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New Dombi aggregation operators on bipolar neutrosophic set with application in multi-attribute decision-making problems

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Abstract. In this paper, we investigate two new Dombi aggregation operators on bipolar neutrosophic set namely bipolar neutrosophic Dombi prioritized weighted geometric aggregation (BNDPWGA) and bipolar neutrosophic Dombi prioritized ordered weighted geometric aggregation (BNDPOWGA) by means of Dombi t-norm (TN) and Dombi t-conorm (TCN). We discuss their properties along with proofs and multi-attribute decision making (MADM) methods in detail. New algorithms based on proposed models are presented to solve multi-attribute decision-making (MADM) problems. In contrast, with existing techniques a comparison analysis of proposed methods are also demonstrated to test their validity, accuracy and significance.

Keywords: Bipolar neutrosophic set, bipolar neutrosophic Dombi prioritized aggregation operators, decision-making environment

1. Introduction

In fuzzy theory, a newly defined model is not accessible unless it reduce (overcome) drawbacks of previously defined related models. Due to vagueness and uncertainties issues in many daily life problems, routine mathematics is not always available. To deal with such issues, various procedures such as hypothesis of probability, rough set hypothesis, and fuzzy set hypotheses have been considered as alternative models. Unfortunately, most of such alternative mathematics has its own down sides and drawbacks. For instance, most of the words like, genuine, lovely, best, renowned are not measurable and are, in fact, ambiguous. The criteria for words like wonderful, best, renowned etc., fluctuate from individual to individual. To handle such type of ambiguous and uncertain information, Zadeh [1] initiated the study of possibility based on participation function, that assigns an enrollment grade in [0, 1], called fuzzy set (FSs). FSs have only one membership degree and cannot handle complex problems. The concept of intuitionistic fuzzy set (IFS) was introduced by Atanassov [2]. The intuitionistic fuzzy set (IFS) includes both membership degree and non-membership degree. Subsequently, the concept of interval-valued intuitionistic fuzzy set (IVIFS) was introduced by Atanassov and Gargov [3]. Note that IVIFS is a generalization of IFS. In decision-making problems, IFS and IVIFS are not good at solving problems regarding inconsistent information. The concept of neutrosophic set (NS) was introduced by

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Smarandache [4]. The neutrosophic set (NS) includes truth, falsity and indeterminacy membership degree which is used to characterize incomplete, inconsistent and uncertain information. Wang et al. [5, 6] introduced the concept of interval neutrosophic set (INS) and the concept of single-valued neutrosophic set (SVNS) to apply NS in daily life decision-making problems.

When it comes to decision-making problems, we think about positive and negative consequences before taking a decision. Positive information explains why a decision is acceptable, permitted, possible, described, or satisfactory . While a negative information explains why a decision is rejected, forbidden, or impossible [7]. Satisfactory or acceptable perceptions are positive preferences, while unsatisfactory or unacceptable perceptions are negative preferences. Positive preferences are related with desires, while negative preferences relate to the constraints [8]. For example, when a decision maker examines an object, he or she need to explain that why he or she considered a satisfaction or an accessible object. Similarly, one has to explain why he or she considered a dissatisfaction or an unaccessible object [9]. In [10], Zhang introduced the concept of bipolar fuzzy set (BFS) consisting of positive membership degree and negative membership degree to describe the words like satisfaction and accessible. Deli et al. introduced the concept of bipolar neutrosophic set (BNS) which describes fuzzy, bipolar, inconsistent and uncertain information in [11]. In [12], Dey et al. proposed the bipolar neutrosophic TOPSIS (BN-TOPSIS) method to solve MADM problems under bipolar neutrosophic fuzzy environments. Zhang et al. proposed the methods based on the Frank Choquet Bonferroni Mean Operators to solve MADM problems under bipolar neutrosophic fuzzy environments in [9]. Many researchers investigated different kind of operators with application in decision-making problems in [14–18].

The MADM model refers to making decisions when there are multiple but a finite list of alternatives and attributes. Dombi [13] introduced new triangular norms which are Dombi TN and Dombi TCN. Dombi TN and Dombi TCN showed good flexibility with operational parameters. Until now, Dombi operations have not been extended to aggregate bipolar neutrosophic fuzzy environments.

The compatibility and validity of a newly defined fuzzy operator over different fuzzy environments is always challenging and worthwhile. Also, a comparative study analysis with perviously defined MADM methods is of great intrust. If a newly defined fuzzy operator can be reduced to MADM method with a batter accuracy then this newly defined operator would be more flexible, charming and useful. The motivation behind this work was to introduce new operators based on Dombi aggregation operator with a batter accuracy in contrast to existing operators.

In this work, we extend Dombi operations to aggregate bipolar neutrosophic fuzzy environments. Also, we establish new aggregation operators based on the combination of bipolar neutrosophic numbers(BNNs) and Dombi operations. We have proposed two new bipolar neutrosophic Dombi aggregation operators to aggregate bipolar neutrosophic fuzzy information, and developed MADM methods based on BNDPWGA, and BNDPOWGA operators to solve MADM problems with bipolar neutrosophic fuzzy information. The flexibility and accuracy over a multi-attributed problem has been demonstrated and verified as an alternative multi-criteria decision making tool. This was the true motivation behind this work.

The rest of the paper is arranged as follows. Section 2 reviews some of the basic definitions and concepts which will be used frequently. Section 3 defines Dombi operations of bipolar neutrosophic numbers (BNNs). Section 4 and Section 5 propose new operators (BNDPWGA and BNDPOWGA) together with their properties and comprehensive MADM methods based on proposed bipolar neutrosophic Dombi aggregation operators. Section 6 provides a numerical example for the selection of cultivating crops. Section 7 confers the effect of parameters and a comparative analysis with existing methods.

2. Preliminaries

In this section, we present a brief survey of few fundamentals of different sorts of sets which will be utilized in sequel.

2.1. Bipolar neutrosophic set and bipolar neutrosophic number

Definition 2.1. [11] Let U be a fixed set. Then, BNS N can be defined as follows: $N(u) = \{< \tau_N^+(u),\ \omega_N^+(u),\ \mho_N^+(u),\ \tau_N^-(u),\ \omega_N^-(u), \mho_N^-(u) > |u \in U\}$. Where $\tau_N^+, \omega_N^+, \mho_N^+ : U \longrightarrow [0, 1]$ and $\tau_N^-, \omega_N^-, \mho_N^- : U \longrightarrow [-1, 0]$. The positive membership degrees $\tau_N^+(u), \omega_N^+(u), \mho_N^+(u)$ are the truth membership, indeterminacy membership degree and falsity membership degree of an element $u \in U$ corresponding to BNS N and the negative membership degrees $\tau_N^-(u), \omega_N^-(u), \mho_N^-(u)$ denote the truth membership degree, indeterminacy membership degree and falsity membership degree of an element $u \in U$ to some implicit counter property corresponding to a BNS N.

In particular, if U has only one element, then $N(u) =< \tau_N^+(u), \omega_N^+(u), \mho_N^+(u), \tau_N^-(u), \omega_N^-(u), \mho_N^-(u) >$ is called bipolar neutrosophic number(BNN). For convenience, BNN $N =< \tau_N^+(u), \omega_N^+(u), \mho_N^+(u), \tau_N^-(u), \omega_N^-(u), \mho_N^-(u) >$ is also denoted as $N =< \tau_N^+, \omega_N^+, \mho_N^+, \tau_N^-, \omega_N^-, \mho_N^- >$.

Deli et al. [11] defined the algebraic operations of BNNs which are as follows:

Definition 2.2. Let $\tilde{\aleph}_1 =< \tau_1^+, \omega_1^+, \mho_1^+, \tau_1^-, \omega_1^-, \mho_1^- >$ and $\tilde{\aleph}_2 =< \tau_2^+, \omega_2^+, \mho_2^+, \tau_2^-, \omega_2^-, \mho_2^- >$ be two BNNs. Then, algebraic operations of BNNs are defined as follows:

(1) $\tilde{\aleph}_1 \oplus \tilde{\aleph}_2 =< \tau_1^+ + \tau_2^+ - \tau_1^+\tau_2^+, \omega_1^+\omega_2^+, \mho_1^+\mho_2^-, -\tau_1^-\tau_2^-, -(-\omega_1^- - \omega_2^- - \omega_1^-\omega_2^-), -(-\mho_1^- - \mho_2^- - \mho_1^-\mho_2^-) >$,

(2) $\tilde{\aleph}_1 \otimes \tilde{\aleph}_2 =< \tau_1^+\tau_2^+, \omega_1^+ + \omega_2^+ - \omega_1^+\omega_2^+, \mho_1^+ + \mho_2^+ - \mho_1^+\mho_2^+, -(-\tau_1^- - \tau_2^- - \tau_1^-\tau_2^-), -\omega_1^-\omega_2^-, -\mho_1^-\mho_2^- >$,

(3) $\ell.\tilde{\aleph}_1 =< 1 - (1 - \tau_1^+)^\ell, (\omega_1^+)^\ell, (\mho_1^+)^\ell, (-\tau_1^-)^\ell, -(1 - (1 - (-\omega_1^-)^\ell)), -(1 - (1 - (-\mho_1^-)^\ell)) > (\ell > 0)$,

(4) $\tilde{\aleph}_1^\ell =< (\tau_1^+)^\ell, 1 - (1 - \omega_1^+)^\ell, 1 - (1 - \mho_1^+)^\ell, -(1 - (1 - (-\tau_1^-)^\ell)), -(-\omega_1^-)^\ell, -(-\mho_1^-)^\ell > (\ell > 0)$.

For comparing two BNNs, Deli et al. [11] developed a comparison method which consists of the score function, accuracy function and certainty function.

Definition 2.3. [11] Let $\tilde{\aleph}_1 =< \tau_1^+, \omega_1^+, \mho_1^+, \tau_1^-, \omega_1^-, \mho_1^- >$ be BNN, then the score function $s(\tilde{\aleph}_1)$, accuracy function $a(\tilde{\aleph}_1)$ and certainty function $c(\tilde{\aleph}_1)$ are defined as:

$$s(\tilde{\aleph}_1) = \frac{\tau_1^+ + 1 - \omega_1^+ + 1 - \mho_1^+ + 1 + \tau_1^- - \omega_1^- - \mho_1^-}{6} \quad (1)$$

$$a(\tilde{\aleph}_1) = (\tau_1^+ - \mho_1^+) + (\tau_1^- - \mho_1^-) \quad (2)$$

$$c(\tilde{\aleph}_1) = 7\tau_1^+ - \mho_1^- \quad (3)$$

The comparison method of BNNs can be obtained based on Equations (1) − (3) as follows.

Definition 2.4. [11] Let $\tilde{\aleph}_1 =< \tau_1^+, \omega_1^+, \mho_1^+, \tau_1^-, \omega_1^-, \mho_1^- >$ and $\tilde{\aleph}_2 =< \tau_2^+, \omega_2^+, \mho_2^+, \tau_2^-, \omega_2^-, \mho_2^- >$ be two BNNs, therefore

(1) if $s(\tilde{\aleph}_1) > s(\tilde{\aleph}_2)$, then $\tilde{\aleph}_1 > \tilde{\aleph}_2$,

(2) if $s(\tilde{\aleph}_1) = s(\tilde{\aleph}_2)$ and $a(\tilde{\aleph}_1) > a(\tilde{\aleph}_2)$, then $\tilde{\aleph}_1 > \tilde{\aleph}_2$,

(3) if $s(\tilde{\aleph}_1) = s(\tilde{\aleph}_2)$, $a(\tilde{\aleph}_1) = a(\tilde{\aleph}_2)$ and $c(\tilde{\aleph}_1) > c(\tilde{\aleph}_2)$, then $\tilde{\aleph}_1 > \tilde{\aleph}_2$,

(4) if $s(\tilde{\aleph}_1) = s(\tilde{\aleph}_2)$, $a(\tilde{\aleph}_1) = a(\tilde{\aleph}_2)$ and $c(\tilde{\aleph}_1) = c(\tilde{\aleph}_2)$, then $\tilde{\aleph}_1 \sim \tilde{\aleph}_2$.

2.2. Dombi operations

The Dombi product and Dombi sum are special cases of TN and TCN respectively, and are given in the following definition.

Definition 2.5. [13] Let $\aleph_1$ and $\aleph_2$ be any two real numbers, then Dombi TN and Dombi TCN are defined in the following expressions:

$$\aleph_1 \otimes_D \aleph_2 = Dom(\aleph_1, \aleph_2) = \frac{1}{1 + \{(\frac{1-\aleph_1}{\aleph_1})^\lambda + (\frac{1-\aleph_2}{\aleph_2})^\lambda\}^{\frac{1}{\lambda}}} \quad (4)$$

and

$$\aleph_1 \oplus_D \aleph_2 = Dom^*(\aleph_1, \aleph_2) = 1 - \frac{1}{1 + \{(\frac{\aleph_1}{1-\aleph_1})^\lambda + (\frac{\aleph_2}{1-\aleph_2})^\lambda\}^{\frac{1}{\lambda}}} \quad (5)$$

where $\lambda \geq 1$ and $(\aleph_1, \aleph_2) \in [0, 1] \times [0, 1]$. Some special cases can be easily proved.

(1) if $\lambda \longrightarrow 1$, then $\aleph_1 \oplus_D \aleph_2 \longrightarrow \frac{\aleph_1+\aleph_2-2\aleph_1\aleph_2}{1-\aleph_1\aleph_2}$ and $\aleph_1 \otimes_D \aleph_2 \longrightarrow \frac{\aleph_1\aleph_2}{\aleph_1+\aleph_2-\aleph_1\aleph_2}$,

(2) If $\lambda \longrightarrow \infty$, then $\aleph_1 \oplus_D \aleph_2 \longrightarrow max(\aleph_1, \aleph_2)$ and $\aleph_1 \otimes_D \aleph_2 \longrightarrow min(\aleph_1, \aleph_2)$. Dombi sum and Dombi product are reduced to simple max-operator and simple min-operator, respectively.

3. Dombi operations of BNNs

In this section, we discuss the Dombi operations of BNNs and discuss their properties.
Dombi operations of BNNs are defined as follows.

Definition 3.1. Let $\tilde{\aleph}_1 =< \tau_1^+, \omega_1^+, \mho_1^+, \tau_1^-, \omega_1^-, \mho_1^- >$ and $\tilde{\aleph}_2 =< \tau_2^+, \omega_2^+, \mho_2^+, \tau_2^-, \omega_2^-, \mho_2^- >$ be two BNNs and $\lambda \geq 1$. Then, Dombi sum and Dombi product of two BNNs $\tilde{\aleph}_1$ and $\tilde{\aleph}_2$ are denoted by $\tilde{\aleph}_1 \otimes_D \tilde{\aleph}_2$ and $\tilde{\aleph}_1 \oplus_D \tilde{\aleph}_2$ respectively and defined as follows:

$$\textbf{(1)}\ \tilde{\aleph}_1 \oplus_D \tilde{\aleph}_2 = \begin{cases} \langle 1 - \frac{1}{1+\{(\frac{\tau_1^+}{1-\tau_1^+})^\lambda + (\frac{\tau_2^+}{1-\tau_2^+})^\lambda\}^{\frac{1}{\lambda}}}, \frac{1}{1+\{(\frac{1-\omega_1^+}{\omega_1^+})^\lambda + (\frac{1-\omega_2^+}{\omega_2^+})^\lambda\}^{\frac{1}{\lambda}}}, \frac{1}{1+\{(\frac{1-\mho_1^+}{\mho_1^+})^\lambda + (\frac{1-\mho_2^+}{\mho_2^+})^\lambda\}^{\frac{1}{\lambda}}}, \\ \frac{-1}{1+\{(\frac{1+\tau_1^-}{-\tau_1^-})^\lambda + (\frac{1+\tau_2^-}{-\tau_2^-})^\lambda\}^{\frac{1}{\lambda}}}, \frac{1}{1+\{(\frac{-\omega_1^-}{1+\omega_1^-})^\lambda + (\frac{-\omega_2^-}{1+\omega_2^-})^\lambda\}^{\frac{1}{\lambda}}} - 1, \frac{1}{1+\{(\frac{-\mho_1^-}{1+\mho_1^-})^\lambda + (\frac{-\mho_2^-}{1+\mho_2^-})^\lambda\}^{\frac{1}{\lambda}}} - 1 \rangle \end{cases}$$

$$\textbf{(2)}\ \tilde{\aleph}_1 \otimes_D \tilde{\aleph}_2 = \begin{cases} \langle \frac{1}{1+\{(\frac{1-\tau_1^+}{\tau_1^+})^\lambda + (\frac{1-\tau_2^+}{\tau_2^+})^\lambda\}^{\frac{1}{\lambda}}}, 1 - \frac{1}{1+\{(\frac{\omega_1^+}{1-\omega_1^+})^\lambda + (\frac{\omega_2^+}{1-\omega_2^+})^\lambda\}^{\frac{1}{\lambda}}}, 1 - \frac{1}{1+\{(\frac{\mho_1^+}{1-\mho_1^+})^\lambda + (\frac{\mho_2^+}{1-\mho_2^+})^\lambda\}^{\frac{1}{\lambda}}}, \\ \frac{1}{1+\{(\frac{-\tau_1^-}{1+\tau_1^-})^\lambda + (\frac{-\tau_2^-}{1+\tau_2^-})^\lambda\}^{\frac{1}{\lambda}}} - 1, \frac{-1}{1+\{(\frac{1+\omega_1^-}{-\omega_1^-})^\lambda + (\frac{1+\omega_2^-}{-\omega_2^-})^\lambda\}^{\frac{1}{\lambda}}}, \frac{-1}{1+\{(\frac{1+\mho_1^-}{-\mho_1^-})^\lambda + (\frac{1+\mho_2^-}{-\mho_2^-})^\lambda\}^{\frac{1}{\lambda}}} \rangle \end{cases}$$

$$\textbf{(3)}\ \ell ._D \tilde{\aleph}_1 = \begin{cases} \langle 1 - \frac{1}{1+\{\ell(\frac{\tau_1^+}{1-\tau_1^+})^\lambda\}^{\frac{1}{\lambda}}}, \frac{1}{1+\{\ell(\frac{1-\omega_1^+}{\omega_1^+})^\lambda\}^{\frac{1}{\lambda}}}, \frac{1}{1+\{\ell(\frac{1-\mho_1^+}{\mho_1^+})^\lambda\}^{\frac{1}{\lambda}}}, \\ \frac{-1}{1+\{\ell(\frac{1+\tau_1^-}{-\tau_1^-})^\lambda\}^{\frac{1}{\lambda}}}, \frac{1}{1+\{\ell(\frac{-\omega_1^-}{1+\omega_1^-})^\lambda\}^{\frac{1}{\lambda}}} - 1, \frac{1}{1+\{\ell(\frac{-\mho_1^-}{1+\mho_1^-})^\lambda\}^{\frac{1}{\lambda}}} - 1 \rangle \qquad (\ell > 0) \end{cases}$$

$$\textbf{(4)}\ \tilde{\aleph}_1^{\bigwedge_D^{\ell}} = \begin{cases} \langle \frac{1}{1+\{\ell(\frac{1-\tau_1^+}{\tau_1^+})^\lambda\}^{\frac{1}{\lambda}}}, 1 - \frac{1}{1+\{\ell(\frac{\omega_1^+}{1-\omega_1^+})^\lambda\}^{\frac{1}{q}}}, 1 - \frac{1}{1+\{\ell(\frac{\mho_1^+}{1-\mho_1^+})^\lambda\}^{\frac{1}{\lambda}}}, \\ \frac{1}{1+\{\ell(\frac{-\tau_1^-}{1+\tau_1^-})^\lambda\}^{\frac{1}{\lambda}}} - 1, \frac{-1}{1+\{\ell(\frac{1+\omega_1^-}{-\omega_1^-})^q\}^{\frac{1}{\lambda}}}, \frac{-1}{1+\{\ell(\frac{1+\mho_1^-}{-\mho_1^-})^\lambda\}^{\frac{1}{\lambda}}} \rangle \qquad (\ell > 0). \end{cases}$$

