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On Single-Valued Neutrosophic Ideals in Šostak Sense

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Abstract: Neutrosophy is a recent section of philosophy. It was initiated in 1980 by Smarandache. It was presented as the study of origin, nature, and scope of neutralities, as well as their interactions with different ideational spectra. In this paper, we introduce the notion of single-valued neutrosophic ideals sets in Šostak's sense, which is considered as a generalization of fuzzy ideals in Šostak's sense and intuitionistic fuzzy ideals. The concept of single-valued neutrosophic ideal open local function is also introduced for a single-valued neutrosophic topological space. The basic structure, especially a basis for such generated single-valued neutrosophic topologies and several relations between different single-valued neutrosophic ideals and single-valued neutrosophic topologies, are also studied here. Finally, for the purpose of symmetry, we also define the so-called single-valued neutrosophic relations.

Keywords: single-valued neutrosophic closure; single-valued neutrosophic ideal; single-valued neutrosophic ideal open local function; single-valued neutrosophic ideal closure; single-valued neutrosophic ideal interior; single-valued neutrosophic ideal open compatible

1. Introduction

The notion of fuzzy sets, employed as an ordinary set generalization, was introduced in 1965 by Zadeh [1]. Later on, using fuzzy sets through the fuzzy topology concept was initially introduced in 1968 by Chang [2]. Afterwards, many properties in fuzzy topological spaces have been explored by various researchers [3–13]

Paradoxically, it is to be emphasized that being fuzzy or what is termed as fuzzy topology in fuzzy openness concept is not highlighted and well-studied. Meanwhile, Samanta et al. [14,15] introduced what is called the graduation of openness of fuzzy sets. Later on, Ramadan [16] introduced smooth continuity, a number of their properties, and smooth topology. Demirci [17] investigated properties and systems of smooth Q-neighborhood and smooth neighborhood alike. It is worth mentioning that Chattopadhyay and Samanta [18] have initiated smooth connectedness and smooth compactness. On the other hand, Peters [19] tackled the notion of primary fuzzy smooth characteristics and structures together with smooth topology in Lowen sense. He [20] further evidenced that smooth topologies collection constitutes a complete lattice. Furthermore, Onassanya and Hošková-Mayerová [21] inspected certain features of subsets of α-level as an integral part of a fuzzy subset topology. Likewise, more specialists in the field like Çoker and Demirci [22], in addition to Samanta and Mondal [23,24], have provided definitions to the concept of graduation intuitionistic openness of fuzzy sets based on Šostak's sense [25] according to Atanassov's [26] intuitionistic fuzzy sets. Essentially, they focused on intuitionistic gradation of openness in light of Chang. On the other hand, Lim et al. [27] examined

Lowen's framework smooth intuitionistic topological spaces. In recent times, Kim et al. [28] considered systems of neighborhood and continuities within smooth intuitionistic topological spaces. Moreover, Choi et al. [29] scrutinized smooth interval-valued topology through graduation of the concept of interval-valued openness of fuzzy sets, as suggested by Gorzalczany [30] and Zadeh [31], respectively. Ying [32] put forward a topology notion termed as fuzzifying topology, taking into consideration the extent of ordinary subset of a set openness. General properties in ordinary smooth topological spaces were elaborated in 2012 by Lim et al. [33]. In addition, they [34–36] inspected compactness, interiors, and closures within normal smooth topological spaces. In 2014, Saber et al. [37] shaped the notion of fuzzy ideal and r-fuzzy open local function in fuzzy topological spaces in view of the definition of Šostak. In addition, they [38,39] inspected intuitionistic fuzzy ideals, fuzzy ideals and fuzzy open local function in fuzzy topological spaces in view of the definition of Chang.

Smarandache [40] determined the notion of a neutrosophic set as intuitionistic fuzzy set generalization. Meanwhile, Salama et al. [41,42] familiarized the concepts of neutrosophic crisp set and neutrosophic crisp relation neutrosophic set theory. Correspondingly, Hur et al. [43,44] initiated classifications NSet(H) and NCSet including neutrosophic crisp and neutrosophic sets, where they examined them in a universe topological position. Furthermore, Salama and Alblowi [45] presented neutrosophic topology as they claimed a number of its characteristics. Salama et al. [46] defined a neutrosophic crisp topology and studied some of its properties. Others, such as Wang et al. [47], defined the single-valued neutrosophic set concept. Currently, Kim et al. [48] has come to grips with a neutrosophic partition single-value, neutrosophic equivalence relation single-value, and neutrosophic relation single-value.

Preliminaries of single-value neutrosophic sets and single-valued neutrosophic topology are reviewed in Section 2. Section 3 is devoted to the concepts of single-valued neutrosophic closure space and single-valued neutrosophic ideal. Some of their characteristic properties are considered. Finally, the concepts of single-valued neutrosophic ideal open local function has been introduced and studied. Several preservation properties and some characterizations concerning single-valued neutrosophic ideal open compatible have been obtained.

2. Preliminaries

In this section, we attempt to cover enough of the fundamental concepts and definitions.

Definition 1 ([49])**.** *A neutrosophic set $\mathcal{H}$ (**NS**, for short) on a nonempty set $\mathcal{S}$ is defined as*

$$\mathcal{H} = \langle \kappa, T_{\mathcal{H}}, I_{\mathcal{H}}, F_{\mathcal{H}} : \kappa \in \mathcal{S} \rangle,$$

where

$$T_{\mathcal{H}} : \mathcal{S} \to]^{-}0, 1^{+}[, \quad I_{\mathcal{H}} : \mathcal{S} \to]^{-}0, 1^{+}[, \quad F_{\mathcal{H}} : \mathcal{S} \to]^{-}0, 1^{+}[$$

and

$$^{-}0 \leq T_{\mathcal{H}}(\kappa) + I_{\mathcal{H}}(\kappa) + F_{\mathcal{H}}(\kappa) \leq 3^{+},$$

representing the degree of membership (namely, $T_{\mathcal{H}}(\kappa)$), the degree of indeterminacy (namely, $I_{\mathcal{H}}(\kappa)$), and the degree of nonmembership (namely, $F_{\mathcal{H}}(\kappa)$); for all $\kappa \in \mathcal{S}$ to the set $\mathcal{H}$.

Definition 2 ([49])**.** *Let $\mathcal{H}$ and $\mathcal{R}$ be fuzzy neutrosophic sets in $\mathcal{S}$. Then, $\mathcal{H}$ is a subset of $\mathcal{R}$ if, for each $\kappa \in \mathcal{S}$,*

$$\inf T_{\mathcal{H}}(x) \leq \inf T_{\mathcal{R}}(\kappa), \quad \inf I_{\mathcal{H}}(x) \geq \inf I_{\mathcal{R}}(\kappa), \quad \inf F_{\mathcal{H}}(x) \geq \inf F_{\mathcal{R}}(\kappa)$$

and

$$\sup T_{\mathcal{H}}(\kappa) \leq \sup T_{\mathcal{R}}(\kappa), \quad \sup I_{\mathcal{H}}(\kappa) \geq \sup I_{\mathcal{R}}(\kappa), \quad \sup F_{\mathcal{H}}(\kappa) \geq \sup F_{\mathcal{R}}(\kappa).$$

Definition 3 ([47]). *Let $\mathcal{H}$ be a space of points (objects) with a generic element in $\mathcal{S}$ denoted by κ. Then, $\mathcal{H}$ is called a single-valued neutrosophic set (in short,* **SVNS***) in $\mathcal{S}$ if $\mathcal{H}$ has the form $\mathcal{H} = \langle T_{\mathcal{H}}, I_{\mathcal{H}}, F_{\mathcal{H}} \rangle$, where $T_{\mathcal{H}}, I_{\mathcal{H}}, F_{\mathcal{H}} : \mathcal{S} \to [0,1]$.*

In this case, $T_{\mathcal{H}}, I_{\mathcal{H}}, F_{\mathcal{H}}$ are called truth-membership function, indeterminacy-membership function, and falsity-membership function, respectively, and we will denote the set of all **SVNS′s** *in $\mathcal{S}$ as* $\mathbf{SVNS}(\mathcal{S})$.

Moreover, we will refer to the Null (empty) **SVNS** *(or the absolute (universe)* **SVNS***) in $\mathcal{S}$ as 0_N (or 1_N) and define by $0_N = (0,1,1)$ (or $1_N = (1,0,0)$) for each $\kappa \in \mathcal{S}$.*

Definition 4 ([47]). *Let $\mathcal{H} = \langle T_{\mathcal{H}}, I_{\mathcal{H}}, F_{\mathcal{H}} \rangle$ be an* **SVNS** *on $\mathcal{S}$. The complement of the set $\mathcal{H}$ ($\mathcal{H}^c$, $for\ short$) and is defined as follows: for every $\kappa \in \mathcal{S}$,*

$$T_{\mathcal{H}^c}(\kappa) = F_{\mathcal{H}}(\kappa), \quad I_{\mathcal{H}^c}(\kappa) = 1 - I_{\mathcal{H}}(\kappa), \quad F_{\mathcal{H}^c}(\kappa) = T_{\mathcal{H}}(\kappa).$$

Definition 5 ([50]). *Suppose that $\mathcal{H} \in \mathbf{SVNS}(\mathcal{S})$. Then,*

(i) $\mathcal{H}$ is said to be contained in $\mathcal{R}$, denoted by $\mathcal{H} \subseteq \mathcal{R}$, if, for every $\kappa \in \mathcal{S}$,

$$T_{\mathcal{H}}(\kappa) \leq T_{\mathcal{R}}(\kappa), \quad I_{\mathcal{H}}(\kappa) \geq I_{\mathcal{R}}(\kappa), \quad F_{\mathcal{H}}(\kappa) \geq F_{\mathcal{R}}(\kappa);$$

(ii) $\mathcal{H}$ is said to be equal to $\mathcal{R}$, denoted by $\mathcal{H} = \mathcal{H}$, if $\mathcal{R} \subseteq \mathcal{R}$ and $\mathcal{H} \supseteq \mathcal{R}$.

Definition 6 ([51]). *Suppose that $\mathcal{H}, \mathcal{R} \in \mathbf{SVNS}(\mathcal{S})$. Then,*

(i) the union of $\mathcal{H}$ and $\mathcal{R}$ ($\mathcal{H} \cup \mathcal{R}$, $for\ short$) is an **SVNS** *in $\mathcal{S}$ defined as*

$$\mathcal{H} \cup \mathcal{R} = (T_{\mathcal{H}} \cup T_{\mathcal{R}}, I_{\mathcal{H}} \cap I_{\mathcal{R}}, F_{\mathcal{H}} \cap F_{\mathcal{R}}),$$

where $(T_{\mathcal{H}} \cup T_{\mathcal{R}})(\kappa) = T_{\mathcal{H}}(\kappa) \cup T_{\mathcal{R}}(\kappa)$ and $(F_{\mathcal{H}} \cap F_{\mathcal{R}})(\kappa) = F_{\mathcal{H}}(\kappa) \cap F_{\mathcal{R}}(\kappa)$, for each $\kappa \in \mathcal{S}$;

(ii) the intersection of $\mathcal{H}$ and $\mathcal{R}$, ($\mathcal{H} \cap \mathcal{R}$, $for\ short$), is an **SVNS** *in $\mathcal{S}$ defined as*

$$\mathcal{H} \cap \mathcal{R} = (T_{\mathcal{H}} \cap T_{\mathcal{R}}, I_{\mathcal{H}} \cup I_{\mathcal{R}}, F_{\mathcal{H}} \cup F_{\mathcal{R}}).$$

Definition 7 ([45]). *Let $\mathcal{H} \in \mathbf{SVNS}(\mathcal{S})$. Then,*

(i) the union of $\{\mathcal{H}_i\}_{i \in J}$ ($\bigcup_{i \in J} \mathcal{H}_i$, $for\ short$) is an **SVNS** *in $\mathcal{S}$ defined as follows: for every $\kappa \in \mathcal{S}$,*

$$(\bigcup_{i \in J} \mathcal{H}_i)(\kappa) = (\bigcup_{i \in J} T_{\mathcal{H}_i}(\kappa), \bigcap_{i \in J} I_{\mathcal{H}_i}(\kappa), \bigcap_{i \in J} F_{\mathcal{H}_i}(\kappa);$$

(ii) the intersection of $\{\mathcal{H}_i\}_{i \in J}$ ($\bigcap_{i \in J} \mathcal{H}_i$, $for\ short$) is an **SVNS** *in $\mathcal{S}$ defined as follows: for every $\kappa \in \mathcal{S}$,*

$$(\bigcap_{i \in J} \mathcal{H}_i)(\kappa) = (\bigcap_{i \in J} T_{\mathcal{H}_i}(\kappa), \bigcup_{i \in J} I_{\mathcal{H}_i}(\kappa), \bigcup_{i \in J} F_{\mathcal{H}_i}(\kappa).$$

Definition 8 ([52]). *A single-valued neutrosophic topology on $\mathcal{S}$ is a map $(\tau^T, \tau^I, \tau^F) : I^{\mathcal{S}} \to I$ satisfying the following three conditions:*

(SVNT1) $\tau^T(\underline{0}) = \tau^T(\underline{1}) = 1$ *and* $\tau^I(\underline{0}) = \tau^I(\underline{1}) = \tau^F(\underline{0}) = \tau^F(\underline{1}) = 0$,
(SVNT2) $\tau^T(\mathcal{H} \cap \mathcal{R}) \geq \tau^T(\mathcal{H}) \cap \tau^T(\mathcal{R}), \quad \tau^I(\mathcal{H} \cap \mathcal{R}) \leq \tau^I(\mathcal{H}) \cup \tau^I(\mathcal{R}),$
$\tau^F(\mathcal{H} \cap \mathcal{R}) \leq \tau^F(\mathcal{H}) \cup \tau^F(\mathcal{R})$, *for any* $\mathcal{H}, \mathcal{R} \in I^{\mathcal{S}}$,

(SVNT3) $\tau^T(\cup_{i\in j}\mathcal{H}_i) \geq \cap_{i\in j}\tau^T(\mathcal{H}_i)$, $\tau^I(\cup_{i\in j}\mathcal{H}_i) \leq \cup_{i\in j}\tau^I(\mathcal{H}_i)$, $\tau^F(\cup_{i\in j}\mathcal{H}_i) \leq \cup_{i\in j}\tau^F(\mathcal{H}_i)$, *for any* $\{\mathcal{H}_i\}_{i\in J} \in I^{\mathcal{S}}$.

The pair $(X, \tau^T, \tau^I, \tau^F)$ is called single-valued neutrosophic topological spaces (**SVNTS**, *for short*). We will occasionally write τ^{TIF} for (τ^T, τ^I, τ^F) and it will cause no ambiguity.

3. Single-Valued Neutrosophic Closure Space and Single-Valued Neutrosophic Ideal in Šostak Sense

This section deals with the definition of single-valued neutrosophic closure space. The researchers examine the connection between single-valued neutrosophic closure space and **SVNTS** based in Šostak sense. Moreover, the researchers focused on the single-valued neutrosophic ideal notion where they obtained fundamental properties. Based on Šostak's sense, where a single-valued neutrosophic ideal takes the form $(\mathcal{S}, \mathcal{L}^T, \mathcal{L}^I, \mathcal{L}^F)$ and the mappings $\mathcal{L}^T, \mathcal{L}^I, \mathcal{L}^F : I^{\mathcal{S}} \to I$, where $(\mathcal{L}^T, \mathcal{L}^I, \mathcal{L}^F)$ are the degree of openness, the degree of indeterminacy, and the degree of non-openness, respectively.

In this paper, $\mathcal{S}$ is used to refer to nonempty sets, whereas I is used to refer to closed interval $[0, 1]$ and I_o is used to refer to the interval $(0, 1]$. Concepts and notations that are not described in this paper are standard, instead, $\mathcal{S}$ is usually used.

Definition 9. *A mapping* $\mathbb{C} : I^{\mathcal{S}} \times I_0 \to I^{\mathcal{S}}$ *is called a single-valued neutrosophic closure operator on* $\mathcal{S}$ *if, for every* $\mathcal{H}, \mathcal{R} \in I^{\mathcal{S}}$ *and* $r, s \in I_0$*, the following axioms are satisfied:*

$(\mathbb{C}_1)$ $\mathbb{C}((0.1.1), s) = (0.1.1)$,
$(\mathbb{C}_2)$ $\mathcal{H} \leq \mathbb{C}(\mathcal{H}, s)$,
$(\mathbb{C}_3)$ $\mathbb{C}(\mathcal{H}, s) \vee \mathbb{C}(\mathcal{R}, s) = \mathbb{C}(\mathcal{H} \vee \mathcal{R}, s)$,
$(\mathbb{C}_4)$ $\mathbb{C}(\mathcal{H}, s) \leq \mathbb{C}(\mathcal{H}, r)$ *if* $s \leq r$,
$(\mathbb{C}_5)$ $\mathbb{C}(\mathbb{C}(\mathcal{H}, s), s) = \mathbb{C}(\mathcal{H}, s)$.

The pair $(X, \mathbb{C})$ is a single-valued neutrosophic closure space ($\mathcal{SVNCS}$, *for short*).

Suppose that $\mathbb{C}_1$ and $\mathbb{C}_2$ are single-valued neutrosophic closure operators on $\mathcal{S}$. Then, $\mathbb{C}_1$ is finer than $\mathbb{C}_2$, denoted by $\mathbb{C}_2 \leq \mathbb{C}_1$ iff $\mathbb{C}_1(\mathcal{H}, s) \leq \mathbb{C}_2(\mathcal{H}, s)$, for every $\mathcal{H} \in I^{\mathcal{S}}$ and $s \in I_0$.

