On neutrosophic uninorms

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Abstract. Uninorm generalizes the notion of t-norm and t-conorm in fuzzy logic theory. They are three increasing, commutative and associate operators having one neutral element. However, such specific value identifies the kind of operator it is; t-norms have the 1 as neutral element, t-conorms have the 0 and uninorms have every number lying between 0 and 1. Uninorms have been applied as aggregators in many fields of Artificial Intelligence and Decision Making. This theory has also been extended to the framework of interval-valued fuzzy sets, intuitionistic fuzzy sets, interval-valued intuitionistic fuzzy sets and L-fuzzy sets. This paper aims to explore neutrosophic uninorms. We demonstrate that it is possible to define uninorms operators from neutrosophic logic. Additionally, we define neutrosophic implicators induced by neutrosophic uninorms. The combination of both, Neutrosophy and uninorms, enriches the applicability of uninorms operators due to the possibility of incorporating indeterminancy as part of the Neutrosophy contribution.

Keywords: neutrosophic uninorm, uninorm, neutrosophic logic, neutrosophic implicator.

1 Introduction

Uninorms generalize the concepts of t-norm and t-conorm in fuzzy set theory, see [17]. Uninorm operators fulfill commutativity, associativity, increasing monotonicity and the existence of a neutral element *e*, in the same way that t-norm and t-conorm do, see [21]. When *e* is 1, the uninorm is a t-norm, when *e* is 0, it is a t-conorm. The generalization consists in widening to [0, 1] the range of values where the neutral element can lie.

Uninorms are not only used to extend theoretically the other aforementioned fuzzy operators, furthermore we can find in literature many fields where they are applied as aggregators, for example, in expert systems, image processing, neural networks, classifiers, among others, see [4, 10, 13, 16, 19, 22, 27]. Moreover, there exists a fuzzy implicator theory based on uninorms, [7].

G. Deschrijver and E. Kerre in [15], extend fuzzy uninorms concepts to interval-valued fuzzy sets, intuitionistic fuzzy sets, interval-valued intuitionistic fuzzy sets and L-fuzzy sets, see [5-6, 14, 18]. They proved in [14], that these four kind of fuzzy sets are isomorphic each another, therefore, it is sufficient to prove uninorm properties in the framework of the L^*-fuzzy set theory.

On the other hand, "Neutrosophy is a new branch of philosophy that studies the origin, nature, and scope of neutralities, as well as their interactions with different ideational spectra", [23-24, 26]. The novelty of this theory is that it includes for the first time the notion of indeterminacy in fuzzy set theory, that is to say, this approach admits the membership and non membership of elements or objects to a set, akin to intuitionistic fuzzy set theory does, as well as a third function which represents indeterminacy. This theory acknowledges that ignorance, contradiction, paradox and other knowledge representation conditions, which are often considered undesirable from the classic logic viewpoint, also should be taken into account.

Neutrosophy has been applied in wide-ranging kinds of areas, e.g., image processing, decision making, clustering, among others. This is due to the nature of this theory, which allows representing and calculating with indeterminacies.

This paper is devoted to introducing neutrosophic uninorms or N-uninorms, for generalizing uninorm operators to the neutrosopic framework. It is worthily to remark that N-uninorms are used to denote neutrosophic uninorms, not n-uninorms, see [2]. To our knowledge, this seems to be the first approach to neutrosophic uninorms. In neutrosophic logic, neutrosophic norms generalize t-norms and neutrosophic conorms generalize t-conorms, hence, N-uninorms extend fuzzy uninorms, uninorms on L^*-fuzzy sets, n-norms and n-conorms.

N-uninorms could replace fuzzy uninorms in the mathematical models where usually the latter one are employed, because this new approach keeps the advantages of uninorms as an esteemed aggregator, which is here improved with the appropriateness of neutrosophy to deal with human reasoning, knowledge representation, vagueness and uncertainty, when indeterminacy is present.

The present paper is organized as follows; the preliminary definitions and results necessaries to develop our work will be given in Section 2. Section 3 is dedicated to exposing the N-uninorm theory, including N-uninorm implicators. Finally, Section 4 draws the conclusions.

2 Preliminaries

This section is devoted to exposing the preliminary definitions and results necessaries to develop the proposed theory of N-uninorms. The first subsection is dedicated to summarizing the basic definitions and results on uninorms. In the second one we recall the definition and aspects concerning neutrosophic logic theory.

2.1 Basic notions of uninorm theory

Definition 2.1. A *uninorm* is a commutative, associative and increasing mapping $U: [0,1]^2 \to [0,1]$, where there exists $e \in [0,1]$, called neutral element, such that $\forall x \in [0,1], U(e,x) = x$, [17].

If $e = 1$, U is a t-norm and if $e = 0$, U is a t-conorm.

Deschrijver and Kerre in [15] extend this definition to the framework of interval-valued fuzzy sets, intuitionistic fuzzy sets, interval-valued intuitionistic fuzzy sets and L-fuzzy sets, which are pairwise isomorphics, therefore they restrict their theory to the set $L^* = \{(x_1, x_2) \in [0,1]^2 \text{ and } x_1 + x_2 \leq 1\}$.

Let us recall two well-known algebraic definitions that we explicitly write for the sake of being self-contained. They are namely, *Partially Ordered Set* or *poset* and *Lattice*, [1, 9, 20].

Definition 2.2. A *Partially Ordered Set* or *poset* is a pair (P, ≤), where P is a set and ≤ is a binary relation over P, which satisfies for every x, y, z ∈ P, the three following conditions:

1. $x \leq x$ (Reflexive).
2. If $x \leq y$ and $y \leq x$, then $x = y$ (Antisymmetry).
3. If $x \leq y$ and $y \leq z$, then $x \leq z$ (Transitivity).