Theorem 3.2. *Let* $\tilde{\aleph}_1 =< \tau_1^+, \omega_1^+, \mho_1^+, \tau_1^-, \omega_1^-, \mho_1^- >$ *and* $\tilde{\aleph}_2 =< \tau_2^+, \omega_2^+, \mho_2^+, \tau_2^-, \omega_2^-, \mho_2^- >$ *be two BNNs and let* $\tilde{e} = \tilde{\aleph}_1 \oplus_D \tilde{\aleph}_2$, $\tilde{f} = \tilde{\aleph}_1 \otimes_D \tilde{\aleph}_2$, $\tilde{g} = \ell ._D \tilde{\aleph}_1$ $(\ell > 0)$ *and* $\tilde{h} = \tilde{\aleph}_1^{\bigwedge_D^{\ell}}$ $(\ell > 0)$. *Therefore,* $\tilde{e}$, $\tilde{f}$, $\tilde{g}$ *and* $\tilde{h}$ *are also BNNs.*

As Theorem 3 can be easily verified. Therefore, the proof is omitted here. The properties of Dombi operations of BNNs are defined as:

Theorem 3.3. *Let* $\tilde{\aleph}_1 =< \tau_1^+, \omega_1^+, \mho_1^+, \tau_1^-, \omega_1^-, \mho_1^- >$ *and* $\tilde{\aleph}_2 =< \tau_2^+, \omega_2^+, \mho_2^+, \tau_2^-, \omega_2^-, \mho_2^- >$ *be two BNNs and* $\ell, \ell_1, \ell_2 > 0$. *Then, the following properties can be proven easily.*

(a) $\tilde{\aleph}_1 \oplus_D \tilde{\aleph}_2 = \tilde{\aleph}_2 \oplus_D \tilde{\aleph}_1$,
(b) $\tilde{\aleph}_1 \otimes_D \tilde{\aleph}_2 = \tilde{\aleph}_2 \otimes_D \tilde{\aleph}_1$,
(c) $\ell ._D(\tilde{\aleph}_1 \oplus_D \tilde{\aleph}_2) = \ell ._D \tilde{\aleph}_2 \oplus_D \ell ._D \tilde{\aleph}_1$,
(d) $(\tilde{\aleph}_1 \otimes_D \tilde{\aleph}_2)^{\bigwedge_D^{\ell}} = \tilde{\aleph}_1^{\bigwedge_D^{\ell}} \otimes_D \tilde{\aleph}_2^{\bigwedge_D^{\ell}}$,
(e) $(\ell_1 + \ell_2)._D \tilde{\aleph}_1 = \ell_1 ._D \tilde{\aleph}_1 \oplus_D \ell_2 ._D \tilde{\aleph}_1$,
(f) $\tilde{\aleph}_1^{\bigwedge_D^{\ell_1+\ell_2}} = \tilde{\aleph}_1^{\bigwedge_D^{\ell_1}} \otimes_D \tilde{\aleph}_1^{\bigwedge_D^{\ell_2}}$.

4. Bipolar neutrosophic dombi prioritized aggregation operators

In this section, we define bipolar neutrosophic Dombi weighted geometric prioritized aggregation (BNDPWGA) and the bipolar neutrosophic Dombi prioritized ordered weighted geometric aggregation (BNDPOWGA) operators and discussed different properties of these aggregation operators (AOs) in detail.

Definition 4.1. Let $C = \{C_1, C_2, ..., C_n\}$ be a collection of attributes and have a prioritization between the attributes followed by the linear ordering $C_1 \succ C_2 \succ ... \succ C_n$, which imply that C_η has a higher priority than C_ρ, if $\eta < \rho$. The value of $C_\rho(u)$ is the performance of any alternative u under attribute $C_\eta(u)$, which satisfies $C_\eta \in [0, 1]$, If

$$PA(C_\eta(u)) = \Sigma_{\rho=1}^{n} \upsilon_\rho C_\rho(u) \tag{6}$$

where $\upsilon_\rho = \frac{\hbar_\rho}{\Sigma_{\rho=1}^{n}\hbar_\rho}$; $\hbar_\rho = \Pi_{\eta=1}^{\rho-1} C_\eta(u)(\rho = 2, 3, ..., n)$, $\hbar_1 = 1$. Then, PA is called the prioritized averaging operator.

4.1. Bipolar neutrosophic dombi prioritized weighted geometric aggregation operator

Definition 4.2. Let $\tilde{\aleph}_\alpha =< \tau_\alpha^+, \omega_\alpha^+, \mho_\alpha^+, \tau_\alpha^-, \omega_\alpha^-, \mho_\alpha^- >$ $(\alpha = 1, , 2, 3, ..., r)$ be a family of BNNs. A mapping BNDPWGA: $U^r \to U$ is called BNDPWGA operator, if it satisfies

$$BNDPWGA(\tilde{\aleph}_1, \tilde{\aleph}_2, ..., \tilde{\aleph}_r) = \bigotimes_{\alpha=1}^{r} \tilde{\aleph}_\alpha^{\bigwedge^{\upsilon_\alpha}} = \tilde{\aleph}_1^{\bigwedge_D^{\upsilon_1}} \otimes_D \tilde{\aleph}_2^{\bigwedge_D^{\upsilon_2}} \otimes_D ... \otimes_D \tilde{\aleph}_r^{\bigwedge_D^{\upsilon_r}} \tag{7}$$

where $\upsilon_\alpha = \frac{\hbar_\alpha \psi_\alpha}{\Sigma_{\alpha=1}^{r}\hbar_\alpha\psi_\alpha}$, $\hbar_\alpha = \prod_{k=1}^{\alpha-1} s(\tilde{\aleph}_k)$ $(\alpha = 2, 3, ..., r)$, $\hbar_1 = 1$ and $s(\tilde{\aleph}_k)$ is the value of score for $\tilde{\aleph}_\alpha$. where $\psi = (\psi_1, \psi_2, ..., \psi_r)^T$ is the weight vector of $\tilde{\aleph}_\alpha$ $(\alpha = 1, 2, ..., r)$, $\psi_\alpha \in [0, 1]$ and $\Sigma_{\alpha=1}\psi_\alpha = 1$.

Theorem 4.3. *Let* $\tilde{\aleph}_\alpha =< \tau_\alpha^+, \omega_\alpha^+, \mho_\alpha^+, \tau_\alpha^-, \omega_\alpha^-, \mho_\alpha^- >$ $(\alpha = 1, 2, 3, ..., r)$ *be a family of BNNs and* $\psi = (\psi_1, \psi_2, ..., \psi_r)^T$ *be the weight vector of* $\tilde{\aleph}_\alpha$, $\psi_\alpha \in [0, 1]$ *and* $\Sigma_{\alpha=1}\psi_\alpha = 1$. *Then, the value aggregated by using BNDPWGA operator is still a BNN, which is calculated by using the following formula*
$BNDPWGA(\tilde{\aleph}_1, \tilde{\aleph}_2, ..., \tilde{\aleph}_r) = \tilde{\aleph}_1^{\bigwedge_D^{\upsilon_1}} \otimes_D \tilde{\aleph}_2^{\bigwedge_D^{\upsilon_2}} \otimes_D ... \otimes_D \tilde{\aleph}_r^{\bigwedge_D^{\upsilon_r}}$

$$= \begin{cases} \langle \frac{1}{1+\{\Sigma_{\alpha=1}^{r} \frac{\hbar_\alpha\psi_\alpha}{\Sigma_{\alpha=1}^{r}\hbar_\alpha\psi_\alpha}(\frac{1-\tau_\alpha^+}{\tau_\alpha^+})^\lambda\}^{\frac{1}{\lambda}}}, 1 - \frac{1}{1+\{\Sigma_{\alpha=1}^{r} \frac{\hbar_\alpha\psi_\alpha}{\Sigma_{j=1}^{r}\hbar_\alpha\psi_\alpha}(\frac{\omega_\alpha^+}{1-\omega_\alpha^+})^\lambda\}^{\frac{1}{\lambda}}}, 1 - \frac{1}{1+\{\Sigma_{\alpha=1}^{r} \frac{\hbar_\alpha\psi_\alpha}{\Sigma_{\alpha=1}^{r}\hbar_\alpha\psi_\alpha}(\frac{\mho_\alpha^+}{1-\mho_\alpha^+})^\lambda\}^{\frac{1}{\lambda}}}, \\ \frac{1}{1+\{\Sigma_{\alpha=1}^{r} \frac{\hbar_\alpha\psi_\alpha}{\Sigma_{\alpha=1}^{r}\hbar_\alpha\psi_\alpha}(\frac{-\tau_\alpha^-}{1+\tau_\alpha^-})^\lambda\}^{\frac{1}{\lambda}}} - 1, \frac{-1}{1+\{\Sigma_{\alpha=1}^{r} \frac{\hbar_\alpha\psi_\alpha}{\Sigma_{\alpha=1}^{r}\hbar_\alpha\psi_\alpha}(\frac{1+\omega_\alpha^-}{-\omega_\alpha^-})^\lambda\}^{\frac{1}{\lambda}}}, \frac{-1}{1+\{\Sigma_{\alpha=1}^{r} \frac{\hbar_\alpha\psi_\alpha}{\Sigma_{\alpha=1}^{r}\hbar_\alpha\psi_\alpha}(\frac{1+\mho_\alpha^-}{-\mho_\alpha^-})^\lambda\}^{\frac{1}{\lambda}}} \rangle \end{cases} \tag{8}$$

Proof. If $r = 2$ based on Definition 4.1 for the Dombi operations of BNNs, the following result can be obtained:

BNDPWGA $(\tilde{\aleph}_1, \tilde{\aleph}_2) = \tilde{\aleph}_1^{\bigwedge_D^{\upsilon_1}} \otimes_D \tilde{\aleph}_2^{\bigwedge_D^{\upsilon_2}}$

$$= \begin{cases} \langle \frac{1}{1+\{\frac{\hbar_1\psi_1}{\hbar_1\psi_1+\hbar_2\psi_2}(\frac{1-t_1^+}{\tau_1^+})^\lambda + \frac{\hbar_2\psi_2}{\hbar_1\psi_1+\hbar_2\psi_2}(\frac{1-\tau_2^+}{\tau_2^+})^\lambda\}^{\frac{1}{\lambda}}}, 1 - \frac{1}{1+\{\frac{\hbar_1\psi_1}{\hbar_1\psi_1+\hbar_2\psi_2}(\frac{\omega_1^+}{1-\omega_1^+})^\lambda + \frac{\hbar_2\psi_2}{\hbar_1\psi_1+\hbar_2\psi_2}(\frac{\omega_2^+}{1-\omega_2^+})^\lambda\}^{\frac{1}{\lambda}}}, \\ 1 - \frac{1}{1+\{\frac{\hbar_1\psi_1}{\hbar_1\psi_1+\hbar_2\psi_2}(\frac{\mho_1^+}{1-\mho_1^+})^\lambda + \frac{\hbar_2\psi_2}{\hbar_1\psi_1+\hbar_2\psi_2}(\frac{\mho_2^+}{1-\mho_2^+})^\lambda\}^{\frac{1}{\lambda}}}, \frac{1}{1+\{\frac{\hbar_1\psi_1}{\hbar_1\psi_1+\hbar_2\psi_2}(\frac{-\tau_1^-}{1+\tau_1^-})^\lambda + \frac{\hbar_2\psi_2}{\hbar_1\psi_1+\hbar_2\psi_2}(\frac{-\tau_2^-}{1+\tau_2^-})^\lambda\}^{\frac{1}{\lambda}}} - 1, \\ \frac{-1}{1+\{\frac{\hbar_1\psi_1}{\hbar_1\psi_1+\hbar_2\psi_2}(\frac{1+\omega_1^-}{-\omega_1^-})^\lambda + \frac{\hbar_2\psi_2}{\hbar_1\psi_1+\hbar_2\psi_2}(\frac{1+\omega_2^-}{-\omega_2^-})^\lambda\}^{\frac{1}{\lambda}}}, \frac{-1}{1+\{\frac{\hbar_1\psi_1}{\hbar_1\psi_1+\hbar_2\psi_2}(\frac{1+\mho_1^-}{-\mho_1^-})^\lambda + \frac{\hbar_2\psi_2}{\hbar_1\psi_1+\hbar_2\psi_2}(\frac{1+\mho_2^-}{-\mho_2^-})^\lambda\}^{\frac{1}{\lambda}}} \rangle, \end{cases}$$

$$
= \left\{ \begin{array}{l}
\langle \frac{1}{1+\{\Sigma_{\alpha=1}^{2}\frac{\hbar_\alpha\psi_\alpha}{\Sigma_{\alpha=1}^{2}\hbar_\alpha\psi_\alpha}(\frac{1-\tau_\alpha^{+}}{\tau_\alpha^{+}})^{\lambda}\}^{\frac{1}{\lambda}}},\ 1-\frac{1}{1+\{\Sigma_{\alpha=1}^{2}\frac{\hbar_\alpha\psi_\alpha}{\Sigma_{\alpha=1}^{2}\hbar_\alpha\psi_\alpha}(\frac{\omega_\alpha^{+}}{1-\omega_\alpha^{+}})^{\lambda}\}^{\frac{1}{\lambda}}}, \\
1-\frac{1}{1+\{\Sigma_{\alpha=1}^{2}\frac{\hbar_\alpha\psi_\alpha}{\Sigma_{\alpha=1}^{2}\hbar_\alpha\psi_\alpha}(\Sigma_{\alpha=1}^{2}\frac{\mho_\alpha^{+}}{1-\mho_\alpha^{+}})^{\lambda}\}^{\frac{1}{\lambda}}},\ \frac{1}{1+\{\Sigma_{\alpha=1}^{2}\frac{\hbar_\alpha\psi_\alpha}{\Sigma_{\alpha=1}^{2}\hbar_\alpha\psi_\alpha}(\frac{-\tau_\alpha^{-}}{1+\tau_\alpha^{-}})^{\lambda}\}^{\frac{1}{\lambda}}}-1, \\
\frac{-1}{1+\{\Sigma_{\alpha=1}^{2}\frac{\hbar_\alpha\psi_\alpha}{\Sigma_{\alpha=1}^{2}\hbar_\alpha\psi_\alpha}(\frac{1+\omega_\alpha^{-}}{-\omega_\alpha^{-}})^{\lambda}\}^{\frac{1}{\lambda}}},\ \frac{-1}{1+\{\Sigma_{\alpha=1}^{2}\frac{\hbar_\alpha\psi_\alpha}{\Sigma_{\alpha=1}^{2}\hbar_\alpha\psi_\alpha}(\frac{1+\mho_\alpha^{-}}{-\mho_\alpha^{-}})^{\lambda}\}^{\frac{1}{\lambda}}} \rangle.
\end{array}\right.
$$

