_{\in \vee q}(A; \beta) = I_{\in}(A; \beta) \cup I_q(A; \beta), \quad (2)$$

$$F_{\in \vee q}(A; \gamma) = F_{\in}(A; \gamma) \cup F_q(A; \gamma). \quad (3)$$

Given $\Phi, \Psi \in \{\in, q, \in \vee q\}$, a neutrosophic set $A = (A_T, A_I, A_F)$ in $X \in \mathcal{B}(X)$ is called a (Φ, Ψ) -neutrosophic subalgebra of X (see [3]) if the following assertions are valid.

$$\begin{aligned} x \in T_{\Phi}(A; \alpha_x), y \in T_{\Phi}(A; \alpha_y) &\Rightarrow x * y \in T_{\Psi}(A; \alpha_x \wedge \alpha_y), \\ x \in I_{\Phi}(A; \beta_x), y \in I_{\Phi}(A; \beta_y) &\Rightarrow x * y \in I_{\Psi}(A; \beta_x \wedge \beta_y), \\ x \in F_{\Phi}(A; \gamma_x), y \in F_{\Phi}(A; \gamma_y) &\Rightarrow x * y \in F_{\Psi}(A; \gamma_x \vee \gamma_y) \end{aligned} \quad (4)$$

for all $x, y \in X$, $\alpha_x, \alpha_y, \beta_x, \beta_y, \in (0, 1]$ and $\gamma_x, \gamma_y \in [0, 1]$.

In what follows, let k_T, k_I and k_F denote arbitrary elements of $[0, 1)$ unless otherwise specified. If k_T, k_I and k_F are the same number in $[0, 1)$, then it is denoted by k , i.e., $k = k_T = k_I = k_F$.

Given a neutrosophic set $A = (A_T, A_I, A_F)$ in a set X , $\alpha, \beta \in (0, 1]$ and $\gamma \in [0, 1)$, we consider the following sets (see [4]):

$$\begin{aligned} T_{q_{k_T}}(A; \alpha) &:= \{x \in X \mid A_T(x) + \alpha + k_T > 1\}, \\ I_{q_{k_I}}(A; \beta) &:= \{x \in X \mid A_I(x) + \beta + k_I > 1\}, \\ F_{q_{k_F}}(A; \gamma) &:= \{x \in X \mid A_F(x) + \gamma + k_F < 1\}, \\ T_{\in \vee q_{k_T}}(A; \alpha) &:= \{x \in X \mid A_T(x) \geq \alpha \text{ or } A_T(x) + \alpha + k_T > 1\}, \\ I_{\in \vee q_{k_I}}(A; \beta) &:= \{x \in X \mid A_I(x) \geq \beta \text{ or } A_I(x) + \beta + k_I > 1\}, \\ F_{\in \vee q_{k_F}}(A; \gamma) &:= \{x \in X \mid A_F(x) \leq \gamma \text{ or } A_F(x) + \gamma + k_F < 1\}. \end{aligned}$$

We say $T_{q_{k_T}}(A; \alpha)$, $I_{q_{k_I}}(A; \beta)$ and $F_{q_{k_F}}(A; \gamma)$ are *neutrosophic q_k -subsets*; and $T_{\in \vee q_{k_T}}(A; \alpha)$, $I_{\in \vee q_{k_I}}(A; \beta)$ and $F_{\in \vee q_{k_F}}(A; \gamma)$ are *neutrosophic $\in \vee q_k$ -subsets*. For $\Phi \in \{\in, q, q_k, q_{k_T}, q_{k_I}, q_{k_F}, \in \vee q, \in \vee q_k, \in \vee q_{k_T}, \in \vee q_{k_I}, \in \vee q_{k_F}\}$, the element of $T_{\Phi}(A; \alpha)$ (resp., $I_{\Phi}(A; \beta)$ and $F_{\Phi}(A; \gamma)$) is called a *neutrosophic T_{Φ} -point* (resp., *neutrosophic I_{Φ} -point* and *neutrosophic F_{Φ} -point*) with value α (resp., β and γ).

It is clear that

$$T_{\in \vee q_{k_T}}(A; \alpha) = T_{\in}(A; \alpha) \cup T_{q_{k_T}}(A; \alpha), \quad (5)$$

$$I_{\in \vee q_{k_I}}(A; \beta) = I_{\in}(A; \beta) \cup I_{q_{k_I}}(A; \beta), \quad (6)$$

$$F_{\in \vee q_{k_F}}(A; \gamma) = F_{\in}(A; \gamma) \cup F_{q_{k_F}}(A; \gamma). \quad (7)$$

