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R-sets, Comprehensive Fuzzy Sets Risk Modeling for Risk-based Information Fusion and Decision-making

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***Abstract*—Fuzzy sets were initially proposed to address ambiguities and uncertainties. However, in certain cases, the fuzzy sets show some degree of uncertainty and risk, when the available data are either obtained from unreliable sources or related to future events. To solve this problem, the R-numbers methodology has been recently developed as a powerful approach to model the risk of fuzzy sets and numbers due to risk factors. In R-numbers, only the variability of *x* values has been taken into account in risk modeling of the fuzzy sets, but not their membership function. Moreover, one source of risk factors related to fuzzy sets and numbers merely has been considered. Therefore, this study presents a new concept called R-sets, in which different risk cases of a membership function due to both future events and unreliable information sources are investigated, and the governing mathematical relations are presented. Subsequently, to overcome previous limitations of R-numbers, the R-sets are applied to develop a decision-making method, and it is tested by using a case study.**

***Index Terms*— R-numbers; decision-making; Risk of information; Future events risk; RS-TOPSIS**

I. Introduction

IN most engineering and decision-making problems, there is no way to avoid uncertainty. In some of these problems, the data to be analyzed are associated with some percentage of risk and error [1]. Risk and reliability are indicative of the accuracy and credibility of certain available information. Although risk gives, a general framework compared to reliability regarding information, both concepts are required for the data accuracy to avoid inappropriate outcomes, but the information reliability is usually determined by the prior performance and knowledge, whereas risks consider not only the earlier performance but also possible unseen situations [2]. Generally, the risk could be due to different reasons such as unreliable information sources or predicted/unpredicted effective factors related to the future events, where the knowledge of present cannot be extrapolated in a definite and reliable way [3]. These sources of error are collectively known as risk factors. The risk factors affect the outcome of an evaluation, causing it to deviate from the predicted values [4]. There is an extended range of risk factors affecting engineering and decision-making problems, which mostly depend on the nature of the problem in question. The different sources of such factors for reliability prediction of a physical system are illustrated in Fig. 1 [5].

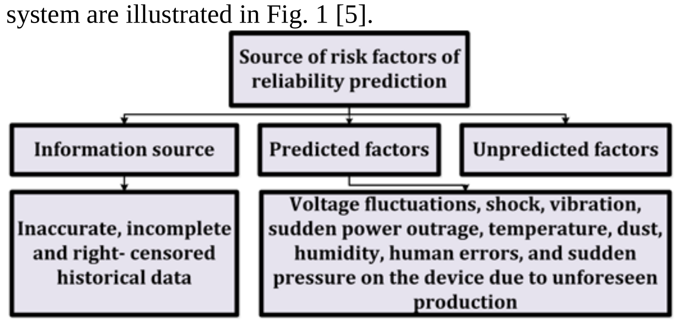

Fig. 1. The main risk factors of reliability prediction

Fuzzy sets [6] are highly effective in addressing problems, in which the available data are not accurate but rather associated with some degree of ambiguity and vagueness. Therefore, in order to implement fuzzy sets, it is crucial to determine the reliability of the available data. So far, the fuzzy sets have been extended to different innovative approaches such as type 2 fuzzy sets [7], interval-valued fuzzy sets (IVFSs) [8], Z-numbers [9], intuitionistic fuzzy sets (IFSs) [10], neutrosophic sets (NSs) [11], hesitant fuzzy sets (HFSs) [12], picture fuzzy sets (PFSs) [13], cloud model [14], plithogenic sets (PLSs) [15], R-numbers [3], etc. Table 1 describes different extensions of fuzzy sets and the proposed R-sets briefly.

The R-numbers proposed by Seiti et al. [3], cope with different risk scenarios of fuzzy sets and numbers. In such an approach, two pessimistic-optimistic type 2 triangular fuzzy numbers (T2 TFNs) were developed as two types of R-numbers for beneficial and non-beneficial values by considering fuzzy negative and positive risks, fuzzy negative and positive acceptable risks, and fuzzy risk perception. In this research, the negative and positive risks were defined as effects of risks, which makes the fuzzy data worse or better. Moreover, the acceptable negative and positive risks coefficients were used to specify the acceptable percentages of negative and positive risks. Finally, risk perception was

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incorporated into R-numbers formulas as the error percentage of risk estimations in the cases where human experts provided the reported risk values.

TABLE I
The comparison of different fuzzy models with the proposed R-sets

Different fuzzy models	How to address uncertainty
HFS	Multiple membership functions
IVFSs	Upper and lower membership functions
IFSs	Membership and non-membership functions
NSs	Truth-membership function, indeterminacy-membership function, falsity-membership function
PLSs	Appurtenance degree of fuzzy, intuitionistic or neutrosophic sets
PFSs	Positive membership function, neutral membership function, negative membership function
Z-numbers	Fuzzy reliability
R-numbers	Type 2 triangular fuzzy numbers through fuzzy negative and positive risks, fuzzy negative and positive acceptable risks and fuzzy risk perception
Proposed R-sets	Type 2 triangular fuzzy membership function through pessimistic and optimistic risks of the information source and influential factors and pessimistic and optimistic acceptable risks

However, the R-numbers methodology presents a few noteworthy research gaps:

1) As it mentioned in [3], the effects of risk factors on a specific fuzzy sets can be investigated in a three different cases, i.e., 1) the effects of risk on the membership function, 2) the risk influences on the X values, 3) the simultaneous impact of risk on the membership function and the X values. Although, the R-numbers can model the risk of membership function or the X values, the proposed methodology, and its relations have been extended mainly for the effects of the risk on the variability of X values rather than the membership function. Therefore, it seems convenient to develop a new mathematical model to correctly study and model the risk scenarios of the membership function.

2) The risk factors come from three different sources, i.e., predicted factors, unpredicted factors, and the unreliability of the information source (Fig. 1). Each risk source has different effects on fuzzy information and should be modeled differently. The R-numbers consider only one type of risk and have no clue for modeling the source and future events risks simultaneously.

Due to the fact that real world decision making problems needs to fill previous gaps, this paper presents a new uncertainty modeling approach so-called R-sets, which can be used to explain and justify the errors and risks associated with membership functions through modeling the risks associated with information sources and future events. Moreover, the R-sets methodology will be employed to extend the Technique for Order of Preference by Similarity to Ideal Solution (TOPSIS) to RS-TOPSIS and applied to failure modes and effect analysis problem. Before introducing R-sets, various risk configurations linked to the information source and future event risks are investigated in different cases such as optimistic and pessimistic modes and acceptable risks and mathematical relations are analytically described with their relationships expressed.

The rest of this manuscript is organized as follows. Section 2 briefly discusses the type 1 and type 2 fuzzy sets and R-numbers. Section 3 comprehensively discusses the different risk modeling of fuzzy sets. Section 4 introduces the concept of R-sets and RS-TOPSIS methodology. In Section 5, an example of FMEA analysis using the proposed R-sets and RS-TOPSIS is are provided to elucidate the applicability of R-sets in real problems. Finally, Section 6 makes conclusions and suggests future research directions.

II. Preliminaries

In this section, we briefly discuss type 1 (T1) and type 2 fuzzy sets (T2 FSs) and R-numbers as they are essential to have an insight of these to grasp the concept of R-sets.

A. Fuzzy sets

Fuzzy set A is a class with the membership grades in a universe of discourse X, where X is a non-empty universe, either finite or infinite and it is defined by a membership function $\mu_A(x)$ which belongs to each element where $x \in X$ and $\mu_A(x) \in [0,1]$. If $\mu_A(x) = 1$ describes that x is the full member of A, while $\mu_A(x) = 0$ means non-membership, unlike the standard sets, other membership degrees are admitted [16]. In fact, FSs are generalizations of the classical sets represented by their membership functions.

Definition 1. Fuzzy set A in the universe of discourse X can be defined as follows [17]:

$$A = \{(x, \mu_A(x)) | \, x \in X\}, \tag{1}$$

where

$$\mu_A(x): X \to [0, 1]. \tag{2}$$

Other definitions have been proposed in the literature for describing A such as $A = \sum_{=1}^{n} \mu_A(x_i)/x_i$ for finite universe of X and $A = \int_X \mu_A(x_i)/x_i$ for an infinite X [17].

Definition 2. The operations of fuzzy sets between $A = \{x, \mu_A(x)\}$ and $B = \{x, \mu_b(x)\}$ could be enlisted as follows [18].

Union:

$$A \cup B \Leftrightarrow \mu_{A\cup B} = \mu_A \vee \mu_B. \tag{3}$$

Intersection:

$$A \cap B \Leftrightarrow \mu_{A\cap B} = \mu_A \wedge \mu_B. \tag{4}$$

Complement:

$$\bar{A} \Leftrightarrow \mu_{\bar{A}} = 1 - \mu_A. \tag{5}$$

Algebraic Product:

$$A.B \Leftrightarrow \mu_{A.B} = \mu_A \mu_B. \tag{6}$$

Algebraic Sum:

$$A + B \Leftrightarrow \mu_{A+B} = \mu_A + \mu_B - \mu_A\mu_B = 1 - (1 - \mu_A)(1 - \mu_B). \tag{7}$$

Fuzzy numbers can be defined as a generalization of classical real numbers and can be considered as a nested stack of intervals [16]. Fuzzy numbers are able to model epistemic uncertainty and are essential in the fuzzy analysis, and have wide applications in fuzzy sets, and fuzzy logic [16]. One of the most widely used fuzzy numbers is triangular fuzzy numbers, which is briefly explained as follows.

Definition 3. A triangular fuzzy number (TFN) $\tilde{A}$ on $\mathbb{R}$ with a membership function $\mu_{\tilde{A}}(x): \mathbb{R} \rightarrow [0,1]$ can be described using (8) [3].

$$\mu_{\tilde{A}}(x) = \begin{cases} (x - a_1)/(a_2 - a_1), & a_1 \leq x < a_2 \\ (a_3 - x)/(a_3 - a_2), & a_2 \leq x \leq a_3 \\ 0, & otherwise \end{cases} . \quad (8)$$

Here, a_1 and a_3 are the lower and upper bounds of the fuzzy number $\tilde{A}$, respectively, and a_2 is the modal value. The TFN can also be defined as $\tilde{A} = (a_1, a_2, a_3)$.

B. *Type-2 fuzzy sets*

Type 2 fuzzy sets are an extension to T1 FSs, which their membership grades are themselves T1 FSs. The main advantage of T2 FSs is their capacity for dealing with the second-order uncertainties due to multiple sources [19]. The T2 FSs usually appear in two forms, i.e., the interval and generalized forms [20]. In interval T2 FS, each secondary membership grade is equal to one, but in the generalized T2 FS (GT2 FS) the secondary membership degree of an element is defined by a value in [0,1] which, handle much more uncertainty that is subtle and sophisticated. Generally, the R-sets methodology is constructed based on GT2 FS [19], [20]. The main definitions and characteristics of GT2 FS are as follows.

Definition 4. A T2 FS A˜ can be defined by $\mu_{\tilde{A}} : X \times U \rightarrow U$, in which x is the primary variable of $\tilde{A}$ and X is the universe of discourse of x. The 3-D membership function of A˜ $(\mu_{\tilde{A}}(x,u))$ can be described through (9) [21].

$$\tilde{A} = \{((x, u), \mu_{\tilde{A}}(x,u)) | \, x \in X, u \in [0,1]\} \quad (9)$$

where u is the secondary variable and $u \in [0,1]$. Besides $\mu_{\tilde{A}}(x,u)$ is recognized as secondary membership grade of x and the primary membership of is defined as $J_x = \{(x,u) | u \in [0,1], \mu_{\tilde{A}}(x,u) > 0\}$

Definition 5. The footprint of uncertainty (FOU) of $\tilde{A}$ is 2-D support of $\mu_{\tilde{A}}(x,u)$ and is defined as follows [21]:

$$FOU(\tilde{A}) = \{(x,u) | x \in X and\ u \in [\underline{\mu}_{\tilde{A}}(x), \bar{\mu}_{\tilde{A}}(x)]\}, \quad (10)$$

where

$$\begin{cases} \underline{\mu}_{\tilde{A}}(x) = \inf \{u | u \in [0,1], \mu_{\tilde{A}}(x,u) > 0\} \\ \bar{\mu}_{\tilde{A}}(x) = \sup \{u | u \in [0,1], \mu_{\tilde{A}}(x,u) > 0\} \end{cases} . \quad (11)$$

Definition 6. The type 2 triangular fuzzy number $\tilde{\tilde{A}}$ can be defined using a triangular fuzzy number with fuzzy elements as follows [3]:

$$\tilde{\tilde{A}} = (\tilde{A}_l, \tilde{A}_m, \tilde{A}_u) \quad (12)$$

where $\tilde{A}_l$ and $\tilde{A}_u$ are the fuzzy lower and upper bounds of $\tilde{\tilde{A}}$ and $\tilde{A}_m$ is the fuzzy modal value.

