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New Results On Pythagorean Neutrosophic Open Sets in Pythagorean Neutrosophic Topological Spaces

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Abstract. In this paper, we introduce and study the notion of Pythagorean neutrosophic $^{\star}b$-open set on Pythagorean neutrosophic topology. Besides, we define the concepts of Pythagorean neutosophic $^{\star}b$-open function, Pythagorean neutrosophic $^{\star}b$-continuous function and Pythagorean neutrosophic $^{\star}b$-homeomorphism. Moreover, we establish some of their properties and characterizations.

Keywords: Pythagorean neutrosophic $^{\star}b$-open sets, Pythagorean neutosophic $^{\star}b$-open function, Pythagorean neutrosophic $^{\star}b$-continuous function, Pythagorean neutrosophic $^{\star}b$-homeomorphism

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1. **Introduction**

L.A. Zadeh [18] introduced the idea of fuzzy set theory in 1965. Then, Chang [4] defined the concept of fuzzy topological space and generalized some basic notions of topology. Besides, Atanassov [2,3] in 1983 introduced the concept of intuitionistic fuzzy set. Furthermore, the notion of neutrosophic set was introduced by Smarandache [14] and Wang et.al. studied te notion of interval neutrosophic set. Moreover, A.A. Salama and S.A. Albowi [13] defined the concept of crisp set and neutrosophic crisp set topological spaces. In 2013, Yager [17] introduced the concept of Pythagorean membership grades in multicriteria decision making. Later, Yager, Zahand and Xu [16] gave some basic operations for Pythagorean fuzzy number. Taking into account mentioned previously, Iswarya et.al. [11] studied the concept of neutrosophic semi-open sets and neutrosophic semi-closed sets. Further, In 2017, Imran et.al. [10]

introduced neutrosophic semi-α-open sets and studied their fundamental properties. Additionally, Arockiarani et.al. [1] defined the notion of neutrosophic semi-open (resp. pre-open and α-open) functions and investigated their relations. Later, Rao et.al. [15] introduced neutrosophic pre-open sets. Then, P.Evanzalin Ebenanjar et al. [6] defined neutrosophic b-open sets in neutrosophic topological space and investigated their properties. Recently Bromi and Smarandache defined the Haudroff distance between neutrosophic sets and

On the other hand, mathematicians have extended the notion of neutrosophic sets. In 2020, Sneha and Nirmala [12] defined the concept of pythagorean neutrosophic b-open sets and pythagorean enutrosophice semi-open sets and established some properties and notions associated to these sets, additionally they defined some variants of continuity. Simultaneously, Granados [7, 8] defined the concept of pythagorean neutrosophice pre-open sets and showed other related notions about pythagorean neutrosophic sets, furthermore he studied and established new variants of continuity on these sets. In this paper, we use these notions to extend the concept of pythagorean neutrosophic open sets and define a new notion of sets which are called Pythagorean neutrosophic $^\star b$-open set. Besides, we show some of their properties. We also define the concept of Pythagorean neutosophic $^\star b$-open function, Pythagorean neutrosophic $^\star b$-continuous function and Pythagorean neutrosophic $^\star b$-homeomorphism. Moreover, we establish some some of their properties and characterizations. Now, we procure some well-known notions which are useful for the developing of this paper. Let A be a subset of X. Then, we will denote the Pythagorean neutrosophic interior and Pythagorean neutrosophic closure of A as follows: $PNInt(A)$ and $PNCl(A)$, respectively. Additionally, if A is a Pythagorean neutrosophic open set in X, then $PNInt(A) = A$. On the other hand, the complement of a Pythagorean neutrosophic open set is called Pythagorean neutrosophic closed set, moreover if B is a Pythagorean neutrosophic closed set in X, then $PNCl(A) = A$. [12] defined the following concepts: Let $f : (X, \tau) \to (Y, \sigma)$ be a function where (X, τ) and (Y, σ) are Pythagorean neutrosophic topological spaces. Then, f is said to be Pythagorean neutrosophic if $f^{-1}(V)$ is a Pythagorean neutrosophic in X for every Pythagorean neutrosophic open set V in Y. Besides, let $f : (X, \tau) \to (Y, \sigma)$ be a function where (X, τ) and (Y, σ) are Pythagorean neutrosophic topological spaces. Then, f is said to be Pythagorean neutrosophic pre-continuous if $f^{-1}(V)$ is a Pythagorean neutrosophic pre-open in X for every Pythagorean neutrosophic open set V in Y.

Definition 1.1. For any Pythagorean neutrosophic set A in a Pythagorean neutrosophic topological space (X, τ), A is said to be Pythagorean neutrosophic pre-open set [7] if $A \subseteq PNInt((PNCl(A))$.

Theorem 1.2. *[7] Every Pythagorean neutrosophic open set is a Pythagorean neutrosophic pre-open set.*

Definition 1.3. [7] Let $f : (X, \tau) \to (Y, \sigma)$ be a function where (X, τ) and (Y, σ) are Pythagorean neutrosophic topological spaces. Then, f is said to be Pythagorean neutrosophic pre-open if $f(A)$ is Pythagorean neutrosophic pre-open set in Y for every Pythagorean neutrosophic open set A in X.