3. GENERALIZATIONS OF NEUTROSOPHIC SUBALGEBRAS

Definition 3.1 ([4]). A neutrosophic set $A = (A_T, A_I, A_F)$ in $X \in \mathcal{B}(X)$ is called an $(\in, \in \vee q_{(k_T, k_I, k_F)})$ -neutrosophic subalgebra of X if

$$\begin{aligned} x \in T_{\in}(A; \alpha_x), y \in T_{\in}(A; \alpha_y) &\Rightarrow x * y \in T_{\in \vee q_{k_T}}(A; \alpha_x \wedge \alpha_y), \\ x \in I_{\in}(A; \beta_x), y \in I_{\in}(A; \beta_y) &\Rightarrow x * y \in I_{\in \vee q_{k_I}}(A; \beta_x \wedge \beta_y), \\ x \in F_{\in}(A; \gamma_x), y \in F_{\in}(A; \gamma_y) &\Rightarrow x * y \in F_{\in \vee q_{k_F}}(A; \gamma_x \vee \gamma_y) \end{aligned} \quad (8)$$

for all $x, y \in X$, $\alpha_x, \alpha_y, \beta_x, \beta_y, \in (0, 1]$ and $\gamma_x, \gamma_y \in [0, 1)$.

An $(\in, \in \vee q_{(k_T, k_I, k_F)})$ -neutrosophic subalgebra with $k_T = k_I = k_F = k$ is called an $(\in, \in \vee q_k)$ -neutrosophic subalgebra.

Lemma 3.1 ([4]). *Given a neutrosophic set $A = (A_T, A_I, A_F)$ in $X \in \mathcal{B}(X)$, the following are equivalent.*

- (1) $A = (A_T, A_I, A_F)$ is an $(\in, \in \vee q_{(k_T, k_I, k_F)})$ -neutrosophic subalgebra of X .
- (2) $A = (A_T, A_I, A_F)$ satisfies the following assertion.

$$(\forall x, y \in X) \left(\begin{array}{l} A_T(x * y) \geq \bigwedge \{A_T(x), A_T(y), \frac{1-k_T}{2}\} \\ A_I(x * y) \geq \bigwedge \{A_I(x), A_I(y), \frac{1-k_I}{2}\} \\ A_F(x * y) \leq \bigvee \{A_F(x), A_F(y), \frac{1-k_F}{2}\} \end{array} \right). \tag{9}$$

Theorem 3.1. *If $A = (A_T, A_I, A_F)$ is an $(\in, \in \vee q_{(k_T, k_I, k_F)})$ -neutrosophic subalgebra of $X \in \mathcal{B}(X)$, then neutrosophic q_k -subsets $T_{q_{k_T}}(A; \alpha)$, $I_{q_{k_I}}(A; \beta)$ and $F_{q_{k_F}}(A; \gamma)$ are subalgebras of X for all $\alpha \in (\frac{1-k_T}{2}, 1]$, $\beta \in (\frac{1-k_I}{2}, 1]$ and $\gamma \in [0, \frac{1-k_F}{2})$ whenever they are nonempty.*

Proof. Assume that $T_{q_{k_T}}(A; \alpha)$, $I_{q_{k_I}}(A; \beta)$ and $F_{q_{k_F}}(A; \gamma)$ are nonempty neutrosophic q_k -subsets for all $\alpha \in (\frac{1-k_T}{2}, 1]$, $\beta \in (\frac{1-k_I}{2}, 1]$ and $\gamma \in [0, \frac{1-k_F}{2})$. Let $x, y \in T_{q_{k_T}}(A; \alpha)$. Then $A_T(x) + \alpha + k_T > 1$ and $A_T(y) + \alpha + k_T > 1$. Using Lemma 3.1 implies that

$$\begin{aligned} A_T(x * y) + \alpha + k_T &\geq \bigwedge \{A_T(x), A_T(y), \frac{1-k_T}{2}\} + \alpha + k_T \\ &= \bigwedge \{A_T(x) + \alpha + k_T, A_T(y) + \alpha + k_T, \frac{1-k_T}{2} + \alpha + k_T\} \\ &> 1 \end{aligned}$$

and so that $x * y \in T_{q_{k_T}}(A; \alpha)$. Hence $T_{q_{k_T}}(A; \alpha)$ is a subalgebra of X . Similarly, we can induce that $I_{q_{k_I}}(A; \beta)$ is a subalgebra of X . Now, let $x, y \in F_{q_{k_F}}(A; \gamma)$. Then $A_F(x) + \gamma + k_F < 1$ and $A_F(y) + \gamma + k_F < 1$. It follows from Lemma 3.1 that

$$\begin{aligned} A_F(x * y) + \gamma + k_F &\leq \bigvee \{A_F(x), A_F(y), \frac{1-k_F}{2}\} + \gamma + k_F \\ &= \bigvee \{A_F(x) + \gamma + k_F, A_F(y) + \gamma + k_F, \frac{1-k_F}{2} + \gamma + k_F\} \\ &< 1. \end{aligned}$$

Thus $x * y \in F_{q_{k_F}}(A; \gamma)$. Therefore $T_{q_{k_T}}(A; \alpha)$, $I_{q_{k_I}}(A; \beta)$ and $F_{q_{k_F}}(A; \gamma)$ are subalgebras of X . □

Corollary 3.1 ([3]). *If $A = (A_T, A_I, A_F)$ is an $(\in, \in \vee q)$ -neutrosophic subalgebra of $X \in \mathcal{B}(X)$, then neutrosophic q -subsets $T_q(A; \alpha)$, $I_q(A; \beta)$ and $F_q(A; \gamma)$ are subalgebras of X for all $\alpha, \beta \in (0.5, 1]$ and $\gamma \in [0, 0.5)$ whenever they are nonempty.*

