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Research article

Interval neutrosophic covering rough sets based on neighborhoods

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Abstract: Covering rough set is a classical generalization of rough set. As covering rough set is a mathematical tool to deal with incomplete and incomplete data, it has been widely used in various fields. The aim of this paper is to extend the covering rough sets to interval neutrosophic sets, which can make multi-attribute decision making problem more tractable. Interval neutrosophic covering rough sets can be viewed as the bridge connecting Interval neutrosophic sets and covering rough sets. Firstly, the paper introduces the definition of interval neutrosophic sets and covering rough sets, where the covering rough set is defined by neighborhood. Secondly, Some basic properties and operation rules of interval neutrosophic sets and covering rough sets are discussed. Thirdly, the definition of interval neutrosophic covering rough sets are proposed. Then, some theorems are put forward and their proofs of interval neutrosophic covering rough sets also be gived. Lastly, this paper gives a numerical example to apply the interval neutrosophic covering rough sets.

Keywords: neutrosophic sets; interval neutrosophic sets; rough sets; covering rough sets; neighborhood
Mathematics Subject Classification: 03-06, 60L70, 68N17

1. Introduction

Rough set theory was initially developed by Pawlak [1] as a new mathematical methodology to deal with the vagueness and uncertainty in information systems. Covering rough set (CRS) theory is a generalization of traditional rough set theory, which is characterized by coverings instead of partitions. Degang Chen et al. [2] proposed belief and plausibility functions to characterize neighborhood-covering rough sets. Essentially, they developed a numerical method for finding reductions using belief functions. Liwen Ma [3] defined the complementary neighborhood of an arbitrary element in the universe and discussed its properties. Based on the concepts of neighborhood and complementary neighborhood, an equivalent definition of a class of CRS is defined or given. Bin Yang and Bao Qing Hu [4] introduced some new definitions of fuzzy-covering approximation spaces and studied the properties of fuzzy-covering approximation spaces and Mas fuzzy covering-based rough set models. On this basis, they proposed three rough set models based on fuzzy coverage as the generalization of Ma model. Yan-Lan Zhang and Mao-Kang Luo [5] studied the relation between relation-based rough sets and covering-based rough sets. In a rough set framework based on relation, they unified five kinds of covering-based rough sets. The equivalence relations of covering-based rough sets and the type of relation-based rough sets were established. Lynn Deer et al. [6] studied 24 such neighborhood operators, which can be derived from a single covering. They also verified the equality between them, reducing the original set to 13 different neighborhood operators. For the latter, they established a partial order showing which operators produce smaller or larger neighborhoods than the others. Li Zhang et al. [7, 8] combined the extended rough set theory with the mature MADM problem solving methods and proposed several types of covering-based general multigranulation intuitionistic fuzzy rough set models by using four types of intuitionistic fuzzy neighborhoods. Sang-Eon Han [9, 10] set a starting point for establishing a CRS for an LFC-Space and developed the notions of accuracy of rough set approximations. Further, he gave two kinds of rough membership functions and two new rough concepts of digital topological rough set . Qingyuan Xu et al. [11] proposed a rough set method to deal with a class of set covering problem, called unicost set covering problem, which is a well-known problem in binary optimization. Liwen Ma [12] considered some types of neighborhood-related covering rough sets by introducing a new notion of complementary neighborhood. Smarandache [13] proposed the concept of neutrosophic sets in 1999, pointing out that neutrosophic sets is a set composed of the truth-membership, indeterminacy-membership and falsity-membership. Compared with previous models, it can better describe the support, neutrality and opposition of fuzzy concepts. Because of the complexity of practical problems in real life, Wang et al. [14] proposed interval neutrosophic sets(INS) and proved various properties of interval neutrosophic sets, which are connected to operations and relations over interval neutrosophic sets. Nguyen Tho Thong et al. [15] presented a new concept called dynamic interval-valued neutrosophic sets for such the dynamic decision-making applications. Irfan Deli [16] defined the notion of the interval valued neutrosophic soft sets, which is a combination of an interval valued neutrosophic sets and a soft sets. And introduced some definition and properties of interval valued neutrosophic soft sets. Hua Ma et al. [17, 18] utilized the INS theory to propose a time-aware trustworthiness ranking prediction approach to selecting the highly trustworthy cloud service meeting the user-specific requirements and a time-aware trustworthy service selection approach with tradeoffs between performance costs and potential risks because of the deficiency of the traditional value prediction approaches. Ye jun [19] defined the Hamming and Euclidean distances between INS and proposed the similarity measures between INS based on the relationship between similarity measures and distances. Hongyu Zhang et al. [20] Defined the operations for INS and put forward a comparison approach based on the related research of interval valued intuitionistic fuzzy sets. Wei Yang et al. [21] developed a new multiple attribute decision-making method based on the INS and linear assignment. Meanwhile he considered the correlation of information by using the Choquet integral. Peide Liu and Guolin Tang [22] combined power average and generalized weighted aggregation operators to INS, and proposed some aggregation operators to apply in decision making problem.

In recent years, many scholars have studied the combined application of rough sets and neutrosophic sets. In order to make a comprehensive overview for neutrosophic fusion of rough set theory Xue Zhan-Ao et al. [23] defifined a new covering rough intuitionistic fuzzy set model in covering approximation space, which is combined by CRS and intuitionistic fuzzy sets. They discussed the properties of lower and upper approximation operators and extended covering rough intuitionistic fuzzy set in rough sets from single-granulation to multi-granulation. Hai-Long Yang et al. [24] proposed single valued neutrosophic rough sets by combining single valued neutrosophic sets and rough sets. They also studied the hybrid model by constructive and axiomatic approaches. Hai-Long Yang et al. [25] combined INS with rough sets and proposed a generalized interval neutrosophic rough sets based on interval neutrosophic relation.They explored the hybrid model through the construction method and the axiomatic method. At the same time, the generalized interval neutrosophic approximation lower and upper approximation operators were defined by the construction method. In this paper we will study the interval neutrosophic covering rough set (INCRS), which is combined by the CRS and INS, and discuss the properties of it. Further we will give the complete proof of them. In order to do so, the remainder of this paper is shown as follows. In Section 2, we briefly review the basic concepts and operational rules of INS and CRS. In Section 3, we propose the definition and the properties of INCRS and give some easy cases to describe it. In Section 4, we discuss some theorems for INCRS and prove them completely. In Section 5, we give a simple application of Interval Neutrosophic Covering Rough Sets. In Section 6, we conclude the paper.

2. Preliminaries

This section gives a brief overview of concepts and definitions of interval neutrosophic sets, and covering rough sets.

2.1. Interval neutrosophic sets

Definition 2.1. [13] Let X be a space of points (objects), with a class of elements in X denoted by x. A neutrosophic set A in X is summarized by a truth-membership function $T_{A(x)}$, an indeterminacy-membership function $I_{A(x)}$, and a falsity-membership function $F_{A(x)}$.The functions $T_{A(x)}$, $I_{A(x)}$, $F_{A(x)}$ are real standard or non-standard subsets of $]0^-, 1^+[$. That is $T_A(x) : X \rightarrow]0^-, 1^+ [I_A(x) : X \rightarrow]\, 0^-, 1^+$ and $F_A(x) : X \rightarrow]0^-, 1^+[$.

There is restriction on the sum of $T_A(x), I_A(x)$ and $F_A(x)$, so $0^- \leq \sup T_A(x) + \sup I_A(x) + \sup F_A(x) \leq 3^+$. As mentioned above, it is hard to apply the neutrosophic set to solve some real problems. Hence, Wang et al presented interval neutrosophic set, which is a subclass of the neutrosophic set and mentioned the definition as follows:

Definition 2.2. [13] Let X be a space of points (objects), with a class of elements in X denoted by x. A single-valued neutrosophic set N in X is summarized by a truth-membership function $T_{N(x)}$, an indeterminacy-membership function $I_{N(x)}$, and a falsity-membership function $F_{N(x)}$. Then an INS A can be denoted as follows:

$$A = \{\langle x, T_A(x), I_A(x), F_A(x)\rangle\; x \in X\} \tag{2.1}$$

where $T_A(x) = \left[T_A^L(x), T_A^U(x)\right]$, $I_A(x) = \left[I_A^L(x), I_A^U(x)\right]$, $F_A(x) = \left[F_A^L(x), F_A^U(x)\right] \subseteq [0, 1]$ for $\forall x \in X$. Meanwhile, the sum of $T_A(x)I_A(x)$, and $F_A(x)$ fulfills the condition $0 \leq T_A(x) + I_A(x) + F_A(x) \leq 3$.

For convenience, we refer to $A = \langle T_A, I_A, F_A\rangle = \left\langle \left[T_A^L, T_A^U\right], \left[I_A^L, I_A^U\right], \left[F_A^L, F_A^U\right]\right\rangle$ as an interval neutrosophic number (INN), which is a basic unit of INS. In addition, let $X = \langle[1, 1], [0, 0], [0, 0]\rangle$ be the biggest interval neutrosophic number, and $\emptyset = \langle[0,0],[1,1],[1,1]\rangle$ be the smallest interval neutrosophic number.

Definition 2.3. [13] The complement of an INS $A = \langle T_A, I_A, F_A\rangle = \left\langle\left[T_A^L, T_A^U\right], \left[I_A^L, I_A^U\right], \left[F_A^L, F_A^U\right]\right\rangle$ is denoted by A^C and which is defined as $A^C = \left\langle\left[F_A^L, F_A^U\right], \left[1 - I_A^U, 1 - I_A^L\right], \left[T_A^L, T_A^U\right]\right\rangle$. For any $x, y \in X$, an INS 1_y and its complement $1_{X-\{y\}}$ are defined as follows:

$$T_{1_y}(x) = \begin{cases} [1,1], x = y \\ [0,0], x \neq y \end{cases}, I_{1_y}(x) = F_{1_y}(x) = \begin{cases} [0,0], x = y \\ [1,1], x \neq y \end{cases}$$

$$T_{1_{x-(y)}}(x) = \begin{cases} [0,0], x = y \\ [1,1], x \neq y \end{cases}, I_{1x_{-1}y}(x) = F_{1_{x-(y)}}(x) = \begin{cases} [1,1], x = y \\ [0,0], x \neq y \end{cases}$$

Definition 2.4. [16] $A = \{\langle x, T_A(x), I_A(x), F_A(x)\rangle\}$ and $B = \{\langle x, T_B(x), I_B(x), F_B(x)\rangle\}$ are two interval neutrosophic sets, where $T_A(x) = \left[T_A^L(x), T_A^U(x)\right]$, $I_A(x) = \left[I_A^L(x), I_A^U(x)\right]$, $F_A(x) = \left[F_A^L(x), F_A^U(x)\right]$, and $T_B(x) = \left[T_B^L(x), T_B^U(x)\right]$, $I_B(x) = \left[I_B^L(x), I_B^U(x)\right]$, $F_B(x) = \left[F_B^L(x), F_B^U(x)\right]$, then

$$\begin{aligned} A \subseteq B &\Leftrightarrow T_A(x) \leq T_B(x), I_A(x) \geq I_B(x), F_A(x) \geq F_B(x) \\ A \supseteq B &\Leftrightarrow T_A(x) \geq T_B(x), I_A(x) \leq I_B(x), F_A(x) \leq F_B(x) \\ A = B &\Leftrightarrow T_A(x) = T_B(x), I_A(x) = I_B(x), F_A(x) = F_B(x) \end{aligned}$$

And it satisfies that:

$$\begin{aligned} T_A(x) \leq T_B(x) &\Leftrightarrow T_A^L(x) \leq T_B^L(x), T_A^U(x) \leq T_B^U(x) \\ T_A(x) \geq T_B(x) &\Leftrightarrow T_A^L(x) \geq T_B^L(x), T_A^U(x) \geq T_B^U(x) \\ T_A(x) = T_B(x) &\Leftrightarrow T_A^L(x) = T_B^L(x), T_A^U(x) = T_B^U(x) \end{aligned}$$

If A and B do not satisfy the above relationship, then they are said to be incompatible.