2. Pythagorean neutrosophic $^\star b$-open sets

In this section we introduce and study the notion of Pythagorean neutrosophic $^\star b$-open sets and we establish some notions associated to them.

Definition 2.1. Let X be a non-empty set. If a, b, c are real standard or non standard subsets of $]0^-, 1^+[$, then the Pythagorean neutrosophic set $x_{a,b,c}$ is said to be Pythagorean neutrosophic point (or simply, PNP) in X and it is given by:

$$x_{a,b,c}(x_p) = \begin{cases} (a, b, c) & \text{if } x = x_p \\ (0, 0, 1) & \text{if } x \neq x_p \end{cases}$$

For each $x_p \in X$ is said to be the support of $x_{a,b,c}$, where a denotes the degree of membership value, b denotes the degree of indeterminacy and c is the degree of non-membership value of $x_{a,b,c}$.

Definition 2.2. For any Pythagorean neutrosophic set A in a Pythagorean neutrosophic topological space (X, τ), A is said to be Pythagorean neutrosophic $^\star b$-open set (or simply, $PN^\star bOS$) if $A \subseteq PNInt((PNCl(A)) \cap PNCl(PNInt(A))$. The complement of a Pythagorean neutrosophic $^\star b$-open set is called Pythagorean neutrosophic $^\star b$-closed set.

Remark 2.3. The collection of all Pythagorean neutrosophic $^\star b$-open sets and Pythagorean neutrosophic $^\star b$-closed sets are denoted by $PN^\star bOS(X, \tau)$ and $PN^\star bCS(X, \tau)$, respectively.

Proposition 2.4. *Let (X, τ) be a Pythagorean neutrosophic topological space and $A \subseteq X$. Then, If A is a Pythagorean neutrosophic $^\star b$-open set, then A is Pythagorean neutrosophic pre-open set.*

The converse of the above proposition need not be true as can be seen in the following example:

Example 2.5. Let $X = \{q, w, e\}$ and $\tau = \{0_N, A, B, 1_N\}$. Then, $A = \langle(0.4, 0.5, 0.2), (0.3, 0.2, 0.1), (0.9, 0.6, 0.8)\rangle$, $B = \langle(0.2, 0.4, 0.5), (0.1, 0.1, 0.2), (0.6, 0.5, 0.8)\rangle$ and $C = \{(0.5, 0.6, 0.1), (0.4, 0.3, 0.1), (0.9, 0.8, 0.5)\rangle$. Then, we can see that C is a Pythagorean neutrosophic pre-open set, but it is not a Pythagorean neutrosophic $^\star b$-open set.

Definition 2.6. A Pythagorean neutrosophic set V in a Pythagorean neutrosophic topological space (X, τ) is said to be Pythagorean neutrosophic $^\star b$-closed (or simply, $PN^\star bCS$)) if $V \supseteq PNInt(PNCl(V)) \cap PNCl(PNInt(V))$.

Definition 2.7. Let (X, τ) be a Pythagorean neutrosophic topological space and V be a Pythagorean neutrosophic set on X. Then we define the Pythagorean neutrosophic $^\star b$-interior and Pythagorean neutrosophic $^\star b$-closure of V as:

(1) Pythagorean neutrosophic $^\star b$-interior of V (or simply, $PN^\star BINT(V)$) as the union of all Pythagorean neutrosophic $^\star b$-open sets of X contained in V. It means that $PN^\star BINT(V) = \bigcup\{A : A \text{ is a } PN^\star bOS \text{ in } X \text{ and } A \subseteq V\}$.

(2) Pythagorean neutrosophic $^\star b$-closure of V (or simply, $PN^\star BCL(V)$) as the intersection of all Pythagorean neutrosophic $^\star b$-closed set of X containing V. It means that $PN^\star BCL(V) = \bigcap\{B : B \text{ is a } PN^\star bCS \text{ in } X \text{ and } V \subseteq B\}$.

Remark 2.8. By the Definition 2.7, we can see that $PN^\star BCL(V)$ is the smallest Pythagorean neutrosophic $^\star b$-closed set of X which contains V. Besides, $PN^\star BINT(V)$ is the largest Pythagorean neutrosophic $^\star b$-open set of X which is contained in V.

Proposition 2.9. *Let V be a Pythagorean neutrosophic set in a Pythagorean neutrosophic topological space (X, τ). Then, the following statements hold:*

(1) *If V is Pythagorean neutrosophic $^\star b$-open set, then $Cl(V)$ is is a Pythagorean neutrosophic $^\star b$-closed set.*

(2) *If V is Pythagorean neutrosophic $^\star b$-closed set, then $Cl(V)$ is is a Pythagorean neutrosophic $^\star b$-open set.*

Proof: The proof is followed by the Definitions 2.2, 2.6 and 2.7.

Theorem 2.10. *Let V be a Pythagorean neutrosophic set in a Pythagorean neutrosophic topological space (X, τ). Then, the following statements hold:*

(1) $Cl(PN^\star BINT(V)) = PN^\star BCL(Cl(V))$.