Theorem 1. *Let* $(\mathcal{S}, \tau^{TIF})$ *be an* **SVNTS**. *Then, for any* $\mathcal{H} \in I^{\mathcal{S}}$ *and* $s \in I_0$*, we define an operator* $\mathbb{C}_{\tau^{TIF}} : I^{\mathcal{S}} \times I_0 \to I^{\mathcal{S}}$ *as follows:*

$$\mathbb{C}_{\tau^{TIF}}(\mathcal{H}, s) = \bigwedge\{\mathcal{R} \in I^X : \mathcal{H} \leq \mathcal{R}, \quad \tau^T(\underline{1} - \mathcal{R}) \geq s, \quad \tau^I(\underline{1} - \mathcal{R}) \leq 1 - s, \quad \tau^F(\underline{1} - \mathcal{R}) \leq 1 - s\}.$$

Then, $(\mathcal{S}, \mathbb{C}_{\tau^{TIF}})$ *is an* $\mathcal{SVNCS}$.

Proof. Suppose that $(\mathcal{S}, \tau^{TIF})$ is an **SVNTS**. Then, $\mathbb{C}_1$, $(\mathbb{C}_2)$ and $(\mathbb{C}_4)$ follows directly from the definition of $\mathbb{C}_{\tau^{TIF}}$.

$(\mathbb{C}_3)$ Since $\mathcal{R}, \mathcal{H} \leq \mathcal{H} \cup \mathcal{R}$, $\mathbb{C}_{\tau^{TIF}}(\mathcal{R}, s) \leq \mathbb{C}_{\tau^{TIF}}(\mathcal{H} \cup \mathcal{R}, s)$ and $\mathbb{C}_{\tau^{TIF}}(\mathcal{H}, s) \leq \mathbb{C}_{\tau^{TIF}}(\mathcal{H} \cup \mathcal{R}, s)$, therefore,

$$\mathbb{C}_{\tau^{TIF}}(\mathcal{H}, s) \cup \mathbb{C}_{\tau^{TIF}}(\mathcal{R}, s) \leq \mathbb{C}_{\tau^{TIF}}(\mathcal{H} \cup \mathcal{R}, s).$$

Let (X, τ^{TIF}) be an **SVNTS**. From $(\mathbb{C}_2)$, we have

$$\mathcal{H} \leq \mathbb{C}_{\tau^{TIF}}(\mathcal{H}, s), \quad \tau^T(\underline{1} - \mathbb{C}_{\tau^{TIF}}(\mathcal{H}, s)) \geq s, \ \tau^I(\underline{1} - \mathbb{C}_{\tau^{TIF}}(\mathcal{H}, s)) \leq 1 - s$$
$$\text{and } \tau^F(\underline{1} - \mathbb{C}_{\tau^{TIF}}(\mathcal{H}, s)) \leq 1 - s,$$

$$\mathcal{R} \leq \mathbb{C}_{\tau^{TIF}}(\mathcal{R}, s), \quad \tau^T(\underline{1} - \mathbb{C}_{\tau^{TIF}}(\mathcal{R}, s)) \geq s, \ \tau^I(\underline{1} - \mathbb{C}_{\tau^{TIF}}(\mathcal{R}, s)) \leq 1 - s$$
$$\text{and } \tau^F(\underline{1} - \mathbb{C}_{\tau^{TIF}}(\mathcal{R}, s)) \leq 1 - s.$$

It implies that $\mathcal{H} \cup \mathcal{R} \leq \mathbb{C}_{\tau^{TIF}}(\mathcal{H}, s) \cup \mathbb{C}_{\tau^{TIF}}(\mathcal{R}, s)$,

$$\begin{aligned} \tau^T(\underline{1} - (\mathbb{C}_{\tau^{TIF}}(\mathcal{H}, s) \cup \mathbb{C}_{\tau^{TIF}}(\mathcal{R}, s))) &= \tau^T((\underline{1} - \mathbb{C}_{\tau^{TIF}}(\mathcal{H}, s)) \cap (\underline{1} - \mathbb{C}_{\tau^{TIF}}(\mathcal{R}, s))) \\ &\geq \tau^T(\underline{1} - \mathbb{C}_{\tau^{TIF}}(\mathcal{H}, s)) \cap \tau^T(\underline{1} - \mathbb{C}_{\tau^{TIF}}(\mathcal{R}, s)) \geq s, \end{aligned}$$

$$\begin{aligned} \tau^I(\underline{1} - (\mathbb{C}_{\tau^{TIF}}(\mathcal{H}, s) \cup \mathbb{C}_{\tau^{TIF}}(\mathcal{R}, s))) &= \tau^I((\underline{1} - \mathbb{C}_{\tau^{TIF}}(\mathcal{H}, s)) \cap (\underline{1} - \mathbb{C}_{\tau^{TIF}}(\mathcal{R}, s))) \\ &\leq \tau^I((\underline{1} - \mathbb{C}_{\tau^{TIF}}(\mathcal{H}, s)) \cup \tau^I(\underline{1} - \mathbb{C}_{\tau^{TIF}}(\mathcal{R}, s)) \leq 1 - s, \end{aligned}$$

$$\begin{aligned} \tau^F(\underline{1} - (\mathbb{C}_{\tau^{TIF}}(\mathcal{H}, s) \cup \mathbb{C}_{\tau^{TIF}}(\mathcal{R}, s))) &= \tau^F((\underline{1} - \mathbb{C}_{\tau^{TIF}}(\mathcal{H}, s)) \cap (\underline{1} - \mathbb{C}_{\tau^{TIF}}(\mathcal{R}, s))) \\ &\leq \tau^F(\underline{1} - \mathbb{C}_{\tau^{TIF}}(\mathcal{H}, s)) \cup \tau^F(\underline{1} - \mathbb{C}_{\tau^{TIF}}(\mathcal{R}, s)) \leq 1 - s. \end{aligned}$$

Hence, $\mathbb{C}_{\tau^{TIF}}(\mathcal{H}, s) \cup \mathbb{C}_{\tau^{TIF}}(\mathcal{H} \cup \mathcal{R}, s) \geq \mathbb{C}_{\tau^{TIF}}(\mathcal{H} \cup \mathcal{R}, s)$. Therefore,

$$\mathbb{C}_{\tau^{TIF}}(\mathcal{H}, s) \cup \mathbb{C}_{\tau^{TIF}}(\mathcal{H} \cup \mathcal{R}, s) = \mathbb{C}_{\tau^{TIF}}(\mathcal{H} \cup \mathcal{R}, s).$$

($\mathbb{C}_5$) Suppose that there exists $s \in I_0$, $\mathcal{H} \in I^{\mathcal{S}}$, and $\kappa \in \mathcal{S}$ such that

$$\mathbb{C}_{\tau^{TIF}}(\mathbb{C}_{\tau^{TIF}}(\mathcal{H}, s), s)(\kappa) > \mathbb{C}_{\tau^{TIF}}(\mathcal{H}, s)(\kappa).$$

By the definition of $\mathbb{C}_{\tau^{TIF}}$, there exists $\mathcal{D} \in I^{\mathcal{S}}$ with $\mathcal{D} \geq \mathcal{H}$, and $\tau^T(\underline{1} - \mathcal{D}) \geq s$, $\tau^I(\underline{1} - \mathcal{D}) \leq 1 - s$ and $\tau^F(\underline{1} - \mathcal{D}) \leq 1 - s$ such that

$$\mathbb{C}_{\tau^{TIF}}(\mathbb{C}_{\tau^{TIF}}(\mathcal{H}, s), s)(\kappa) > \mathcal{D}(\kappa) \geq \mathbb{C}_{\tau^{TIF}}(\mathcal{H}, s)(\kappa).$$

Since $\mathbb{C}_{\tau^{TIF}}(\mathcal{H}, s) \leq \mathcal{D}$ and $\tau^T(\underline{1} - \mathcal{D}) \geq s$, $\tau^I(\underline{1} - \mathcal{D}) \leq 1 - s$, and $\tau^F(\underline{1} - \mathcal{D}) \leq 1 - s$, by the definition of $\mathbb{C}_{\tau^{TIF}}(\mathbb{C}_{\tau^{TIF}})$, we have

$$\mathbb{C}_{\tau^{TIF}}(\mathbb{C}_{\tau^{TIF}}(\mathcal{H}, s), s) \leq \mathcal{D}.$$

It is a contradiction. Thus, $\mathbb{C}_{\tau^{TIF}}(\mathbb{C}_{\tau^{TIF}}(\mathcal{H}, s), s) = \mathbb{C}_{\tau^{TIF}}(\mathcal{H}, s)$. Hence, $\mathbb{C}_{\tau^{TIF}}$ is a single-valued neutrosophic closure operator on $\mathcal{S}$. □

Theorem 2. *Let* $(\mathcal{S}, \mathbb{C})$ *be an* $\mathcal{SVNCS}$ *and* $\mathcal{H} \in \mathcal{S}$ *. Define the mapping* $\tau_C^{TIF} : I^{\mathcal{S}} \to I$ *on* $\mathcal{S}$ *by*

$$\tau_{\mathbb{C}}^T(\mathcal{H}) = \bigcup \{s \in I_0 \mid \mathbb{C}(\overline{1} - \mathcal{H}, s) = \overline{1} - \mathcal{H}\},$$

$$\tau_{\mathbb{C}}^I(\mathcal{H}) = \bigcap \{1 - s \in I_0 \mid \mathbb{C}(\overline{1} - \mathcal{H}, s) = \overline{1} - \mathcal{H}\},$$

$$\tau_{\mathbb{C}}^F(\mathcal{H}) = \bigcap \{1 - s \in I_0 \mid \mathbb{C}(\overline{1} - \mathcal{H}, s) = \overline{1} - \mathcal{H}\},$$

Then,

(1) $\tau_{\mathbb{C}}^{TIF}$ *is an* **SVNTS** *on* $\mathcal{S}$*;*
(2) $\mathbb{C}_{\tau_{\mathbb{C}}^{TIF}}$ *is finer than* $\mathbb{C}$*.*

Proof. (SVNT1) Let $(\mathcal{S}, \mathbb{C})$ be an $\mathcal{SVNCS}$. Since $\mathbb{C}((0.1.1), r) = (0.1.1)$ and $\mathbb{C}(1,0,0), r) = (1,0,0)$ for every $s \in I_0$, (SVNT1).

(SVNT2) Let $(\mathcal{S}, \mathbb{C})$ be an $\mathcal{SVNCS}$. Suppose that there exists $\mathcal{H}_1, \mathcal{H}_2 \in I^{\mathcal{S}}$ such that

$$\tau^T_{\mathbb{C}}(\mathcal{H}_1 \cap \mathcal{H}_2) < \tau^T_{\mathbb{C}}(\mathcal{H}_1) \cap \tau^T_{\mathbb{C}}(\mathcal{H}_2), \quad \tau^I_{\mathbb{C}}(\mathcal{H}_1 \cap \mathcal{H}_2) > \tau^I_{\mathbb{C}}(\mathcal{H}_1) \cup \tau^I_{\mathbb{C}}(\mathcal{H}_2),$$

$$\tau^F_{\mathbb{C}}(\mathcal{H}_1 \cap \mathcal{H}_2) > \tau^F_{\mathbb{C}}(\mathcal{H}_1) \cup \tau^F_{\mathbb{C}}(\mathcal{H}_2).$$

There exists $s \in I_0$ such that

$$\tau^T_{\mathbb{C}}(\mathcal{H}_1 \cap \mathcal{H}_2) < s < \tau^T_{\mathbb{C}}(\mathcal{H}_1) \cap \tau^T_{\mathbb{C}}(\mathcal{H}_2), \qquad \tau^I_{\mathbb{C}}(\mathcal{H}_1 \cap \mathcal{H}_2) > 1 - s > \tau^I_{\mathbb{C}}(\mathcal{H}_1) \cup \tau^I_{\mathbb{C}}(\mathcal{H}_2),$$

$$\tau^F_{\mathbb{C}}(\mathcal{H}_1 \cap \mathcal{H}_2) > 1 - s > \tau^F_{\mathbb{C}}(\mathcal{H}_1) \cup \tau^F_{\mathbb{C}}(\mathcal{H}_2).$$

For each $i \in \{1,2\}$, there exists $s \in I_0$ with $\mathbb{C}(\mathcal{H}_i, s_i) = \overline{1} - \mathcal{H}_i$ such that

$$s < s_i \leq \tau^T_{\mathbb{C}}(\mathcal{H}_i), \quad \tau^I_{\mathbb{C}}(\mathcal{H}_i) \leq 1 - s_i < 1 - s, \quad \tau^F_{\mathbb{C}}(\mathcal{H}_i) \leq 1 - s_i < 1 - s.$$

In addition, since $(\overline{1} - \mathcal{H}_i, r) = \overline{1} - \mathcal{H}_i$ by $\mathbb{C}_2$ and $\mathbb{C}_4$ of Definition 9, for any $i \in \{1,2\}$,

$$\mathbb{C}((\overline{1} - \mathcal{H}_1) \cup (\overline{1} - \mathcal{H}_2), s) = (\overline{1} - \mathcal{H}_1) \cup (\overline{1} - \mathcal{H}_2).$$

It follows that $\tau^T_{\mathbb{C}}(\mathcal{H}_1 \cap \mathcal{H}_2) \geq s$, $\tau^I_{\mathbb{C}}(\mathcal{H}_1 \cap \mathcal{H}_2) \leq 1 - s$, and $\tau^F_{\mathbb{C}}(\mathcal{H}_1 \cap \mathcal{H}_2) \leq 1 - s$. It is a contradiction. Thus, for every $\mathcal{H}, \mathcal{R} \in I^{\mathcal{S}}$, $\tau^T_{\mathbb{C}}(\mathcal{H} \cap \mathcal{R}) \geq \tau^T_{\mathbb{C}}(\mathcal{H}) \cap \tau^T_{\mathbb{C}}(\mathcal{B})$, $\tau^I_{\mathbb{C}}(\mathcal{H} \cap \mathcal{R}) \leq \tau^I_{\mathbb{C}}(\mathcal{H}) \cup \tau^I_{\mathbb{C}}(\mathcal{R})$, and $\tau^F_{\mathbb{C}}(\mathcal{H} \cap \mathcal{R}) \leq \tau^F_{\mathbb{C}}(\mathcal{H}) \cup \tau^F_{\mathbb{C}}(\mathcal{R})$.

(SVNT3) Suppose that there exists $\mathcal{H} = \bigcup_{i \in I} \mathcal{H}_i \in I^{\mathcal{S}}$ such that

$$\tau^T_{\mathbb{C}}(\mathcal{H}) < \bigcup_{i \in I} \tau^T_{\mathbb{C}}(\mathcal{H}_i), \quad \tau^I_{\mathbb{C}}(\mathcal{H}) > \bigcup_{i \in I} \tau^I_{\mathbb{C}}(\mathcal{H}_i), \quad \tau^F_{\mathbb{C}}(\mathcal{H}) > \bigcup_{i \in I} \tau^F_{\mathbb{C}}(\mathcal{H}_i).$$

There exists $s_0 \in I_0$ such that

$$\tau^T_{C}(\mathcal{H}) < s_0 < \bigcup_{i \in I} \tau^T_{\mathbb{C}}(\mathcal{H}_i), \quad \tau^I_{\mathbb{C}}(\mathcal{H}) > 1 - s_0 > \bigcup_{i \in I} \tau^I_{\mathbb{C}}(\mathcal{H}_i), \quad \tau^F_{\mathbb{C}}(\mathcal{H}) > 1 - s_0 > \bigcup_{i \in I} \tau^F_{\mathbb{C}}(\mathcal{H}_i).$$

For every $i \in I$, there exists $\mathbb{C}(\mathcal{H}_i, s_i) = \overline{1} - \mathcal{H}_i$ and $s_i \in I_0$ such that

$$s_0 < s_i \leq \tau^T_{\mathbb{C}}(\mathcal{H}_i), \quad 1 - s_0 > 1 - s_i \geq \tau^I_{\mathbb{C}}(\mathcal{H}_i), \quad 1 - s_i > 1 - s_0 \geq \tau^F_{\mathbb{C}}(\mathcal{H}_i).$$

In addition, since $\mathbb{C}(\overline{1} - \mathcal{H}_i, r_0) \leq \mathbb{C}(\overline{1} - \mathcal{H}_i, s_i) = \overline{1} - \mathcal{H}_i$, by $\mathbb{C}_2$ of Definition 9,

$$\mathbb{C}(\overline{1} - \mathcal{H}_i, s_0) = \overline{1} - \mathcal{H}_i.$$

It implies, for all $i \in I$,

$$\mathbb{C}(\overline{1} - \mathcal{H}, s_0) \leq \mathbb{C}(\overline{1} - \mathcal{H}_i, s_0) = \overline{1} - \mathcal{H}_i.$$

It follows that

$$\mathbb{C}(\overline{1} - \mathcal{H}, r_0) \leq \bigcap_{i \in J} (\overline{1} - \mathcal{H}_i) = \overline{1} - \mathcal{H}.$$

Thus, $\mathbb{CI}(\overline{1}-\mathcal{H},s_0)=\overline{1}-\mathcal{H}$, that is, $\tau^T_{\mathbb{C}}(\mathcal{H})\geq s_0$, $\tau^I_{\mathbb{C}}(\mathcal{H})\leq 1-s_0$, and $\tau^F_{\mathbb{C}}(\mathcal{H})\leq 1-s_0$. It is a contradiction. Hence, $\tau^{TIF}_{\mathbb{C}}$ is an **SVNTS** on $\mathcal{S}$.

