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A Note on Neutrosophic Bitopological Spaces

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Abstract: In this paper, we have introduced the idea on neutrosophic bitopological space and studied its properties with examples. We have defined several definitions of neutrosophic interior, closure and boundary also we have studied all of its properties.
Keywords: Neutrosophic Closed set; Neutrosophic Open set; (τ_i, τ_j) - N-Interior; (τ_i, τ_j) - N-Closure ; (τ_i, τ_j)- N- Boundary; Neutrosophic Bitopological Space.

1. Introduction

In 1995 neutrosophic set has been proposed by F. Smarandache [3, 4] as a new branch of philosophy dealing with ancient roots, origin, nature and scope of neutralities as well as their interactions with different ideational spectra. The term "neutron-sophy" means knowledge of neutral thoughts with natural represents the main distinction between fuzzy set and intuitionistic fuzzy set.

In 1965, L. A. Zadeh defined the concept of membership function and discovered the fuzzy set [1]. With the help of fuzzy set [1] Zadeh explained the idea of uncertainty. In 1989, K. T. Atanassov [2] generalized the concepts of fuzzy set and introduced the degree of non-membership as an independent component and proposed the intuitionistic fuzzy set.

After the introduction of fuzzy sets, several researches were conducted on the generalizations of the notions of fuzzy set. After the generalization of fuzzy sets, many researchers have applied generalization of fuzzy set theory in many branches of science and technology. Chang [5] introduced fuzzy topology. Coker (1997) defined the notion of intuitionistic fuzzy topological spaces. In 1963, J.C. Kely [12] defined the study of Bitopological spaces. A. Kandil et al.[13] discussed on fuzzy bitopological spaces. Lee et al. [14] discussed on some properties of Intuitionistic Fuzzy Bitopological Spaces. Now a day many researchers have studied topology on neutrosophic sets, such as Lupianez [7–10] and Salama [11]. Abdel-Baset et al. [17] discussed on Hybride plitogenic decision-making approach with quality function deployment for selecting supply chain sustainability metrics. Recently Abdel-Baset et al. [18] studied on Novel plithogenic TOPSIS-CRITIC model for sustainable supply chain risk management.

In this paper, we introduce the concept of Netrosophic Bitopological Spaces. Next, we introduce the concepts of neutrosophic interior set, neutrosophic closure set and neutrosophic boundary set. Also, we have discussed some propositions related to neutrosophic interior set, neutrosophic closure set and neutrosophic boundary set.

2. Basic operations

Definition 2.1 [20] A neutrosophic set A on the universe of discourse X is defined as

$A = \{< x, \mu_{A,}\ \sigma_A,\ \gamma_A >: x \epsilon X\}$

Where $\mu_{A,}\ \sigma_A,\ \gamma_A : X \longrightarrow]0^-, 1^+[$and $0^- \leq \mu_A + \sigma_A + \gamma_A \leq 3^+$, μ_A represents degrees of membership function, $\sigma_A$is the degree of indeterminacy and γ_A is the degree of non-membership function.

Let $= \{< x, \mu_{A,}\ \sigma_A,\ \gamma_A >: x \epsilon X\}$ and $B = \{< x, \mu_{B,}\ \sigma_B,\ \gamma_B >: x \epsilon X\}$ be two neutrosophic sets on X. Then

i. Neutrosophic subset: A≤B if $\mu_A \leq \mu_B$, $\sigma_A \geq \sigma_B$ and $\gamma_A \geq \gamma_B$, That is A is neutrosophic subset of B
ii. Neutrosophic equality: If A ≤B and A ≥B then A=B
iii. Neutrosophic intersection : A∧B = $\{< x, \mu_A \wedge \mu_A,\ \sigma_A \vee \sigma_A,\ \gamma_A \vee \gamma_B >: x \epsilon X\}$
iv. Neutrosophic union: A∨B = $\{< x, \mu_A \vee \mu_A,\ \sigma_A \wedge \sigma_A,\ \gamma_A \wedge \gamma_B >: x \epsilon X\}$
v. Neutrosophic complement: $A^C = \{< x,\ \gamma_A,\ 1 - \sigma_A,\ \mu_{A,}\ >: x \epsilon X\}$
vi. Neutrosophic universal set: $1_X = \{< x,\ 1,\ 0,\ 0 >: x \epsilon X\}$
vii. Neutrosophic empty set: $0_X = \{< x,\ 0,\ 1,\ 1 >: x \epsilon X\}$

