On Finite and Infinite NeutroRings of Type-NR[8,9]

A.A.A. Agboola◇
Department of Mathematics, Federal University of Agriculture, Abeokuta, Nigeria.
agboolaaaa@funaab.edu.ng
◇ In commemoration of the 60th birthday of the author

Abstract

NeutroRings are alternatives to the classical rings and they are of different types. NeutroRings in some cases exhibit different algebraic properties, and in some cases they exhibit algebraic properties similar to the classical rings. The objective of this paper is to revisit the concept of NeutroRings and study finite and infinite NeutroRings of type-NR[8,9]. In NeutroRings of type-NR[8,9], the left and right distributive axioms are taking to be either partially true or partially false for some elements; while all other classical laws and axioms are taking to be totally true for all the elements. Several examples and properties of NeutroRings of type-NR[8,9] are presented. NeutroSubrings, NeutroIdeals, NeutroQuotientRings and NeutroRingHomomorphisms of the NeutroRings of type-NR[8,9] are studied with several interesting examples and their basic properties are presented. It is shown that in NeutroRings of type-NR[8,9], the fundamental theorem of homomorphisms of the classical rings holds.
Keywords: NeutroRing, AntiRing, NeutroSubring, NeutroIdeal, NeutroQuotientRing, NeutroRingHomomorphism.

1 Introduction and Preliminaries

In the classical rings $(R, +, .)$, addition and multiplication closure laws are 100% true for all the elements of R. Also, associative and distributive axioms over R are 100% true for all the elements of R. There are no provisions in the classical ring R to have addition and multiplication laws to be either partially true or partially indeterminate or partially false for the elements of R. Also, there are no provisions for associative and distributive axioms over R to be either partially true or partially indeterminate or partially false for the elements of R. Lack of these provisions in the classical rings posses problems because such rings cannot be used to model the real life situations accurately. These problems were addressed by Smarandache in [10] by introducing the concepts of NeutroAlgebraicStructures and AntiAlgebraicStructures. Smarandache further studied these new concepts in [9] and [8] respectively. With these new concepts, a lot of research activities have begun with some papers already published. For instance in [7], Rezaei and Smarandache studied Neutro-BE-algebras and Anti-BE-algebras and they showed that any classical algebra S with n operations (laws and axioms) where $n \geq 1$ will have $(2^n - 1)$ NeutroAlgebras and $(3^n - 2^n)$ AntiAlgebras. In [3], Agboola et al. studied NeutroAlgebras and AntiAlgebras viz-a-viz the classical number systems, in [4], Agboola studied NeutroGroups and in [5], he studied NeutroRings. Also in [2], Agboola revisited NeutroGroups and in [1], he studied AntiGroups. In the present paper, the concept of NeutroRings introduced in [5] is revisited. It is shown that there are 511 types of NeutroRings and 19171 types of AntiRings. In particular, finite and infinite NeutroRings of type-NR[8,9] are studied. In NeutroRings of type-NR[8,9], the left and right distributive axioms are taking to be either partially true or partially false for some elements; while all other classical laws and axioms are taking to be totally true for all the elements. Several examples and properties of NeutroRings of type-NR[8,9] are presented. NeutroSubrings, NeutroIdeals, NeutroQuotientRings and NeutroRingHomomorphisms of the NeutroRings of type-NR[8,9] are studied with several interesting examples and their basic properties are presented. It is shown that in NeutroRings of type-NR[8,9], the fundamental theorem of homomorphisms of the classical rings holds.

Definition 1.1. [8]

(i) A classical operation is an operation well defined for all the set's elements.

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(ii) A NeutroOperation is an operation partially well defined, partially indeterminate, and partially outer defined on the given set.

(iii) An AntiOperation is an operation that is outer defined for all set's elements.

(iv) A classical law/axiom defined on a nonempty set is a law/axiom that is totally true (i.e. true for all set's elements).

(v) A NeutroLaw/NeutroAxiom (or Neutrosophic Law/Neutrosophic Axiom) defined on a nonempty set is a law/axiom that is true for some set's elements [degree of truth (T)], indeterminate for other set's elements [degree of indeterminacy (I)], or false for the other set's elements [degree of falsehood (F)], where $T, I, F \in [0, 1]$, with $(T, I, F) \neq (1, 0, 0)$ that represents the classical axiom, and $(T, I, F) \neq (0, 0, 1)$ that represents the AntiAxiom.

(vi) An AntiLaw/AntiAxiom defined on a nonempty set is a law/axiom that is false for all set's elements.

(vii) A NeutroAlgebra is an algebra that has at least one NeutroOperation or one NeutroAxiom (axiom that is true for some elements, indeterminate for other elements, and false for other elements).

(viii) An AntiAlgebra is an algebra endowed with at least one AntiOperation or at least one AntiAxiom.

Theorem 1.2. *[7] Let $\mathbb{U}$ be a nonempty finite or infinite universe of discourse and let S be a finite or infinite subset of $\mathbb{U}$. If n classical operations (laws and axioms) are defined on S where $n \geq 1$, then there will be $(2^n - 1)$ NeutroAlgebras and $(3^n - 2^n)$ AntiAlgebras.*

2 NeutroRings Revisited

Definition 2.1. [Classical ring][6]
Let R be a nonempty set and let $+, . : R \times R \to R$ be binary operations of the usual addition and multiplication respectively defined on R. The triple $(R, +, .)$ is called a classical ring if the following conditions $(R1 - R9)$ hold:

(R1) $x + y \in R \; \forall x, y \in R$ [closure law of addition].

(R2) $x + (y + z) = (x + y) + z \; \forall x, y, z \in R$ [axiom of associativity].

(R3) There exists $e \in R$ such that $x + e = e + x = x \; \forall x \in R$ [axiom of existence of neutral element].

(R4) There exists $-x \in R$ such that $x + (-x) = (-x) + x = e \; \forall x \in G$ [axiom of existence of inverse element]

(R5) $x + y = y + x \; \forall x, y \in R$ [axiom of commutativity].

(R6) $x.y \in R \; \forall x, y \in R$ [closure law of multiplication].

(R7) $x.(y.z) = (x.y).z \; \forall x, y, z \in R$ [axiom of associativity].