Definition 7. Let us consider two T2 TFNs $\tilde{\tilde{A}} = \begin{pmatrix} (a_{11}, a_{12}, a_{13}), \\ (a_{21}, a_{22}, a_{23}), \\ (a_{31}, a_{32}, a_{33}) \end{pmatrix}$ and $\tilde{\tilde{B}} = \begin{pmatrix} (b_{11}, b_{12}, b_{13}), \\ (b_{21}, b_{22}, b_{23}), \\ (b_{31}, b_{32}, b_{33}) \end{pmatrix}$, then we have:

$$\tilde{\tilde{A}} \oplus \tilde{\tilde{B}} = \begin{pmatrix} (a_{11} + b_{11}, a_{12} + b_{12}, a_{13} + b_{13}), \\ (a_{21} + b_{21}, a_{22} + b_{22}, a_{23} + b_{23}), \\ (a_{31} + b_{31}, a_{32} + b_{32}, a_{33} + b_{33}) \end{pmatrix}, \quad (13)$$

$$\tilde{\tilde{A}} \ominus \tilde{\tilde{B}} = \begin{pmatrix} (a_{11} - b_{13}, a_{12} - b_{12}, a_{13} - b_{11}), \\ (a_{21} - b_{23}, a_{22} - b_{22}, a_{23} - b_{21}), \\ (a_{31} - b_{33}, a_{32} - b_{32}, a_{33} - b_{31}) \end{pmatrix}, \quad (14)$$

$$\tilde{\tilde{A}} \otimes \tilde{\tilde{B}} = \begin{pmatrix} (a_{11}b_{11}, a_{12}b_{12}, a_{13}b_{13}), \\ (a_{21}b_{21}, a_{22}b_{22}, a_{23} \cdot b_{23}), \\ (a_{31}b_{31}, a_{32}b_{32}, a_{33}b_{33}) \end{pmatrix}, \quad (15)$$

$$\tilde{\tilde{A}} \oslash \tilde{\tilde{B}} = \begin{pmatrix} \left(\frac{a_{11}}{b_{33}}, \frac{a_{12}}{b_{32}}, \frac{a_{13}}{b_{31}}\right), \\ \left(\frac{a_{21}}{b_{23}}, \frac{a_{22}}{b_{22}}, \frac{a_{23}}{b_{21}}\right), \\ \left(\frac{a_{31}}{b_{13}}, \frac{a_{32}}{b_{12}}, \frac{a_{33}}{b_{11}}\right) \end{pmatrix}. \quad (16)$$

C. *R-numbers*

R numbers method has been developed to explain or justify the errors and risks of future events associated with fuzzy numbers, although its application is not limited only to fuzzy numbers. The following are the main definitions of R-numbers methodology.

Definition 8. The R-numbers for arbitrary fuzzy number $\tilde{B}$ with lower limit $l_{\tilde{B}}$ and upper limit $u_{\tilde{B}}$ in beneficial and non-beneficial modes are denoted by $R_b(\tilde{B})$ and $R_c(\tilde{B})$ and can be described as follows [3]:

$$R_b(\tilde{B}) = (R_{1b}(\tilde{A}), R_{2b}(\tilde{B}), R_{3b}(\tilde{B})), \text{ where} \quad (17)$$

$$\begin{cases} R_{1b}(\tilde{B}) = max\left(\tilde{B} \otimes \left(1 \ominus min\left(\frac{\tilde{r}^-}{1 \ominus \widetilde{RP}^-}, \tau\right) \otimes (1 \ominus \widetilde{AR}^-)\right), l_{\tilde{B}}\right) \\ R_{2b}(\tilde{B}) = \tilde{B} \\ R_{3b}(\tilde{B}) = min\left(\tilde{B} \otimes \left(1 \oplus \frac{\tilde{r}^+}{1 \ominus \widetilde{RP}^+} \otimes (1 \ominus \widetilde{AR}^+\right), u_{\tilde{B}}\right) \\ 0 < \tilde{r}^- < 1, \ \tilde{r}^+ > 0 \end{cases}, \quad (18)$$

and

$$R_c(\tilde{B}) = (R_{1c}(\tilde{B}), R_{2c}(\tilde{B}), R_{3c}(\tilde{B})), \text{ where} \quad (19)$$

$$\begin{cases} R_{1c}(\tilde{B}) = max\left(\tilde{B} \otimes \left(1 \ominus min\left(\frac{\tilde{r}^+}{1 \ominus \widetilde{RP}^+}, \tau\right) \otimes (1 \ominus \widetilde{AR}^+)\right), l_{\tilde{B}}\right) \\ R_{2c}(\tilde{B}) = \tilde{B} \\ R_{3c}(\tilde{B}) = min\left(\tilde{B} \otimes \left(1 \oplus \frac{\tilde{r}^-}{1 \ominus \widetilde{RP}^-} \otimes (1 \ominus \widetilde{AR}^-)\right), u_{\tilde{B}}\right) \\ \tilde{r}^- > 1, \ 1 < \tilde{r}^+ < 0 \end{cases}, \quad (20)$$

in which τ is a number infinitely close to one and $\tilde{r}^-$ and $\tilde{r}^+$ are fuzzy negative and positive risks and are defined as the fuzzy error and risk, which leads the fuzzy numbers becomes worse or better in the future. The fuzzy amount of positive and negative risks that can be tolerated by decision-makers are denoted by $\widehat{AR}^-$ and $\widehat{AR}^+$, respectively, and $\widehat{RP}^-$ and $\widehat{RP}^+$ are experts' fuzzy risk perceptions related to negative and positive risks. In these relations, $\widetilde{AR^-}$ and $\widetilde{AR^+}$ values are always between zero and one but the possible range of $\widetilde{RP}^-$ and $\widetilde{RP}^+$ is defined as follows [3]:

$$\begin{cases} Optimist\ Experts & 0<\widetilde{RP}^-,\ \widetilde{RP}<1 \\ Neutral\ Experts & 0 \\ Pessimist\ Experts & -\infty<\widetilde{RP}^-,\ \widetilde{RP}<0 \end{cases}. \quad (21)$$

Definition 9. The main algebraic operations between two R-numbers $R(\tilde{A})$ and $R(\tilde{B})$ can be performed using (22) [3].

$$\begin{cases} R(\tilde{A})\oplus R(\tilde{B})=\left(R_1(\tilde{A})\oplus R_1(\tilde{B}),\ R_2(\tilde{A})\oplus R_2(\tilde{B}),\ R_3(\tilde{A})\oplus R_3(B)\right) \\ R(\tilde{A})\ominus R(\tilde{B})=\left(R_1(\tilde{A})\ominus R_3(\tilde{B}), R_2(\tilde{A})\ominus R_2(\tilde{B}), R_3(\tilde{A})\ominus R_1(\tilde{B})\right) \\ R(A)\otimes R(\tilde{B})=\left(R_1(\tilde{A})\otimes R_1(\tilde{B}), R_2(A)\otimes R_2(\tilde{B}), R_3(\tilde{A})\otimes R_3(\tilde{B})\right) \\ R(\tilde{A})\oslash R(\tilde{B})=\left(R_1(\tilde{A})\oslash R_3(\tilde{B}), R_2(\tilde{A})\oslash R_2(\tilde{B}), R_3(\tilde{A})\oslash R_1(\tilde{B})\right) \end{cases}, \quad (22)$$

III. RISK MODELING OF FUZZY SETS

This section explores different schemes of risk modeling of fuzzy set membership functions considering two different cases, i.e., risk modeling of a membership function due to the risk of the information source and risk modeling of membership function due to future influential factors. Additionally, different configurations, arising, e.g., due to pessimistic and optimistic risks, in the two mentioned cases are also exemplified. Risk modeling due to the information source is discussed in Section 3.1, while due to the future influential factors is outlined in Section 3.2, risk modeling by taking simultaneously both information source risk and future events risk is investigated in Section 3.3, and other schemes of risk modeling of fuzzy sets are elaborated in Section 3.4.

A. Risk modeling of fuzzy sets due to risk and error of information source

Depending on the nature of the problem, a set of different cases may be considered in risk modeling of the specific membership function due to the existence of risk and error in the information source, which is comprehensively investigated in this section. For this purpose, three cases such as pessimistic, optimistic, and pessimistic-optimistic are considered in coming subsections. Depending on the nature of the information source and effective variables on the information source, either of the cases may occur. In a pessimistic case, the existence of risk and error of information source results in the obtained membership degree would be higher than the actual one, while in an optimistic case, the existence of risk leads to a decrease in the reported value of the source in contrast to the actual one, and the pessimistic-optimistic mode represents the general form of the membership risk modeling, thus existing as a combination of the two previous cases.

1) The pessimistic interval of the membership function thorough considering the pessimistic risk of the information source

Assume that the presence of risk affecting the information source results in a higher output value than the actual value. In this case, the obtained value is higher than the exact value; thus, the data need to be discarded.

Theorem 1. Let the maximum risk of an information source S in pessimistic mode be denoted as $r^{PS}{}_{max}$. Thus, for a fuzzy set $\tilde{A} = \{(x,\mu_{\tilde{A}}(x)) | \forall x\epsilon X\}$ that is obtained by an information source S, the discounted membership function, i.e., $\mu^{PS}{}_{\tilde{A}}(x)$, considering the risk of $r^{PS}{}_{max}$, is depicted as:

$$\begin{cases} \mu^{PS}{}_{\tilde{A}}(x) = \frac{\mu_{\tilde{A}}(x)}{1+r^{PS}{}_{max}}, \\ r^{PS}{}_{max} > 0 \end{cases} \quad (23)$$

where $\mu^{PS}{}_{\tilde{A}}(x)$ is the discounted (pessimistic) $\mu_{\tilde{A}}(x)$.

Proof. The risk or error of specific data can be defined as the deviation of the exact parameter from the obtained value, and it can be obtained as follows [22]:

$$r = \frac{|Approximate\ value - Exact\ value|}{Exact\ value}. \quad (24)$$

In the pessimistic case, as the obtained membership degree is greater than the exact membership degree, we obtain from (24),

$\mu_{\tilde{A}}(x) > \mu^{PS}{}_{\tilde{A}}(x) \rightarrow$

$$r^{PS}{}_{max} = \frac{\mu_{\tilde{A}}(x)-\mu^{PS}{}_{\tilde{A}}}{\mu^{PS}{}_{\tilde{A}}} \rightarrow \mu^{PS}{}_{\tilde{A}} = \frac{\mu_{\tilde{A}}(x)}{(1+r^{PS}{}_{max})}. \quad (25)$$

This proves the proposed relation.■

Furthermore, as the exact parameter is not known and the derived value of $\mu^{PS}{}_{\tilde{A}}(x)$ is determined from the maximum possible risk, which does not always occur in reality, we can describe an interval of pessimistic values for membership function that indicates the possible range of exact membership function in the presence of $r^{PS}{}_{max}$.

Definition 10. If the pessimistic interval of $\mu_{\tilde{A}}(x)$ is indicated with $I\mu^{PS}{}_{\tilde{A}}(x)$, the pessimistic fuzzy set $\tilde{A}^{PS}$ is obtained in the same way as explained below:

$$\tilde{A}^{PS} = \left\{\left(x,\ I\mu^{PS}{}_{\tilde{A}}(x)\right) \middle| \forall x\epsilon\ X\right\}, \quad (26)$$

where

$$I\mu^{PS}{}_{\tilde{A}}(x) = [\mu^{PS}{}_{\tilde{A}}, \mu_{\tilde{A}}(x)]. \quad (27)$$

Example 1. Let X be a set and $\tilde{A}$ a fuzzy set as:

$X = \{1, 2, 3, 4\}$,

$\tilde{A} = \{(1,0.2), (2,0.4), (3,0.5)\}$.

By assuming a risk of $r^{PS}{}_{max} = 0.2$, the pessimistic fuzzy sets $\tilde{A}^{PS}$ can be derived using (26) as follows:

$\tilde{A}^{PS} = \{(1, [0.167,0.2]), (2, [0.33,0.4]), (3, [0.416,0.5])\}$.

2) The optimistic interval of membership function based on the optimistic risk of an information source

Let us assume that depending on the historical data and other influential factors, the obtained membership function from the information source, S, is lower than the actual values. In this case, if the aim is to obtain correct output values, then the reported values must be enhanced similarly as per the error.

Theorem 2. Similar to the pessimistic mode, assume the maximum optimistic risk value of the membership degree by source S to be $r^{OS}{}_{max}$, thus taking into account the optimistic mode for the membership function, the inflated mode can be described as follows:

$$\begin{cases}\mu^{OS}{}_{\tilde{A}}(x) = min\left(\frac{\mu_{\tilde{A}}(x)}{1-r^{OS}{}_{max}}, 1\right), \\ 0 \le r^{OS}{}_{max} < 1\end{cases} \quad (28)$$

where $\mu^{OS}{}_{\tilde{A}}(x)$ is the inflated (optimistic) $\mu_{\tilde{A}}(x)$.