(2) Since $\mathcal{H}\leq\mathbb{C}(\mathcal{H},r)$,

$$\tau^T_{\mathbb{C}}(\overline{1}-\mathbb{C}(\mathcal{H},s))\geq s,\ \tau^I_{\mathbb{C}}(\overline{1}-\mathbb{C}(\mathcal{H},s))\leq 1-s,\ \tau^F_{\mathbb{C}}(\overline{1}-\mathbb{C}(\mathcal{H},s))\leq 1-s.$$

From $\mathbb{C}_5$ of Definition 9, we have $\mathbb{C}_{\tau^{TIF}_{\mathbb{C}}}(\mathcal{H},s)\leq\mathbb{C}(\mathcal{H},s)$. Thus, $\mathbb{C}_{\tau^{TIF}_{\mathbb{C}}}$ is finer than $\mathbb{C}$. □

Example 1. *Let $\mathcal{S}=\{a,b\}$. Define $\mathcal{B},\mathcal{H},\mathcal{A}\in I^{\mathcal{S}}$ as follows:*

$$\mathcal{B}=\langle(0.2,0.2),(0.3,0.3),(0.3,0.3)\rangle;\mathcal{H}=\langle(0.5,0.5),(0.1,0.1),(0.1,0.1)\rangle.$$

We define the mapping $\mathbb{C}:I^{\mathcal{S}}\times I_0\to I^{\mathcal{S}}$ as follows:

$$\mathbb{C}(\mathcal{A},s)=\begin{cases}(0.1.1), & \text{if}\quad \mathcal{A}=(0.1.1),\quad s\in I_0,\\ \mathcal{B}\cap\mathcal{H}, & \text{if}\quad 0\neq\mathcal{A}\leq\mathcal{B}\cap\mathcal{H},\ 0<r<\frac{1}{2},\\ \mathcal{B}, & \text{if}\quad \mathcal{A}\leq\mathcal{B},\mathcal{A}\not\leq\mathcal{H},\quad 0<r<\frac{1}{2},\\ & \text{or } 0\neq\mathcal{A}\leq\mathcal{B}\quad \frac{1}{2}<r<\frac{2}{3},\\ \mathcal{H}, & \text{if}\quad \mathcal{A}\leq\mathcal{H},\mathcal{A}\not\leq\mathcal{B},\ 0<r<\frac{1}{2},\\ \mathcal{B}\cup\mathcal{H}, & \text{if}\quad 0\neq\mathcal{A}\leq\mathcal{B}\cup\mathcal{H},\ 0<r<\frac{1}{2},\\ \overline{1}, & \text{otherwise.}\end{cases}$$

Then, $\mathbb{C}$ is a single-valued neutrosophic closure operator.
From Theorem 2, we have a single-valued neutrosophic topology $(\tau^T_{\mathbb{C}},\tau^I_{\mathbb{C}},\tau^F_{\mathbb{C}})$ on $\mathcal{S}$ as follows:

$$\tau^T_{\mathbb{C}}(\mathcal{A})=\begin{cases}1, & \text{if } \mathcal{A}=(1,0,0) \text{ or } (0,1,1),\\ \frac{2}{3}, & \text{if } \mathcal{A}=\mathcal{B}^c,\\ \frac{1}{2}, & \text{if } \mathcal{A}=\mathcal{H}^c,\\ \frac{1}{2}, & \text{if } \mathcal{A}=\mathcal{B}^c\cup\mathcal{H}^c,\\ \frac{1}{2}, & \text{if } \mathcal{A}=\mathcal{B}^c\cap\mathcal{H}^c,\\ 0, & \text{otherwise.}\end{cases}$$

$$\tau^I_{\mathbb{C}}(\mathcal{A})=\begin{cases}0, & \text{if } \mathcal{A}=(1,0,0) \text{ or } (0,1,1),\\ \frac{1}{3}, & \text{if } \mathcal{A}=\mathcal{B}^c,\\ \frac{1}{2}, & \text{if } \mathcal{A}=\mathcal{H}^c,\\ \frac{1}{2}, & \text{if } \mathcal{A}=\mathcal{B}^c\cup\mathcal{H}^c,\\ \frac{1}{2}, & \text{if } \mathcal{A}=\mathcal{B}^c\cap\mathcal{H}^c,\\ 1, & \text{otherwise.}\end{cases}$$

$$\tau^F_{\mathbb{C}}(\mathcal{A})=\begin{cases}0, & \text{if } \mathcal{A}=(1,0,0) \text{ or } (0,1,1),\\ \frac{1}{3}, & \text{if } \mathcal{A}=\mathcal{B}^c,\\ \frac{1}{2}, & \text{if } \mathcal{A}=\mathcal{H}^c,\\ \frac{1}{2}, & \text{if } \mathcal{A}=\mathcal{B}^c\cup\mathcal{H}^c,\\ \frac{1}{2}, & \text{if } \mathcal{A}=\mathcal{B}^c\cap\mathcal{H}^c,\\ 1, & \text{otherwise.}\end{cases}$$

Thus, the $\tau^{TIF}_{\mathbb{C}}$ is a single-valued neutrosophic topology on $\mathcal{S}$.

Definition 10. *A single-valued neutrosophic ideal (**SVNI**) on $\mathcal{S}$ in Šostak's sense on a nonempty set $\mathcal{S}$ is a family $\mathcal{L}^T,\mathcal{L}^I,\mathcal{L}^F$ of single-valued neutrosophic sets in $\mathcal{S}$ satisfying the following axioms:*

(L_1) $\mathcal{L}^T(\underline{0}) = 1$ *and* $\mathcal{L}^I(\underline{0}) = \mathcal{L}^F(\underline{0}) = 0$.

(L_2) *If* $\mathcal{H} \leq \mathcal{B}$, *then* $\mathcal{L}^T(\mathcal{R}) \leq \mathcal{L}^T(\mathcal{H})$, $\mathcal{L}^I(\mathcal{R}) \geq \mathcal{L}^I(\mathcal{H})$, *and* $\mathcal{L}^F(\mathcal{R}) \geq \mathcal{L}^F(\mathcal{H})$, *for each single-valued neutrosophic set* $\mathcal{R}, \mathcal{H}$ *in* $I^{\mathcal{S}}$.

(L_3) $\mathcal{L}^T(\mathcal{R} \cup \mathcal{H}) \geq \mathcal{L}^T(\mathcal{R}) \cap \mathcal{L}^T(\mathcal{H})$, $\mathcal{L}^I(\mathcal{R} \cup \mathcal{H}) \leq \mathcal{L}^I(\mathcal{R}) \cup \mathcal{L}^I(\mathcal{H})$, *and* $\mathcal{L}^F(\mathcal{R} \cup \mathcal{H}) \leq \mathcal{L}^F(\mathcal{R}) \cup \mathcal{L}^F(\mathcal{H})$, *for each single-valued neutrosophic set* $\mathcal{R}, \mathcal{H}$ *in* $I^{\mathcal{S}}$.

If $\mathcal{L}_1$ *and* $\mathcal{L}_2$ *are* **SVNI** *on* $\mathcal{S}$, *we say that* $\mathcal{L}_1$ *is finer than* $\mathcal{L}_2$, *denoted by* $\mathcal{L}_1 \leq \mathcal{L}_2$, *iff* $\mathcal{L}_1^T(\mathcal{H}) \leq \mathcal{L}_2^T(\mathcal{H})$, $\mathcal{L}_1^I(\mathcal{H}) \geq \mathcal{L}_2^I(\mathcal{A})$, *and* $\mathcal{L}_1^F(\mathcal{H}) \geq \mathcal{L}_2^F(\mathcal{H})$, *for* $\mathcal{H} \in I^{\mathcal{S}}$.

The triable $(X, (\tau^T, \tau^I, \tau^F), (\mathcal{L}^T, \mathcal{L}^I, \mathcal{L}^F)$ *is called a single-valued neutrosophic ideal topological space in Šostak sense* (**SVNITS**, $for\ short$).

We will occasionally write $\mathcal{L}^{TIF}$, $\mathcal{L}_i^{TIF}$, *and* $\mathcal{L}^{TIF} : I^X \to I$ *for* $(\mathcal{L}^T, \mathcal{L}^I, \mathcal{L}^F)$, $(\mathcal{L}_i^T, \mathcal{L}_i^I, \mathcal{L}_i^F)$, *and* $\mathcal{L}^T, \mathcal{L}^I, \mathcal{L}^F : I^{\mathcal{S}} \to I$, *respectively.*

Remark 1. *The conditions* (L_2) *and* (L_3), *which are given in Definition 10, are equivalent to the following axioms:* $\mathcal{L}^T(\mathcal{H} \cup \mathcal{R}) = \mathcal{L}^T(\mathcal{H}) \cap \mathcal{L}^T(\mathcal{R})$, $\mathcal{L}^I(\mathcal{H} \cup \mathcal{R}) \neq \mathcal{L}^I(\mathcal{H}) \cup \mathcal{L}^I(\mathcal{R})$, *and* $\mathcal{L}^F(\mathcal{H} \cup \mathcal{R}) \neq \mathcal{L}^F(\mathcal{H}) \cup \mathcal{L}^F(\mathcal{R})$, *for every* $\mathcal{R}, \mathcal{H} \in I^{\mathcal{S}}$.

Example 2. *Let* $\mathcal{S} = \{a, b\}$. *Define the single-valued neutrosophic sets* $\mathcal{R}, \mathcal{C}, \mathcal{H}, \mathcal{A}$ *and* $(\mathcal{L}^T, \mathcal{L}^T, \mathcal{L}^T) : I^{\mathcal{S}} \to I$ *as follows:*

$$\mathcal{R} = \langle (0.3, 0.5), (0.4, 0.5), (0.5, 0.5) \rangle; \quad \mathcal{C} = \langle (0.3, 0.4), (0.5, 0.5), (0.3, 0.4) \rangle,$$

$$\mathcal{H} = \langle (0.1, 0.2), (0.5, 0.5), (0.5, 0.5) \rangle.$$

$$\mathcal{L}^T(\mathcal{A}) = \begin{cases} 1, & if\ \mathcal{B} = (0.1.1), \\ \frac{1}{2}, & if\ \mathcal{A} = \mathcal{R}, \\ \frac{2}{3}, & if\ (0.1.1) < \mathcal{A} < \mathcal{R}, \\ 0, & otherwise. \end{cases}$$

$$\mathcal{L}^I(\mathcal{A}) = \begin{cases} 0, & if\ \mathcal{A} = (0.1.1), \\ \frac{1}{2}, & if\ \mathcal{A} = \mathcal{C}, \\ \frac{1}{4}, & if\ (0.1.1) < \mathcal{A} < \mathcal{C}, \\ 1, & otherwise. \end{cases}$$

$$\mathcal{L}^T(\mathcal{B}) = \begin{cases} 0, & if\ \mathcal{A} = (0, 1, 1), \\ \frac{1}{2}, & if\ \mathcal{A} = \mathcal{H}, \\ \frac{1}{4}, & if\ (0.1.1) < \mathcal{A} < \mathcal{H}, \\ 1, & otherwise. \end{cases}$$

Then, $\mathcal{L}^{TIF}$ *is an* **SVNI** *on* $\mathcal{S}$.

Remark 2. *(i) If* $\mathcal{L}^T(\underline{1}) = 1$, $\mathcal{L}^I(\underline{1}) = 0$, *and* $\mathcal{L}^F(\underline{1}) = 0$, *then* $\mathcal{L}^{TIF}$ *is called a single-valued neutrosophic proper ideal.*

(ii) If $\mathcal{L}^T(\underline{1}) = 0$, $\mathcal{L}^I(\underline{1}) = 1$, *and* $\mathcal{L}^F(\underline{1}) = 1$, *then* $\mathcal{L}^{TIF}$ *is called a single-valued neutrosophic improper ideal.*

Proposition 1. *Let* $\{\mathcal{L}_i^{TIF}\}_{i \in J}$ *be a family of* **SVNI** *on* $\mathcal{S}$. *Then, their intersection* $\bigcap_{i \in J} \mathcal{L}_i^{TIF}$ *is also* **SVNI**.

Proof. Directly from Definition 7. □

Proposition 2. *Let* $\{\mathcal{L}_i^{TIF}\}_{i\in J}$ *be a family of* **SVNI** *on* $\mathcal{S}$*. Then, their union* $\bigcup_{i\in J}\mathcal{L}_i^{TIF}$ *is also an* **SVNI***.*

Proof. Directly from Definition 7. □

4. Single-Valued Neutrosophic Ideal Open Local Function in Šostak Sense

In this section, we study the single-valued neutrosophic ideal open local function in Šostak's sense and present some of their properties. Additionally, properties preserved by single-valued neutrosophic ideal open compatible are examined.

Definition 11. *Let* $s, t, p \in I_0$ *and* $s + t + p \leq 3$*. A single-valued neutrosophic point* $x_{s,t,r}$ *of* $\mathcal{S}$ *is the single-valued neutrosophic set in* $I^{\mathcal{S}}$ *for each* $\kappa \in \mathcal{H}$*, defined by*

$$x_{s,t,p}(\kappa) = \begin{cases} (s,t,p), & \text{if } x = \kappa, \\ (0,1,1), & \text{if } x \neq \kappa. \end{cases}$$

A single-valued neutrosophic point $x_{s,t,p}$ *is said to belong to a single-valued neutrosophic set* $\mathcal{H} = \langle T_{\mathcal{H}}, I_{\mathcal{H}}, F_{\mathcal{H}}\rangle \in I^{\mathcal{S}}$*, denoted by* $x_{s,t,p} \in \mathcal{H}$ *iff* $s < T_{\mathcal{H}}$*,* $t \geq I_{\mathcal{H}}$ *and* $p \geq F_{\mathcal{H}}$*. 1. We indicate the set of all single-valued neutrosophic points in* $\mathcal{S}$ *as* $\mathbf{SVNP}(\mathcal{S})$*.*

For every $x_{s,t,p} \in \mathbf{SVNP}(\mathcal{S})$ and $\mathcal{H} \in I^{\mathcal{S}}$ we shall write $x_{s,t,p}$ quasi-coincident with $\mathcal{H}$, denoted by $x_{s,t,p}q\mathcal{H}$, if

$$s + T_{\mathcal{H}}(\kappa) > 1, \qquad t + I_{\mathcal{H}}(\kappa) \leq 1, \qquad p + F_{\mathcal{H}}(\kappa) \leq 1.$$

For every $\mathcal{R}, \mathcal{H} \in \mathcal{S}$ we shall write $\mathcal{H}\bar{q}\mathcal{R}$ to mean that $\mathcal{H}$ is quasi-coincident with $\mathcal{R}$ if there exists $\kappa \in \mathcal{S}$ such that

$$T_{\mathcal{H}}(\kappa) + T_{\mathcal{R}}(\kappa) > 1, \quad I_{\mathcal{H}}(\kappa) + I_{\mathcal{R}}(\kappa) \leq 1, \quad F_{\mathcal{H}}(\kappa) + F_{\mathcal{R}}(\kappa) \leq 1.$$

Definition 12. *Let* $(\mathcal{S}, \tau^{TIF})$ *be an* **SVNTS***. For each* $r \in I_0$*,* $\mathcal{H} \in I^{\mathcal{S}}$*,* $x_{s,t,p} \in \mathbf{SVNP}(\mathcal{S})$*, a single-valued neutrosophic open* $Q_{\tau^{TIF}}$*-neighborhood of* $x_{s,t,p}$ *is defined as follows:*

$$Q_{\tau^{TIF}}(x_{s,t,p}, r) = \{\mathcal{H} | (x_{s,t,p})q\mathcal{H}, \qquad \tau^T(\mathcal{H}) \geq r, \qquad \tau^I(\mathcal{H}) \leq 1 - r, \qquad \tau^F(\mathcal{H}) \leq 1 - r\}.$$

Lemma 1. *A single-valued neutrosophic point* $x_{s,t,p} \in \mathbb{C}_{\tau^{TIF}}(\mathcal{R}, r)$ *iff every single-valued neutrosophic open* $Q_{\tau^{TIF}}$*-neighborhood of* $x_{s,t,p}$ *is quasi-coincident with* $\mathcal{H}$*.*

Definition 13. *Let* $(\mathcal{S}, \tau^{TIF})$ *be an* **SVNTS** *for each* $\mathcal{H} \in I^{\mathcal{S}}$*. Then, the single-valued neutrosophic ideal open local function* $\mathcal{H}_r^{\star}(\tau^{TIF}, \mathcal{L}^{TIF})$ *of* $\mathcal{H}$ *is the union of all single-valued neutrosophic points* $x_{s,t,p}$ *such that if* $\mathcal{R} \in Q_{\tau^{TIF}}(x_{s,t,p}, r)$ *and* $\mathcal{L}^T(\mathcal{C}) \geq r$*,* $\mathcal{L}^I(\mathcal{C}) \leq 1 - r$*,* $\mathcal{L}^F(\mathcal{C}) \leq 1 - r$*, then there is at least one* $\kappa \in \mathcal{S}$ *for which* $T_{\mathcal{R}}(\kappa) + T_{\mathcal{H}}(\kappa) - 1 > T_{\mathcal{C}}(\kappa)$*,* $I_{\mathcal{R}}(\kappa) + I_{\mathcal{H}}(\kappa) - 1 \leq I_{\mathcal{C}}(\kappa)$*, and* $F_{\mathcal{R}}(\kappa) + F_{\mathcal{H}}(\kappa) - 1 \leq F_{\mathcal{C}}(\kappa)$*.*

Occasionally, we will write $\mathcal{H}_r^{\star}$ for $\mathcal{H}_r^{\star}(\tau^{TIF}, \mathcal{L}^{TIF})$ and it will have no ambiguity.