(2) $Cl(PN^\star BCL(V)) = PN^\star BINT(Cl(V))$.

Proof: We begin proving (1): Let V be a Pythagorean neutrosophic set. Now, by the Definition 2.7 part (1), $PN^\star BINT(V) = \bigcup\{A : A \text{ is a } PN^\star bOS \text{ in } X \text{ and } A \subseteq V\}$, this implies that $Cl(PN^\star BINT(V)) = Cl(\bigcup\{A : A \text{ is a } PN^\star bOS \text{ in } X \text{ and } A \subseteq V\})=$ $\bigcap\{Cl(A) : Cl(A) \text{ is a } PN^\star bCS \text{ in } X \text{ and } Cl(V) \subseteq Cl(A)\}$. Now, we will replace $Cl(A)$ by B, then we have that $Cl(PN^\star BINT(V) = \bigcap\{B : B \text{ is a } PN^\star bCS \text{ in } X \text{ and } Cl(V) \subseteq B\}$, and so $Cl(PN^\star BINT(V)) = PN^\star BCL(Cl(V))$.

The proof of (2) is made similarly to (1).

Theorem 2.11. *For a Pythagorean neutrosophic topological space (X, τ) and $A, B \subseteq X$. The following statements hold:*

(1) *Every Pythagorean neutrosophic set is Pythagorean neutrosophic* $\star$*b-open set.*
(2) $PN^\star BINT(PN^\star BINT(A)) = PN^\star BINT(A)$.
(3) $PN^\star BCL(PN^\star BCL((A)) = PN^\star BCL(A)$.
(4) *Let* A, B *be two Pythagorean neutrosophic* $\star$*b-open sets, then* $PN^\star bOS(A) \cup PN^\star bOS(B) = PN^\star bOS(A \cup B)$.
(5) *Let* A, B *be two Pythagorean neutrosophic* $\star$*b-closed sets, then* $PN^\star bCS(A) \cap PN^\star bCS(B) = PN^\star bCS(A \cap B)$.
(6) *For any two sets* A, B, $PN^\star BINT(A) \cap PN^\star BINT(B) = PN^\star BInt(A \cap B)$.
(7) *For any two sets* A, B, $PN^\star BCL(A) \cup PN^\star BCL(B) = PN^\star BCL(A \cup B)$.
(8) *If* A *is* $PN^\star bOS(X, \tau)$, *then* $A = PN^\star BINT(A)$.
(9) *If* $A \subseteq B$, *then* $PN^\star BINT(A) \subseteq PN^\star BINT(B)$.
(10) *For any two sets* A, B, $PN^\star BINT(A) \cup PN^\star BINT(B) \subseteq PN^\star BINT(A \cup B)$.
(11) *If* A *is* $PN^\star bCS(X, \tau)$, *then* $A = PN^\star BCL(A)$.
(12) *If* $A \subseteq B$, *then* $PN^\star BCL(A) \subseteq PN^\star BCL(B)$.
(13) *For any two sets* A, B, $PN^\star BCL(A \cap B) \subseteq PN^\star BCL(A) \cap PN^\star BCL(B)$.

Proof: The proofs of (1), (2), (3), (4), (5), (9), (11) and (12) are followed by the Definitions 2.2 and 2.6. The proofs of (6), (7) and (8) are followed by the Definition 2.7 and the proofs of (10) and (13) are followed by the Definition 2.7 and parts (9) and (12) of this Theorem.

The following example shows that the equality (10) need not be hold in Theorem 2.11 .

Example 2.12. Let $X = \{q, w, e\{$ and $\tau = \{0_N$, A, B, C, D, $1_N\}$ where $A = \langle($ 0.4, 0.7, 0.1), (0.5, 0.6, 0.2), (0.9, 0.7, 0.3)$\rangle$, $B = \langle($ 0.4, 0.6, 0.1), (0.7, 0.7, 0.2), (0.9, 0.5, 0.1) $\rangle$, $C = \langle($ 0.4, 0.7, 0.1), (0.7, 0.7, 0.2), (0.9, 0.7, 0.1)$\rangle$, $D = \langle($ 0.4, 0.6, 0.1), (0.5, 0.6, 0.2), (0.9, 0.5, 0.3) $\rangle$. Then, τ is Pythagorean neutrosophic topological space. Now, Consider $E = \langle($ 0.7, 0.6, 0.1), (0.7, 0.6, 0.1), (0.9, 0.5, 0)$\rangle$ and $F = \langle($ 0.4, 0.6, 0.1), (0.5, 0.7, 0.2), (1, 0.7, 0.1)$\rangle$. Then, $PN^\star BINT(E) = D$ and $PN^\star BINT(F) = D$. This implies that $PN^\star BINT(E) \cup PN^\star BINT(F) = D$. Now, $E \cup F = \langle($ 0.7, 0.6, 0.1), (0.7, 0.7, 0.1), (1, 0.7, 0) $\rangle$, it follows that $PN^\star BINT(E \cup F) = B$. Then, $PN^\star BINT(E \cup F) \nsubseteq PN^\star BINT(E) \cup PN^\star BINT(F)$.