Theorem 2.1 [20] Let A and B be two neutrosophic sets on X then

i. A∨A= A and A∧A=A
ii. A∨B= B∨A and A∧B = B ∧ A
iii. A ∨ 0_X= A and A ∨ 1_X=1_X
iv. A∧ 0_X= 0_X and A∧ 1=A
v. A ∨ (B∨C)= (A∨B) ∨C and A ∧ (B∧C)= (A∧B) ∧C
vi. $(A^C)^C$=A

Theorem 2.2 [20] Let A and B be two neutrosophic sets on X then De Morgan's law is valid.

i. $[\vee_{i\in I} A_{i\in I}]^C = \wedge_{i\in I} {A_i}^C$
ii. $[\wedge_{i\in I} A_i]^C = \vee_{i\in I} {A_i}^C$

Definition 2.2 [7] Neutrosophic topological spaces

Let τ be a collection of all neutrosophic subsets on X. Then τ is called a neutrosophic topology in X if the following conditions hold

i. 0_X and 1_X is belong to τ.
ii. Union of any number of neutrosophic sets in τ is again belong to τ.
iii. Intersection of any two neutrosophic set in τ is belong to τ.

Then the pair (X, τ) is called neutrosophic topology on X.

Definition 2.3 [7, 8, 9]

Let (X, τ) be a neutrosophic topological space over X and A is neutrosophic subset on X. Then, the neutrosophic interior of A is the union of all neutrosophic open subsets of A. Clearly neutrosophic interior of A is the biggest neutrosophic open set over X which containing A.

Definition 2.4 [7, 8, 9]

Let (X, τ) be a neutrosophic topological space over X and A is neutrosophic subset on X. Then, the neutrosophic closure of A is the intersection of all neutrosophic closed super sets of A. Clearly neutrosophic closure of A is the smallest neutrosophic closed set over X which contains A.

3. Main Results

Definition 3.1

A system (X, τ_i, τ_j) consisting of a set X with two neutrosophic topologies τ_i and τ_j on X is called Neutrosophic Bitopological space. Throughout in this paper the indices i, j take the value $\in \{1, 2\}$ and $i \neq j$.

Example 3.1

Let X= {a,b} and $A = \{< a, 0.5, 0.5, 0.5 >, < b, 0.4, 0.4, 0.4 >\}$,
$B = \{< a, 0.6, 0.6, 0.6 >, < b, 0.3, 0.3, 0.3 >\}$, $C = \{< a, 0.6, 0.6, 0.6 >, < b, 0.2, 0.2, 0.2 >\}$,
$D = \{< a, 0.7, 0.7, 0.7 >, < b, 0.4, 0.4, 0.4 >\}$. Then $\tau_1 = \{0_X,\ 1_X,\ A,\ B, A \wedge B, A \vee B\}$ and $\tau_2 = \{0_X,\ 1_X,$ $C,\ D, C \wedge D, C \vee D\}$ then (X, τ_1, τ_2) is neutrosophic bitopological space

Definition 3.2

Let (X, τ_i, τ_j) be a neutroscopic bitopological space. Then for a set $A = \{< x, \mu_{ij}, \ \sigma_{ij},\ \gamma_{ij} >: x \epsilon X\}$, neutrosophic (τ_i, τ_j)- N-interior of A is the union of all (τ_i, τ_j)-N-open sets of X contained in A and is defined as follows

(τ_i, τ_j)-N-Int(A) = $\{< x, \vee_{\tau_i} \vee_{\tau_j} \mu_{ij},\ \wedge_{\tau_i} \wedge_{\tau_j} \sigma_{ij},\ \wedge_{\tau_i} \wedge_{\tau_j} \gamma_{ij} >: x \epsilon X\}$

Note : Here $\mu_{ij,}$ represents degrees of membership function, σ_{ij} is the degree of indeterminacy and γ_{ij} is the degree of non-membership function of a neutrosophic set and i is interrelated with neutrosophic topology τ_i and j is interrelated with neutrosophic topologie τ_j when we discussed on (τ_i, τ_j)-N-Int(A).

Example 3.2

Let X={a, b} and $A = \{< a, 0.5, 0.5, 0.5 >, < b, 0.4, 0.4, 0.4 >\}$, $B = \{< a, 0.6, 0.6, 0.6 >, < b, 0.3, 0.3, 0.3 >\}$, $C = \{< a, 0.6, 0.6, 0.6 >, < b, 0.2, 0.2, 0.2 >\}$, $D = \{< a, 0.7, 0.7, 0.7 >, < b, 0.4, 0.4, 0.4 >\}$. Then $\tau_1 = \{0_X,\ 1_X,\ A,\ B, A \wedge B, A \vee B\}$ and $\tau_2 = \{0_X,\ 1_X,\ C,\ D, C \wedge D, C \vee D\}$ then (X, τ_1, τ_2) is neutrosophic bitopological space
Let $Q = \{< a, 0.6, 0.4, 0.4 >, < b, 0.3, 0.3, 0.4 >\}$
τ_2-N-Int (Q) = 0_X and τ_1-N-Int (0_X) =0_X
Hence (τ_1, τ_2)-N-Int(Q)= 0_X