Proof. Since the obtained membership degree is smaller than the exact value from (24), we obtain,

$$\mu_{\tilde{A}}(x) < \mu^{OS}{}_{\tilde{A}}(x) \rightarrow r^{OS}{}_{max} = \frac{\mu^{OS}{}_{\tilde{A}} - \mu_{\tilde{A}}(x)}{\mu^{OS}{}_{\tilde{A}}} \quad (29)$$

$$\rightarrow \mu^{OS}{}_{\tilde{A}} = \frac{\mu_{\tilde{A}}(x)}{(1-r^{OS}{}_{max})}.$$

Moreover, $\mu^{OS}{}_{\tilde{A}}$, could not be higher than one, so $min\left(\frac{\mu_{\tilde{A}}(x)}{1-r^{OS}{}_{max}}, 1\right)$ should be considered to satisfy this property. ■

Definition 11. According to Definition 10, the optimistic interval of $\mu_{\tilde{A}}(x)$ and the optimistic set of $\tilde{A}$, as denoted by $I\mu^{OS}{}_{\tilde{A}}$ and $\tilde{A}^{OS}$ are derived as follows:

$$\tilde{A}^{OS} = \left\{\left(x,\ I\mu^{OS}{}_{\tilde{A}}(x)\right) \middle| \forall x \epsilon X\right\}, \quad (30)$$

where

$$I\mu^{OS}{}_{\tilde{A}}(x) = \left[\mu_{\tilde{A}}(x),\ \mu^{OS}{}_{\tilde{A}}\ \right]. \quad (31)$$

Example 2. If we take like from Example 1 with a risk value of $r^{OS}{}_{max} = 0.2$, $\tilde{A}^{OS}$ using (30), the final optimistic value can be obtained as follows.

$$\tilde{A}^{OS} = \{(1, [0.2,0.25]), (2, [0.4,0.5]), (3, [0.5,0.825])\}.$$

3) The pessimistic-optimistic interval of membership function based on the risks of an information source

Considering the various two cases, assume that a problem where fuzzy information obtained by n different sources has a different percentage of error for each of them. For such a problem, in total, a set of 2^n different cases can be considered, wherein each case may end up with a different result. Moreover, for many problems, in order to achieve robust results and decrease the number of cases to be examined, we need to consider a pessimistic-optimistic range of output values in the presence of risk. According to Definitions 10 and 11, for the pessimistic-optimistic scenario, we can consider a range of pessimistic to optimistic values having the same degree of possibility, and it further contains the reported value.

Definition 12. The pessimistic-optimistic interval of the fuzzy sets $\tilde{A}$ and the membership function $\mu_{\tilde{A}}$ can be defined by considering the risks of $r^{PS}{}_{max}$ and $r^{OS}{}_{max}$ as follows:

$$\tilde{A}^{(P-O)S} = \left\{\left(x,\ I\mu^{(P-O)S}{}_{\tilde{A}}(x)\right) \middle| \forall x \epsilon X\right\}, \quad (32)$$

$$I\mu^{(P-O)S}{}_{\tilde{A}}(x) = \left[\mu^{PS}{}_{\tilde{A}}(x), \mu^{OS}{}_{\tilde{A}}(x)\right], \quad (33)$$

$$r^{PS}{}_{max} \ge 0, \quad (34)$$

$$0 \le r^{OS}{}_{max} < 1, \quad (35)$$

where $I\mu^{(P-O)S}{}_{\tilde{A}}(x)$ is the pessimistic-optimistic interval for $\mu_{\tilde{A}}(x)$, the pessimistic-optimistic fuzzy set is $\tilde{A}^{(P-O)S}$, and $\mu^{PS}{}_{\tilde{A}}$ and $\mu^{OS}{}_{\tilde{A}}$ are obtained using (23) and (28), respectively.

Definition 13. The membership function $I\mu^{(P-O)S}{}_{\tilde{A}}(x)$ and the related fuzzy sets $\tilde{A}^{(P-O)S}$ can be described by using TFNs while taking the risks of $r^{PS}{}_{max}$ and $r^{OS}{}_{max}$ into account, so we have:

$$\tilde{A}^{(P-O)ST} = \left\{\left(x, \tilde{\mu}^{(P-O)ST}{}_{\tilde{A}}(x)\right) \middle| \forall x \epsilon X\right\}, \quad (36)$$

$$\tilde{\mu}^{TP-O}{}_{\tilde{A}}(x) = \left(\mu^{PS}{}_{\tilde{A}}(x), \mu_{\tilde{A}}(x), \mu^{OS}{}_{\tilde{A}}(x)\right), \quad (37)$$

$$r^{PS}{}_{max} \ge 0, \quad (38)$$

$$0 \le r^{OS}{}_{max} < 1, \quad (39)$$

where $\tilde{\mu}^{TP-O}{}_{\tilde{A}}(x)$ and $\tilde{A}^{(P-O)ST}$ are the triangular fuzzy pessimistic-optimistic membership function and fuzzy set, respectively.

Proof. In an interval-valued output set, all values of an interval have the same possibility of occurrence, though it is not a logical assumption. Indeed, the pessimistic and optimistic reported risks of the information source are determined based on maximum possible risks, which do not always occur in reality and can be observed only in the worst case. Therefore, we can say that the interval close to the reported value has a higher possibility. Thus, we can say that the initial evaluation value will have a higher degree of possibility, and when risk is increased, the values determined only by the same risk will eventually have a lower possibility, and at the end, the optimistic and pessimistic values will have the lowest possibilities to occur. In another approach, if these terms need to be expressed in degrees of possibility, one can utilize a TFN.■

One may consider a special case, where $r^{PS}{}_{max}$ and $r^{OS}{}_{max}$ values are identical; in this case, the effective factors may equally improve or worsen the results. Moreover, the optimistic and pessimistic approaches represent special cases with zero $r^{PS}{}_{max}$ or $r^{OS}{}_{max}$ value, respectively.

Example 3. Let us suppose again Example 1 with a risk value of $r^{OS}{}_{max} = 0.2$ and $r^{PS}{}_{max}$, $\tilde{A}^{(P-O)ST}$ can be obtained as follows.

$$\tilde{A}^{(P-O)ST} = \left\{\begin{matrix}(1, (0.167,0.2,0.25)), \\ (2, (0.33,0.4,0.5)), \\ (3, (0.417,0.5,0.825))\end{matrix}\right\}.$$

B. Risk modeling of fuzzy sets due to the presence of future influential factors

This section investigates different schemes of risk modeling of fuzzy sets. In this case, the risk may be caused due to the influential factors, which may be prevalent in the future. For this purpose, similar to Section 3.1, we consider three cases for risk modeling, i.e., pessimistic, optimistic, and pessimistic-optimistic.

1) Pessimistic set of membership function by considering future events risks

If we assume that the values of the membership function are only worsened by a future risk or only pessimistic values are considered by the decision-maker, the risk values ranging from 0 to 1 can be taken into account, and one may end up with a pessimistic mode of fuzzy sets.

Theorem 3. Suppose $\tilde{A}$ represents a fuzzy set of X, and $r^{PF}{}_{max}$ represents the maximum possible pessimistic risk of possible future events, in this case, the pessimistic membership function $\mu_{\tilde{A}}{}^{PF}(x)$ can then be derived by considering $r^{PF}{}_{max}$ as follows:

$$\begin{cases}\mu_{\tilde{A}}{}^{PF}(x) = (1 - r^{PF}{}_{max}).\mu_{\tilde{A}}(x) \\ 0 \le r^{PF}{}_{max} < 1\end{cases}. \quad (40)$$

Proof. The risk and error (r) values of E_2 with respect to E_1 can be obtained using (41) [3]:

$$r = \frac{|E_2 - E_1|}{E_1}, \quad (41)$$

In the pessimistic case, the aim is to obtain $\mu_{\tilde{A}}{}^{PF}(x)$ caused by $r^{PF}{}_{max}$, since $\mu_{\tilde{A}}{}^{PF}(x)$ will be lower than $\mu_{\tilde{A}}(x)$, by using (41), we have:

$$\mu_{\tilde{A}}{}^{PF}(x) < \mu_{\tilde{A}}(x) \rightarrow r^{PF}{}_{max} = \frac{\mu_{\tilde{A}}(x) - \mu_{\tilde{A}}{}^{PF}(x)}{\mu_{\tilde{A}}(x)} \quad (42)$$

$$\rightarrow \mu_{\tilde{A}}{}^{PF}(x) = (1 - r^{PF}{}_{max}).\mu_{\tilde{A}}(x). \blacksquare$$

Definition 14. As described in Section 3.1.1, the pessimistic interval for membership function and the pessimistic fuzzy sets arising from $r^{PF}{}_{max}$ which are denoted by $I\mu_{\tilde{A}}{}^{PF}(x)$ and $\tilde{A}^{PF}$ can be obtained as follows:

$$\tilde{A}^{PF} = \left\{\left(x, I\mu_{\tilde{A}}{}^{PF}(x)\right) \middle| \forall x \epsilon X\right\}, \quad (43)$$

where

$$I\mu_{\tilde{A}}{}^{PF}(x) = \left[\mu_{\tilde{A}}{}^{PF}(x),\ \mu_{\tilde{A}}(x)\right]. \quad (44)$$

Example 4. Let the risks associated with the membership function of fuzzy set $\tilde{A}$ in Example 1 be $r^{PF}{}_{max} = 0.2$; the pessimistic set $\tilde{A}^{PF}$ can be obtained as follows.

$\tilde{A} = \{(1,0.2), (2,0.4), (3,0.5)\}$,

$\tilde{A}^{PF} = \{(1, [0.16,0.2]), (2, [0.32,0.4]), (3, [0.4,0.5])\}$.

2) The optimistic interval of membership function considering future events risks

Now we take another case where future effective factors can only improve the values, which are usually dependent on the degree to which the decision-maker is optimistic. In this mode, the associated risk with the future factors that can result in better solutions is called optimistic risk.

Theorem 4. If the maximum optimistic risk is shown by $r^{OF}{}_{max}$, the optimistic value of the membership function, i.e., $\mu_{\tilde{A}}{}^{OF}(x)$ considering $r^{OF}{}_{max}$ could be obtained as follows:

$$\mu_{\tilde{A}}{}^{OF}(x) = min\left((1 + r^{OF}{}_{max}).\mu_{\tilde{A}}(x), 1\right), \quad (45)$$

$$r^{OF}{}_{max} \ge 0. \quad (46)$$

Proof. Since $\mu_{\tilde{A}}{}^{OF}(x)$ is greater than $\mu_{\tilde{A}}(x)$, from (40), we have,

$$\mu_{\tilde{A}}{}^{OF}(x) < \mu_{\tilde{A}}(x) \rightarrow r^{OF}{}_{max} = \frac{\mu_{\tilde{A}}{}^{OF}(x) - \mu_{\tilde{A}}(x)}{\mu_{\tilde{A}}(x)} \quad (47)$$

$$\rightarrow \mu_{\tilde{A}}{}^{OF}(x) = (1 + r^{OF}{}_{max}).\mu_{\tilde{A}}(x).$$

Since max $\mu_{\tilde{A}}{}^{OF}(x)$ could not be greater than one; thus, we should consider $min\left((1 + r^{OF}{}_{max}).\mu_{\tilde{A}}(x), 1\right)$, so the proposed relation is proved. ■

Definition 15. The optimistic fuzzy set and optimistic interval of $\mu_{\tilde{A}}(x)$ are derived using (48) and (49), respectively.

$$\tilde{A}^{OF} = \left\{\left(x, I\mu_{\tilde{A}}{}^{OF}(x)\right) \middle| \forall x \epsilon X\right\}, \quad (48)$$

where

$$I\mu_{\tilde{A}}{}^{OF}(x) = \left[\mu_{\tilde{A}}(x), \mu_{\tilde{A}}{}^{OF}(x)\right], \quad (49)$$

in which, $I\mu_{\tilde{A}}{}^{OF}(x)$ is an optimistic interval of the membership function, and $\tilde{A}^{OF}$ is optimistic fuzzy set.

Example 5. Referring to the reference set X and fuzzy set $\tilde{A}$ in Example 1 and $r^{OF}{}_{max} = 0.2$, the optimistic set $\tilde{A}^{OF}$ is obtained as follows:

$\tilde{A} = \{(1,0.2), (2,0.4), (3,0.5)\}$,

$\tilde{A}^{OF} = \{(1, [0.2,0.24]), (2, [0.4,0.48]), (3, [0.5,0.6])\}$.

3) Pessimistic-optimistic approach

In this condition, risk factors are expected to influence the membership function both pessimistically ($r^{PF}{}_{max}$) and optimistically ($r^{OF}{}_{max}$). As in the presence of risk, the existing and evaluated membership degree showed the highest degree of possibility, and the remaining points within the interval exhibit the minimum degree of possibility. Therefore, the pessimistic-optimistic range can be described using Definition 16.

Definition 16. The pessimistic-optimistic fuzzy sets and pessimistic-optimistic interval for the membership function using a TFN are obtained using (50) and (51), respectively.

$$\tilde{A}^{(P-O)FT} = \left\{\left(x, \tilde{\mu}^{(P-O)FT}{}_{\tilde{A}}(x)\right) \middle| \forall x \epsilon X\right\}, \quad (50)$$

$$\tilde{\mu}^{(P-O)FT}{}_{\tilde{A}}(x) = \left(\mu^{PF}{}_{\tilde{A}}(x), \mu_{\tilde{A}}(x), \mu^{OF}{}_{\tilde{A}}(x)\right), \quad (51)$$

$$0 \le r^{PF}{}_{max} < 1, \quad (52)$$

$$r^{OF}{}_{max} \ge 0, \quad (53)$$

where $\tilde{\mu}^{(P-O)FT}{}_{\tilde{A}}$ and $\tilde{A}^{(P-O)FT}$ are the triangular fuzzy-valued pessimistic-optimistic set of $\tilde{A}$ and membership function $\mu_{\tilde{A}}(x)$, respectively.

Example 6. Regarding Example 1 and by considering $r^{PF}{}_{max} = r^{OF}{}_{max} = 0.2$, $\tilde{A}^{(P-O)FT}$ can be written as follows.