If $r = s$, based on equation (8), then we have got the following equation:

$$
\mathrm{BNDPWGA}(\tilde{\aleph}_1, \tilde{\aleph}_2, ..., \tilde{\aleph}_s) = \bigotimes_{\alpha=1}^{s} \tilde{\aleph}_\alpha^{\bigwedge^{\upsilon_\alpha}}
$$

$$
= \left\{ \begin{array}{l}
\langle \frac{1}{1+\{\Sigma_{\alpha=1}^{s}\frac{\hbar_\alpha\psi_\alpha}{\Sigma_{\alpha=1}^{s}\hbar_\alpha\psi_\alpha}(\frac{1-\tau_\alpha^{+}}{\tau_\alpha^{+}})^{\lambda}\}^{\frac{1}{\lambda}}},\ 1-\frac{1}{1+\{\Sigma_{\alpha=1}^{s}\frac{\hbar_\alpha\psi_\alpha}{\Sigma_{\alpha=1}^{s}\hbar_\alpha\psi_\alpha}(\frac{\omega_\alpha^{+}}{1-\omega_\alpha^{+}})^{\lambda}\}^{\frac{1}{\lambda}}},\ 1-\frac{1}{1+\{\Sigma_{\alpha=1}^{s}\frac{\hbar_\alpha\psi_\alpha}{\Sigma_{\alpha=1}^{s}\hbar_\alpha\psi_\alpha}(\frac{\mho_\alpha^{+}}{1-\mho_\alpha^{+}})^{\lambda}\}^{\frac{1}{\lambda}}}, \\
\frac{1}{1+\{\Sigma_{\alpha=1}^{s}\frac{\hbar_\alpha\psi_\alpha}{\Sigma_{\alpha=1}^{s}\hbar_\alpha\psi_\alpha}(\frac{-\tau_\alpha^{-}}{1+\tau_\alpha^{-}})^{\lambda}\}^{\frac{1}{\lambda}}}-1,\ \frac{-1}{1+\{\Sigma_{\alpha=1}^{s}\frac{\hbar_\alpha\psi_\alpha}{\Sigma_{\alpha=1}^{s}\hbar_\alpha\psi_\alpha}(\frac{1+\omega_\alpha^{-}}{-\omega_\alpha^{-}})^{\lambda}\}^{\frac{1}{\lambda}}},\ \frac{-1}{1+\{\Sigma_{\alpha=1}^{s}\frac{\hbar_\alpha\psi_\alpha}{\Sigma_{\alpha=1}^{s}\hbar_\alpha\psi_\alpha}(\frac{1+\mho_\alpha^{-}}{-\mho_\alpha^{-}})^{\lambda}\}^{\frac{1}{\lambda}}} \rangle.
\end{array}\right.
$$

If $r = s + 1$, then there is following result:

$$
\mathrm{BNDPWGA}(\tilde{\aleph}_1, \tilde{\aleph}_2, ..., \tilde{\aleph}_s, \tilde{\aleph}_{s+1}) = \bigotimes_{\alpha=1}^{s} \tilde{\aleph}_\alpha^{\bigwedge^{\upsilon_\alpha}} \bigotimes \tilde{\aleph}_{s+1}^{\bigwedge^{\upsilon_{s+1}}}
$$

$$
= \left\{ \begin{array}{l}
\langle \frac{1}{1+\{\Sigma_{\alpha=1}^{s}\frac{\hbar_\alpha\psi_\alpha}{\Sigma_{\alpha=1}^{s}\hbar_\alpha\psi_\alpha}(\frac{1-\tau_\alpha^{+}}{\tau_\alpha^{+}})^{\lambda}\}^{\frac{1}{\lambda}}},\ 1-\frac{1}{1+\{\Sigma_{\alpha=1}^{s}\frac{\hbar_\alpha\psi_\alpha}{\Sigma_{\alpha=1}^{s}\hbar_\alpha\psi_\alpha}(\frac{\omega_\alpha^{+}}{1-\omega_\alpha^{+}})^{\lambda}\}^{\frac{1}{\lambda}}},\ 1-\frac{1}{1+\{\Sigma_{\alpha=1}^{s}\frac{\hbar_\alpha\psi_\alpha}{\Sigma_{\alpha=1}^{s}\hbar_\alpha\psi_\alpha}(\frac{\mho_\alpha^{+}}{1-\mho_\alpha^{+}})^{\lambda}\}^{\frac{1}{\lambda}}}, \\
\frac{1}{1+\{\Sigma_{\alpha=1}^{s}\frac{\hbar_\alpha\psi_\alpha}{\Sigma_{\alpha=1}^{s}\hbar_\alpha\psi_\alpha}(\frac{-\tau_\alpha^{-}}{1+\tau_\alpha^{-}})^{\lambda}\}^{\frac{1}{\lambda}}}-1,\ \frac{1}{1+\{\Sigma_{\alpha=1}^{s}\frac{\hbar_\alpha\psi_\alpha}{\Sigma_{\alpha=1}^{s}\hbar_\alpha\psi_\alpha}(\frac{1+\omega_\alpha^{-}}{-\omega_\alpha^{-}})^{\lambda}\}^{\frac{1}{\lambda}}},\ \frac{1}{1+\{\Sigma_{\alpha=1}^{s}\frac{\hbar_\alpha\psi_\alpha}{\Sigma_{\alpha=1}^{s}\hbar_\alpha\psi_\alpha}(\frac{1+\mho_\alpha^{-}}{-\mho_\alpha^{-}})^{\lambda}\}^{\frac{1}{\lambda}}} \rangle
\end{array}\right.
$$

$$
\otimes_D \left\{ \begin{array}{l}
\langle \frac{1}{1+\{\frac{\hbar_{s+1}\psi_{s+1}}{\Sigma_{\alpha=1}^{s+1}\hbar_\alpha\psi_\alpha}(\frac{1-\tau_{s+1}^{+}}{\tau_{s+1}^{+}})^{\lambda}\}^{\frac{1}{\lambda}}},\ 1-\frac{1}{1+\{\frac{\hbar_{s+1}\psi_{s+1}}{\Sigma_{\alpha=1}^{s+1}\hbar_\alpha\psi_\alpha}(\frac{\omega_{s+1}^{+}}{1-\omega_{s+1}^{+}})^{\lambda}\}^{\frac{1}{\lambda}}},\ 1-\frac{1}{1+\{\frac{\hbar_{s+1}\psi_{s+1}}{\Sigma_{\alpha=1}^{s+1}\hbar_\alpha\psi_\alpha}(\frac{\mho_{s+1}^{+}}{1-\mho_{s+1}^{+}})^{\lambda}\}^{\frac{1}{\lambda}}}, \\
\frac{1}{1+\{\frac{\hbar_{s+1}\psi_{s+1}}{\Sigma_{\alpha=1}^{s+1}\hbar_\alpha\psi_\alpha}(\frac{-\tau_\alpha^{-}}{1+\tau_{s+1}^{-}})^{\lambda}\}^{\frac{1}{\lambda}}}-1,\ \frac{1}{1+\{\frac{\hbar_{s+1}\psi_{s+1}}{\Sigma_{\alpha=1}^{s+1}\hbar_\alpha\psi_\alpha}(\frac{1+\omega_{s+1}^{-}}{-\omega_{s+1}^{-}})^{\lambda}\}^{\frac{1}{\lambda}}},\ \frac{1}{1+\{\frac{\hbar_{s+1}\psi_{s+1}}{\Sigma_{\alpha=1}^{s+1}\hbar_\alpha\psi_\alpha}(\frac{1+\mho_{s+1}^{-}}{-\mho_{s+1}^{-}})^{\lambda}\}^{\frac{1}{\lambda}}} \rangle
\end{array}\right.
$$

$$
= \left\{ \begin{array}{l}
\langle \frac{1}{1+\{\Sigma_{\alpha=1}^{s+1}\frac{\hbar_\alpha\psi_\alpha}{\Sigma_{\alpha=1}^{s+1}\hbar_\alpha\psi_\alpha}(\frac{1-\tau_\alpha^{+}}{\tau_\alpha^{+}})^{\lambda}\}^{\frac{1}{\lambda}}},\ 1-\frac{1}{1+\{\Sigma_{\alpha=1}^{s+1}\frac{\hbar_\alpha\psi_\alpha}{\Sigma_{\alpha=1}^{s+1}\hbar_\alpha\psi_\alpha}(\frac{\omega_\alpha^{+}}{1-\omega_\alpha^{+}})^{\lambda}\}^{\frac{1}{\lambda}}},\ 1-\frac{1}{1+\{\Sigma_{\alpha=1}^{s+1}\frac{\hbar_\alpha\psi_\alpha}{\Sigma_{\alpha=1}^{s+1}\hbar_\alpha\psi_\alpha}(\frac{\mho_\alpha^{+}}{1-\mho_\alpha^{+}})^{\lambda}\}^{\frac{1}{\lambda}}}, \\
\frac{1}{1+\{\Sigma_{\alpha=1}^{s+1}\frac{\hbar_\alpha\psi_\alpha}{\Sigma_{\alpha=1}^{s+1}\hbar_\alpha\psi_\alpha}(\frac{-\tau_\alpha^{-}}{1+\tau_\alpha^{-}})^{\lambda}\}^{\frac{1}{\lambda}}}-1,\ \frac{1}{1+\{\Sigma_{\alpha=1}^{s+1}\frac{\hbar_\alpha\psi_\alpha}{\Sigma_{\alpha=1}^{s+1}\hbar_\alpha\psi_\alpha}(\frac{1+\omega_\alpha^{-}}{-\omega_\alpha^{-}})^{\lambda}\}^{\frac{1}{\lambda}}},\ \frac{1}{1+\{\Sigma_{\alpha=1}^{s+1}\frac{\hbar_\alpha\psi_\alpha}{\Sigma_{\alpha=1}^{s+1}\hbar_\alpha\psi_\alpha}(\frac{1+\mho_\alpha^{-}}{-\mho_\alpha^{-}})^{\lambda}\}^{\frac{1}{\lambda}}} \rangle
\end{array}\right.
$$

Hence, Theorem (4.3) is true for $r = s + 1$. Thus, equation (8) holds for all r. □

The BNDPWGA operator also has the following properties:

(1) Idomopotency: Let all the BNNs be $\tilde{b_\alpha} =< \tau_\alpha^+, \omega_\alpha^+, \mho_\alpha^+, \tau_\alpha^-, \omega_\alpha^-, \mho_\alpha^- >= \tilde{\aleph}$ $(\alpha = 1, 2, 3, ..., r)$, where $\psi = (\psi_1, \psi_2, ..., \psi_r)^T$ be the weight vector of $\tilde{\aleph}_\alpha, \psi_\alpha \in [0, 1]$ and $\Sigma_{\alpha=1}\psi_\alpha = 1$. Then, $\mathrm{BNDPWGA}(\tilde{\aleph}_1, \tilde{\aleph}_2, ..., \tilde{\aleph}_r) = \tilde{\aleph}$.

(2) Monotonicity: Let $\tilde{\aleph}_\alpha (\alpha = 1, 2, ..., r)$ and $\tilde{\aleph}'_\alpha (\alpha = 1, 2, ..., r)$ be two families of BNNs, where $\psi = (\psi_1, \psi_2, ..., \psi_r)^T$ be the weight vector of $\tilde{\aleph}_\alpha$ and $\tilde{\aleph}'_\alpha$, $\psi_\alpha \in [0, 1]$ and $\Sigma_{\alpha=1}\psi_\alpha = 1$. For all α ,if $\tilde{\aleph}_\alpha \geq \tilde{\aleph}'_\alpha$, then
$\mathrm{BNDPWGA}(\tilde{\aleph}_1, \tilde{\aleph}_2, ..., \tilde{\aleph}_r) \geq \mathrm{BNDPWGA}(\tilde{\aleph}'_1, \tilde{\aleph}'_2, ..., \tilde{\aleph}'_r)$.

(3) Boundedness: Let $\tilde{\aleph}_\alpha =< \tau_\alpha^+, \omega_\alpha^+, \mho_\alpha^+, \tau_\alpha^-, \omega_\alpha^-, \mho_\alpha^- > (\alpha = 1, , 2, 3, ..., r)$ be a family of BNNs, where $\psi = (\psi_1, \psi_2, ..., \psi_r)^T$ be the weight vector of $\tilde{\aleph}_\alpha$, $\psi_\alpha \in [0, 1]$ and $\Sigma_{\alpha=1}\psi_\alpha = 1$. Therefore, we have BNDPWGA$(\tilde{\aleph}^-, \tilde{\aleph}^-, ..., \tilde{\aleph}^-)$

$$\leq BNDPWGA(\tilde{\aleph}_1, \tilde{\aleph}_2, ..., \tilde{\aleph}_r) \leq BNDPWGA(\tilde{\aleph}^+, \tilde{\aleph}^+, ..., \tilde{\aleph}^+),$$

where $\tilde{\aleph}^- =< \tau_{\tilde{\aleph}^-}^+, \omega_{\tilde{\aleph}^-}^+, \mho_{\tilde{\aleph}^-}^+, \tau_{\tilde{\aleph}^-}^-, \omega_{\tilde{\aleph}^-}^-, \mho_{\tilde{\aleph}^-}^- >$

$$= \begin{cases} < min(\tau_1^+, \tau_2^+, ..., \tau_r^+), max(\omega_1^+, \omega_2^+, ..., \omega_r^+), max(\mho_1^+, \mho_2^+, ..., \mho_r^+), \\ max(\tau_1^-, \tau_2^-, ..., \tau_r^-), min(\omega_\alpha^-, \omega_\alpha^-, ..., \omega_\alpha^-), min(\mho_1^-, \mho_2^-, ..., \mho_r^-) > \end{cases}$$

and

$\tilde{\aleph}^+ =< \tau_{\tilde{\aleph}^+}^+, \omega_{\tilde{\aleph}^+}^+, \mho_{\tilde{\aleph}^+}^+, \tau_{\tilde{\aleph}^+}^-, \omega_{\tilde{\aleph}^+}^-, \mho_{\tilde{\aleph}^+}^- >$

$$= \begin{cases} < max(\tau_1^+, \tau_2^+, ..., \tau_r^+), min(\omega_1^+, \omega_2^+, ..., \omega_r^+), min(\mho_1^+, \mho_2^+, ..., \mho_r^+), \\ min(\tau_1^-, \tau_2^-, ..., \tau_r^-), max(\omega_\alpha^-, \omega_\alpha^-, ...\omega_\alpha^-), max(\mho_1^-, \mho_2^-, ..., \mho_r^-) > . \end{cases}$$

Proof.

(1) Since $\tilde{\aleph}_\alpha =< \tau_\alpha^+, \omega_\alpha^+, \mho_\alpha^+, \tau_\alpha^-, \omega_\alpha^-, \mho_\alpha^- >= \tilde{\aleph}(\alpha = 1, , 2, 3, ..., r)$. Then, the following result can be obtained by using equation (8).

$$\text{BNDPWGA}(\tilde{\aleph}_1, \tilde{\aleph}_2, ..., \tilde{\aleph}_r) = \bigotimes_{\alpha=1}^r \tilde{\aleph}_\alpha^{\bigwedge^{\upsilon_\alpha}}$$

$$= \begin{cases} \langle \frac{1}{1+\{\Sigma_{\alpha=1}^r \frac{\hbar_\alpha\psi_\alpha}{\Sigma_{\alpha=1}^r \hbar_\alpha\psi_\alpha}(\frac{1-\tau_\alpha^+}{\tau_\alpha^+})^\lambda\}^{\frac{1}{\lambda}}}, 1 - \frac{1}{1+\{\Sigma_{\alpha=1}^r \frac{\hbar_\alpha\psi_\alpha}{\Sigma_{\alpha=1}^r \hbar_\alpha\psi_\alpha}(\frac{\omega_\alpha^+}{1-\omega_\alpha^+})^\lambda\}^{\frac{1}{\lambda}}}, 1 - \frac{1}{1+\{\Sigma_{\alpha=1}^r \frac{\hbar_\alpha\psi_\alpha}{\Sigma_{\alpha=1}^r \hbar_\alpha\psi_\alpha}(\frac{\mho_\alpha^+}{1-\mho_\alpha^+})^\lambda\}^{\frac{1}{\lambda}}}, \\ \frac{1}{1+\{\Sigma_{\alpha=1}^r \frac{\hbar_\alpha\psi_\alpha}{\Sigma_{\alpha=1}^r \hbar_\alpha\psi_\alpha}(\frac{-\tau_\alpha^-}{1+\tau_\alpha^-})^\lambda\}^{\frac{1}{\lambda}}} - 1, \frac{-1}{1+\{\Sigma_{\alpha=1}^r \frac{\hbar_\alpha\psi_\alpha}{\Sigma_{\alpha=1}^r \hbar_\alpha\psi_\alpha}(\frac{1+\omega_\alpha^-}{-\omega_\alpha^-})^\lambda\}^{\frac{1}{\lambda}}}, \frac{-1}{1+\{\Sigma_{\alpha=1}^r \frac{\hbar_\alpha\psi_\alpha}{\Sigma_{\alpha=1}^r \hbar_\alpha\psi_\alpha}(\frac{1+\mho_\alpha^-}{-\mho_\alpha^-})^\lambda\}^{\frac{1}{\lambda}}} \rangle \end{cases}$$

$$= \begin{cases} \langle \frac{1}{1+\{(\frac{1-\tau^+}{\tau^+})^\lambda\}^{\frac{1}{\lambda}}}, 1 - \frac{1}{1+\{(\frac{\omega^+}{1-\omega^+})^\lambda\}^{\frac{1}{\lambda}}}, 1 - \frac{1}{1+\{(\frac{\mho^+}{1-\mho^+})^\lambda\}^{\frac{1}{\lambda}}}, \\ \frac{1}{1+\{(\frac{-\tau^-}{1+\tau^-})^\lambda\}^{\frac{1}{\lambda}}} - 1, \frac{-1}{1+\{(\frac{1+\omega^-}{-\omega^-})^\lambda\}^{\frac{1}{\lambda}}}, \frac{-1}{1+\{(\frac{1+\mho^-}{-\mho^-})^\lambda\}^{\frac{1}{\lambda}}} \rangle \end{cases}$$

$=< \tau^+, \omega^+, \mho^+, \tau^-, \omega^-, \mho^- >= \tilde{\aleph}$.