An *upper bound* of X, $X \subseteq P$, is an element $a \in P$, such that $\forall x \in X$ it holds $x \leq a$. Equivalently, a *lower bound* is an element $b \in P$, such that $\forall x \in X$, $b \leq x$. The *supremum* of X is the least upper bound and the *infimum* is the greater lower bound.

Definition 2.3. A *lattice* $(L, \leq_L)$ is a poset, where every pair of elements x and y in L have an infimum or 'meet', denoted by $x \wedge y$ and a supremum or 'join' denoted by $x \vee y$.

L is a *complete lattice* if every of its subsets has an infimum and a supremum in L.

The lattice $(L^*, \leq_{L^*})$ is defined by the following poset:

$(x_1, x_2) \leq_{L^*} (y_1, y_2) \Leftrightarrow x_1 \leq y_1$ and $x_2 \geq y_2$, $\forall (x_1, x_2), (y_1, y_2) \in L^*$. The units of L^* are $0_{L^*} = (0,1)$ and $1_{L^*} = (1,0)$. See that $x = (x_1, x_2)$ and $y = (y_1, y_2)$ can be incomparable with regard to $\leq_{L^*}$, where either $x_1 < y_1$ and $x_2 < y_2$, or $x_1 > y_1$ and $x_2 > y_2$. It is denoted by $x \parallel_{L^*} y$.

Evidently, $(x_1, x_2) \geq_{L^*} (y_1, y_2)$ if and only if $(y_1, y_2) \leq_{L^*} (x_1, x_2)$. If $(x_1, x_2) \leq_{L^*} (y_1, y_2)$ and $(x_1, x_2) \geq_{L^*} (y_1, y_2)$ then $(x_1, x_2) =_{L^*} (y_1, y_2)$.

Formally, the uninorm on L^* is defined as follows:

Definition 2.4. A *uninorm on L^** is a commutative, associative and increasing mapping $\mathbf{U}: {L^*}^2 \to L^*$, where there exists $e \in L^*$, called neutral element, such that $\forall x \in L^*, \mathbf{U}(e,x) = x$, [15].

Here, if $e = 1_{L^*}$, **U** defines a t-norm on L^* and if $e = 0_{L^*}$, it is a t-conorm on L^*. Nevertheless, the most interesting cases of uninorms are those where e satisfies $0_{L^*} <_{L^*} e <_{L^*} 1_{L^*}$.

In [15] we can find properties and their demonstrations concerning uninorms on L^* that generalize the properties of fuzzy uninorms, including those of the uninorm-based R-implicators and S-implicators. Further, we shall guide the exposition of N-uninorms theory through the theory developed in that paper. Our goal is to prove that N-uninorms extend uninorms on L^*.

2.2 Basic notions of neutrosophic logic

Definition 2.5. Given X, a universe of discourse containing elements or objects. A is a *neutrosophic set* ([25-26]) if it has the form: $A = \{(x: T_A(x), I_A(x), F_A(x)),\ x \in X\}$, where $T_A(x), I_A(x), F_A(x) \subseteq]^-0, 1^+[$, i.e., they are three functions over either the standard or nonstandard subsets of $]^-0, 1^+[$. $T_A(x)$ represents the degree of membership of x to A, $I_A(x)$ represents its degree of indeterminacy and $F_A(x)$ its degree of non-membership. They do not satisfy any restriction, i.e., $\forall x \in X,\ ^-0 \leq \inf T_A(x) + \inf I_A(x) + \inf F_A(x) \leq \sup T_A(x) + \sup I_A(x) + \sup F_A(x) \leq 3^+$.

Another particular definition is that of *Single-valued Neutrosophic set*, which is formally defined as follows:

Definition 2.6. Given X, a universe of discourse which contains elements or objects. A is a *single-valued neutrosophic set* (*SVNS*) [25] if it has the form: $A = \{(x: T_A(x), I_A(x), F_A(x)), x \in X\}$, where $T_A(x), I_A(x), F_A(x) \in [0, 1]$. $T_A(x)$ represents the degree of membership of x to A, $I_A(x)$ represents its degree of indeterminacy and $F_A(x)$ its degree of non-membership. $\forall x \in X, 0 \le T_A(x) + I_A(x) + F_A(x) \le 3$.

See that SVNS is derived from the definition of neutrosophic sets. In the present paper we prefer to use the former one.

In neutrosophic set theory a lattice can be defined as follows:

Given the universe of discourse X and $x(T_x, I_x, F_x)$, $y(T_y, I_y, F_y)$ two SVNS, we say that $x \le_N y$ if and only if $T_x \le T_y$, $I_x \ge I_y$ and $F_x \ge F_y$, $(X, \le_N)$ is a poset. Whereas, $(L, \wedge, \vee)$ is a lattice, because it is a triple direct product of lattices, see [9]. $x \wedge y = (\min\{T_x, T_y\}, \max\{I_x, I_y\}, \max\{F_x, F_y\})$ and $x \vee y = (\max\{T_x, T_y\}, \min\{I_x, I_y\}, \min\{F_x, F_y\})$. Moreover, it is easy to prove that it is complete.

Let us remark that this definition is valid for interval-valued neutrosophic sets, when we substitute their operators by interval-valued operators.

See also that there exist two special elements, viz., $0_N = (0, 1, 1)$ and $1_N = (1, 0, 0)$, which are the infimum and the supremum respectively, of every SVNS with regard to $\le_N$.