The following example shows that the equality (13) need not be hold in Theorem 2.11 .

Example 2.13. Let $X = \{q, w, e\}$, $\tau = \{0_N$, A, B, C, D, $1_N\}$ and $C_\tau = \{1_N$, E, F, G, H, $0_N\}$ where $A = \langle($ 0.5, 0.6, 0.1), (0.6, 0.7, 0.1), (0.9, 0.5, 0.2)$\rangle$, $B = \langle($ 0.4, 0.5, 0.2), (0.8, 0.6, 0.3), (0.9, 0.7, 0.3)$\rangle$, $C = \langle($ 0.4, 0.5, 0.2), (0.6, 0.6, 0.3), (0.9, 0.5, 0.3)$\rangle$, $D = \langle($ 0.5, 0.6, 0.1), (0.8, 0.7, 0.1), (0.9, 0.7, 0.2)$\rangle$, $E = \langle($ 0.1, 0.4, 0.5), (0.1, 0.3, 0.6), (0.2, 0.5, 0.9)$\rangle$, $F = \langle($ 0.2, 0.5, 0.4), (0.3, 0.4, 0.8), (0.3, 0.3, 0.9)$\rangle$, $G = \langle($ 0.2, 0.5, 0.4), (0.3, 0.4, 0.6), (

0.3, 0.5, 0.9)⟩, $H = \langle($ 0.1, 0.4, 0.5), (0.1, 0.3, 0.8), (0.2, 0.3, 0.9)⟩. Then τ is Pythagorean neutrosophic topological space. Now, Consider $I = \langle($ 0.1, 0.2, 0.5), (0.2, 0.3, 0.7), (0.3, 0.3, 1)⟩ and $J = \langle($ 0.2, 0.4, 0.8), (0.1, 0.2, 0.8), (0.2, 0.5, 0.9)⟩. Then $PN^\star BCL(I) = G$ and $PN^\star BCL(J) = G$. This implies that $PN^\star BCL(I) \cap PN^\star BCL(J) = G$. Now, $I \cap J = \langle$ (0.1, 0.2, 0.8), (0.1, 0.2, 0.8), (0.2, 0.3, 1)⟩, it follows that $PN^\star BCL(I \cap G) = H$. Then $PN^\star BCL(I) \cap PN^\star BCL(J) \nsubseteq PN^\star BCL(I \cap G)$.

The following example shows that the intersection of two Pythagorean neutrosophic $^\star b$-open sets need not be a Pythagorean neutrosophic $^\star b$-open set.

Example
2.14. Let $X = \{q, w\}$, $A = \langle(0.1, 0.3, 0.5), (0.3, 0.5, 0.7)\rangle$, $B = \langle(0.1, 0.1, 0.4), (0.7, 0.5, 0.3)\rangle$, $C = \langle(0.4, 0.6, 0.9), (0.6, 03, 0.3)\rangle$ and $D = \langle(0.3, 0.5, 0.3), (0.9, 0.5, 0.9)\rangle$. Then, τ is a Pythagorean neutrosophic topological space. Now, choose $A_1 = \langle(0.3, 0.5, 0.3), (1.0, 0.1, 0.1)\rangle$ and $A_2 = \langle(1.0, 1.0, 0.4), (0.9, 0.4, 0.6)\rangle$. We can see that $A_1 \cap A_2$ is not a Pythagorean neutrosophic $^\star b$-open set of (X, τ).

The following examples show that the union of two Pythagorean neutrosophic $^\star b$-closed sets need not be a Pythagorean neutrosophic $^\star b$-closed set.

Example 2.15. By the example 2.14, we can imply that $A_1^c \cup A_2^c$ is not a Pythagorean neutrosophic $^\star b$-closed set of (X, τ).

Example 2.16. Let $X = \{q\}$ and $A = \langle(1, 0.5, 0.7)\rangle$, $B = \langle(0, 0.9, 0.2)\rangle$, $C = \langle(1, 0.9, 0.2)\rangle$ and $D = \langle(0, 0.5, 0.7)\rangle$. Then, τ is a Pythagorean neutrosophic topological space. Now, choose $A_1^c = \langle(0.4, 0.5, 1)\rangle$ and $A_2^c = \langle(0.2, 0, 0.8)\rangle$. We can see that $A_1 \cup A_2$ is not a Pythagorean neutrosophic $^\star b$-closed set of (X, τ).