$\tilde{A} = \{(1,0.2), (2,0.4), (3,0.5)\}$,

$$\tilde{A}^{(P-O)FT} = \left\{\begin{matrix}(1, (0.14,0.2,0.24)), \\ (2, (0.28,0.4,0.48)), \\ (3, (0.35,0.5,0.6))\end{matrix}\right\}.$$

Similar to Section 3.1, in a special case, $r^{PF}{}_{max}$ and $r^{OF}{}_{max}$ values could be equal. In this scenario, it is equally possible that the effective factors may improve or worsen the results. The next section presents a discussion on risk modeling of the fuzzy sets considering both simultaneously, the information risk and future risks.

C. Modeling fuzzy pessimistic-optimistic interval considering both information and future event risk

As outlined in the previous discourse, two kinds of risks can be considered for a specific membership function. The proposed relations between these two cases in a pessimistic-optimistic mode are discussed in Sections 3.1.3 and 3.2.3.

Definition 17. Let a problem be with the maximum pessimistic and optimistic risks of the source and the future events taken as ${r^{PS}}_{max}$, ${r^{OS}}_{max}$ ${r^{PF}}_{max}$ and ${r^{OF}}_{max}$, respectively, now both risks can be modeled simultaneously on the membership function. If we define the model for a pessimistic-optimistic mode, T2 TFN with TFN elements for membership function and fuzzy sets can be obtained using (54) and (55), which are denoted by $\tilde{\tilde{A}}^{(P-O)T}$ and ${\tilde{\mu}^{(P-O)T}}_{\tilde{A}}(x)$, respectively.

$$\tilde{\tilde{A}}^{(P-O)T} = \left\{\left(x, {\tilde{\mu}^{(P-O)T}}_{\tilde{A}^k}(x)\right) \middle| \forall x \epsilon X\right\}, \tag{54}$$

$${\tilde{\mu}^{(P-O)T}}_{\tilde{A}}(x) = \left(\begin{matrix} \left(\begin{matrix} (1-{r^{PF}}_{max}).\left(\frac{\mu_{\tilde{A}}(x)}{(1+{r^{PS}}_{max})}\right), \\ (1-{r^{PF}}_{max}).\mu_{\tilde{A}}(x), \\ min\left((1-{r^{PF}}_{max}).\left(\frac{\mu_{\tilde{A}}(x)}{(1-{r^{OS}}_{max})}\right),1\right) \end{matrix}\right), \\ \left(\begin{matrix} \left(\frac{\mu_{\tilde{A}}(x)}{(1+{r^{PS}}_{max})}\right), \\ \mu_{\tilde{A}}(x), \\ min\left(\left(\frac{\mu_{\tilde{A}}(x)}{(1-{r^{OS}}_{max})}\right),1\right) \end{matrix}\right), \\ \left(\begin{matrix} min\left((1+{r^{OF}}_{max}).\left(\frac{\mu_{\tilde{A}}(x)}{(1+{r^{PS}}_{max})}\right),1\right), \\ min\left((1+{r^{OF}}_{max}).\mu_{\tilde{A}}(x),1\right), \\ min\left((1+{r^{OF}}_{max}).\left(\frac{\mu_{\tilde{A}}(x)}{(1-{r^{OS}}_{max})}\right),1\right) \end{matrix}\right) \end{matrix}\right), \tag{55}$$

where

$${r^{PS}}_{max}, {r^{OF}}_{max} \geq 0, \tag{56}$$

$$0 \leq {r^{PF}}_{max}, {r^{OS}}_{max} < 1. \tag{57}$$

Proof. Let the obtained information from source S be $\mu_{\tilde{A}}(x)$, since the obtained information from source S is risky, we can obtain ${\tilde{\mu}^{(P-O)ST}}_{\tilde{A}}(x)$ by considering ${r^{PS}}_{max}$ and ${r^{OS}}_{max}$ as follows:

$${\tilde{\mu}^{(P-O)ST}}_{\tilde{A}}(x) = \left(\begin{matrix} \left(\frac{\mu_{\tilde{A}}(x)}{(1+{r^{PS}}_{max})}\right), \\ \mu_{\tilde{A}}(x), \\ min\left(\left(\frac{\mu_{\tilde{A}}(x)}{(1-{r^{OS}}_{max})}\right),1\right) \end{matrix}\right). \tag{58}$$

Now ${\tilde{\mu}^{(P-O)ST}}_{\tilde{A}}(x)$ itself faces to pessimistic and optimistic risks of future events, i.e., ${r^{PF}}_{max}$ and ${r^{OF}}_{max}$, so by using (51) and inserting (58) instead of ${\tilde{\mu}^{(P-O)ST}}_{\tilde{A}}(x)$, (55) is proved. ∎

Example 7. The T2 TFN-described fuzzy sets ($\tilde{\tilde{A}}^{(P-O)T}$) of Example 1 considering ${r^{PS}}_{max} = {r^{OS}}_{max} = {r^{PF}}_{max} = {r^{OF}}_{max} = 0.2$ is as follows:

$\tilde{A} = \{(1,0.2),(2,0.4),(3,0.5)\}$,

${r^{PS}}_{max} = {r^{OS}}_{max} = {r^{PF}}_{max} = {r^{OF}}_{max} = 0.2$,

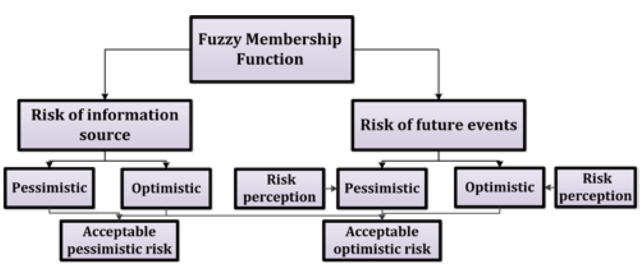

Fig. 2. All possible configurations of taking risks of a membership function

$$\tilde{\tilde{A}}^{(P-O)T} = \left\{\begin{matrix} (1,(0.133,0.16,0.2),(0.166,0.2,0.25),(0.2,0.24,0.3)), \\ (2,(0.267,0.32,0.4),(0.33,0.4,0.55),(0.4,0.48,0.6)), \\ (3,(0.33,0.4,0.5),(0.417,0.5,0.625),(0.5,0.6,0.75)) \end{matrix}\right\}$$

D. *Other configurations of risk modeling of fuzzy sets*

When it comes to the risk of a membership function, there might be other possible cases due to the nature of risk itself. In this section, we consider two additional concepts similar to R-numbers. One of them is the acceptable risk, which shows how much risk is tolerable by the decision-makers for each evaluation, and another one is the risk perception of experts in evaluating the future-events risks which are used to modify the obtained risks by the experts. The comprehensive discussion about these issues can be found in [3]. Let us denote the acceptable risk by AR and the pessimistic and optimistic degrees of risk-taking by AR^P and AR^O, respectively. Now, if we show the pessimistic and optimistic risk perceptions of the experts for pessimistic and optimistic future-events risks by RP^{PF} and RP^{OF}, according to (18) and (20) we can rewrite (55) as follows.

$${\tilde{\tilde{\mu}}^{P-O}}_{\tilde{A}^k}(x) = \left(\begin{matrix} \left(\begin{matrix} \left(1-min\left(\frac{{r^{PF}}_{max}}{1-RP^{PF}},\tau\right)(1-AR^P)\right).\left(\frac{\mu_{\tilde{A}}(x)}{(1+{r^{PS}}_{max}(1-AR^P))}\right), \\ \left(1-min\left(\frac{{r^{PF}}_{max}}{1-RP^{PF}},\tau\right)(1-AR^P)\right).\mu_{\tilde{A}}(x), \\ min\left(\left(1-min\left(\frac{{r^{PF}}_{max}}{1-RP^{PF}},\tau\right)(1-AR^P)\right).\left(\frac{\mu_{\tilde{A}}(x)}{(1-{r^{O^k}}_{max}(1-AR^O))}\right),1\right) \end{matrix}\right), \\ \left(\begin{matrix} \left(\frac{\mu_{\tilde{A}}(x)}{(1+{r^{PS}}_{max}(1-AR^P))}\right), \\ \mu_{\tilde{A}}(x), \\ min\left(\left(\frac{\mu_{\tilde{A}}(x)}{(1-{r^{OS}}_{max}(1-AR^O))}\right),1\right) \end{matrix}\right), \\ \left(\begin{matrix} min\left(\left(1+\frac{{r^{OF}}_{max}}{1-RP^{OF}}(1-AR^O)\right).\left(\frac{\mu_{\tilde{A}}(x)}{(1+{r^{PS}}_{max}(1-AR^P))}\right),1\right), \\ min\left(\left(1+\frac{{r^{OF}}_{max}}{1-RP^{OF}}(1-AR^O)\right).\mu_{\tilde{A}}(x),1\right), \\ min\left(\left(1+\frac{{r^{OF}}_{max}}{1-RP^{OF}}(1-AR^O)\right).\left(\frac{\mu_{\tilde{A}}(x)}{(1-{r^{OS}}_{max}(1-AR^O))}\right),1\right) \end{matrix}\right) \end{matrix}\right), \tag{59}$$

The AR values are expressed as a number between 0 and 1 and τ is a value close to one. It should be noted that risk perceptions are used only when the risk values of future-events are reported by human experts. The different values of risk perceptions for different types of experts are defined by using (21).

All of the possible scenarios of the risk modeling of a membership function are illustrated in Fig. 2.

IV. R-SETS

As discussed previously, in many fuzzy decision-making problems, the risk due to an information source or future risk factors results in deviations of fuzzy information from the actual data. The different scenarios of risk modeling in fuzzy sets are discussed in Section 3. This section sets out a general concept designated as R-sets to be applied on fuzzy sets in a pessimistic-optimistic scheme that considers all risk configurations. This novel concept is expected to be useful in decision-making problems, where the membership degrees are strongly linked with risks and errors. The new algebraic operations for R-sets based on T2 FS are given in Section 4.1. The capabilities of the proposed approach are demonstrated in Section 4.2 by developing a multi-criteria framework based on TOPSIS method and the concept of R-sets.

A. R-sets: definitions and operations

For an arbitrary fuzzy set $\{\langle x, \mu\rangle\}$, the corresponding R-set ($RS(x)$) on X are defined on the basis of a pessimistic-optimistic membership function, $\tilde{\mu}_{RS}$, considering the entire set of risks R, containing risk parameters, i.e., maximum pessimistic and optimistic risks of information sources and future events, pessimistic and optimistic acceptable risks, and risk perceptions, i.e., we have

$$R = \{r^{PS}{}_{max}, r^{OS}{}_{max}, r^{PF}{}_{max}, r^{OF}{}_{max}, AR^P, AR^O, RP^{PF}, RP^{OF}\}, \quad (60)$$

$$RS(x) = \{x, \mu, R\} = \{\langle x, \tilde{\tilde{\mu}}_{RS}\rangle\} \quad \text{for } x \in X, \quad (61)$$

where

$$\tilde{\tilde{\mu}}_{RS} = ((\mu_{RS11}, \mu_{RS12}, \mu_{RS13}), (\mu_{RS21}, \mu_{RS22}, \mu_{RS23}), (\mu_{RS31}, \mu_{RS32}, \mu_{RS33})), \quad (62)$$

and

$$\begin{cases} \mu_{RS11} = \left(1 - min\left(\frac{r^{PF}{}_{max}}{1-RP^{PF}}, \tau\right).(1-AR^P)\right).\left(\frac{\mu}{(1+r^{PS}{}_{max}.(1-AR^P))}\right) \\ \mu_{RS12} = \left(1 - min\left(\frac{r^{PF}{}_{max}}{1-RP^{PF}}, \tau\right).(1-AR^P)\right).\mu \\ \mu_{RS13} = min\left(\left(1 - min\left(\frac{r^{PF}{}_{max}}{1-RP^{PF}}, \tau\right).(1-AR^P)\right).\left(\frac{\mu}{(1-r^{OS}{}_{max}.(1-AR^O))}\right), 1\right) \\ \mu_{RS21} = \left(\frac{\mu}{(1+r^{PS}{}_{max}.(1-AR^P))}\right) \\ \mu_{RS22} = \mu \\ \mu_{RS23} = min\left(\left(\frac{\mu}{(1-r^{OS}{}_{max}.(1-AR^O))}\right), 1\right) \\ \mu_{RS31} = min\left(\left(1 + \frac{r^{OF}{}_{max}}{1-RP^{OF}}.(1-AR^O)\right).\left(\frac{\mu}{(1+r^{PS}{}_{max}.(1-AR^P))}\right), 1\right) \\ \mu_{RS32} = min\left(\left(1 + \frac{r^{OF}{}_{max}}{1-RP^{OF}}.(1-AR^O)\right).\mu, 1\right) \\ \mu_{RS33} = min\left(\left(1 + \frac{r^{OF}{}_{max}}{1-RP^{OF}}.(1-AR^O)\right).\left(\frac{\mu}{(1-r^{OS}{}_{max}.(1-AR^O))}\right), 1\right) \end{cases} \quad (63)$$

also $\mu_{RS1}, \mu_{RS12}, \mu_{RS13}, \mu_{RS21}, \mu_{RS22}, \mu_{RS23}, \mu_{RS31}, \mu_{RS32}, \mu_{RS33} : X \rightarrow [0,1]$.