Given two neutrosophic sets, A and B, three basic operations over them are the following [25]:

1. $A \cap B = A \wedge B$ (Conjunction).

2. $A \cup B = A \vee B$ (Disjunction).

3. $\bar{A} = (F_A, 1 - I_A, T_A)$ (Complement).

Definition 2.7. A *neutrosophic norm* or *n-norm* N_n [25], is a mapping $N_n: (]^-0, 1^+[\times]^-0, 1^+[\times]^-0, 1^+[)^2 \to]^-0, 1^+[\times]^-0, 1^+[\times]^-0, 1^+[$, such that $N_n\left(x(T_x, I_x, F_x), y(T_y, I_y, F_y)\right) = (N_nT(x, y), N_nI(x, y), N_nF(x, y))$, where N_nT means the degree of membership, N_nI the degree of indeterminacy and N_nF the degree of non-membership of the conjunction of both, x and y.

For every x, y and z belonging to the universe of discourse, N_n must satisfy the following axioms:

1. $N_n(x, 0_N) = 0_N$ and $N_n(x, 1_N) = x$ (Boundary conditions).
2. $N_n(x,y) = N_n(y,x)$ (Commutativity).
3. If $x \le_N y$, then $N_n(x,z) \le_N N_n(y,z)$ (Monotonicity).
4. $N_n(N_n(x,y), z) = N_n(x, N_n(y,z))$ (Associativity).

Definition 2.8. A *neutrosophic conorm* or *n-conorm* N_c [25], is a mapping $N_c: (]^-0, 1^+[\times]^-0, 1^+[\times]^-0, 1^+[)^2 \to]^-0, 1^+[\times]^-0, 1^+[\times]^-0, 1^+[$, such that $N_c\left(x(T_x, I_x, F_x), y(T_y, I_y, F_y)\right) = (N_cT(x, y), N_cI(x, y), N_cF(x, y))$, where N_cT means the degree of membership, N_cI the degree of indeterminacy and N_cF the degree of non-membership of the disjunction of x with y.

For every x, y and z belonging to the universe of discourse, N_c must satisfy the following axioms:

1. $N_c(x, 0_N) = x$ and $N_c(x, 1_N) = 1_N$ (Boundary conditions).
2. $N_c(x,y) = N_c(y,x)$ (Commutativity).
3. If $x \le_N y$, then $N_c(x,z) \le_N N_c(y,z)$ (Monotonicity).
4. $N_c(N_c(x,y),z) = N_c(x, N_c(y,z))$ (Associativity).

According to [8] a Singled-valued neutrosophic negator is defined as follows:

Definition 2.9. a *singled-valued neutrosophic negator* is a decreasing unary neutrosophic operator $N_N: [0, 1]^3 \longrightarrow [0, 1]^3$, satisfying the following boundary conditions:

1. $N_N(0_N) = 1_N$.
2. $N_N(1_N) = 0_N$.

It is called *involutive* if and only if $N_N(N_N(x)) = x$ for every $x \in [0, 1]^3$.

In the following, we show the neutrosophic negators that we shall consider hereunder, extracted from the literature, see [25]. Given a SVNS $A(T_A, I_A, F_A)$, we have:

1. $N_N((T_A, I_A, F_A)) = (1 - T_A, 1 - I_A, 1 - F_A)$, $N_N((T_A, I_A, F_A)) = (1 - T_A, I_A, 1 - F_A)$, $N_N((T_A, I_A, F_A)) = (F_A, I_A, T_A)$and $N_N((T_A, I_A, F_A)) = (F_A, 1 - I_A, T_A)$ (Involutive negators).
2. $N_N((T_A, I_A, F_A)) = \left(F_A, \frac{F_A + I_A + T_A}{3}, T_A\right)$ and $N_N((T_A, I_A, F_A)) = \left(1 - T_A, \frac{F_A + I_A + T_A}{3}, 1 - F_A\right)$ (Non-involutive negators).

In literature, we found neutrosophic implicators, which extend only the notion of S-implications [11]. Moreover, we did not find a general definition on neutrosophic implications except in [8]. In the following, we conclude this section with such definition and properties.

Definition 2.10. A *singled-valued neutrosophic implicator* is an operator $I_N: [0,1]^3 \times [0,1]^3 \to [0,1]^3$ which satisfies the following conditions, for all $x, x', y, y' \in [0,1]^3$:

1. If $x' \leq_N x$, then $I_N(x,y) \leq_N I_N(x',y)$.
2. If $y \leq_N y'$, then $I_N(x,y) \leq_N I_N(x,y')$.
3. $I_N(0_N, 0_N) = I_N(0_N, 1_N) = I_N(1_N, 1_N) = 1_N$.
4. $I_N(1_N, 0_N) = 0_N$.

Herein we use the term neutrosophic implicator or n-implicator to mean singled-valued neutrosophic implicator.

It can satisfy the following properties for every $x, y, z \in [0,1]^3$:

1. $I_N(1_N, x) = x$ (Neutrality principle)
2. $I_N(x,y) = I_N(N_{IN}(y), N_{IN}(x))$, where $N_{IN}(x) = I_N(x, 0_N)$ is an n-negator (Contrapositivity).
3. $I_N(x, I_N(y,z)) = I_N(y, I_N(x,z))$ (Interchangeability principle).
4. $x \leq_N y$ if and only if $I_N(x,y) = 1_N$ (Confinement principle).
5. I_N is a continuous mapping (Continuity).

