n- Refined Neutrosophic Rings

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Abstract

The aim of this paper is to introduce the concept of n-refined neutrosophic ring as a generalization of refined neutrosophic ring. Also, wepresent concept of n-refined polynomial ring. We study some basic concepts related to these rings such as AH-subrings, AH-ideals, AH-factors, and AH-homomorphisms.

Keywords: n-Refined neutrosophic ring, AH-ideal, AHS-ideal, AH-homomorphism, n-Refined neutrosophic polynomial ring.

1.Introduction

Neutrosophy as a new branch of philosophy founded by F.Smarandache became a useful tool in algebraic studies.Many neutrosophic algebraic structures were defined and studied such as neutrosophic groups, neutrosophic rings, and neutrosophic vector spaces. (See [1,2,3,4,5,6]).Refined neutrosophic theory was introduced by Smarandache in 2013 when he extended the neutrosophic set / logic / probability to refined [n-valued] neutrosophic set / logic / probability respectively, i.e. the truth value T is refined/split into types of sub-truths such as (T_1, T_2, …), similarly indeterminacy I is refined/split into types of sub-indeterminacies (I_1, I_2, …) and the falsehood F is refined/split into sub-falsehood (F_1, F_2,..)[10]. In [9], Smarandache proposed a way to split the Indeterminacy element I into n sub-indeterminacies $I_1, I_2, \dots, I_n$. This idea is very interesting and helps to define new generalizations of refined neutrosophic algebraic structures.

For our purpose we define multiplication operation between indeterminacies $I_1, I_2, \dots, I_n$ as follows:

$I_m I_s = I_{\min(m,s)}$. For examples if $n = 4$ we get

$I_4 I_2 = I_2, I_1 I_2 = I_1, I_2 I_3 = I_2$. If $n = 6$ we get $I_2 I_4 = I_2, I_1 I_4 = I_1, I_4 I_5 = I_4$. If $n = 2$ we get $I_1 I_2 = I_1$ (2-refined neutrosophic ring).

AH-subtructures were firstly defined in [1]. AH-ideal in a neutrosophic ring R(I) has the form P+QI, where P,Q are ideals in the ring R. We can understand these substructures as two sections, each one is ideal (in rings). These ideals are interesting since they have properties which are similar to classical ideals and they lead us to study the concept of AHS-homomorphisms which are ring homomorphisms but not neutrosophic homomorphisms. In this article we aim to define these ideals in n-refined neutrosophic rings too.

2. Preliminaries

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Definition 2.1: [7]

Let (R,+,×) be a ring, $R(I) = \{a + bI : a, b \in R\}$ is called the neutrosophic ring where I is a neutrosophic element with condition $I^2 = I$.

Remark 2.2: [4]

The element I can be split into two indeterminacies I_1 , I_2 with conditions:

$I_1^2 = I_1$, $I_2^2 = I_2$, $I_1 I_2 = I_2 I_1 = I_1$.

Definition 2.3: [4]

If X is a set then $X(I_1, I_2) = \{(a, bI_1, cI_2): a , b , c \in X \}$ is called the refined neutrosophic set generated by X , I_1, I_2.

Definition 2.4: [4]

Let (R,+,×) be a ring, $(R(I_1, I_2) , + ,\times)$ is called a 2-refined neutrosophic ring generated by R ,I_1, I_2.

Theorem 2.5: [4]

Let $(R(I_1, I_2) , + ,\times)$ be a 2-refined neutrosophic ring then it is a ring.

In the following we remind the reader about some AH-substructures.

Definition 2.6: [2]

Let $(R(I_1, I_2) , + ,.)$ be a refined neutrosophic ring and P_0, P_1, P_2 be ideals in the ring R then the set $P = (P_0, P_1 I_1, P_2 I_2) = \{(a, bI_1, cI_2): a \in P_0, b \in P_1, c \in P_2\}$ is called a refined neutrosophic AH-ideal.

If $P_0 = P_1 = P_2$ then P is called a refined neutrosophic AHS-ideal.

Definition 2.7: [1]

Let R be a ring and R(I) be the related neutrosophic ring and

$P = P_0 + P_1 I = \{a_0 + a_1 I ; a_0 \in P_0 , a_1 \in P_1\}$; P_0, P_1 are two subsets of R.

(a) We say that P is an AH-ideal if P_0, P_1 are ideals in the ring R.

(b) We say that P is an AHS-ideal if $P_0 = P_1$.