Definition 2.5. A and B are two INNs , we have the following basic properties of INNs.

$$\begin{aligned} &(1)\ A \subseteq A \cup B, B \subseteq A \cup B \\ &(2)\ A \cap B \subseteq A, A \cap B \subseteq B \\ &(3)\ (A \cup B)^c = A^C \cap B^C; \\ &(4)\ \left(A^C\right)^C = A \end{aligned}$$

2.2. *Covering rough sets*

Definition 2.6. [25] Let X be a finite set space of points (objects), and R be an equivalence relation on X. Denote by X/R the family of all equivalence classes induced by R. Obviously X/R gives a partition of X. (X,R) is called an interval neutrosophic approximation space. For $x \in X$, the lower and upper approximations of A are defined as below:

$$R^-(A) = \{x \in X | [x]_R \subseteq A\}, R^+(A) = \left\{x \in X | [x]_R \bigcap A \neq \emptyset\right\},$$

where

$$[x]_R = \{y \in X | (x,y) \in R\}. \text{ It follows that } R^-(A) \subseteq A \subseteq R^+(A)$$

If $R^-(A) \neq R^+(A)$, A is called a rough set.

Definition 2.7. [3] Let X be a space of points (objects) and $C = \{C_1, C_2, \cdots, C_m\}$ be a family of subsets of X. If none of the elements in C is empty and $\bigcup_{i=1}^{m} C_i = X$, then C is called a covering of X, and (X, C) is called a covering approximation space.
Definition 2.8. [3] Let (X, C) be a covering approximation space. For any $x \in X$, the neighborhood of x is defined as $\bigcap_{i=1}^{m} \{C_i \in C | x \in C_i\}$, which is denoted by N_x.
Definition 2.9. [24] Let (X, C) be a covering approximation space. For any $x \in X$, the lower and upper approximations of A are defined as below:

$$C^-(A) = \{x \in X | N_x \subseteq A\}, C^+(A) = \{x \in X | N_x \cap A \neq \emptyset\}$$

Based on the definition of neighborhood, the new covering rough models can be obtained.

3. The notion of interval neutrosophic covering rough sets

We will give the definition of interval neutrosophic covering rough sets in this section, meanwhile we'll also use some examples for the sake of intuition. In addition, we will given some properties and their proofs of INCRS.
Definition 3.1. Let X be a space of points (objects). For any $[s, t] \in [0, 1]$ and $C = \{C_1, C_2, \cdots, C_m\}$, where $C_i = \{T_{c,i} I_{c_i}, F_{c_i}\}$ and $C_i \in INS(i = 1, 2, \cdots, m)$. For $\forall x \in X, \exists C_k \in C$, then $C_k(x) \geq [s, t]$, where $T_{C_k}(x) \geq [s, t], I_{C_k}(x) \leq [1 - t, 1 - s]$,$F_{C_k}(x) \leq [1 - t, 1 - s]$. Then C is called a interval neutrosophic $[s, t]$ covering of X.
Definition 3.2. Let $C = \{C_1, C_2, \cdots, C_m\}$ be an interval neutrosophic $[s, t]$ covering of X. If $0 \leq [s', t'] \leq [s, t]$, C is an interval neutrosophic $[s', t']$covering of X.
Proof. $C = \{C_1, C_2, \cdots, C_m\}$ is a interval neutrosophic $[s, t]$ covering of X. Thus$C_k(x) \geq [s, t]$, and satisfy $T_{C_k}(x) \geq [s, t], I_{C_k}(x) \leq [1 - t, 1 - s]$,$F_{C_k}(x) \leq [1 - t, 1 - s]$. when $0 \leq [s', t'] \leq [s, t]$, we can get $0 \leq [s', t'] \leq [s, t] \leq T_{C_k}(x)$ and $0 \leq I_{C_k}(x) \leq [1 - s, 1 - t] \leq [1 - s', 1 - t'], 0 \leq F_{C_k}(x) \leq [1 - s, 1 - t] \leq [1 - s', 1 - t']$. So C is a interval neutrosophic left$[s', t']$ covering of X.
Definition 3.3. [26] Suppose $C = \{C_1, C_2, \cdots, C_m\}$ is an interval neutrosophic $[s, t]$ covering of X. If$s = t = \beta$, then C is called a interval neutrosophic β covering of X.
Definition 3.4. Suppose $C = \{C_1, C_2, \cdots, C_m\}$ is an interval neutrosophic $[s, t]$ covering of X, where $C_i = \{T_{c,i} I_{c_i}, F_{c_i}\}$ and $C_i \in INS(i = 1, 2, \cdots, m)$. For $\forall x \in X$, the interval neutrosophic $[s, t]$ neighborhood of x is defined as follows:

$$N_x^{[s,t]}(y) = \bigcap \{C_i \in C | T_{C_i}(x) \geq [s, t], I_{C_i}(x) \leq [1 - t, 1 - s], F_{C_i}(x) \leq [1 - t, 1 - s]\}.$$

Definition 3.5. [26] Let $C = \{C_1, C_2, \cdots, C_m\}$ be an interval neutrosophic $[s, t]$ covering of X, where $C_i = \{T_{c,i} I_{c_i}, F_{c_i}\}$ and $C_i \in INS(i = 1, 2, \cdots, m)$. If $s = t = \beta$, then the interval neutrosophic $[s, t]$ neighborhood of x is degraded as the interval neutrosophic β neighborhood of x.
Theorem 3.6. Let $C = \{C_1, C_2, \cdots, C_m\}$ be an interval neutrosophic $[s, t]$ covering of X, where $C_i = \{T_{c,i} I_{c_i}, F_{c_i}\}$ and $C_i \in INS(i = 1, 2, \cdots, m)$. $\forall x, y, z \in X$, some propositions are shown as follows:
(1) $N_x^{[s,t]}(x) \geq [s, t]$;
(2) if $N_x^{[s,t]}(y) \geq [s, t]$ and $N_y^{[s,t]}(z) \geq [s, t]$, then $N_x^{[s,t]}(z) \geq [s, t]$;
(3) $C_i \supseteq \bigcup_{x \in X} \left\{N_x^{[s,t]} | C_i(x) \geq [s, t]\right\}, i \in \{1, 2, \cdots, m\}$;
(4) if $[s_1, t_1] \leq [s_2, t_2] \leq [s, t]$, then $N_x^{[s_1,t_1]} \subseteq N_x^{[s_2,t_2]}$.

Proof. (1)

$$N_x^{[s,t]}(x) = \left(\bigcap_{T_{C_i}(x)\geq[s,t], I_{C_i}(x)\leq[1-t,1-s], F_{C_i}(x)\leq[1-t,1-s]} \right)(x) = \left(\bigcap_{C_i(x)\geq[s,t]} C_i \right)(x)$$
$$= \bigwedge_{C_i(x)\geq[s,t]} C_i(x) \geq [s,t].$$

(2)

If $N_x^{[s,t]}(y) \geq [s,t]$, then $N_x^{[s,t]}(y) = (\bigcap_{T_{C_i}(x)\geq[s,t], I_{C_i}(x)\leq[1-t,1-s], F_{C_i}(x)\leq[1-t,1-s]} C_i)(y) = (\bigcap_{C_i(x)\geq[s,t]} C_i)(y)$ $= \wedge_{C_i(x)\geq[s,t]} C_i(y) \geq [s,t]$, thus $C_i(x) \geq [s,t] \Rightarrow C_i(y) \geq [s,t]$,similarly, it can be obtained that $C_i(y) \geq [s,t] \Rightarrow C_i(z) \geq [s,t]$. So $C_i(x) \geq [s,t] \Rightarrow C_i(z) \geq [s,t]$, thus $N_x^{[s,t]}(z) = (\bigcap_{T_{C_i}(x)\geq[s,t], I_{C_i}(x)\leq[1-t,1-s], F_{C_i}(x)\leq[1-t,1-s]} C_i)(z) = (\bigcap_{C_i(x)\geq[s,t]} C_i)(z) = \wedge_{C_i(x)\geq[s,t]} C_i(z) \geq [s,t]$

(3)

$N_x^{[s,t]} = \bigcap\{C_i \in C | T_{C_i}(x) \geq [s,t], I_{C_i}(x) \leq [1-t, 1-s], F_{C_i}(x) \leq [1-t, 1-s]\} = \left(\bigcap_{C_i(x)\geq[s,t]} C_i\right) \subseteq C_i$, hence for any $x \in X$, it can be obtained that $C_i \supseteq \bigcup_{x\in X}\{N_x^{[s,t]}(x) | C_i(x) \geq [s,t]\}, (i = 1, 2, \cdots, m)$

(4)

$\{C_i \in C | T_{C_i}(x) \geq [s_1, t_1], I_{C_i}(x) \leq [1-t_1, 1-s_1], F_{C_i}(x) \leq [1-t_1, 1-s_1]\} = \{C_i \in C | C_i(x) \geq [s_1, t_1]\}$. When $[s_1, t_1] \leq [s_2, t_2]$, it is obvious that $\{C_i \in C | C_i(x) \geq [s_1, t_1]\} \subseteq \{C_i \in C | C_i(x) \geq [s_2, t_2]\}$, then $\bigcap\{C_i \in C | C_i(x) \geq [s_1, t_1]\} \subseteq \bigcap\{C_i \in C | C_i(x) \geq [s_2, t_2]\}$,,that is $N_x^{[s_1,t_1]} \geq N_x^{[s_2,t_2]}$.