3 Neutrosophic uninorms

This section is the core of the present paper, because here we explain the neutrosophic uninorm theory. We start defining this concept formally.

3.1 N-uninorms

Definition 3.1. A *neutrosophic uninorm* or *N-uninorm* $\mathbf{U}_N$, is a commutative, increasing and associative mapping, $\mathbf{U}_N: (]^-0,1^+[\times]^-0,1^+[\times]^-0,1^+[)^2 \to]^-0,1^+[\times]^-0,1^+[\times]^-0,1^+[$, such that:
$\mathbf{U}_N\left(x(T_x, I_x, F_x), y(T_y, I_y, F_y)\right) = (\mathbf{U}_N T(x,y), \mathbf{U}_N I(x,y), \mathbf{U}_N F(x,y))$, where $\mathbf{U}_N T$ means the degree of membership, $\mathbf{U}_N I$ the degree of indeterminacy and $\mathbf{U}_N F$ the degree of non-membership of both, x and y. Additionally, there exists a neutral element $e \in]^-0,1^+[\times]^-0,1^+[\times]^-0,1^+[$, where $\forall x \in]^-0,1^+[\times]^-0,1^+[\times]^-0,1^+[$, $\mathbf{U}_N(e, x) = x$.

Remark 3.1. See that Def. 3.1, extends Def. 2.4 in two ways, according to the differences between L* fuzzy sets and neutrosophic sets. First, $\mathbf{U}_N$ includes the third function representing indeterminacy and secondly, there not exists constraints in the relationship among T, I and F. In addition, Def. 3.1 extends Def. 2.7 when $e = 1_N$ and Def 2.8., when $e = 0_N$.

Remark 3.2. For the sake of simplicity, we shall develop the theory only for singled-valued neutrosophic uninorms.

A trivial consequence of Def. 3.1 is that the neutral element is unique, which is a uninorm property in Def. 2.1 and Def. 2.4.

In the following, we explore the formulas of N-uninorms related to those corresponding to n-norms and n-conorms. For this end, first we need to describe two kinds of sets, namely, $E_1 = \{x \in [0,1]^3 : x \leq_N e\}$ and $E_2 = \{x \in [0,1]^3 : x \geq_N e\}$.

Lemma 3.1. Let $e \in]0,1] \times [0,1[\times [0,1[$. The mapping $\phi_e: [0,1]^3 \to [0,1]^3$, defined by:

$$\phi_e(x) = \left(e_1 x_1, x_2 + e_2(1 - x_2), x_3 + e_3(1 - x_3)\right) \quad (1)$$

for every $x \in [0,1]^3$ is an increasing bijection from $[0,1]^3$ to E_1 and ϕ_e^{-1} is increasing as well.

Proof. To prove ϕ_e is injective, let $x, y \in [0,1]^3$ and suppose $\phi_e(x) = \phi_e(y)$.Then, clearly the equation $\left(e_1 x_1, x_2 + e_2(1 - x_2), x_3 + e_3(1 - x_3)\right) = \left(e_1 y_1, y_2 + e_2(1 - y_2), y_3 + e_3(1 - y_3)\right)$ is fulfilled only if x = y, and the injection is proved, also taking into account that we excluded the cases $e_1 = 0$, $e_2 = 1$ and $e_3 = 1$.

Let us take any $y \in E_1$ and define $x = (x_1, x_2, x_3)$, such that $x_1 = \frac{y_1}{e_1}$, $x_2 = \frac{y_2 - e_2}{1 - e_2}$ and $x_3 = \frac{y_3 - e_3}{1 - e_3}$. Then, $\phi_e(x) = y$ and $x_1, x_2, x_3 \in [0,1]$, which can be proved applying $y \leq_N e$. Therefore, ϕ_e is surjective and evidently it is increasing. The equation of the inverse is the following:

$$\phi_e^{-1}(x) = \left(\frac{x_1}{e_1}, \frac{x_2 - e_2}{1 - e_2}, \frac{x_3 - e_3}{1 - e_3}\right) \quad (2)$$

□

Lemma 3.2. Let $e \in [0,1[\times]0,1]\times]0,1]$. The mapping $\psi_e\colon [0,1]^3 \to [0,1]^3$, defined by:

$$\psi_e(x) = (e_1 + x_1 - e_1x_1, e_2x_2, e_3x_3) \quad (3)$$

for every $x \in [0,1]^3$ is an increasing bijection from $[0,1]^3$ to E_2 as well as ψ_e^{-1} is increasing.

Proof. This lemma can be proved similarly to the proof carried out in the Lemma 3.1. The equation of the inverse is as follows:

$$\psi_e^{-1}(x) = \left(\frac{x_1 - e_1}{1 - e_1}, \frac{x_2}{e_2}, \frac{x_3}{e_3}\right) \quad (4)$$

□

Theorem 3.3. Given $\mathbf{U}_N$ an N-uninorm with neutral element $e \in]0,\ 1[^3$. Then the following two conditions are satisfied:

i. The mapping $N_{n,\mathbf{U}_N}\colon [0,1]^3 \times [0,1]^3 \to [0,1]^3$ defined for all $x, y \in [0,1]^3$ by the equation:

$$N_{n,\mathbf{U}_N}(x,y) = \phi_e^{-1}\left(\mathbf{U}_N\big(\phi_e(x), \phi_e(y)\big)\right) \quad (5)$$

is an n-norm.

ii. The mapping $N_{c,\mathbf{U}_N}\colon [0,1]^3 \times [0,1]^3 \to [0,1]^3$ defined for all $x, y \in [0,1]^3$ by the equation:

$$N_{c,\mathbf{U}_N}(x,y) = \psi_e^{-1}\left(\mathbf{U}_N\big(\psi_e(x), \psi_e(y)\big)\right) \quad (6)$$

is an n-conorm.