Definition 2.8: [2]

(a) Let $f: R(I_1, I_2) \to T(I_1, I_2)$ be an AHS-homomorphism we define AH-Kernel of f by : $AH - Kerf = \{ (a, bI_1, cI_2); a , b , c \in Kerf_R \} = (Kerf_R , Kerf_R I_1 , Kerf_R I_2)$

(b) let $S=(S_0 , S_1 I_1, S_2 I_2)$ be a subset of $R(I_1, I_2)$, then : $f(S) = (f_R(S_0), f_R(S_1) I_1, f_R(S_2) I_2) = \{(f_R(a_0), f_R(a_1) I_1, f_R(a_2) I_2); a_i \in S_i\}$.

(c) let $S=(S_0 , S_1 I_1, S_2 I_2)$ be a subset of $T(I_1, I_2)$. Then

$f^{-1}(S) = (f_T^{-1}(S_0), f_T^{-1}(S_1) I_1, f_T^{-1}(S_2) I_2)$.

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Definition 2.9: [2]

Let $f: \mathrm{R}(I_1, I_2) \rightarrow \mathrm{T}(I_1, I_2)$ be an AHS-homomorphism we say that f is an AHS-isomorphism if it is a bijective map and $\mathrm{R}(I_1, I_2)$, $\mathrm{T}(I_1, I_2)$ are called AHS-isomorphic refined neutrosophic rings.

It is easy to see that f_R will be an isomorphism between R , T .

Theorem 2.10 :

Let $f: \mathrm{R}(I_1, I_2) \rightarrow \mathrm{T}(I_1, I_2)$ be an AHS-homomorphism then we have :

(a) AH-Kerf is an AHS-ideal of $\mathrm{R}(I_1, I_2)$.

(b) If P is a refined neutrosophic AH-ideal of $\mathrm{R}(I_1, I_2)$, f(P) is a refined neutrosophic AH-ideal of $\mathrm{T}(I_1, I_2)$.

(c) If P is a refined neutrosophic AHS-ideal of $\mathrm{R}(I_1, I_2)$, f(P) is a refined neutrosophic AHS-ideal of $\mathrm{T}(I_1, I_2)$.

3. n-Refined neutrosophic rings

Definition 1.3:

Let (R,+,×) be a ring and $I_k; 1 \leq k \leq n$ be n indeterminacies. We define R_n(I)=$\{a_0 + a_1 I + \cdots + a_n I_n;\ a_i \in R\}$ to be n-refined neutrosophic ring. If n=2 we get a ring which is isomorphic to 2-refined neutrosophic ring $R(I_1, I_2)$.

Additionand multiplication on R_n(I)are defined as:

$\sum_{i=0}^{n} x_i I_i + \sum_{i=0}^{n} y_i I_i = \sum_{i=0}^{n} (x_i + y_i) I_i$, $\sum_{i=0}^{n} x_i I_i \times \sum_{i=0}^{n} y_i I_i = \sum_{i,j=0}^{n} (x_i \times y_j) I_i I_j$.

Where × is the multiplication defined on the ring R.

It is easy to see that R_n(I) is a ring in the classical concept and contains a proper ring R.

Definition 2.3:

Let R_n(I) be an n-refined neutrosophic ring, it is said to be commutative if $xy = yx$ for each x , y ∈ R_n(I), if there is 1∈ R_n(I) such $1.x = x.1 = x$, then it is called an n-refined neutrosophic ring with unity.

Theorem 3.3:

Let R_n(I) be an n-refinedneutrosophic ring. Then

(a) R is commutative if and only if R_n(I) is commutative,

(b) R has unity if and only if R_n(I) has unity,

(c) R_n(I) = $\sum_{i=0}^{n} R I_i$ = $\{\sum_{i=0}^{n} x_i I_i : x_i \in R\}$.

Proof:

(a) Holds directly from the definition of multiplication on R_n(I).

(b) If 1 is a unity of R then for each $a_0 + a_1 I + \cdots + a_n I^n \in R_n$(I) we have

1·$(a_0 + a_1 I + \cdots + a_n I_n) = (a_0 + a_1 I + \cdots + a_n I_n)$·1=$a_0 + a_1 I + \cdots + a_n I_n$ so 1 is the unity of R_n(I).