Proof. It follows from taking $k_T = k_I = k_F = 0$ in Theorem 3.1. □

Definition 3.2 ([4]). *A neutrosophic set $A = (A_T, A_I, A_F)$ in $X \in \mathcal{B}(X)$ is called a $(q_{(k_T, k_I, k_F)}, \in \vee q_{(k_T, k_I, k_F)})$ -neutrosophic subalgebra of X if*

$$\begin{aligned} x \in T_{q_{k_T}}(A; \alpha_x), y \in T_{q_{k_T}}(A; \alpha_y) &\Rightarrow x * y \in T_{\in \vee q_{k_T}}(A; \alpha_x \wedge \alpha_y), \\ x \in I_{q_{k_I}}(A; \beta_x), y \in I_{q_{k_I}}(A; \beta_y) &\Rightarrow x * y \in I_{\in \vee q_{k_I}}(A; \beta_x \wedge \beta_y), \\ x \in F_{q_{k_F}}(A; \gamma_x), y \in F_{q_{k_F}}(A; \gamma_y) &\Rightarrow x * y \in F_{\in \vee q_{k_F}}(A; \gamma_x \vee \gamma_y) \end{aligned} \tag{10}$$

for all $x, y \in X$, $\alpha_x, \alpha_y, \beta_x, \beta_y \in (0, 1]$ and $\gamma_x, \gamma_y \in [0, 1)$.

Theorem 3.2. *If $A = (A_T, A_I, A_F)$ is a $(q_{(k_T, k_I, k_F)}, \in \vee q_{(k_T, k_I, k_F)})$ -neutrosophic subalgebra of $X \in \mathcal{B}(X)$, then the nonempty neutrosophic $\in \vee q_k$ -subsets $T_{\in \vee q_{k_T}}(A; \alpha)$, $I_{\in \vee q_{k_I}}(A; \beta)$ and $F_{\in \vee q_{k_F}}(A; \gamma)$ are subalgebras of X for all $\alpha \in (\frac{1-k_T}{2}, 1]$, $\beta \in (\frac{1-k_I}{2}, 1]$ and $\gamma \in [0, \frac{1-k_F}{2})$.*

Proof. Assume that $T_{\in \vee q_{k_T}}(A; \alpha)$, $I_{\in \vee q_{k_I}}(A; \beta)$ and $F_{\in \vee q_{k_F}}(A; \gamma)$ are nonempty neutrosophic $\in \vee q_k$ -subsets for all $\alpha \in (\frac{1-k_T}{2}, 1]$, $\beta \in (\frac{1-k_I}{2}, 1]$ and $\gamma \in [0, \frac{1-k_F}{2})$. Let $x, y \in I_{\in \vee q_{k_I}}(A; \beta)$. Then

$$A_I(x) \geq \beta \text{ or } A_I(x) + \beta + k_I > 1$$

and

$$A_I(y) \geq \beta \text{ or } A_I(y) + \beta + k_I > 1.$$

If $A_I(x) + \beta + k_I > 1$ and $A_I(y) + \beta + k_I > 1$, then obviously $x * y \in I_{\in \vee q_{k_I}}(A; \beta)$. Assume that $A_I(x) \geq \beta$ and $A_I(y) + \beta + k_I > 1$. Then $A_I(x) + \beta + k_I \geq 2\beta + k_I > 1$. Hence $x * y \in I_{\in \vee q_{k_I}}(A; \beta)$. By the similar way, if $A_I(y) \geq \beta$ and $A_I(x) + \beta + k_I > 1$, then $x * y \in I_{\in \vee q_{k_I}}(A; \beta)$. Suppose that $A_I(x) \geq \beta$ and $A_I(y) \geq \beta$. Then $A_I(x) + \beta + k_I \geq 2\beta + k_I > 1$ and $A_I(y) + \beta + k_I \geq 2\beta + k_I > 1$. It follows that $x * y \in I_{\in \vee q_{k_I}}(A; \beta)$. Hence $I_{\in \vee q_{k_I}}(A; \beta)$ is a subalgebra of X . Similarly, we can verify that $T_{\in \vee q_{k_T}}(A; \alpha)$ is a subalgebra of X . Now, let $x, y \in F_{\in \vee q_{k_F}}(A; \gamma)$. Then