Example 3. *Let* $(\mathcal{S}, \tau^{TIF}, \mathcal{L}^{TIF})$ *be an* **SVNITS***. The simplest single-valued neutrosophic ideal on* $\mathcal{S}$ *is* $\mathcal{L}_0^{TIF} : I^{\mathcal{S}} \to I$*, where*

$$\mathcal{L}_0^{TIF}(\mathcal{R}) = \begin{cases} 1, & \text{if } \mathcal{R} = (1,0,0), \\ 0, & \text{otherwise}. \end{cases}$$

If we take $\mathcal{L}^{TIF} = \mathcal{L}_0^{TIF}$*, for each* $\mathcal{H} \in I^{\mathcal{S}}$ *we have* $\mathcal{H}_r^{\star} = \mathbb{C}_{\tau^{TIF}}(\mathcal{H}, r)$*.*

Theorem 3. *Let* $(\mathcal{S}, \tau^{TIF})$ *be an* **SVNTS** *and* $\mathcal{L}_1^{TIF}, \mathcal{L}_2^{TIF} \in$ **SVNI**$(\mathcal{S})$*. Then, for any* $\mathcal{H}, \mathcal{R} \in I^{\mathcal{S}}$ *and* $r \in I_0$*, we have*

(1) *If* $\mathcal{H} \leq \mathcal{R}$*, then* $\mathcal{H}_r^\star \leq \mathcal{R}_r^\star$*;*
(2) *If* $\mathcal{L}_1^T \leq \mathcal{L}_2^T$*,* $\mathcal{L}_1^I \geq \mathcal{L}_2^I$ *and* $\mathcal{L}_1^F \geq \mathcal{L}_2^F$*, then* $\mathcal{H}_r^\star(\mathcal{L}_1^{TIF}, \tau^{TIF}) \geq \mathcal{H}_r^\star((\mathcal{L}_2^{TIF}, \tau^{TIF})$*;*
(3) $\mathcal{H}_r^\star = C_{\tau^{TIF}}(\mathcal{A}_r^\star, r) \leq \mathbb{C}_{\tau^{TIF}}(\mathcal{H}, r)$*;*
(4) $(\mathcal{H}_r^\star)_r^\star \leq \mathcal{H}_r^\star$*;*
(5) $(\mathcal{H}_r^\star \vee \mathcal{R}_r^\star) = (\mathcal{H} \vee \mathcal{R})_r^\star$*;*
(6) *If* $\mathcal{L}^T(\mathcal{H}) \geq r$*,* $\mathcal{L}^I(\mathcal{R}) \leq 1 - r$*, and* $\mathcal{L}^F(\mathcal{R}) \leq 1 - r$ *then* $(\mathcal{H} \vee \mathcal{R})_r^\star = \mathcal{A}_r^\star \vee \mathcal{R}_r^\star = \mathcal{H}_r^\star$*;*
(7) *If* $\tau^T(\mathcal{R}) \geq r$*,* $\tau^I(\mathcal{R}) \leq 1 - r$*, and* $\tau^F(\mathcal{R}) \leq 1 - r$*, then* $(\mathcal{R} \wedge \mathcal{H}_r^\star) \leq (\mathcal{R} \wedge \mathcal{H})_r^\star$*;*
(8) $(\mathcal{H}_r^\star \wedge \mathcal{R}_r^\star) \geq (\mathcal{H} \wedge \mathcal{R})_r^\star$*.*

Proof. (1) Suppose that $\mathcal{H} \in I^{\mathcal{S}}$ and $\mathcal{H}_r^\star \not\leq \mathcal{R}_r^\star$. Then, there exists $\kappa \in \mathcal{S}$ and $s, t, p \in I_0$ such that

$$T_{\mathcal{H}_r^\star}(\kappa) \geq s > T_{\mathcal{R}_r^\star}(\kappa), \quad I_{\mathcal{H}_r^\star}(\kappa) < t \leq I_{\mathcal{R}_r^\star}(\kappa), \quad F_{\mathcal{H}_r^\star}(\kappa) < p \leq F_{\mathcal{R}_r^\star}(\kappa). \tag{1}$$

Since $T_{\mathcal{R}_r^\star}(\kappa) < s$, $I_{\mathcal{R}_r^\star}(\kappa) \geq t$, and $F_{\mathcal{R}_r^\star}(\kappa) \geq p$. Then, there exists $\mathcal{D} \in Q_{(\tau^{TIF})}(x_{s,t,p}, r)$, $\mathcal{L}^T(\mathcal{C}) \geq r$, $\mathcal{L}^I(\mathcal{C}) \leq 1 - r$, and $\mathcal{L}^F(\mathcal{C}) \leq 1 - r$ such that for any $\kappa_1 \in \mathcal{S}$,

$$T_{\mathcal{D}}(\kappa_1) + T_{\mathcal{R}}(\kappa_1) - 1 \leq T_{\mathcal{C}}(\kappa_1), \quad I_{\mathcal{D}}(\kappa_1) + I_{\mathcal{R}}(\kappa_1) - 1 > I_{\mathcal{C}}(\kappa_1), \quad F_{\mathcal{D}}(\kappa_1) + F_{\mathcal{R}}(\kappa_1) - 1 > F_{\mathcal{C}}(\kappa_1).$$

Since $\mathcal{H} \leq \mathcal{R}$,

$$T_{\mathcal{D}}(\kappa_1) + T_{\mathcal{H}}(\kappa_1) - 1 \leq T_{\mathcal{C}}(\kappa_1), \quad I_{\mathcal{D}}(\kappa_1) + I_{\mathcal{H}}(\kappa_1) - 1 > I_{\mathcal{C}}(\kappa_1), \quad F_{\mathcal{D}}(\kappa_1) + F_{\mathcal{H}}(\kappa_1) - 1 > F_{\mathcal{C}}(\kappa_1).$$

So, $T_{\mathcal{H}_r^\star}(\kappa) < s$, $I_{\mathcal{H}_r^\star}(\kappa) \geq t$, and $F_{\mathcal{H}_r^\star}(\kappa) \geq p$ and we arrive at a contradiction for Equation (1). Hence, $\mathcal{H}_r^\star \leq \mathcal{R}_r^\star$.

(2) Suppose $\mathcal{H}_r^\star(\mathcal{L}_1^{TIF}, \tau^{TIF}) \not\geq \mathcal{H}_r^\star(\mathcal{L}_2^{TIF}, \tau^{TIF})$. Then, there exists $s, t, p \in I_0$ and $\kappa \in \mathcal{S}$ such that

$$T_{\mathcal{H}_r^\star(\mathcal{L}_1^{TIF}, \tau^{TIF})}(\kappa) < s \leq T_{\mathcal{H}_r^\star(\mathcal{L}_2^{TIF}, \tau^{TIF})}(\kappa),$$

$$I_{\mathcal{H}_r^\star(\mathcal{L}_1^{TIF}, \tau^{TIF})}(\kappa) \geq t > I_{\mathcal{H}_r^\star(\mathcal{L}_2^{TIF}, \tau^{TIF})}(\kappa), \tag{2}$$

$$F_{\mathcal{H}_r^\star(\mathcal{L}_1^{TIF}, \tau^{TIF})}(\kappa) \geq p > F_{\mathcal{H}_r^\star(\mathcal{L}_2^{TIF}, \tau^{TIF})}(\kappa).$$

Since $T_{\mathcal{H}_r^\star(\mathcal{L}_1^{TIF}, \tau^{TIF})}(\kappa) < s$, $I_{\mathcal{H}_r^\star(\mathcal{L}_1^{TIF}, \tau^{TIF})}(\kappa) \geq t$, and $F_{\mathcal{H}_r^\star(\mathcal{L}_1^{TIF}, \tau^{TIF})}(\kappa) \geq p$, $\mathcal{D} \in Q_{\tau^{TIF}}(x_{s,t,p}, r)$ with $\mathcal{L}_1^T(\mathcal{C}) \geq r$, $\mathcal{L}_1^I(\mathcal{C}) \leq 1 - r$ and $\mathcal{L}_1^F(\mathcal{C}) \leq 1 - r$. Thus, for every $\kappa_1 \in \mathcal{S}$,

$$T_{\mathcal{D}}(\kappa_1) + T_{\mathcal{H}}(\kappa_1) - 1 \leq T_{\mathcal{C}}(\kappa_1), \quad I_{\mathcal{D}}(\kappa_1) + I_{\mathcal{H}}(\kappa_1) - 1 > I_{\mathcal{C}}(\kappa_1), \quad F_{\mathcal{D}}(\kappa_1) + F_{\mathcal{H}}(\kappa_1) - 1 > F_{\mathcal{C}}(\kappa_1).$$

Since $\mathcal{L}_2^T(\mathcal{C}) \geq \mathcal{L}_1^T(\mathcal{C})) \geq r$, $\mathcal{L}_2^I(\mathcal{C}) \leq \mathcal{L}_1^I(\mathcal{C})) \leq 1 - r$, and $\mathcal{L}_2^F(\mathcal{C}) \leq \mathcal{L}_1^F(\mathcal{C})) \leq 1 - r$,

$$T_{\mathcal{D}}(\kappa_1) + T_{\mathcal{H}}(\kappa_1) - 1 \leq T_{\mathcal{C}}(\kappa_1), \quad I_{\mathcal{D}}(\kappa_1) + I_{\mathcal{H}}(\kappa_1) - 1 > I_{\mathcal{C}}(\kappa_1), \quad F_{\mathcal{D}}(\kappa_1) + F_{\mathcal{H}}(\kappa_1) - 1 > F_{\mathcal{C}}(\kappa_1).$$

Thus, $T_{\mathcal{H}_r^\star(\mathcal{L}_2^{TIF}, \tau^{TIF})}(\kappa) < s$, $I_{\mathcal{H}_r^\star(\mathcal{L}_2^{TIF}, \tau^{TIF})}(\kappa) \geq t$, and $F_{\mathcal{H}_r^\star(\mathcal{L}_2^{TIF}, \tau^{TIF})}(\kappa) \geq p$. This is a contradiction for Equation (2). Hence, $\mathcal{H}_r^\star((\mathcal{L}_1^{TIF}, \tau^{TIF})) \geq \mathcal{H}_r^\star((\mathcal{L}_2^{TIF}, \tau^{TIF}))$.

(3)($\Rightarrow$) Suppose $\mathcal{H}_r^\star \not\leq \mathbb{C}_{\tau^{TIF}}(\mathcal{H}, r)$. Then, there exists $s, t, p \in I_0$ and $\kappa \in \mathcal{S}$ such that

$$T_{\mathcal{H}_r^\star}(\kappa) \geq s > T_{\mathbb{C}_{\tau^{TIF}}(\mathcal{H},r)}(\kappa), \quad I_{\mathcal{H}_r^\star}(\kappa) < t \leq I_{\mathbb{C}_{\tau^{TIF}}(\mathcal{H},r)}(\kappa), \quad F_{\mathcal{H}_r^\star}(\kappa) < p \leq F_{\mathbb{C}_{\tau^{TIF}}(\mathcal{H},r)}(\kappa). \tag{3}$$

Since $T_{\mathcal{H}_r^\star}(\kappa) \geq s$, $I_{\mathcal{H}_r^\star}(\kappa) < t$ and $F_{\mathcal{H}_r^\star}(\kappa) < p$, $x_{s,t,p} \in \mathcal{H}_r^\star$. So there is at least one $\kappa_1 \in \mathcal{S}$ for every $\mathcal{D} \in Q_{\tau^{TIF}}(x_{s,t,p}, r)$ with $\mathcal{L}_1^T(\mathcal{C}) \geq r$, $\mathcal{L}_1^I(\mathcal{C}) \leq 1 - r$, $\mathcal{L}_1^F(\mathcal{C}) \leq 1 - r$ such that

$$T_{\mathcal{D}}(\kappa_1) + T_{\mathcal{H}}(\kappa_1) > T_{\mathcal{C}}(\kappa_1) + 1, \quad I_{\mathcal{D}}(\kappa_1) + I_{\mathcal{H}}(\kappa_1) \leq I_{\mathcal{C}}(\kappa_1) + 1, \quad F_{\mathcal{D}}(\kappa_1) + F_{\mathcal{H}}(\kappa_1) \leq F_{\mathcal{C}}(\kappa_1) + 1.$$

Therefore, by Lemma 1, $x_{s,t,p} \in \mathbb{C}_{\tau^{TIF}}(\mathcal{H}, r)$ which is a contradiction for Equation (3). Hence, $\mathcal{H}_r^\star \leq \mathbb{C}_{\tau^{TIF}}(\mathcal{H}, r)$.

($\Leftarrow$) Suppose $\mathcal{H}_r^\star \not\geq \mathbb{C}_{\tau^{TIF}}(\mathcal{H}_r^\star, r)$. Then, there exists $s, t, p \in I_0$ and $\kappa \in \mathcal{S}$ such that

$$T_{\mathcal{H}_r^\star}(\kappa) < s \leq T_{\mathbb{C}_{\tau^{TIF}}(\mathcal{H}_r^\star, r)}(\kappa), \quad I_{\mathcal{H}_r^\star}(\kappa) \geq t > I_{\mathbb{C}_{\tau^{TIF}}(\mathcal{H}_r^\star, r)}(\kappa), \quad F_{\mathcal{H}_r^\star}(\kappa) \geq p > F_{\mathbb{C}_{\tau^{TIF}}(\mathcal{H}_r^\star, r)}(\kappa). \tag{4}$$

Since $T_{\mathbb{C}_{\tau^{TIF}}(\mathcal{H}_r^\star, r)}(\kappa) \geq t$, $I_{\mathbb{C}_{\tau^{TIF}}(\mathcal{H}_r^\star, r)}(\kappa) < s$, $\mathbb{C}_{\tau^{TIF}}(\mathcal{H}_r^\star, r)(\kappa) < p$ we have $x_{s,t,p} \in \mathbb{C}_{\tau^{TIF}}(\mathcal{H}_r^\star, r)$. So, there is at least one $\kappa_1 \in \mathcal{S}$ with $\mathcal{R} \in Q_{\tau^{TIF}}(x_{s,t,p}, r)$ such that

$$T_{\mathcal{R}}(\kappa_1) + T_{\mathcal{H}_r^\star}(\kappa_1) > 1, \quad I_{\mathcal{R}}(\kappa_1) + I_{\mathcal{H}_r^\star}(\kappa_1) \leq 1, \quad F_{\mathcal{R}}(\kappa_1) + F_{\mathcal{H}_r^\star}(\kappa_1) \leq 1.$$

Therefore, $\mathcal{H}_r^\star(\kappa_1) \neq 0$. Let $s_1 = T_{\mathcal{H}_r^\star}(\kappa_1)$, $t_1 = I_{\mathcal{H}_r^\star}(\kappa_1)$, and $p_1 = F_{\mathcal{H}_r^\star}(\kappa_1)$. Then, $(\kappa_1)_{s_1,t_1,p_1} \in \mathcal{H}_r^*$ and $s_1 + T_{\mathcal{R}}(\kappa_1) > 1$, $t_1 + I_{\mathcal{R}}(\kappa_1) \leq 1$, and $p_1 + F_{\mathcal{R}}(\kappa_1) \leq 1$ so that $\mathcal{R} \in Q_{\tau^{TIF}}((\kappa_1)_{s_1,t_1,p_1}, r)$. Now, $(\kappa_1)_{s_1,t_1,p_1} \in \mathcal{H}_r^\star$ implies there is at least one $\kappa' \in \mathcal{S}$ such that $T_{\mathcal{D}}(\kappa') + T_{\mathcal{H}}(\kappa') - 1 > T_{\mathcal{C}}(\kappa')$, $I_{\mathcal{D}}(\kappa') + I_{\mathcal{H}}(\kappa') - 1 \leq I_{\mathcal{C}}(\kappa')$, and $F_{\mathcal{D}}(\kappa') + F_{\mathcal{H}}(\kappa') - 1 \leq F_{\mathcal{C}}(\kappa')$, for all $\mathcal{L}^T(\mathcal{C}) \geq r$, $\mathcal{L}^I(\mathcal{C}) \leq 1 - r$, $\mathcal{L}^F(\mathcal{C}) \leq 1 - r$, and $\mathcal{D} \in Q_{\tau^{TIF}}((\kappa_1)_{s_1,t_1,p_1}, r)$. That is also true for $\mathcal{R}$. So there is at least one $\kappa'' \in \mathcal{S}$ such that $T_{\mathcal{R}}(\kappa'') + T_{\mathcal{H}}(\kappa'') - 1 > T_{\mathcal{C}}(\kappa'')$, $I_{\mathcal{R}}(\kappa'') + I_{\mathcal{H}}(\kappa'') - 1 \leq I_{\mathcal{C}}(\kappa'')$, and $F_{\mathcal{R}}(x'') + F_{\mathcal{H}}(\kappa'') - 1 \leq F_{\mathcal{C}}(\kappa'')$. Since $\mathcal{R} \in Q_{\tau^{TIF}}(\kappa_{s,t,p}, r)$ and $\mathcal{R}$ is arbitrary; then $T_{\mathcal{H}_r^\star}(\kappa) > s$, $I_{\mathcal{H}_r^\star}(\kappa) \leq t$ and $T_{\mathcal{H}_r^\star}(\kappa) \leq p$. It is a contradiction for (4). Thus, $\mathcal{H}_r^\star \geq \mathbb{C}_{\tau^{TIF}}(\mathcal{H}_r^\star, r)$.

(4) ($\Rightarrow$) Can be easily established using standard technique.

(5) ($\Rightarrow$) Since $\mathcal{H}, \mathcal{R} \leq \mathcal{H} \cup \mathcal{R}$. By (1), $\mathcal{H}_r^\star \leq (\mathcal{H} \cup \mathcal{R})_r^\star$ and $\mathcal{R}_r^\star \leq (\mathcal{H} \cup \mathcal{R})_r^\star$. Hence, $\mathcal{H}_r^\star \cup \mathcal{B}_r^\star \leq (\mathcal{H} \cup \mathcal{R})_r^\star$.