(R8) $x.(y + z) = (x.y) + (x.z) \; \forall x, y, z \in R$ [axiom of left distributivity].

(R9) $(y + z).x = (y.x) + (z.x) \; \forall x, y, z \in R$ [axiom of right distributivity].

If in addition we have,

(R10) $x.y = y.x \; \forall x, y \in R$ [axiom of commutativity],

then $(R, +, .)$ is called a commutative ring.

Definition 2.2. [NeutroSophication of the laws and axioms of the classical ring]

(NR1) There exist at least three duplets $(x, y), (u, v), (p, q) \in R$ such that $x + y \in R$ (degree of truth T) and [$u + v =$ outer-defined/indeterminate (degree of indeterminacy I) or $p + q \notin R$] (degree of falsehood F) [NeutroClosure law of addition].

(NR2) There exist at least three triplets $(x, y, z), (p, q, r), (u, v, w) \in R$ such that $x + (y + z) = (x + y) + z$ (degree of truth T) and [$[p + (q + r)]$or$[(p + q) + r] =$ outer-defined/indeterminate (degree of indeterminacy I) or $u + (v + w) \neq (u + v) + w$] (degree of falsehood F) [NeutroAxiom of associativity (NeutroAssociativity)].

(NR3) There exists an element $e \in R$ such that $x+e = x+e = x$ and [$[x+e]$or$[e+x]$ = outer-defined/indeterminate or $x + e \neq x \neq e + x$] for at least one $x \in R$ [NeutroAxiom of existence of neutral element (NeutroNeutralElement)].

(NR4) There exists $-x \in R$ such that $x+(-x) = (-x)+x = e$ and [$[-x+x]$or$[x+(-x)]$ = outer-defined/indeterminate or $-x+x \neq e \neq x+(-x)$] for at least one $x \in R$ [NeutroAxiom of existence of inverse element (NeutroInverseElement)].

(NR5) There exist at least three duplets $(x, y), (u, v), (p, q) \in R$ such that $x+y = y+x$ and [$[p+q]$or$[q+p]$ = outer-defined/indeterminate (degree of indeterminacy I) or $u + v \neq v + u$] (degree of falsehood F) [NeutroAxiom of commutativity (NeutroCommutativity)].

(NR6) There exist at least three duplets $(x, y), (p, q), (u, v) \in R$ such that $x.y \in R$ (degree of truth T) and [$u.v$ = outer-defined/indeterminate (degree of indeterminacy I) or $p.q \notin R$] (degree of falsehood F) NeutroClosure law of multiplication].

(NR7) There exist at least three triplets $(x, y, z), (p, q, r), (u, v, w) \in R$ such that $x.(y.z) = (x.y).z$ (degree of truth T) and [$[p.(q.r)]$or$[(p.q).r]$ = outer-defined/indeterminate (degree of indeterminacy I) or $u.(v.w) \neq (u.v).w$] (degree of falsehood F) [NeutroAxiom of associativity (NeutroAssociativity)].

(NR8) There exist at least three triplets $(x, y, z), (p, q, r), (u, v, w) \in R$ such that $x.(y + z) = (x.y) + (x.z)$ (degree of truth T) and [$[p.(q + r)]$or$[(p.q) + (p.r)]$ = outer-defined/indeterminate (degree of indeterminacy I) or $u.(v + w) \neq (u.v) + (u.w)$] (degree of falsehood F) [NeutroAxiom of left distributivity (NeutroLeftDistributivity)].

(NR9) There exist at least three triplets $(x, y, z), (p, q, r), (u, v, w) \in R$ such that $(y + z).x = (y.x) + (z.x)$ (degree of truth T) and [$[(v + w).u]$or$[(v.u) + (w.u)]$ = outer-defined/indeterminate (degree of indeterminacy I) or $(v + w).u \neq (v.u) + (w.u)$] (degree of falsehood F) [NeutroAxiom of right distributivity (NeutroRightDistributivity)].

(NR10) There exist at least three duplets $(x, y), (p, q), (u, v) \in R$ such that $x.y = y.x$ (degree of truth T) and [$[p.q]$or$[q.p]$ = outer-defined/indeterminate (degree of indeterminacy I) or $u.v \neq v.u$] (degree of falsehood F) [NeutroAxiom of commutativity (NeutroCommutativity)].

Definition 2.3. [AntiSophication of the law and axioms of the classical ring]

(AR1) For all the duplets $(x, y) \in R$, $x + y \notin R$ [AntiClosure law of addition].

(AR2) For all the triplets $(x, y, z) \in R$, $x + (y + z) \neq (x + y) + z$ [AntiAxiom of associativity (AntiAssociativity)].

(AR3) There doest not exist an element $e \in R$ such that $x + e = x + e = x \; \forall x \in R$ [AntiAxiom of existence of neutral element (AntiNeutralElement)].

(AR4) There does not exist $-x \in R$ such that $x + (-x) = (-x) + x = e \; \forall x \in R$ [AntiAxiom of existence of inverse element (AntiInverseElement)].

(AR5) For all the duplets $(x, y) \in R$, $x + y \neq y + x$ [AntiAxiom of commutativity (AntiCommutativity)].

(AR6) For all the duplets $(x, y) \in R$, $x.y \notin R$ [AntiClosure law of multiplication].

(AR7) For all the triplets $(x, y, z) \in R$, $x.(y.z) \neq (x.y).z$ [AntiAxiom of associativity (AntiAssociativity)].

(AR8) For all the triplets $(x, y, z) \in R$, $x.(y + z) \neq (x.y) + (x.z)$ [AntiAxiom of left distributivity (AntiLeftDistributivity)].

(AR9) For all the triplets $(x, y, z) \in R$, $(y + z).x \neq (y.x) + (z.x)$ [AntiAxiom of right distributivity (AntiRightDistributivity)].

(AR10) For all the duplets $(x, y) \in R$, $x.y \neq y.x$ [AntiAxiom of commutativity (AntiCommutativity)].

Definition 2.4. [NeutroRing]
A NeutroRing NR is an alternative to the classical ring R that has at least one NeutroLaw or at least one of $\{NR1, NR2, NR3, NR4, NR5, NR6, NR7, NR8, NR9\}$ with no AntiLaw or AntiAxiom.