Example 1. Let X be a space of a points(objects), with a class of elements in X denoted by x, $C = \{C_1, C_2, C_3, C_4\}$ is a interval neutrosophic covering of X, which is shown in Table 1. Set $[s,t] = [0.4, 0.5]$, and it can be gotten that C is a interval neutrosophic $[0.4, 0.5]$ covering of X.

Table 1. The interval neutrosophic $[0.4, 0.5]$ covering of X.

	C_1	C_2	C_3	C_4
x_1	⟨[0.4, 0.5], [0.2, 0.3], [0.3, 0.4]⟩	⟨[0.4, 0.6], [0.1, 0.3], [0.2, 0.4]⟩	⟨[0.7, 0.9], [0.2, 0.3], [0.4, 0.5]⟩	⟨[0.4, 0.5], [0.3, 0.4], [0.5, 0.7]⟩
x_2	⟨[0.6, 0.7], [0.1, 0.2], [0.2, 0.3]⟩	⟨[0.6, 0.7], [0.1, 0.2], [0.2, 0.3]⟩	⟨[0.3, 0.6], [0.2, 0.3], [0.3, 0.4]⟩	⟨[0.5, 0.7], [0.2, 0.3], [0.1, 0.3]⟩
x_3	⟨[0.3, 0.6], [0.3, 0.5], [0.8, 0.9]⟩	⟨[0.5, 0.6], [0.2, 0.3], [0.3, 0.4]⟩	⟨[0.4, 0.5], [0.2, 0.4], [0.7, 0.9]⟩	⟨[0.3, 0.5], [0.0, 0.2], [0.2, 0.4]⟩
x_4	⟨[0.7, 0.8], [0.0, 0.1], [0.1, 0.2]⟩	⟨[0.6, 0.7], [0.1, 0.2], [0.1, 0.3]⟩	⟨[0.6, 0.7], [0.3, 0.4], [0.8, 0.9]⟩	⟨[0.4, 0.5], [0.5, 0.6], [0.3, 0.4]⟩

$N_{x_1}^{[0.4,0.5]} = C_1 \cap C_2 \cap C_3, N_{x_2}^{[0.4,0.5]} = C_1 \cap C_2 \cap C_4, N_{x_3}^{[0.4,0.5]} = C_2 \cap C_4, N_{x_4}^{[0.4,0.5]} = C_1 \cap C_2.$

The interval neutrosophic $[0.4, 0.5]$ neighborhood of $x_i(i = 1, 2, 3, 4)$ is shown in Table 2. Obviously, the interval neutrosophic $[0.4, 0.5]$ neighborhood of $x_i(i = 1, 2, 3, 4)$ is covering of X.

Table 2. The interval neutrosophic $[0.4, 0.5]$ neighborhood of $x_i(i = 1, 2, 3, 4)$.

	x_1	x_2	x_3	x_4
$N_{x_1}^{[0.4,0.5]}$	⟨[0.4, 0.5], [0.2, 0.3], [0.4, 0.5]⟩	⟨[0.3, 0.6], [0.3, 0.5], [0.8, 0.9]⟩	⟨[0.3, 0.5], [0.2, 0.4], [0.7, 0.9]⟩	⟨[0.6, 0.7], [0.3, 0.4], [0.8, 0.9]⟩
$N_{x_2}^{[0.4,0.5]}$	⟨[0.4, 0.5], [0.3, 0.4], [0.5, 0.7]⟩	⟨[0.5, 0.7], [0.2, 0.3], [0.2, 0.3]⟩	⟨[0.3, 0.5], [0.2, 0.4], [0.3, 0.4]⟩	⟨[0.4, 0.5], [0.5, 0.6], [0.3, 0.4]⟩
$N_{x_3}^{[0.4,0.5]}$	⟨[0.4, 0.5], [0.3, 0.4], [0.5, 0.7]⟩	⟨[0.5, 0.7], [0.2, 0.3], [0.2, 0.3]⟩	⟨[0.3, 0.5], [0.2, 0.3], [0.3, 0.4]⟩	⟨[0.4, 0.5], [0.5, 0.6], [0.3, 0.4]⟩
$N_{x_4}^{[0.4,0.5]}$	⟨[0.4, 0.5], [0.2, 0.3], [0.3, 0.4]⟩	⟨[0.6, 0.7], [0.1, 0.2], [0.2, 0.3]⟩	⟨[0.3, 0.6], [0.2, 0.3], [0.3, 0.4]⟩	⟨[0.6, 0.7], [0.1, 0.2], [0.1.0.3]⟩

The interval neutrosophic $[s,t]$ covering was presented in the previous section. Based on this, the coverage approximation space can be obtained.

Definition 3.7. [26] Let $C = \{C_1, C_2, \cdots, C_m\}$ be an interval neutrosophic $[s,t]$ covering of X, where $C_i = \{T_{c_i} I_{c_i}, F_{c_i}\}$ and $C_i \in INS(i = 1, 2, \cdots, m)$. Then (X, C) is called a interval neutrosophic $[s,t]$ covering approximation space.

Definition 3.8. Let (X, C) be an interval neutrosophic $[s, t]$ covering approximation space, for any $A \in$ INS, the lower approximation operator $\underline{C}^{[s,t]}(A)$ and the upper approximation operator $\overline{C}^{[s,t]}(A)$ of interval neutrosophic A are defined as follows: $\underline{C}^{[s,t]}(A) = \left\{T_{\underline{C}^{[s,t]}(A)}, I_{\underline{C}^{[s,t]}(A)}, F_{\underline{C}^{[s,t]}(A)}\right\}, \overline{C}^{[s,t]}(A) = \left\{T_{\overline{C}^{[s,t]}(A)}, I_{\overline{C}^{[s,t]}(A)}, F_{\overline{C}^{[s,t]}(A)}\right\}$, where

$$T_{\underline{C}^{[s,t]}(A)} = \wedge\left\{T_A(y) \vee F_{N_x^{[s,t]}}(y)|y \in X\right\}, I_{\underline{C}^{[s,t]}(A)} = \vee\left\{I_A(y) \wedge ([1,1] - I_{N_x^{[s,t]}}(y))|y \in X\right\},$$

$$F_{\underline{C}^{[s,t]}(A)} = \vee\left\{F_A(y) \wedge T_{N_x^{[s,t]}}(y)|y \in X\right\}, T_{\overline{C}^{[s,t]}(A)} = \vee\left\{T_A(y) \wedge T_{N_x^{[s,t]}}(y)|y \in X\right\},$$

$$I_{\overline{C}^{[s,t]}(A)} = \wedge\left\{I_A(y) \vee I_{N_x^{[s,t]}}(y))T|y \in X\right\}, F_{\overline{C}^{[s,t]}(A)} = \wedge\left\{F_A(y) \vee F_{N_x^{[s,t]}}(y)|y \in X\right\}.$$

For any $x \in X$, then A is called an interval neutrosophic $[s, t]$ covering rough set, if $\underline{C}^{[s,t]}(A) \neq \overline{C}^{[s,t]}(A)$.

Example 2. Let A be a interval neutrosophic set, where

$$A(x_1) = \langle[0.4, 0.6], [0.2, 0.4], [0.3, 0.4]\rangle, A(x_2) = \langle[0.4, 0.5], [0.1, 0.3], [0.2, 0.4]\rangle,$$

$$A(x_3) = \langle[0.4, 0.5], [0.2, 0.5], [0.3, 0.6]\rangle, A(x_4) = \langle[0.3, 0.5], [0.2, 0.4], [0.4, 0.6]\rangle.$$

Then the lower approximation operator $\underline{C}^{[0.4,0.5]}(A)$ and the upper approximation operator $\overline{C}^{[0.4,0.5]}(A)$ of interval neutrosophic A can be calculated by Definition 3.8.

$$\underline{C}^{[0.4,0.5]}(A)(x_1) = \langle[0.4, 0.6], [0.2, 0.5], [0.4, 0.6]\rangle, \underline{C}^{[0.4,0.5]}(A)(x_2) = \langle[0.3, 0.5], [0.2, 0.5], [0.4, 0.5]\rangle,$$

$$\underline{C}^{[0.4,0.5]}(A)(x_3) = \langle[0.3, 0.5], [0.2, 0.5], [0.4, 0.5]\rangle, \underline{C}^{[0.4,0.5]}(A)(x_4) = \langle[0.3, 0.5], [0.2, 0.5], [0.4, 0.6]\rangle.$$

$$\overline{C}^{[0.4,0.5]}(A)(x_1) = \langle[0.4, 0.5], [0.2, 0.4], [0.4, 0.5]\rangle, \overline{C}^{[0.4,0.5]}(A)(x_2) = \langle[0.4, 0.5], [0.2, 0.3], [0.2, 0.4]\rangle,$$

$$\overline{C}^{[0.4,0.5]}(A)(x_1) = \langle[0.4, 0.5], [0.2, 0.3], [0.2, 0.4]\rangle, \overline{C}^{[0.4,0.5]}(A)(x_2) = \langle[0.4, 0.5], [0.1, 0.3], [0.2, 0.4]\rangle.$$

4. Some theorems and their proofs of interval neutrosophic covering rough sets

In this section we'll give you some theorems about INCRS and a complete proof of them.

Theorem 1. (1) $\underline{C}^{[s,t]}(X) = X, \overline{C}^{[s,t]}(\emptyset) = \emptyset$;
(2) $\underline{C}^{[s,t]}(A^C) = (\overline{C}^{[s,t]}(A))^C, \overline{C}^{[s.t]}(A^C) = (\underline{C}^{[s,t]}(A))^C$;
(3) $\underline{C}^{[s,t]}(A \bigcap B) = \underline{C}^{[s,t]}(A) \bigcap \underline{C}^{[s,t]}(B), \overline{C}^{[s,t]}(A \bigcup B) = \overline{C}^{[s,t]}(A) \bigcup \overline{C}^{[s,t]}(B)$;
(4) If $A \subseteq B$, then $\underline{C}^{[s,t]}(A) \subseteq \underline{C}^{[s,t]}(B), \overline{C}^{[s,t]}(A) \subseteq \overline{C}^{[s,t]}(B)$;
(5) $\underline{C}^{[s,t]}(A \bigcup B) \supseteq \underline{C}^{[s,t]}(A) \bigcup \underline{C}^{[s,t]}(B), \overline{C}^{[s,t]}(A \bigcap B) \subseteq \overline{C}^{[s,t]}(A) \bigcap \overline{C}^{[s,t]}(B)$;
(6) If $0 \le [s', t'] \le [s, t]$, then $\underline{C}^{[s',t']}(A) \supseteq \underline{C}^{[s,t]}(A), \overline{C}^{[s',t']}(A) \subseteq \overline{C}^{[s,t]}(A)$.