Proposition 2.17. *Let A be a Pythagorean neutrosophic set in Pythagorean neutrosophic topological space (X, τ). If B is a Pythagorean neutrosophic $^\star b$-open set and $B \subseteq A \subseteq PNInt(PNCl(A)) \cap PNCl(PNInt(A))$, then A is a Pythagorean neutrosophic $^\star b$-open set.*

Theorem 2.18. *Arbitrary union of Pythagorean neutrosophic $^\star b$-open sets is a Pythagorean neutrosophic $^\star b$-open set.*

Proof: Let $A_1, A_2, ...A_n$ be a collection of Pythagorean neutrosophic $^\star b$-open sets, then by the Definition 2.2, $A_1 \subseteq PNInt(PNCl(A_1)) \cap PNCl(PNInt(A_1))$, $A_2 \subseteq PNInt(PNCl(A_2)) \cap PNCl(PNInt(A_2))$, ..., $A_n \subseteq PNInt(PNCl(A_n)) \cap PNCl(PNInt(A_n))$. Now, $A_1 \cup A_2 \cup ... \cup A_n \subseteq (PNInt(PNCl(A_1)) \cap PNCl(PNInt(A_1))) \cup (PNInt(PNCl(A_2)) \cap PNCl(PNInt(A_2))) \cup ... \cup (PNInt(PNCl(A_n)) \cap PNCl(PNInt(A_N)))$, by the Theorem 2.11 parts (7) and (10), $A_1 \cup A_2 \cup ... \cup A_n \subseteq PNInt(PNCl(A_1 \cup A_2 \cup ... \cup$

$A_n)) \cap PNCl(PNInt(A_1 \cup A_2 \cup ... \cup A_n))$. This proofs that $A_1 \cup A_2 \cup ... \cup A_n$ is a Pythagorean neutrosophic $^\star b$-open set.

Th following example shows that the intersection of two Pythagorean neutrosophic $^\star b$-open sets need not be a Pythagorean neutrosophic $^\star b$-open set.

Example 2.19.
Let $X = \{z, x\}$ and $A = \langle(0.3, 0.5, 0.4), (0.6, 0.2, 0.5)\rangle$, $B = \langle(0.2, 0.6, 0.7), (0.5, 0.3, 0.1)\rangle$, $C = \langle(0.3, 0.6, 0.4), (0.6, 0.3, 0.1)\rangle$ and $D = \langle(0.2, 0.5, 0.7), (0.5, 0.2, 0.5)\rangle$. Then, τ is a Pythagorean neutrosophic topological space. Now, take $A_1 = \langle(0.4, 0.6, 0.4), (0.8, 0.3, 0.4)\rangle$ and $A_2 = \langle(1.0, 0.9, 0.2), (0.5, 0.7, 0)\rangle$. We can see that $A_1 \cup A_2$ is not a Pythagorean neutrosophic $^\star b$-open set of (X, τ).

Remark 2.20. By the Example 2.14, we support that the arbitrary intersection of Pythagorean neotrosophic $^\star b$-open sets need not be a Pythagorean neotrosophic $^\star b$-open set.

Proposition 2.21. *Arbitrary intersection of Pythagorean neutrosophic $^\star b$-closed sets is a Pythagorean neutrosophic $^\star b$-closed set.*

Remark 2.22. By the Example 2.15, the arbitrary union of Pythagorean neotrosophic $^\star b$-closed sets need not be a Pythagorean neotrosophic $^\star b$-closed set.

Theorem 2.23. *A Pythagorean neutrosophic set A in a Pythagorean neutrosophic topological space (X, τ) is Pythagorean neutrosophic b-open if and only for every Pythagorean neutrosophic point $x_{a,b,c} \in A$ there exits a Pythagorean neutrosophic $^\star b$-open $B_{x_{a,b,c}}$ such that $x_{a,b,c} \in B_{x_{a,b,c}} \subseteq A$.*

Proof: **Necessary**: Let A be a Pythagorean neutrosophic b-open set. Then, we have that $B_{x_{a,b,c}} = A$ for each $x_{a,b,c}$.
Sufficiency: Suppose that for every Pythagorean neutrosophic point $x_{a,b,c} \in A$, there exits a neutrosophic $^\star b$-open set $B_{x_{a,b,c}}$ such that $x_{a,b,c} \in B_{x_{a,b,c}} \subseteq A$. Thus, $A = \bigcup\{x_{a,b,c} : x_{a,b,c} \in A\} \subseteq \{B_{x_{a,b,c}} : x_{a,b,c} \in A\} \subseteq A$ and then, $A = \bigcup\{B_{x_{a,b,c}} : x_{a,b,c} \in A\}$. Therefore, by the Theorem 2.18, it is a Pythagorean neutrosophic $^\star b$-open set

Definition 2.24. Let $f : (X, \tau) \to (Y, \sigma)$ be a function where (X, τ) and (Y, σ) are Pythagorean neutrosophic topological spaces. Then, f is said to be Pythagorean neutrosophic $^\star b$-open if $f(A)$ is Pythagorean neutrosophic $^\star b$-open set in Y for every Pythagorean neutrosophic open set A in X.

Proposition 2.25. *Let $f : (X, \tau) \to (Y, \sigma)$ be a function where (X, τ) and (Y, σ) are Pythagorean neutrosophic topological spaces. If f is Pythagorean neutrosophic $^\star b$-open, then, f is Pythagorean neutrosophic pre-open*

Proof: Let f be a Pythagorean neutrosophic $^\star b$-open and A be a Pythagorean neutrosophic open set in X. Then, by hypothesis $f(A)$ is a Pythagorean neutrosophic $^\star b$-open set in Y, by the Proposition 2.4, $f(A)$ is a Pythagorean neutrosophic pre-open set in X. Therefore, f is a Pythagorean neutrosophic pre-open function.