Theorem 3.1

Let (X, τ_i, τ_j) be neutrosophic bitopological space then

i. (τ_i, τ_j)-N-Int(0_X) = 0_X, (τ_i, τ_j)-N-Int(1_X) = 1_X
ii. (τ_i, τ_j)-N-Int(A)≤ A.
iii. A is neutroscopic open set if and only if A=(τ_i, τ_j)-N-Int(A)
iv. (τ_i, τ_j)-N-Int [(τ_i, τ_j)-N-Int(A)] = A
v. A≤B implies (τ_i, τ_j)-N-Int(A)≤(τ_i, τ_j)-N-Int(B)

vi. (τ_i, τ_j)-N-Int(A)∨ (τ_i, τ_j)-N-Int(B) ≤ (τ_i, τ_j)-N-Int(A∨B)

vii.(τ_i, τ_j)-N-Int(A ∧ B) = (τ_i, τ_j)-N-Int(A) ∧ (τ_i, τ_j)-N-Int(B).

Proof of the theorems are straightforward.

Remark 3.1: (τ_i, τ_j)-N-Int(A)≠(τ_j, τ_i)-N-Int(A) when i≠ j. For this we cite an example.

Example 3.3

Let X={a, b} and $A = \{< a, 0.5, 0.6, 0.7 >, < b, 0.4, 0.5, 0.6 >\}$,
$B = \{< a, 0.6, 0.6, 0.7 >, < b, 0.6, 0.4, 0.5 >\}$, $C = \{< a, 0.6, 0.6, 0.7 >, < b, 0.3, 0.2, 0.3 >\}$, $D = \{< a, 0.7, 0.6, 0.7 >, < b, 0.7, 0.2, 0.3 >\}$. Then $\tau_1 = \{0_X, 1_X, A, B, A \wedge B, A \vee B\}$ and $\tau_2 = \{ 0_X, 1_X, C, D, C \wedge D, C \vee D\}$ then (X, τ_1, τ_2) is neutrosophic bitopological space.

Let $P = \{< a, 0.8, 0.4, 0.5 >, < b, 0.7, 0.1, 0.2 >\}$

Then τ_2-N-Int(P) = D and (τ_1, τ_2)-N-Int(P) = B.

Now τ_1-N-Int(P) = B and (τ_2, τ_1)-N-Int(P) = C.

Hence the result that is (τ_1, τ_2)-N-Int(A)≠((τ_2, τ_1)-N-Int(A).

Definition 3.3

Let (X, τ_i, τ_j) be a neutrosophic bitopological space. Then for a set $A = \{< x, \mu_{ij}, \sigma_{ij}, \gamma_{ij} >: x\epsilon X\}$, neutrosophic (τ_i, τ_j)- N-closure of A is the intersection of all (τ_i, τ_j)-N-closed sets of X contained in A and is defined as follows

(τ_i, τ_j)-N-Cl(A) = $\{< x, \wedge_{\tau_i}\wedge_{\tau_j}\mu_{ij}\ , \vee_{\tau_i}\vee_{\tau_j}\sigma_{ij}, \ \vee_{\tau_i}\vee_{\tau_j}\gamma_{ij} >: x\epsilon X\}$

Example 3.4

Let X={a, b} and $A = \{< a, 0.5, 0.5, 0.5 >, < b, 0.4, 0.4, 0.4 >\}$,
$B = \{< a, 0.6, 0.6, 0.6 >, < b, 0.3, 0.3, 0.3 >\}$, $C = \{< a, 0.6, 0.6, 0.6 >, < b, 0.2, 0.2, 0.2 >\}$, $D = \{< a, 0.7, 0.7, 0.7 >, < b, 0.4, 0.4, 0.4 >\}$. Then $\tau_1 = \{0_X, 1_X, A, B, A \wedge B, A \vee B\}$ and $\tau_2 = \{0_X, 1_X, C, D, C \wedge D, C \vee D\}$ then (X, τ_1, τ_2) is neutroscopic bitopological space

Let $P = \{< a, 0.6, 0.5, 0.4 >, < b, 0.4, 0.3, 0.2 >\}$

$P^C = \{< a, 0.4, 0.5, 0.6 >, < b, 0.2, 0.7, 0.4 >\}$.

Now τ_2-N-Cl(P)= 1_X and τ_1-N-Cl(1_X) =1_X

Hence (τ_1, τ_2)-N-Cl(P)= 1_X.