$$x \in F_{\in}(A; \gamma) \text{ or } x \in F_{q_{k_F}}(A; \gamma)$$

and

$$y \in F_{\in}(A; \gamma) \text{ or } y \in F_{q_{k_F}}(A; \gamma).$$

If $x \in F_{q_{k_F}}(A; \gamma)$ and $y \in F_{q_{k_F}}(A; \gamma)$, then clearly $x * y \in F_{\in \vee q_{k_F}}(A; \gamma)$. If $x \in F_{\in}(A; \gamma)$ and $y \in F_{\in}(A; \gamma)$, then $A_F(x) + \gamma + k_F \leq 2\gamma + k_F < 1$ and $A_F(y) + \gamma + k_F \leq 2\gamma + k_F < 1$, that is, $x, y \in F_{q_{k_F}}(A; \gamma)$ which implies that $x * y \in F_{\in \vee q_{k_F}}(A; \gamma)$. Suppose that $x \in F_{\in}(A; \gamma)$ and $y \in F_{q_{k_F}}(A; \gamma)$. Then $A_F(x) + \gamma + k_F \leq 2\gamma + k_F < 1$, i.e., $x \in F_{q_{k_F}}(A; \gamma)$. It follows that $x * y \in F_{\in \vee q_{k_F}}(A; \gamma)$. Similarly, if $x \in F_{q_{k_F}}(A; \gamma)$ and $y \in F_{\in}(A; \gamma)$, then $x * y \in F_{\in \vee q_{k_F}}(A; \gamma)$. Therefore $F_{\in \vee q_{k_F}}(A; \gamma)$ is a subalgebra of X . \square

Corollary 3.2 ([3]). *If $A = (A_T, A_I, A_F)$ is a $(q, \in \vee q)$ -neutrosophic subalgebra of $X \in \mathcal{B}(X)$, then the nonempty neutrosophic $\in \vee q$ -subsets $T_{\in \vee q}(A; \alpha)$, $I_{\in \vee q}(A; \beta)$ and $F_{\in \vee q}(A; \gamma)$ are subalgebras of X for all $\alpha, \beta \in (0.5, 1]$ and $\gamma \in [0, 0.5)$.*

Given a neutrosophic set $A = (A_T, A_I, A_F)$ in a set X , consider the following sets:

$$X_{k_T} := \{x \in X \mid A_T(x) > k_T\},$$

$$X_{k_I} := \{x \in X \mid A_I(x) > k_I\},$$

and

$$X_{k_F} := \{x \in X \mid A_F(x) < k_F\}.$$

Theorem 3.3. *Let $A = (A_T, A_I, A_F)$ be an $(\in, q_{(k_T, k_I, k_F)})$ -neutrosophic subalgebra of $X \in \mathcal{B}(X)$. If $k_T \in [0, \frac{1-A_T(x) \wedge A_T(y)}{2}]$, $k_I \in [0, \frac{1-A_I(x) \wedge A_I(y)}{2}]$ and $k_F \in (\frac{1-A_F(x) \vee A_F(y)}{2}, 1)$, then the sets X_{k_T} , X_{k_I} and X_{k_F} are subalgebras of X .*

Proof. Let $x, y \in X_{k_T}$. Then $A_T(x) > k_T$ and $A_T(y) > k_T$. If $A_T(x * y) \leq k_T$, then

$$A_T(x * y) + \alpha + k_T \leq 2k_T + \alpha \leq 1$$

where $\alpha = A_T(x) \wedge A_T(y)$. Hence $x * y \notin T_{q_{k_T}}(A; \alpha)$, a contradiction since $x \in T_{\in}(A; A_T(x))$ and $y \in T_{\in}(A; A_T(y))$. Thus $A_T(x * y) > k_T$, that is, $x * y \in X_{k_T}$. Similarly, if $x, y \in X_{k_I}$,

then $x * y \in X_{k_I}$. Let $x, y \in X_{k_F}$. Then $A_F(x) < k_F$ and $A_F(y) < k_F$. If $A_F(x * y) \geq k_F$, then

$$A_F(x * y) + \gamma + k_F \geq 2k_F + \gamma \geq 1$$

where $\gamma = A_F(x) \vee A_F(y)$, and so $x * y \notin F_{q_{k_F}}(A; \gamma)$. This is a contradiction, and thus $A_F(x * y) < k_F$, i.e., $x * y \in X_{k_F}$. Therefore X_{k_T} , X_{k_I} and X_{k_F} are subalgebras of X . \square

Corollary 3.3. *Let $A = (A_T, A_I, A_F)$ be an $(\in, q_{(k_T, k_I, k_F)})$ -neutrosophic subalgebra of $X \in \mathcal{B}(X)$. If $k_T \in [0, \frac{1-A_T(x) \wedge A_T(y)}{2}]$, $k_I \in [0, \frac{1-A_I(x) \wedge A_I(y)}{2}]$ and $k_F \in (\frac{1-A_F(x) \vee A_F(y)}{2}, 1)$, then $X_{k_T} \cap X_{k_I} \cap X_{k_F}$ is a subalgebra of X .*

Theorem 3.4. *If $A = (A_T, A_I, A_F)$ is a $(q_{(k_T, k_I, k_F)}, \in)$ -neutrosophic subalgebra of $X \in \mathcal{B}(X)$ with $k_F \in (0, \frac{1}{2}]$, then the sets X_{k_T} , X_{k_I} and X_{k_F} are subalgebras of X .*