3. Pythagorean neutrosophic $^\star b$-continuous functions

In this section, we use the notion of Pythagorean neutrosophic $^\star b$-open sets to introduce and study the concepts of Pythagorean neutrosophic $^\star b$-continuous function and Pythagorean neutrosophic $^\star b$-homeomorphism. Moreover, we establish some of their properties.

Definition 3.1. Let $f : (X, \tau) \to (Y, \sigma)$ be a function where (X, τ) and (Y, σ) are Pythagorean neutrosophic topological spaces. Then, f is said to be Pythagorean neutrosophic $^\star b$-continuous if $f^{-1}(V)$ is a Pythagorean neutrosophic $^\star b$-open set in X for every Pythagorean neutrosophic open set V in Y.

Proposition 3.2. *Every Pythagorean neutrosophic continuous function is Pythagorean neutrosophic $^\star b$-continuous function.*

Definition 3.3. Let $x_{a,b,c}$ be a Pythagorean neutrosophic point of a Pythagorean neutrosophic topological space (X, τ). A Pythagorean neutrosophic set D of X is said to be Pythagorean neutrosophic neighbourhood of $x_{a,b,c}$ if there exits a Pythagorean neutrosophic open set V in X such that $x_{a,b,c} \in V \subseteq D$

Proposition 3.4. *Let $f : (X, \tau) \to (Y, \sigma)$ be a function where (X, τ) and (Y, σ) are Pythagorean neutrosophic topological spaces. Then, the following statements are equivalent:*

(1) *f is a Pythagorean neutrosophic $^\star b$-continuous function.*
(2) *For each Pythagorean neutrosophic point $x_{a,b,c}$ and every Pythagorean neutrosophic A of $f(x_{a,b,c})$, there exits a Pythagorean neutrosophic $^\star b$-open set B of X such that $x_{a,b,c} \in B \subseteq f^{-1}(A)$.*
(3) *For each Pythagorean neutrosophic point $x_{a,b,c} \in X$ and every Pythagorean neutrosophic neighbourhood A of $f(x_{a,b,c})$, there exits a Pythagorean neutrosophic $^\star b$-open set B of X such that $x_{a,b,c} \in B$ and $f(B) \subseteq A$.*

Proof: (1) $\Rightarrow$ (2): Let $x_{a,b,c}$ be a Pythagorean neutrosophic point of X and let A be a Pythagorean neutrosophic neighbourhood of $f(x_{a,b,c})$. Then, there exits a Pythagorean neutrosophic open set B of Y such that $f(x_{a,b,c}) \in B \subseteq A$. Now, since f is a Pythagorean neutrosophic $^\star b$-continuous function, we have that $f^{-1}(B)$ is a Pythagorean neutrosophic $^\star b$-open set of X and $x_{a,b,c} \in f^{-1}(f(x_{a,b,c})) \subseteq f^{-1}(B) \subseteq f^{-1}(A)$ and this ends the proof.

(2) $\Rightarrow$ (3): Let $x_{a,b,c}$ be a Pythagorean neutrosophic point of X and let A be a Pythagorean neutrosophic neighbourhood of $f(x_{a,b,c})$. By hypothesis, there exits a Pythagorean neutrosophic $^{\star}b$-open set B of X such that $x_{a,b,c} \in B \subseteq f^{-1}(A)$ and then $x_{a,b,c} \in B$ of X such that $f(B) \subseteq f(f^{-1}(A)) \subseteq A$ and this ends the proof.

(3) $\Rightarrow$ (1): Let B be a Pythagorean neutrosophic open set of Y and let $x_{a,b,c} \in f^{-1}(B)$ and so $f(x_{a,b,c}) \in B$ and then B is a Pythagorean neutrosophic neighbourhood of $f(x_{a,b,c})$. Now, since B is a Pythagorean neutrosophic open set and by hypothesis, there exits a Pythagorean neutrosophic $^{\star}b$-open set A of X such that $x_{a,b,c} \in A$ and $f(A) \subseteq B$. Indeed, $x_{a,b,c} \in A \subseteq f^{-1}(f(A)) \subseteq f^{-1}(B)$ and this implies that $f^{-1}(B)$ is a Pythagorean neutrosophic b-open set of X. Therefore, f is a Pythagorean neutrosophic $^{\star}b$-open continuous function.

Proposition 3.5. *Let $f : (X, \tau) \to (Y, \sigma)$ be a function where (X, τ) and (Y, σ) are Pythagorean neutrosophic topological spaces. If f is a Pythagorean neutrosophic $^{\star}b$-open function, then f is a Pythagorean neutrosophic pre-continuous function.*

Definition 3.6. Let $f : (X, \tau) \to (Y, \sigma)$ be a bijection function where (X, τ) and (Y, σ) are Pythagorean neutrosophic topological spaces. Then, f is said to be Pythagorean nuetrosophic $^{\star}b$-homeomorphism if f and f^{-1} are Pythagorean neutrosophic $^{\star}b$-continuous functions.