Theorem 3.2 Let (X, τ_i, τ_j) be neutrosophic bitopological space then

i. (τ_i, τ_j)-N-Cl(0_X) = 0_X, (τ_i, τ_j)-N-Cl(1_X) = 1_X

ii. A ≤(τ_i, τ_j)-N-Cl(A).

iii. A is neutrosophic closed set if and only if A=(τ_i, τ_j)-N-Cl(A)

iv. (τ_i, τ_j)-N-Cl [(τ_i, τ_j)-N-Cl(A)] = A

v. A≤B implies (τ_i, τ_j)-N-Cl(A) ≤(τ_i, τ_j)-N-Cl(B).

vi. (τ_i, τ_j)-N-Cl(A∨B) = (τ_i, τ_j)-N-Cl(A)∨(τ_i, τ_j)-N-Cl(B)

vii.(τ_i, τ_j)-N-Cl(A∧B)≤(τ_i, τ_j)-N-Cl(A) ∧ (τ_i, τ_j)-N-Cl(B).

Prove of the theorems are straightforward.

Remark 3.2 (τ_i, τ_j)-N-Cl(A)≠(τ_j, τ_i)-N-Cl(A) when i≠ j. For this we cite an example.

Example 3.5

Let X={a, b} and $A = \{< a, 0.5, 0.5, 0.5 >, < b, 0.4, 0.4, 0.4 >\}$,
$B = \{< a, 0.4, 0.6, 0.6 >, < b, 0.2, 0.8, 0.4 >\}$, $C = \{< a, 0.6, 0.6, 0.6 >, < b, 0.2, 0.2, 0.2 >\}$,
$D = \{< a, 0.7, 0.7, 0.7 >, < b, 0.4, 0.4, 0.4 >\}$. Then $\tau_1 = \{0_X, 1_X, A, B, A \wedge B, A \vee B\}$ and $\tau_2 = \{0_X, 1_X, C, D, C \wedge D, C \vee D\}$ then (X, τ_1, τ_2) is neutroscopic bitopological space
Let $P = \{< a, 0.6, 0.5, 0.7 >, < b, 0.3, 0.7, 0.4 >\}$
τ_2-N-Cl(P)= D^C and τ_1-N-Cl(D^C)=1_X and (τ_1, τ_2)-N-Cl(P)= 1_X.
Now, τ_1-N-Cl(P)= B^C and τ_2-N-Cl(B^C)=D^C and (τ_1, τ_2)-N-Cl(P)= D^C.
Hence (τ_i, τ_j)-N-Cl(A) ≠ (τ_j, τ_i)-N-Cl(A).

Theorem 3.3

Let (X, τ_i, τ_j) be neutrosophic bitopological space then

i. (τ_i, τ_j)-N-Int(A^C) =[(τ_i, τ_j)-N-Cl(A)]^C
ii. (τ_i, τ_j)-N-Cl(A^C) =[(τ_i, τ_j)-N-Int(A)]^C.
iii. (τ_i, τ_j)-N-Int(A) =[(τ_i, τ_j)-N-Cl(A^C)]^C
iv. (τ_i, τ_j)-N-Cl(A) =[(τ_i, τ_j)-N-Int(A^C)]^C

Proof of (i)
Let $A = \{< x, \mu_{ij}, \sigma_{ij}, \gamma_{ij} >: x \epsilon X\}$.
Then $A^C = \{< x, \gamma_{ij}, 1 - \sigma_{ij}, \mu_{ij} >: x \epsilon X\}$.
Now (τ_i, τ_j)-N-Int(A^C) = $\{< x, \vee_{\tau_i} \vee_{\tau_j} \gamma_{ij}, \wedge_{\tau_i} \wedge_{\tau_j} (1 - \sigma_{ij}), \wedge_{\tau_i} \wedge_{\tau_j} \mu_{ij} >: x \epsilon X\}$
$= \{< x, \vee_{\tau_i} \vee_{\tau_j} \gamma_{ij}, 1 - \vee_{\tau_i} \vee_{\tau_j} \sigma_{ij}, \wedge_{\tau_i} \wedge_{\tau_j} \mu_{ij} >: x \epsilon X\}$
(τ_i, τ_j)-N-Cl(A)= $\{< x, \wedge_{\tau_i} \wedge_{\tau_j} \mu_{ij}, \vee_{\tau_i} \vee_{\tau_j} \sigma_{ij}, \vee_{\tau_i} \vee_{\tau_j} \gamma_{ij} >: x \epsilon X\}$
[(τ_i, τ_j)-N-Cl(A)]^C=$\{< x, \vee_{\tau_i} \vee_{\tau_j} \gamma_{ij}, 1 - \vee_{\tau_i} \vee_{\tau_j} \sigma_{ij}, \wedge_{\tau_i} \wedge_{\tau_j} \mu_{ij} >: x \epsilon X\}$
Hence (τ_i, τ_j)-N-Int(A^C) =[(τ_i, τ_j)-N-Cl(A)]^C.