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(c) It is obvious that $\sum_{i=0}^{n} RI_i \leq R(I)$. Conversely assume that $a_0 + a_1I + \cdots + a_nI_n \in R_n$(I) then by the definition we have that $a_0 + a_1I + \cdots + a_nI_n \in \sum_{i=0}^{n} RI_i$.Thus the proof is complete.

Definition 4.3:

(a) Let R_n(I) be an n-refined neutrosophic ring and P = $\sum_{i=0}^{n} P_iI_i$ = { $a_0 + a_1I + \cdots + a_nI_n$: $a_i \in P_i$} where P_i is a subset of R, we define P to be an AH-subring if P_i is a subring of R for all i, AHS-subring is defined by the condition $P_i = P_j$ for all i , j.

(b)P is an AH-ideal if P_i is an two sides ideal of R for all i, the AHS-ideal is defined by the condition $P_i = P_j$ for all i , j.

(c) The AH-ideal P is said to be null if $P_i = R$ or $P_i = \{0\}$ for all i.

Theorem 5.3:

Let R_n(I) be an n-refined neutrosophic ring and P is an AH-ideal,(P,+) is an abelian neutrosophic group with $k \leq n$ and r.p$\in P$ for all p $\in P$ and r $\in R$.

Proof :

Since P_i is abelian subgroup of $(R, +)$ and $r.x \in P_i$ for all $r \in R$, $x \in P_i$, the proof holds.

Remark 6.3:

We can define the right AH-ideal by the condition that P_i is a right ideal of R, the left AH-ideal can be defined as the same.

Definition 7.3:

Let R_n(I) be an n-refined neutrosophic ring and P = $\sum_{i=0}^{n} P_iI_i$, Q = $\sum_{i=0}^{n} Q_iI_i$be two AH-ideals then we define:

P+Q = $\sum_{i=0}^{n}(P_i + Q_i)I_i$, P∩Q = $\sum_{i=0}^{n}(P_i \cap Q_i)I_i$.

Theorem 8.3:

Let R_n(I) be an n-refined neutrosophic ring and P = $\sum_{i=0}^{n} P_iI_i$, Q = $\sum_{i=0}^{n} Q_iI_i$ be two AH- ideals then P+Q, P∩Q are AH-ideals. If P, Q are AHS-ideals then P+Q, P∩Q are AHS-ideals.

Proof :

Since $P_i + Q_i$, $P_i \cap Q_i$ are ideals of R then P+Q, P∩Q are AH-ideals of R_n(I).

Definition 9.3:

Let R_n(I) be an n-refined neutrosophic ring and P = $\sum_{i=0}^{n} P_iI_i$be an AH-ideal then the AH- radical of P can be defined as $H - rad(P) = \sum_{i=0}^{n}(\sqrt{P_i})I_i$.

Theorem 10.3:

The AH-radical of an AH-ideal is an AH-ideal.

Proof :

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Since $\sqrt{P_i}$ is an ideal of R then $AH - Rad(P)$ is an AH-ideal of R_n(I).

Definition 11.3:

Let R_n(I) be an n-refined neutrosophic ring and P = $\sum_{i=0}^{n} P_i I_i$ be an AH-ideal, we define AH-factor R(I)/P = $\sum_{i=0}^{n}(R/P_i)I_i = \sum_{i=0}^{n}(x_i + P_i)I_i \,; x_i \in R$.

Theorem 12.3:

Let R_n(I) be an n-refined neutrosophic ring and P = $\sum_{i=0}^{n} P_i I_i$ be an AH-ideal:

R_n(I)/P is aring with the following two binary operations

$\sum_{i=0}^{n}(x_i + P_i)I_i + \sum_{i=0}^{n}(y_i + P_i)I_i = \sum_{i=0}^{n}(x_i + y_i + P_i)I_i$,

$\sum_{i=0}^{n}(x_i + P_i)I_i \times \sum_{i=0}^{n}(y_i + P_i)I_i = \sum_{i=0}^{n}(x_i \times y_i + P_i)I_i$.

Proof :

Proof is similar tothat of Theorem 3.9 in [1].

Definition 13.3:

(a) Let R_n(I), T_n(I) be two n-refined neutrosophic rings respectively, and $f_R\colon R \to T$ be a ring homomorphism. We define n-refined neutrosophic AHS-homomorphism as :

$f\colon R_n$(I)→ T_n(I); $f(\sum_{i=0}^{n} x_i I_i) = \sum_{i=0}^{n} f_R(x_i)I_i$.

(b)f is an n-refined neutrosophic AHS-isomorphism if it is a bijective n-refined neutrosophic AHS-homomorphism.