proof. (1) $T_{\underline{C}^{[s,t]}(X)} = \wedge\left\{T_X(y) \vee F_{N_x^{[s,t]}}(y)|y \in X\right\} = [1, 1]$,
$I_{\underline{C}^{[s,t]}(X)} = \vee\left\{I_X(y) \wedge ([1,1] - I_{N_x^{[s,t]}}(y))|y \in X\right\} = [0, 0]$,
$F_{\underline{C}^{[s,t]}(X)} = \vee\left\{F_X(y) \wedge T_{N_x^{[s,t]}}(y)|y \in X\right\} = [0, 0]$,
$\underline{C}^{[s,t]}(X) = \left\langle T_{\underline{C}^{[s,t]}(X)}, I_{\underline{C}^{[s,t]}(X)}, F_{\underline{C}^{[s,t]}(X)}\right\rangle = \langle[1, 1], [0, 0], [0, 0]\rangle = X$;
$T_{\overline{C}^{[s,t]}(\emptyset)} = \vee\left\{T_\emptyset(y) \wedge T_{N_x^{[s,t]}}(y)|y \in X\right\} = [0, 0]$,

$I_{\overline{C}^{[s,t]}(\emptyset)} = \wedge\left\{I_{\emptyset}(y) \vee I_{N_x^{[s,t]}}(y)|y \in X\right\} = [1,1]$,
$F_{\overline{C}^{[s,t]}(\emptyset)} = \wedge\left\{F_{\emptyset}(y) \vee F_{N_x^{[s,t]}}(y)|y \in X\right\} = [1,1]$,
$\overline{C}^{[s,t]}(\emptyset) = \left\langle T_{\overline{C}^{[s,t]}(\emptyset)}, I_{\overline{C}^{[s,t]}(\emptyset)}, F_{\overline{C}^{[s,t]}(\emptyset)}\right\rangle = \langle[0,0],[1,1],[1,1]\rangle = \emptyset$.

(2) $A^C = \langle F_A, [1,1] - I_A, T_A\rangle$,
$T_{\underline{C}^{[s,t]}(A^C)} = \wedge\left\{F_A(y) \vee F_{N_x^{[s,t]}}(y)|y \in X\right\} = F_{\overline{C}^{[s,t]}(A)}$,
$I_{\underline{C}^{[s,t]}(A^C)} = \vee\left\{([1,1] - I_A(y)) \wedge ([1,1] - I_{N_x^{[s,t]}}(y))|y \in X\right\}$
$= [1,1] - \wedge\left\{I_A(y) \vee I_{N_x^{[s,t]}}(y)|y \in X\right\} = [1,1] - I_{\overline{C}^{[s,t]}(A)}$
$F_{\underline{C}^{[s,t]}(A^C)} = \vee\left\{T_A(y) \wedge T_{N_x^{[s,t]}}(y)|y \in X\right\} = T_{\overline{C}^{[s,t]}(A)}$,
$\underline{C}^{[s,t]}(A^C) = \left\{T_{\underline{C}^{[s,t]}(A^C)}, I_{\underline{C}^{[s,t]}(A^C)}, F_{\underline{C}^{[s,t]}(A^C)}\right\} = \left\{F_{\overline{C}^{[s,t]}(A)}, [1,1] - I_{\overline{C}^{[s,t]}(A)}, T_{\overline{C}^{[s,t]}(A)}\right\}$
$(\overline{C}^{[s,t]}(A))^C = \left\{F_{\overline{C}^{[s,t]}(A)}, [1,1] - I_{\overline{C}^{[s,t]}(A)}, T_{\overline{C}^{[s,t]}(A)}\right\} = \underline{C}^{[s,t]}(A^C)$.
Similarly, it can be gotten that $\overline{C}^{[s,t]}(A^C) = (\underline{C}^{[s,t]}(A))^C$

(3) $A \bigcap B = \{T_A \bigcap T_B, I_A \bigcup I_B, F_A \bigcup F_B\}$,
$T_{\underline{C}^{[s,t]}(A \bigcap B)} = \wedge\left\{(T_A(y) \bigcap T_B(y)) \vee F_{N_x^{[s,t]}}(y)|y \in X\right\}$
$= \wedge\left\{(T_A(y) \vee F_{N_x^{[s,t]}}(y)) \bigcap (T_B(y) \vee F_{N_x^{[s,t]}}(y))|y \in X\right\} = T_{\underline{C}^{[s,t]}(A)} \bigcap T_{\underline{C}^{[s,t]}(B)}$,
$I_{\underline{C}^{[s,t]}(A \bigcap B)} = \vee\left\{(I_A(y) \bigcup I_B(y)) \wedge ([1,1] - I_{N_x^{[s,t]}}(y))|y \in X\right\}$
$= \vee\left\{(I_A(y) \wedge ([1,1] - I_{N_x^{[s,t]}}(y))) \bigcup (I_B(y) \wedge ([1,1] - I_{N_x^{[s,t]}}(y)))|y \in X\right\} = I_{\underline{C}^{[s,t]}(A)} \bigcup I_{\underline{C}^{[s,t]}(A)}$,
$F_{\underline{C}^{[s,t]}(A \bigcap B)} = \vee\left\{(F_A(y) \bigcup F_B(y)) \wedge T_{N_x^{[s,t]}}(y)|y \in X\right\}$
$= \vee\left\{(F_A(y) \wedge (T_{N_x^{[s,t]}}(y))) \bigcup (F_B(y) \wedge (T_{N_x^{[s,t]}}(y)))|y \in X\right\} = F_{\underline{C}^{[s,t]}(A)} \bigcup F_{\underline{C}^{[s,t]}(A)}$,
$\underline{C}^{[s,t]}(A \bigcap B) = \left\{T_{\underline{C}^{[s,t]}(A \bigcap B)}, I_{\underline{C}^{[s,t]}(A \bigcap B)}, F_{\underline{C}^{[s,t]}(A \bigcap B)}\right\}$
$= \left\{T_{\underline{C}^{[s,t]}(A)} \bigcap T_{\underline{C}^{[s,t]}(B)}, I_{\underline{C}^{[s,t]}(A)} \bigcup I_{\underline{C}^{[s,t]}(A)}, F_{\underline{C}^{[s,t]}(A)} \bigcup F_{\underline{C}^{[s,t]}(A)}\right\} = \underline{C}^{[s,t]}(A) \bigcap \underline{C}^{[s,t]}(B)$.
Similarly, it can be gotten that $\overline{C}^{[s,t]}(A \bigcup B) = \overline{C}^{[s,t]}(A) \bigcup \overline{C}^{[s,t]}(B)$

(4) If $A \subseteq B$, then $T_A \subseteq T_B, I_A \supseteq I_B, F_A \supseteq F_B$.
When $T_A \subseteq T_B$, then $\left\{T_A(y) \vee F_{N_x^{[s,t]}}(y)|y \in X\right\} \subseteq \left\{T_B(y) \vee F_{N_x^{[s,t]}}(y)|y \in X\right\}$,
thus $\wedge\left\{T_A(y) \vee F_{N_x^{[s,t]}}(y)|y \in X\right\} \subseteq \left\{T_B(y) \vee F_{N_x^{[s,t]}}(y)|y \in X\right\}$,
hence $\left\{T_B(y) \vee F_{N_x^{[s,t]}}(y)|y \in X\right\}, \subseteq \wedge\left\{T_B(y) \vee F_{N_x^{[s,t]}}(y)|y \in X\right\}$, that is $T_{\underline{C}^{[s,t]}(A)} \subseteq T_{\underline{C}^{[s,t]}(B)}$.
When $I_A \supseteq I_B$, then $\left\{I_A(y) \wedge (1 - I_{N_x^{[s,t]}}(y))|y \in X\right\} \supseteq \left\{I_B(y) \wedge ([1,1] - I_{N_x^{[s,t]}}(y))|y \in X\right\}$.
Thus $\vee\left\{I_A(y) \wedge ([1,1] - I_{N_x^{[s,t]}}(y))|y \in X\right\} \supseteq \vee\left\{I_B(y) \wedge ([1,1] - I_{N_x^{[s,t]}}(y))|y \in X\right\}$,
hence $I_{\underline{C}^{[s,t]}(A)} \supseteq I_{\underline{C}^{[s,t]}(A)}$.
When $F_A \supseteq F_B$, then $\left\{F_A(y) \wedge T_{N_x^{[s,t]}}(y)|y \in X\right\} \supseteq \left\{F_B(y) \wedge T_{N_x^{[s,t]}}(y)|y \in X\right\}$,
thus $\vee\left\{F_A(y) \wedge T_{N_x^{[s,t]}}(y)|y \in X\right\} \supseteq \vee\left\{F_B(y) \wedge T_{N_x^{[s,t]}}(y)|y \in X\right\}$, so $F_{\underline{C}^{[s,t]}(A)} \supseteq F_{\underline{C}^{[s,t]}(A)}, \underline{C}^{[s,t]}(A) \subseteq \underline{C}^{[s,t]}(B)$.
Similarly, it can be gotten that $\overline{C}^{[s,t]}(A) \subseteq \overline{C}^{[s,t]}(B)$.

(5) It is obvious that $A \subseteq A \bigcup B, B \subseteq A \bigcup B, A \bigcap B \subseteq A, A \bigcap B \subseteq B$.
So $\underline{C}^{[s,t]}(A) \subseteq \underline{C}^{[s,t]}(A \bigcup B), \underline{C}^{[s,t]}(B) \subseteq \underline{C}^{[s,t]}(A \bigcup B), \overline{C}^{[s,t]}(A \bigcap B) \subseteq \overline{C}^{[s,t]}(A), \overline{C}^{[s,t]}(A \bigcap B) \subseteq \overline{C}^{[s,t]}(B)$.

Hence $\underline{C}^{[s,t]}(A) \bigcup \underline{C}^{[s,t]}(B) \subseteq C^{[s,t]}(A \bigcup B)$, $\overline{C}^{[s,t]}(A \bigcap B) \subseteq \overline{C}^{[s,t]}(A) \bigcap \overline{C}^{[s,t]}(B)$.

(6) If $0 \leq [s', t'] \leq [s,t]$, then $N_x^{[s',t']} \subseteq N_x^{[s,t]}$. Thus $T_{N_x^{[s',t']}} \subseteq T_{N_x^{[s,t]}}$, $I_{N_x^{[s',t']}} \supseteq I_{N_x^{[s,t]}}$, $F_{N_x^{[s',t']}} \supseteq F_{N_x^{[s,t]}}$, hence $\wedge \left\{T_A(y) \vee F_{N_x^{[s',t']}}(y)|y \in X\right\} \supseteq \wedge \left\{T_A(y) \vee F_{N_x^{[s,t]}}(y)|y \in X\right\}$,
$\vee \left\{I_A(y) \wedge ([1,1] - I_{N_x^{[s',t']}}(y))|y \in X\right\} \subseteq \vee \left\{I_A(y) \wedge ([1,1] - I_{N_x^{[s,t]}}(y))|y \in X\right\}$,
$\vee \left\{F_A(y) \wedge T_{N_x^{[s',t']}}(y)|y \in X\right\} \subseteq \vee \left\{F_A(y) \wedge T_{N_x^{[s,t]}}(y)|y \in X\right\}$.
That is $\underline{C}^{[s',t']}(A) \supseteq \underline{C}^{[s,t]}(A)$. Similarly, it can be gotten that $\overline{C}^{[s',t']}(A) \subseteq \overline{C}^{[s,t]}(A)$.