Example 3.6

From the **Example 3.4,** we have
τ_2-N-Int(P^C) = 0_X , (τ_1, τ_2)-N-Int(P^C)= 0_X.
τ_2-N-Cl(P) = 1_X, (τ_1, τ_2)-N-Cl(P)= 1_X and [(τ_1, τ_2)-N-Cl(P)]^C= 0_X.
Hence (τ_i, τ_j)-N-Int(A^C) =[(τ_i, τ_j)-N-Cl(A)]^C.
Proof of (ii) is straight forward
Proof of (iii)
Let $A = \{< x, \mu_{ij}, \sigma_{ij}, \gamma_{ij} >: x \epsilon X\}$.
Then $A^C = \{< x, \gamma_{ij}, 1 - \sigma_{ij}, \mu_{ij} >: x \epsilon X\}$ and

(τ_i, τ_j)-N-Int(A) = $\{< x, \vee_{\tau_i} \vee_{\tau_j} \mu_{ij}, \wedge_{\tau_i} \wedge_{\tau_j} \sigma_{ij}, \wedge_{\tau_i} \wedge_{\tau_j} \gamma_{ij} >: x \epsilon X\}$

Now

$$(\tau_i, \tau_j) - \mathrm{N} - \mathrm{Cl}(A^C) = \{< x, \wedge_{\tau_i} \wedge_{\tau_j} \gamma_{ij}, \quad 1 - \wedge_{\tau_i} \wedge_{\tau_j} \sigma_{ij}, \vee_{\tau_i} \vee_{\tau_j} \mu_{ij} >: x \epsilon X\}$$

So,

$$[(\tau_i, \tau_j) - \mathrm{N} - \mathrm{Cl}(A^C)]^C = \{< x, \vee_{\tau_i} \vee_{\tau_j} \mu_{ij}, \wedge_{\tau_i} \wedge_{\tau_j} \sigma_{ij}, \wedge_{\tau_i} \wedge_{\tau_j} \gamma_{ij} >: x \epsilon X\}$$

Hence (τ_i, τ_j)-N-Int(A) =[(τ_i, τ_j)-N-Cl(A^C)]^C

Example 3.7 Let X={a, b} and $A=\{<a,0.5,0.5,0.5>,<b,0.4,0.4,0.4>\}$, $B=\{<a,0.6,0.6,0.6>,<b,0.3,0.3,0.3>\}$,$C=\{<a,0.6,0.6,0.6>,<b,0.2,0.2,0.2>\}$, $D=\{<a,0.7,0.7,0.7>,<b,0.4,0.4,0.4>\}$. Then $\tau_1=\{0_X,\ 1_X,\ A,B,A\wedge B,A\vee B\}$ and $\tau_2=\{0_X,\ 1_X,\ C,\ D,C\wedge D,C\vee D\}$ then (X, τ_1,τ_2) is neutrosophic bitopological space

Let $P=\{<a,0.6,0.5,0.4>,<b,0.2,0.3,0.2>\}$ and Let $P^C=\{<a,0.4,0.5,0.6>,<b,0.2,0.7,0.2>\}$

τ_2-N-Int(P) = 0_X , (τ_1,τ_2)-N-Int(P)= 0_X.

τ_2-N-Cl(P^C) = 1_X, (τ_1,τ_2)-N-Cl(P^C)= 1_X and [(τ_1,τ_2)-N-Cl(P^C)]^C= 0_X.

Hence (τ_i,τ_j)-N-Int(A) =[(τ_i,τ_j)-N-Cl(A^C)]^C.

Proof of (iv) is straight forward

Definition 3.4

Let A be a neutrosophic set in (X, τ_i,τ_j), then (τ_i,τ_j)-N-neutrosophic boundary of A is defined as (τ_i,τ_j)-N-Bd(A) = (τ_i,τ_j)-N-Cl(A) ∧(τ_i,τ_j)-N-Cl(A^C).

Proposition 3.1

Let A be neutrosophic set in (X, τ_i,τ_j). Then (τ_i,τ_j)-N-Bd(A) ∨A ≤($\tau_i,,\tau_j$)-N-Cl(A).