Hence, BNDPWGA$(\tilde{\aleph}_1, \tilde{\aleph}_2, ..., \tilde{\aleph}_r) = \tilde{\aleph}$ holds.

(2) The property is obvious based on the equation (8).

(3) Let $\tilde{\aleph}^- =< \tau_{\tilde{\aleph}^-}^+, \omega_{\tilde{\aleph}^-}^+, \mho_{\tilde{\aleph}^-}^+, \tau_{\tilde{\aleph}^-}^-, \omega_{\tilde{\aleph}^-}^-, \mho_{\tilde{\aleph}^-}^-$ and $\tilde{\aleph}^+ =< \tau_{\tilde{\aleph}^+}^+, \omega_{\tilde{\aleph}^+}^+, \mho_{\tilde{\aleph}^+}^+, \tau_{\tilde{\aleph}^+}^-, \omega_{\tilde{\aleph}^+}^-, \mho_{\tilde{\aleph}^+}^- >$. There are the following inequalities:

$$\frac{1}{1+\{\Sigma_{\alpha=1}^r \frac{\hbar_\alpha\psi_\alpha}{\Sigma_{\alpha=1}^r \hbar_\alpha\psi_\alpha}(\frac{1-\tau_{\tilde{\aleph}^-}^+}{1_{\tilde{\aleph}^-}^+})^\lambda\}^{\frac{1}{\lambda}}} \leq \frac{1}{1+\{\Sigma_{\alpha=1}^r \frac{\hbar_\alpha\psi_\alpha}{\Sigma_{\alpha=1}^r \hbar_\alpha\psi_\alpha}(\frac{1-\tau_\alpha^+}{\tau_\alpha^+})^\lambda\}^{\frac{1}{\lambda}}} \leq \frac{1}{1+\{\Sigma_{\alpha=1}^r \frac{\hbar_\alpha\psi_\alpha}{\Sigma_{\alpha=1}^r \hbar_\alpha\psi_\alpha}(\frac{1-\tau_{\tilde{\aleph}^+}^+}{\tau_{\tilde{\aleph}^+}^+})^\lambda\}^{\frac{1}{\lambda}}},$$

$$1 - \frac{1}{1+\{\Sigma_{\alpha=1}^r \frac{\hbar_\alpha\psi_\alpha}{\Sigma_{\alpha=1}^r \hbar_\alpha\psi_\alpha}(\frac{\omega_{\tilde{\aleph}^+}^+}{1-\omega_{\tilde{\aleph}^+}^+})^\lambda\}^{\frac{1}{\lambda}}} \leq 1 - \frac{1}{1+\{\Sigma_{\alpha=1}^r \frac{\hbar_\alpha\psi_\alpha}{\Sigma_{\alpha=1}^r \hbar_\alpha\psi_\alpha}(\frac{\omega_\alpha^+}{1-\omega_\alpha^+})^\lambda\}^{\frac{1}{\lambda}}} \leq 1 - \frac{1}{1+\{\Sigma_{\alpha=1}^r \frac{\hbar_\alpha\psi_\alpha}{\Sigma_{\alpha=1}^r \hbar_\alpha\psi_\alpha}(\frac{\omega_{\tilde{\aleph}^-}^+}{1-\omega_{\tilde{\aleph}^-}^+})^\lambda\}^{\frac{1}{\lambda}}},$$

$$1 - \frac{1}{1+\{\Sigma_{\alpha=1}^r \frac{\hbar_\alpha\psi_\alpha}{\Sigma_{\alpha=1}^r \hbar_\alpha\psi_\alpha}(\frac{\mho_{\tilde{\aleph}^+}^+}{1-\mho_{\tilde{\aleph}^+}^+})^\lambda\}^{\frac{1}{\lambda}}} \leq 1 - \frac{1}{1+\{\Sigma_{\alpha=1}^r \frac{\hbar_\alpha\psi_\alpha}{\Sigma_{\alpha=1}^r \hbar_\alpha\psi_\alpha}(\frac{\mho_\alpha^+}{1-\mho_\alpha^+})^\lambda\}^{\frac{1}{\lambda}}} \leq 1 - \frac{1}{1+\{\Sigma_{\alpha=1}^r \frac{\hbar_\alpha\psi_\alpha}{\Sigma_{\alpha=1}^r \hbar_\alpha\psi_\alpha}(\frac{\mho_{\tilde{\aleph}^-}^+}{1-\mho_{\tilde{\aleph}^-}^+})^\lambda\}^{\frac{1}{\lambda}}},$$

$$\frac{1}{1+\{\Sigma_{\alpha=1}^{r}\frac{\hbar_\alpha\psi_\alpha}{\Sigma_{\alpha=1}^{r}\hbar_\alpha\psi_\alpha}(\frac{-\tau^{-}_{\tilde{\aleph}^{+}}}{1+\tau^{-}_{\tilde{\aleph}^{+}}})^{\lambda}\}^{\frac{1}{\lambda}}}-1$$
$$\leq \frac{1}{1+\{\Sigma_{\alpha=1}^{r}\frac{\hbar_\alpha\psi_\alpha}{\Sigma_{\alpha=1}^{r}\hbar_\alpha\psi_\alpha}(\frac{-\tau^{-}_{\alpha}}{1+\tau^{-}_{\alpha}})^{\lambda}\}^{\frac{1}{\lambda}}}-1 \leq \frac{1}{1+\{\Sigma_{\alpha=1}^{r}\frac{\hbar_\alpha\psi_\alpha}{\Sigma_{\alpha=1}^{r}\hbar_\alpha\psi_\alpha}(\frac{-\tau^{-}_{\tilde{\aleph}^{-}}}{1+\tau^{-}_{\tilde{\aleph}^{-}}})^{\lambda}\}^{\frac{1}{\lambda}}}-1,$$
$$\frac{-1}{1+\{\Sigma_{\alpha=1}^{r}\frac{\hbar_\alpha\psi_\alpha}{\Sigma_{\alpha=1}^{r}\hbar_\alpha\psi_\alpha}(\frac{1+\omega^{-}_{\tilde{\aleph}^{-}}}{-\omega^{-}_{\tilde{\aleph}^{-}}})^{\lambda}\}^{\frac{1}{\lambda}}} \leq \frac{-1}{1+\{\Sigma_{\alpha=1}^{r}\frac{\hbar_\alpha\psi_\alpha}{\Sigma_{\alpha=1}^{r}\hbar_\alpha\psi_\alpha}(\frac{1+\omega^{-}_{\alpha}}{-\omega^{-}_{\alpha}})^{\lambda}\}^{\frac{1}{\lambda}}} \leq \frac{-1}{1+\{\Sigma_{\alpha=1}^{r}\frac{\hbar_\alpha\psi_\alpha}{\Sigma_{\alpha=1}^{r}\hbar_\alpha\psi_\alpha}(\frac{1+\omega^{-}_{\tilde{\aleph}^{+}}}{-\omega^{-}_{\tilde{\aleph}^{+}}})^{\lambda}\}^{\frac{1}{\lambda}}},$$
$$\frac{-1}{1+\{\Sigma_{\alpha=1}^{r}\frac{\hbar_\alpha\psi_\alpha}{\Sigma_{\alpha=1}^{r}\hbar_\alpha\psi_\alpha}(\frac{1+\mho^{-}_{\tilde{\aleph}^{-}}}{-\mho^{-}_{\tilde{\aleph}^{-}}})^{\lambda}\}^{\frac{1}{\lambda}}} \leq \frac{-1}{1+\{\Sigma_{\alpha=1}^{r}\frac{\hbar_\alpha\psi_\alpha}{\Sigma_{\alpha=1}^{r}\hbar_\alpha\psi_\alpha}(\frac{1+\mho^{-}_{\alpha}}{-\mho^{-}_{\alpha}})^{\lambda}\}^{\frac{1}{\lambda}}} \leq \frac{-1}{1+\{\Sigma_{\alpha=1}^{r}\frac{\hbar_\alpha\psi_\alpha}{\Sigma_{\alpha=1}^{r}\hbar_\alpha\psi_\alpha}(\frac{1+\mho^{-}_{\tilde{\aleph}^{+}}}{-\mho^{-}_{\tilde{\aleph}^{+}}})^{\lambda}\}^{\frac{1}{\lambda}}}.$$

Hence, $\text{BNDPWGA}(\tilde{\aleph}^-, \tilde{\aleph}^-, ..., \tilde{\aleph}^-) \leq \text{BNDPWGA}(\tilde{\aleph}_1, \tilde{\aleph}_2, ..., \tilde{\aleph}_r) \leq \text{BNDPWGA}(\tilde{\aleph}^+, \tilde{\aleph}^+, ..., \tilde{\aleph}^+)$ holds.

□

Example 4.4. Let $\tilde{\aleph}_1 =< 0.6, 0.7, 0.3, -0.6, -0.3, -0.5 >$, $\tilde{\aleph}_2 =< 0.5, 0.4, 0.6, -0.6, -0.7, -0.3 >$, $\tilde{\aleph}_3 =< 0.6, 0.7, 0.4, -0.9, -0.7, -0.7 >$ and $\tilde{\aleph}_4 =< 0.2, 0.6, 0.8, -0.6, -0.3, -0.9 >$ be four BNNs and let the weight vector of BNNs $\tilde{\aleph}_\alpha(\alpha = 1, 2, 3, 4)$ be $\psi = (\frac{1}{2}, \frac{1}{4}, \frac{1}{8}, \frac{1}{8})^T$. Now, by Definition 4.1, $\hbar_1 = 1$, $\hbar_2 = s(\tilde{\aleph}_1) = 0.4667$, $\hbar_3 = s(\tilde{\aleph}_1)s(\tilde{\aleph}_2) = 0.2256$ and $\hbar_4 = s(\tilde{\aleph}_1)s(\tilde{\aleph}_2)s(\tilde{\aleph}_3) = 0.1128$. $\psi_1 = \frac{1}{2}$, $\psi_2 = \frac{1}{4}$, $\psi_3 = \frac{1}{8}$ and $\psi_4 = \frac{1}{8}$ are the weight of $\tilde{\aleph}_\alpha(\alpha = 1, 2, 3, 4)$ such that $\Sigma_{\alpha=1}\psi_\alpha = 1$. By Definition 4.2, $\frac{\hbar_1\psi_1}{\Sigma_{\alpha=1}^{4}\hbar_\alpha\psi_\alpha} = 0.7588$, $\frac{\hbar_2\psi_2}{\Sigma_{\alpha=1}^{4}\hbar_\alpha\psi_\alpha} = 0.1771$, $\frac{\hbar_3\psi_3}{\Sigma_{\alpha=1}^{4}\hbar_\alpha\psi_\alpha} = 0.0428$, $\frac{\hbar_4\psi_4}{\Sigma_{\alpha=1}^{4}\hbar_\alpha\psi_\alpha} = 0.0214$. Then, by Theorem 4.3, for $\lambda = 3$
$BNDPWGA(\tilde{\aleph}_1, \tilde{\aleph}_2, \tilde{\aleph}_3, \tilde{\aleph}_4) =$

$$\left\{\begin{array}{l}
\langle\frac{1}{1+\{0.7588(\frac{1-0.6}{0.6})^3+0.1771(\frac{1-0.5}{0.5})^3+0.0428(\frac{1-0.6}{0.6})^3+0.0214(\frac{1-0.2}{0.2})^3\}^{\frac{1}{3}}},\\
1-\\
\frac{1}{1+\{0.7588(\frac{0.7}{1-0.7})^3+0.1771(\frac{0.4}{1-0.4})^3+0.0428(\frac{0.7}{1-0.7})^3+0.0214(\frac{0.6}{1-0.6})^3\}^{\frac{1}{3}}},\\
1-\\
\frac{1}{1+\{0.7588(\frac{0.3}{1-0.3})^3+0.1771(\frac{0.6}{1-0.6})^3+0.0428(\frac{0.4}{1-0.4})^3+0.0214(\frac{0.8}{1-0.8})^3\}^{\frac{1}{3}}},\\
\frac{1}{1+\{0.7588(\frac{0.6}{1-0.6})^3+0.1771(\frac{0.6}{1-0.6})^3+0.0428(\frac{0.9}{1-0.9})^3+0.0214(\frac{0.6}{1-0.6})^3\}^{\frac{1}{3}}}\\
-1,\\
\frac{-1}{1+\{0.7588(\frac{1-0.3}{0.3})^3+0.1771(\frac{1-0.7}{0.7})^3+0.0428(\frac{1-0.7}{0.7})^3+0.0214(\frac{1-0.3}{0.3})^3\}^{\frac{1}{3}}},\\
\frac{-1}{1+\{0.7588(\frac{1-0.5}{0.5})^3+0.1771(\frac{1-0.3}{0.3})^3+0.0428(\frac{1-0.7}{0.7})^3+0.0214(\frac{1-0.9}{0.9})^3\}^{\frac{1}{3}}}\rangle,
\end{array}\right.$$

$BNDPWGA(\tilde{\aleph}_1, \tilde{\aleph}_2, \tilde{\aleph}_3, \tilde{\aleph}_4) = \langle 0.4519, 0.6852, 0.5591, -0.7649, -0.3175, -0.4092\rangle$.

4.2. Bipolar neutrosophic Dombi prioritized ordered weighted geometric aggregation operator

In this section, we propose bipolar neutrosophic Dombi prioritized ordered weighted geometric aggregation operator (BNDPOWGA) which is more useful to pervious defined bipolar neutrosophic Dombi prioritized weighted geometric aggregation operator (BNDPWGA). Because in this operator one more parameter added, called an ordered parameter. This mean that BNDPOWGA is more informative than (BNDPWGA).

Definition 4.5. Let $\tilde{\aleph}_\alpha =< \tau_\alpha^+, \omega_\alpha^+, \mho_\alpha^+, \tau_\alpha^-, \omega_\alpha^-, \mho_\alpha^- > (\alpha = 1, 2, 3, ..., r)$ be a family of BNNs. A mapping BNDPOWGA: $U^r \to U$ is called BNDPOWGA operator, if it satisfies

$$BNDPOWGA(\tilde{\aleph}_1, \tilde{\aleph}_2, ..., \tilde{\aleph}_r) = \bigotimes_{\beta=1}^r \tilde{\aleph}_{\sigma(\beta)}^{\bigwedge^{\upsilon_\beta}} = \tilde{\aleph}_{\sigma(1)}^{\bigwedge_D^{\upsilon_1}} \otimes_D \tilde{\aleph}_{\sigma(2)}^{\bigwedge_D^{\upsilon_2}} \otimes_D ... \otimes_D \tilde{\aleph}_{\sigma(r)}^{\bigwedge_D^{\upsilon_r}} \tag{9}$$

where $\upsilon_\beta = \frac{\hbar_\beta \psi_\beta}{\Sigma_{\beta=1}^r \hbar_\beta \psi_\beta}$, $\hbar_\beta = \prod_{\alpha=1}^{\beta-1} s(\tilde{\aleph}_\alpha)\ (\alpha = 2, 3, ..., r)$, $\hbar_1 = 1$ and $s(\tilde{\aleph}_\alpha)$ is the value of score for $\tilde{\aleph}_\alpha$. where σ is permutation that orders the elements: $\tilde{\aleph}_{\sigma(1)} \geq \tilde{\aleph}_{\sigma(2)} \geq ... \geq \tilde{\aleph}_{\sigma(r)}$. where $\psi = (\psi_1, \psi_2, ..., \psi_r)^T$ is the weight vector of $\tilde{\aleph}_\alpha\ (\alpha = 1, 2, ..., r)$, $\psi_\alpha \in [0, 1]$ and $\Sigma_{\alpha=1} \psi_\alpha = 1$.