Theorem 2. Let (X, C) be an interval neutrosophic $[s,t]$ covering approximation space, then the following statements are equivalent:
(1) $\underline{C}^{[s,t]}(\emptyset) = \emptyset$;
(2) $\overline{C}^{[s,t]}(X) = X$;
(3) For any$x \in X$,$\{y \in X|\forall C_i \in C((C_i(x) \geq [s,t]) \Rightarrow (C_i(y) = X))\} \neq \emptyset$.

Proof. $\{y \in X|\forall C_i \in C((C_i(x) \geq [s,t]) \Rightarrow (C_i(y) = X))\} \neq \emptyset$ means for each $x \in X$ and $C_i(x) \geq [s,t]$, $\exists y \in X$ such that $C_i(y) = X$, satisfying $N_x^{[s,t]}(y) = X$.
(1) $\Rightarrow$ (3) If $\underline{C}^{[s,t]}(\emptyset) = \emptyset$, then
$\underline{C}^{[s,t]}(\emptyset) = \left\{\wedge F_{N_x^{[s,t]}}(y), \vee([1,1] - I_{N_x^{[s,t]}}(y)), \vee T_{N_x^{[s,t]}}(y)|y \in X\right\} = \emptyset \Rightarrow \exists y \in X$,
$\wedge F_{N_x^{[s,t]}}(y) = [0,0]$, $\wedge I_{N_x^{[s,t]}}(y) = [0,0]$, $\vee T_{N_x^{[s,t]}}(y) = [1,1]$, that is $N_x^{[s,t]}(y) = X$.
(3) $\Rightarrow$ (2) If $N_x^{[s,t]}(y) = X$, then
$\overline{C}^{[s,t]}(X) = \left\{\vee T_{N_x^{[s,t]}}(y), \wedge I_{N_x^{[s,t]}}(y), \wedge F_{N_x^{[s,t]}}(y)|y \in X\right\} = \{[1,1],[0,0],[0,0]\} = X$.
(2) $\Rightarrow$ (1) It is proved by the rotation of $\underline{C}$ and $\overline{C}$. So they are equivalent.

Theorem 3. Let (X, C) be an interval neutrosophic $[s,t]$ covering approximation space. $A$is an INS and B is an constant interval neutrosophic set, where $B = \langle[\alpha^-, \alpha^+], [\beta^-, \beta^+], [\gamma^-, \gamma^+]\rangle$. It satisfies that for any $x \in X$, $\langle[\alpha^-, \alpha^+], [\beta^-, \beta^+], [\gamma^-, \gamma^+]\rangle(x) = \langle[\alpha^-, \alpha^+], [\beta^-, \beta^+], [\gamma^-, \gamma^+]\rangle$.
If $\{y \in X|\forall C_i \in C((C_i(x) \geq [s,t]) \Rightarrow (C_i(y) = X))\} \neq \emptyset$, then
(1) $\underline{C}^{[s,t]}(B) = B$, $\overline{C}^{[s,t]}(B) = B$;
(2) $\underline{C}^{[s,t]}(A \bigcup B) = \underline{C}^{[s,t]}(A) \bigcup B$, $\overline{C}^{[s,t]}(A \bigcap B) = \overline{C}^{[s,t]}(A) \bigcap B$.

Proof. (1) $\{y \in X|\forall C_i \in C((C_i(x) \geq [s,t]) \Rightarrow (C_i(y) = X))\} \neq \emptyset$ means for each $x \in X$ and $C_i(x) \geq [s,t]$,$\exists y \in X$,such that $C_i(y) = X$, then $N_x^{[s,t]}(y) = X$.

$T_{\underline{B}^{[s,t]}} = \wedge \left\{[\alpha^-, \alpha^+] \vee F_{N_x^{[s,t]}}(y)|y \in X\right\} = [\alpha^-, \alpha^+]$,
$I_{\underline{B}^{[s,t]}} = \vee \left\{[\beta^-, \beta^+] \wedge ([1,1] - I_{N_x^{[s,t]}}(y))|y \in X\right\} = [\beta^-, \beta^+]$,
$F_{\underline{B}^{[s,t]}} = \vee \left\{[\gamma^-, \gamma^+] \wedge T_{N_x^{[s,t]}}(y)|y \in X\right\} = [\gamma^-, \gamma^+]$.
So that $\underline{C}^{[s,t]}(B) = B$. Similarly, it can be gotten that $\overline{C}^{[s,t]}(B) = B$.

(2) $T_{\underline{C}^{[s,t]}(A \bigcup B)} = \wedge \left\{(T_A(y) \bigcup [\alpha^-, \alpha^+]) \vee F_{N_x^{[s,t]}}(y)|y \in X\right\}$
$= \wedge \left\{T_A(y) \vee F_{N_x^{[s,t]}}(y)|y \in X\right\} \bigcup [\alpha^-, \alpha^+]$,
$I_{\underline{C}^{[s,t]}(A \bigcup B)} = \vee \left\{(I_A(y) \bigcup [\beta^-, \beta^+]) \wedge ([1,1] - I_{N_x^{[s,t]}}(y))|y \in X\right\}$
$= \vee \left\{I_A(y) \wedge ([1,1] - I_{N_x^{[s,t]}}(y))|y \in X\right\} \bigcup [\beta^-, \beta^+]$,

$F_{\underline{C}^{[s,t]}(A\bigcup B)} = \vee\left\{(F_A(y)\bigcup[\gamma^-,\gamma^+])\wedge T_{N_x^{[s,t]}}(y)|y\in X\right\} = \vee\left\{F_A(y)\wedge T_{N_x^{[s,t]}}(y)|y\in X\right\}\bigcup[\gamma^-,\gamma^+]$,
Thus $\underline{C}^{[s,t]}(A\bigcup B) = \underline{C}^{[s,t]}(A)\bigcup B$. Similarly, it can be proofed that $\overline{C}^{[s,t]}(A\bigcap B) = \overline{C}^{[s,t]}(A)\bigcap B$.
Corollary. When $\alpha^- = \alpha^+ = \alpha, \beta^- = \beta^+ = \beta, \gamma^- = \gamma^+ = \gamma, B = \langle\alpha,\beta,\gamma\rangle$ It can be gotten that
(1) $\underline{C}^{[s,t]}\langle\alpha,\beta,\gamma\rangle = \langle\alpha,\beta,\gamma\rangle, \overline{C}^{[s,t]}\langle\alpha,\beta,\gamma\rangle = \langle\alpha,\beta,\gamma\rangle$;
(2) $\underline{C}^{[s,t]}(A\bigcup\langle\alpha,\beta,\gamma\rangle) = \underline{C}^{[s,t]}(A)\bigcup\langle\alpha,\beta,\gamma\rangle, \overline{C}^{[s,t]}(A\bigcap\langle\alpha,\beta,\gamma\rangle = \overline{C}^{[s,t]}(A)\bigcap\langle\alpha,\beta,\gamma\rangle$.
The proof is omitted.
Theorem 4. Let (X, C) be an interval neutrosophic $[s,t]$ covering approximation space. $A$is an INS and $A\in X$, for any $x\in X$, there are
(1) $\overline{C}^{[s,t]}(1_y)(x) = N_x^{[s,t]}(y)$;
(2) $\underline{C}^{[s,t]}(1_{X-\{y\}})(x) = (N_x^{[s,t]}(y))^C$.
Proof. $T_{\overline{C}^{[s,t]}(1_y)}(x) = \vee\left\{T_{1_y}(z)\wedge T_{N_x^{[s,t]}}(z)|z\in X\right\}$
$= (T_{1_y}(y)\wedge T_{N_x^{[s,t]}}(y))\vee(\vee_{z\in X-\{y\}}(T_{1_y}(z)\wedge T_{N_x^{[s,t]}}(z)))$
$= ([1,1]\wedge T_{N_x^{[s,t]}}(y))\vee([0,0]\wedge T_{N_x^{[s,t]}}(z)) = T_{N_x^{[s,t]}}(y)$,
$I_{\overline{C}^{[s,t]}(1_y)}(x) = \wedge\left\{I_{1_y}(z)\vee I_{N_x^{[s,t]}}(z)|z\in X\right\}$
$= (I_{1_y}(y)\vee I_{N_x^{[s,t]}}(y))\wedge(\wedge_{z\in X-\{y\}}(I_{1_y}(z)\vee I_{N_x^{[s,t]}}(z)))$
$= ([0,0]\vee I_{N_x^{[s,t]}}(y))\wedge([1,1]\vee I_{N_x^{[s,t]}}(z)) = I_{N_x^{[s,t]}}(y)$,
$F_{\overline{C}^{[s,t]}(1_y)}(x) = \wedge\left\{F_{1_y}(z)\vee F_{N_x^{[s,t]}}(z)|z\in X\right\}$
$= (F_{1_y}(y)\vee F_{N_x^{[s,t]}}(y))\wedge(\wedge_{z\in X-\{y\}}(F_{1_y}(z)\vee F_{N_x^{[s,t]}}(z)))$
$= ([0,0]\vee F_{N_x^{[s,t]}}(y))\wedge([1,1]\vee F_{N_x^{[s,t]}}(z)) = F_{N_x^{[s,t]}}(y)$.
So $\overline{C}^{[s,t]}(1_y)(x) = N_x^{[s,t]}(y)$.
Similarly, it can be gotten that $\underline{C}^{[s,t]}(1_{X-\{y\}})(x) = (N_x^{[s,t]}(y))^C$, and the proof process is omitted.
Theorem 5. Let (X, C) be an interval neutrosophic $[s,t]$ covering approximation space. $A$is an INS and $A\in X$, for any $x\in X$, if $\left(N_x^{[s,t]}\right)^C \le A \le N_x^{[s,t]}$, then $\underline{C}^{[s,t]}(\underline{C}^{[s,t]}(A)) \subseteq \underline{C}^{[s,t]}(A) \subseteq A \subseteq \overline{C}^{[s,t]}(A) \subseteq \overline{C}^{[s,t]}(\overline{C}^{[s,t]}(A))$.
Proof. $(N_x^{[s,t]})^C = \left\langle F_{N_x^{[s,t]}}, ([1,1]-I_{N_x^{[s,t]}}), T_{N_x^{[s,t]}}\right\rangle$.
When $(N_x^{[s,t]})^C \le A$,thus $F_{N_x^{[s,t]}} \le T_A$, $[1,1]-I_{N_x^{[s,t]}} \ge I_A, T_{N_x^{[s,t]}} \ge F_A$,
so $T_{\underline{C}^{[s,t]}(A)} = \wedge\left\{T_A(y)\vee F_{N_x^{[s,t]}}(y)|y\in X\right\} = \wedge\{T_A(y)|y\in X\} \le T_A$,
$I_{\underline{C}^{[s,t]}(A)} = \vee\left\{I_A(y)\wedge([1,1]-I_{N_x^{[s,t]}}(y))|y\in X\right\} = \vee\{I_A(y)|y\in X\} \ge I_A$,
$F_{\underline{C}^{[s,t]}(A)} = \vee\left\{F_A(y)\wedge T_{N_x^{[s,t]}}(y)|y\in X\right\} = \vee\{F_A(y)|y\in X\} \ge F_A$.
That is $\underline{C}^{[s,t]}(A) \subseteq A$. Similarly,$A \subseteq \overline{C}^{[s,t]}(A)$.
According to theorem 1(4),$\underline{C}^{[s,t]}(\underline{C}^{[s,t]}(A)) \subseteq \underline{C}^{[s,t]}(A) \subseteq A \subseteq \overline{C}^{[s,t]}(A) \subseteq \overline{C}^{[s,t]}(\overline{C}^{[s,t]}(A))$.