Proof. This theorem is a consequence of Lemmas 3.1 and 3.2.□

Remark 3.3. Some cases of e were excluded in Lemmas 3.1, 3.2 and Theorem 3.3, for instance, $e = (0, \beta, \gamma)$, where $0 \le \beta, \gamma \le 1$ in Lemma 3.1. It is easy to prove that when e is one of them, there not exist any increasing bijection from $[0,1]^3$ to E_1 or E_2, because E_1 or E_2 have one constant component, and therefore they only depend on at most two components, however, $[0,1]^3$ depends on three, and that contradicts the injection. For example, if $e = (0, \beta, \gamma)$, then $E_1 = \{0\} \times [\beta, 1] \times [\gamma, 1]$, and there not exists a bijective mapping from $[0,1]^3$ to E_1.

Corollary 3.4. Given $\mathbf{U}_N$ an N-uninorm with neutral element $e \in]0,\ 1[^3$. Then the following two conditions are satisfied:

i. For every $x, y \in E_1$, $\mathbf{U}_N(x,y) = \phi_e\left(N_{n,\mathbf{U}_N}\big(\phi_e^{-1}(x), \phi_e^{-1}(y)\big)\right)$.

ii. For every $x, y \in E_2$, $\mathbf{U}_N(x,y) = \psi_e\left(N_{c,\mathbf{U}_N}\big(\psi_e^{-1}(x), \psi_e^{-1}(y)\big)\right)$.

Proof. The proof is obtained immediately from Theorem 3.3.□

Remark 3.4. See that Theorem 3.3 and Corollary 3.4 mean that we can define N-uninorms from n-norms and n-conorms, and vice versa.

Remark 3.5. Comparing the precedent issues with their similar ones appeared in [15], we can find few differences and numerous similarities. Indeed, so far we have proved that N-uninorms extend the approach to structures of uninorms on L^* fuzzy sets, which is valid to interval-valued fuzzy sets, intuitionistic fuzzy sets, interval-valued intuitionistic fuzzy sets and Goguen's L-fuzzy sets.

Definition 3.2. We say that $N_n(x,y)$ is an *Archimedean n-norm* respect to $<_N$ if for every $x \in [0,1]^3$it satisfies: $N_n(x,x) <_N x$.

Definition 3.3. We say that $N_c(x,y)$ is an *Archimedean n-conorm* respect to $<_N$ if for every $x \in [0,1]^3$it satisfies: $N_c(x,x) >_N x$.

Definition 3.4. $\mathbf{U}_N(x,y)$ is an *Archimedean N-uninorm* respect to $<_N$ if it satisfies the following conditions:

1. $\mathbf{U}_N(x,x) <_N x$ for every $0 <_N x <_N e$.
2. $\mathbf{U}_N(x,x) >_N x$ for every $e <_N x <_N 1_N$.

Proposition 3.5. Given $\mathbf{U}_N$ an N-uninorm with neutral element $e \in]0,\ 1[^3$. It is Archimedean if and only if the n-norm and n-conorm defined in Eq. 5 and 6, respectively, are Archimedean.

Proof Let $0 <_N x <_N e$, and $\mathbf{U}_N(x,y)$ an Archimedean N-uninorm, i.e., $\mathbf{U}_N(x,x) <_N x$, then taking into account that ϕ_e and ϕ_e^{-1} are increasing bijections, we have $N_{n,\mathbf{U}_N}(x,x) = \phi_e^{-1}\left(\mathbf{U}_N\big(\phi_e(x), \phi_e(x)\big)\right) <_N \phi_e^{-1}\big(\phi_e(x)\big) = x$. Equivalently, it is easy to prove that $N_{n,\mathbf{U}_N}(x,x) <_N x$ implies $\mathbf{U}_N(x,x) <_N x$. The proof for the n-conorm is similar.□

Proposition 3.6. Given $\mathbf{U}_N$ an N-uninorm with neutral element e, and $x, y \in [0,1]^3$ are two elements such that either $x \leq_N e \leq_N y$ or $y \leq_N e \leq_N x$, then the following two inequalities hold:

$\min(x,y) \leq_N \mathbf{U}_N(x,y) \leq_N \max(x,y)$.

Proof. Without loss of generality, suppose $x \leq_N e \leq_N y$, then because of the monotonicity of the N-uninorms $\mathbf{U}_N(x,y) \leq_N \mathbf{U}_N(e,y) = y = \max(x,y)$ and $\mathbf{U}_N(x,y) \geq_N \mathbf{U}_N(x,e) = x = \min(x,y)$.□

The proposition above means that there exists a domain where $\mathbf{U}_N$ is compensatory with regard to $\leq_N$.

Let us note that there exists other sets where $x \parallel_{\leq_N} y$ or $x \parallel_{\leq_N} e$.

Example 3.1. Two examples of N-uninorms are the following:

Recalling the well-known weakest and strongest fuzzy uninorms, respectively, defined as follows:

$$\underline{U}_{e_1}(x_1,y_1) := \begin{cases} 0 & \text{if } 0 \leq x_1, y_1 < e_1 \\ \max\{x_1,y_1\} & \text{if } e_1 \leq x_1, y_1 \leq 1 \\ \min\{x_1,y_1\} & \text{otherwise} \end{cases} \text{ and } \overline{U}_{e_1}(x_1,y_1) := \begin{cases} \min\{x_1,y_1\} & \text{if } 0 \leq x_1, y_1 \leq e_1 \\ 1 & \text{if } e_1 < x_1, y_1 \leq 1 \\ \max\{x_1,y_1\} & \text{otherwise} \end{cases}$$

For every $x_1, y_1 \in [0,1]$ and $e_1 \in]0,1[$.