Example 3.7. Let $X = \{q, w\}$ and $Y = \{e, r\}$. Then, $\tau = \{0_N, U_1, U_2, 1_N\}$ and $\sigma = \{0_N, V, 1_N\}$ are Pythagorean neutrosophic topological spaces on X and Y respectively, where $U_1 = \langle x, (0.2, 0.4, 0.7), (0.4, 0.4, 0.4)\rangle$, $U_2 = \langle x, (0.3, 0.5, 0.6), (0.5, 0.4, 0.6)\rangle$ and $V = \langle y, (0.3, 0.5, 0.6), (0.5, 0.2, 0.7)\rangle$. Then, we define the function $f : (X, \tau) \to (Y, \sigma)$ as $f(q) = e$ and $f(w) = w$. We can see that f and f^{-1} are Pythagorean neutrosophic $^{\star}b$-continuous and then f is Pythagorean neutrosophic $^{\star}b$-homeomorphism.

Definition 3.8. Let $f : (X, \tau) \to (Y, \sigma)$ be a bijection function where (X, τ) and (Y, σ) are Pythagorean neutrosophic topological spaces. Then, f is said to be Pythagorean nuetrosophic homeomorphism if f and f^{-1} are Pythagorean neutrosophic continuous functions.

Theorem 3.9. *Each Pythagorean neutrosophic homeomorphism is Pythagorean neustrosophic $^{\star}b$-homeomorphism.*

Proof: Let $f : (X, \tau) \to (Y, \sigma)$ be a bijection and Pythagorean neutrosophic homeomorphism function in which f and f^{-1} are Pythagorean neutrosophic continuous functions. Since that every Pythagorean neutrosophic continuous function is Pythagorean neutrosophic $^{\star}b$-continuous, this implies that f and f^{-1} are Pythagorean neutrosophic $^{\star}b$-continuous functions. Therefore, f is a Pythagorean neutrosophic $^{\star}b$-homeomorphism. **Proof**: The following example shows that the converse of the above Theorem need not be true.

Example 3.10. Let $X = \{q, w\}$ and $Y = \{e, r\}$. Then, $\tau = \{0_N, U_1, U_2, 1_N\}$ and $\sigma = \{0_N, V, 1_N\}$ are Pythagorean neutrosophic topological spaces on X and Y respectively, where $U_1 = \langle x, (0.3, 0.5, 0.8), (0.4, 0.4, 0.4)\rangle$, $U_2 = \langle x, (0.1, 0.3, 0.8), (0.1, 0.5, 0.8)\rangle$ and $V = \langle y, (0.4, 0.5, 0.6), (0.1, 0.3, 0.6)\rangle$. Then, we define the function $f : (X, \tau) \to (Y, \sigma)$ as $f(q) = e$ and $f(w) = w$. We can see that f is a Pythagorean neutrosophic b-homeomorphism, but it is not a Pythagorean neutrosophic homeomorphism.

Theorem 3.11. *Let $f : (X, \tau) \to (Y, \sigma)$ be a bijection function where (X, τ) and (Y, σ) are Pythagorean neutrosophic topological spaces. Then, the following statements hold:*

(1) *f is Pythagorean neutrosophic $^\star b$-closed.*
(2) *f is Pythagorean neutrosophic $^\star b$-open.*
(3) *f is Pythagorean neutrosophic $^\star b$-homeomorphism.*

Proof: $(1) \Rightarrow (2)$: Let f be a bijection Pythagorean neutrosophic $^\star b$-closed function. Then, f^{-1} is Pythagorean neutrosophic $^\star b$-continuous function. Now, since every Pythagorean neutrosophic open set of (X, τ) is a Pythagorean neutrosophic b-open set of (X, τ), this implies that f is a Pythagorean neutrosophic $^\star b$-open function.

$(2) \Rightarrow (3)$: Let f be a bijective Pythagorean neutrosophic $^\star b$-open function. Then, f^{-1} is a Pythagorean neutrosophic $^\star b$-continuous function. Indeed, f and f^{-1} are Pythagorean neutrosophic $^\star b$-continuous functions. Therefore, f is a Pythagorean neutrosophic $^\star b$-homeomorphism.

$(3) \Rightarrow (1)$: Let f be a Pythagorean neutrosophic $^\star b$-homeomorphism. Then, f and f^{-1} are Pythagorean neutrosophic $^\star b$-continuous functions. Since every Pythagorean neutrosophic closed set of (X, τ) is a Pythagorean neutrosophic $^\star b$-closed set of (X, τ), this implies that f is a Pythagorean neutrosophic $^\star b$-closed function.

The following example shows that the composition of two Pythagorean neutrosophic $^\star b$-homeomorphisms need not be a Pythagorean neutrosophic $^\star b$-homeomorphism.