Theorem 5 gives a sufficient condition for $\underline{C}^{[s,t]}(A) \subseteq A \subseteq \overline{C}^{[s,t]}(A)$, and then theorem 6 will give the necessary condition.

Theorem 6. Let (X, C) be an interval neutrosophic $[s,t]$ covering approximation space. $A\in X$, if $\forall x\in X, C_i(x) \ge [s,t] \Rightarrow C_i(x) = X(i = \{1,2,\cdots m\})$, and then
$\underline{C}^{[s,t]}(A) \subseteq A \subseteq \bar{C}^{[s,t]}(A)$.
Proof. $\forall x\in X, C_i(x) \ge [s,t] \Rightarrow C_i(x) = X(i = \{1,2,\cdots m\})$, which means $\forall x\in X, N_x^{[s,t]} = X = \langle[1,1],[0,0],[0,0]\rangle$.

$T_{\underline{C}^{[s,t]}(A)} = \wedge \left\{T_A(y) \vee F_{N_x^{[s,t]}}(y) | y \in X\right\} = \wedge \{T_A(y) \vee [0,0] | y \in X\} = \wedge \{T_A(y) | y \in X\} \leq T_A,$
$I_{\underline{C}^{[s,t]}(A)} = \vee \left\{I_A(y) \wedge ([1,1] - I_{N_x^{[s,t]}}(y)) | y \in X\right\} = \vee \{I_A(y) \wedge [1,1] | y \in X\} = \vee \{I_A(y) | y \in X\} \geq I_A,$
$F_{\underline{C}^{[s,t]}(A)} = \vee \left\{F_A(y) \wedge T_{N_x^{[s,t]}}(y) | y \in X\right\} = \vee \{F_A(y) \wedge [1,1] | y \in X\} = \vee \{F_A(y) | y \in X\} \geq F_A.$
So $\underline{C}^{[s,t]}(A) \subseteq A$.
$T_{\overline{C}^{[s,t]}(A)} = \vee \left\{T_A(y) \wedge T_{N_x^{[s,t]}}(y) | y \in X\right\} = \vee \{T_A(y) | y \in X\} \geq T_A,$
$I_{\overline{C}^{[s,t]}(A)} = \wedge \left\{I_A(y) \vee I_{N_x^{[s,t]}}(y) | y \in X\right\} = \wedge \{I_A(y) | y \in X\} \leq I_A,$
$F_{\overline{C}^{[s,t]}(A)} = \wedge \left\{F_A(y) \vee F_{N_x^{[s,t]}}(y) | y \in X\right\} = \wedge \{F_A(y) | y \in X\} \leq F_A.$
So $A \subseteq \overline{C}^{[s,t]}(A)$.
Hence $\underline{C}^{[s,t]}(A) \subseteq A \subseteq \overline{C}^{[s,t]}(A)$.

Theorem 7. Let $C = \{C_1, C_2, \cdots, C_m\}$ be an interval neutrosophic $[s,t]$ covering of $X. A \in INS, \overline{C}$ and $\underline{C}$ are the upper and lower approximation operator, which are defined in defination 3.8. Then we can get that:

(1) C is serial $\Leftrightarrow \underline{C}^{[s,t]}\langle\alpha,\beta,\lambda\rangle = \langle\alpha,\beta,\lambda\rangle, \forall\alpha,\beta,\lambda \in [0,1],$
$\Leftrightarrow \underline{C}^{[s,t]}(\emptyset) = \emptyset,$
$\Leftrightarrow \bar{C}^{[s,t]}\langle\alpha,\beta,\lambda\rangle = \langle\alpha,\beta,\lambda\rangle, \forall\alpha,\beta,\lambda \in [0,1],$
$\Leftrightarrow \bar{C}^{[s,t]}(X) = X;$
(2) C is reflexive $\Leftrightarrow \underline{C}^{[s,t]}(A) \subseteq A,$
$\Leftrightarrow A \subseteq \bar{C}^{[s,t]}(A);$
(3) C is symmetric $\Leftrightarrow \underline{C}^{[s,t]}\left(1_{X-(y\})}\right)(x) = \underline{C}^{[s,t]}\left(1_{X-\{x\}}\right)(y), \forall x, y \in X,$
$\Leftrightarrow \bar{C}^{[s,t]}\left(1_y\right)(x) = \bar{C}^{[s,t]}\left(1_x\right)(y), \forall x, y \in X;$
(4) C is transitive $\Leftrightarrow \underline{C}^{[s,t]}(A) \subseteq \underline{C}^{[s,t]}\left(\underline{C}^{[s,t]}(A)\right),$
$\Leftrightarrow \bar{C}^{[s,t]}\left(\bar{C}^{[s,t]}(A)\right) \subseteq \bar{C}^{[s,t]}(A).$

Proof. (1) When C is serial, then it satisfies $\exists y \in X$ and $N_x^{[s,t]}(y) = X$. So it can be proved by Theorem 3, Theorem 4 and Deduction.
(2) $\Rightarrow$ When C is reflexive, then $N_x^{[s,t]}(x) = X = \langle[1,1],[0,0],[0,0]\rangle$
$T_{\underline{C}^{[s,t]}(A)}(x) = \wedge \left\{T_A(y) \vee F_{N_x^{[s,t]}}(y) | y \in X\right\} \leq T_A(x) \vee F_{N_x^{[s,t]}}(x) = T_A(x),$
$I_{\underline{C}^{[s,t]}(A)}(x) = \vee \left\{I_A(y) \wedge ([1,1] - I_{N_x^{[s,t]}}(y)) | y \in X\right\} \geq I_A(x) \wedge [1,1] = I_A(x),$
$F_{\underline{C}^{[s,t]}(A)}(x) = \vee \left\{F_A(y) \wedge T_{N_x^{[s,t]}}(y) | y \in X\right\} \geq F_A(x) \wedge [1,1] = F_A(x).$
That is $\underline{C}^{[s,t]}(A) \subseteq A$.

$\Leftarrow$ If $\underline{C}^{[s,t]}(A) \subseteq A$, let $A = 1_{X-(x)}$, and $\forall x, y \in X$, then
$T_{N_x^{[s,t]}}(x) = (T_{N_x^{[s,t]}}(x) \wedge [1,1]) \vee [0,0]$
$= (T_{N_x^{[s,t]}}(x) \wedge F_{(1_{X-\{x\}})}(x)) \vee (\vee_{y \in X-\{x\}}(T_{N_x^{[s,t]}}(y) \wedge F_{(1_{X-\{x\}})}(y)))$
$= \vee \left\{T_{N_x^{[s,t]}}(y) \wedge F_{(1_{X-\{x\}})}(y) | y \in X\right\}$
$= F_{\underline{C}^{[s,t]}(1_{X-\{x\}})}(x) \geq F_{(1_{X-\{x\}})}(x) = [1,1],$
$[1,1] - I_{N_x^{[s,t]}}(x) = \left\{([1,1] - I_{N_x^{[s,t]}}(x)) \wedge [1,1]\right\} \vee [0,0]$
$= \left\{([1,1] - I_{N_x^{[s,t]}}(x)) \wedge I_{(1_{X-\{x\}})}(x)\right\} \vee \left\{\vee_{y \in X-\{x\}}(([1,1] - I_{N_x^{[s,t]}}(y)) \wedge I_{(1_{X-\{x\}})}(y))\right\}$
$= \vee \left\{I_{(1_{X-\{x\}})}(x) \wedge ([1,1] - I_{N_x^{[s,t]}}(y)) | y \in X\right\}$