Proof. Let $x, y \in X_{k_I}$. Then $A_I(x) > k_I$ and $A_I(y) > k_I$, which imply that $A_I(x) + k_I + 1 > 1$ and $A_I(y) + k_I + 1 > 1$. Hence $x, y \in I_{q_{k_I}}(A; 1)$, and so $x * y \in I_{\in}(A; 1)$. If $x * y \notin X_{k_I}$, then $A_I(x * y) \leq k_I < 1 = 1 \wedge 1$, that is, $x * y \notin I_{\in}(A; 1 \wedge 1) = I_{\in}(A; 1)$. This is a contradiction, and thus $x * y \in X_{k_I}$. By the similar way, we can verify that if $x, y \in X_{k_T}$, then $x * y \in X_{k_T}$. Now, let $x, y \in X_{k_F}$. Then $A_F(x) < k_F$ and $A_F(y) < k_F$. Since $k_F \leq \frac{1}{2}$, it follows that $A_F(x) + k_F + 0 < 1$ and $A_F(y) + k_F + 0 < 1$, that is, $x, y \in F_{q_{k_F}}(A; 0)$. Thus $x * y \in F_{\in}(A; 0)$, and so $A_F(x * y) = 0 < k_F$, i.e., $x * y \in X_{k_F}$. Therefore X_{k_T} , X_{k_I} and X_{k_F} are subalgebras of X . \square

Theorem 3.5. *Given a a neutrosophic set $A = (A_T, A_I, A_F)$ in $X \in \mathcal{B}(X)$, the nonempty neutrosophic \in -subsets $T_{\in}(A; \alpha)$, $I_{\in}(A; \beta)$ and $F_{\in}(A; \gamma)$ are subalgebras of X for all $\alpha \in (\frac{1-k_T}{2}, 1]$, $\beta \in (\frac{1-k_I}{2}, 1]$ and $\gamma \in [0, \frac{1-k_F}{2})$ if and only if the following assertion is valid.*

$$(\forall x, y \in X) \left(\begin{array}{l} A_T(x * y) \vee \frac{1-k_T}{2} \geq A_T(x) \wedge A_T(y) \\ A_I(x * y) \vee \frac{1-k_I}{2} \geq A_I(x) \wedge A_I(y) \\ A_F(x * y) \wedge \frac{1-k_F}{2} \leq A_F(x) \vee A_F(y) \end{array} \right). \tag{11}$$

Proof. Suppose that the nonempty neutrosophic \in -subsets $T_{\in}(A; \alpha)$, $I_{\in}(A; \beta)$ and $F_{\in}(A; \gamma)$ are subalgebras of X for all $\alpha \in (\frac{1-k_T}{2}, 1]$, $\beta \in (\frac{1-k_I}{2}, 1]$ and $\gamma \in [0, \frac{1-k_F}{2})$. If there are $a, b \in X$ such that $A_T(a * b) \vee \frac{1-k_T}{2} < A_T(a) \wedge A_T(b) := \alpha$, then $\alpha \in (\frac{1-k_T}{2}, 1]$ and $a, b \in T_{\in}(A; \alpha)$. It follows that $a * b \in T_{\in}(A; \alpha)$, that is, $A_T(a * b) \geq \alpha$ since $T_{\in}(A; \alpha)$ is a subalgebra of X . This is a contradiction, and so $A_T(x * y) \vee \frac{1-k_T}{2} \geq A_T(x) \wedge A_T(y)$ for all $x, y \in X$. By the similar way, we know that $A_I(x * y) \vee \frac{1-k_I}{2} \geq A_I(x) \wedge A_I(y)$ for all $x, y \in X$. Now, assume that $A_F(a * b) \wedge \frac{1-k_F}{2} > A_F(a) \vee A_F(b)$ for some $a, b \in X$. Then $a, b \in F_{\in}(A; \gamma)$ and $\gamma \in [0, \frac{1-k_F}{2})$ where $\gamma = A_F(a) \vee A_F(b)$. But $a * b \notin F_{\in}(A; \gamma)$, a contradiction. Thus $A_F(x * y) \wedge \frac{1-k_F}{2} \leq A_F(x) \vee A_F(y)$ for all $x, y \in X$.

Conversely, let $A = (A_T, A_I, A_F)$ be a neutrosophic set in $X \in \mathcal{B}(X)$ which satisfies the condition (11). Let $a, b, x, y \in X$, $\alpha \in (\frac{1-k_T}{2}, 1]$ and $\beta \in (\frac{1-k_I}{2}, 1]$ be such that $x, y \in T_{\in}(A; \alpha)$ and $a, b \in I_{\in}(A; \beta)$. Then

$$\begin{aligned} A_T(x * y) \vee \frac{1-k_T}{2} &\geq A_T(x) \wedge A_T(y) \geq \alpha > \frac{1-k_T}{2}, \\ A_I(a * b) \vee \frac{1-k_I}{2} &\geq A_I(a) \wedge A_I(b) \geq \beta > \frac{1-k_I}{2}. \end{aligned}$$

