Maximality and Minimality of Ideals In Some Neutrosophic Rings

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Abstract: This paper studied the condition of maximal and minimal ideals in neutrosophic ring theory.

Keywords: Neutrosophic ring, refined neutrosophic ring, maximal ideal, minimal ideal, AH-ideal.

3. Ideals in Neutrosophic rings

Remark 3.1:

Since every neutrosophic ring R(I) can be understood as $R(I) = R + RI = (a + bI; a, b \in R\}$,

Then each subset of R(I) has the form $M = P + SI$; P,S are two subsets of R. We call P the real part, S the neutrosophic part of M.

An important question arises here. This question is:

When M is a neutrosophic ideal of R(I)?. In other words, what conditions on the real part P and neutrosophic part S which make M be an ideal?.

The following theorem clarifies the necessary and sufficient condition to answer the previous question.

Theorem 3.2:

Let R(I) be a neutrosophic ring, $M = P + SI$ be any subset of R(I), then

M is a neutrosophic ideal if and only if the following conditions are true:

(a) P is an ideal on R.

(b) P is contained in S.

(c) S is an ideal of R.

Proof:

Firstly, we assume that (a),(b), and (c) are true, we have:

$(M, +)$ is a subgroup of $(R(I), +)$, that is because if $a + bI, c + dI \in M; a, c \in P, b, d \in S$, we find

$(a+bI)-(c+dI)=(a-c)+(b-d)I \in M; a-c \in P, b-d \in S$..

Now, suppose that $a+bI \in M$ *and* $r=m+nI \in R(I)$, we have

$r.(a+bI)=m.a+I[m.b+n.b+n.a]$, by the assumption, we regard that $m.b+n.b \in S$, *and* $n.a \in P \leq S$, thus $r.(a+bI)=m.a+I[m.b+n.b+n.a] \in P+SI=M$, which means that M is a neutrosophic ideal of R(I).

Conversely, we suppose that $M=P+SI$ is a neutrosophic ideal of R(I). Let a, c be two arbitrary elements in P, and b, d be two arbitrary elements in S, we have $a+bI, c+dI \in M$, by using the assumption we have M as an ideal, hence $(a+bI)-(c+dI)=(a-c)+(b-d)I \in M=P+SI$, thus

$a-c \in P$, *and* $b-d \in S$, thus $(P,+),(S,+)$ are two subgroups of $(R,+)$.

For every $r \in R$, *we have* $r=r+0I \in R(I)$, *and* $r.(a+bI)=r.a+r.bI \in M=P+SI$, thus

$r.a \in P, r.b \in S$, this means that P, S are ideals in the classical ring R.

Now, we prove that P is contained in S. We have $(1-I) \in R(I)$, that is because R(I) has a unity 1. On the other hand, we can write $(1-I)(a+bI)=(a-aI) \in M=P+SI$, and that is because M is an ideal of R(I), hence $-a \in S$, thus $a \in S$, by regarding that a is an arbitrary element of P, we get that $P \leq S$.

The previous theorem ensures that each ideal is an AH-ideal, since P,S are supposed to be classical ideals of R.

Example 3.3:

Let $R=Z$ be the ring of integers, $R(I)=Z(I)=\{a+bI;\ a,b \in Z\}$ be the corresponding neutrosophic ring, we have:

(a) $P=<2>, Q=<4>, S=<3>$, are three ideals of R, with $Q \leq P$.

(b) $M=Q+PI=\{4m+2nI; m,n \in Z\}$ is an ideal of R(I).

(c) $N=P+SI=\{2m+3nI; m,n \in Z\}$ is not a neutrosophic ideal, that is because P is not contained in S.

Example 3.4:

Let $R = Z_8$ be the ring of integers modulo 8. $R(I) = \{a + bI; a, b \in Z_8\}$, be the corresponding neutrosophic ring. Consider the set $M = \{0,4,2I,4I,6I,4 + 2I,4 + 6I,4 + 4I\}$. We have M as an ideal of R(I), that is because $M =< 4 > +< 2 > I \; and < 4 >\leq< 2 >$.

Theorem 3.5:

The following theorem determines the form of maximal ideals in R(I).