Definition 2.5. [AntiRing]
An AntiRing AR is an alternative to the classical ring R that has at least one AntiLaw or at least one of $\{AR1, AR2, AR3, AR4, AR5, AR6, AR7, AR8, AR9\}$.

Definition 2.6. [NeutroCommutativeRing]
A NeutroNoncommutativeRing NR is an alternative to the classical noncommutative ring R that has at least one NeutroLaw or at least one of $\{NR1, NR2, NR3, NR4, NR5, NR6, NR7, NR8, NR9\}$ and $NR10$ with no AntiLaw or AntiAxiom.

Definition 2.7. [AntiCommutativeRing]
An AntiCommutativeRing AR is an alternative to the classical commutative ring R that has at least one AntiLaw or at least one of $\{AR1, AR2, AR3, AR4, AR5, AR6, AR7, AR8, AR9\}$ and $AR10$.

Proposition 2.8. *Let $(R, +, .)$ be a finite or infinite classical ring. Then:*

(i) There are 511 types of NeutroRings.

(ii) There are 19171 types of AntiRings.

Proof. Follows from Theorem 1.2. □

Proposition 2.9. *Let $(R, +, .)$ be a finite or infinite classical commutative ring. Then:*

(i) There are 1023 types of NeutroCommutativeRings.

(ii) There are 58025 types of AntiCommutativeRings.

Proof. Follows from Theorem 1.2. □

Remark 2.10. It is evident from Proposition 2.8 and Proposition 2.9 that there are many types of NeutroRings and NeutroCommutativeRings. The type of NeutroRings studied by Agboola in [5] are those for which $NR1-NR10$ are all true.

Example 2.11.

(i) Let $NR = \mathbb{Z}$ and let $\oplus$ be a binary operation of ordinary addition and for all $x, y \in NR$, let $\odot$ be a binary operation defined on NR as $x \odot y = \sqrt{xy}$. Then $(NR, \oplus, \odot)$ is a NeutroRing.

(ii) Let $NR = \mathbb{Q}$ and let $\oplus$ be a binary operation of ordinary addition and for all $x, y \in NR$, let $\odot$ be a binary operation defined on NR as $x \odot y = x/y$. Then $(NR, \oplus, \odot)$ is a NeutroRing.

(iii) Let $AR = \mathbb{N}$ and let $\ominus$ and $\otimes$ be two binary operations of ordinary subtraction and ordinary multiplication respectively defined on AR. Then $(AR, \ominus, \otimes)$ is an AntiRing.

(iv) Let $AR = \mathbb{N}$ and let $\oplus$ and $\otimes$ be two binary operations of ordinary addition and ordinary multiplication respectively defined on AR. Then $(AR, \oplus, \otimes)$ is an AntiRing.

Definition 2.12. Let $(NR, +, .)$ be a NeutroRing.

(i) NR is called a finite NeutroRing of order n if the cardinality of NR is n that is $o(NR) = n$. Otherwise, NR is called an infinite NeutroRing and we write $o(NR) = \infty$.

(ii) NR is called a NeutroRing with unity if there exists a multiplicative unit element $u \in NR$ such that $ux = xu = x$ for at least one $x \in R$.

(iii) If there exists a least positive integer n such that $nx = e$ for at least one $x \in NR$, then NR is called a NeutroRing of characteristic n. If no such n exists, then NR is called a NeutroRing of characteristic zero.

(iv) An element $x \in NR$ is called an idempotent element if $x^2 = x$.

(v) An element $x \in NR$ is called a nilpotent element if for the least positive integer n, we have $x^n = e$.

(vi) An element $e \neq x \in NR$ is called a zero divisor element if there exists an element $e \neq y \in NR$ such that $xy = e$ or $yx = e$.

(vii) An element $x \in NR$ is called a multiplicative inverse element if there exists at least one $y \in NR$ such that $xy = yx = u$ where u is the multiplicative unity element in NR.

Definition 2.13. Let $(NR, +, .)$ be a NeutroCommutativeRing with unity. Then

(i) NR is called a NeutroIntegralDomain if NR has no at least one zero divisor element.

(ii) NR is called a NeutroField if NR has at least one multiplicative inverse element.

Example 2.14. Let $NR = \mathbb{Z}_5 = \{0, 1, 2, 3, 4\}$ and let $\oplus$ and $\odot$ be two binary operations defined on NR by

$$x \oplus y = x + y - 1, \ \ x \odot y = x + xy \ \ \forall\, x, y \in NR.$$

It is clear that $(NR, \oplus)$ is an abelian group.

(1) **NeutroAssociativity:** Let $x, y, z \in NR$. Then

$$\begin{aligned} x \odot (y \odot z) &= x + xy + xyz, \\ (x \odot y) \odot z &= x + xy + xz + xyz, \\ \therefore \quad x + xy + xyz &= x + xy + xz + xyz \\ \Rightarrow \quad xz &= 0 \\ \therefore \quad x &= 0 \ \text{ or } \ z = 0. \end{aligned}$$

This shows that only the triplets $(0, y, z), (x, y, 0), (0, y, 0)$ can verify associativity with 60% degree of associativity.

(2) **NeutroLeftDistributivity:** Let $x, y, z \in NR$. Then

$$\begin{aligned} x \odot (y \oplus z) &= x + xy + xz - x, \\ (x \odot y) \oplus (x \odot z) &= x + xy + x + xz - 1, \\ \therefore \quad x + xy + xz - x &= x + xy + x + xz - 1 \\ \Rightarrow \quad 2x &= 1 \\ \therefore \quad x &= 3. \end{aligned}$$

This shows that only the triplet $(3, y, z)$ can verify left distributivity with 20% degree of left distributivity.

(3) **NeutroRightDistributivity:** Let $x, y, z \in NR$. Then

$$\begin{aligned} (y \oplus z) \odot x &= y + z - 1 + yx + zx - x, \\ (y \odot x) \oplus (z \odot x) &= y + yx + z + zx - 1, \\ \therefore \quad y + z - 1 + yx + zx - x &= y + yx + z + zx - 1 \\ \Rightarrow \quad -x &= 0 \\ \therefore \quad x &= 0. \end{aligned}$$

This shows that only the triplet $(0, y, z)$ can verify right distributivity with 20% degree of right distributivity.