($\Leftarrow$) Suppose $(\mathcal{H}_r^\star \cup \mathcal{R}_r^\star) \not\geq (\mathcal{H} \cup \mathcal{R})_r^\star$. Then, there exists $s, t, p \in I_0$ and $\kappa \in \mathcal{S}$ such that

$$T_{(\mathcal{H}_r^\star \cup \mathcal{R}_r^\star)}(\kappa) < s \leq T_{(\mathcal{H} \cup \mathcal{R})_r^\star}(\kappa),\ I_{(\mathcal{H}_r^\star \cup \mathcal{R}_r^\star)}(\kappa) \geq t > I_{(\mathcal{H} \cup \mathcal{R})_r^\star}(\kappa),\ F_{(\mathcal{H}_r^\star \cup \mathcal{R}_r^\star)}(\kappa) \geq p > F_{(\mathcal{H} \cup \mathcal{R})_r^\star}(\kappa). \tag{5}$$

Since $T_{(\mathcal{H}_r^\star \cup \mathcal{R}_r^\star)}(\kappa) < s$, $I_{(\mathcal{H}_r^\star \cup \mathcal{R}_r^\star)}(\kappa) \geq t$, and $F_{(\mathcal{H}_r^\star \cup \mathcal{R}_r^\star)}(\kappa) \geq p$, we have $T_{\mathcal{H}_r^\star}(\kappa) < s$, $I_{\mathcal{H}_r^\star}(\kappa) \geq t$, $F_{\mathcal{H}_r^\star}(\kappa) \geq p$ or $T_{\mathcal{R}_r^\star}(\kappa) < t$, $I_{\mathcal{R}_r^\star}(\kappa) \geq t$, $F_{\mathcal{R}_r^\star}(\kappa) \geq t$. So, there exists $\mathcal{D}_1 \in Q_{\tau^{TIF}}(x_{s,t,p}, r)$ such that for every $\kappa_1 \in \mathcal{S}$ and for some $\mathcal{L}^T(\mathcal{C}_1) \geq r$, $\mathcal{L}^I(\mathcal{C}_1) \leq 1 - r$, $\mathcal{L}^F(\mathcal{C}_1) \leq 1 - r$, we have

$$T_{\mathcal{D}_1}(\kappa_1) + T_{\mathcal{H}}(\kappa_1) - 1 \leq T_{\mathcal{C}_1}(\kappa_1), \quad I_{\mathcal{D}_1}(\kappa_1) + I_{\mathcal{H}}(\kappa_1) - 1 > I_{\mathcal{C}_1}(\kappa_1), \quad F_{\mathcal{D}_1}(\kappa_1) + F_{\mathcal{H}}(\kappa_1) - 1 > F_{\mathcal{C}_1}(\kappa_1).$$

Similarly, there exists $\mathcal{D}_2 \in Q_{\tau^{TIF}}(x_{s,t,p}, r)$ such that for every $\kappa_1 \in \mathcal{S}$ and for some $\mathcal{L}^T(\mathcal{C}_2) \geq r$, $\mathcal{L}^I(\mathcal{C}_2) \leq 1 - r$, $\mathcal{L}^F(\mathcal{C}_2) \leq 1 - r$, we have

$$T_{\mathcal{D}_2}(\kappa_1) + T_{\mathcal{H}}(\kappa_1) - 1 \leq T_{\mathcal{C}_2}(\kappa_1), \quad I_{\mathcal{D}_2}(\kappa_1) + I_{\mathcal{H}}(\kappa_1) - 1 > I_{\mathcal{C}_2}(\kappa_1), \quad F_{\mathcal{D}_2}(\kappa_1) + F_{\mathcal{H}}(\kappa_1) - 1 > F_{\mathcal{C}_2}(\kappa_1).$$

Since $\mathcal{D} = \mathcal{D}_1 \wedge \mathcal{D}_2 \in Q_{\tau^{TIF}}(x_{s,t,p}, r)$ and by (L_3), $\mathcal{L}^T(\mathcal{C}_1 \cup \mathcal{C}_2) \geq \mathcal{L}^T(\mathcal{C}_1) \cap \mathcal{L}^T(\mathcal{C}_2) \geq r$, $\mathcal{L}^I(\mathcal{C}_1 \cup \mathcal{C}_2) \leq \mathcal{L}^I(\mathcal{C}_1) \cup \mathcal{L}^I(\mathcal{C}_2) \leq 1 - r$, and $\mathcal{L}^F(\mathcal{C}_1 \cup \mathcal{C}_2) \leq \mathcal{L}^T(\mathcal{C}_1) \cup \mathcal{L}^T(\mathcal{C}_2) \leq 1 - r$. Thus, for every $\kappa_1 \in \mathcal{S}$,

$$\begin{aligned} T_{\mathcal{D}}(\kappa_1) + T_{\mathcal{R} \cup \mathcal{H}}(\kappa_1) - 1 &\leq T_{\mathcal{C}_1 \cup \mathcal{C}_2}(\kappa_1), \\ I_{\mathcal{D}}(\kappa_1) + I_{\mathcal{R} \cup \mathcal{H}}(\kappa_1) - 1 &\geq I_{\mathcal{C}_1 \cup \mathcal{C}_2}(\kappa_1), \\ F_{\mathcal{D}}(\kappa_1) + F_{\mathcal{R} \cup \mathcal{H}}(\kappa_1) &\geq F_{\mathcal{C}_1 \cup \mathcal{C}_2}(\kappa_1). \end{aligned}$$

Therefore, $T_{(\mathcal{H} \cup \mathcal{R})_r^\star}(\kappa) < s$, $I_{(\mathcal{H} \cup \mathcal{R})_r^\star}(\kappa) \geq t$, and $F_{(\mathcal{H} \cup \mathcal{R})_r^\star}(\kappa) \geq p$. So, we arrive at a contradiction for (5). Hence, $(\mathcal{H}_r^\star \cup \mathcal{R}_r^\star) \geq (\mathcal{H} \cup \mathcal{R})_r^\star$.

(6), (7), and (8) can be easily established using the standard technique. □

Example 4. *Let $\mathcal{S} = \{a, b\}$. Define $\mathcal{R}, \mathcal{C}, \mathcal{H} \in \mathcal{S}$ as follows:*

$$\mathcal{R}_1 = \langle (0.5, 0.5, 0.5), (0.5, 0.5, 0.5), (0.5, 0.5, 0.5) \rangle; \quad \mathcal{R}_2 = \langle (0.4, 0.4, 0.4), (0.1, 0.1, 0.1), (0.1, 0.1, 0.1) \rangle;$$

$$\mathcal{R}_3 = \langle (0.3, 0.3, 0.3), (0.1, 0.1, 0.1), (0.1, 0.1, 0.1); \quad \mathcal{C}_1 = \langle (0.3, 0.3, 0.3), (0.3, 0.3, 0.3), (0.1, 0.1, 0.1) \rangle;$$

$$\mathcal{C}_2 = \langle (0.2, 0.2, 0.2), (0.2, 0.2, 0.2), (0.1, 0.1, 0.1); \quad \mathcal{C}_3 = \langle (0.1, 0.1, 0.1), (0.1, 0.1, 0.1), (0.1, 0.1, 0.1) \rangle.$$

Define $\tau^{TIF}, \mathcal{L}^{TIF} : I^X \to I$ as follows:

$$\tau^T(\mathcal{H}) = \begin{cases} 1, & \text{if } \mathcal{H} = (0,1,1), \\ 1, & \text{if } \mathcal{H} = (1,0,0), \\ \frac{1}{2}, & \text{if } \mathcal{H} = \mathcal{R}_1; \end{cases} \qquad \mathcal{L}^T(\mathcal{H}) = \begin{cases} 1, & \text{if } \mathcal{H} = (0,1,1), \\ \frac{1}{2}, & \text{if } \mathcal{H} = \mathcal{C}_1, \\ \frac{2}{3}, & \text{if } \underline{0} < \mathcal{H} < \mathcal{C}_1; \end{cases}$$

$$\tau^I(\mathcal{H}) = \begin{cases} 0, & \text{if } \mathcal{H} = (0,1,1), \\ 0, & \text{if } \mathcal{H} = (1,0,0), \\ \frac{1}{2}, & \text{if } \mathcal{H} = \mathcal{R}_2; \end{cases} \qquad \mathcal{L}^I(\mathcal{R}) = \begin{cases} 0, & \text{if } \mathcal{H} = (0,1,1), \\ \frac{1}{2}, & \text{if } \mathcal{H} = \mathcal{C}_2, \\ \frac{1}{4}, & \text{if } \underline{0} < \mathcal{H} < \mathcal{C}_2; \end{cases}$$

$$\tau^F(\mathcal{H}) = \begin{cases} 0, & \text{if } \mathcal{H} = (0,1,1), \\ 0, & \text{if } \mathcal{H} = (1,0,0), \\ \frac{1}{2}, & \text{if } \mathcal{H} = \mathcal{R}_3; \end{cases} \qquad \mathcal{L}^F(\mathcal{H}) = \begin{cases} 0, & \text{if } \mathcal{H} = (0,1,1), \\ \frac{1}{2}, & \text{if } \mathcal{H} = \mathcal{C}_3, \\ \frac{1}{4}, & \text{if } \underline{0} < \mathcal{H} < \mathcal{C}_3. \end{cases}$$

Let $\mathcal{G} = \langle (0.4, 0.4, 0.4), (0.4, 0.4, 0.4), (0.4, 0.4, 0.4) \rangle$. Then, $\mathcal{G}^\star_{\frac{1}{2}} = \mathcal{R}_1$.

Theorem 4. *Let $\{\mathcal{H}_i\}_{i \in J} \subset I^{\mathcal{S}}$ be a family of single-valued neutrosophic sets on $\mathcal{S}$ and $(\mathcal{S}, \tau^{TIF}, \mathcal{L}^{TIF})$ be an* **SVNITS**. *Then,*

(1) $(\bigcup (\mathcal{H}_i)^\star_r : i \in J) \leq (\bigcup \mathcal{H}_i : i \in J)^\star_r$;
(2) $(\bigcap (\mathcal{H}_i)^\star_r : i \in J) \geq (\bigcap \mathcal{H}_i : i \in J)^\star_r$.

Proof. (1) Since $\mathcal{H}_i \leq \bigcup \mathcal{H}_i$ for all $i \in J$, and by Theorem 3 (1), we obtain $(\bigcup (\mathcal{H}_i)^\star_r, i \in J) \leq (\bigcup \mathcal{H}_i, i \in J)^\star_r$. Then, (1) holds.

(2) Easy, so omitted. □

Remark 3. *Let $(\mathcal{S}, \tau^{TIF}, \mathcal{L}^{TIF})$ be an* **SVNITS** *and $\mathcal{H} \in I^{\mathcal{S}}$, we can define*

$$\mathbb{C}^\star_{\tau^{TIF}}(\mathcal{H}, r) = \mathcal{H} \cup \mathcal{H}^\star_r, \qquad int^\star_{\tau^{TIF}}(\mathcal{H}, r) = \mathcal{H} \wedge [\underline{1} - (\underline{1} - \mathcal{H})^\star_r].$$

It is clear, $\mathbb{C}^\star_{\tau^{TIF}}$ is a single-valued neutrosophic closure operator and $(\tau^{T\star}(\mathcal{L}^T), \tau^{I\star}(\mathcal{L}^I), \tau^{F\star}(\mathcal{L}^F)$ is the single-valued neutrosophic topology generated by $\mathbb{C}^\star_{\tau^{TIF}}$, i.e.,

$$\tau^\star(\mathcal{I})(\mathcal{H}) = \bigcup \{r | \ \mathbb{C}^\star_{\tau^{TIF}}(\underline{1} - \mathcal{H}, r) = \underline{1} - \mathcal{H}\}.$$

Now, if $\mathcal{L}^{TIF} = \mathcal{L}^{TIF}_0$, then, $\mathbb{C}^\star_{\tau^{TIF}}(\mathcal{H}, r) = \mathcal{H}^*_r \cup \mathcal{H} = \mathbb{C}^\star_{\tau^{TIF}}(\mathcal{H}, r) \cup \mathcal{H} = \mathbb{C}_{\tau^{TIF}}(\mathcal{H}, r)$, for $\mathcal{H} \in I^{\mathcal{S}}$. So, $\tau^{TIF\star}(\mathcal{L}^{TIF}_0) = \tau^{TIF}$.

Proposition 3. *Let $(\mathcal{S}, \tau^{TIF}, \mathcal{L}^{TIF})$ be an* **SVNITS**, *$r \in I_0$, and $\mathcal{H} \in I^{\mathcal{S}}$. Then,*

(1) $\mathbb{C}^\star_{\tau^{TIF}}(\underline{1}, r) = \underline{1}$;

(2) $\mathbb{C}^{\star}_{\tau^{TIF}}(\underline{0}, r) = \underline{0}$;
(3) $int^{\star}_{\tau^{TIF}}(\mathcal{H} \cup \mathcal{R}, r) \leq int^{\star}_{\tau^{TIF}}(\mathcal{H}, r) \cup int^{\star}_{\tau^{TIF}}(\mathcal{R}, r)$;
(4) $int^{\star}_{\tau^{TIF}}(\mathcal{H}, r) \leq \mathcal{H} \leq \mathbb{C}^{\star}_{\tau^{TIF}}(\mathcal{H}, r) \leq \mathbb{C}_{\tau^{TIF}}(\mathcal{H}, r)$;
(5) $\mathbb{C}^{\star}_{\tau^{TIF}}(\underline{1} - \mathcal{H}, r) = \underline{1} - int^{\star}_{\tau^{TIF}}(\mathcal{H}, r)$ *and* $\underline{1} - \mathbb{C}^{\star}_{\tau^{TIF}}\star(\mathcal{H}, r) = int^{\star}_{\tau^{TIF}}(\underline{1} - \mathcal{H}, r)$;
(6) $int^{\star}_{\tau^{TIF}}(\mathcal{H} \cap \mathcal{R}, r) = int^{\star}_{\tau^{TIF}}(\mathcal{H}, r) \cap int^{\star}_{\tau^{TIF}}(\mathcal{R}, r)$.

Proof. Follows directly from definitions of $\mathbb{C}^{\star}_{\tau^{TIF}}$, $int^{\star}_{\tau^{TIF}}$, $\mathbb{C}_{\tau^{TIF}}$, and Theorem 3 (5). □

Theorem 5. *Let* $(\mathcal{S}, \tau_1^{TIF}, \mathcal{L}^{TIF})$ *and* $(\mathcal{S}, \tau_2^{TIF}, \mathcal{L}^{TIF})$ *be* **SVNTS′s** *and* $\tau_1^{TIF} \leq \tau_2^{TIF}$. *Then,* $\mathcal{H}_r^{\star}(\tau_2^{TIF}, \mathcal{L}^{TIF}) \leq \mathcal{H}_r^{\star}(\tau_1^{TIF}, \mathcal{L}^{TIF})$.

Proof. Suppose $\mathcal{H}_r^{\star}(\tau_2^{TIF}, \mathcal{L}^{TIF}) \not\leq \mathcal{H}_r^{\star}(\tau_1^{TIF}, \mathcal{L}^{TIF})$. Then, there exists $s, t, p \in I_0$, $\kappa \in \mathcal{S}$ such that

$$T_{\mathcal{H}_r^{\star}(\tau_2^{TIF}, \mathcal{L}^{TIF})}(\kappa) \geq s > T_{\mathcal{H}_r^{\star}(\tau_1^{TIF}, \mathcal{L}^{TIF})}(\kappa),$$

$$I_{\mathcal{H}_r^{\star}(\tau_2^{TIF}, \mathcal{L}^{TIF})}(\kappa) < t \leq I_{\mathcal{H}_r^{\star}(\tau_1^{TIF}, \mathcal{L}^{TIF})}(\kappa), \tag{6}$$

$$F_{\mathcal{H}_r^{\star}(\tau_2^{TIF}, \mathcal{L}^{TIF})}(\kappa) < t \leq F_{\mathcal{H}_r^{\star}(\tau_1^{TIF}, \mathcal{L}^{TIF})}(\kappa).$$

Since $T_{\mathcal{H}_r^{\star}(\tau_1^{TIF}, \mathcal{L}^{TIF})}(\kappa) < s$, $I_{\mathcal{H}_r^{\star}(\tau_1^{TIF}, \mathcal{L}^{TIF})}(\kappa) \geq t$, $F_{\mathcal{H}_r^{\star}(\tau_1^{TIF}, \mathcal{L}^{TIF})}(\kappa) \geq p$, there exists $\mathcal{D} \in Q_{\tau_1^{TIF}}(x_{s,t,p}, r)$ with $\mathcal{L}^T(\mathcal{C}_1) \geq r$, $\mathcal{L}^I(\mathcal{C}_1) \leq 1 - r$ and $\mathcal{L}^F(\mathcal{C}_1) \leq 1 - r$, such that for any $\kappa_1 \in \mathcal{S}$,

$$T_{\mathcal{D}}(\kappa_1) + T_{\mathcal{H}}(\kappa_1) - 1 \leq T_{\mathcal{C}}\kappa_1), \quad I_{\mathcal{D}}(\kappa_1) + I_{\mathcal{H}}(\kappa_1) - 1 > I_{\mathcal{C}}(\kappa_1), \quad F_{\mathcal{D}}(\kappa_1) + F_{\mathcal{H}}(\kappa_1) - 1 > F_{\mathcal{C}}(\kappa_1).$$

Since $\tau_1^{TIF} \leq \tau_2^{TIF}$, $\mathcal{D} \in Q_{\tau_2^{TIF}}(x_{s,t,p}, r)$. Thus, $T_{\mathcal{H}_r^{\star}(\tau_2^{TIF}, \mathcal{L}^{TIF})}(\kappa) < s$, $I_{\mathcal{H}_r^{\star}(\tau_2^{TIF}, \mathcal{L}^{TIF}))}(\kappa) \geq t$, $F_{\mathcal{H}_r^{\star}(\tau_2^{TIF}, \mathcal{L}^{TIF})}(\kappa) \geq p$. It is a contradiction for Equation (6). □

Theorem 6. *Let* $(\mathcal{S}, \tau^{TIF}, \mathcal{L}_1^{TIF})$ *and* $(\mathcal{S}, \tau^{TIF}, \mathcal{L}_2^{TIF})$ *be* **SVNTS′s** *and* $\mathcal{L}_1^{TIF} \leq \mathcal{L}_2^{TIF}$. *Then,* $\mathcal{H}_r^{\star}(\mathcal{L}_1^{TIF}, \tau^{TIF}) \geq \mathcal{H}_r^{\star}(\mathcal{L}_2^{TIF}, \tau^{TIF})$.