Now, since $\tilde{\tilde{\mu}}_{RS}$ is a T2 TFN, we can obtain its secondary membership function. Let us suppose $\tilde{\tilde{\mu}}_{RS} = (\tilde{\mu}_{RS1}, \tilde{\mu}_{RS2}, \tilde{\mu}_{RS3})$, where

$$u = \begin{cases} \mu_{\tilde{\mu}_{RS1}}(x) = \begin{cases} (x-\mu_{RS11})/(\mu_{RS12}-\mu_{RS11}), & \mu_{RS11} \le x < \mu_{RS12} \\ (\mu_{RS13}-x)/(\mu_{RS13}-\mu_{RS12}), & \mu_{RS12} \le x \le \mu_{RS13} \\ 0, & otherwise \end{cases} \\ \mu_{\tilde{\mu}_{RS2}}(x) = \begin{cases} (x-\mu_{RS21})/(\mu_{RS22}-\mu_{RS21}), & \mu_{RS21} \le x < \mu_{RS22} \\ (\mu_{RS23}-x)/(\mu_{RS23}-\mu_{RS22}), & \mu_{RS22} \le x \le \mu_{RS23} \\ 0, & otherwise \end{cases} \\ \mu_{\tilde{\mu}_{RS3}}(x) = \begin{cases} (x-\mu_{RS31})/(\mu_{RS32}-\mu_{RS31}), & \mu_{RS31} \le x < \mu_{RS32} \\ (\mu_{RS33}-x)/(\mu_{RS33}-\mu_{RS32}), & \mu_{RS32} \le x \le \mu_{RS33} \\ 0, & otherwise \end{cases} \end{cases} . \quad (64)$$

The secondary membership function of $\tilde{\tilde{\mu}}_{RS}$ which is denoted by $\mu_{\tilde{\tilde{\mu}}_{RS}}(x,u)$ can be defined by using (65) as follows

$$\mu_{\tilde{\tilde{\mu}}_{RS}}(x,u) = \begin{cases} \begin{cases} (x-\mu_{RS13})/(\mu_{RS21}-\mu_{RS13}), & \mu_{RS13} \le x \le \mu_{RS21} \\ 1, & \mu_{RS21} \le x \le \mu_{RS23} \\ (\mu_{RS31}-x)/(\mu_{RS31}-\mu_{RS23}), & \mu_{RS23} \le x \le \mu_{RS31} \\ 0, & otherwise \end{cases} & \forall u, 0 \le u \le 1 \text{ and } \tilde{\mu}_{RS1} \cap \tilde{\mu}_{RS2} \cap \tilde{\mu}_{RS3} = \emptyset \\ \begin{cases} 1, & \mu_{RS21} \le x \le \mu_{RS23} \\ (\mu_{RS31}-x)/(\mu_{RS31}-\mu_{RS23}), & \mu_{RS23} \le x \le \mu_{RS31} \\ 0, & otherwise \end{cases} & \forall u, 0 \le u \le 1 \text{ and } \tilde{\mu}_{RS1} \cap \tilde{\mu}_{RS2} \ne \emptyset, \tilde{\mu}_{RS2} \cap \tilde{\mu}_{RS3} = \emptyset \\ \begin{cases} (x-\mu_{RS13})/(\mu_{RS21}-\mu_{RS13}), & \mu_{RS13} \le x \le \mu_{RS21} \\ 1 & \mu_{RS21} \le x \le \mu_{RS23} \\ 0, & otherwise \end{cases} & \forall u, 0 \le u \le 1 \text{ and } \tilde{\mu}_{RS1} \cap \tilde{\mu}_{RS2} = \emptyset, \tilde{\mu}_{RS2} \cap \tilde{\mu}_{RS3} \ne \emptyset \\ \begin{cases} 1, & \mu_{RS21} \le x \le \mu_{RS23} \\ 0, & otherwise \end{cases} & \forall u, 0 \le u \le 1 \text{ and } \tilde{\mu}_{RS1} \cap \tilde{\mu}_{RS2} \cap \tilde{\mu}_{RS3} \ne \emptyset \end{cases} . \quad (65)$$

Different cases in (65) are the special cases of $\tilde{\mu}_{RS1} \cap \tilde{\mu}_{RS2} \cap \tilde{\mu}_{RS3} = \emptyset$ which is displayed in Fig. (3)

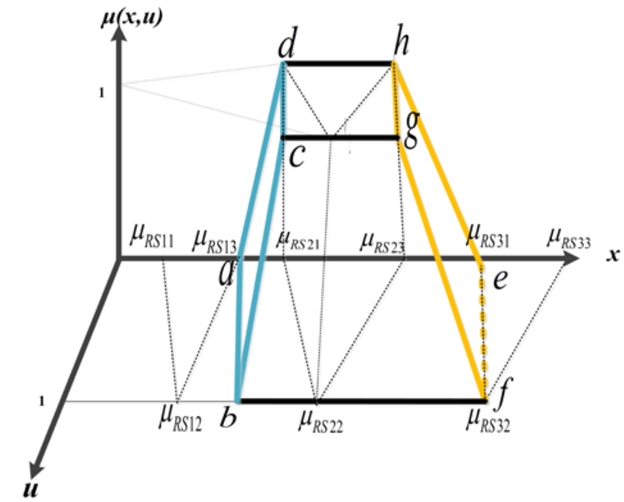

Fig. 3. The secondary membership function of R-sets when $\tilde{\mu}_{RS1} \cap \tilde{\mu}_{RS2} \cap \tilde{\mu}_{RS3} = \emptyset$

The same algebraic operations as those applied to fuzzy sets (e.g., union, intersection, etc.) can also be derived for R-sets according to operation between T2 TFNs ((13)-(16)) and classic fuzzy sets ((3)-(7)). For R-sets A and B, denoted as $RS_A(x)$ and $RS_B(x)$, respectively, the operations can be put forward as follows.

Union:

$$RS_A(x) \cup RS_B(x) = \{\langle x, max(\tilde{\tilde{\mu}}_{RSA}, \tilde{\tilde{\mu}}_{RSB})\rangle\} =$$

$$\left\{\left\langle x, \begin{pmatrix} (max(\mu_{RS11A}, \mu_{RS11B}), max(\mu_{RS12A}, \mu_{RS12B}), max((\mu_{RS13A}, \mu_{RS13B}))), \\ (max(\mu_{RS21A}, \mu_{RS21B}), max(\mu_{RS22A}, \mu_{RS22B}), max((\mu_{RS23A}, \mu_{RS23B}))), \\ (max(\mu_{RS31A}, \mu_{RS31B}), max(\mu_{RS32A}, \mu_{RS32B}), max((\mu_{RS33A}, \mu_{RS33B}))) \end{pmatrix}\right\rangle\right\}. \quad (66)$$

Intersection:

$$RS_A(x) \cap RS_B(x) = \{\langle x, min(\tilde{\tilde{\mu}}_{RSA}, \tilde{\tilde{\mu}}_{RSB})\rangle\} =$$

$$\left\{\left\langle \begin{matrix} x, ((min(\mu_{RS11A}, \mu_{RS11B}), min(\mu_{RS12A}, \mu_{RS12B}), min((\mu_{RS13A}, \mu_{RS13B})))) \\ (min(\mu_{RS21A}, \mu_{RS21B}), min(\mu_{RS22A}, \mu_{RS22B}), min((\mu_{RS23A}, \mu_{RS23B}))) \\ (min(\mu_{RS31A}, \mu_{RS31B}), min(\mu_{RS32A}, \mu_{RS32B}), min((\mu_{RS33A}, \mu_{RS33B}))) \end{matrix}\right\rangle\right\}. \quad (67)$$

Complement:

$$\overline{RS_A(x)} = \{\langle x, \bar{\tilde{\tilde{\mu}}}_{RSA}\rangle\} = \left\{\left\langle x, \begin{pmatrix} (1-\mu_{RS33}, 1-\mu_{RS32}, 1-\mu_{RS31}), (1-\mu_{RS23}, 1-\mu_{RS22}, 1-\mu_{RS21}), \\ (1-\mu_{RS13}, 1-\mu_{RS12}, 1-\mu_{RS11}) \end{pmatrix}\right\rangle\right\}. \quad (68)$$

$\oplus$ −Union:

$$RS_A(x) \oplus RS_B(x) = \{\langle x, \tilde{\tilde{\mu}}_{RSA} + \tilde{\tilde{\mu}}_{RSB} - \tilde{\tilde{\mu}}_{RSA}\tilde{\tilde{\mu}}_{RSB}\rangle\} =$$

$$\left\{\left\langle x, \begin{pmatrix} (\mu_{RS11A} + \mu_{RS11B} - \mu_{RS11A}\mu_{RS11B}, \mu_{RS12A} + \mu_{RS12B} - \mu_{RS12A}\mu_{RS12B}, \mu_{RS13A} + \mu_{RS13B} - \mu_{RS13A}\mu_{RS13B}) \\ (\mu_{RS21A} + \mu_{RS21B} - \mu_{RS21A}\mu_{RS21B}, \mu_{RS22A} + \mu_{RS22B} - \mu_{RS22A}\mu_{RS22B}, \mu_{RS23A} + \mu_{RS23B} - \mu_{RS23A}\mu_{RS23B}) \\ (\mu_{RS31A} + \mu_{RS31B} - \mu_{RS31A}\mu_{RS31B}, \mu_{RS32A} + \mu_{RS32B} - \mu_{RS32A}\mu_{RS32B}, \mu_{RS33A} + \mu_{RS33B} - \mu_{RS33A}\mu_{RS33B}) \end{pmatrix}\right\rangle\right\}. \quad (69)$$

$\otimes$ −Intersection:

$$RS_A(x) \otimes RS_B(x) = \{\langle x, \tilde{\tilde{\mu}}_{RSA}.\tilde{\tilde{\mu}}_{RSB}\rangle\} =$$

$$\left\{\left\langle \begin{pmatrix} x, (\mu_{RS11A}.\mu_{RS11B}, \mu_{RS12A}.\mu_{RS12B}, \mu_{RS13A}.\mu_{RS13B}) \\ (\mu_{RS21A}.\mu_{RS21B}, \mu_{RS22A}.\mu_{RS22B}, \mu_{RS23A}.\mu_{RS23B}) \\ (\mu_{RS31A}.\mu_{RS31B}, \mu_{RS32A}.\mu_{RS32B}, \mu_{RS33A}.\mu_{RS33B}) \end{pmatrix}\right\rangle\right\}. \quad (70)$$

Defuzzification

For the defuzzification process of $\tilde{\tilde{\mu}}_{RSA}$, (71) and (72) can be employed. The defuzzification of $\tilde{\tilde{\mu}}_{RSA}$ can be done first time using (68), and we can obtain [3]:

$$COA_1(\tilde{\tilde{\mu}}_{RSA}) = \frac{[((\mu_{RS31}, \mu_{RS32}, \mu_{RS33}) - (\mu_{RS11}, \mu_{RS12}, \mu_{RS13})) + ((\mu_{RS21}, \mu_{RS22}, \mu_{RS23}) - (\mu_{RS11}, \mu_{RS12}, \mu_{RS13}))]}{3} + (\mu_{RS11}, \mu_{RS12}, \mu_{RS13}) \quad (71)$$

Since the obtained value is somewhat fuzzy, another defuzzification operation is performed to determine a final crisp value, which is as follows [3]:

$$COA_2(\tilde{\mu}_{RSA}) = \frac{1}{9}(\mu_{RS11} + \mu_{RS12} + \mu_{RS13} + \mu_{RS21} + \mu_{RS22} + \mu_{RS23} + \mu_{RS31} + \mu_{RS32} + \mu_{RS33}). \quad (72)$$

All of the operations mentioned above can easily be proven considering the operations applicable to basic fuzzy sets and TFNs.

Example 8. Let $\tilde{A} = \{(1,0.2)\}$, $\tilde{B} = \{(1,0.5)\}$, $R_A = \{0.3, 0.4, 0.2, 0.1, 0.2, 0.3, 0, 0\}$, and $R_B = \{0.2, 0.1, 0.2, 0.1, 0.2, 03, 0, 0\}$, then we have:

$\tilde{A} = \{(1,0.2)\}$,

${r_A}^{PS} = 0.3, {r_A}^{OS} = 0.4, {r_A}^{PF} = 0.2, {r_A}^{OF} = 0.1, {AR_A}^{P} = 0.2, {AR_A}^{O} = 0.3, RP^{PF} = 0$ and $RP^{OF} = 0$,

$$\rightarrow RS_A(x) = \begin{Bmatrix} 1, (0.135, 0.168, 0.233), \\ (0.161, 0.2, 0.278), \\ (0.172, 0.214, 0.297) \end{Bmatrix},$$

$\tilde{B} = \{(1,0.5)\}$,

${r_B}^{PS} = 0.3, {r_B}^{OS} = 0.4, {r_B}^{PF} = 0.2, {r_B}^{OF} = 0.1, {AR_B}^{P} = 0.2,$ ${AR_B}^{O} = 0.3$ $RP^{PF} = 0$ and $RP^{OF} = 0$,

$$\rightarrow RS_B(x) = \begin{Bmatrix} 1, (0.362, 0.42, 0.452), \\ (0.431, 0.5, 0.538), \\ (0.461, 0.535, 0.575) \end{Bmatrix},$$

$$RS_A(x) \cup RS_B(x) = \begin{Bmatrix} 1, (0.362, 0.42, 0.452), \\ (0.431, 0.5, 0.538), \\ (0.461, 0.535, 0.575) \end{Bmatrix},$$

$$RS_A(x) \cap RS_B(x) = \begin{Bmatrix} 1, (0.135, 0.168, 0.233), \\ (0.161, 0.2, 0.278), \\ (0.172, 0.214, 0.297) \end{Bmatrix},$$

$$\overline{RS_A(x)} = \begin{Bmatrix} 1, (0.703, 0.786, 0.827), \\ (0.722, 0.8, 0.839), \\ (0.767, 0.832, 0.864) \end{Bmatrix},$$

$$RS_A(x) \oplus RS_B(x) = \begin{Bmatrix} 1, (0.448, 0.517, 0.579), \\ (0.523, 0.6, 0.666), \\ (0.554, 0.634, 0.701) \end{Bmatrix},$$

$$RS_A(x) \otimes RS_B(x) = \begin{Bmatrix} 1, (0.049, 0.07, 0.105), \\ (0.069, 0.1, 0.149), \\ (0.079, 0.114, 0.171) \end{Bmatrix}.$$

B. RS-TOPSIS method

A novel TOPSIS method, namely RS-TOPSIS, is proposed in this section. This method is based on R-sets methodology and includes followings different steps.