$= I_{\underline{C}^{[s,t]}(1_{X-\{x\}})}(x) \geq I_{(1_{X-\{x\}})}(x) = [1,1]$,
so $I_{N_x^{[s,t]}}(x) = [0,0]$.
$F_{N_x^{[s,t]}}(x) = \left\{F_{N_x^{[s,t]}}(x)) \vee [0,0]\right\} \wedge [1,1]$
$= \left\{F_{N_x^{[s,t]}}(x) \vee T_{(1_{X-\{x\}})}(x)\right\} \wedge \left\{\wedge_{y\in X-\{x\}}(F_{N_x^{[s,t]}}(y) \vee T_{(1_{X-\{x\}})}(y))\right\}$
$= \wedge \left\{T_{(1_{X-\{x\}})}(x) \vee F_{N_x^{[s,t]}}(y)|y \in X\right\}$
$= T_{\underline{C}^{[s,t]}(1_{X-\{x\}})}(x) \leq T_{(1_{X-\{x\}})}(x) = [0,0]$.
That is $N_x^{[s,t]}(x) = \langle [1,1],[0,0],[0,0]\rangle = X$. So C is reflexive. Meanwhile, it is easy to prove the other part by the same way.
(3) $T_{\underline{C}^{[s,t]}(1_{X-\{x\}})}(y) = \wedge \left\{T_{(1_{X-\{x\}})}(z) \vee F_{N_y^{[s,t]}}(z)|z \in X\right\}$
$= \left\{F_{N_y^{[s,t]}}(x) \vee T_{(1_{X-\{x\}})}(x)\right\} \wedge \left\{\wedge_{z\in X-\{x\}}(F_{N_y^{[s,t]}}(z) \vee T_{(1_{X-\{x\}})}(z))\right\}$
$= \left\{F_{N_y^{[s,t]}}(x) \vee [0,0]\right\} \wedge [1,1]$
$= F_{N_y^{[s,t]}}(x)$,
$T_{\underline{C}^{[s,t]}(1_{X-\{y\}})}(x) = \wedge \left\{T_{(1_{X-\{y\}})}(z) \vee F_{N_x^{[s,t]}}(z)|z \in X\right\}$
$= \left\{F_{N_x^{[s,t]}}(y) \vee T_{(1_{X-\{y\}})}(y)\right\} \wedge \left\{\wedge_{z\in X-\{y\}}(F_{N_x^{[s,t]}}(z) \vee T_{(1_{X-\{y\}})}(z))\right\}$
$= \left\{F_{N_x^{[s,t]}}(y) \vee [0,0]\right\} \wedge [1,1]$
$= F_{N_x^{[s,t]}}(y)$,
$I_{\underline{C}^{[s,t]}(1_{X-\{x\}})}(y) = \vee \left\{I_{(1_{X-\{x\}})}(z) \wedge ([1,1] - I_{N_y^{[s,t]}}(z))|z \in X\right\}$
$= \left\{([1,1] - I_{N_y^{[s,t]}}(x)) \wedge I_{(1_{X-\{x\}})}(x)\right\} \vee \left\{\vee_{z\in X-\{x\}}(([1,1] - I_{N_y^{[s,t]}}(z)) \wedge I_{(1_{X-\{x\}})}(z))\right\}$
$= \left\{([1,1] - I_{N_y^{[s,t]}}(x)) \wedge [1,1]\right\} \vee [0,0]$
$= [1,1] - I_{N_y^{[s,t]}}(x)$,
$I_{\underline{C}^{[s,t]}(1_{X-\{y\}})}(x) = \vee \left\{I_{(1_{X-\{y\}})}(z) \wedge ([1,1] - I_{N_x^{[s,t]}}(z))|z \in X\right\}$
$= \left\{([1,1] - I_{N_x^{[s,t]}}(y)) \wedge I_{(1_{X-\{y\}})}(y)\right\} \vee \left\{\vee_{z\in X-\{y\}}(([1,1] - I_{N_x^{[s,t]}}(z)) \wedge I_{(1_{X-\{y\}})}(z))\right\}$
$= \left\{([1,1] - I_{N_x^{[s,t]}}(y)) \wedge [1,1]\right\} \vee [0,0]$
$= [1,1] - I_{N_x^{[s,t]}}(y)$,
$F_{\underline{C}^{[s,t]}(1_{X-\{x\}})}(y) = \vee \left\{F_{(1_{X-\{x\}})}(z) \wedge T_{N_y^{[s,t]}}(z)|z \in X\right\}$
$= \left\{T_{N_y^{[s,t]}}(x) \wedge F_{(1_{X-\{x\}})}(x)\right\} \vee \left\{\vee_{z\in X-\{x\}}(T_{N_y^{[s,t]}}(z) \wedge F_{(1_{X-\{x\}})}(z))\right\}$
$= \left\{T_{N_y^{[s,t]}}(x) \wedge [1,1]\right\} \vee [0,0]$
$= T_{N_y^{[s,t]}}(x)$,
$F_{\underline{C}^{[s,t]}(1_{X-\{y\}})}(x) = \vee \left\{F_{(1_{X-\{y\}})}(z) \wedge T_{N_x^{[s,t]}}(z)|z \in X\right\}$
$= \left\{T_{N_x^{[s,t]}}(y) \wedge F_{(1_{X-\{y\}})}(y)\right\} \vee \left\{\vee_{z\in X-\{y\}}(T_{N_x^{[s,t]}}(z) \wedge F_{(1_{X-\{y\}})}(z))\right\}$
$= \left\{T_{N_x^{[s,t]}}(y) \wedge [1,1]\right\} \vee [0,0]$
$= T_{N_x^{[s,t]}}(y)$.
So when is symmetric, it satisfies $T_{N_x^{[s,t]}}(y) = T_{N_y^{[s,t]}}(x), I_{N_x^{[s,t]}}(y) = I_{N_y^{[s,t]}}(x)$, $F_{N_x^{[s,t]}}(y) = F_{N_y^{[s,t]}}(x)$, that is $N_x^{[s,t]}(y) = N_y^{[s,t]}(x)$, then
$T_{\underline{C}^{[s,t]}(1_{X-\{x\}})}(y) = T_{\underline{C}^{[s,t]}(1_{X-\{y\}})}(x)$,
$I_{\underline{C}^{[s,t]}(1_{X-\{x\}})}(y) = I_{\underline{C}^{[s,t]}(1_{X-\{y\}})}(x)$,
$F_{\underline{C}^{[s,t]}(1_{X-\{x\}})}(y) = F_{\underline{C}^{[s,t]}(1_{X-\{y\}})}(x)$
That is $\underline{C}^{[s,t]}(1_{X-\{x\}})(y) = \underline{C}^{[s,t]}(1_{X-\{y\}})(x)$.

It is similar to get $\overline{C}^{[s,t]}(1_y)(x) = \overline{C}^{[s,t]}(1_x)(y)$, and the proof is omitted.
(4) $\Rightarrow$ If C is transitive, then $\vee\left\{T_{N_x^{[s,t]}}(y) \wedge T_{N_y^{[s,t]}}(z)|y \in X\right\} \leq T_{N_x^{[s,t]}}(z)$,
$\wedge\left\{I_{N_x^{[s,t]}}(y) \vee I_{N_y^{[s,t]}}(z)|y \in X\right\} \geq I_{N_x^{[s,t]}}(z)$, $\wedge\left\{F_{N_x^{[s,t]}}(y) \vee F_{N_y^{[s,t]}}(z)|y \in X\right\} \geq F_{N_x^{[s,t]}}(z)$.
$T_{\underline{C}^{[s,t]}(\underline{C}^{[s,t]}(A))}(x) = \wedge\left\{T_{\underline{C}^{[s,t]}(A)}(y) \vee F_{N_x^{[s,t]}}(y)|y \in X\right\} = \wedge\left\{\wedge\left\{T_A(z) \vee F_{N_y^{[s,t]}}(z)|z \in X\right\} \vee F_{N_x^{[s,t]}}(y)|y \in X\right\}$
$= \wedge_{y\in X}\wedge_{z\in X}(T_A(z) \vee F_{N_x^{[s,t]}}(z) \vee F_{N_x^{[s,t]}}(y)) = \wedge_{z\in X}(\wedge_{y\in X}(F_{N_y^{[s,t]}}(z) \vee F_{N_x^{[s,t]}}(y)) \vee T_A(z))$
$\geq \wedge_{z\in X}(F_{N_x^{[s,t]}}(z) \vee T_A(z)) = T_{\underline{C}^{[s,t]}(A)}(x)$,
$I_{\underline{C}^{[s,t]}(\underline{C}^{[s,t]}(A))}(x) = \vee\left\{I_{\underline{C}^{[s,t]}(A)}(z) \wedge (1 - I_{N_x^{[s,t]}}(y))|y \in X\right\}$
$= \vee\left\{\vee\left\{I_A(z) \wedge (1 - I_{N_y^{[s,t]}}(z))|z \in X\right\} \wedge (1 - I_{N_x^{[s,t]}}(y))|y \in X\right\}$
$= \vee_{y\in X}\vee_{z\in X}(I_A(z) \wedge (1 - I_{N_y^{[s,t]}}(z)) \wedge (1 - I_{N_x^{[s,t]}}(y)) = \vee_{z\in X}((1 - \wedge_{y\in X}(I_{N_y^{[s,t]}}(z) \vee I_{N_x^{[s,t]}}(y)) \wedge I_A(z)$
$\leq \vee_{z\in X}(1 - I_{N_x^{[s,t]}}(z)) \wedge I_A(z) = I_{\underline{C}^{[s,t]}(A)}(x)$,
$F_{\underline{C}^{[s,t]}(\underline{C}^{[s,t]}(A))}(x) = \vee\left\{F_{\underline{C}^{[s,t]}(A)}(y) \wedge T_{N_x^{[s,t]}}(y)|y \in X\right\} = \vee\left\{\vee\left\{F_A(z) \wedge T_{N_y^{[s,t]}}(z)|z \in X\right\} \wedge T_{N_x^{[s,t]}}(y)|y \in X\right\}$
$= \vee_{y\in X}\vee_{z\in X}(F_A(z) \wedge T_{N_y^{[s,t]}}(z) \wedge T_{N_x^{[s,t]}}(y)) = \vee_{z\in X}(\vee_{y\in X}(T_{N_y^{[s,t]}}(z) \wedge T_{N_x^{[s,t]}}(y))) \wedge F_A(z)$
$\leq \vee_{z\in X}(T_{N_x^{[s,t]}}(z) \wedge F_A(z)) = F_{\underline{C}^{[s,t]}(A)}(x)$,
so $\underline{C}^{[s,t]}(A) \subseteq \underline{C}^{[s,t]}(\underline{C}^{[s,t]}(A))$.
Similarly, it can be gotten that $\overline{C}^{[s,t]}(\overline{C}^{[s,t]}(A)) \subseteq \overline{C}^{[s,t]}(A)$. $\Leftarrow$ If $\underline{C}^{[s,t]}(A) \subseteq \underline{C}^{[s,t]}\left(\underline{C}^{[s,t]}(A)\right)$, let $A = 1_{X-\{x\}}$ and $\forall x, y, z \in X$,
from the proving process of (3), we have
$T_{N_x^{[s,t]}}(z) = F_{\underline{C}^{[s,t]}(1_{X-\{z\}})}(x) \geq F_{\underline{C}^{[s,t]}(\underline{C}^{[s,t]}(1_{X-\{z\}})}(x) = \vee\left\{F_{\underline{C}^{[s,t]}(1_{X-\{z\}})}(y) \wedge T_{N_x^{[s,t]}}(y)|y \in X\right\}$
$= \vee\left\{T_{N_y^{[s,t]}}(z) \wedge T_{N_x^{[s,t]}}(y)|y \in X\right\}$,
$[1, 1] - I_{N_x^{[s,t]}}(z) = I_{\underline{C}^{[s,t]}(1_{X-\{z\}})}(x) \geq I_{\underline{C}^{[s,t]}(\underline{C}^{[s,t]}(1_{X-\{z\}})}(x) \vee \left\{I_{\underline{C}^{[s,t]}(1_{X-\{z\}})}(y) \wedge ([1, 1] - I_{N_x^{[s,t]}}(y))|y \in X\right\}$
$= \vee\left\{([1, 1] - I_{N_y^{[s,t]}}(z) \wedge ([1, 1] - I_{N_x^{[s,t]}}(y)|y \in X\right\}$,
Thus $I_{N_x^{[s,t]}}(z) \leq \wedge\left\{I_{N_y^{[s,t]}}(z) \vee I_{N_x^{[s,t]}}(y)|y \in X\right\}$.
$F_{N_x^{[s,t]}}(z) = T_{\underline{C}^{[s,t]}(1_{X-\{z\}})}(x) \leq T_{\underline{C}^{[s,t]}(\underline{C}^{[s,t]}(1_{X-\{z\}})}(x) = \wedge\left\{T_{\underline{C}^{[s,t]}(1_{X-\{z\}})}(y) \vee F_{N_x^{[s,t]}}(y)|y \in X\right\}$
$= \wedge\left\{F_{N_y^{[s,t]}}(z) \vee F_{N_x^{[s,t]}}(y)|y \in X\right\}$,
Therefore C is transitive. When $\overline{C}^{[s,t]}(\overline{C}^{[s,t]}(A)) \subseteq \overline{C}^{[s,t]}(A)$, it can be proved C is transitive by the same way.