Proof. Clear. □

Definition 14. *Let* Θ *be a subset of* $I^{\mathcal{S}}$, *and* $\underline{0} \notin \Theta$. *A mapping* $\beta^T, \beta^I, \beta^F : \Theta \to I$ *is called a single-valued neutrosophic base on* $\mathcal{S}$ *if it satisfies the following conditions:*

(1) $\beta^T(\underline{1}) = 1$ *and* $\beta^I(\underline{1}) = \beta^F(\underline{1}) = 0$;
(2) *For all* $\mathcal{H}, \mathcal{R} \in \Theta$,

$$\beta^T(\mathcal{H} \cap \mathcal{R}) \geq \beta^T(\mathcal{H}) \cap \beta^T(\mathcal{R}), \quad \beta^I(\mathcal{H} \cap \mathcal{R}) \leq \beta^I(\mathcal{H}) \cup \beta^I(\mathcal{R}), \quad \beta^F(\mathcal{H} \cap \mathcal{R}) \leq \beta^F(\mathcal{H}) \cup \beta^F(\mathcal{R}).$$

Theorem 7. *Define a mapping* $\beta : \Theta \to I$ *on* $\mathcal{S}$ *by*

$$\beta^I(\mathcal{H}) = \bigcup\{\tau^T(\mathcal{R}) \cap \mathcal{I}^T(\mathcal{C}) |\ \mathcal{H} = \mathcal{R} \cap (\underline{1} - \mathcal{C})\},$$

$$\beta^I(\mathcal{H}) = \bigcap\{\tau^I(\mathcal{R}) \cup \mathcal{I}^I(\mathcal{C}) |\ \mathcal{H} = \mathcal{R} \cap (\underline{1} - \mathcal{C})\},$$

$$\beta^F(\mathcal{H}) = \bigcap\{\tau^F(\mathcal{R}) \cup \mathcal{I}^F(\mathcal{C}) |\ \mathcal{H} = \mathcal{R} \cap (\underline{1} - \mathcal{C})\}.$$

Then, β^{TIF} is a base for the single-valued neutrosophic topology $\tau^{TIF\star}$.

Proof. (1) Since $\mathcal{L}^T(\underline{0}) = 1$ and $\mathcal{L}^I(\underline{0}) = \mathcal{L}^F(\underline{0}) = 0$, we have $\beta^T(\underline{1}) = 1$ and $\beta^I(\underline{1}) = \beta^F(\underline{1}) = 0$;
(2) Suppose that there exists $\mathcal{H}_1, \mathcal{H}_2 \in \Theta$ such that

$$\begin{aligned} \beta^T(\mathcal{H}_1 \cap \mathcal{H}_2) &\ngeq \beta^T(\mathcal{H}_1) \cap \beta^T(\mathcal{H}_2), \\ \beta^I(\mathcal{H}_1 \cap \mathcal{H}_2) &\nleq \beta^I(\mathcal{H}_1) \cup \beta^I(\mathcal{H}_2), \\ \beta^F(\mathcal{H}_1 \cap \mathcal{H}_2) &\nleq \beta^F(\mathcal{H}_1) \cup \beta^F(\mathcal{H}_2). \end{aligned}$$

There exists $s, t, p \in I_0$ and $\kappa \in \mathcal{S}$ such that

$$\begin{aligned} &\beta^T(\mathcal{H}_1 \cap \mathcal{H}_2)(\kappa) < s \le \beta^T(\mathcal{H}_1)(x) \cap \beta^T(\mathcal{H}_2)(\kappa), \\ &\beta^I(\mathcal{H}_1 \cap \mathcal{H}_2)(\kappa) \ge t > \beta^I(\mathcal{H}_1)(\kappa) \cap \beta^I(\mathcal{H}_2)(\kappa), \\ &\beta^F(\mathcal{H}_1 \cap \mathcal{H}_2)(\kappa) \ge p > \beta^F(\mathcal{H}_1)(\kappa) \cup \beta^F(\mathcal{H}_2)(\kappa). \end{aligned} \tag{7}$$

Since $\beta^T(\mathcal{H}_1)(\kappa) \ge s$, $\beta^I(\mathcal{H}_1)(\kappa) < t$, $\beta^F(\mathcal{H}_1)(\kappa) < p$, and $\beta^T(\mathcal{H}_2)(\kappa) \ge s$, $\beta^I(\mathcal{H}_2)(\kappa) < t$, $\beta^F(\mathcal{H}_2)(\kappa) < p$, then there exists $\mathcal{R}_1, \mathcal{R}_1, \mathcal{C}_1, \mathcal{C}_2 \in \Theta$ with $\mathcal{H}_1 = \mathcal{R}_1 \cap (\underline{1} - \mathcal{C}_1)$ and $\mathcal{H}_2 = \mathcal{R}_2 \cap (\underline{1} - \mathcal{C}_2)$, such that $\beta^T(\mathcal{H}_1) \ge \tau^T(\mathcal{R}_1) \cap \mathcal{L}^T(\mathcal{C}_1) \ge s$, $\beta^I(\mathcal{H}_1) \le \tau^I(\mathcal{R}_1) \cup \mathcal{L}^I(\mathcal{C}_1) < t$, $\beta^F(\mathcal{H}_1) \le \tau^F(\mathcal{R}_1) \cup \mathcal{L}^F(\mathcal{C}_1) < p$, and $\beta^T(\mathcal{H}_2) \ge \tau^T(\mathcal{R}_2) \cap \mathcal{L}^T(\mathcal{C}_2) \ge s$, $\beta^I(\mathcal{H}_2) \le \tau^I(\mathcal{R}_2) \cup \mathcal{L}^I(\mathcal{C}_2) < t$, $\beta^F(\mathcal{H}_2) \le \tau^F(\mathcal{R}_2) \cup \mathcal{L}^F(\mathcal{C}_2) < p$. Therefore,

$$\begin{aligned} \mathcal{H}_1 \cap \mathcal{H}_2 &= (\mathcal{R}_1 \cap (\underline{1} - \mathcal{C}_1)) \cap (\mathcal{R}_2 \cap (\underline{1} - \mathcal{C}_2)) \\ &= (\mathcal{R}_1 \cap \mathcal{R}_2) \cap ((\underline{1} - \mathcal{C}_1) \cap (\underline{1} - \mathcal{C}_2)) \\ &= (\mathcal{R}_1 \cap \mathcal{R}_2) \cap (\underline{1} - (\mathcal{C}_1 \cup \mathcal{C}_2)). \end{aligned}$$

Hence, from Definition 14, we have

$$\begin{aligned} \beta^T(\mathcal{H}_1 \cap \mathcal{H}_2) &\ge \tau^T(\mathcal{R}_1 \cap \mathcal{R}_2) \cap \mathcal{L}^T(\mathcal{C}_1 \cup \mathcal{C}_2) \\ &\ge \tau^T(\mathcal{R}_1) \cap \tau^T(\mathcal{R}_2) \cap \mathcal{L}^T(\mathcal{C}_1) \cap \mathcal{L}^T(\mathcal{C}_2) \\ &= (\tau^T(\mathcal{R}_1) \cap \mathcal{L}^T(\mathcal{C}_1)) \cap (\tau^T(\mathcal{R}_2) \cap \mathcal{L}^T(\mathcal{C}_2)) \ge s, \end{aligned}$$

$$\begin{aligned} \beta^I(\mathcal{H}_1 \cap \mathcal{H}_2) &\le \tau^I(\mathcal{R}_1 \cap \mathcal{R}_2) \cup \mathcal{L}^I(\mathcal{C}_1 \cup \mathcal{C}_2) \\ &\le \tau^I(\mathcal{R}_1) \cup \tau^I(\mathcal{R}_2) \cup \mathcal{L}^I(\mathcal{C}_1) \cup \mathcal{L}^I(\mathcal{C}_2) \\ &= (\tau^I(\mathcal{R}_1) \cup \mathcal{L}^F(\mathcal{C}_1)) \cup (\tau^I(\mathcal{R}_2) \cup \mathcal{L}^I(\mathcal{C}_2)) < t, \end{aligned}$$

$$\begin{aligned} \beta^F(\mathcal{H}_1 \cap \mathcal{H}_2) &\le \tau^F(\mathcal{R}_1 \cap \mathcal{R}_2) \cup \mathcal{L}^F(\mathcal{C}_1 \cup \mathcal{C}_2) \\ &\le \tau^F(\mathcal{R}_1) \cup \tau^F(\mathcal{R}_2) \cup \mathcal{L}^F(\mathcal{C}_1) \cup \mathcal{L}^F(\mathcal{C}_2) \\ &= (\tau^F(\mathcal{R}_1) \cup \mathcal{L}^F(\mathcal{C}_1)) \cup (\tau^F(\mathcal{R}_2) \cup \mathcal{L}^F(\mathcal{C}_2)) < p. \end{aligned}$$

It is a contradiction for Equation (7). Thus,

$$\beta^T(\mathcal{H}_1 \cap \mathcal{H}_2) \ge \beta^T(\mathcal{H}_1) \cap \beta^T(\mathcal{H}_2), \beta^I(\mathcal{H}_1 \cap \mathcal{H}_2) \le \beta^I(\mathcal{H}_1) \cup \beta^I(\mathcal{H}_2), \beta^F(\mathcal{H}_1 \cap \mathcal{H}_2) \le \beta^F(\mathcal{H}_1) \cup \beta^F(\mathcal{H}_2).$$

□

Theorem 8. *Let $(\mathcal{S}, \tau^{TIF})$ be an* **SVNTS**, *and $\mathcal{L}_1^{TIF}$ and $\mathcal{L}_1^{TIF}$ be two single-valued neutrosophic ideals on $\mathcal{S}$. Then, for every $r \in I_0$ and $\mathcal{H} \in I^{\mathcal{S}}$,*

(1) $\mathcal{H}_r^\star(\mathcal{L}_1^{TIF} \cap \mathcal{L}_2^{TIF}, \tau^{TIF}) = \mathcal{H}_r^\star(\mathcal{L}_1^{TIF}, \tau^{TIF}) \cup \mathcal{H}_r^\star(\mathcal{L}_2^{TIF}, \tau^{TIF})$,
(2) $\mathcal{H}_r^\star(\mathcal{L}_1^{TIF} \cup \mathcal{L}_2^{TIF}, \tau) = \mathcal{H}_r^\star(\mathcal{L}_1^{TIF}, \tau^{T\star}(\mathcal{L}_2^{TIF})) \cap \mathcal{H}^\star(\mathcal{L}_2^{TIF}, \tau^{T\star}(\mathcal{L}_1^{TIF}))$.

Proof. (1) Suppose that $\mathcal{H}_r^\star(\mathcal{L}_1^{TIF} \cap \mathcal{L}_2^{TIF}, \tau^{TIF}) \not\leq \mathcal{H}_r^\star(\mathcal{L}_1^{TIF}, \tau^{TIF}) \cup \mathcal{H}_r^\star(\mathcal{L}_2^{TIF}, \tau^{TIF})$, there exists $\kappa \in \mathcal{S}$ and $s, t, p \in I_0$ such that

$$T_{\mathcal{H}_r^\star(\mathcal{L}_1^T \cap \mathcal{L}_2^T, \tau^T)}(\kappa) \geq s > T_{\mathcal{H}_r^\star(\mathcal{L}_1^T, \tau^T)}(\kappa) \cup T_{\mathcal{H}_r^\star(\mathcal{L}_2^T, \tau^T)}(\kappa), \tag{8}$$
$$I_{\mathcal{H}_r^\star(\mathcal{L}_1^I \cap \mathcal{L}_2^I, \tau^I)}(\kappa) < t \leq I_{\mathcal{H}_r^\star(\mathcal{L}_1^I, \tau^I)}(\kappa) \cup I_{\mathcal{H}_r^\star(\mathcal{L}_2^I, \tau^I)}(\kappa),$$

$$F_{\mathcal{H}_r^\star(\mathcal{L}_1^F \cap \mathcal{L}_2^F, \tau^F)}(\kappa) < p \leq F_{\mathcal{H}_r^\star(\mathcal{L}_1^F, \tau^F)}(\kappa) \cap F_{\mathcal{H}_r^\star(\mathcal{L}_2^F, \tau^F)}(\kappa).$$

Since $T_{\mathcal{H}_r^\star(\mathcal{L}_1^T, \tau^T)}(\kappa) \cup T_{\mathcal{H}_r^\star(\mathcal{L}_2^T, \tau^T)}(\kappa) < s$, $I_{\mathcal{H}_r^\star(\mathcal{L}_1^I, \tau^I)}(\kappa) \cap I_{\mathcal{H}_r^\star(\mathcal{L}_2^I, \tau^I)}(\kappa) \geq t$, $F_{\mathcal{H}_r^\star(\mathcal{L}_1^F, \tau^F)}(\kappa) \cap F_{\mathcal{H}_r^\star(\mathcal{L}_2^F, \tau^F)}(\kappa) \geq p$, we have, $T_{\mathcal{H}_r^\star(\mathcal{L}_1^T, \tau^T)}(\kappa) < s$, $I_{\mathcal{H}_r^\star(\mathcal{L}_1^I, \tau^I)}(\kappa) \geq t$, $F_{\mathcal{H}_r^\star(\mathcal{L}_1^F, \tau^F)}(\kappa) \geq p$, and $I_{\mathcal{H}_r^\star(\mathcal{L}_2^I, \tau^I)}(\kappa) < s$, $I_{\mathcal{H}_r^\star(\mathcal{L}_2^I, \tau^I)}(\kappa) \geq t$, $F_{\mathcal{H}_r^\star(\mathcal{L}_2^F, \tau^F)}(\kappa) \geq p$.

Now, $T_{\mathcal{H}_r^\star(\mathcal{L}_1^T, \tau^T)}(\kappa) < s$, $I_{\mathcal{H}_r^\star(\mathcal{L}_1^I, \tau^I)}(\kappa) \geq t$, $F_{\mathcal{H}_r^\star(\mathcal{L}_1^F, \tau^F)}(\kappa) \geq p$ implies that there exists $\mathcal{D}_1 \in Q_{\tau^{TIF}}(x_{s,t,p}, r)$ and for some $\mathcal{L}_1^T(\mathcal{C}_1) \geq r$, $\mathcal{L}_1^I(\mathcal{C}_1) \leq 1 - r$ and $\mathcal{L}_1^F(\mathcal{C}_1) \leq 1 - r$ such that for every $\kappa_1 \in \mathcal{S}$,

$$T_{\mathcal{D}_1}(\kappa_1) + T_{\mathcal{H}}(\kappa_1) - 1 \leq T_{\mathcal{C}_1}(\kappa_1), \quad I_{\mathcal{D}_1}(\kappa_1) + I_{\mathcal{H}}(\kappa_1) - 1 \geq I_{\mathcal{C}_1}(\kappa_1), \quad F_{\mathcal{D}_1}(\kappa_1) + F_{\mathcal{H}}(\kappa_1) - 1 \geq F_{\mathcal{C}_1}(\kappa_1).$$

Once again, $T_{\mathcal{H}_r^\star(\mathcal{L}_2^T, \tau^T)}(\kappa) < s$, $I_{\mathcal{H}_r^\star(\mathcal{L}_2^I, \tau^I)}(\kappa) \geq t$, $F_{\mathcal{H}_r^\star(\mathcal{L}_2^F, \tau^F)}(\kappa) \geq p$, implies there exists $\mathcal{D}_2 \in Q_{\tau^{TIF}}(x_{s,t,p}, r)$ and for some $\mathcal{L}_2^T(\mathcal{C}_2) \geq r$, $\mathcal{L}_2^I(\mathcal{C}_2) \leq 1 - r$ and $\mathcal{L}_2^F(\mathcal{C}_2) \leq 1 - r$, such that for $\kappa_1 \in \mathcal{S}$,

$$T_{\mathcal{D}_2}(\kappa_1) + T_{\mathcal{H}}(\kappa_1) - 1 \leq T_{\mathcal{C}_2}(\kappa_1), \quad I_{\mathcal{D}_2}(\kappa_1) + I_{\mathcal{H}}(\kappa_1) - 1 \geq I_{\mathcal{C}_2}(\kappa), \quad F_{\mathcal{D}_2}(\kappa_1) + F_{\mathcal{H}}(\kappa_1) - 1 \geq F_{\mathcal{C}_2}(\kappa_1),$$

Therefore, for every $\kappa_1 \in \mathcal{S}$, we have

$$T_{\mathcal{D}_1 \cap \mathcal{D}_2}(\kappa_1) + T_{\mathcal{H}}(\kappa_1) - 1 \leq T_{\mathcal{C}_1 \cap \mathcal{C}_2}(\kappa_1), \quad I_{\mathcal{D}_1 \cup \mathcal{D}_2}(\kappa_1) + I_{\mathcal{H}}(\kappa_1) - 1 \geq I_{\mathcal{C}_1 \cup \mathcal{C}_2}(\kappa_1),$$

$$F_{\mathcal{D}_1 \cup \mathcal{D}_2}(\kappa_1) + F_{\mathcal{H}}(\kappa_1) - 1 \geq F_{\mathcal{C}_1 \cup \mathcal{C}_2}(\kappa_1).$$