Let us define two N-uninorms as follows: for every $x, y \in [0,1]^3$ and $e \in [0,1]^3$ is the neutral element:

$$\underline{\mathbf{U}}_e(x,y) := \left(\underline{U}_{e_1}(x_1,y_1), \overline{U}_{e_2}(x_2,y_2), \overline{U}_{e_3}(x_3,y_3)\right) \quad (7)$$

and

$$\overline{\mathbf{U}}_e(x,y) := \left(\overline{U}_{e_1}(x_1,y_1), \underline{U}_{e_2}(x_2,y_2), \underline{U}_{e_3}(x_3,y_3)\right) \quad (8)$$

Both $\underline{\mathbf{U}}_e(x,y)$ and $\overline{\mathbf{U}}_e(x,y)$, are N-uninorms, because every one of the components are uninorms, thus, they are commutative, associative and increasing. The neutral element components are formed by the neutral elements of every individual uninorm.

Moreover, $\underline{\mathbf{U}}_e(x,y)$ is a conjunctive N-uninorm and $\overline{\mathbf{U}}_e(x,y)$ is a disjunctive N-uninorm, i.e., $\underline{\mathbf{U}}_e(0_N, 1_N) = 0_N$ and $\overline{\mathbf{U}}_e(0_N, 1_N) = 1_N$.

See that $\mathbf{U}_e(x,y) = \left(\underline{U}_{e_1}(x_1,y_1), \underline{U}_{e_2}(x_2,y_2), \underline{U}_{e_3}(x_3,y_3)\right)$ is also an N-uninorm, nevertheless, it is neither conjunctive nor disjunctive, $\mathbf{U}_e(0_N, 1_N) = (0,0,0)$.

Definition 3.5. An N-uninorm $\mathbf{U}_N$ is said to be *t-representable* if there exist three fuzzy uninorms, $U_{e_1}(x_1,y_1)$, $U_{e_2}(x_2,y_2)$ and $U_{e_3}(x_3,y_3)$, such that for all $x = (x_1,x_2,x_3)$ and $y = (y_1,y_2,y_3)$ it has the form $\mathbf{U}_N(x,y) = \left(U_{e_1}(x_1,y_1), U_{e_2}(x_2,y_2), U_{e_3}(x_3,y_3)\right)$.

Proposition 3.7. Let $\mathbf{U}_N$ be an N-uninorm with neutral element e and $x \in [0,1]^3$, then the following properties hold:

i. $\mathbf{U}_N(0_N, 0_N) = 0_N$ and $\mathbf{U}_N(1_N, 1_N) = 1_N$.
ii. If $e \in [0,1]^3 \setminus \{0_N, 1_N\}$, we have $\mathbf{U}_N(0_N, 1_N) = \mathbf{U}_N(\mathbf{U}_N(0_N, 1_N), x)$, for every $x \in [0,1]^3$.
iii. If $e \in [0,1]^3 \setminus \{0_N, 1_N\}$, then either $\mathbf{U}_N(0_N, 1_N) = 0_N$ or $\mathbf{U}_N(0_N, 1_N) = 1_N$ or $\mathbf{U}_N(0_N, 1_N) \parallel_{\leq_N} e$.

Proof.

i. See that $\mathbf{U}_N(e, 0_N) = 0_N$, $\mathbf{U}_N(e, 1_N) = 1_N$ and apply the increasing axiom of N-uninorm.

ii. If $x \leq_N e$ then because $\mathbf{U}_N$ is increasing, we have $\mathbf{U}_N(0_N, x) \leq_N \mathbf{U}_N(0_N, e) = 0_N$, thus, $\mathbf{U}_N(0_N, x) = 0_N$ and $\mathbf{U}_N(0_N, 1_N) = \mathbf{U}_N(\mathbf{U}_N(0_N, x), 1_N)$. Because of the commutativity and the associativity, $\mathbf{U}_N(0_N, 1_N) = \mathbf{U}_N(\mathbf{U}_N(0_N, 1_N), x)$.

If $x \geq_N e$ then $\mathbf{U}_N(1_N, x) \geq_N \mathbf{U}_N(1_N, e) = 1_N$ and therefore, $\mathbf{U}_N(1_N, x) = 1_N$. $\mathbf{U}_N(0_N, 1_N) = \mathbf{U}_N(0_N, \mathbf{U}_N(1_N, x))$, and finally due to the commutativity and associativity, we obtain $\mathbf{U}_N(0_N, 1_N) = \mathbf{U}_N(\mathbf{U}_N(0_N, 1_N), x)$.

If $x \parallel_{\leq_N} e$ then $x \wedge e \leq_N x \leq_N x \vee e$. We have $x \wedge e \leq_N e$ and $e \leq_N x \vee e$, thus according to the precedent results $\mathbf{U}_N(0_N, 1_N) = \mathbf{U}_N(\mathbf{U}_N(0_N, 1_N), x \wedge e) = \mathbf{U}_N(\mathbf{U}_N(0_N, 1_N), x \vee e)$. Applying the increasing axiom of N-uninorms we obtain $\mathbf{U}_N(0_N, 1_N) = \mathbf{U}_N(\mathbf{U}_N(0_N, 1_N), x)$.

iii. Suppose $\mathbf{U}_N(0_N, 1_N) \nparallel_{\leq_N} e$, that implies either $\mathbf{U}_N(0_N, 1_N) \leq_N e$ or $e \leq_N \mathbf{U}_N(0_N, 1_N)$.