It follows that $A_T(x * y) \geq \alpha$ and $A_I(a * b) \geq \beta$, that is, $x * y \in T_{\in}(A; \alpha)$ and $a * b \in I_{\in}(A; \beta)$. Now, let $x, y \in F_{\in}(A; \gamma)$ for $x, y \in X$ and $\gamma \in [0, \frac{1-k_F}{2})$. Then

$$A_F(x * y) \wedge \frac{1-k_F}{2} \leq A_F(x) \vee A_F(y) \leq \gamma < \frac{1-k_F}{2},$$

and so $A_F(x * y) \leq \gamma$. Hence $x * y \in F_{\in}(A; \gamma)$. Therefore $T_{\in}(A; \alpha)$, $I_{\in}(A; \beta)$ and $F_{\in}(A; \gamma)$ are subalgebras of X for all $\alpha \in (\frac{1-k_T}{2}, 1]$, $\beta \in (\frac{1-k_I}{2}, 1]$ and $\gamma \in [0, \frac{1-k_F}{2}]$. \square

Theorem 3.6. *Given a neutrosophic set $A = (A_T, A_I, A_F)$ in $X \in \mathcal{B}(X)$, if the nonempty neutrosophic q_k -subsets $T_{q_{k_T}}(A; \alpha)$, $I_{q_{k_I}}(A; \beta)$ and $F_{q_{k_F}}(A; \gamma)$ are subalgebras of X for all $\alpha \in (0, \frac{1-k_T}{2}]$, $\beta \in (0, \frac{1-k_I}{2}]$ and $\gamma \in [\frac{1-k_F}{2}, 1)$, then the following assertion is valid.*

$$(\forall x, y \in X) \left(\begin{array}{l} x \in T_{q_{k_T}}(A; \alpha_x), y \in T_{q_{k_T}}(A; \alpha_y) \Rightarrow x * y \in T_{\in}(A; \alpha_x \vee \alpha_y) \\ x \in I_{q_{k_I}}(A; \beta_x), y \in I_{q_{k_I}}(A; \beta_y) \Rightarrow x * y \in I_{\in}(A; \beta_x \vee \beta_y) \\ x \in F_{q_{k_F}}(A; \gamma_x), y \in F_{q_{k_F}}(A; \gamma_y) \Rightarrow x * y \in F_{\in}(A; \gamma_x \wedge \gamma_y) \end{array} \right). \quad (12)$$

for all $x, y \in X$, $\alpha_x, \alpha_y \in (0, \frac{1-k_T}{2}]$, $\beta_x, \beta_y \in (0, \frac{1-k_I}{2}]$ and $\gamma_x, \gamma_y \in [\frac{1-k_F}{2}, 1)$.

Proof. Let $x, y \in X$ and $\alpha_x, \alpha_y \in (0, \frac{1-k_T}{2}]$ be such that $x \in T_{q_{k_T}}(A; \alpha_x)$ and $y \in T_{q_{k_T}}(A; \alpha_y)$. Then $x, y \in T_{q_{k_T}}(A; \alpha_x \vee \alpha_y)$. Since $\alpha_x \vee \alpha_y \in (0, \frac{1-k_T}{2}]$, it follows from the hypothesis that $x * y \in T_{q_{k_T}}(A; \alpha_x \vee \alpha_y)$. Hence

$$A_T(x * y) > 1 - (\alpha_x \vee \alpha_y) - k_T \geq \alpha_x \vee \alpha_y,$$

and so $x * y \in T_{\in}(A; \alpha_x \vee \alpha_y)$. Similarly, we can verify that if $x \in I_{q_{k_I}}(A; \beta_x)$ and $y \in I_{q_{k_I}}(A; \beta_y)$, then $x * y \in I_{\in}(A; \beta_x \vee \beta_y)$. Now, let $x, y \in X$ and $\gamma_x, \gamma_y \in [\frac{1-k_F}{2}, 1)$ be such that $x \in F_{q_{k_F}}(A; \gamma_x)$ and $y \in F_{q_{k_F}}(A; \gamma_y)$. Then $x, y \in F_{q_{k_F}}(A; \gamma_x \wedge \gamma_y)$ since $\gamma_x \wedge \gamma_y \in [\frac{1-k_F}{2}, 1)$, which implies from hypothesis that $x * y \in F_{q_{k_F}}(A; \gamma_x \wedge \gamma_y)$. Thus

$$A_F(x * y) < 1 - (\gamma_x \wedge \gamma_y) - k_F \leq \frac{1-k_F}{2} \leq \gamma_x \wedge \gamma_y$$

and hence $x * y \in F_{\in}(A; \gamma_x \wedge \gamma_y)$. \square

Corollary 3.4 ([1]). *Given a neutrosophic set $A = (A_T, A_I, A_F)$ in $X \in \mathcal{B}(X)$, if the nonempty neutrosophic q -subsets $T_q(A; \alpha)$, $I_q(A; \beta)$ and $F_q(A; \gamma)$ are subalgebras of X for all $\alpha, \beta \in (0, 0.5]$ and $\gamma \in [0.5, 1)$, then the following assertion is valid.*

$$(\forall x, y \in X) \left(\begin{array}{l} x \in T_q(A; \alpha_x), y \in T_q(A; \alpha_y) \Rightarrow x * y \in T_{\in}(A; \alpha_x \vee \alpha_y) \\ x \in I_q(A; \beta_x), y \in I_q(A; \beta_y) \Rightarrow x * y \in I_{\in}(A; \beta_x \vee \beta_y) \\ x \in F_q(A; \gamma_x), y \in F_q(A; \gamma_y) \Rightarrow x * y \in F_{\in}(A; \gamma_x \wedge \gamma_y) \end{array} \right).$$

for all $x, y \in X$, $\alpha_x, \alpha_y \in (0, 0.5]$ and $\gamma_x, \gamma_y \in [0.5, 1)$.