(4) **NeutroCommutativity:** Let $x, y \in NR$. Then

$$\begin{aligned} x \odot y &= x + xy, \\ y \odot x &= y + yx, \\ \therefore \quad x + xy &= y + yx \\ \Rightarrow \quad x &= y \\ \therefore \quad x &= y. \end{aligned}$$

This shows that only the duplet (x, x) can verify commutativity with 20% degree of commutativity.

We have just shown according to Definition 2.6 that $(NR, \oplus, \odot)$ is a NeutroRing.

Example 2.15. Let $NR = \{a, b, c, d\}$ and let $''+''$ and $''.''$ be binary operations defined on NR as shown in the Cayley tables below:

$+$	a	b	c	d
a	a	b	c	d
b	b	c	d	a
c	c	d	a	b
d	d	a	b	c

$.$	a	b	c	d
a	a	b	c	d
b	a	c	b	c
c	c	d	c	d
d	d	a	d	a

It is clear that $(NR, +)$ is an abelian group. From the tables we have:

(1) **NeutroAssociativity:**

$$\begin{aligned} a(bc) &= (ab)c = b, \\ b(bb) &= b \text{ but } (bb)b = d \neq b. \end{aligned}$$

This shows NeutroAssociativity of $''+''$.

(2) **NeutroLeftDistributivity:**

$$\begin{aligned} a(b+c) &= ab + ac = d, \\ b(c+d) &= c \text{ but } bc + bd = d \neq c. \end{aligned}$$

This shows NeutroLeftDistributivity of $''.''$ over $''+''$.

(3) **NeutroRightDistributivity:**

$$\begin{aligned} (b+c)c &= bc + cc = d, \\ (b+c)a &= d \text{ but } ba + ca = c \neq d. \end{aligned}$$

This shows NeutroRightDistributivity of $''.''$ over $''+''$.

(4) **NeutroCommutativity:**

$$\begin{aligned} ac &= ca = a, \\ bc &= b \text{ but } cb = d \neq b. \end{aligned}$$

This shows NeutroCommutativity of $''.''$.

We have just shown according to Definition 2.6 that $(NR, +, .)$ is a NeutroRing.

Example 2.16. From Example 2.15, we note that $e = a$ is the additive neutral element. We now have the following:

(i) NR is a NeutroCommutativeRing with unity since $aa = a, ac = ca = c, ad = da = d$.

(ii) $\{a, c\}$ are idempotent elements.

(iii) $\{d\}$ is a nilpotent element.

(iv) $\{b, d\}$ are zero divisor elements.

(v) $\{a, d\}$ are invertible elements.

(vi) NR is not a NeutroIntegralDomain.

(vii) NR is a NeutroField.

(viii) NR is a NeutroCommutativeRing of characteristic 2.

Example 2.17. Let $\mathbb{U} = \{e, a, b, c, d, f\}$ be a universe of discourse and let $NR = \{e, a, b, c\}$. Suppose that $*$ and $\circ$ are two binary operations defined on NR as shown in the Cayley tables below:

$\circ$	e	a	b	c
e	e	a	b	c
a	a	b or e	c	b
b	b	c	c or e	a
c	c	b	a	f

$*$	e	a	b	c
e	e	b	c	a or b or e
a	a	c	e	d
b	b	e	a	c
c	c	a	b	e

It is clear that $(NR, \circ)$ is a NeutroGroup. Now consider the following:

(i) **NeutroAssociativity of** $*$**:** $a*(b*b) = (a*b)*b = c$, $b*(a*b) = b$ but $(b*a)*b = c \neq b$, $a*(b*c) = d$ (outer-defined), $(a*b)*c = e*c =$ indeterminate.

(ii) **NeutroLeftDistributivity of** $*$ **over** $\circ$**:** $e*(e\circ e) = (e*e)\circ(e*e) = e$, $a*(b\circ e) = e$ but $(a*b)\circ(a*e) = a \neq e$, $a*(b\circ c) = e$ but $(a*b)\circ(a*c) = e\circ d = ?$.

(iii) **NeutroRightDistributivity of** $*$ **over** $\circ$**:** $(e\circ e)*e) = (e*e)\circ(e*e) = e$, $(b\circ c)*a = c$ but $(b*a)\circ(c*a) = a \neq c$, $(e\circ e)*c = e*e = ?$ and $(e*c)\circ(e*c) = ?$.

(iv) **NeutroCummutativity of** $*$**:** $e*e = c*c = e$, $b*c = c$ but $c*b = b \neq c$, $e*c =$ indeterminate but $c*e = c$.

Hence $(NR, \circ, *)$ is a NeutroRing.

3 Finite and Infinite NeutroRings of Type-NR[8,9]

In this section, we are going to study a type of NeutroRings $(NR, \circ, *)$ where $R1, R2, R3, R4, R5, R6, R7, R10$ are totally true for all the elements of NR, and where $R8$ and $R9$ are either partially true or partially false for some elements of NR. This type of NeutroRings will be called NeutroRings of type-NR[8,9].

Example 3.1. Let $NR = \mathbb{Z}_6 = \{0,1,2,3,4,5\}$ and let $\circ$ and $*$ be two binary operations defined on NR by

$$x\circ y = x+y, \quad x*y = x+y+xy \quad \forall x,y \in NR$$

where $''+''$ is addition modulo 6. Then $(NR, \circ, *)$ is a finite NeutroRing of type-NR[8,9]. To see this, consider the Cayley tables below.

$\circ$	0	1	2	3	4	5
0	0	1	2	3	4	5
1	1	2	3	4	5	0
2	2	3	4	5	0	1
3	3	4	5	0	1	2
4	4	5	0	1	2	3
5	5	0	1	2	3	4

$*$	0	1	2	3	4	5
0	0	1	2	3	4	5
1	1	3	5	1	3	5
2	2	5	2	5	2	5
3	3	1	5	3	1	5
4	4	3	2	1	0	5
5	5	5	5	5	5	5

It is clear from the tables that $(NR, \circ)$ is an abelian group with $e = 0$ as the identity element, and that $(NR, *)$ is a commutative semigroup. It remains to show that the two distributive axioms are NeutroAxioms.