Theorem 4.6. *Let $\tilde{\aleph}_\alpha =< \tau_\alpha^+, \omega_\alpha^+, \mho_\alpha^+, \tau_\alpha^-, \omega_\alpha^-, \mho_\alpha^- > (\alpha = 1, 2, 3, ..., r)$ be a family of BNNs and $\psi = (\psi_1, \psi_2, ..., \psi_r)^T$ be the weight vector of $\tilde{\aleph}_\alpha$, $\psi_\alpha \in [0, 1]$ and $\Sigma_{\alpha=1} \psi_\alpha = 1$. Then, the value aggregated by using BNDPOWGA operator is still a BNN, which is calculated by using the following formula*

$$BNDPOWGA(\tilde{\aleph}_1, \tilde{\aleph}_2, ..., \tilde{\aleph}_r) = \tilde{\aleph}_{\sigma(1)}^{\bigwedge_D^{\upsilon_1}} \otimes_D \tilde{\aleph}_{\sigma(2)}^{\bigwedge_D^{\upsilon_2}} \otimes_D ... \otimes_D \tilde{\aleph}_{\sigma(r)}^{\bigwedge_D^{\upsilon_r}}$$

$$= \begin{cases} \langle \frac{1}{1+\{\Sigma_{\beta=1}^r \frac{\hbar_\beta\psi_\beta}{\Sigma_{\beta=1}^r \hbar_\beta\psi_\beta}(\frac{1-\tau_\beta^+}{\tau_\beta^+})^\lambda\}^{\frac{1}{\lambda}}}, 1 - \frac{1}{1+\{\Sigma_{\beta=1}^r \frac{\hbar_\beta\psi_\beta}{\Sigma_{\beta=1}^r \hbar_\beta\psi_\beta}(\frac{\omega_\beta^+}{1-\omega_\beta^+})^\lambda\}^{\frac{1}{\lambda}}}, 1 - \frac{1}{1+\{\Sigma_{\beta=1}^r \frac{\hbar_\beta\psi_\beta}{\Sigma_{\beta=1}^r \hbar_\beta\psi_\beta}(\frac{\mho_\beta^+}{1-\mho_\beta^+})^\lambda\}^{\frac{1}{\lambda}}}, \\ \frac{1}{1+\{\Sigma_{\beta=1}^r \frac{\hbar_\beta\psi_\beta}{\Sigma_{\beta=1}^r \hbar_\beta\psi_\beta}(\frac{-\tau_\beta^-}{1+\tau_\beta^-})^\lambda\}^{\frac{1}{\lambda}}} - 1, \frac{-1}{1+\{\Sigma_{\beta=1}^r \frac{\hbar_\beta\psi_\beta}{\Sigma_{\beta=1}^r \hbar_\beta\psi_\beta}(\frac{1+\omega_\beta^-}{-\omega_\beta^-})^\lambda\}^{\frac{1}{\lambda}}}, \frac{-1}{1+\{\Sigma_{\beta=1}^r \frac{\hbar_\beta\psi_\beta}{\Sigma_{\beta=1}^r \hbar_\beta\psi_\beta}(\frac{1+\mho_\beta^-}{-\mho_\beta^-})^\lambda\}^{\frac{1}{\lambda}}} \rangle \end{cases} \tag{10}$$

Proof. If $r = 2$ based on Definition 4.2 for the Dombi operations of BNNs , the following result can be obtained:

$$\text{BNDPOWGA } (\tilde{\aleph}_1, \tilde{\aleph}_2) = \tilde{\aleph}_{\sigma(1)}^{\bigwedge_D^{\upsilon_1}} \otimes_D \tilde{\aleph}_{\sigma(2)}^{\bigwedge_D^{\upsilon_2}}$$

$$= \begin{cases} \langle \frac{1}{1+\{\frac{\hbar_1\psi_1}{\hbar_1\psi_1+\hbar_2\psi_2}(\frac{1-\tau_1^+}{\tau_1^+})^\lambda + \frac{\hbar_2\psi_2}{\hbar_1\psi_1+\hbar_2\psi_2}(\frac{1-\tau_2^+}{\tau_2^+})^\lambda\}^{\frac{1}{\lambda}}}, 1 - \frac{1}{1+\{\frac{\hbar_1\psi_1}{\hbar_1\psi_1+\hbar_2\psi_2}(\frac{\omega_1^+}{1-\omega_1^+})^\lambda + \frac{\hbar_2\psi_2}{\hbar_1\psi_1+\hbar_2\psi_2}(\frac{\omega_2^+}{1-\omega_2^+})^\lambda\}^{\frac{1}{\lambda}}}, \\ 1 - \frac{1}{1+\{\frac{\hbar_1\psi_1}{\hbar_1\psi_1+\hbar_2\psi_2}(\frac{\mho_1^+}{1-\mho_1^+})^\lambda + \frac{\hbar_2\psi_2}{\hbar_1\psi_1+\hbar_2\psi_2}(\frac{\mho_2^+}{1-\mho_2^+})^\lambda\}^{\frac{1}{\lambda}}}, \frac{1}{1+\{\frac{\hbar_1\psi_1}{\hbar_1\psi_1+\hbar_2\psi_2}(\frac{-\tau_1^-}{1+\tau_1^-})^\lambda + \frac{\hbar_2\psi_2}{\hbar_1\psi_1+\hbar_2\psi_2}(\frac{-\tau_2^-}{1+\tau_2^-})^\lambda\}^{\frac{1}{\lambda}}} - 1, \\ \frac{-1}{1+\{\frac{\hbar_1\psi_1}{\hbar_1\psi_1+\hbar_2\psi_2}(\frac{1+\omega_1^-}{-\omega_1^-})^\lambda + \frac{\hbar_2\psi_2}{\hbar_1\psi_1+\hbar_2\psi_2}(\frac{1+\omega_2^-}{-\omega_2^-})^\lambda\}^{\frac{1}{\lambda}}}, \frac{-1}{1+\{\frac{\hbar_1\psi_1}{\hbar_1\psi_1+\hbar_2\psi_2}(\frac{1+\mho_1^-}{-\mho_1^-})^\lambda + \frac{\hbar_2\psi_2}{\hbar_1\psi_1+\hbar_2\psi_2}(\frac{1+\mho_2^-}{-\mho_2^-})^\lambda\}^{\frac{1}{\lambda}}} \rangle, \end{cases}$$

$$= \begin{cases} \langle \frac{1}{1+\{\Sigma_{\beta=1}^2 \frac{\hbar_\beta\psi_\beta}{\Sigma_{\beta=1}^2 \hbar_\beta\psi_\beta}(\frac{1-\tau_\beta^+}{\tau_\beta^+})^\lambda\}^{\frac{1}{\lambda}}}, 1 - \frac{1}{1+\{\Sigma_{\beta=1}^2 \frac{\hbar_\beta\psi_\beta}{\Sigma_{\beta=1}^2 \hbar_\beta\psi_\beta}(\frac{\omega_\beta^+}{1-\omega_\beta^+})^\lambda\}^{\frac{1}{\lambda}}}, 1 - \frac{1}{1+\{\Sigma_{\beta=1}^2 \frac{\hbar_\beta\psi_\beta}{\Sigma_{\beta=1}^2 \hbar_\beta\psi_\beta}(\Sigma_{\beta=1}^2 \frac{\mho_\beta^+}{1-\mho_\beta^+})^\lambda\}^{\frac{1}{\lambda}}}, \\ \frac{1}{1+\{\Sigma_{\beta=1}^2 \frac{\hbar_\beta\psi_\beta}{\Sigma_{\beta=1}^2 \hbar_\beta\psi_\beta}(\frac{-\tau_\beta^-}{1+\tau_\beta^-})^\lambda\}^{\frac{1}{\lambda}}} - 1, \frac{-1}{1+\{\Sigma_{\beta=1}^2 \frac{\hbar_\beta\psi_\beta}{\Sigma_{\beta=1}^2 \hbar_\beta\psi_\beta}(\frac{1+\omega_\beta^-}{-\omega_\beta^-})^\lambda\}^{\frac{1}{\lambda}}}, \frac{-1}{1+\{\Sigma_{\beta=1}^2 \frac{\hbar_\beta\psi_\beta}{\Sigma_{\beta=1}^2 \hbar_\beta\psi_\beta}(\frac{1+\mho_\beta^-}{-\mho_\beta^-})^\lambda\}^{\frac{1}{\lambda}}} \rangle. \end{cases}$$

If $r = s$, based on equation (10), then we have got the following equation:

$$\text{BNDPOWGA}(\tilde{\aleph}_1, \tilde{\aleph}_2, ..., \tilde{\aleph}_s) = \bigotimes_{\beta=1}^s \tilde{\aleph}_\beta^{\bigwedge^{\upsilon_\beta}}$$

$$= \begin{cases} \langle \frac{1}{1+\{\Sigma_{\beta=1}^s \frac{\hbar_\beta\psi_\beta}{\Sigma_{\beta=1}^s \hbar_\beta\psi_\beta}(\frac{1-\tau_\beta^+}{\tau_\beta^+})^\lambda\}^{\frac{1}{\lambda}}}, 1 - \frac{1}{1+\{\Sigma_{\beta=1}^s \frac{\hbar_\beta\psi_\beta}{\Sigma_{\beta=1}^s \hbar_\beta\psi_\beta}(\frac{\omega_\beta^+}{1-\omega_\beta^+})^\lambda\}^{\frac{1}{\lambda}}}, 1 - \frac{1}{1+\{\Sigma_{\beta=1}^s \frac{\hbar_\beta\psi_\beta}{\Sigma_{\beta=1}^s \hbar_\beta\psi_\beta}(\frac{\mho_\beta^+}{1-\mho_\beta^+})^\lambda\}^{\frac{1}{\lambda}}}, \\ \frac{1}{1+\{\Sigma_{\beta=1}^s \frac{\hbar_\beta\psi_\beta}{\Sigma_{\beta=1}^s \hbar_\beta\psi_\beta}(\frac{-\tau_\beta^-}{1+\tau_\beta^-})^\lambda\}^{\frac{1}{\lambda}}} - 1, \frac{-1}{1+\{\Sigma_{\beta=1}^s \frac{\hbar_\beta\psi_\beta}{\Sigma_{\beta=1}^s \hbar_\beta\psi_\beta}(\frac{1+\omega_\beta^-}{-\omega_\beta^-})^\lambda\}^{\frac{1}{\lambda}}}, \frac{-1}{1+\{\Sigma_{\beta=1}^s \frac{\hbar_\beta\psi_\beta}{\Sigma_{\beta=1}^s \hbar_\beta\psi_\beta}(\frac{1+\mho_\beta^-}{-\mho_\beta^-})^\lambda\}^{\frac{1}{\lambda}}} \rangle. \end{cases}$$

If $r = s + 1$, then there is following result:

$$\text{BNDPOWGA}(\tilde{\aleph}_1, \tilde{\aleph}_2, ..., \tilde{\aleph}_s, \tilde{\aleph}_{s+1}) = \bigotimes_{\beta=1}^{s} \tilde{\aleph}_{\sigma(\beta)}^{\bigwedge^{\upsilon_\beta}} \bigotimes \tilde{\aleph}_{\sigma(s+1)}^{\bigwedge^{\upsilon_{s+1}}}$$

$$= \begin{cases} \langle \frac{1}{1+\{\Sigma_{\beta=1}^{s} \frac{\hbar_\beta \psi_\beta}{\Sigma_{\beta=1}^{s} \hbar_\beta \psi_\beta} (\frac{1-\tau_\beta^+}{\tau_\beta^+})^\lambda\}^{\frac{1}{\lambda}}}, 1 - \frac{1}{1+\{\Sigma_{\beta=1}^{s} \frac{\hbar_\beta \psi_\beta}{\Sigma_{\beta=1}^{s} \hbar_\beta \psi_\beta} (\frac{\omega_\beta^+}{1-\omega_\beta^+})^\lambda\}^{\frac{1}{\lambda}}}, 1 - \frac{1}{1+\{\Sigma_{\beta=1}^{s} \frac{\hbar_\beta \psi_\beta}{\Sigma_{\beta=1}^{s} \hbar_\beta \psi_\beta} (\frac{\mho_\beta^+}{1-\mho_\beta^+})^\lambda\}^{\frac{1}{\lambda}}}, \\ \frac{1}{1+\{\Sigma_{\beta=1}^{s} \frac{\hbar_\beta \psi_\beta}{\Sigma_{\beta=1}^{s} \hbar_\beta \psi_\beta} (\frac{-\tau_\beta^-}{1+\tau_\beta^-})^\lambda\}^{\frac{1}{\lambda}}} - 1, \frac{1}{1+\{\Sigma_{\beta=1}^{s} \frac{\hbar_\beta \psi_\beta}{\Sigma_{\beta=1}^{s} \hbar_\beta \psi_\beta} (\frac{1+\omega_\beta^-}{-\omega_\beta^-})^\lambda\}^{\frac{1}{\lambda}}}, \frac{1}{1+\{\Sigma_{\beta=1}^{s} \frac{\hbar_\beta \psi_\beta}{\Sigma_{\beta=1}^{s} \hbar_\beta \psi_\beta} (\frac{1+\mho_\beta^-}{-\mho_\beta^-})^\lambda\}^{\frac{1}{\lambda}}} \rangle \end{cases}$$

$$\otimes_D \begin{cases} \langle \frac{1}{1+\{\frac{\hbar_{s+1} \psi_{s+1}}{\Sigma_{\beta=1}^{s+1} \hbar_\beta \psi_\beta} (\frac{1-\tau_{s+1}^+}{\tau_{s+1}^+})^\lambda\}^{\frac{1}{\lambda}}}, 1 - \frac{1}{1+\{\frac{\hbar_{s+1} \psi_{s+1}}{\Sigma_{\beta=1}^{s+1} \hbar_\beta \psi_\beta} (\frac{\omega_{s+1}^+}{1-\omega_{s+1}^+})^\lambda\}^{\frac{1}{\lambda}}}, 1 - \frac{1}{1+\{\frac{\hbar_{s+1} \psi_{s+1}}{\Sigma_{\beta=1}^{s+1} \hbar_\beta \psi_\beta} (\frac{\mho_{s+1}^+}{1-\mho_{s+1}^+})^\lambda\}^{\frac{1}{\lambda}}}, \\ \frac{1}{1+\{\frac{\hbar_{s+1} \psi_{s+1}}{\Sigma_{\beta=1}^{s+1} \hbar_\beta \psi_\beta} (\frac{-\tau_{s+1}^-}{1+\tau_{s+1}^-})^\lambda\}^{\frac{1}{\lambda}}} - 1, \frac{1}{1+\{\frac{\hbar_{s+1} \psi_{s+1}}{\Sigma_{\beta=1}^{s+1} \hbar_\beta \psi_\beta} (\frac{1+\omega_{s+1}^-}{-\omega_{s+1}^-})^\lambda\}^{\frac{1}{\lambda}}}, \frac{1}{1+\{\frac{\hbar_{s+1} \psi_{s+1}}{\Sigma_{\beta=1}^{s+1} \hbar_\beta \psi_\beta} (\frac{1+\mho_{s+1}^-}{-\mho_{s+1}^-})^\lambda\}^{\frac{1}{\lambda}}} \rangle \end{cases}$$

$$= \begin{cases} \langle \frac{1}{1+\{\Sigma_{\beta=1}^{s+1} \frac{\hbar_\beta \psi_\beta}{\Sigma_{\beta=1}^{s+1} \hbar_\beta \psi_\beta} (\frac{1-\tau_\beta^+}{\tau_\beta^+})^\lambda\}^{\frac{1}{\lambda}}}, 1 - \frac{1}{1+\{\Sigma_{\beta=1}^{s+1} \frac{\hbar_\beta \psi_\beta}{\Sigma_{\beta=1}^{s+1} \hbar_\beta \psi_\beta} (\frac{\omega_\beta^+}{1-\omega_\beta^+})^\lambda\}^{\frac{1}{\lambda}}}, 1 - \frac{1}{1+\{\Sigma_{\beta=1}^{s+1} \frac{\hbar_\beta \psi_\beta}{\Sigma_{\beta=1}^{s+1} \hbar_\beta \psi_\beta} (\frac{\mho_\beta^+}{1-\mho_\beta^+})^\lambda\}^{\frac{1}{\lambda}}}, \\ \frac{1}{1+\{\Sigma_{\beta=1}^{s+1} \frac{\hbar_\beta \psi_\beta}{\Sigma_{\beta=1}^{s+1} \hbar_\beta \psi_\beta} (\frac{-\tau_\beta^-}{1+\tau_\beta^-})^\lambda\}^{\frac{1}{\lambda}}} - 1, \frac{1}{1+\{\Sigma_{\beta=1}^{s+1} \frac{\hbar_\beta \psi_\beta}{\Sigma_{\beta=1}^{s+1} \hbar_\beta \psi_\beta} (\frac{1+\omega_\beta^-}{-\omega_\beta^-})^\lambda\}^{\frac{1}{\lambda}}}, \frac{1}{1+\{\Sigma_{\beta=1}^{s+1} \frac{\hbar_\beta \psi_\beta}{\Sigma_{\beta=1}^{s+1} \hbar_\beta \psi_\beta} (\frac{1+\mho_\beta^-}{-\mho_\beta^-})^\lambda\}^{\frac{1}{\lambda}}} \rangle \end{cases}$$

Hence, Theorem 4.6 is true for $r = s + 1$. Thus, Equation (10) holds for all r. □

The BNDPOWGA operator also has the following properties:

(1) Idompotency: Let all the BNNs be $\tilde{\aleph}_\alpha =< \tau_\alpha^+, \omega_\alpha^+, \mho_\alpha^+, \tau_\alpha^-, \omega_\alpha^-, \mho_\alpha^- >= \tilde{\aleph}$ $(\alpha = 1, 2, 3, ..., r)$, where $\psi = (\psi_1, \psi_2, ..., \psi_r)^T$ be the weight vector of $\tilde{\aleph}_\alpha$, $\psi_\alpha \in [0, 1]$ and $\Sigma_{\alpha=1}\psi_\alpha = 1$.
Then, $\text{BNDPOWGA}(\tilde{\aleph}_1, \tilde{\aleph}_2, ..., \tilde{\aleph}_r) = \tilde{\aleph}$.

(2) Monotonicity: Let $\tilde{\aleph}_\alpha(\alpha = 1, 2, ..., r)$ and $\tilde{\aleph}'_\alpha(\alpha = 1, 2, ..., r)$ be two families of BNNs, where $\psi = (\psi_1, \psi_2, ..., \psi_r)^T$ be the weight vector of $\tilde{\aleph}_\alpha$ and $\tilde{\aleph}'_\alpha$, $\psi_\alpha \in [0, 1]$ and $\Sigma_{\alpha=1}\psi_\alpha = 1$. For all α ,if $\tilde{\aleph}_\alpha \geq \tilde{\aleph}'_\alpha$, then $\text{BNDPOWGA}(\tilde{\aleph}_1, \tilde{\aleph}_2, ..., \tilde{\aleph}_r) \geq \text{BNDPOWGA}(\tilde{\aleph}'_1, \tilde{\aleph}'_2, ..., \tilde{\aleph}'_r)$.