Theorem 3.7. *Given a neutrosophic set $A = (A_T, A_I, A_F)$ in $X \in \mathcal{B}(X)$, if the nonempty neutrosophic $\in \vee q_k$ -subsets $T_{\in \vee q_{k_T}}(A; \alpha)$, $I_{\in \vee q_{k_I}}(A; \beta)$ and $F_{\in \vee q_{k_F}}(A; \gamma)$ are subalgebras of X for all $\alpha \in (0, \frac{1-k_T}{2}]$, $\beta \in (0, \frac{1-k_I}{2}]$ and $\gamma \in [\frac{1-k_F}{2}, 1)$, then the following assertion is valid.*

$$(\forall x, y \in X) \left(\begin{array}{l} x \in T_{q_{k_T}}(A; \alpha_x), y \in T_{q_{k_T}}(A; \alpha_y) \Rightarrow x * y \in T_{\in \vee q_{k_T}}(A; \alpha_x \vee \alpha_y) \\ x \in I_{q_{k_I}}(A; \beta_x), y \in I_{q_{k_I}}(A; \beta_y) \Rightarrow x * y \in I_{\in \vee q_{k_I}}(A; \beta_x \vee \beta_y) \\ x \in F_{q_{k_F}}(A; \gamma_x), y \in F_{q_{k_F}}(A; \gamma_y) \Rightarrow x * y \in F_{\in \vee q_{k_F}}(A; \gamma_x \wedge \gamma_y) \end{array} \right). \quad (13)$$

for all $x, y \in X$, $\alpha_x, \alpha_y \in (0, \frac{1-k_T}{2}]$, $\beta_x, \beta_y \in (0, \frac{1-k_I}{2}]$ and $\gamma_x, \gamma_y \in [\frac{1-k_F}{2}, 1)$.

Proof. Assume that $x \in I_{q_{k_I}}(A; \beta_x)$ and $y \in I_{q_{k_I}}(A; \beta_y)$ for $x, y \in X$ and $\beta_x, \beta_y \in (0, \frac{1-k_I}{2}]$. Then $x, y \in I_{\in \vee q_{k_I}}(A; \beta_x \vee \beta_y)$ where $\beta_x \vee \beta_y \in (0, \frac{1-k_I}{2}]$. It follows from the assumption that $x * y \in I_{\in \vee q_{k_I}}(A; \beta_x \vee \beta_y)$. By the similar way, we know that if $x \in T_{q_{k_T}}(A; \alpha_x)$ and $y \in T_{q_{k_T}}(A; \alpha_y)$, then $x * y \in T_{\in \vee q_{k_T}}(A; \alpha_x \vee \alpha_y)$. Let $x, y \in X$ and $\gamma_x, \gamma_y \in [\frac{1-k_F}{2}, 1)$

be such that $x \in F_{q_{k_F}}(A; \gamma_x)$ and $y \in F_{q_{k_F}}(A; \gamma_y)$. Then $x, y \in F_{\in \vee q_{k_F}}(A; \gamma_x \wedge \gamma_y)$ with $\gamma_x \wedge \gamma_y \in [\frac{1-k_F}{2}, 1)$. Since $F_{\in \vee q_{k_F}}(A; \gamma_x \wedge \gamma_y)$ is a subalgebra of X by hypothesis, we have $x * y \in F_{\in \vee q_{k_F}}(A; \gamma_x \wedge \gamma_y)$. \square

Corollary 3.5 ([1]). *Given a neutrosophic set $A = (A_T, A_I, A_F)$ in $X \in \mathcal{B}(X)$, if the nonempty neutrosophic $\in \vee q$ -subsets $T_{\in \vee q}(A; \alpha)$, $I_{\in \vee q}(A; \beta)$ and $F_{\in \vee q}(A; \gamma)$ are subalgebras of X for all $\alpha, \beta \in (0, 0.5]$ and $\gamma \in [0.5, 1)$, then the following assertion is valid.*

$$(\forall x, y \in X) \left(\begin{array}{l} x \in T_q(A; \alpha_x), y \in T_q(A; \alpha_y) \Rightarrow x * y \in T_{\in \vee q}(A; \alpha_x \vee \alpha_y) \\ x \in I_q(A; \beta_x), y \in I_q(A; \beta_y) \Rightarrow x * y \in I_{\in \vee q}(A; \beta_x \vee \beta_y) \\ x \in F_q(A; \gamma_x), y \in F_q(A; \gamma_y) \Rightarrow x * y \in F_{\in \vee q}(A; \gamma_x \wedge \gamma_y) \end{array} \right).$$

for all $x, y \in X$, $\alpha_x, \alpha_y, \beta_x, \beta_y \in (0, 0.5]$ and $\gamma_x, \gamma_y \in [0.5, 1)$.