(i) **NeutroLeftDistributivity of** $*$ **over** $\circ$**:** Let $x,y,z \in NR$. Then $x*(y\circ z) = x+y+z+xy+xz$ and $(x*y)\circ(x*z) = 2x+y+z+xy+xz$. For left distributivity to hold, we must have $2x = x$ from which we obtain $x = 0$. Hence, only the triplets $(0,y,z), (0,y,0), (0,0,z), (0,0,0)$ can verify the left distributivity of $*$ over $\circ$ with 66.67% degree of distributivity. Hence, $*$ is NeutroLeftDistributive over $\circ$ in NR.

(ii) **NeutroRightDistributivity of** $*$ **over** $\circ$**:** It can similarly be shown that $*$ is NeutroRightDistributive over $\circ$ with 66.67% degree of distributivity.

Hence, $(NR, \circ, *)$ is a finite NeutroRing of type-NR[8,9].

Example 3.2. Let $NR = \mathbb{Z}$ or $\mathbb{Q}$ or $\mathbb{R}$ or $\mathbb{C}$ and let $\circ$ and $*$ be two binary operations defined on NR by

$$x\circ y = x+y, \quad x*y = x+y+xy \quad \forall x,y \in NR$$

where $''+''$ is the ordinary addition of integers or rationals or reals or complex numbers. It is clear that $(NR, \circ)$ is an abelian group with $e = 0$ as the identity element, and that $(NR, *)$ is a commutative semigroup. It remains to show that the two distributive axioms are NeutroAxioms.

(i) **NeutroLeftDistributivity of** $*$ **over** $\circ$**:** Let $x,y,z \in NR$. Then $x*(y\circ z) = x+y+z+xy+xz$ and $(x*y)\circ(x*z) = 2x+y+z+xy+xz$. For left distributivity to hold, we must have $2x = x$ from which we obtain $x = 0$. Hence, only the triplets $(0,y,z), (0,y,0), (0,0,z), (0,0,0)$ can verify the left distributivity of $*$ over $\circ$. Hence, $*$ is NeutroLeftDistributive over $\circ$ in NR.

(ii) **NeutroRightDistributivity of $*$ over $\circ$:** It can similarly be shown that that $*$ is NeutroRightDistributive over $\circ$.

Hence, $(NR, \circ, *)$ is an infinite NeutroRing of type-NR[8,9].

Proposition 3.3. *Let $(NR_i, \circ, *), i = 1, 2$ be NeutroRings of type-NR[8,9]. In the Cartesian product $NR_1 \times NR_2$ of NR_i, let $\oplus$ and $\odot$ be two binary operations defined $\forall (w, x), (y, z) \in NR_1 \times NR_2$ as follows:*

$$\begin{aligned} (w, x) \oplus (y, z) &= (w \circ y, x \circ z) \\ (w, x) \odot (y, z) &= (w * y, x * z). \end{aligned}$$

Then $(NR_1 \times NR_2, \oplus, \odot)$ is a NeutroRing of type-NR[8,9].

Proof. Follows from Definition 2.2. □

Proposition 3.4. *Let $(NR, +, .)$ be a NeutroRing of type-NR[8,9] and let e be the identity element in NR with respect to $''+''$. Then for some $x, y, z \in NR$, we have:*

(i) $x.e \neq e$.

(ii) $x.(-y) \neq -(x.y) \neq (-x).y$.

(iii) $x.(y - z) \neq x.y - x.z$.

(iv) $(y - z).x \neq y.x - z.x$.

Proof. Since $''.''$ is NeutroDistributive over $''+''$, the required results follow. □

Proposition 3.5. *Let $(NR, +, .)$ be a NeutroRing of type-NR[8,9]. Then for some $w, x, y, z \in NR$, we have:*

(i) $(w + x).(y + z) \neq (w.y + w.z) + (x.y + x.z)$.

(ii) $(w + x).(y - z) \neq (w.y + x.y) - (w.z + x.z)$.

(iii) $(w - x).(y - z) \neq (w.y + x.z) - (w.z + x.y)$.

(iv) $(w + x).(w - x) \neq (w.w - x.x) + (x.w - w.x)$.

Proof. Since $''.''$ is NeutroDistributive over $''+''$, the required results follow. □

Proposition 3.6. *Let $(NR, +, .)$ be a NeutroRing of type-NR[8,9] and let $m, n \in \mathbb{N}$. Then $\forall x \in NR$, we have:*

(i) $x^m.x^n = x^{m+n}$.

(ii) $(x^m)^n = (x^n)^m = x^{mn}$.

Proof. Since $''.''$ is associative, the required results follow. □

Definition 3.7. Let $(NR, \circ, *)$ be a NeutroRing of type-NR[8,9] and let NS be nonempty subset of NR.

(i) NS is called a NeutroSubring of NR if $(NS, \circ, *)$ is also a NeutroRing of type-NR[8,9].

(ii) NS is called a QuasiNeutroSubring of NR if $(NS, \circ, *)$ is a NeutroRing of the type different from the type of the parent NeutroRing NR.

The only trivial NeutroSubring of NR is NR.

Proposition 3.8. *There exist NeutroRings of type-NR[8,9] with only trivial NeutroSubrings.*

Proof. Consider the structure $(NR, \circ, *)$ such that $NR = \mathbb{Z}_6$ and $\forall\ x, y \in NR$, we have $x \circ y = x + y + 1, x * y = x + y + 3xy$ and consider the structure $(NS, \circ, *)$ where $NS = \mathbb{Z}$ and $\forall\ x, y \in NS$, $x \circ y = x + y - 7, x * y = x + y - 3xy$. It can be shown that NR and NS are NeutroRings of type-NR[8,9] with only trivial NeutroSubrings. □

Example 3.9. Let $(NR, \circ, *)$ be the NeutroRing of Example 3.1 and let $NS_1 = \{0, 3\}$ and $NS_2 = \{0, 2, 4\}$ be two subsets of NR. It can easily be shown that $(NS_1, \circ, *)$ and $(NS_2, \circ, *)$ are NeutroRings of type-NR[8,9] and consequently they are NeutroSubrings of NR. It is observed that $NS_1 \cap NS_2 = \{0\}$ and $NS_1 \cup NS_2 = \{0, 2, 3, 4\}$ are not NeutroSubrings of NR. Also, $NS_1 \times NS_2 = \{(0, 0), (0, 2), (0, 4), (3, 0), (3, 3), (3, 4)\}$ is a NeutroSubring of $NR \times NR$.