Proof : We have from the definition (τ_i,τ_j)-N-Bd(A)≤($\tau_i,,\tau_j$)-N-Cl(A) and A≤($\tau_i,,\tau_j$)-N-Cl(A) and hence (τ_i,τ_j)-N-Bd(A)∨A≤ ($\tau_i,\ \tau_j$)-N-Cl(A).

Remark 3.3: The converse part of the proposition is not true. For this we cite an example.

Example 3.8

Let X={a, b} and $A=\{<a,0.8,0.7,0.8>,<b,0.5,0.4,0.5>\}$, $B=\{<a,0.6,0.6,0.6>,<b,0.3,0.3,0.3>\}$, $C=\{<a,0.6,0.6,0.6>,<b,0.2,0.2,0.2>\}$,
$D=\{<a,0.7,0.7,0.7>,<b,0.4,0.4,0.4>\}$. Then $\tau_1=\{0_X,\ 1_X,\ A,B,A\wedge B,A\vee B\}$ and $\tau_2=\{0_X,\ 1_X,\ C,\ D,C\wedge D,C\vee D\}$ then (X, τ_1,τ_2) is neutrosophic bitopological space

Let $P=\{<a,0.7,0.4,0.7>,<b,0.4,0.4,0.3>\}$

P^C= $\{<a,0.7,0.6,0.7>,<b,0.3,0.6,0.4>\}$.

Now τ_2-N-Cl(P)= 1_X and (τ_i,τ_j)-N-Cl(P) = 1_X

τ_2-N-Cl(P^C)= (C∧D)^C and (τ_i,τ_j)-N-Cl((C∧D)^C) = (A∧B)^C

Now (τ_1,τ_2)-N-Bd(P)= (A∧B)^C and

(τ_1,τ_2)-N-Bd(P) ∨P ={$<a,0.8,0.3,0.6>,<b,0.5,0.6,0.3>$}

Hence (τ_i,τ_j)-N-Bd(A) ∨ A≠(τ_i,τ_j)-N-Cl(A).

Propositions 3.2

Let A and B be neutrosophic sets in (X, τ_i,τ_j). Then

i. (τ_i,τ_j)-N-Bd(A) = ($\tau_i,\ \tau_j$)-N-Bd(A^C).
ii. If A be (τ_i,τ_j)-N- neutrosophic closed set then (τ_i,τ_j)-N-Bd(A) ≤A
iii. If A be (τ_i,τ_j)-N- neutrosophic open set then (τ_i,τ_j)-N-Bd(A) ≤ A^C

Proof of (i)

(τ_i,τ_j)-N-Bd(A) =(τ_i,τ_j)-N-cl(A) ∧ (τ_i,τ_j)-N-Cl(A^C)

$$=\{<x,\ \wedge_{\tau_i}\wedge_{\tau_j}\mu_{ij}\ ,\ \vee_{\tau_i}\vee_{\tau_j}\sigma_{ij},\ \ \vee_{\tau_i}\vee_{\tau_j}\gamma_{ij}>: x\epsilon X\}\wedge\{<x,\ \wedge_{\tau_i}\wedge_{\tau_j}\gamma_{ij},\ 1-\ \wedge_{\tau_i}\wedge_{\tau_j}\sigma_{ij},\ \ \vee_{\tau_i}\vee_{\tau_j}\mu_{ij}\ \ >: x\epsilon X\}$$

Also ($\tau_i,,\tau_j$)-N-Bd(A^C) = (τ_i,τ_j)-N-Cl(A^C) ∧ (τ_i,τ_j)-N-Cl(A)

$=\{< x, \wedge_{\tau_i}\wedge_{\tau_j}\gamma_{ij}, 1- \wedge_{\tau_i}\wedge_{\tau_j}\sigma_{ij}, \vee_{\tau_i}\vee_{\tau_j}\mu_{ij} >: x\epsilon X\}\wedge$

$\{< x, \wedge_{\tau_i}\wedge_{\tau_j}\mu_{ij} , \vee_{\tau_i}\vee_{\tau_j}\sigma_{ij}, \vee_{\tau_i}\vee_{\tau_j}\gamma_{ij} >: x\epsilon X\}$

Hence (τ_i, τ_j)-N-Bd(A) = (τ_i, τ_j)-N-Bd(A^C).

Proof of (ii)

Let A be (τ_i, τ_j)-N- neutrosophic closed set then (τ_i, τ_j)-N-Cl(A) = A

Now (τ_i, τ_j)-N-Bd(A) = (τ_i, τ_j)-N-Cl(A) ∧ (τ_i, τ_j)-N-Cl(A^C) ≤ (τ_i, τ_j)-N-Cl(A) = A

Hence (τ_i, τ_j)-N-Bd(A) ≤ A.

Converse part is not true.