Step 1. Forming the decision and weight matrices

In this step, m alternatives with respect to n criteria are evaluated considering the fuzzy values and the fuzzy decision matrix and, in last, the fuzzy criteria weights are determined, which are denoted as $\widetilde{D}_k$ and $\widetilde{W}$, respectively.

$$\widetilde{D}_k = \left[\tilde{s}^k{}_{ij}\right]_{m\times n} \quad i = 1, \dots, m, j = 1, \dots, n, \quad k = 1, 2, 3 \dots, K, \quad (73)$$

$$\widetilde{W} = \left[\widetilde{w}_j\right]_{1\times n} \; j = 1, \dots, n. \quad (74)$$

Step 2. Determining the risk matrices

In the second step, the pessimistic and optimistic risk matrices of K experts are defined considering the past performance and other factors such as age, education, etc., of the decision-makers. Moreover, the pessimistic and optimistic risk matrices of m alternatives concerning n criteria are determined by considering the opinions of each decision-maker:

$$R^{PS} = \left[r^{PS^k}\right]_{1\times K} \; k = 1, 2, \dots, K, \quad (75)$$

$$R^{OS} = \left[r^{OS^k}\right]_{1\times K} \; k = 1, 2, \dots, K, \quad (76)$$

$$R_k{}^{PF} = \left[r^{PF}{}_{ij}{}^k\right]_{m\times n} \quad i = 1, \dots, m, j = 1, \dots, n, \quad (77)$$

$$R_k{}^{OF} = \left[r^{OF}{}_{ij}{}^k\right]_{m\times n} \quad i = 1, \dots, m, j = 1, \dots, n, \quad (78)$$

where R^{PS} and R^{OS} are the pessimistic and optimistic risk matrices of K experts and the pessimistic and optimistic risks matrices of evaluation of m alternatives respect to n criteria by kth decision-maker are shown with $R_k{}^{PF}$ and $R_k{}^{OF}$, respectively.

Step 3. Determining AR matrix

Now, we focus on the expert opinions and organizational goals and prescribe pessimistic and optimistic AR matrices of n criteria. In this case, AR^P and AR^O indicate the pessimistic and optimistic AR matrices, respectively.

$$AR^P = \left[AR^P{}_j\right]_{1\times n} \quad j = 1, \dots, n, \quad (79)$$

$$AR^O = \left[AR^O{}_j\right]_{1\times n} \quad j = 1, \dots, n. \quad (80)$$

Step 4. Defining the R-sets matrix (${R_S}^k$)

In the fourth step, the R-sets decision matrix of kth decision-maker is constructed. Given $\tilde{s}^k{}_{ij}$ in (78), risk matrices (75)–(78), and acceptable risk matrices (79) and (80), the R-sets matrix, i.e., R_k, is obtained.

$${R_S}^k = \left[{R_S}^k(\tilde{s}^k{}_{ij})\right]_{m\times n} \quad (81)$$

where ${R_S}^k(\tilde{s}_{ij})$ indicates the R-sets value related to $\tilde{s}^k{}_{ij}$.

Step 5. Aggregating the R-sets matrix ${R_S}^k$

Subsequently, ${R_S}^k$ should be aggregated to give ${R_S}^T$ using (82) as follows:

$${R_S}^T = \oplus_{k=1}^{K} {R_S}^k, \quad (82)$$

where ${R_S}^T = \left[{R_S}^T(\tilde{s}_{ij})\right]_{m\times n}$ and $\oplus$ is the algebraic sum of R-sets, which is described by employing (66).

Step 6. Normalizing the decision matrix

In this step, the weighted matrix elements are normalized. If ${R_S}^T(\tilde{s}_{ij}) = \begin{pmatrix} (s_{ij11}, s_{ij12}, s_{ij13}), \\ (s_{ij21}, s_{ij22}, s_{ij23}), \\ (s_{ij31}, s_{ij32}, s_{ij33}) \end{pmatrix}$, the normalized values are defined through (83) [3]:

$$\begin{cases} {R_S}^N(\tilde{s}_{ij}) = \begin{pmatrix} \frac{(s_{ij11}, s_{ij12}, s_{ij13})}{c_i^+}, \\ \frac{(s_{ij21}, s_{ij22}, s_{ij23})}{c_i^+}, \\ \frac{(s_{ij31}, s_{ij32}, s_{ij33})}{c_i^+} \end{pmatrix}; \\ c_i^+ = max_j\, s_{4ij} \text{ for beneficial criterion} \\ {R_S}^N(\tilde{s}_{ij}) = \begin{pmatrix} \frac{a_i^-}{(s_{ij33}, s_{ij32}, s_{ij31})}, \\ \frac{a_i^-}{(s_{ij23}, s_{ij22}, s_{ij21})}, \\ \frac{a_i^-}{(s_{ij13}, s_{ij12}, s_{ij11})} \end{pmatrix}; \\ a_i^- = min_j\, s_{1ij} \text{ for non} - \text{beneficial criterion}, \end{cases} \quad (83)$$

where ${R_S}^N$ denotes the normalized matrix.

Step 7. Weighting the normalized matrix

Now, the weighted aggregated matrix ($R_S{}^W$) is obtained by employing (84) and (85).

$$R_S{}^W = \left[R_S{}^W(\tilde{s}_{ij})\right]_{m\times n}, \tag{84}$$

$$\forall i, j \;\; R^W{}_S(\tilde{s}_{ij}) = \widetilde{w}_j \otimes R_S{}^N(\tilde{s}_{ij}). \tag{85}$$

Step 8. Determining the FPISs and FNISs

In this step, the fuzzy positive ideal solutions (FPISs) set, A^+, and fuzzy negative ideal solution (FNISs) set, A^-, for each criterion are defined. Let's suppose $R_S{}^N(\tilde{s}_{ij}) = \begin{pmatrix} (s^N{}_{ij11}, s^N{}_{ij12}, s^N{}_{ij13}), \\ (s^N{}_{ij21}, s^N{}_{ij22}, s^N{}_{ij23}), \\ (s^N{}_{ij31}, s^N{}_{ij32}, s^N{}_{ij33}) \end{pmatrix}$, FPISs and FNISs sets can be determined according to the following relations:

$$\begin{cases} A^+ = \{\tilde{v}_1^+, \tilde{v}_2^+, \dots \tilde{v}_n^+\} \\ A^- = \{\tilde{v}_1^-, \tilde{v}_2^-, \dots \tilde{v}_n^-\} \end{cases}, \tag{86}$$

where $\tilde{v}_j^+$ and $\tilde{v}_j^-$ are FIPS and FNIS for jth criterion and are obtained according to the minimum and maximum fuzzy values of alternatives as follows [5]:

$$\begin{cases} \tilde{v}_j^+ = max_i \, s^N{}_{ij33} \\ \tilde{v}_j^- = min_i \, s^N{}_{ij11} \end{cases}. \tag{87}$$

Step 9. Distance calculation of each option from FPISs and FNISs

The distances between each of the alternatives from each FPIS and each FNIS are denoted by d_i^+ and d_i^- and obtained as follows:

$$\begin{cases} d_i^+ = \sum_{j=1}^{n} d\left(\tilde{u}_{ij}, \tilde{v}_j^+\right) \quad , \; i = 1,2, \dots, m \\ d_i^- = \sum_{j=1}^{n} d\left(\tilde{u}_{ij}, \tilde{v}_j^-\right) \quad , \; i = 1,2, \dots, m \end{cases}, \tag{88}$$

The distance between two T2 TFNs $\tilde{\tilde{A}} = \begin{pmatrix} (a_{11}, a_{12}, a_{13}), \\ (a_{21}, a_{22}, a_{23}), \\ (a_{31}, a_{32}, a_{33}) \end{pmatrix}$ and $\tilde{\tilde{B}} = \begin{pmatrix} (b_{11}, b_{12}, b_{13}), \\ (b_{21}, b_{22}, b_{23}), \\ (b_{31}, b_{32}, b_{33}) \end{pmatrix}$ can be determined as follows [5]:

$$d\left(\tilde{\tilde{A}}, \tilde{\tilde{B}}\right) = \sqrt{\frac{1}{6}\begin{pmatrix} \left(\sqrt{\left(\frac{1}{6}\right)\left((a_{11}-b_{11})^2 + 4(a_{12}-b_{12})^2 + (a_{13}-b_{13})^2\right)}\right) + \\ \left(4\sqrt{\left(\frac{1}{6}\right)\left((a_{21}-b_{21})^2 + 4(a_{22}-b_{22})^2 + (a_{23}-b_{23})^2\right)}\right) + \\ \left(\sqrt{\left(\frac{1}{6}\right)\left((a_{31}-b_{31})^2 + 4(a_{32}-b_{32})^2 + (a_{33}-b_{33})^2\right)}\right) \end{pmatrix}}. \tag{89}$$

Step 9. Determining the closeness coefficient

Eventually, following the below relationship, we can obtain closeness coefficient (CC_i) for i-th option:

$$CC_i = \frac{d_i^-}{d_i^- + d_i^+} \qquad i = 1,2, \dots, m. \tag{90}$$

Step 10. Ranking the alternatives

In this step, the alternatives are ranked in descending order according to CC_i. The steps of the proposed method are illustrated in Fig. 4.

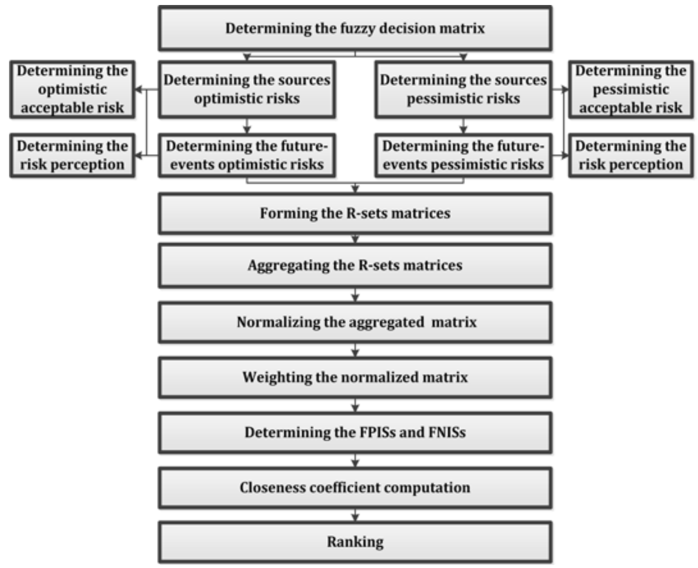

Fig. 4. Flowchart of the proposed R-TOPSIS model

V. Illustrative Example and Comparisons

In this section, a case studies failure mode and effect analysis (FMEA) is presented with the results of different scenarios compared which demonstrates the capabilities of the proposed RS-TOPSIS method in solving decision-making problems under risk and uncertainty.

A. *Case study (FMEA analysis)*

One of the most popular industrial analysis techniques is FMEA, which works by ranking potential failure modes based on the Risk Priority Number (RPN) measure. The RPN is evaluated by multiplying the major risk factors, namely, occurrence (O), severity (S), and detection (D) [2]. Since this analysis has to be performed before actually designing the process or product, the risk factors cannot be predicted accurately [2]. Various FMEA models have been proposed to deal with data uncertainty, including rough set-based FMEA model [23], neutrosophic FMEA model [24], hesitant FMEA model [25], intuitionistic FMEA model [26], FMEA cloud model [27], etc.

In this part, the proposed RS-TOPSIS model was evaluated on an ocean-fishing vessel as a case study. For this purpose, FMEA was applied for structure, propulsion, electrical, and auxiliary systems of the vessel. A possible failure of any of these systems may end up with an injury to the crew and other negative impacts. For each failure mode, the entire spectrum of signals and warnings needs to be investigated. The fuzzy assessments for different failure modes with respect to different risk factors by two experts are given in Table 5. In case of doubt, due to information deficiency, the experts assign two values (i.e., pessimistic and optimistic risks) to each risk factor. Moreover, according to past performance of the experts and their education and expertise, two risk factors as maximum optimistic and pessimistic risks are ascribed to each expert (Table 2). Further, for each evaluation, two pessimistic and optimistic risk values arising during a future event are also specified by each expert (Table 2). The pessimistic and optimistic acceptable risks are also determined on the basis of the organizational goals and expert opinions (Table 3). In this example, the risk perceptions of experts considered being zero.