(c) AH-Ker f = $\sum_{i=0}^{n} Ker(f_R)I_i$= $\{\sum_{i=0}^{n} x_i I_i \,;\ x_i \in Ker\ f_R\}$.

Theorem 14.3:

Let R_n(I), T_n(I) be two n-refined neutrosophic rings respectively and f be an n-refined neutrosophic AHS-homomorphism $f\colon R_n$(I)→ T_n(I). Then

(a) If P = $\sum_{i=0}^{n} P_i I_i$is an AH- subring of R_n(I) then f(P) is an AH- subring of T_n(I),

(b) If P = $\sum_{i=0}^{n} P_i I_i$ is an AHS- subring of R_n(I) then f(P) is an AHS- subring of T_n(I),

(c) If P = $\sum_{i=0}^{n} P_i I_i$ is an AH-ideal of R_n(I) then f(P) is an AH-ideal of f(R_n(I)),

(d)P = $\sum_{i=0}^{n} P_i I_i$ is an AHS-ideal of R_n(I) then f(P) is an AHS-ideal of f(R_n(I)),

(e) R_n(I)$/AH - Ker(f)\ is\ AHS - isomorphic\ to\ f\big(R(I)\big)$,

(f) Inverse image of an AH-ideal P in T_n(I) is an AH-ideal in R(I).

Proof :

(a) Since $f(P_i)$ is a subring of T then f(P) is an AH- subring of T_n(I).

(b) Holds by a similar way to (a).

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(c) Since $f(P_i)$ is an ideal of f(R) then f(P) is an AH- ideal of f(R(I)).

(d) It is similar to (c).

(e) We have $R/Ker(f_R) \cong f(R)$, by definition of AH-factor and $AH - Ker(f)$ we find that $R(I)/P \cong f(R(I))$.

(f) It is similar to the classical case.

Definition15.3:

(a) Let R(I) be a commutative n-refined neutrosophic ring, and P = $\sum_{i=0}^{n} P_i I_i$ be an AH- ideal, we define P to be a weak prime AH-ideal if P_i is a prime ideal of R for all i .

(b) P is called a weak maximal AH-ideal if P_i is a maximal ideal of R for all i.

(c) P is called a weak principal AH-ideal if P_i is a principal ideal of R for all i.

Theorem16.3:

Let R_n(I), T_n(I) be two commutative n-refined neutrosophic rings with an n-refined neutrosophic AHS-homomorphism $f: R_n$(I)→ T_n(I):

(a) If P = $\sum_{i=0}^{n} P_i I_i$ is an AHS- ideal of R_n(I) and Ker(f_R) $\leq P_i \neq R_n$(I):

(a) P is a weak prime AHS-ideal if and only if f(P) is a weak prime AHS-ideal in f(R_n(I)).

(b) P is a weak maximal AHS-ideal if and only if f(P) is a weak maximal AHS-ideal in f(R_n(I)).

(c) If Q = $\sum_{i=0}^{n} Q_i I_i$ is an AHS-ideal of T_n(I)then it is a weak prime AHS-ideal if and only if $f^{-1}(Q)$ is a weak prime inR_n(I).

(d)if Q = $\sum_{i=0}^{n} Q_i I_i$ is an AHS-ideal of T_n(I) then it is a weak maximal AHS-ideal if and only if $f^{-1}(Q)$ is a weak maximal in R_n(I).

Proof :

Proof is similar to that of Theorem 3.8 in [1].

Example 17.3:

Let $R = Z$ be the ring of integers, $T = Z_6$ be thering of integers modulo 6 with multiplication and addition modulo 6, we have:

(a)$f_R: R \to T; f(x) = x \, mod 6$ is a ring homomorphism, $ker(f_R) = 6Z,$ the corresponding AHS-homomorphism between R_4(I), T_4(I) is:

$f: R_4(\mathrm{I}) \to T_4(\mathrm{I}); f(a + bI_1 + cI_2 + dI_3 + eI_4) = (a \, mod 6) + (b \, mod 6)I_1 + (c \, mod 6)I_2 + (d \, mod 6)I_3 + (e \, mod 6)I_4; a, b, c, d, e \in Z.$

(b) $P =< 2 >, Q =< 3 >$are two prime and maximal and principal ideals in R,

M= $P + PI_1 + QI_2 + QI_3 + PI_4 = \{(2a + 2bI_1 + 3cI_2 + 3dI_3 + 2eI_4; a, b, c, d, e \in Z\}$ is a weak prime/ maximal AH-ideal of R_4(I).