Let R(I) be a neutrosophic ring, $M = P + SI$ be an ideal of R(I), then M is maximal if and only if P is maximal in R with $S = R$ or $M = R(I)$.

Proof:

Suppose that M is maximal of R(I), let $N = V + WI$ be any ideal of R(I) with the property $M \leq N$, then $P \leq V$ and $S \leq W$, by the assumption of the maximality of M, we find that $N = M \; or \; N = R(I)$, this implies that

$(V = P \; with \; W = R) or \; (V = W = R)$, which means that P is maximal in R or $P = R$. On the other hand $P \leq S$ and P is maximal, thus $S = P \; or \; S = R$. Since $P + SI \leq P + RI$, hence the only non trivial maximal ideal is $M = P + RI$, with P as a maximal ideal in R.

The converse is clear.

Theorem 3.6:

The following theorem describes minimal ideals in R(I).

Let R(I) be a neutrosophic ring, $M = P + SI$ be an ideal of R(I), then M is minimal if and only if S is minimal in R and $P = \{0\}$.

Proof:

Suppose that M is minimal of R(I), let $N = V + WI$ be any ideal of R(I) with the property $N \leq M$, then $V \leq P$ and $W \leq S$, by the assumption of the minimality of M, we find that $N = M \; or \; N = \{0\}$, this implies that

$(V = P \; with \; W = S) or \; (W = N = \{0\})$, which means that P, S are minimal in R. On the other hand $P \leq S$ and S is minimal, thus $S = P \; or \; P = \{0\}$. Since SI is a sub-ideal of P+SI, hence $P = \{0\}$.

The converse is clear.

Remark 3.7:

According to Theorem 5.1 and Theorem 6.1, we get a full description of the structure of maximal and minimal ideals in the neutrosophic ring R(I).

(a) Non trivial Maximal ideals in R(I) has the form {P+RI}, where P is maximal in R.

(b) Non trivial minimal ideals have the form {{0}+SI} where S is minimal in R.

Example 3.8:

Let Z(I) be the neutrosophic ring of integers, non trivial maximal ideals in Z(I) are

$\{< p > +ZI\}$, where p is any prime number.

4. Ideals in refined neutrosophic rings

Remark 4.1:

Since every refined neutrosophic ring R(I_1, I_2) can be understood as $R(I_1, I_2) = (R, RI_1, RI_2) = \{(a, bI_1, cI_2); a, b, c \in R\}$,

Then each subset of $R(I_1, I_2)$ has the form $M = (P, QI_1, SI_2)$; P, Q, S are two subsets of R.

An important question arises here. This question is:

When M is a refined neutrosophic ideal of $R(I_1, I_2)$?. In other words, what conditions on P, Q, S which make M an ideal?.

The following theorem clarifies the necessary and sufficient condition to answer the previous question.

Theorem 4.2:

Consider the following:

$R(I_1, I_2) = \{(a, bI_1, cI_2); a, b, c \in R\}$ be a refined neutrosophic ring, $M = (P, QI_1, SI_2)$ be a subset of $R(I_1, I_2)$

M is an ideal of $R(I_1, I_2)$ if and only if .

(a) P, Q, S are ideals on R

(b)$P \leq S \leq Q$.

Proof:

Suppose that M is an ideal, then we have for every $a, m \in P and\ b, n \in Q\ and\ c, t \in S$,

$x = (a, bI_1, cI_2), y = (m, nI_1, tI_2)$, are two elements of $R(I_1, I_2)$.

$x - y = (a - m, [b - n]I_1, [c - t]I_2) \in M, thus\ a - m \in P, b - n \in Q, c - t \in S, hence$(P, +), (Q, +), (S, +) are subgroups of $(R, +)$.

For every $r \in R, we\ have\ (r, 0,0) \in R(I_1, I_2) and (r, 0,0). (a, bI_1, cI_2) = (r.a, r.bI_1, r.cI_2) \in M$, thus $r.a \in P, r.b \in Q, r.c \in S$, thus P, Q, S are ideals of R.

On the other hand, we have $(1,0, -I_2) \in R(I_1, I_2), thus\ (1,0, -I_2).(a, bI_1, cI_2) = (a, 0, -aI_2) \in M, hence - a \in S\ and\ P \leq S$, that is because a is an arbitrary element in P.