Theorem 3.8. *Given a neutrosophic set $A = (A_T, A_I, A_F)$ in $X \in \mathcal{B}(X)$, if the nonempty neutrosophic $\in \vee q_{k_T}$ -subsets $T_{\in \vee q_{k_T}}(A; \alpha)$, $I_{\in \vee q_{k_I}}(A; \beta)$ and $F_{\in \vee q_{k_F}}(A; \gamma)$ are subalgebras of X for all $\alpha \in (\frac{1-k_T}{2}, 1]$, $\beta \in (\frac{1-k_I}{2}, 1]$ and $\gamma \in [0, \frac{1-k_F}{2})$, then $A = (A_T, A_I, A_F)$ satisfies (13) for all $\alpha_x, \alpha_y \in (\frac{1-k_T}{2}, 1]$, $\beta_x, \beta_y \in (\frac{1-k_I}{2}, 1]$ and $\gamma_x, \gamma_y \in [0, \frac{1-k_F}{2})$.*

Proof. It is similar to the proof of Theorem 3.7. \square

Corollary 3.6 ([1]). *Given a neutrosophic set $A = (A_T, A_I, A_F)$ in $X \in \mathcal{B}(X)$, if the nonempty neutrosophic $\in \vee q$ -subsets $T_{\in \vee q}(A; \alpha)$, $I_{\in \vee q}(A; \beta)$ and $F_{\in \vee q}(A; \gamma)$ are subalgebras of X for all $\alpha, \beta \in (0.5, 1]$ and $\gamma \in [0, 0.5)$, then the following assertion is valid.*

$$(\forall x, y \in X) \left(\begin{array}{l} x \in T_q(A; \alpha_x), y \in T_q(A; \alpha_y) \Rightarrow x * y \in T_{\in \vee q}(A; \alpha_x \vee \alpha_y) \\ x \in I_q(A; \beta_x), y \in I_q(A; \beta_y) \Rightarrow x * y \in I_{\in \vee q}(A; \beta_x \vee \beta_y) \\ x \in F_q(A; \gamma_x), y \in F_q(A; \gamma_y) \Rightarrow x * y \in F_{\in \vee q}(A; \gamma_x \wedge \gamma_y) \end{array} \right).$$

for all $x, y \in X$, $\alpha_x, \alpha_y, \beta_x, \beta_y \in (0.5, 1]$ and $\gamma_x, \gamma_y \in [0, 0.5)$.

4. CONCLUSION

In this paper, we investigate further properties of $(\in, \in \vee q_{(k_T, k_I, k_F)})$ -neutrosophic subalgebra, $(q_{(k_T, k_I, k_F)}, \in \vee q_{(k_T, k_I, k_F)})$ -neutrosophic subalgebra, $(\in, q_{(k_T, k_I, k_F)})$ -neutrosophic subalgebra and $(q_{(k_T, k_I, k_F)}, \in)$ -neutrosophic subalgebra in BCK/BCI -algebras. We hope that this work will provide a deep impact on the upcoming research in this field and other related areas to open up new horizons of interest and innovations. Indeed, this work may serve as a foundation for further study of neutrosophic subalgebras in BCK/BCI -algebras. To extend these results, one can further study the neutrosophic set theory of different algebras such as MTL-algebras, BL-algebras, MV-algebras, EQ-algebras, R0-algebras and Q-algebras etc. One may also apply this concept to study some applications in many fields like decision making, knowledge base systems, medical diagnosis, data analysis and graph theory etc.

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G. Muhiuddin is an Associate Professor at the Department of Mathematics, University of Tabuk, Saudi Arabia. He received his Ph.D. and M. Phil. in Pure Mathematics. His mathematical research areas include Algebras related to logic (*BCK*, *BCI*, *BCC*-algebras, Hilbert algebras, and implication algebras), soft algebraic structures, Algebraic Hyperstructures, Theory of derivations in algebraic structures and Category theory. He serves as an expert referee for several research papers in these areas of his expertise. He also serves as reviewer for Mathematics Reviews of the American Math Society, USA (Mathematical Reviews/ MathSciNet Reviewer Number: 105900). He has published a number of research papers in internationally reputed mathematical journals.



Young Bae Junis is an Emeritus Professor at Department of Mathematics Education, Gyeongsang National University. He received his Ph. D. Degree at Kyung Hee University in pure Mathematics. He has published a book, ***BCK*-algebras**, with Professor J. Meng, and published many research papers. His mathematical research areas are *BCK/BCI*-algebra, fuzzy algebraic structure and soft (rough) algebraic structure, and he is reviewing many papers in this areas. His Awards and Honors are: Academic Achievement Award (7 July 2006), Busan-Kyeongnam Branch of the Korean Mathematical Society, and listed in 8th edition of "Marquis Who's Who in Science and Engineering".