Remark 3.4: The converse part of the proposition is not true. For this we cite an example.

Example 3.9

Let X={a, b} and $A = \{< a, 0.8, 0.7, 0.8 >, < b, 0.5, 0.4, 0.5 >\}$, $B = \{< a, 0.6, 0.6, 0.6 >, < b, 0.2, 0.3, 0.3 >\}$, $C = \{< a, 0.6, 0.6, 0.6 >, < b, 0.2, 0.2, 0.2 >\}$, $D = \{< a, 0.7, 0.7, 0.7 >, < b, 0.4, 0.4, 0.4 >\}$. Then $\tau_1 = \{0_X, 1_X, A, B, A \wedge B, A \vee B\}$ and $\tau_2 = \{0_X, 1_X, C, D, C \wedge D, C \vee D\}$ then (X, τ_1, τ_2) is neutroscopic bitopological space

Let $S = \{< a, 0.9, 0.3, 0.2 >, < b, 0.6, 0.2, 0.3 >\}$

S^C= $\{< a, 0.2, 0.7, 0.9 >, < b, 0.3, 0.8, 0.6 >\}$.

Now τ_2-N-Cl(S)= 1_X and (τ_i, τ_j)-N-Cl(1_X) = 1_X

τ_2-N-Cl(S^C)= $(C\wedge D)^C$ and (τ_i, τ_j)-N-Cl($(C\wedge D)^C$) = $(A\wedge B)^C$

Now (τ_1, τ_2)-N-Bd(S)= $(A\wedge B)^C$ ≤ S.

But S is not a (τ_i, τ_j)-N-closed set.

Hence the converse part is not true.

Proof of (iii) is straight forward.

Proposition 3.3

Let A be neutrosophic set in (X, τ_i, τ_j), then

$[(\tau_i, \tau_j) - \mathrm{N} - \mathrm{Bd(A)}]^C$ = (τ_i, τ_j)-N-Int(A) ∨(τ_i, τ_j)-N-Int(A^C)

Proof:

From the definition we have (τ_i, τ_j)-N-Bd(A) = (τ_i, τ_j)-N-Cl(A) ∧ (τ_i, τ_j)-N-Cl(A^C)

[(τ_i, τ_j)-N-Bd(A)$]^C$= [(τ_i, τ_j)-N-Cl(A)$]^C$ ∨ [(τ_i, τ_j)-N-Cl(A^C)$]^C$

= (τ_i, τ_j)-N-Int(A) ∨ [(τ_i, τ_j)-N-Int(A^C).

Example 3.10 Let X={a, b} and $A = \{< a, 0.8, 0.7, 0.8 >, < b, 0.5, 0.4, 0.5 >\}$, $B = \{< a, 0.6, 0.6, 0.6 >, < b, 0.2, 0.3, 0.3 >\}$, $C = \{< a, 0.6, 0.6, 0.6 >, < b, 0.2, 0.2, 0.2 >\}$, $D = \{< a, 0.7, 0.7, 0.7 >, < b, 0.4, 0.4, 0.4 >\}$. Then $\tau_1 = \{0_X, 1_X, A, B, A \wedge B, A \vee B\}$ and $\tau_2 = \{ 0_X, 1_X, C, D, C \wedge D, C \vee D\}$ then (X, τ_1, τ_2) is neutrosophic bitopological space

Let $P = \{< a, 0.9, 0.3, 0.2 >, < b, 0.6, 0.2, 0.3 >\}$

P^C= $\{< a, 0.2, 0.7, 0.9 >, < b, 0.3, 0.8, 0.6 >\}$.

Now τ_2-N-Cl(P)= 1_X and (τ_1, τ_2)-N-Cl(1_X) = 1_X

τ_2-N-Cl(P^C^)= (C∧D)^C^ and (τ_1, τ_2)-N-Cl((C∧D)^C^) = (A∧B)^C^

So, (τ_1, τ_2)-N-Bd(P)= $(A\wedge B)^C$ and [(τ_1, τ_2)-N-Bd(P)]^C= A∧B.

Now τ_2-N-Int(P)= C∨D, (τ_1, τ_2)-N-Int(P)= A∧B
τ_2-N-Int(P^C)= ϕ, (τ_1, τ_2)-N-Int(P)= ϕ and (τ_1, τ_2)-N-Int(A)∨[(τ_1, τ_2)-N-Int(A^C)= A∧B.
Thus $[(\tau_1, \tau_2) - \mathrm{N} - \mathrm{Bd(A)}]^C =$ (τ_1, τ_2)-N-Int(A) ⋁(τ_1, τ_2)-N-Int(A^C) .