(3) Boundedness: Let $\tilde{\aleph}_\alpha =< \tau_\alpha^+, \omega_\alpha^+, \mho_\alpha^+, \tau_\alpha^-, \omega_\alpha^-, \mho_\alpha^- > (\alpha = 1, , 2, 3, ..., r)$ be a family of BNNs, where $\psi = (\psi_1, \psi_2, ..., \psi_r)^T$ be the weight vector of $\tilde{\aleph}_\alpha$, $\psi_\alpha \in [0, 1]$ and $\Sigma_{\alpha=1}\psi_\alpha = 1$. Therefore, we have $\text{BNDPOWGA}(\tilde{\aleph}^-, \tilde{\aleph}^-, ..., \tilde{\aleph}^-) \leq \text{BNDPOWGA}(\tilde{\aleph}_1, \tilde{\aleph}_2, ..., \tilde{\aleph}_r) \leq \text{BNDPOWGA}(\tilde{\aleph}^+, \tilde{\aleph}^+, ..., \tilde{\aleph}^+)$, where $\tilde{\aleph}^- =< \tau_{\tilde{\aleph}^-}^+, \omega_{\tilde{\aleph}^-}^+, \mho_{\tilde{\aleph}^-}^+, \tau_{\tilde{\aleph}^-}^-, \omega_{\tilde{\aleph}^-}^-, \mho_{\tilde{\aleph}^-}^- >$

$$= \begin{cases} < min(\tau_1^+, \tau_2^+, ..., \tau_r^+), max(\omega_1^+, \omega_2^+, ..., \omega_r^+), max(\mho_1^+, \mho_2^+, ..., \mho_r^+), \\ max(\tau_1^-, \tau_2^-, ..., \tau_r^-), min(\omega_\alpha^-, \omega_\alpha^-, ..., \omega_\alpha^-), min(\mho_1^-, \mho_2^-, ..., \mho_r^-) > \end{cases}$$

and

$$\tilde{\aleph}^+ =< \tau_{\tilde{\aleph}^+}^+, \omega_{\tilde{\aleph}^+}^+, \mho_{\tilde{\aleph}^+}^+, t_{\tilde{\aleph}^+}^-, \omega_{\tilde{\aleph}^+}^-, \mho_{\tilde{\aleph}^+}^- >$$

$$= \begin{cases} < max(\tau_1^+, \tau_2^+, ..., \tau_r^+), min(\omega_1^+, \omega_2^+, ..., \omega_r^+), min(\mho_1^+, \mho_2^+, ..., \mho_r^+), \\ min(\tau_1^-, \tau_2^-, ..., \tau_r^-), max(\omega_\alpha^-, \omega_\alpha^-, ..., \omega_\alpha^-), max(\mho_1^-, \mho_2^-, ..., \mho_r^-) > . \end{cases}$$

Proof.

(1) Since $\tilde{\aleph}_\alpha =< \tau^+_\alpha, \omega^+_\alpha, \mho^+_\alpha, \tau^-_\alpha, \omega^-_\alpha, \mho^-_\alpha >= \tilde{\aleph}(\alpha = 1, , 2, 3, ..., r)$. Then, the following result can be obtained by using Equation (10):

$$\mathrm{BNDPOWGA}(\tilde{\aleph}_1, \tilde{\aleph}_2, ..., \tilde{\aleph}_r) = \bigotimes_{\beta=1}^{r} \tilde{\aleph}_\beta^{\bigwedge^{\upsilon_\beta}}$$

$$= \begin{cases} \langle \frac{1}{1+\{\Sigma_{\beta=1}^r \frac{\hbar_\beta\psi_\beta}{\Sigma_{\beta=1}^r \hbar_\beta\psi_\beta}(\frac{1-\tau^+_\beta}{\tau^+_\beta})^\lambda\}^{\frac{1}{\lambda}}}, 1-\frac{1}{1+\{\Sigma_{\beta=1}^r \frac{\hbar_\beta\psi_\beta}{\Sigma_{\beta=1}^r \hbar_\beta\psi_\beta}(\frac{\omega^+_\beta}{1-\omega^+_\beta})^\lambda\}^{\frac{1}{\lambda}}}, 1-\frac{1}{1+\{\Sigma_{\beta=1}^r \frac{\hbar_\beta\psi_\beta}{\Sigma_{\beta=1}^r \hbar_\beta\psi_\beta}(\frac{\mho^+_\beta}{1-\mho^+_\beta})^\lambda\}^{\frac{1}{\lambda}}}, \\ \frac{1}{1+\{\Sigma_{\beta=1}^r \frac{\hbar_\beta\psi_\beta}{\Sigma_{\beta=1}^r \hbar_\beta\psi_\beta}(\frac{-\tau^-_\beta}{1+\tau^-_\beta})^\lambda\}^{\frac{1}{\lambda}}}-1, \frac{-1}{1+\{\Sigma_{\beta=1}^r \frac{\hbar_\beta\psi_\beta}{\Sigma_{\beta=1}^r \hbar_\beta\psi_\beta}(\frac{1+\omega^-_\beta}{-\omega^-_\beta})^\lambda\}^{\frac{1}{\lambda}}}, \frac{-1}{1+\{\Sigma_{\beta=1}^r \frac{\hbar_\beta\psi_\beta}{\Sigma_{\beta=1}^r \hbar_\beta\psi_\beta}(\frac{1+\mho^-_\beta}{-\mho^-_\beta})^\lambda\}^{\frac{1}{\lambda}}} \rangle \end{cases}$$

$$= \begin{cases} \langle \frac{1}{1+\{(\frac{1-\tau^+}{\tau^+})^\lambda\}^{\frac{1}{\lambda}}}, 1-\frac{1}{1+\{(\frac{\omega^+}{1-\omega^+})^\lambda\}^{\frac{1}{\lambda}}}, 1-\frac{1}{1+\{(\frac{\mho^+}{1-\mho^+})^\lambda\}^{\frac{1}{\lambda}}}, \\ \frac{1}{1+\{(\frac{-\tau^-}{1+\tau^-})^\lambda\}^{\frac{1}{\lambda}}}-1, \frac{-1}{1+\{(\frac{1+\omega^-}{-\omega^-})^\lambda\}^{\frac{1}{\lambda}}}, \frac{-1}{1+\{(\frac{1+\mho^-}{-\mho^-})^\lambda\}^{\frac{1}{\lambda}}} \rangle \end{cases} =< \tau^+, \omega^+, \mho^+, \tau^-, \omega^-, \mho^- >= \tilde{\aleph}.$$

Hence, $\mathrm{BNDPOWGA}(\tilde{\aleph}_1, \tilde{\aleph}_2, ..., \tilde{\aleph}_r) = \tilde{\aleph}$ holds.

(2) The property is obvious based on the equation (10).

(3) Let $\tilde{\aleph}^- =< \tau^+_{\tilde{\aleph}^-}, \omega^+_{\tilde{\aleph}^-}, \mho^+_{\tilde{\aleph}^-}, \tau^-_{\tilde{\aleph}^-}, \omega^-_{\tilde{\aleph}^-}, \mho^-_{\tilde{\aleph}^-} >$ and $\tilde{\aleph}^+ =< \tau^+_{\tilde{\aleph}^+}, \omega^+_{\tilde{\aleph}^+}, \mho^+_{\tilde{\aleph}^+}, \tau^-_{\tilde{\aleph}^+}, \omega^-_{\tilde{\aleph}^+}, \mho^-_{\tilde{\aleph}^+} >$. There are the following inequalities:

$$\frac{1}{1+\{\Sigma_{\beta=1}^r \frac{\hbar_\beta\psi_\beta}{\Sigma_{\beta=1}^r \hbar_\beta\psi_\beta}(\frac{1-\tau^+_{\tilde{\aleph}^-}}{1^+_{\tilde{\aleph}^-}})^\lambda\}^{\frac{1}{\lambda}}} \leq \frac{1}{1+\{\Sigma_{\beta=1}^r \frac{\hbar_\beta\psi_\beta}{\Sigma_{\beta=1}^r \hbar_\beta\psi_\beta}(\frac{1-\tau^+_\beta}{\tau^+_\beta})^\lambda\}^{\frac{1}{\lambda}}} \leq \frac{1}{1+\{\Sigma_{\beta=1}^r \frac{\hbar_\beta\psi_\beta}{\Sigma_{\beta=1}^r \hbar_\beta\psi_\beta}(\frac{1-\tau^+_{\tilde{\aleph}^+}}{\tau^+_{\tilde{\aleph}^+}})^\lambda\}^{\frac{1}{\lambda}}},$$

$$1-\frac{1}{1+\{\Sigma_{\beta=1}^r \frac{\hbar_\beta\psi_\beta}{\Sigma_{\beta=1}^r \hbar_\beta\psi_\beta}(\frac{\omega^+_{\tilde{\aleph}^+}}{1-\omega^+_{\tilde{\aleph}^+}})^\lambda\}^{\frac{1}{\lambda}}} \leq 1-\frac{1}{1+\{\Sigma_{\beta=1}^r \frac{\hbar_\beta\psi_\beta}{\Sigma_{\beta=1}^r \hbar_\beta\psi_\beta}(\frac{\omega^+_\beta}{1-\omega^+_\beta})^\lambda\}^{\frac{1}{\lambda}}} \leq 1-\frac{1}{1+\{\Sigma_{\beta=1}^r \frac{\hbar_\beta\psi_\beta}{\Sigma_{\beta=1}^r \hbar_\beta\psi_\beta}(\frac{\omega^+_{\tilde{\aleph}^-}}{1-\omega^+_{\tilde{\aleph}^-}})^\lambda\}^{\frac{1}{\lambda}}},$$

$$1-\frac{1}{1+\{\Sigma_{\beta=1}^r \frac{\hbar_\beta\psi_\beta}{\Sigma_{\beta=1}^r \hbar_\beta\psi_\beta}(\frac{\mho^+_{\tilde{\aleph}^+}}{1-\mho^+_{\tilde{\aleph}^+}})^\lambda\}^{\frac{1}{\lambda}}} \leq 1-\frac{1}{1+\{\Sigma_{\beta=1}^r \frac{\hbar_\beta\psi_\beta}{\Sigma_{\beta=1}^r \hbar_\beta\psi_\beta}(\frac{\mho^+_\beta}{1-\mho^+_\beta})^\lambda\}^{\frac{1}{\lambda}}} \leq 1-\frac{1}{1+\{\Sigma_{\beta=1}^r \frac{\hbar_\beta\psi_\beta}{\Sigma_{\beta=1}^r \hbar_\beta\psi_\beta}(\frac{\mho^+_{\tilde{\aleph}^-}}{1-\mho^+_{\tilde{\aleph}^-}})^\lambda\}^{\frac{1}{\lambda}}},$$

$$\frac{1}{1+\{\Sigma_{\beta=1}^r \frac{\hbar_\beta\psi_\beta}{\Sigma_{\beta=1}^r \hbar_\beta\psi_\beta}(\frac{-\tau^-_{\tilde{\aleph}^+}}{1+\tau^-_{\tilde{\aleph}^+}})^\lambda\}^{\frac{1}{\lambda}}}-1 \leq \frac{1}{1+\{\Sigma_{\beta=1}^r \frac{\hbar_\beta\psi_\beta}{\Sigma_{\beta=1}^r \hbar_\beta\psi_\beta}(\frac{-\tau^-_\beta}{1+\tau^-_\beta})^\lambda\}^{\frac{1}{\lambda}}}-1 \leq \frac{1}{1+\{\Sigma_{\beta=1}^r \frac{\hbar_\beta\psi_\beta}{\Sigma_{\beta=1}^r \hbar_\beta\psi_\beta}(\frac{-\tau^-_{\tilde{\aleph}^-}}{1+\tau^-_{\tilde{\aleph}^-}})^\lambda\}^{\frac{1}{\lambda}}}-1,$$

$$\frac{-1}{1+\{\Sigma_{\beta=1}^r \frac{\hbar_\beta\psi_\beta}{\Sigma_{\beta=1}^r \hbar_\beta\psi_\beta}(\frac{1+\omega^-_{\tilde{\aleph}^-}}{-\omega^-_{\tilde{\aleph}^-}})^\lambda\}^{\frac{1}{\lambda}}} \leq \frac{-1}{1+\{\Sigma_{\beta=1}^r \frac{\hbar_\beta\psi_\beta}{\Sigma_{\beta=1}^r \hbar_\beta\psi_\beta}(\frac{1+\omega^-_\beta}{-\omega^-_\beta})^\lambda\}^{\frac{1}{\lambda}}}$$
$$\leq \frac{-1}{1+\{\Sigma_{\beta=1}^r \frac{\hbar_\beta\psi_\beta}{\Sigma_{\beta=1}^r \hbar_\beta\psi_\beta}(\frac{1+\omega^-_{\tilde{\aleph}^+}}{-\omega^-_{\tilde{\aleph}^+}})^\lambda\}^{\frac{1}{\lambda}}},$$

$$\frac{-1}{1+\{\Sigma_{\beta=1}^r \frac{\hbar_\beta\psi_\beta}{\Sigma_{\beta=1}^r \hbar_\beta\psi_\beta}(\frac{1+\mho^-_{\tilde{\aleph}^-}}{-\mho^-_{\tilde{\aleph}^-}})^\lambda\}^{\frac{1}{\lambda}}} \leq \frac{-1}{1+\{\Sigma_{\beta=1}^r \frac{\hbar_\beta\psi_\beta}{\Sigma_{\beta=1}^r \hbar_\beta\psi_\beta}(\frac{1+\mho^-_\beta}{-\mho^-_\beta})^\lambda\}^{\frac{1}{\lambda}}} \leq \frac{-1}{1+\{\Sigma_{\beta=1}^r \frac{\hbar_\beta\psi_\beta}{\Sigma_{\beta=1}^r \hbar_\beta\psi_\beta}(\frac{1+\mho^-_{\tilde{\aleph}^+}}{-\mho^-_{\tilde{\aleph}^+}})^\lambda\}^{\frac{1}{\lambda}}}.$$

Hence, $\mathrm{BNDPOWGA}(\tilde{\aleph}^-, \tilde{\aleph}^-, ..., \tilde{\aleph}^-) \leq \mathrm{BNDPOWGA}(\tilde{\aleph}_1, \tilde{\aleph}_2, ..., \tilde{\aleph}_r) \leq \mathrm{BNDPOWGA}(\tilde{\aleph}^+, \tilde{\aleph}^+, ..., \tilde{\aleph}^+)$ holds. □

Example 4.7 Let $\tilde{\aleph}_1 =< 0.6, 0.7, 0.3, -0.6, -0.3, -0.5 >$, $\tilde{\aleph}_2 =< 0.5, 0.4, 0.6, -0.6, -0.7, -0.3 >$, $\tilde{\aleph}_3 =< 0.6, 0.7, 0.4, -0.9, -0.7, -0.7 >$ and $\tilde{\aleph}_4 =< 0.2, 0.6, 0.8, -0.6, -0.3, -0.9 >$ be four BNNs and let the weight vector of BNNs $\tilde{\aleph}_\alpha(\alpha = 1, 2, 3, 4)$ be $\psi = (\frac{1}{2}, \frac{1}{4}, \frac{1}{8}, \frac{1}{8})^T$. Now, by Definition 4.2, $\hbar_1 = 1$, $\hbar_2 = s(\tilde{\aleph}_1) = 0.4667$, $\hbar_3 = s(\tilde{\aleph}_1)s(\tilde{\aleph}_2) = 0.2256$ and $\hbar_4 = s(\tilde{\aleph}_1)s(\tilde{\aleph}_2)s(\tilde{\aleph}_3) = 0.1128$. $\psi_1 = \frac{1}{2}$, $\psi_2 = \frac{1}{4}$, $\psi_3 = \frac{1}{8}$ and $\psi_4 = \frac{1}{8}$ are the weights of $\tilde{\aleph}_\alpha(\alpha = 1, 2, 3, 4)$ such that $\Sigma_{\alpha=1}\psi_\alpha = 1$. By Definition 4.2, $\frac{\hbar_1\psi_1}{\Sigma_{\beta=1}^4 \hbar_\beta\psi_\beta} = 0.7588$, $\frac{\hbar_2\psi_2}{\Sigma_{\beta=1}^4 \hbar_\beta\psi_\beta} = 0.1771$, $\frac{\hbar_3\psi_3}{\Sigma_{\beta=1}^4 \hbar_\beta\psi_\beta} = 0.0428$, $\frac{\hbar_4\psi_4}{\Sigma_{\beta=1}^4 \hbar_\beta\psi_\beta} = 0.0214$. Then, by Theorem 4.2, for $\lambda = 3$

$BNDPOWGA(\tilde{\aleph}_1, \tilde{\aleph}_2, \tilde{\aleph}_3, \tilde{\aleph}_4) =$

$$\begin{cases} \langle \frac{1}{1+\{0.7588(\frac{1-0.6}{0.6})^3+0.1771(\frac{1-0.5}{0.5})^3+0.0428(\frac{1-0.6}{0.6})^3+0.0214(\frac{1-0.2}{0.2})^3\}^{\frac{1}{3}}}, \\ 1- \\ \frac{1}{1+\{0.7588(\frac{0.7}{1-0.7})^3+0.1771(\frac{0.4}{1-0.4})^3+0.0428(\frac{0.7}{1-0.7})^3+0.0214(\frac{0.6}{1-0.6})^3\}^{\frac{1}{3}}}, \\ 1- \\ \frac{1}{1+\{0.7588(\frac{0.4}{1-0.4})^3+0.1771(\frac{0.6}{1-0.6})^3+0.0428(\frac{0.3}{1-0.3})^3+0.0214(\frac{0.8}{1-0.8})^3\}^{\frac{1}{3}}}, \\ \frac{1}{1+\{0.7588(\frac{0.9}{1+0.9})^3+0.1771(\frac{0.6}{1+0.6})^3+0.0428(\frac{0.6}{1+0.6})^3+0.0214(\frac{0.6}{1+0.6})^3\}^{\frac{1}{3}}} \\ -1, \\ \frac{-1}{1+\{0.7588(\frac{1-0.7}{0.7})^3+0.1771(\frac{1-0.7}{0.7})^3+0.0428(\frac{1-0.3}{0.3})^3+0.0214(\frac{1-0.3}{0.3})^3\}^{\frac{1}{3}}}, \\ \frac{-1}{1+\{0.7588(\frac{1-0.7}{0.7})^3+0.1771(\frac{1-0.3}{0.3})^3+0.0428(\frac{1-0.5}{0.5})^3+0.0214(\frac{1-0.9}{0.9})^3\}^{\frac{1}{3}}} \rangle, \end{cases}$$

$BNDPOWGA(\tilde{\aleph}_1, \tilde{\aleph}_2, \tilde{\aleph}_3, \tilde{\aleph}_4) = \langle 0.4519, 0.6852, 0.5651, -0.8915, -0.5098, -0.4292\rangle$.