Example 3.10. Let $(NR, \circ, *)$ be the NeutroRing of Example 3.2 and let $NS_1 = 2\mathbb{Z}$, $NS_2 = 3\mathbb{Z}$ and $NS_3 = 4\mathbb{Z}$ be three subsets of NR. It can easily be shown that NS_1, NS_2 and NS_3 are NeutroSubrings of NR. Generally for positive integers $n \geq 2$, it can be shown that $NS = n\mathbb{Z}$ are NeutroSubrings of NR. It is observed that $NS_1 \cap NS_2 = 6\mathbb{Z}$, $NS_1 \cap NS_3 = 4\mathbb{Z}$, $NS_2 \cap NS_3 = 12\mathbb{Z}$ and $NS_1 \cup NS_3 = 2\mathbb{Z}$ are NeutroSubrings of NR. However, $NS_1 \cup NS_2$ and $NS_2 \cup NS_3$ are not NeutroSubrings of NR.

Proposition 3.11. *Let $(NR, \circ, *)$ be a NeutroRing of type-NR[8,9] and let $\{NS_i\}, i = 1, 2$ be NeutroSubrings of NR. Then*

(i) $NS = NS_1 \cap NS_2$ is not necessarily a NeutroSubring of NR.

(ii) $NS = NS_1 \times NS_2$ is a NeutroSubring of $NR \times NR$.

(iii) $NS = NS_1 \cup NS_2$ is not necessarily a NeutroSubring of NR.

Definition 3.12. Let $(NR, \circ, *)$ be a NeutroRing of type-NR[8,9]. A nonempty subset NI of NR is called a NeutroIdeal of NR if the following conditions hold:

(i) NI is a NeutroSubring of NR.

(ii) $x \in NI$ and $r \in NR$ imply that at least one $r * x$ or $x * r \in NI$ for all $r \in NR$.

Definition 3.13. Let $(NR, \circ, *.)$ be a NeutroRing of type-NR[8,9]. A nonempty subset NI of NR is called a QuasiNeutroIdeal of NR if the following conditions hold:

(i) NI is a QuasiNeutroSubring of NR.

(ii) $x \in NI$ and $r \in NR$ imply that at least one $x * r$ or $r * x \in NI$ for all $r \in NR$.

Example 3.14. Let $NI_1 = NS_1 = \{0, 3\}$ and $NI_2 = NS_2 = \{0, 2, 4\}$ be NeutroSubrings of Example 3.9. Then for NI_1, we have $0 * 0 = 0, 1 * 0 = 1, 2 * 0 = 2, 3 * 0 = 3, 4 * 0 = 4, 5 * 0 = 5$ and $0 * 3 = 3, 1 * 3 = 1, 2 * 3 = 5, 3 * 3 = 3, 4 * 3 = 1, 5 * 3 = 5$. Accordingly, NI_1 is a NeutroIdeal.

Also for NI_2, we have $0 * 0 = 0, 1 * 0 = 1, 2 * 0 = 2, 3 * 0 = 3, 4 * 0 = 4, 5 * 0 = 5, 0 * 2 = 2, 1 * 2 = 5, 2 * 2 = 2, 3 * 2 = 5, 4 * 2 = 2, 5 * 2 = 5$ and $0 * 4 = 4, 1 * 4 = 3, 2 * 4 = 2, 3 * 4 = 1, 4 * 4 = 0, 5 * 4 = 5$. Accordingly, NI_2 is a NeutroIdeal.

Example 3.15. Let $NI_1 = NS_1 = 2\mathbb{Z}$, $NI_2 = NS_2 = 3\mathbb{Z}$ and $NI_3 = NS_3 = 4\mathbb{Z}$ be NeutroSubrings of Example 3.10. It can easily be shown that NI_1, NI_2 and NI_3 are NeutroIdeals. Generally, $NI = n\mathbb{Z}$ are NeutroIdeals for $n \geq 2$.

Definition 3.16. Let $(NR, \circ, *)$ be a NeutroRing of type-NR[8,9] and let NI be a NeutroIdeal of NR. The set NR/NI is defined by

$$NR/NI = \{x \circ NI : x \in NR\}.$$

For $x \circ NI, y \circ NI \in NR/NI$ with $x, y \in NR$, let $\oplus$ and $\odot$ be binary operations on NR/NI defined as follows:

$$\begin{aligned} (x \circ NI) \oplus (y \circ NI) &= (x \circ y) \circ NI, \\ (x \circ NI) \odot (y \circ NI) &= (x * y) \circ NI. \end{aligned}$$

If the triple $(NR/NI, \oplus, \odot)$ is a NeutroRing of type-NR[8,9], it will be called a NeutroQuotientRing.

Example 3.17. Let $NI_1 = \{0, 3\}$ and $NI_2 = \{0, 2, 4\}$ be NeutroIdeals of Example 3.14. For NI_1, we have

$$NR/NI_1 = \{NI_1, 1 + NI_1, 2 + NI_1\}$$

and the compositions of elements of NR/NI_1 according to Definition 3.16 are given in the Cayley tables:

$\oplus$	NI_1	$1+NI_1$	$2+NI_1$
NI_1	NI_1	$1+NI_1$	$2+NI_1$
$1+NI_1$	$1+NI_1$	$2+NI_1$	NI_1
$2+NI_1$	$2+NI_1$	NI_1	$1+NI_1$

$\odot$	NI_1	$1+NI_1$	$2+NI_1$
NI_1	NI_1	$1+NI_1$	$2+NI_1$
$1+NI_1$	$1+NI_1$	NI_1	$2+NI_1$
$2+NI_1$	$2+NI_1$	$2+NI_1$	$2+NI_1$

It can easily be deduced from the Cayley tables that $(NR/NI_1, \oplus, \odot)$ is a NeutroRing of type-NR[8,9] with $e = NI_1$ as the identity element.