Also, $(1, -I_1, 0) \in R(I_1, I_2), thus\ (1, -I_1, 0).(0, bI_1, cI_2) = (0, -cI_1, cI_2) \in M, hence - c \in Q\ and\ S \leq Q$. That is because c is an arbitrary element in S.

For the converse, we suppose that (a) and (b) are true, we have (M,+) as a subgroup of $R(I_1, I_2)$.

Let $r = (m, nI_1, tI_2) \in R(I_1, I_2) and x = (a, bI_1, cI_2) \in M$, we have

$r.x = (m.a, [m.b + n.a + n.b + n.c + t.b]I_1, [m.c + t.a + t.c]I_2)$, it is clear that

$m.c + t.c \in S, t.a \in P \leq S, thus\ m.a + t.a + t.c \in S$. Also,

$m.b + n.b + t.b \in Q, and\ n.a + n.c \in S \leq Q, thus\ m.b + n.a + n.b + n.c + t.b \in Q$. This implies that $r.x \in M$, hence M is an ideal.

Example 4.3:

Let $Z(I_1, I_2)$ be the refined neutrosophic ring of integers, we have

$(< 8 >, < 2 > I_1, < 4 > I_2) = \{(8a, 2bI_1, 4cI_2); a, b, c \in Z\}$ is an ideal in $Z(I_1, I_2)$. That is because

$< 8 > \leq < 4 > \leq < 2 >$.

Example 4.4:

Let $Z_{20}(I_1, I_2)$ be the refined neutrosophic ring of integers modulo 20, we have

$(0, < 5 > I_1, < 10 > I_2) = \{(0,0,0), (0,5I_1, 0), (0,5I_1, 10I_2), (0,10I_1, 0), (0,10I_1, 10I_2), (0,15I_1, 0), (0,15I_1, 10I_2), (0,0,10I_2\}$.

Theorem 4.5:

Consider the following:

$R(I_1, I_2) = \{(a, bI_1, cI_2); a, b, c \in R\}$ be a refined neutrosophic ring, $M = (P, QI_1, SI_2)$ be any non trivial maximal ideal of $R(I_1, I_2)$

M has the following form:

(P, RI_1, RI_2). Where P is any maximal ideal of R.

Proof:

We assume that M is a maximal ideal, and $N = (X, YI_1, ZI_2)$ *is an ideal of* $R(I_1, I_2)$ *with* $M \leq N$, hence $M = N$ *or* $N = R(I_1, I_2)$, we have $P = X, Q = Y, S = Z$, or $X = Y = Z = R$. This implies that P, S, Q should be maximal; but we have that

$P \leq S \leq Q$, hence $(R = S, Q = R; P \text{ is maximal in } R)$.

The converse is clear.

Theorem 4.6:

Consider the following:

$R(I_1, I_2) = \{(a, bI_1, cI_2); a, b, c \in R\}$ be a refined neutrosophic ring, $M = (P, QI_1, SI_2)$ be any non trivial minimal ideal of $R(I_1, I_2)$

M has the following form:

$(0, PI_1, 0)$. Where P is any minimal ideal of R.

Proof:

The proof is similar to Theorem 5.2.

Example 4.7:

(a) Consider $Z_8(I_1, I_2)$ the refined neutrosophic ring of integers modulo 8, we have $< 4 >= \{0,4\}$ is a minimal ideal of Z_8. Hence $(0, < 4 > I_1, 0) = \{(0,0,0), (0,4I_1, 0)\}$ is a minimal ideal of $Z_8(I_1, I_2)$.

(b) $< 2 >= \{2,4,6,0\}$ is maximal in $\boldsymbol{Z_8}$. Hence $(< 2 >, \boldsymbol{Z_8 I_1}, \boldsymbol{Z_8 I_2}) = \{(\boldsymbol{a}, \boldsymbol{bI_1}, \boldsymbol{cI_2}); \boldsymbol{a} \in < 2 >$ $\boldsymbol{\textit{and } b, c} \in \boldsymbol{Z_8}\}$ is maximal in $\boldsymbol{Z_8(I_1, I_2)}$

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