5. Model for MADM using bipolar neutrosophic information

In this section, three comprehensive MADM methods are extended based on the proposed BNDPWGA and BNDPOWGA operators.

For MADM model with bipolar neutrosophic fuzzy information, let $A = \{A_1, A_2, ..., A_r\}$ be a set of alternatives and $C = \{C_1, C_2, ..., C_r\}$ be a set of attributes. For BNDPWGA and BNDPOWGA operators, there is a prioritization between the attributes expressed by the linear ordering $C_1 \succ C_2 \succ ... \succ C_r$, indicates attribute C_η has a higher priority than C_ρ, if $\eta < \rho$. Let $\psi = (\psi_1, \psi_2, ..., \psi_r)^T$ be the weight vector of attributes such that $\psi_\beta > 0$, $\Sigma_{\beta=1}\psi_\beta = 1(\beta = 1, 2, ..., r)$ and ψ_β refers to the weight of attribute C_β. Suppose that $N = (\tilde{\aleph}_{\alpha\beta})_{s\times r} = (\tau^+_{\alpha\beta}, \omega^+_{\alpha\beta}, \mho^+_{\alpha\beta}, \tau^-_{\alpha\beta}, \omega^-_{\alpha\beta}, \mho^-_{\alpha\beta})_{s\times r}(\alpha = 1, 2, ..., s)(\beta = 1, 2, ..., r)$ is BNN decision matrix, where $\tau^+_{\alpha\beta}, \omega^+_{\alpha\beta}, \mho^+_{\alpha\beta}$ indicates the truth membership degree, indeterminacy membership degree and falsity membership degree of alternative A_α under attribute C_β with respect to positive preferences and $\tau^-_{\alpha\beta}, \omega^-_{\alpha\beta}, \mho^-_{\alpha\beta}$ indicates the truth membership degree, indeterminacy membership degree and falsity membership degree of alternative A_α under attribute C_β with respect to negative preferences. We have conditions $\tau^+_{\alpha\beta}, \omega^+_{\alpha\beta}, \mho^+_{\alpha\beta} \in [0, 1]$ and $\tau^-_{\alpha\beta}, \omega^-_{\alpha\beta}, \mho^-_{\alpha\beta} \in [-1, 0]$ such that $0 \leq \tau^+_{\alpha\beta} + \omega^+_{\alpha\beta} + \mho^+_{\alpha\beta} - \tau^-_{\alpha\beta} - \omega^-_{\alpha\beta} - \mho^-_{\alpha\beta} \leq 6$ for $(\alpha = 1, 2, ..., s)$ $(\beta = 1, 2, ..., r)$.

An algorithm is constructed based on proposed bipolar neutrosophic Dombi aggregation operators which solves MADM problems.

Algorithm

Step 1 Collect information on the bipolar neutrosophic evaluation .

Step 2 Calculate score and the accuracy values of collected information.

The score values $s(\tilde{\aleph}_{\alpha\beta})$ and accuracy values $a(\tilde{\aleph}_{\alpha\beta})$ of alternatives A_α can be calculated by using Equations (1) and (2).

Step 3 Reordering the value of each attribute by using the comparison method.

Step 4 Derive the collective BNN $\tilde{\aleph}_\alpha$ $(\alpha = 1, 2, ..., s)$ for the alternative $A_\alpha(\alpha = 1, 2, ..., s)$.

By using method (1), calculate the values of $\hbar_{\alpha\beta}$ $(\alpha = 1, 2, ..., s)$ $(\beta, \delta = 2, 3, ..., r)$ as follows:

$$\hbar_{\alpha\beta} = \prod_{\alpha\delta=1}^{\alpha\beta-1} s(\tilde{\aleph}_{\alpha\delta})(\alpha = 1, 2, ..., s)(\beta = 2, 3, ..., r), \tag{11}$$

$$\hbar_{\alpha 1} = 1(\alpha = 1, 2, ..., s), \tag{12}$$

and utilize BNDPWGA operator to calculate the collective BNN for each alternative, then

$$\tilde{\aleph}_\alpha = \text{BNDPWGA}(\tilde{\aleph}_{\alpha 1}, \tilde{\aleph}_{\alpha 2}, ..., \tilde{\aleph}_{\alpha r}) = \bigotimes_{\beta=1}^{r} \tilde{\aleph}_{\alpha\beta}^{\bigwedge^{\upsilon_{\alpha\beta}}}$$

$$= \begin{cases} \langle \frac{1}{1+\{\Sigma_\beta^r \frac{\hbar_{\alpha\beta}\psi_{\alpha\beta}}{\Sigma_\beta^r \hbar_{\alpha\beta}\psi_{\alpha\beta}}(\frac{1-\tau_{\alpha\beta}^+}{\tau_{\alpha\beta}^+})^\lambda\}^{\frac{1}{\lambda}}}, 1 - \frac{1}{1+\{\Sigma_\beta^r \frac{\hbar_{\alpha\beta}\psi_{\alpha\beta}}{\Sigma_\beta^r \hbar_{\alpha\beta}\psi_{\alpha\beta}}(\frac{\omega_{\alpha\beta}^+}{1-\omega_{\alpha\beta}^+})^\lambda\}^{\frac{1}{\lambda}}}, 1 - \frac{1}{1+\{\Sigma_\beta^r \frac{\hbar_{\alpha\beta}\psi_{\alpha\beta}}{\Sigma_\beta^r \hbar_{\alpha\beta}\psi_{\alpha\beta}}(\frac{\mho_{\alpha\beta}^+}{1-\mho_{\alpha\beta}^+})^\lambda\}^{\frac{1}{\lambda}}}, \\ \frac{1}{1+\{\Sigma_\beta^r \frac{\hbar_{\alpha\beta}\psi_{\alpha\beta}}{\Sigma_\beta^r \hbar_{\alpha\beta}\psi_{\alpha\beta}}(\frac{-\tau_{\alpha\beta}^-}{1+\tau_{\alpha\beta}^-})^\lambda\}^{\frac{1}{\lambda}}} - 1, \frac{-1}{1+\{\Sigma_\beta^r \frac{\hbar_{\alpha\beta}\psi_{\alpha\beta}}{\Sigma_\beta^r \hbar_{\alpha\beta}\psi_{\alpha\beta}}(\frac{1+\omega_{\alpha\beta}^-}{-\omega_{\alpha\beta}^-})^\lambda\}^{\frac{1}{\lambda}}}, \frac{-1}{1+\{\Sigma_\beta^r \frac{\hbar_{\alpha\beta}\psi_{\alpha\beta}}{\Sigma_\beta^r \hbar_{\alpha\beta}\psi_{\alpha\beta}}(\frac{1+\mho_{\alpha\beta}^-}{-\mho_{\alpha\beta}^-})^\lambda\}^{\frac{1}{\lambda}}} \rangle, \end{cases} \tag{13}$$

where $\upsilon_{\alpha\beta} = \frac{\hbar_{\alpha\beta}\psi_{\alpha\beta}}{\Sigma_{j=1}^n \hbar_{\alpha\beta}\psi_{\alpha\beta}}$ and $\psi = (\psi_1, \psi_2, ..., \psi_r)^T$ is the weight vector $(\alpha = 1, 2, ..., s)$ such that $\psi_\beta \in [0, 1]$ and $\Sigma_{\beta=1}\psi_\beta = 1$.

By using method (2) calculate the values of $\hbar_{\alpha\gamma}$ $(\alpha = 1, 2, ..., s)$ $(\gamma = 1, 2, ..., r)$ as follows:

$$\hbar_{\alpha\gamma} = \prod_{\alpha\delta=1}^{\alpha\delta-1} s(\tilde{\aleph}_{\alpha\delta})(\alpha = 1, 2, ..., s)(\beta = 2, 3, ..., r), \tag{14}$$

$$\hbar_{\alpha 1} = 1(\alpha = 1, 2, ..., s), \tag{15}$$

and utilize BNDPOWGA operator to calculate the collective BNN for each alternative, then

$$\tilde{\aleph}_\alpha = \text{BNDPOWGA}(\tilde{\aleph}_{\alpha 1}, \tilde{\aleph}_{\alpha 2}, ..., \tilde{\aleph}_{\alpha r}) = \bigotimes_{\gamma=1}^{r} \tilde{\aleph}_{\alpha\gamma}^{\bigwedge^{\upsilon_{\alpha\gamma}}}$$

$$= \begin{cases} \langle \frac{1}{1+\{\Sigma_{\gamma=1}^r \frac{\hbar_{\alpha\gamma}\psi_{\alpha\gamma}}{\Sigma_{\gamma=1}^r \hbar_{\alpha\gamma}\psi_{\alpha\gamma}}(\frac{1-\tau_{\alpha\gamma}^+}{\tau_{\alpha\gamma}^+})^\lambda\}^{\frac{1}{\lambda}}}, 1 - \frac{1}{1+\{\Sigma_{\gamma=1}^r \frac{\hbar_{\alpha\gamma}\psi_{\alpha\gamma}}{\Sigma_{\gamma=1}^r \hbar_{\alpha\gamma}\psi_{\alpha\gamma}}(\frac{\omega_{\alpha\gamma}^+}{1-\omega_{\alpha\gamma}^+})^\lambda\}^{\frac{1}{\lambda}}}, 1 - \frac{1}{1+\{\Sigma_{\gamma=1}^r \frac{\hbar_{\alpha\gamma}\psi_{\alpha\gamma}}{\Sigma_{\gamma=1}^r \hbar_{\alpha\gamma}\psi_{\alpha\gamma}}(\frac{\mho_{\alpha\gamma}^+}{1-\mho_{\alpha\gamma}^+})^\lambda\}^{\frac{1}{\lambda}}}, \\ \frac{1}{1+\{\Sigma_{\gamma=1}^r \frac{\hbar_{\alpha\gamma}\psi_{\alpha\gamma}}{\Sigma_{\gamma=1}^r \hbar_{\alpha\gamma}\psi_{\alpha\gamma}}(\frac{-\tau_{\alpha\gamma}^-}{1+\tau_{\alpha\gamma}^-})^\lambda\}^{\frac{1}{\lambda}}} - 1, \frac{-1}{1+\{\Sigma_{\gamma=1}^r \frac{\hbar_{\alpha\gamma}\psi_{\alpha\gamma}}{\Sigma_{\gamma=1}^r \hbar_{\alpha\gamma}\psi_{\alpha\gamma}}(\frac{1+\omega_{\alpha\gamma}^-}{-\omega_{\alpha\gamma}^-})^\lambda\}^{\frac{1}{\lambda}}}, \frac{-1}{1+\{\Sigma_{\gamma=1}^r \frac{\hbar_{\alpha\gamma}\psi_{\alpha\gamma}}{\Sigma_{\gamma=1}^r \hbar_{\alpha\gamma}\psi_{\alpha\gamma}}(\frac{1+\mho_{\alpha\gamma}^-}{-\mho_{\alpha\gamma}^-})^\lambda\}^{\frac{1}{\lambda}}} \rangle, \end{cases} \tag{16}$$

where $\upsilon_{\alpha\gamma} = \frac{\hbar_{\alpha\gamma}\psi_{\alpha\gamma}}{\Sigma_{\gamma=1}^r \hbar_{\alpha\gamma}\psi_{\alpha\gamma}}$ and σ is permutation that orders the elements: $\tilde{\aleph}_{\sigma(\alpha 1)} \geq \tilde{\aleph}_{\sigma(\alpha 2)} \geq ... \geq \tilde{\aleph}_{\sigma(\alpha r)}$. where $\psi = (\psi_1, \psi_2, ..., \psi_r)^T$ is the weight vector such that $\psi_\gamma \in [0, 1]$ and $\Sigma_{\gamma=1}\psi_\gamma = 1$.

Step 5 Calculate the score values $s(\tilde{\aleph}_\alpha)$ $(\alpha = 1, 2, ..., s)$ of BNNs $\tilde{\aleph}_\alpha$ $(\alpha = 1, 2, ..., s)$ to rank all the alternatives A_α $(\alpha = 1, 2, ..., s)$ and then select favorable one(s). If score values $\tilde{\aleph}_\alpha$ and $\tilde{\aleph}_\beta$ of BNNs are equal, then we calculate accuracy values $a(\tilde{\aleph}_\alpha)$ and $a(\tilde{\aleph}_\beta)$ of BNNs $\tilde{\aleph}_\alpha$ and $\tilde{\aleph}_\beta$, respectively and then rank the alternatives A_α and A_β as accuracy values $a(\tilde{\aleph}_\alpha)$ and $a(\tilde{\aleph}_\beta)$.

Step 6 Rank all the alternatives $A_\alpha(\alpha = 1, 2, ..., s)$ and select favorable one(s).

Step 7 End.

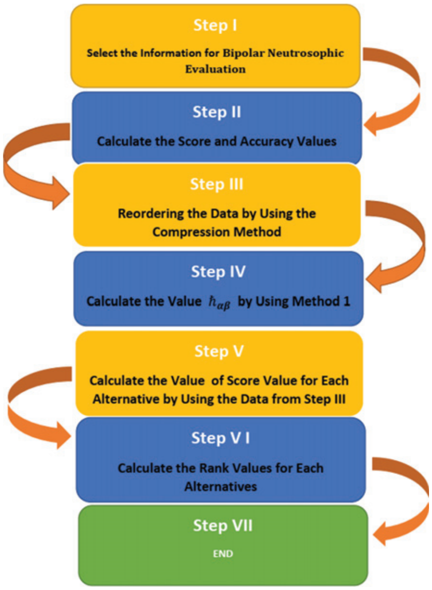

Fig. 1. Flow Chart.

The flow chart of proposed algorithm is given below in Figure 1.

6. Numerical example

A numerical example of the selection of cultivating crops taken from Deli et al. [10] is provided here. Furthermore, parametric analysis and comparative analysis confirm the flexibility and effectiveness of the proposed methods. In order to increase the production of agriculture, an agriculture department considers selecting a crop for cultivating in the farm. Four cultivating crops A_1, A_2, A_3, and A_4 are selected for further evaluation through preliminary screening. The agriculture department decided to invite four group experts to evaluate information. The expert group consists of investment experts, land experts, weather experts and labour experts. The four cultivating crops are evaluated by experts on the basis of four attributes or criteria: cost (C_1), nutrition of land (C_2), effect of weather (C_3), labor (C_4). For the proposed methods BNDPWGA and BNDPOWGA operators, the prioritization relation for the attributes is given as: $C_1 \succ C_2 \succ C_3 \succ C_4$. These attributes are interactive and interlinked.

Step 1 Collect information on bipolar neutrosophic evaluation. The information collected from expert discussion on evaluation is given in Table 1.

Step 2 Calculate score and accuracy values of collected information.

For each alternative A_α under attribute C_β, the score values $s(\tilde{\aleph}_{\alpha\beta})$ and accuracy values $a(\tilde{\aleph}_{\alpha\beta})$ can be calculated based on equations (1) and (2). The score values $s(\tilde{\aleph}_{\alpha\beta})$ and accuracy values $a(\tilde{\aleph}_{\alpha\beta})$ are shown in Tables 2 and 3, respectively.

Step 3 Reordering information on evaluation under each attribute.