For NI_2, we have

$$NR/NI_2 = \{NI_2, 1 + NI_2\}$$

and the compositions of elements of NR/NI_2 according to Definition 3.16 are given in the Cayley tables:

$\oplus$	NI_2	$1+NI_2$
NI_2	NI_2	$1+NI_2$
$1+NI_2$	$1+NI_2$	NI_2

$\odot$	NI_2	$1+NI_2$
NI_2	NI_2	$1+NI_2$
$1+NI_2$	$1+NI_2$	$1+NI_2$

It can easily be deduced from the Cayley tables that $(NR/NI_2, \oplus, \odot)$ is a NeutroRing of type-NR[8,9] with $e = NI_2$ as the identity element.

Example 3.18. Let $NI_1 = 2\mathbb{Z}$, $NI_2 = 3\mathbb{Z}$ and $NI_3 = 4\mathbb{Z}$ be NeutroIdeals of Example 3.15. For NI_1, we have

$$NR/NI_1 = \{NI_1, 1 + NI_1\}$$

and the compositions of elements of NR/NI_1 according to Definition 3.16 are given in the Cayley tables:

$\oplus$	NI_1	$1+NI_1$
NI_1	NI_1	$1+NI_1$
$1+NI_1$	$1+NI_1$	NI_1

$\odot$	NI_1	$1+NI_1$
NI_1	NI_1	$1+NI_1$
$1+NI_1$	$1+NI_1$	$1+NI_1$

It can easily be deduced from the Cayley tables that $(NR/NI_1, \oplus, \odot)$ is a NeutroRing of type-NR[8,9] with $e = NI_1$ as the identity element.

For NI_2, we have

$$NR/NI_2 = \{NI_2, 1 + NI_2, 2 + NI_2\}$$

and the compositions of elements of NR/NI_2 according to Definition 3.16 are given in the Cayley tables:

$\oplus$	NI_2	$1+NI_2$	$2+NI_2$
NI_2	NI_2	$1+NI_2$	$2+NI_2$
$1+NI_2$	$1+NI_2$	$2+NI_2$	NI_2
$2+NI_2$	$2+NI_2$	NI_2	$1+NI_2$

$\odot$	NI_2	$1+NI_2$	$2+NI_2$
NI_2	NI_2	$1+NI_2$	$2+NI_2$
$1+NI_2$	$1+NI_2$	NI_2	$2+NI_2$
$2+NI_2$	$2+NI_2$	$2+NI_2$	$2+NI_2$

It can easily be deduced from the Cayley tables that $(NR/NI_2, \oplus, \odot)$ is a NeutroRing of type-NR[8,9] with $e = NI_2$ as the identity element.

For NI_3, we have

$$NR/NI_3 = \{NI_3, 1 + NI_3, 2 + NI_3, 3 + NI_3\}$$

and the compositions of elements of NR/NI_3 according to Definition 3.16 are given in the Cayley tables:

$\oplus$	NI_3	$1+NI_3$	$2+NI_3$	$3+NI_3$
NI_3	NI_3	$1+NI_3$	$2+NI_3$	$3+NI_3$
$1+NI_3$	$1+NI_3$	$3+NI_3$	$1+NI_3$	$3+NI_3$
$2+NI_3$	$2+NI_3$	$1+NI_3$	NI_3	$3+NI_3$
$3+NI_3$	$3+NI_3$	$3+NI_3$	$3+NI_3$	$3+NI_3$

$\odot$	NI_3	$1+NI_3$	$2+NI_3$	$3+NI_3$
NI_3	NI_3	$1+NI_3$	$2+NI_3$	$3+NI_3$
$1+NI_3$	$1+NI_3$	$2+NI_3$	$3+NI_3$	NI_3
$2+NI_3$	$2+NI_3$	$3+NI_3$	NI_3	$1+NI_3$
$3+NI_3$	$3+NI_3$	NI_3	$1+NI_3$	$2+NI_3$

It can easily be deduced from the Cayley tables that $(NR/NI_3, \oplus, \odot)$ is a NeutroRing of type-NR[8,9] with $e = NI_3$ as the identity element.

Proposition 3.19. *Let $(NR, +, .)$ be a NeutroRing of type-NR[8,9] and let NI be a NeutroIdeal of NR. For $x+NI, y+NI \in NR/NI$ with $x, y \in NR$, let $\oplus$ and $\odot$ be binary operations on NR/NI defined as follows:*

$$\begin{aligned} (x + NI) \oplus (y + NI) &= (x + y) + NI, \\ (x + NI) \odot (y + NI) &= (xy) + NI. \end{aligned}$$

Then the triple $(NR/NI, \oplus, \odot)$ is a NeutroRing of type-NR[8,9] with $e = NI$ as the identity element.

Proof. Suppose that $(NR, +, .)$ is a NeutroRing of type-NR[8,9] and suppose that NI is NeutroIdeal of NR. That the binary operations $\oplus$ and $\odot$ on NR/NI are well-defined are the same as for the classical rings. It is clear that $(NR/NI, \oplus)$ is an abelian group with $e = NI$ as the identity element and that $(NR/NI, \odot)$ is a commutative semigroup. Since NR is of type-NR[8,9], it follows that there exists at least a triplet $(x, y, z) \in NR$ such that $x(y + z) \neq xy + xz$ and $(y + z)x \neq yx + zx$. Consequently,

$$\begin{aligned} (x + NI) \odot ((y + NI) \oplus (z + NI)) &= x(y + z) + NI \\ &\neq (xy + xz) + NI \\ &= [(x + NI) \odot (y + NI)] \oplus [(x + NI) \odot (z + NI)] \text{ and} \\ ((y + NI) \oplus (z + NI)) \odot (x + NI) &= (y + z)x + NI \\ &\neq (yx + zx) + NI \\ &= [(y + NI) \odot (x + NI)] \oplus [(z + NI) \odot (x + NI)]. \end{aligned}$$

Hence, $(NR/NI, \oplus, \odot)$ is a NeutroRing of type-NR[8,9] with $e = NI$ as the identity element. □