Since $(\mathcal{D}_1 \wedge \mathcal{D}_2) \in Q_{\tau^{TIF}}(x_{s,t,p}, r)$ and $(\mathcal{L}_1^T \cap \mathcal{L}_2^T)(\mathcal{C}_1 \cap \mathcal{C}_2) \geq r$, $(\mathcal{L}_1^I \cap \mathcal{L}_2^I)(\mathcal{C}_1 \cup \mathcal{C}_2) \leq 1 - r$, and $(\mathcal{L}_1^F \cap \mathcal{L}_2^F)(\mathcal{C}_1 \cup \mathcal{C}_2) \geq 1 - r$ we have $T_{\mathcal{H}_r^\star(\mathcal{L}_1^T \cap \mathcal{L}_2^T, \tau^T)}(\kappa) \leq s$, $I_{\mathcal{H}_r^\star(\mathcal{L}_1^I \cap \mathcal{L}_2^I, \tau^I)}(\kappa) > t$, and $F_{\mathcal{H}_r^\star(\mathcal{L}_1^F \cap \mathcal{L}_2^F, \tau^F)}(\kappa) > t$ and this is a contradiction for Equation (8). So that

$$\mathcal{H}_r^\star(\mathcal{L}_1^{TIF} \cap \mathcal{L}_2^{TIF}, \tau^{TIF}) \leq \mathcal{H}_r^\star(\mathcal{L}_1^{TIF}, \tau^{TIF}) \cup \mathcal{H}_r^\star(\mathcal{L}_2^{TIF}, \tau^{TIF}).$$

On the opposite direction, $\mathcal{L}_1^{TIF} \geq \mathcal{L}_1^{TIF} \cap \mathcal{L}_2^{TIF}$ and $\mathcal{L}_2^{TIF} \geq \mathcal{L}_1^{TIF} \cap \mathcal{L}_2^{TIF}$, so by Theorem 3 (2),

$$\mathcal{H}_r^\star(\mathcal{L}_1^{TIF} \cap \mathcal{L}_2^{TIF}, \tau^T) \geq \mathcal{H}_r^\star(\mathcal{L}_1^{TIF}, \tau^{TIF}) \cup \mathcal{H}_r^\star(\mathcal{L}_2^{TIF}, \tau^{TIF}).$$

Then,

$$\mathcal{H}_r^\star(\mathcal{L}_1^{TIF} \cap \mathcal{L}_2^{TIF}, \tau^{TIF}) = \mathcal{H}_r^\star(\mathcal{L}_1^{TIF}, \tau^{TIF}) \cup \mathcal{H}_r^\star(\mathcal{L}_2^{TIF}, \tau^{TIF}).$$

(2) Straightforward. □

The above theorem results in an important consequence. $\tau^{TIF\star}(\mathcal{L}^{TIF})$ and $[\tau^{TIF\star}(\mathcal{L}^{TIF})]^\star(\mathcal{L}^{TIF})$ (in short $\tau^{\star\star}$) are equal for any single-valued neutrosophic ideal on $\mathcal{S}$.

Corollary 1. *Let $(\mathcal{S}, \tau^{TIF}, \mathcal{L}^{TIF})$ be an* **SVNITS**. *For every $r \in I_0$ and $\mathcal{H} \in I^X$, $\mathcal{H}_r^\star(\mathcal{L}^{TIF}) = \mathcal{H}_r^\star(\mathcal{L}^{TIF}, \tau^{TIF\star})$ and $\tau^{TIF\star}(\mathcal{L}^{TIF}) = \tau^{TIF\star\star}$.*

Proof. Putting $\mathcal{L}_1^{TIF} = \mathcal{L}_2^{TIF}$ in Theorem 8 (2), we have the required result. □

Corollary 2. *Let $(\mathcal{S}, \tau^{TIF})$ be an* **SVNTS**, *and $\mathcal{L}_1^{TIF}$ and $\mathcal{L}_1^{TIF}$ be two single-valued neutrosophic ideals on $\mathcal{S}$. Then, for any $\mathcal{H} \in I^{\mathcal{S}}$ and $r \in I_0$,*

(1) $\tau^{T\star}(\mathcal{L}_1^{TIF} \cup \mathcal{I}_2^{TIF}) = (\tau^{TIF\star}(\mathcal{L}_2^{TIF}))^\star(\mathcal{L}_1^{T}) = (\tau^{TIF\star}(\mathcal{L}_1^{TIF}))^\star(\mathcal{L}_2^{T})$,
(2) $\tau^{T\star}(\mathcal{L}_1^{TIF} \cap \mathcal{L}_2^{TIF}) = \tau^{TIF\star}(\mathcal{L}_1^{TIF}) \cap \tau^{T\star}(\mathcal{L}_2^{TIF})$.

Proof. Straightforward. □

Definition 15. *For an* **SVNTS** *$(\mathcal{S}, \tau^{TIF})$ with a single-valued neutrosophic ideal $\mathcal{I}^{TIF}$, τ^{TIF} is said to be single-valued neutrosophic ideal open compatible with $\mathcal{I}^{TIF}$, denoted by $\tau^{TIF} \sim \mathcal{L}^{TIF}$, if for each $\mathcal{H}, \mathcal{C} \in I^{\mathcal{S}}$ and $x_{s,t,p} \in \mathcal{H}$ with $\mathcal{L}^T(\mathcal{C}) \geq r$, $\mathcal{L}^I(\mathcal{C}) \leq 1 - r$, and $\mathcal{L}^F(\mathcal{C}) \leq 1 - r$, there exists $\mathcal{D} \in Q_{\tau^{TIF}}(x_t, r)$ such that $T_{\mathcal{D}}(\kappa) + T_{\mathcal{H}}(\kappa) - 1 \leq T_{\mathcal{C}}(\kappa)$, $I_{\mathcal{D}}(\kappa) + I_{\mathcal{H}}(\kappa) - 1 > I_{\mathcal{C}}(\kappa)$, and $F_{\mathcal{D}}(\kappa) + F_{\mathcal{H}}(\kappa) - 1 > F_{\mathcal{C}}(\kappa)$ holds for any $\kappa \in \mathcal{S}$, then $\mathcal{L}^T(\mathcal{H}) \geq r$, $\mathcal{L}^I(\mathcal{H}) \leq 1 - r$ and $\mathcal{L}^F(\mathcal{H}) \leq 1 - r$.*

Definition 16. *Let $\{\mathcal{R}_j\}_{j \in J}$ be an indexed family of a single-valued neutrosophic set of $\mathcal{S}$ such that $\mathcal{R}_j q \mathcal{H}$ for each $j \in J$, where $\mathcal{H} \in I^{\mathcal{S}}$. Then, $\{\mathcal{R}_j\}_{j \in J}$ is said to be a single-valued neutrosophic quasi-cover of $\mathcal{H}$ iff $T_{\mathcal{H}}(\kappa) + T_{\bigvee_{j \in J}(\mathcal{R}_j)}(\kappa) \geq 1$, $I_{\mathcal{H}}(\kappa) + I_{\bigvee_{j \in J}(\mathcal{R}_j)}(\kappa) < 1$, and $F_{\mathcal{H}}(\kappa) + F_{\bigvee_{j \in J}(\mathcal{R}_j)}(\kappa) < 1$, for every $\kappa \in \mathcal{S}$.*

Further, let $(\mathcal{S}, \tau^{TIF})$ be an **SVNTS**, for each $\tau^T(\mathcal{R}_j) \geq r$, $\tau^I(\mathcal{R}_j) \leq 1 - r$, and $\tau^F(\mathcal{R}_j) \leq 1 - r$. Then, any single-valued neutrosophic quasi-cover will be called single-valued neutrosophic quasi open-cover of $\mathcal{H}$.

Theorem 9. *Let $(\mathcal{S}, \tau^{TIF})$ be an* **SVNTS** *with single-valued neutrosophic ideal $\mathcal{L}^{TIF}$ on $\mathcal{S}$. Then, the following conditions are equivalent:*

(1) $\tau \sim \mathcal{L}$.
(2) *If for every $\mathcal{H} \in I^{\mathcal{S}}$ has a single-valued neutrosophic quasi open-cover of $\{\mathcal{R}_j\}_{j \in J}$ such that for each j, $T_{\mathcal{H}}(\kappa) + T_{R_j}(\kappa) - 1 \leq T_{\mathcal{C}}(\kappa)$, $I_{\mathcal{H}}(\kappa) + I_{R_j}(\kappa) - 1 > I_{\mathcal{C}}(\kappa)$, and $F_{\mathcal{H}}(\kappa) + F_{R_j}(\kappa) - 1 > F_{\mathcal{C}}(\kappa)$ for every $\kappa \in \mathcal{S}$ and for some $\mathcal{L}^T(\mathcal{C}) \geq r$, $\mathcal{L}^I(\mathcal{C}) \leq 1 - r$, and $\mathcal{L}^F(\mathcal{C}) \leq 1 - r$, then $\mathcal{L}^T(\mathcal{H}) \geq r$, $\mathcal{L}^I(\mathcal{H}) \leq 1 - r$, and $\mathcal{L}^F(\mathcal{H}) \leq 1 - r$,*
(3) *For every $\mathcal{H} \in I^{\mathcal{S}}$, $\mathcal{H} \wedge \mathcal{H}_r^\star = (0, 1, 1)$ implies $\mathcal{L}^T(\mathcal{H}) \geq r$, $\mathcal{L}^I(\mathcal{H}) \leq 1 - r$, and $\mathcal{L}^F(\mathcal{H}) \leq 1 - r$,*
(4) *For every $\mathcal{H} \in I^{\mathcal{S}}$, $\mathcal{L}^T(\widetilde{\mathcal{H}}) \geq r$, $\mathcal{L}^I(\widetilde{\mathcal{H}}) \leq 1 - r$, and $\mathcal{L}^F(\widetilde{\mathcal{H}}) \leq 1 - r$, where $\widetilde{\mathcal{H}} = \bigvee x_{s,t,p}$ such that $x_{s,t,p} \in \mathcal{H}$ but $x_{s,t,p} \notin \mathcal{H}_r^*$,*
(5) *For every $\tau^{T\star}(\underline{1} - \mathcal{H}) \geq r$, $\tau^{I\star}(\underline{1} - \mathcal{H}) \leq 1 - r$, and $\tau^{F\star}(\underline{1} - \mathcal{H}) \leq 1 - r$ we have $\mathcal{L}^T(\widetilde{\mathcal{H}}) \geq r$, $\mathcal{L}^I(\widetilde{\mathcal{H}}) \leq 1 - r$, and $\mathcal{L}^F(\widetilde{\mathcal{H}}) \leq 1 - r$,*
(6) *For every $\mathcal{H} \in I^{\mathcal{S}}$, if $\mathcal{A}$ contains no $\mathcal{R} \neq (0, 1, 1)$ with $\mathcal{R} \leq \mathcal{R}_r^\star$, then $\mathcal{L}^T(\mathcal{H}) \geq r$, $\mathcal{L}^I(\mathcal{H}) \leq 1 - r$, and $\mathcal{L}^F(\mathcal{H}) \leq 1 - r$.*

Proof. It is proved that most of the equivalent conditions ultimately prove the all the equivalence.

(1)⇒(2): Let $\{\mathcal{R}_j\}_{j \in J}$ be a single-valued neutrosophic quasi open-cover of $\mathcal{H} \in I^{\mathcal{S}}$ such that for $j \in J$, $T_{\mathcal{H}}(\kappa) + T_{R_j}(\kappa) - 1 \leq T_{\mathcal{C}}(\kappa)$, $I_{\mathcal{H}}(\kappa) + I_{R_j}(\kappa) - 1 > I_{\mathcal{C}}(\kappa)$, and $F_{\mathcal{H}}(\kappa) + F_{R_j}(\kappa) - 1 > F_{\mathcal{C}}(\kappa)$ for every $\kappa \in \mathcal{R}$ and for some $\mathcal{L}^T(\mathcal{C}) \geq r$, $\mathcal{L}^I(\mathcal{C}) \leq 1 - r$, and $\mathcal{L}^F(\mathcal{C}) \leq 1 - r$. Therefore, as $\{\mathcal{R}_j\}_{j \in J}$ is a single-valued neutrosophic quasi open-cover of $\mathcal{R}$, for each $x_{s,t,p} \in \mathcal{H}$, there exists at least one $\mathcal{R}_{j\circ}$ such that $x_{s,t,p} q \mathcal{R}_{j\circ}$ and for every $\kappa \in \mathcal{S}$, $T_{\mathcal{H}}(\kappa) + T_{R_{j\circ}}(\kappa) - 1 \leq T_{\mathcal{C}}(\kappa)$, $I_{\mathcal{H}}(\kappa) + I_{R_{j\circ}}(\kappa) - 1 > I_{\mathcal{C}}(\kappa)$, and $F_{\mathcal{H}}(\kappa) + F_{R_{j\circ}}(\kappa) - 1 > F_{\mathcal{C}}(\kappa)$ for every $\kappa \in \mathcal{S}$ and for some $\mathcal{L}^T(\mathcal{C}) \geq r$, $\mathcal{L}^I(\mathcal{C}) \leq 1 - r$ and $\mathcal{L}^F(\mathcal{C}) \leq 1 - r$. Obviously, $\mathcal{R}_{j\circ} \in Q_{\tau^{TIF}}(x_{s,t,p}, r)$. By (1), we have $\mathcal{L}^T(\mathcal{H}) \geq r$, $\mathcal{L}^I(\mathcal{H}) \leq 1 - r$, and $\mathcal{L}^F(\mathcal{H}) \leq 1 - r$.

(2)⇒(1): Clear from the fact that a collection of $\{\mathcal{R}_j\}_{j\in J}$, which contains at least one $\mathcal{R}_{j\circ} \in Q_{\tau^TIF}(x_{s,t,p}, r)$ of each single-valued neutrosophic point of $\mathcal{H}$, constitutes a single-valued neutrosophic quasi-open cover of $\mathcal{H}$.

(1)⇒(3): Let $\mathcal{H} \cap \mathcal{H}^\star_r = (0, 1, 1)$, for every $\kappa \in \mathcal{S}$, $x_t \in \mathcal{H}$ implies $x_{s,t,p} \notin \mathcal{H}^\star_r$. Then, there exists $\mathcal{D} \in Q_{\tau^{TIF}}(x_{s,t,p}, r)$ and $\mathcal{L}^T(\mathcal{C}) \geq r$, $\mathcal{L}^I(\mathcal{C}) \leq 1 - r$, $\mathcal{L}^F(\mathcal{C}) \leq 1 - r$ such that for every $\kappa \in \mathcal{S}$, $T_{\mathcal{D}}(\kappa) + T_{\mathcal{H}}(\kappa) - 1 \leq T_{\mathcal{C}}(\kappa)$, $I_{\mathcal{D}}(\kappa) + I_{\mathcal{H}}(\kappa) - 1 > I_{\mathcal{C}}(\kappa)$, and $F_{\mathcal{D}}(\kappa) + F_{\mathcal{H}}(\kappa) - 1 > F_{\mathcal{C}}(\kappa)$. Since $\mathcal{D} \in Q_{\tau^{TIF}}(x_{s,t,p}, r)$, By (1), we have $\mathcal{L}^T(\mathcal{H}) \geq r$, $\mathcal{L}^I(\mathcal{H}) \leq 1 - r$, and $\mathcal{L}^F(\mathcal{H}) \leq 1 - r$.

(3)⇒(1): For every $x_{s,t,p} \in \mathcal{H}$, there exists $\mathcal{D} \in Q_{\tau^{TIF}}(x_{s,t,p}, r)$ such that for every $\kappa \in \mathcal{S}$, $T_{\mathcal{D}}(\kappa) + T_{\mathcal{H}}(\kappa) - 1 \leq T_{\mathcal{C}}(\kappa)$, $I_{\mathcal{D}}(\kappa) + I_{\mathcal{H}}(\kappa) - 1 > I_{\mathcal{C}}(\kappa)$, and $F_{\mathcal{D}}(\kappa) + F_{\mathcal{H}}(\kappa) - 1 > F_{\mathcal{C}}(\kappa)$, for some $\mathcal{L}^T(\mathcal{C}) \geq r$, $\mathcal{L}^I(\mathcal{C}) \leq 1 - r$, $\mathcal{L}^F(\mathcal{C}) \leq 1 - r$. This implies $x_{s,t,p} \notin \mathcal{H}^\star_r$. Now, there are two cases: either $\mathcal{H}^\star_r = (0, 1, 1)$ or $\mathcal{H}^\star_r \neq (0, 1, 1)$ but $s > T_{\mathcal{H}^\star_r}(\kappa) \neq 0$, $t \leq I_{\mathcal{H}^\star_r}(\kappa) \neq 1$, and $p \leq F_{\mathcal{H}^\star_r}(\kappa) \neq 1$. Let, if possible, $x_{s,t,p} \in \mathcal{H}$ such that $t > T_{\mathcal{H}^\star_r}(\kappa) \neq 0$, $t \leq I_{\mathcal{H}^\star_r}(\kappa) \neq 1$, and $t \leq F_{\mathcal{H}^\star_r}(\kappa) \neq 1$. Let $s' = T_{\mathcal{H}^\star_r}(\kappa) \neq 0$, $t' = I_{\mathcal{H}^\star_r}(\kappa) \neq 1$, and $p' = F_{\mathcal{H}^\star_r}(\kappa) \neq 1$. Then, $x_{s',t',p'} \in \mathcal{H}^\star_r(\kappa)$. In addition, $x_{s',t',p'} \in \mathcal{H}$. Thus, for every $\mathcal{V} \in Q_{\tau^{TIF}}(x_{s,t,p}, r)$, for every $\mathcal{L}^T(\mathcal{C}) \geq r$, $\mathcal{L}^I(\mathcal{C}) \leq 1 - r$, and $\mathcal{L}^F(\mathcal{C}) \leq 1 - r$, there is at least one $\kappa \in \mathcal{S}$ such that $T_{\mathcal{V}}(\kappa) + T_{\mathcal{H}}(\kappa) - 1 > T_{\mathcal{C}}(\kappa)$, $I_{\mathcal{V}}(\kappa) + I_{\mathcal{H}}(\kappa) - 1 \leq I_{\mathcal{C}}(\kappa)$, and $F_{\mathcal{V}}(\kappa) + F_{\mathcal{H}}(\kappa) - 1 \leq F_{\mathcal{C}}(\kappa)$. Since $x_{s,t,p} \in \mathcal{H}$, this contradicts the assumption for every single-valued neutrosophic point of $\mathcal{H}$. So, $\mathcal{H}^\star_r = (0, 1, 1)$. That means $x_{s,t,p} \in \mathcal{H}$ implies $x_{s,t,p} \notin \mathcal{H}^\star_r$. Now this is true for every $\mathcal{H} \in I^{\mathcal{S}}$. So, for any $\mathcal{H} \in I^{\mathcal{S}}$, $\mathcal{H} \cap \mathcal{H}^\star_r = (0, 1, 1)$. Hence, by (3), we have $\mathcal{L}^T(\mathcal{H}) \geq r$, $\mathcal{L}^I(\mathcal{H}) \leq 1 - r$, $\mathcal{L}^F(\mathcal{H}) \leq 1 - r$, which implies $\tau^{TIF} \sim \mathcal{L}^{TIF}$.