Step 4 Derive the collective BNN $\tilde{\aleph}_\alpha$ ($\alpha = 1, 2, ..., s$) for the alternative $A_\alpha(\alpha = 1, 2, ..., s)$. Method (1) Calculate the values of $\hbar_{\alpha\beta}$ ($\alpha = 1, 2, ..., s$) ($\beta = 1, 2, ..., r$) using equations (11) and (12) as follows:

$$\hbar_{\alpha\beta} = \begin{pmatrix} 1.0000 & 0.4667 & 0.2333 & 0.1167 \\ 1.0000 & 0.4667 & 0.1711 & 0.1141 \\ 1.0000 & 0.5167 & 0.3014 & 0.1507 \\ 1.0000 & 0.4833 & 0.2256 & 0.1278 \end{pmatrix}$$

and utilize BNDPWGA operator using equation (13) and supporting $\lambda = 7$ to calculate the collective BNN for each alternative, then

$$\tilde{\aleph}_1 = \langle 0.1608, 0.6933, 0.5760, -0.7305, -0.2840, -0.4601\rangle;$$

$$\tilde{\aleph}_2 = \langle 0.6325, 0.6923, 0.7578, -0.6983, -0.1610, -0.1007\rangle;$$

$$\tilde{\aleph}_3 = \langle 0.2387, 0.4263, 0.3979, -0.5917, -0.2807, -0.2631\rangle;$$

$$\tilde{\aleph}_4 = \langle 0.3533, 0.6914, 0.7013, -0.7934, -0.5494, -0.1037\rangle.$$

By using method (2) calculate the values of $\hbar_{\alpha\beta}$ ($\alpha = 1, 2, ..., s$) ($\beta = 1, 2, ..., r$) using equations (14) and (15) as follows:

$$\hbar_{\alpha\beta} = \begin{pmatrix} 1.0000 & 0.4667 & 0.2333 & 0.1167 \\ 1.0000 & 0.4667 & 0.1711 & 0.1141 \\ 1.0000 & 0.5167 & 0.3014 & 0.1507 \\ 1.0000 & 0.4833 & 0.2256 & 0.1278 \end{pmatrix}$$

and utilize BNDPOWGA operator using equation (16) and supporting $\lambda = 7$ to calculate the collective BNN for each alternative, then

$$\tilde{\aleph}_1 = \langle 0.1608, 0.6933, 0.5788, -0.7937, -0.2999, -0.4601\rangle;$$

$$\tilde{\aleph}_2 = \langle 0.5612, 0.5900, 0.7046, -0.6067, -0.1244, -0.1441\rangle;$$

Table 1
Bipolar neutrosophic evaluation information

	C_1	C_2	C_3	C_4
A_1	$\langle 0.5, 0.7, 0.2, -0.7, -0.3, -0.6\rangle$	$\langle 0.4, 0.4, 0.5, -0.7, -0.8, -0.4\rangle$	$\langle 0.7, 0.7, 0.5, -0.8, -0.7, -0.6\rangle$	$\langle 0.1, 0.5, 0.7, -0.5, -0.2, -0.8\rangle$
A_2	$\langle 0.9, 0.7, 0.5, -0.7, -0.7, -0.1\rangle$	$\langle 0.7, 0.6, 0.8, -0.7, -0.5, -0.1\rangle$	$\langle 0.9, 0.4, 0.6, -0.1, -0.7, -0.5\rangle$	$\langle 0.5, 0.2, 0.7, -0.5, -0.1, -0.9\rangle$
A_3	$\langle 0.3, 0.4, 0.2, -0.6, -0.3, -0.7\rangle$	$\langle 0.2, 0.2, 0.2, -0.4, -0.7, -0.4\rangle$	$\langle 0.9, 0.5, 0.5, -0.6, -0.5, -0.2\rangle$	$\langle 0.7, 0.5, 0.3, -0.4, -0.2, -0.2\rangle$
A_4	$\langle 0.9, 0.7, 0.2, -0.8, -0.6, -0.1\rangle$	$\langle 0.3, 0.5, 0.2, -0.5, -0.5, -0.2\rangle$	$\langle 0.5, 0.4, 0.5, -0.1, -0.7, -0.2\rangle$	$\langle 0.4, 0.2, 0.8, -0.5, -0.5, -0.6\rangle$

Table 2
Score values $s(\tilde{\aleph}_{\alpha\beta})$

	C_1	C_2	C_3	C_4
A_1	0.4667	0.5000	0.5000	0.4000
A_2	0.4667	0.3667	0.6667	0.5167
A_3	0.5167	0.5833	0.5000	0.4833
A_4	0.4833	0.4667	0.5667	0.5000

Table 3
Accuracy values $a(\tilde{\aleph}_{\alpha\beta})$

	C_1	C_2	C_3	C_4
A_1	0.2000	−0.4000	0	−0.3000
A_2	−0.2000	−0.7000	0.7000	0.2000
A_3	0.2000	0	0	0.2000
A_4	0	−0.2000	0.1000	−0.3000

$$\tilde{\aleph}_3 = \langle 0.4097, 0.5978, 0.7582, -0.7181, -0.5555, -0.1441\rangle;$$

$$\tilde{\aleph}_4 = \langle 0.2072, 0.4162, 0.3979, -0.5511, -0.2903, -0.2629\rangle.$$

Step 5 Calculate the score values $s(\tilde{\aleph}_\alpha)$ $(\alpha = 1, 2, ..., s)$ of BNNs $\tilde{\aleph}_\alpha$ $(\alpha = 1, 2, ..., s)$ for each alternatives A_α $(\alpha = 1, 2, ..., s)$. The score values $s(\tilde{\aleph}_\alpha)$ is calculated by using equation (1).

Method 1 The following score values are obtained by using the BNDPWGA operator.
$s(\tilde{\aleph}_1) = 0.3175$; $s(\tilde{\aleph}_2) = 0.2910$; $s(\tilde{\aleph}_3) = 0.3944$; $s(\tilde{\aleph}_4) = 0.3034$.

Method 2 The following score values are obtained by using the BNDPOWGA operator.
$s(\tilde{\aleph}_1) = 0.3091$; $s(\tilde{\aleph}_2) = 0.3214$; $s(\tilde{\aleph}_3) = 0.3992$; $s(\tilde{\aleph}_4) = 0.3398$.
Step 6 Rank all the alternatives $A_\alpha(\alpha = 1, 2, ..., s)$ and select favorable one(s).

The alternative can be ranked in descending order based on the comparison method, and favorable alternative can be selected.
Method 1 The ranking order based on score values is obtained by using BNDPWGA operator: $A_3 \succ A_1 \succ A_4 \succ A_2$. Thus, A_3 is favorable.
Method 2 The ranking order based on score values is obtained by using BNDPOWGA operator: $A_3 \succ A_4 \succ A_2 \succ A_1$. Thus, A_3 is favorable.
Step 7 End.

7. Parametric analysis and comparative analysis

This section describes effect of parametric λ on decision making results and comparison between proposed methods and existing methods.

7.1. Analysis on the effect of parameter λ on decision making results

This subsection discusses the effect of parameter λ in detail.
First, effect of parameter λ on the proposed operators is as follows.
Table 5 shows that the corresponding ranking orders with respect to the BNDPWGA operator are changed as the value of λ changing from 1 to 10.
Table 5 shows that ranking order is stable and the corresponding favorable alternative remains identical, when the value of λ is changed for BNDPWGA operator. For, $1 \le \lambda \le 10$ the corresponding ranking order is $A_3 \succ A_1 \succ A_4 \succ A_2$, then favorable one is A_3. As a result, favorable stable alternative is A_3. The behavior of BNDPWGA operator is shown in Figure 2.

Table 4
Reordering bipolar neutrosophic evaluation information

	C_1	C_2	C_3	C_4
A_1	$\langle 0.7, 0.7, 0.5, -0.8, -0.7, -0.6\rangle$	$\langle 0.4, 0.4, 0.5, -0.7, -0.8, -0.4\rangle$	$\langle 0.5, 0.7, 0.2, -0.7, -0.3, -0.6\rangle$	$\langle 0.1, 0.5, 0.7, -0.5, -0.2, -0.8\rangle$
A_2	$\langle 0.9, 0.4, 0.6, -0.1, -0.7, -0.5\rangle$	$\langle 0.5, 0.2, 0.7, -0.5, -0.1, -0.9\rangle$	$\langle 0.9, 0.7, 0.5, -0.7, -0.7, -0.1\rangle$	$\langle 0.7, 0.6, 0.8, -0.7, -0.5, -0.1\rangle$
A_3	$\langle 0.2, 0.2, 0.2, -0.4, -0.7, -0.4\rangle$	$\langle 0.3, 0.4, 0.2, -0.6, -0.3, -0.7\rangle$	$\langle 0.9, 0.5, 0.5, -0.6, -0.5, -0.2\rangle$	$\langle 0.7, 0.5, 0.3, -0.4, -0.2, -0.2\rangle$
A_4	$\langle 0.5, 0.4, 0.5, -0.1, -0.7, -0.2\rangle$	$\langle 0.4, 0.2, 0.8, -0.5, -0.5, -0.6\rangle$	$\langle 0.9, 0.7, 0.2, -0.8, -0.6, -0.1\rangle$	$\langle 0.3, 0.5, 0.2, -0.5, -0.5, -0.2\rangle$

Table 5
Ranking orders with parameter of BNDPWGA operator

λ	$s(\tilde{\aleph}_1)$	$s(\tilde{\aleph}_2)$	$s(\tilde{\aleph}_3)$	$s(\tilde{\aleph}_4)$	$BNDPWGA$
1	0.4426	0.4250	0.4947	0.4380	$A_3 \succ A_1 \succ A_4 \succ A_2$
2	0.4090	0.3784	0.4653	0.3852	$A_3 \succ A_1 \succ A_4 \succ A_2$
3	0.3773	0.3423	0.4422	0.3499	$A_3 \succ A_1 \succ A_4 \succ A_2$
4	0.3547	0.3210	0.4249	0.3299	$A_3 \succ A_1 \succ A_4 \succ A_2$
5	0.3388	0.3073	0.4121	0.3176	$A_3 \succ A_1 \succ A_4 \succ A_2$
6	0.3269	0.2979	0.4023	0.3093	$A_3 \succ A_1 \succ A_4 \succ A_2$
7	0.3175	0.2910	0.3944	0.3034	$A_3 \succ A_1 \succ A_4 \succ A_2$
8	0.3099	0.2857	0.3880	0.2988	$A_3 \succ A_1 \succ A_4 \succ A_2$
9	0.3036	0.2817	0.3826	0.2953	$A_3 \succ A_1 \succ A_4 \succ A_2$
10	0.2984	0.2784	0.3781	0.2924	$A_3 \succ A_1 \succ A_4 \succ A_2$

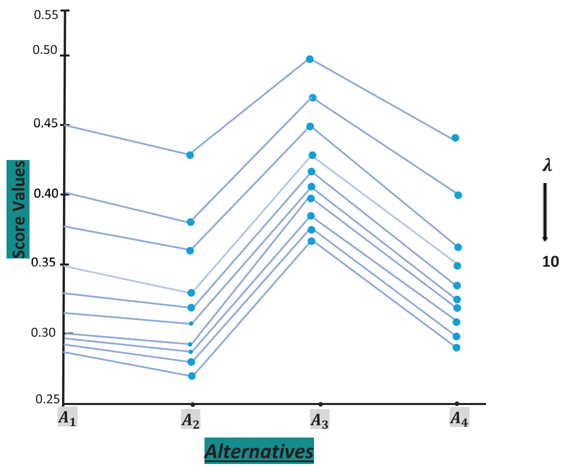

Fig. 2. BNDPWGA operator

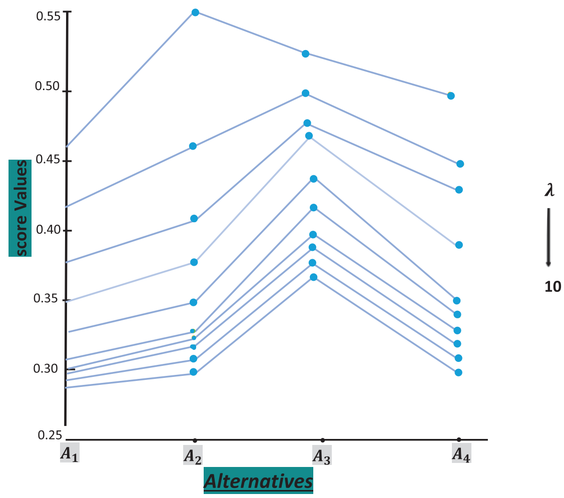

Fig. 3. BNDPOWGA operator

Table 6 shows that the corresponding ranking orders with respect to the BNDPOWGA operator are changed as the value of λ changing from 1 to 10.
Table 6 shows that the ranking order is different when the value of λ is changed for BNDPOWGA operator. For, $1 \leq \lambda \leq 2$, the corresponding ranking orders are $A_2 \succ A_3 \succ A_4 \succ A_1$ and $A_3 \succ A_2 \succ A_4 \succ A_1$. It follows that the favorable alternatives are A_2 and A_3, respectively. For, $3 \leq \lambda \leq 10$, the corresponding ranking order is $A_3 \succ A_4 \succ A_2 \succ A_1$. As a result, favorable one is A_3. The behavior of BNDPOWGA operator is shown in Figure 3.

7.2. Comparative analysis

In this subsection, a comparative analysis of the proposed methods based on proposed bipolar neutro-

Table 6
Ranking orders with parameter of BNDPOWGA operator

λ	$s(\tilde{\aleph}_1)$	$s(\tilde{\aleph}_2)$	$s(\tilde{\aleph}_3)$	$s(\tilde{\aleph}_4)$	$BNDPOWGA$
1	0.4655	0.5416	0.5292	0.5041	$A_2 \succ A_3 \succ A_4 \succ A_1$
2	0.4159	0.4545	0.4947	0.4522	$A_3 \succ A_4 \succ A_2 \succ A_1$
3	0.3747	0.4031	0.4647	0.4140	$A_3 \succ A_4 \succ A_2 \succ A_1$
4	0.3490	0.3710	0.4408	0.3894	$A_3 \succ A_4 \succ A_2 \succ A_1$
5	0.3317	0.3491	0.4229	0.3663	$A_3 \succ A_4 \succ A_2 \succ A_1$
6	0.3189	0.3332	0.4094	0.3512	$A_3 \succ A_4 \succ A_2 \succ A_1$
7	0.3091	0.3214	0.3398	0.3992	$A_3 \succ A_4 \succ A_2 \succ A_1$
8	0.3015	0.3123	0.3912	0.3308	$A_3 \succ A_4 \succ A_2 \succ A_1$
9	0.2955	0.3052	0.3849	0.3237	$A_3 \succ A_4 \succ A_2 \succ A_1$
10	0.2907	0.2994	0.3798	0.3179	$A_3 \succ A_4 \succ A_2 \succ A_1$

Table 7
Ranking orders obtained by different methods

Methods	Rankings
A_ψ operator [11]	$A_3 \succ A_4 \succ A_2 \succ A_1$
G_ψ operator [11]	$A_3 \succ A_4 \succ A_2 \succ A_1$
BN-TOPSIS method(ψ_1) [12]	$A_4 \succ A_2 \succ A_3 \succ A_1$
BN-TOPSIS method(ψ_2) [12]	$A_3 \succ A_2 \succ A_4 \succ A_1$
FBNCWBM operator ($s, t = 1$) [9]	$A_3 \succ A_4 \succ A_2 \succ A_1$
FBNCGBM operator ($s, t = 1$) [9]	$A_3 \succ A_4 \succ A_2 \succ A_1$
BNDPWGA operator ($\lambda = 7$)	$A_3 \succ A_1 \succ A_4 \succ A_2$
BNDPOWGA operator ($\lambda = 7$)	$A_3 \succ A_4 \succ A_2 \succ A_1$

Table 8
Characteristic comparison of different methods

Methods	Flexible measure easier	prioritized
A_ψ operator [11]	No	No
G_ψ operator [11]	No	No
BN-TOPSIS method(ψ_1) [12]	No	No
BN-TOPSIS method(ψ_2) [12]	No	No
FBNCWBM operator ($s, t = 1$) [9]	Yes	No
FBNCGBM operator ($s, t = 1$) [9]	Yes	No
BNDPWGA operator ($\lambda = 7$)	Yes	Yes
BNDPOWGA operator ($\lambda = 7$)	Yes	Yes

sophic Dombi prioritized aggregation operators with existing methods will be discussed.

The ranking orders obtained by the proposed methods show that A_3 is favorable alternative.

In contrast to A_ψ, G_ψ, BN-TOPSIS(ψ_1) and BN-TOPSIS(ψ_2), FBNCWBM ($s, t = 1$) and FBNCGBM ($s, t = 1$) methods, the proposed method based on proposed BNDPWGA operator considers the prioritized relationship among the attributes by establishing the prioritized aggregation operator. In some practical MADM problems, the prioritization relationship exists among attributes. Then, DMs can use the proposed method based on the proposed BNDPWGA operator to solve MADM problems which have prioritization relationship among attributes. In contrast to A_ψ, G_ψ, BN-TOPSIS(ψ_1) and BN-TOPSIS(ψ_2), FBNCWBM ($s, t = 1$) and FBNCGBM ($s, t = 1$) methods, the proposed method based on proposed BNDPOWGA operator considers the prioritized relationship among the attributes by establishing the prioritized aggregation operator, and the interaction and interrelationship among attributes by using ordered weighted geometric aggregation operator. In some practical MADM problems, the prioritization relationship, interaction and interrelationship exist among the attributes. Then, DMs can use the proposed method based on the proposed BNDPOWGA operator to solve the MADM problems which have the prioritization relationship, interaction and interrelationship among the attributes. Thus, the proposed methods based on the proposed bipolar neutrosophic Dombi prioritized aggregation operators are more reliable and flexible. The DMs can use these proposed methods based on the proposed bipolar neutrosophic Dombi prioritized aggregation operators according to their requirements in practical MADM problems.

8. Conclusion

BNSs describe fuzzy, bipolar, inconsistent and uncertain information. BNDPWGA and BNDPOWGA operators were proposed based on Dombi operations to make sure that the bipolar neutrosophic Dombi prioritized aggregation operators are reliable and flexible. Furthermore, a numerical example was given to verify proposed methods. The reliability and flexibility of the proposed methods were further illustrated through a parameter analysis and a comparative analysis with existing methods. The contribution of this paper is as follows. First, BNSs were used to present the decision-making information based on evaluation. Secondly, Dombi operations were put forward to bipolar neutrosophic fuzzy environments. Thirdly, BNDPWGA and BNDPOWGA operators are proposed under the bipolar neutrosophic fuzzy environment. Fourthly, MADM methods based on proposed bipolar neutrosophic Dombi prioritized aggregation operators were developed. Finally, the flexibility and reliability of proposed methods were verified by a numerical example. In future, BNDPWGA and BNDPOWGA operators can be put forward to other fuzzy environments such as bipolar neutrosophic soft expert sets and bipolar interval neutrosophic sets. Also it could be seen for MAGDM with multi-granular hesitant fuzzy linguistic term sets by means of different kinds of fuzzy environments.

Data Availability

The data used to support the findings of this study are available from the corresponding author upon request.

Conflict of interests

The authors declare that there is no conflict of interests regarding the publication of this paper.

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