TABLE II
The fuzzy evaluation matrix and related risks reported by two experts

Expert 1 $r^{PS}_{max}=0.2, r^{OS}_{max}=0.5$									
ID	***O***	r^{PF}_{max}	r^{OF}_{max}	***S***	r^{PF}_{max}	r^{OF}_{max}	***D***	r^{PF}_{max}	r^{OF}_{max}
F_1	0.1	0.1	0.2	0.15	0.4	0.2	0.3	0.3	0.1
F_2	0.3	0.2	0.3	0.35	0.25	0.3	0.5	0.2	0.5
F_3	0.2	0.2	0.5	0.25	0.1	0.3	0.3	0	0
F_4	0.5	0.3	0	0.15	0.3	0.3	0.1	0.2	0.2
F_5	0.6	0.4	0.05	0.4	0.4	0.2	0.35	0.1	0.1
F_6	0.3	0.5	0.1	0.6	0.5	0.5	0.1	0.25	0
Expert 2 $r^{PS}_{max}=0.3, r^{OS}_{max}=0.4$									
F_1	0.3	0.2	0	0.3	0	0.1	0.2	0.2	0.1
F_2	0.3	0.6	0.1	0.35	0.25	0	0.3	0.25	0.1
F_3	0.25	0	0	0.25	0.1	0.1	0.2	0.1	0.25
F_4	0.2	0.05	0.1	0.15	0.1	0.25	0.5	0	0
F_5	0.6	0.25	0	0.3	0.05	0.2	0.6	0.25	0.3
F_6	0.4	0	0.2	0.5	0.3	0.4	0.3	0.2	0

TABLE III
The acceptable risk matrix

AR	***O***	***S***	***D***
Pessimistic acceptable risk (AR^P)	0.2	0.3	0.3
Optimistic acceptable risk (AR^O)	0.1	0.1	0.2

As the first step, R-sets are determined for each evaluation using (62) and (63) (Table 4), and then the evaluations are integrated via (69). Table 5 shows the aggregated results for the failure probabilities of all periods. Subsequently, CC_i values are obtained by calculating the distance from PNISs and FNISs, respectively. The results of the ranking procedure are enlisted in Table 6.

TABLE IV
The obtained R-sets matrices

	O	***S***	***D***
Expert 1			
F_1	((0.079,0.092,0.167), (0.086,0.1,0.182), (0.102,0.118,0.214))	((0.094,0.108,0.196), (0.131,0.15,0.272), (0.155,0.177,0.321))	((0.208,0.237,0.397), (0.263,0.3,0.5), (0.284,0.324,0.54))
F_2	((0.217,0.252,0.458), (0.259,0.3,0.545), (0.328,0.381,0.692))	((0.253,0.288,0.525), (0.307,0.35,0.636), (0.390,0.444,0.808))	((0.377,0.43,0.716), (0.439,0.5,0.833), (0.614,0.7,1))
F_3	((0.145,0.168,0.305), (0.172,0.2,0.363), (0.25,0.29,0.527))	((0.203,0.232,0.422), (0.219,0.25,0.454), (0.278,0.317,0.577))	((0.263,0.3,0.5), (0.263,0.3,0.5), (0.263,0.3,0.5))
F_4	((0.327,0.38,0.690), (0.431,0.5,0.909), (0.547,0.635,1))	((0.103,0.118,0.215), (0.131,0.15,0.272), (0.167,0.190,0.346))	((0.075,0.086,0.143), (0.088,0.1,0.167), (0.101,0.116,0.193))
F_5	((0.351,0.408,0.741), (0.517,0.6,1), (0.540,0.627,1))	((0.252,0.288,0.524), (0.350,0.4,0.727), (0.414,0.472,0.858))	((0.285,0.325,0.542), (0.307,0.35,0.583), (0.331,0.378,0.63))
F_6	((0.155,0.18,0.327), (0.259,0.3,0.545), (0.281,0.327,0.595))	((0.342,0.39,0.709), (0.526,0.6,1), (0.763,0.87,1))	((0.072,0.083,0.138), (0.088,0.1,0.167), (0.088,0. 1,0.167))
Expert 2			

This article has been accepted for publication in a future issue of this journal, but has not been fully edited. Content may change prior to final publication. Citation information: DOI 10.1109/TFUZZ.2019.2955061, IEEE Transactions on Fuzzy Systems

F_1	((0.148,0.184,0.288), (0.161,0.2,0.312), (0.161,0.2,0.312))	((0.248,0.3,0.469), (0.248,0.3,0.469), (0.293,0.354,0.553))	((0.213,0.258,0.379), (0.248,0.3,0.441), (0.267,0.324,0.476))
F_2	((0.242,0.3,0.469), (0.242,0.3,0.469), (0.296,0.368,0.574))	((0.269,0.326,0.509), (0.289,0.35,0.547), (0.445,0.539,0.842))	((0.205,0.248,0.346), (0.248,0.3,0.441), (0.267, 0.324,0.476))
F_3	((0.074,0.092,0.144), (0.080,0.1,0.156), (0.102,0.127,0.198))	((0.207,0.25,0.390), (0.207,0.25,0.390), (0.207,0.25,0.390))	((0.192,0.232,0.342), (0.207,0.25,0.368), (0.248, 0.3,0.441))
F_4	((0.323,0.4,0.625), (0.403,0.5,0.781), (0.440,0.545,0.851))	((0.115,0.140,0.218), (0.124,0.15,0.234), (0.130,0.157,0.245))	((0.059,0.072,0.106), (0.083,0.1,0.147), (0.096, 0.116,0.170))
F_5	((0.406,0.504,0.787), (0.484,0.6,0.938), (0.506,0.627,0.980))	((0.248,0.3,0.469), (0.248,0.3,0.469), (0.304,0.368,0.574))	((0.409,0.495,0.727), (0.496,0.6,0.882), (0.614, 0.744,1.094))
F_6	((0.165,0.204,0.319), (0.242,0.3,0.469), (0.307,0.381,0.595))	((0.355,0.43,0.672), (0.413,0.5,0.781), (0.413,0.5,0.781))	((0.284,0.344,0.505), (0.330,0.4,0.588), (0.330, 0.4,0.588))

TABLE V
The aggregated matrix

	O	*S*	*D*
F_1	((0.216,0.256,0.406), (0.233,0.28,0.437), (0.246, 0.294,0.46))	((0.319,0.376,0.573), (0.347,0.405,0.614), (0.402, 0.468,0.697))	((0.376,0.433,0.625), (0.446,0.51,0.721), (0.476, 0.543,0.759))
F_2	((0.406,0.476,0.712), (0.438,0.51,0.759), (0.527, 0.608,0.869))	((0.454,0.520,0.766), (0.507,0.577,0.835), (0.661, 0.744,0.970))	((0.504,0.571,0.820), (0.578,0.65,0.907), (0.717, 0.798,1))
F_3	((0.208,0.244,0.405), (0.239,0.28,0.463), (0.327, 0.380,0.621))	((0.368,0.424,0.648), (0.380,0.437,0.668), (0.428, 0.488,0.742))	((0.404,0.463,0.671), (0.416,0.475,0.684), (0.446, 0.51,0.721))
F_4	((0.544,0.628,0.884), (0.660,0.75,0.980), (0.746, 0.833,1))	((0.207,0.241,0.386), (0.239,0.278,0.443), (0.275, 0.317,0.506))	((0.130,0.152,0.234), (0.163,0.19,0.289), (0.187, 0.218,0.331))
F_5	((0.615,0.706,0.945), (0.750,0.84,1), (0.772, 0.860,1))	((0.438,0.502,0.747), (0.512,0.58,0.855), (0.592, 0.666,0.940))	((0.578,0.659,0.875), (0.651,0.74,0.950), (0.742, 0.840,0.035))
F_6	((0.294,0.347,0.541), (0.438,0.51,0.759), (0.502, 0.583,0.836))	((0.576,0.652,0.905), (0.722,0.8,1), (0.861, 0.935,1))	((0.336,0.398,0.573), (0.389,0.46,0.657), (0.389, 0.46,0.657))

The effectiveness of the proposed model in various scenarios of the problem was tested by using the following four other cases:

***Case 1*:** Risks of experts are not considered (${r^{PS}}_{max} = {r^{OS}}_{max} = 0$).

Case 2: Risks of evaluations are not considered (${r^{PF}}_{max} = {r^{OF}}_{max} = 0$).

***Case 3*:** AR is considered to be zero ($AR^P = AR^O = 0$).

***Case 4*:** The problem is risk-free (${r^{PS}}_{max} = {r^{OS}}_{max} = {r^{PF}}_{max} = {r^{OF}}_{max} = 0$).

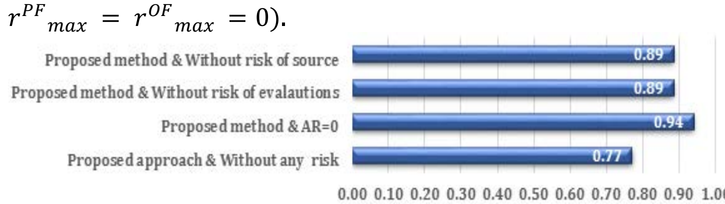

Fig. 5. Spearman’s coefficient

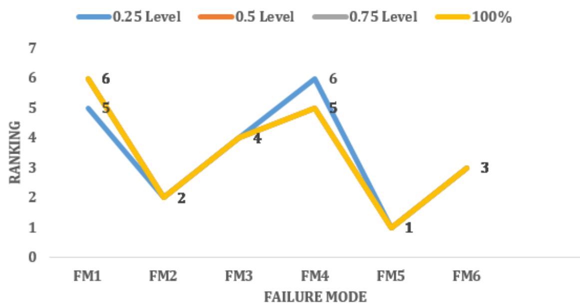

Fig. 6. Results of sensitivity analysis for $r^{OS}{}_{max}$

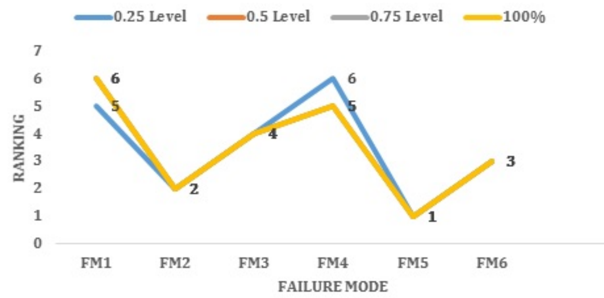

Fig. 7. Results of sensitivity analysis for $r^{PS}{}_{max}$

TABLE VI

Comparison of different scenarios

ID	**Default case**		**Without considering the source risks**		**Without considering the evaluations risks**		$\boldsymbol{AR=0}$		**Without risk**	
	CC_i	Rank	CC_i	Rank	CC_i	Rank	CC_i	Rank	CC_i	Rank
F_1	1.273	6	1.2	6	1.268	5	1.289	5	1.195	5
F_2	1.842	2	1.768	3	1.814	3	1.85	2	1.375	3
F_3	1.291	4	1.207	5	1.276	4	1.312	4	1.292	6
F_4	1.273	5	1.224	4	1.267	6	1.283	6	1.217	4
F_5	2.202	1	2.168	1	2.196	1	2.193	1	2.16	1
F_6	1.829	3	1.775	2	1.825	2	1.831	3	1.177	2

Table 6 present the results of the above four described cases, indicating discrepancies emerging from different risk configurations in the problem. Further, Spearman's correlation coefficients between the results of the proposed models and the four scenarios, namely, without considering the source risks, without considering the evaluations risks, without *AR*, and without risks are illustrated in Fig. 5.

1) Sensitivity analysis of different risk values

In this section, the aim is to investigate the sensitivity of different values of risk parameters ($r^{PS}{}_{max}$, $r^{OS}{}_{max}$, $r^{PF}{}_{max}$, and $r^{OF}{}_{max}$) on the ranking results, hence, for each risk parameter three different sensitivity levels (α), i.e., $\alpha = 0.25$, $\alpha = 0.5$, and $\alpha = 0.75$ are considered while other parameters are assumed to be constant. Now, for each risk level by multiplication of each sensitivity level into the reported risk value, the corresponding closeness coefficient and ranking values are calculated, and the results of all cases are compared with each other. It is noteworthy that $\alpha = 1$ is the same as the default case study. The related outcomes for each level of sensitivity case are shown in Table 7. Moreover, the different ranking results of each risk parameters in each sensitivity level are depicted by Figs. 6-9.

B. Discussion and managerial implications

- Risk factors behave as influential operators and cause deviations between the assessed numbers and the real ones. The proposed *R-sets* can consider the deviated and primary evaluated values of fuzzy membership function due to risks of the information source and the future events simultaneously by taking into account different degrees of possibility through pessimistic-optimistic T2 TFNs. It can be employed that when the membership values posed to great risk or the outputs of different sources are conflicting; the organizations might select the robust results using the *R-sets* methodology.
- In real-world problems, the maximum pessimistic and optimistic risks of an information source can be determined based on static or dynamic measures. In the static approach, the mentioned risks can be determined using past historical records of the studied source. In the dynamic method, the risks can be quantified by investigating the affecting risk factors on the source or using dynamic measures when there are multiple information sources such as distance [28] or entropy measures [29].
- The R-sets methodology has been developed based on T2 TFNs and R-numbers concepts such as future-events risks, acceptable risks, and risk perception. Moreover, it uses some R-numbers operations such as distance measures and defuzzification techniques for fuzzy ranking and decision-making, similar what has been proposed in RS-TOPSIS method. As it discussed in Introduction the main differences between proposed R-sets and R-numbers are that R-sets has been proposed to model variability of membership function instead of the x values and the R-sets consider the risks of the information sources too. Moreover, unlike R-numbers, in R-sets the pessimistic and optimistic risks are defined as the influential factors, which make the membership function becomes lesser and higher,

respectively, so different relations are not defined for beneficial and non-beneficial values. Hence, the decision-makers should define the pessimistic and optimistic risks according to this definition.