Example 3.12. Let $X = \{q, w\}$, $Y = \{e, r\}$ and $Z = \{t, y\}$. Then, $\tau = \{0_N, U, 1_N\}$, $\sigma = \{0_N, V, 1_N\}$ and $\omega = \{0_N, W, 1_N\}$ are Pythagorean neutrosophic topological spaces on X, Y and Z respectively, where $U = \langle x, (0.1, 0.3, 0.5), (0.3, 0.5, 0.7)\rangle$, $V = \langle y, (0.2, 0.7, 0.9), (0.3, 0.6, 0.7)\rangle$ and $W = \langle z, (0.7, 0.5, 0.2), (0.7, 0.7, 0.2)\rangle$. We define the function $f : (X, \tau) \to (Y, \sigma)$ as $f(q) = e$ and $f(w) = r$. Besides, we define the function $g : (Y, \sigma) \to (Z, \omega)$ as $g(e) = t$ and $g(r) = y$. We can see that f and g are Pythagorean neutrosophic $^\star b$-homeomorphism, but $g \circ f$ is not a Pythagorean neutrosophic $^\star b$-homeomorphism.

Definition 3.13. Let $f : (X, \tau) \rightarrow (Y, \sigma)$ be a function where (X, τ) and (Y, σ) are Pythagorean neutrosophic topological spaces. Then, f is said to be Pythagorean neutrosophic $^\star b$-irresolute if $f^{-1}(V)$ is a Pythagorean neutrosophic $^\star b$-open set in X for every Pythagorean neutrosophic $^\star b$-open set V in Y.

Definition 3.14. Let $f : (X, \tau) \rightarrow (Y, \sigma)$ be a bijection function where (X, τ) and (Y, σ) are Pythagorean neutrosophic topological spaces. Then, f is said to be Pythagorean neutrosophic $^\star b$i-homeomorphism if f and f^{-1} are Pythagorean neutrosophic $^\star b$-irresolute functions.

Theorem 3.15. *Every Pythagorean neutrosophic $^\star bi$-homeomorphism is a Pythagorean neutrosophic $^\star b$-homeomorphism.*

Proof: Let $f : (X, \tau) \rightarrow (Y, \sigma)$ be a bijection and Pythagorean neutrosophic $^\star b$i-homeomorphism function. Suppose that B is a Pythagorean neutrosophic closed set of (Y, σ), this implies that B is a Pythagorean neutrosophic $^\star b$-closed set of (Y, σ). Now, since f is Pythagorean neutrosophic irresolute, $f^{-1}(B)$ is a Pythagorean neutrosophic $^\star b$-closed set of (X, τ). Indeed, f is a Pythagorean neutrosophic $^\star b$-continuous function. therefore, f and f^{-1} are Pythagorean neutrosophi $^\star b$-continuous functions and then f is Pythagorean neutrosophic $^\star b$-homeomorphism.

The following example shows that the converse of the above Theorem need not be true.

Example 3.16. Let $X = \{q, w\}$ and $Y = \{e, r\}$. Then, $\tau = \{0_N, U_1, U_2, 1_N\}$ and $\sigma = \{0_N, V, 1_N\}$ are Pythagorean neutrosophic topological spaces on X and Y respectively, where $U_1 = \langle x, (0.2, 0.4, 0.6), (0.3, 0.3, 0.3)\rangle$, $U_2 = \langle x, (0.4, 0.7, 0.9), (0.1, 0.1, 0.3)\rangle$ and $V = \langle y, (0.4, 0.7, 0.9), (0.1, 0.2, 0.3)\rangle$. Then, we define the function $f : (X, \tau) \rightarrow (Y, \sigma)$ as $f(q) = e$ and $f(w) = w$. We can see that f is a Pythagorean neutrosophic $^\star b$-homeomorphism, but it is not a Pythagorean neutrosophic $^\star b$i-homeomorphism.

Theorem 3.17. *If $f : (X, \tau) \rightarrow (Y, \sigma)$ and $g : (Y, \sigma) \rightarrow (Z, \omega$ are Pythagorean neutrosophic $^\star bi$-homeomorphisms, then $g \circ f : (X, \tau) \rightarrow (Z, \omega)$ is a Pythagorean neutrosophic $^\star bi$-homeomorphism.*

Proof: Let f and g be two Pythagorean neutrosophic b-homeomorphisms. Now, suppose that B is a Pythagorean neutrosophic $^\star b$-closed set of (Z, ω), then $g^{-1}(B)$ is a Pythagorean neutrosophic $^\star b$-closed set of (Y, σ). Then by hypothesis, $f^{-1}(g^{-1}(B))$ is a Pythagorean neutrosophic $^\star b$-closed set of (X, τ). Therefore, $g \circ f$ is a Pythagorean neutrosophic $^\star b$-irresolute function Now, let β be a Pythagorean neutrosophic $^\star b$-closed set of (X, τ). By assumption, $f(\beta)$ is a Pythagorean neutrosophic $^\star b$-closed set of (Y, σ). Then, by hypothesis, $g(f(\beta))$ is a Pythagorean neutrosophic $^\star b$-closed set of (Z, ω). This implies that $g \circ f$ is a

Pythagorean neutrosophic $^\star b$-irresolute function and then $g \circ f$ is a Pythagorean neutrosophic $^\star b$i-homeomorphism.

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