5. Application of interval neutrosophic covering rough sets

In medicine, a combination of drugs is usually used to cure a disease. Suppose, $X = \left\{x_j, j = 1, 2, \cdots, n\right\}$ is a collection of n drugs,$V = \{y_i, i = 1, 2, \cdots, m\}$ are m important symptom (such as fever, cough, fatigue, phlegm, etc.) of diseases (such as: 2019-NCOV, etc.), and $C_i(x_j)$ represents the effective value of medication for the treatment of symptoms.

Let $[s, t]$ be the evaluation range. For each drug$x_j \in X$, if there is at least one symptom $y_i \in V$ that causes the effective value of drug x_j for the treatment of symptom y_i to be in the $[s, t]$ interval, then$C = \{C_i : i = 1, 2, \cdots, m\}$ is the interval neutrosophic $[s, t]$ covering on X. Thus, for each drug x_j, we consider the set of symptoms $\left\{y_i : C_i(x_j) \geq [s, t]\right\}$.

The interval neutrosophic $[s, t]$ neighborhood of x_j is $N_{x_j}^{[s,t]} = \bigcap\{C_i \in C|T_{C_i}\left(x_j\right) \geq [s, t], I_{C_i}\left(x_j\right) \leq [1 - t, 1 - s], F_{C_i}\left(x_j\right) \leq [1 - t, 1 - s]\}(x_k) = \left(\bigcap_{C_i(x)\geq[s,t]} C_i\right) \subseteq C_i(x_k, k = 1, 2, \ldots, n$. This represents the effective value interval for each drugx_k for all symptoms in the symptom set $\left\{y_i : C_i(x_j) \geq [s,t]\right\}$. We consider as the upper and lower thresholds of effective values of s and t. If they are lower than the lower threshold, there will be no therapeutic effect; if they are higher than the upper threshold, the therapeutic effect will be too strong, and it is easy to cause other side effects to the body during the treatment (regardless of the situation of reducing the usage). Let an interval neutrosophic set of A represent the therapeutic ability of all drugs in X that can cure disease X. Since $A$is imprecise, we consider the approximation of A, that is, the lower approximation and the upper approximation of interval neutrosophic covering rough .

Example 3. Let$X$ be a space of a points (objects), with a class of elements in X denoted by x, being a interval neutrosophic covering of X, which is shown in Table 3. Set $[s,t] = [0.4, 0.5]$, and it can be gotten that C is a interval neutrosophic $[s,t]$ covering of X. $N_{x_1}^{[0.4,0.5]} = C_1 \bigcap C_2 \bigcap C_3, N_{x_2}^{[0.4,0.5]} = C_1 \bigcap C_4, N_{x_3}^{[0.4,0.5]} = C_2 \bigcap C_4, N_{x_4}^{[0.4,0.5]} = C_2 \bigcap C_3$. The interval neutrosophic $[s,t]$neighborhood of $x_i(i = 1,2,3,4)$ is shown in Table 4. Obviously, the interval neutrosophic$[s,t]$neighborhood of $x_i(i = 1,2,3,4)$ is covering of X.

Table 3. The interval neutrosophic $[0.4, 0.5]$ neighborhood of $x_i(i = 1,2,3,4)$.

	C_1	C_2	C_3	C_4
x_1	$\langle[0.4,0.5],[0.2,0.3],[0.4,0.5]\rangle$	$\langle[0.4,0.6],[0.1,0.3],[0.3,0.5]\rangle$	$\langle[0.7,0.9],[0.2,0.3],[0.4,0.5]\rangle$	$\langle[0.4,0.5],[0.3,0.4],[0.6,0.7]\rangle$
x_2	$\langle[0.6,0.7],[0.1,0.2],[0.2,0.3]\rangle$	$\langle[0.2,0.4],[0.1,0.2],[0.2,0.3]\rangle$	$\langle[0.3,0.6],[0.3,0.5],[0.8,0.9]\rangle$	$\langle[0.5,0.7],[0.2,0.3],[0.4,0.6]\rangle$
x_3	$\langle[0.3,0.5],[0.2,0.3],[0.4,0.5]\rangle$	$\langle[0.5,0.6],[0.2,0.3],[0.3,0.4]\rangle$	$\langle[0.3,0.5],[0.2,0.4],[0.3,0.4]\rangle$	$\langle[0.5,0.6],[0.0,0.2],[0.3,0.4]\rangle$
x_4	$\langle[0.7,0.8],[0.6,0.7],[0.1,0.2]\rangle$	$\langle[0.6,0.7],[0.1,0.2],[0.1,0.3]\rangle$	$\langle[0.6,0.7],[0.3,0.4],[0.3,0.5]\rangle$	$\langle[0.3,0.5],[0.5,0.6],[0.6,0.7]\rangle$

Table 4. The interval neutrosophic $[0.4, 0.5]$ covering of X.

	C_1	C_2	C_3	C_4
x_1	$\langle[0.4,0.5],[0.2,0.3],[0.4,0.5]\rangle$	$\langle[0.2,0.4],[0.3,0.5],[0.8,0.9]\rangle$	$\langle[0.3,0.5],[0.2,0.4],[0.4,0.5]\rangle$	$\langle[0.6,0.7],[0.6,0.7],[0.3,0.5]\rangle$
x_2	$\langle[0.4,0.5],[0.3,0.4],[0.6,0.7]\rangle$	$\langle[0.5,0.7],[0.2,0.3],[0.4,0.6]\rangle$	$\langle[0.3,0.5],[0.2,0.3],[0.4,0.5]\rangle$	$\langle[0.3,0.5],[0.6,0.7],[0.6,0.7]\rangle$
x_3	$\langle[0.4,0.5],[0.3,0.4],[0.6,0.7]\rangle$	$\langle[0.2,0.4],[0.2,0.3],[0.4,0.6]\rangle$	$\langle[0.5,0.6],[0.2,0.3],[0.3,0.5]\rangle$	$\langle[0.3,0.5],[0.5,0.6],[0.6,0.7]\rangle$
x_4	$\langle[0.4,0.6],[0.2,0.3],[0.4,0.5]\rangle$	$\langle[0.2,0.4],[0.3,0.5],[0.8,0.9]\rangle$	$\langle[0.3,0.5],[0.2,0.4],[0.3,0.5]\rangle$	$\langle[0.6,0.7],[0.3,0.4],[0.3,0.5]\rangle$

Let A be an interval neutrosophic set, and
$A(x_1) = \langle[0.2,0.4],[0.2,0.4],[0.3,0.4]\rangle, A(x_2) = \langle[0.5,0.7],[0.1,0.3],[0.2,0.4]\rangle,$
$A(x_3) = \langle[0.3,0.4],[0.2,0.5],[0.3,0.5]\rangle, A(x_4) = \langle[0.5,0.6],[0.2,0.4],[0.4,0.6]\rangle.$
The lower approximation operator $\underline{C}^{[0.4,0.5]}(A)$ and the upper approximation operator $\overline{C}^{[0.4,0.5]}(A)$ of the intelligent set A in the interval can be obtained by definition 3.9.
$\underline{C}^{[0.4,0.5]}(A)(x_1) = \langle[0.4,0.5],[0.2,0.5],[0.4,0.6]\rangle, \underline{C}^{[0.4,0.5]}(A)(x_2) = \langle[0.4,0.5],[0.2,0.5],[0.3,0.5]\rangle,$
$\underline{C}^{[0.4,0.5]}(A)(x_3) = \langle[0.3,0.5],[0.2,0.5],[0.3,0.5]\rangle, \underline{C}^{[0.4,0.5]}(A)(x_4) = \langle[0.3,0.5],[0.2,0.5],[0.4,0.6]\rangle.$
$\overline{C}^{[0.4,0.5]}(A)(x_1) = \langle[0.5,0.6],[0.2,0.4],[0.4,0.5]\rangle, \overline{C}^{[0.4,0.5]}(A)(x_2) = \langle[0.5,0.7],[0.2,0.3],[0.4,0.5]\rangle,$
$\overline{C}^{[0.4,0.5]}(A)(x_3) = \langle[0.3,0.5],[0.2,0.3],[0.3,0.5]\rangle, \overline{C}^{[0.4,0.5]}(A)(x_4) = \langle[0.5,0.6],[0.2,0.4],[0.3,0.5]\rangle.$
Then A is the interval neutrosophic $[s,t]$ covering of X.
And we can get that
(1) $A(x_2) \geq [0.4.0.5], \underline{C}^{[0.4,0.5]}(A)(x_2) \geq [0.4.0.5], \overline{C}^{[0.4,0.5]}(A)(x_2) \geq [0.4.0.5]$. Therefore, drug $x_2$plays an important role in the treatment of disease$A$.
(2) $A(x_3) < [0.4.0.5], \underline{C}^{[0.4,0.5]}(A)(x_3) < [0.4.0.5], \overline{C}^{[0.4,0.5]}(A)(x_3) < [0.4.0.5]$. So drug x_3 has no effect on the treatment of disease$A$.
(3) $A(x_1) < [0.4.0.5], \underline{C}^{[0.4,0.5]}(A)(x_1) \geq [0.4.0.5], \overline{C}^{[0.4,0.5]}(A)(x_1) \geq [0.4.0.5]$. Therefore, drug x_1 has

less effect on the treatment of disease A than drug x_2 and drug x_4.

6. Conclusions

In this paper, we propose the interval neutrosophic covering rough sets by combining the CRS and INS. Firstly, the paper introduces the definition of interval neutrosophic sets and covering rough sets, where the covering rough set is defined by neighborhood. Secondly, Some basic properties and operation rules of interval neutrosophic sets and covering rough sets are discussed. Thirdly, the definition of interval neutrosophic covering rough sets are proposed. Then, this paper put forward some theorems and give their proofs of interval neutrosophic covering rough sets. Lastly, we give the numerical example to apply the interval neutrosophic covering rough sets in the real life.

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Conflict of interest

The authors declare that there is no conflict of interest.

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