- As proposed in this study, *R-sets* can be used to address various risk scenarios. Accordingly, the proposed methodology can be adopted, and organizations and decision-makers can define these parameters considering the acceptable degree of optimistic/pessimistic risk in real applications.
- Similar to R-number methodology, different outcomes were obtained in the presented case studies, when the *R-sets* methodology was used. This highlights the flexibility and efficiency of the method in terms of risk-related problems. Therefore, the outcome of the *R-sets* can be considered as a robust solution.
- The effects of different risk levels in the second case study were investigated by performing sensitivity analysis and comparing them; it is observed that different values of risk parameters (the pessimistic and optimistic source and future events risks) lead to different results. Moreover, the results show that the most discrepancies between the results in various sensitivity analyses have occurred in the case of ${r^{OS}}_{max}$ and ${r^{OF}}_{max}$, due to the high level of these risk parameters respect to ${r^{PS}}_{max}$ and ${r^{PF}}_{max}$. Besides, it is seen that in all cases and for all risk values, FM_5 is the most critical failure mode.
- The *R-sets* methodology could be best for problems with fuzzy membership functions and high levels of precision, such as safety or environment-related problems where a wrong decision could have serious consequences.

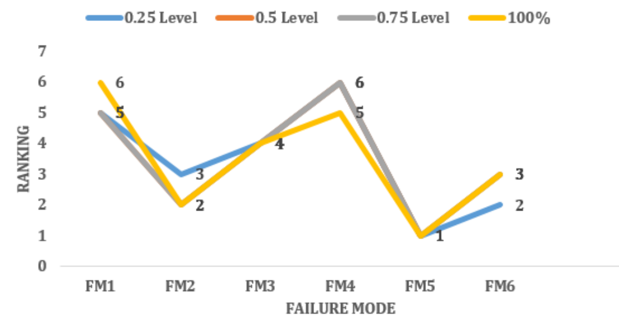

Fig. 8. Results of sensitivity analysis for ${r^{OF}}_{max}$

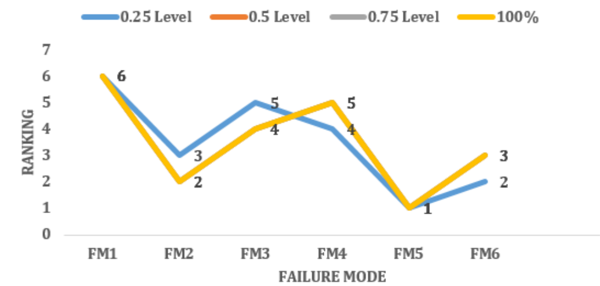

Fig. 9. Results of sensitivity analysis for ${r^{PF}}_{max}$

TABLE VII
Results of different sensitivity levels

Sensitivity level (α)	Type of risk	Results	F_1	F_2	F_3	F_4	F_5	F_{16}
0.25	${r^{OS}}_{max}$	CC_i	1.188	1.751974	1.197912	1.211905	2.141294	1.754751
		Rank	6	3	5	4	1	2
	${r^{PS}}_{max}$	CC_i	1.297179	1.871297	1.314066	1.296927	2.237908	1.861651
		Rank	5	2	4	6	1	3
	${r^{OF}}_{max}$	CC_i	1.268812	1.81745	1.281396	1.266719	2.187541	1.818137
		Rank	5	3	4	6	1	2
	${r^{PF}}_{max}$	CC_i	1.192231	1.756597	1.199564	1.217275	2.149283	1.764965
		Rank	6	3	5	4	1	2
0.5	${r^{OS}}_{max}$	CC_i	1.211957	1.779708	1.222724	1.230366	2.163281	1.778338
		Rank	6	2	5	4	1	3
	${r^{PS}}_{max}$	CC_i	1.289513	1.860712	1.306829	1.289684	2.224872	1.850661
		Rank	6	2	4	5	1	3
	${r^{OF}}_{max}$	CC_i	1.271292	1.825731	1.285672	1.269991	2.191701	1.822224
		Rank	5	2	4	6	1	3
	${r^{PF}}_{max}$	CC_i	1.278517	1.844268	1.295258	1.279875	2.204683	1.836986
		Rank	6	2	4	5	1	3
0.75	${r^{OS}}_{max}$	CC_i	1.240555	1.80981	1.254001	1.251402	2.183015	1.803572
		Rank	6	2	4	5	1	3
	${r^{PS}}_{max}$	CC_i	1.282536	1.850708	1.300197	1.282811	2.212072	1.840164
		Rank	6	2	4	5	1	3
	${r^{OF}}_{max}$	CC_i	1.273742	1.833667	1.289907	1.273175	2.19572	1.826247
		Rank	5	2	4	6	1	3
	${r^{PF}}_{max}$	CC_i	1.277348	1.84278	1.29468	1.278118	2.202192	1.833654
		Rank	6	2	4	5	1	3

1	All the risks	CC_i	1.273	1.842	1.291	1.272	2.202	1.829
		Rank	6	2	4	5	1	3

VI. Conclusion

The decision-making problems, particularly those are based on unreliable information or future events, usually, carry some level of risk and error, which emphasizes the need for defining a confidence factor in such cases. As fuzzy data can capture well the ambiguities and uncertainties, in this work, we propose a new concept called R-sets to describe the risk linked with the fuzzy membership function. The proposed R-sets model considers all possible risk scenarios of the fuzzy sets membership function, and take both information source and future event risks into account and the other control parameters, such as risk appetite (risk-taking degree) to achieve better and robust results. The novelty of this concept is that the pessimistic and optimistic risks inherent in an information source and the effective factors of the pessimistic and optimistic risks of a membership function can be well defined. The concept of R-sets was grounded on the pessimistic and optimistic intervals in the form of T2 TFNs, and its general mathematical relationships were presented. Finally, a new approach called RS-TOPSIS methodology was proposed taking the R-sets method as the framework to solve decision-making problems. Further, a case study of FMEA risk analysis was presented to demonstrate the reliability of the proposed methodology.

The proposed concept can be used to address the problems, where membership function contains risks and errors. However, our model does not address the determination and quantification of such risks, which can be explored in future research. Moreover, due to the fact of the relations of the extensions of fuzzy sets [30, 31], *R-sets* can be applied to other fuzzy models (e.g., intuitionistic fuzzy sets, hesitant fuzzy sets, and neutrosophic sets), future studies may explore the development of *R-sets* for them.

References

[1] Seiti H, Tagipour R, Hafezalkotob A, Asgari F. Maintenance strategy selection with risky evaluations using RAHP. Journal of Multi-Criteria Decision Analysis. 2017 Sep;24(5-6):257-74.

[2] Seiti H, Hafezalkotob A, Najafi SE, Khalaj M. A risk-based fuzzy evidential framework for FMEA analysis under uncertainty: An interval-valued DS approach. Journal of Intelligent & Fuzzy Systems. 2018 Jan 1(Preprint):1-2.

[3] Seiti H, Hafezalkotob A, Martínez L. R-numbers, a new risk modeling associated with fuzzy numbers and its application to decision making. Information Sciences. 2019 May 1;483:206-31.

[4] Seiti H, Hafezalkotob A, Fattahi R. Extending a pessimistic–optimistic fuzzy information axiom based approach considering acceptable risk: Application in the selection of maintenance strategy. Applied Soft Computing. 2018 Jun 1;67:895-909.

[5] Seiti H, Hafezalkotob A. Developing the R-TOPSIS methodology for risk-based preventive maintenance planning: A case study in rolling mill company. Computers & Industrial Engineering. 2019 Feb 1;128:622-36.

[6] Zadeh LA. Fuzzy sets. Information and control. 1965 Jun 1;8(3):338-53.

[7] Mendel JM. Type-2 fuzzy sets and systems: an overview. IEEE computational intelligence magazine. 2007 Feb;2(1):20-9.

[8] Dubois D, Prade H. Interval-valued Fuzzy Sets, Possibility Theory and Imprecise Probability. InEUSFLAT Conf. 2005 Sep 7 (pp. 314-319).

[9] Zadeh LA. A note on Z-numbers. Information Sciences. 2011 Jul 15;181(14):2923-32.

[10] Hao Z, Xu Z, Zhao H, Zhang R. Novel intuitionistic fuzzy decision making models in the framework of decision field theory. Information Fusion. 2017 Jan 1;33:57-70.

[11] Wang H, Smarandache F, Sunderraman R, Zhang YQ. interval neutrosophic sets and logic: theory and applications in computing: Theory and applications in computing. Infinite Study; 2005.

[12] Rodríguez RM, Bedregal B, Bustince H, Dong YC, Farhadinia B, Kahraman C, Martínez L, Torra V, Xu YJ, Xu ZS, Herrera F. A position and perspective analysis of hesitant fuzzy sets on information fusion in decision making. Towards high quality progress. Information Fusion. 2016 May 1;29:89-97.

[13] Cuong BC, Kreinovich V. Picture Fuzzy Sets-a new concept for computational intelligence problems. In2013 Third World Congress on Information and Communication Technologies (WICT 2013) 2013 Dec 15 (pp. 1-6). IEEE.

[14] Li D, Liu C, Gan W. A new cognitive model: Cloud model. International Journal of Intelligent Systems. 2009 Mar;24(3):357-75.

[15] Smarandache F. Plithogenic Set, an Extension of Crisp, Fuzzy, Intuitionistic Fuzzy, and Neutrosophic Sets–Revisited. Neutrosophic Sets and Systems. 2018 Jul 1;21:153-66.

[16] Bede B. Fuzzy clustering. In Mathematics of Fuzzy Sets and Fuzzy Logic 2013 (pp. 213-219). Springer, Berlin, Heidelberg.

[17] Bustince H, Barrenechea E, Pagola M, Fernandez J, Xu Z, Bedregal B, Montero J, Hagras H, Herrera F, De Baets B. A historical account of types of fuzzy sets and their relationships. IEEE Transactions on Fuzzy Systems. 2015 Jul 1;24(1):179-94.

[18] Mizumoto M, Tanaka K. Fuzzy sets and their operations. Information and Control. 1981 Jan 1;48(1):30-48.

[19] Greenfield S, Chiclana F. The Collapsing Defuzzifier for discretised generalised type-2 fuzzy sets. International Journal of Approximate Reasoning. 2018 Nov 1;102:21-40.

[20] Castro JR, Sanchez MA, Gonzalez CI, Melin P, Castillo O. A New Method for Parameterization of General Type-2 Fuzzy Sets. Fuzzy Information and Engineering. 2018 Jan 2;10(1):31-57.

[21] Mendel JM, Rajati MR, Sussner P. On clarifying some definitions and notations used for type-2 fuzzy sets as well as some recommended changes. Information Sciences. 2016 May 1;340:337-45.

[22] Seiti H, Hafezalkotob A. Developing pessimistic–optimistic risk-based methods for multi-sensor fusion: An interval-valued evidence theory approach. Applied Soft Computing. 2018 Nov 1;72:609-23.

[23] Li J, Fang H, Song W. Failure Mode and Effects Analysis Using Variable Precision Rough Set Theory and TODIM Method. IEEE Transactions on Reliability. 2019 Aug 1.

[24] Ayber S, Erginel N. Developing the Neutrosophic Fuzzy FMEA Method as Evaluating Risk Assessment Tool. InInternational Conference on Intelligent and Fuzzy Systems 2019 Jul 23 :. Springer, Cham.

[25] Chang KH, Wen TC, Chung HY. Soft failure mode and effects analysis using the OWG operator and hesitant fuzzy linguistic term sets. Journal of Intelligent & Fuzzy Systems. 2018 Jan 1;34(4):2625-39.

[26] Mirghafoori SH, Izadi MR, Daei A. Analysis of the barriers affecting the quality of electronic services of libraries by VIKOR, FMEA and entropy combined approach in an intuitionistic-fuzzy environment. Journal of Intelligent & Fuzzy Systems. 2018 Jan 1;34(4):2441-51.

[27] Liu HC, Wang LE, Li Z, Hu YP. Improving risk evaluation in FMEA with cloud model and hierarchical TOPSIS method. IEEE Transactions on Fuzzy Systems. 2018 Jul 31;27(1):84-95.

[28] Deng X, Xiao F, Deng Y. An improved distance-based total uncertainty measure in belief function theory. Applied Intelligence. 2017 Jun 1;46(4):898-915.

[29] Xiao F. Multi-sensor data fusion based on the belief divergence measure of evidences and the belief entropy. Information Fusion. 2019 Mar 1;46:23-32.

[30] Deschrijver G, Kerre EE. On the relationship between some extensions of fuzzy set theory. Fuzzy sets and systems. 2003 Jan 16;133(2):227-35.

[31] Rodríguez RM, Martínez L, Torra V, Xu ZS, Herrera F. Hesitant fuzzy sets: state of the art and future directions. International Journal of Intelligent Systems. 2014 Jun;29(6):495-524.