(3)⇒(4): Let $x_{s,t,p} in \widetilde{\mathcal{H}}$. Then, $x_{s,t,p} \in \mathcal{H}$ but $x_{s,t,p} \notin \mathcal{H}^\star_r$. So, there exists a $\mathcal{D} \in Q_{\tau^{TIF}}(x_{s,t,p}, r)$ such that for every $\kappa \in \mathcal{S}$, $T_{\mathcal{D}}(\kappa) + T_{\mathcal{H}}(\kappa) - 1 \leq T_{\mathcal{C}}(\kappa)$, $I_{\mathcal{D}}(\kappa) + I_{\mathcal{H}}(\kappa) - 1 > I_{\mathcal{C}}(\kappa)$, and $F_{\mathcal{D}}(\kappa) + F_{\mathcal{H}}(\kappa) - 1 > F_{\mathcal{C}}(\kappa)$, for some $\mathcal{L}^T(\mathcal{C}) \geq r$, $\mathcal{L}^I(\mathcal{C}) \leq 1 - r$, $\mathcal{L}^F(\mathcal{C}) \leq 1 - r$. Since $\widetilde{\mathcal{H}} \leq \mathcal{H}$, for every $\kappa \in \mathcal{S}$, $T_{\mathcal{D}}(\kappa) + T_{\widetilde{\mathcal{H}}}(\kappa) - 1 \leq T_{\mathcal{C}}(\kappa)$, $I_{\mathcal{D}}(\kappa) + I_{\widetilde{\mathcal{H}}}(\kappa) - 1 > I_{\mathcal{C}}(\kappa)$, and $F_{\mathcal{D}}(\kappa) + F_{\widetilde{\mathcal{H}}}(\kappa) - 1 > F_{\mathcal{C}}(\kappa)$, for some $\mathcal{L}^T(\mathcal{C}) \geq r$, $\mathcal{L}^I(\mathcal{C}) \leq 1 - r$ and $\mathcal{L}^F(\mathcal{C}) \leq 1 - r$. Therefore, $x_{s,t,p} \notin \widetilde{\mathcal{H}}^\star_r$ implies that $\widetilde{\mathcal{H}}^\star_r = (0, 1, 1)$ or $\widetilde{\mathcal{H}}^\star_r \neq (0, 1, 1)$ but $s > T_{\widetilde{\mathcal{H}}^\star_r}$, $t \leq I_{\widetilde{\mathcal{H}}^\star_r}$, and $p \leq F_{\widetilde{\mathcal{H}}^\star_r}$. Let $x_{s',t',p'}$ in $SVNP(\mathcal{S})$ such that $s' \leq T_{\widetilde{\mathcal{H}}^\star_r}(\kappa) < s$, $t \leq I_{\widetilde{\mathcal{A}}^\star_r}(\kappa) < t'$, and $p \leq F_{\widetilde{\mathcal{H}}^\star_r}(\kappa) < p'$, i.e., $x_{s',t',p'} \in \widetilde{\mathcal{H}}^\star_r$. Then, for each $\mathcal{V} \in Q_{\tau^{TIF}}(x_{s',t',p'}, r)$ and for each $\mathcal{L}^T(\mathcal{C}) \geq r$, $\mathcal{L}^I(\mathcal{C}) \leq 1 - r$, $\mathcal{L}^F(\mathcal{C}) \leq 1 - r$, there is at least one $\kappa \in \mathcal{S}$ such that $T_{\mathcal{V}}(\kappa) + T_{\widetilde{\mathcal{H}}}(\kappa) - 1 > T_{\mathcal{C}}(\kappa)$, $I_{\mathcal{V}}(\kappa) + I_{\widetilde{\mathcal{H}}}(\kappa) - 1 \leq I_{\mathcal{C}}(\kappa)$, and $F_{\mathcal{V}}(\kappa) + F_{\widetilde{\mathcal{H}}}(\kappa) - 1 \leq F_{\mathcal{C}}(\kappa)$. Since $\widetilde{\mathcal{H}} \leq \mathcal{H}$, then for each $\mathcal{V} \in Q_{\tau^{TIF}}(x_{s',t',p'}, r)$ and for each $\mathcal{L}^T(\mathcal{C}) \geq r$, $\mathcal{L}^I(\mathcal{C}) \leq 1 - r$, $\mathcal{L}^F(\mathcal{C}) \leq 1 - r$, there is at least one $\kappa \in \mathcal{S}$ such that $T_{\mathcal{V}}(\kappa) + T_{\mathcal{H}}(\kappa) - 1 > T_{\mathcal{C}}(\kappa)$, $I_{\mathcal{V}}(\kappa) + I_{\mathcal{H}}(\kappa) - 1 \leq I_{\mathcal{C}}(\kappa)$, and $F_{\mathcal{V}}(\kappa) + F_{\mathcal{H}}(\kappa) - 1 \leq F_{\mathcal{C}}(\kappa)$. This implies $x_{s',t',p'} \in \mathcal{H}^\star_r$. But as $s' < s$, $t' < t$, and $p' < p$, $x_{s,t,p} \in \widetilde{\mathcal{H}}$ implies $x_{s',t',p'} \in \widetilde{\mathcal{H}}$, and therefore, $x_{s',t',p'} \notin \mathcal{H}^\star_r$. This is a contradiction. Hence, $\mathcal{H}^\star_r = (0, 1, 1)$, so that $x_{s,t,p} \in \widetilde{\mathcal{H}}$ implies $x_{s,t,p} \notin \widetilde{\mathcal{H}}^\star_r$ with $\widetilde{\mathcal{H}}^\star_r = (0, 1, 1)$. Thus, $\widetilde{\mathcal{H}} \cap \widetilde{\mathcal{H}}^*_r = \underline{0}$, for every $\mathcal{H} \in I^X$. Hence, by (3), $\mathcal{L}^T(\widetilde{\mathcal{H}}) \geq r$, $\mathcal{L}^I(\widetilde{\mathcal{H}}) \leq 1 - r$, and $\mathcal{L}^F(\widetilde{\mathcal{H}}) \leq 1 - r$.

(4)⇒(5): Straightforward.

(4)⇒(6): Let $\mathcal{H} \in I^{\mathcal{S}}$ and $\mathcal{H} \leq R \neq (0, 1, 1)$ with $\mathcal{R} \leq \mathcal{R}^\star_r$. Then, for any $\mathcal{H} \in I^{\mathcal{S}}$, $\mathcal{H} = \widetilde{\mathcal{H}} \cup (\mathcal{H} \cap \mathcal{H}^\star_r)$. Therefore, $\mathcal{H}^\star_r = (\widetilde{\mathcal{A}} \cup (\mathcal{H} \cap \mathcal{H}^\star_r))^\star_r = \widetilde{\mathcal{H}}^\star_r \cup (\mathcal{H} \cap \mathcal{H}^\star_r)^\star_r$. by Theorem 3 (5).

Now, by (4), we have $\mathcal{L}^T(\widetilde{\mathcal{H}}) \geq r$, $\mathcal{L}^I(\widetilde{\mathcal{H}}) \leq 1 - r$, and $\mathcal{L}^F(\widetilde{\mathcal{H}}) \leq 1 - r$, then $\widetilde{\mathcal{H}}^\star_r = (0, 1, 1)$. Hence, $(\mathcal{H} \cap \mathcal{H}^\star_r)^\star_r = \mathcal{H}^\star_r$ but $\mathcal{H} \cap \mathcal{H}^\star_r \leq \mathcal{H}^\star_r$, then $\mathcal{H} \cap \mathcal{A}^\star_r \leq (\mathcal{H} \cap \mathcal{H}^\star_r)^\star_r$. This contradicts the hypothesis about every single-valued neutrosophic set $\mathcal{H} \in I^{\mathcal{S}}$, if $(0, 1, 1) \neq \mathcal{R} \leq \mathcal{H}$ with $\mathcal{R} \leq \mathcal{R}^\star_r$. Therefore, $\mathcal{H} \cap \mathcal{H}^\star_r = (0, 1, 1)$, so that $\mathcal{H} = \widetilde{\mathcal{H}}$ by (4), we have $\mathcal{L}^T(\mathcal{H}) \geq r$, $\mathcal{L}^I(\mathcal{H}) \leq 1 - r$, and $\mathcal{L}^F(\mathcal{H}) \leq 1 - r$.

(6)⇒(4): Since, for every $\mathcal{H} \in I^{\mathcal{S}}$, $\mathcal{H} \cap \mathcal{H}^\star_r = (0, 1, 1)$. Therefore, by (6), as $\mathcal{H}$ contains no non-empty single-valued neutrosophic subset $\mathcal{R}$ with $\mathcal{R} \leq \mathcal{R}^\star_r$, $\mathcal{L}^T(\mathcal{H}) \geq r$, $\mathcal{L}^I(\mathcal{H}) \leq 1 - r$, and $\mathcal{L}^F(\mathcal{H}) \leq 1 - r$.

(5)⇒(1): For every $\mathcal{H} \in I^{\mathcal{S}}$, $x_{s,t,p} \in \mathcal{H}$, there exists an $\mathcal{D} \in Q_{\tau^{TIF}}(x_{s,t,p}, r)$ such that $T_{\mathcal{D}}(\kappa) + T_{\mathcal{H}}(\kappa) - 1 \leq T_{\mathcal{C}}(\kappa)$, $I_{\mathcal{D}}(\kappa) + I_{\mathcal{H}}(\kappa) - 1 > I_{\mathcal{C}}(\kappa)$, and $F_{\mathcal{D}}(\kappa) + F_{\mathcal{H}}(\kappa) - 1 > F_{\mathcal{C}}(\kappa)$ holds for every $\kappa \in \mathcal{S}$ and for some $\mathcal{L}^T(\mathcal{H}) \geq r$, $\mathcal{L}^I(\mathcal{H}) \leq 1 - r$, and $\mathcal{L}^F(\mathcal{H}) \leq 1 - r$. This implies $x_{s,t,p} \notin \mathcal{H}^\star_r$. Let $\mathcal{R} =$

$\mathcal{H} \cup \mathcal{H}^*_r$. Then, $\mathcal{R}^*_r = (\mathcal{H} \cup \mathcal{H}^*_r)^\star_r = \mathcal{H}^\star_r \cup (\mathcal{H}^\star_r)^\star_r = \mathcal{H}^\star_r$ by Theorem 3(4). So, $C^\star_{\tau^{TIF}}(\mathcal{R}, r) = \mathcal{R} \cup \mathcal{R}^\star_r = \mathcal{R}$. That means $\tau^{T\star}(\underline{1} - \mathcal{R}) \geq r$, $\tau^{I\star}(\underline{1} - \mathcal{R}) \leq 1 - r$, and $\tau^{F\star}(\underline{1} - \mathcal{R}) \leq 1 - r$. Therefore, by (5), we have $\mathcal{L}^T(\mathcal{R}) \geq r$, $\mathcal{L}^I(\mathcal{R}) \leq 1 - r$, and $\mathcal{L}^F(\mathcal{R}) \leq 1 - r$.

Once again, for any $x_{s,t,p}$ in $SVNP(X)$, $x_{s,t,p} \notin \widetilde{\mathcal{R}}^\star_r$ implies $x_{s,t,p} \in \mathcal{R}$ but $x_{s,t,p} \notin \mathcal{R}^\star_r = \mathcal{H}^\star_r$ So, as $\mathcal{B} = \mathcal{H} \vee \mathcal{H}^\star_r$, $x_{s,t,p} \in \mathcal{H}$. Now, by hypothesis about $\mathcal{H}$. Then, for any $x_{s,t,p} \in \mathcal{H}^\star_r$. So, $\widetilde{\mathcal{R}} = \mathcal{H}$. Hence, $\mathcal{L}^T(\mathcal{H}) \geq r$, $\mathcal{L}^I(\mathcal{H}) \leq 1 - r$, and $\mathcal{L}^F(\mathcal{H}) \leq 1 - r$, i.e., $\tau^{TIF} \sim \mathcal{L}^{TIF}$. □

Theorem 10. *Let $(\mathcal{S}, \tau^{TIF})$ be an* **SVNTS** *with single-valued neutrosophic ideal $\mathcal{L}^{TIF}$ on $\mathcal{S}$. Then, the following are equivalent and implied by $\tau \sim \mathcal{L}$.*

(1) For every $\mathcal{H} \in I^{\mathcal{S}}$, $\mathcal{H} \wedge \mathcal{H}^\star_r = (0,1,1)$ implies $\mathcal{H}^*_r = (0,1,1)$;
(2) For any $\mathcal{H} \in I^{\mathcal{S}}$, $\widetilde{\mathcal{H}}^\star_r = (0,1,1)$;
(3) For every $\mathcal{H} \in I^{\mathcal{S}}$, $\mathcal{H} \wedge \mathcal{H}^\star_r = \mathcal{H}^\star_r$.

Proof. Clear from Theorem 9. □

The following corollary is an important consequence of Theorem 10.

Corollary 3. *Let $\tau^{TIF} \sim \mathcal{L}^{TIF}$. Then, $\beta(\tau^{TIF}, \mathcal{L}^{TIF})$ is a base for $\tau^{TIF\star}$ and also $\beta(\tau^{TIF}, \mathcal{L}^{TIF}) = \tau^{TIF\star}$.*

Definition 17. *Let $\mathcal{H}, \mathcal{R} \in$* **SVNS** *on $\mathcal{S}$. If $\mathcal{H}$ is a single-valued neutrosophic relation on a set $\mathcal{S}$, then $\mathcal{H}$ is called a single-valued neutrosophic relation on $\mathcal{B}$ if, for every $\kappa, \kappa_1 \in \mathcal{S}$,*
$T_{\mathcal{R}}(\kappa, \kappa_1) \leq \min(T_{\mathcal{H}}(\kappa), T_{\mathcal{H}}(\kappa_1))$,
$I_{\mathcal{R}}(\kappa, \kappa_1) \geq \max(I_{\mathcal{H}}(\kappa), I_{\mathcal{H}}(\kappa_1))$, *and*
$F_{\mathcal{R}}(\kappa, \kappa_1) \geq \max(F_{\mathcal{H}}(\kappa), F_{\mathcal{H}}(\kappa_1))$.

A single-valued neutrosophic relation $\mathcal{H}$ on $\mathcal{S}$ is called symmetric if, for every $\kappa, \kappa_1 \in \mathcal{S}$,

$$T_{\mathcal{H}}(\kappa, \kappa_1) = T_{\mathcal{H}}(\kappa_1, \kappa), \quad I_{\mathcal{H}}(\kappa, \kappa_1) = I_{\mathcal{H}}(\kappa_1, \kappa), \quad F_{\mathcal{H}}(\kappa, \kappa_1) = F_{\mathcal{H}}(\kappa_1, \kappa); \text{ and}$$

$$T_{\mathcal{R}}(\kappa, \kappa_1) = T_{\mathcal{R}}(\kappa_1, \kappa) \quad I_{\mathcal{R}}(\kappa, \kappa_1) = I_{\mathcal{R}}(\kappa_1, \kappa), \quad F_{\mathcal{R}}(\kappa, \kappa_1) = F_{\mathcal{R}}(\kappa_1, \kappa).$$

In the purpose of symmetry, we can replace Definition 3 with Definition 17.

5. Conclusions

In this paper, we defined a single-valued neutrosophic closure space and single-valued neutrosophic ideal to study some characteristics of neutrosophic sets and obtained some of their basic properties. Next, the single-valued neutrosophic ideal open local function, single-valued neutrosophic ideal closure, single-valued neutrosophic ideal interior, single-valued neutrosophic ideal open compatible, and ordinary single-valued neutrosophic base were introduced and studied.

Discussion for further works:
We can apply the following ideas to the notion of single-valued ideal topological spaces.

(a) The collection of bounded single-valued sets [53];
(b) The concept of fuzzy bornology [54];
(c) The notion of boundedness in topological spaces. [54].

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