If $\mathbf{U}_N(0_N, 1_N) \leq_N e$, then $\mathbf{U}_N(0_N, 1_N) = \mathbf{U}_N(\mathbf{U}_N(0_N, 1_N), 0_N) = 0_N$, according to ii.

If $\mathbf{U}_N(0_N, 1_N) \geq_N e$, then $\mathbf{U}_N(0_N, 1_N) = \mathbf{U}_N(\mathbf{U}_N(0_N, 1_N), 1_N) = 1_N$, according to ii.□

Let us note that the precedent issues are similar to the ones obtained in [15].

3.2 Implicators induced by N-uninorms

This subsection is dedicated to explore the notion of n-implicators induced by N-uninorms. First of all we define the concept of neutrosophic R-implicator, which is new in this framework, at least in the scope of our knowledge.

Definition 3.6. A *neutrosophic R- implicator* or *n-R-implicator* is an n-implicator defined as follows:

Given N_n an n-norm, for every $x, y \in [0,1]^3$, $RI_N(x,y) = \sup\{t \in [0,1]^3 : N_n(x,t) \leq_N y\}$.

Let us note that this definition extends both, the definition of fuzzy R-implicator, see [7], and that of L* fuzzy implicator, [15]. As well as others appeared in [3, 12].

Indeed, it is an actual n-implicator. Taking into account the properties of $\leq_N$, and the increasing property of n-norms with regard to $\leq_N$, we have that $RI_N(x,\cdot)$ is decreasing and $RI_N(\cdot,y)$ is increasing. Additionally, the satisfaction of the boundary conditions by RI_N can be verified straightforwardly.

Example 3.2. Let $a = (0.6,\ 0.2,\ 0.4)$, $b = (0.7,\ 0.1,\ 0.3)$ and $c = (0.5,\ 0.3,\ 0.5)$ be three SVNS. Observe that $c \leq_N a \leq_N b$. Consider the n-norm, $N_{n-\min}(x,y) = (\min\{T_x, T_y\}, \max\{I_x, I_y\}, \max\{F_x, F_y\})$.

Then, $RI_N(a,b) = 1_N$, $RI_N(a,c) = (0.5,\ 0.3,\ 0.5)$, $RI_N(b,a) = (0.6, 0.2, 0.4)$ and $RI_N(c,a) = 1_N$. See that $RI_N(a,c) \leq_N RI_N(a,b)$ and $RI_N(b,a) \leq_N RI_N(c,a)$.

Proposition 3.8. Let RI_N be an n-R-implicator induced by the n-norm N_n, then the two following properties hold:

i. $RI_N(1_N, y) = y$ for every $y \in [0,1]^3$ (Neutrality principle).
ii. $RI_N(x,x) = 1_N$ for every $x \in [0,1]^3$ (Identity principle).
iii. $x, y \in [0,1]^3$ and $x \leq_N y$ if and only if $RI_N(x,y) = 1_N$ (Confinement principle).

Proof.

i. For $y \in [0,1]^3$, we have $RI_N(1_N, y) = \sup\{t \in [0,1]^3 : N_n(1_N, t) = t \leq_N y\} = y$.
ii. For $x \in [0,1]^3$, we have $RI_N(x,x) = \sup\{t \in [0,1]^3 : N_n(x,t) \leq_N x\} = 1_N$, because N_n is increasing and $N_n(x, 1_N) = x$.
iii. For $x, y \in [0,1]^3$ and $x \leq_N y$, taking into account the inequalities $N_n(x,t) \leq_N N_n(x,1_N) = x \leq_N y$ for every $t \in [0,1]^3$, we have $RI_N(x,y) = 1_N$. On the other hand, $RI_N(x,y) = 1_N$ evidently implies $x \leq_N y$, from the definition.

Theorem 3.9. Let $\mathbf{U}_N$ be an N-uninorm with neutral element $e \in]0,\ 1[^3$. Let us establish the mapping $RI_{\mathbf{U}_N} : [0,1]^3 \times [0,1]^3 \to [0,1]^3$ defined as follows:

$RI_{\mathbf{U}_N}(x,y) = \sup\{t \in [0,1]^3 : \mathbf{U}_N(x,t) \leq_N y\}$ for every $x, y \in [0,1]^3$.

It is an n-implicator if and only if there exists $\tilde{x} >_N 0_N$ such that every $x \geq_N \tilde{x}$ satisfies $\mathbf{U}_N(0_N, x) = 0_N$.

Proof. It is easy to verify that $RI_{\mathbf{U}_N}(x,\cdot)$ is decreasing and $RI_{\mathbf{U}_N}(\cdot,y)$ is increasing.

On the other hand, $RI_{\mathbf{U}_N}(0_N, 1_N) = RI_{\mathbf{U}_N}(1_N, 1_N) = 1_N$, because $\mathbf{U}_N$ is increasing and 1_N is the supremum.

See that for every $t \in [0,1]^3$, $\mathbf{U}_N(1_N, t) \geq_N \mathbf{U}_N(e, t) = t$, then $\mathbf{U}_N(1_N, t) >_N 0_N$ if and only if $t >_N 0_N$, therefore $RI_{\mathbf{U}_N}(1_N, 0_N) = 0_N$.