Definition 3.20. Let $(NR, +, .)$ and $(NS, +', .')$ be any two NeutroRings of type-NR[8,9]. The mapping $\phi : NR \to NS$ is called a NeutroRingHomomorphism if ϕ preserves the binary operations of NR and NS that is if for at least a duplet $(x, y) \in NR$, we have:

$$\begin{aligned} \phi(x + y) &= \phi(x) +' \phi(y), \\ \phi(x.y) &= \phi(x).'\phi(y). \end{aligned}$$

The kernel of ϕ denoted by $Ker\phi$ is defined as

$$Ker\phi = \{x : \phi(x) = e_{NR}\}.$$

The image of ϕ denoted by $Im\phi$ is defined as

$$Im\phi = \{y \in NS : y = \phi(x) \text{ for at least one } y \in NS\}.$$

If in addition ϕ is a NeutroBijection, then ϕ is called a NeutroRingIsomorphism and we write $NR \cong NS$. NeutroRingEpimorphism, NeutroRingMonomorphism, NeutroRingEndomorphism and NeutroRingAutomorphism are defined similarly.

Example 3.21. Let $(NR, +, *)$ be the NeutroRing of Example 3.1.

(i) Let $\phi : NR \to NR$ be a mapping defined by

$$\phi(x) = 2 * x \quad \forall x \in NR.$$

Then, ϕ is not a NeutroRingHomomorphism. Since $*$ is NeutroDistributive over $''+''$, we have for $x, y \in NR$,

$$\begin{aligned} \phi(x + y) &= 2 * (x + y) \\ &\neq 2 * x + 2 * y \\ &= \phi(x) + \phi(y). \end{aligned}$$

This shows that ϕ does not preserve $''+''$. However since $*$ is associative and $2 * 2 = 2$, we have $\forall x, y \in NR$

$$\begin{aligned} \phi(x * y) &= 2 * (x * y) \\ &= (2 * x) * (2 * y) \\ &= \phi(x) * \phi(y). \end{aligned}$$

This shows that ϕ preserves $*$. Accordingly, ϕ is not a NeutroRingHomomorphism.

(ii) Let $\phi : NR \times NR \to NR$ be a projection defined by

$$\phi(x, y) = x \quad \forall x, y \in NR.$$

It can easily be shown that ϕ is a NeutroRingHomomorphism with

$$\begin{aligned} Ker\phi &= \{(0,0),(0,1),(0,2),(0,3),(0,4),(0,5)\} \quad \text{and} \\ Im\phi &= \{0,1,2,3,4,5\} = NR. \end{aligned}$$

It can be shown that $Ker\phi$ is a NeutroIdeal of $NR \times NR$.

Example 3.22. Let $NR/NI_1 = \{NI_1, 1 + NI_1, 2 + NI_1\}$ be the NeutroQuotientRing of Example 3.17 and let $\phi : NR \to NR/NI_1$ be a mapping defined by $\phi(x) = x + NI_1\ \forall x \in NR$. Then

$$\begin{aligned} \phi(0) &= \phi(3) = NI_1, \\ \phi(1) &= \phi(4) = 1 + NI_1, \\ \phi(2) &= \phi(5) = 2 + NI_1. \end{aligned}$$

It can easily be shown that ϕ is a NeutroRingHomomorphism with $Ker\phi = \{0, 3\} = NI_1$.

Example 3.23. Let $NR/NI_3 = \{NI_3, 1 + NI_3, 2 + NI_3, 3 + NI_3\}$ be the NeutroQuotientRing of Example 3.18 and let $\phi : NR \to NR/NI_3$ be a mapping defined by $\phi(x) = x + NI_3\ \forall x \in NR$. Then

$$\begin{aligned} \phi(0) &= NI_3, \\ \phi(1) &= 1 + NI_3, \\ \phi(2) &= 2 + NI_3, \\ \phi(3) &= 3 + NI_3. \end{aligned}$$

It can easily be shown that ϕ is a NeutroRingHomomorphism with $Ker\phi = 4\mathbb{Z} = NI_3$.

Proposition 3.24. *Let NR and NS be two NeutroRings of type-NR[8,9] and suppose that $\phi : NR \to NS$ is a NeutroRingHomomorphism. Then:*

(i) $\phi(e_{NR}) = e_{NS}$.

(ii) $Ker\phi$ *is a NeutroIdeal of* NR.

(iii) $Im\phi$ *is a NeutroSubring of* NS.

(iv) ϕ *is NeutroInjective if and only if* $Ker\phi = \{e_{NR}\}$.

Proof. The proof is the same as for the classical rings and so omitted. □

Proposition 3.25. *Let NI be a NeutroIdeal of the NeutroRing NR of type-NR[8,9]. The mapping $\psi : NR \to NR/NI$ defined by*

$$\psi(x) = x + NI \quad \forall x \in NR$$

is a NeutroRingEpimomorphism and the $Ker\psi = NI$.

Proof. The proof is the same as for the classical rings and so omitted. □

Proposition 3.26. [Fundamental Theorem of NeutroRingHomomorphisms]. *Let NR and NS be NeutroRings of type-NR[8,9] and let $\phi : NR \to NS$ be a NeutroRingHomomorpism with $K = Ker\phi$. Then the mapping $\psi : NR/K \to Im\phi$ defined by*

$$\psi(x + K) = \phi(x) \quad \forall x \in NR$$

is a NeutroRingIsomorphism.

Proof. The proof is the same as for the classical rings and so omitted. □

4 Conclusion

We have in this paper revisited the concept of NeutroRings introduced by Agboola in [5]. It was shown that there are 511 types of NeutroRings and 19171 types of AntiRings. In particular, we have studied finite and infinite NeutroRings of type-NR[8,9]. In the class of NeutroRings of type-NR[8,9], the left and right distributive axioms were taking to be either partially true or partially false for some elements; while all other classical laws and axioms were taking to be totally true for all the elements. Several examples and properties of NeutroRings of type-NR[8,9] were presented. NeutroSubrings, NeutroIdeals, NeutroQuotientRings and NeutroRingHomomorphisms of the NeutroRings of type-NR[8,9] were studied with several interesting examples and their basic properties were presented. It was shown that in the class of NeutroRings of type-NR[8,9], the fundamental theorem of homomorphisms of the classical rings holds.

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