(c) $Ker(f_R) = 6Z \leq P, Q, f_R(P) = \{0,2,4\}, f_R(Q) = \{0,3\}$,

$f(M) = f(P) + f(P)I_1 + f(Q)I_2 + f(Q)I_3 + f(P)I_4$ which is a weak maximal/ prime/principal AH-ideal of T_4(I).

(d) $AH - Ker(f) = 6Z + 6ZI_1 + 6ZI_2 + 6ZI_3 + 6ZI_4$ which is an AHS-ideal of $R_4(I)$.

(e) $R_4(I)/AH - Kerf = R/6Z + R/6Z\,I_1 + R/6Z\,I_2 + R/6Z\,I_3 + R/6Z\,I_4$ which is AHS-isomorphic to $f\big(R_4(I)\big) = T_4(I)$, since $R/6Z \cong T$.

Example 18.3:

Let $R = Z_8$ be a ring with addition and multiplication modulo 8.

(a) 3-refined neutrosophic ring related with R is Z_{8_3}(I)={a+$bI_1 + cI_2 + dI_3; a, b, c, d \in Z_8$}.

(b) P={0,4} is an ideal of R, $\sqrt{P}$={0,2,4,6}, $M = P + PI_1 + PI_2 + PI_3$ is an AHS-ideal of Z_{8_3}(I),

$AH - Rad(M) = \sqrt{P} + \sqrt{P}I_1 + \sqrt{P}I_2 + \sqrt{P}I_3$ which is an AHS-ideal of Z_{8_3}(I).

Example 19.3:

Let R=Z_2 the ring of integers modulo 2, let $n = 3$. The corresponding 3-refined neutrosophic ring is

Z_{2_3}(I)={$0, 1, I_1, I_2, I_3, 1 + I_1, 1 + I_2, 1 + I_3, I_1 + I_2, I_1 + I_3, I_1 + I_2 + I_3, I_2 + I_3, 1 + I_1 + I_2 + I_3, 1 + I_2 + I_3, 1 + I_1 + I_3, 1 + I_1 + I_2$}.

4. n-Refined neutrosophic polynomial rings

Definition1.4:

Let R_n(I) be a commutative n-refined neutrosophic ring and $P: R_n$(I) → R_n(I) is a function defined as $P(x) = \sum_{i=0}^{m} a_i x^i$ such $a_i \in R_n$(I), we call P a neutrosophic polynomial on R_n(I).

We denote byR_n(I)[x] to the ring of neutrosophic polynomials over R_n(I).

Since R_n(I) is a classical ring then R_n(I)[x] is a classical ring.

Theorem 2.4:

Let R(I) be a commutative n-refined neutrosophic ring. Then R_n(I)[x] = $\sum_{i=0}^{n}$ R$[x]I_i$.

Proof :

Let P(x) = $\sum_{i=0}^{n} P_i(x)I^i \in \sum_{i=0}^{n}$ R$[x]I^i$,by rearranging the previous sum we can write it as P(x) = $\sum_{i=0}^{m} a_i x^i \in R_n(I)[x]$.

Conversely, if P(x) = $\sum_{i=0}^{n} a_i x^i \in R_n(I)[x]$, then we can write it as

P(x) = $\sum_{i=0}^{n} P_i(x)I_i \in \sum_{i=0}^{n} R[x]I_i$, by the previous argument we find the proof.

Example 3.4:

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Let $Z_3(I)$ be a 3-refined neutrosophic ring and P(x) = I_1 + (2+I_1) x + (I_1+I_3)x^2 a polynomial over ${Z_3}_n(I)$, then we can write P(x) = 2x + I_1 (1 + x + x^2) + I_2x^2.

It is obvious that R_n(I)≤ R_n(I)$[x]$.

Definition4.4:

Let P(x) = $\sum_{i=0}^{n} P_i(x)I^i$ a neutrosophic polynomial over R_n(I) we define the degree of P by deg P = max(deg P_i).

5. Conclusion

In this paper we have defined the n-refined neutrosophic ring and n-refined neutrosophic polynomial ring, we have introduced and studied AH-structures such as:

AH-ideal, AHS-ideal, AH-weak principal ideal, AH-weak prime ideal. Authors hope that other n-refined neutrosophic algebraic structures will be defined in future research.

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