Additionally, if there exists $\tilde{x} >_N 0_N$ such that every $x \geq_N \tilde{x}$ satisfies $\mathbf{U}_N(0_N, x) = 0_N$, then because $\mathbf{U}_N$ is increasing and 1_N is the supremum of that set, $\mathbf{U}_N(0_N, 1_N) = 0_N$ and $RI_{\mathbf{U}_N}(0_N, 0_N) = 1_N$.□

Remark 3.6. The Theorem 3.9 is valid when $\mathbf{U}_N$ is a conjuctive N-uninorm.

Example 3.3. Given again $a = (0.6,\ 0.2,\ 0.4)$, $b = (0.7,\ 0.1,\ 0.3)$ and $c = (0.5,\ 0.3,\ 0.5)$, three SVNS, as in Example 3.2. Let us consider $\underline{\mathbf{U}}_e$ of the Example 3.1, where $e = (0.5, 0.5, 0.5)$. Recall that $\underline{\mathbf{U}}_e(0_N, 1_N) = 0_N$. Then, $RI_{\underline{\mathbf{U}}_e}(a,b) = (0.7, 0.1, 0.3)$, $RI_{\underline{\mathbf{U}}_e}(a,c) = (0.5, 0.5, 0.5)$, $RI_{\underline{\mathbf{U}}_e}(b,a) = (0.5, 0.5, 0.5)$ and $RI_{\underline{\mathbf{U}}_e}(c,a) = (0.6, 0.2, 0.4)$.

Proposition 3.10. Given $\mathbf{U}_N$ an N-uninorm with $e \in [0,1]^3 \setminus \{0_N, 1_N\}$. Then, $RI_{\mathbf{U}_N}(e, x) = x$, for every $x \in [0,1]^3$.

Proof. Let us fix $x \in [0,1]^3$, $RI_{\mathbf{U}_N}(e, x) = \sup\{t \in [0,1]^3 : \mathbf{U}_N(e,t) = t \leq_N x\} = x$.□

Proposition 3.11. Given $\mathbf{U}_N$ an N-uninorm with $e \in [0,1]^3 \setminus \{0_N, 1_N\}$. $RI_{\mathbf{U}_N}(x, 1_N) = 1_N$, for every $x \in [0,1]^3$ (Right boundary condition).

Proof. Taking into account $\mathbf{U}_N$ is increasing and 1_N is the supremum of the elements of the lattice, then, $RI_{\mathbf{U}_N}(x, 1_N) = \sup\{t \in [0,1]^3 : \mathbf{U}_N(x,t) \leq_N 1_N\} = 1_N$.□

Proposition 3.12. Given $\mathbf{U}_N$ an N-uninorm with $e \in [0,1]^3 \setminus \{0_N, 1_N\}$. If it is contrapositive respect to a negator N_N, which satisfies $N_N(e) = e$, then $N_N(x) = N_{NI_{\mathbf{U}_N}}(x) = RI_{\mathbf{U}_N}(x, e)$ for every $x \in [0,1]^3$ and $N_{NI_{\mathbf{U}_N}}$ is involutive.

Proof. Reproduce the similar proof in [15] adapted to N-uninorms.□

Proposition 3.13. Given $\mathbf{U}_N$ an N-uninorm and N_N an n-negator. The mapping $SI_{\mathbf{U}_N}(x,y) = \mathbf{U}_N(N_N(x), y)$ is an n-implicator if and only if $\mathbf{U}_N$ is disjunctive.

Proof. Reproduce the similar proof in [15] adapted to N-uninorms.□

Example 3.4. Revisiting Examples 3.2 and 3.3, where $a = (0.6,\ 0.2,\ 0.4)$, $b = (0.7,\ 0.1,\ 0.3)$ and $c = (0.5,\ 0.3,\ 0.5)$. Now we consider the n-negator $N_N\big((T_x, I_x, F_x)\big) = (F_x, I_x, T_x)$ and from the Example 3.1, $\overline{\mathbf{U}}_e(x,y)$ with $e = (0.5, 0.5, 0.5)$. There, we proved it is disjunctive.

Then, we have $SI_{\overline{\mathbf{U}}_e}(a,b) = (0.7, 0, 0.3)$, $SI_{\overline{\mathbf{U}}_e}(a,c) = (0.4,\ 0, 0.6)$, $SI_{\overline{\mathbf{U}}_e}(b,a) = (0.6,\ 0, 0.4)$ and $SI_{\overline{\mathbf{U}}_e}(c,a) = (0.6, 0, 0.4)$.

Proposition 3.14. Given $\mathbf{U}_N$ an N-uninorm and N_N an n-negator. The mapping $SI_{\mathbf{U}_N}$ satisfies the Interchangeability Principle:

$SI_{\mathbf{U}_N}\left(x, SI_{\mathbf{U}_N}(y,z)\right) = SI_{\mathbf{U}_N}\left(y, SI_{\mathbf{U}_N}(x,z)\right)$ for every $x, y, z \in [0,1]^3$.

Proof. It is proved by using the commutativity and associativity of N-uninorms.□

Conclusion

The proposed paper was devoted to define and study a new operator called neutrosophic uninorm or N-uninorm. We demonstrated that it is possible to extend the notion of uninorm to the framework of neutrosophy logic theory. In addition, we defined new neutrosophic implicators induced by N-uninorms. Moreover, we introduced a new neutrosophic implicator which generalizes the fuzzy notion of R-implicator. The importance of this new theory is that the appreciated quality of fuzzy uninorms as aggregators is enriched with the capacity of neutrosophy to deal with indeterminacy.

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Received: May 1, 2021. Accepted: August 30, 2021