Proposition 3.4
Let A be neutrosophic set in (X, τ_i, τ_j), then

(τ_i, τ_j)-N-Bd(A) = (τ_i, τ_j)-N-Cl(A) - (τ_i, τ_j)-N-Int(A)

Proof: From the definition of (τ_i, τ_j)-N-Bd(A) we have

(τ_i, τ_j)-N-Bd(A) = (τ_i, τ_j)-N-Cl(A)∧(τ_i, τ_j)-N-Cl(A^C)

= (τ_i, τ_j)-N-Cl(A) – [(τ_i, τ_j)-N-Cl(A^C)]^C
= (τ_i, τ_j)-N-Cl(A) –(τ_i, τ_j)-N-Int(A).

Example 3.11
From the **Example 3.10** we have
τ_2-N-Int(P)= C∨D, (τ_1, τ_2)-N-Int(P)= A∧B and 1_X - A∧B = $(A\wedge B)^C$.

Hence (τ_1, τ_2)-N-Bd(A) = (τ_1, τ_2)-N-Cl(A) - (τ_1, τ_2)-N-Int(A).

Proposition 3.5
Let A be neutrosophic set in (X, τ_i, τ_j), then
(τ_i, τ_j)-N-Bd(Int(A)) ≤ (τ_i, τ_j)-N-Bd(A).
Proof :
(τ_i, τ_j)-N-Bd(Int(A))= (τ_i, τ_j)-N-Cl(Int A)∧(τ_i, τ_j)-N-Cl(Int A^C)
= (τ_i, τ_j)-N-Cl(Int A) – [(τ_i, τ_j)-N-Cl(Int A^C)]^C
= (τ_i, τ_j)-N-Cl(Int A) – (τ_i, τ_j)-N-Int(A)
≤(τ_i, τ_j)-N-Cl(A) – (τ_i, τ_j)-N-Int(A)
= (τ_i, τ_j)-N-Bd(A).

Remark 3.5: The converse of the proposition is not true. For this we cite an example.

Example 3.12
From Example 3.10, we have
τ_2-N-Int(P^C)= 0_X, (τ_1, τ_2)-N-Int(P^C)= 0_X
(τ_1, τ_2)-N-Bd(Int(A))= (τ_1, τ_2)-N-Cl(0_X) ∧(τ_1, τ_2)-N-Cl(Int 1_X)
= 0_X
Also Now τ_2-N-Cl(P)= 1_X and (τ_1, τ_2)-N-Cl(1_X) = 1_X
τ_2-N-Cl(P^C)= $(C\wedge D)^C$ and (τ_1, τ_2)-N-Cl($(C\wedge D)^C$) = $(A\wedge B)^C$
Now (τ_1, τ_2)-N-Bd(P)= $(A\wedge B)^C$.
Hence (τ_1, τ_2)-N-Bd(Int(A))≤(τ_1, τ_2)-N-Bd(A) but (τ_1, τ_2)-N-Bd(Int(A)) ≠(τ_1, τ_2)-N-Bd(A).

Proposition 3.6

Let A be neutrosophic set in (X, τ_i, τ_j), then
(τ_i, τ_j)-N-Bd(Cl(A)) ≤(τ_i, τ_j)-N-Bd(A).
Proof : Straightforward.

Remark 3.6: The converse of the proposition is not true. For this we cite an example.
Example 3.13
From Example 3.10, we have
(τ_1, τ_2)-N-Bd(Cl(P)) = (τ_i, τ_j)-N-Cl(1_X)∧(τ_i, τ_j)-N-Cl(0_X)
= 0_X
Also Now τ_2-N-Cl(P)= 1_X and (τ_1, τ_2)-N-Cl(1_X) = 1_X
τ_2-N-Cl(P^C)= $(C∧D)^C$ and (τ_1, τ_2)-N-Cl($(C∧D)^C$) = $(A∧B)^C$
Now (τ_1, τ_2)-N-Bd(P)= $(A∧B)^C$.
Hence (τ_1, τ_2)-N-Bd(Cl(A)) ≤(τ_1, τ_2)-N-Bd(A) but (τ_1, τ_2)-N-Bd(Int(A)) ≠(τ_1, τ_2)-N-Bd(A).
Proposition 3.7
Let A be neutrosophic set in (X, τ_i, τ_j), then

(τ_i, τ_j)-N-Int (A) =A - (τ_i, τ_j)-N-Bd(A)

Proof: Straightforward.
Proposition 3.8
Let A and B be neutrosophic set in (X, τ_i, τ_j). Then
(τ_i, τ_j)-N-Bd(A∨B)≤(τ_i, τ_j)-N-Bd(A) ∨ (τ_i, τ_j)-N-Bd(B)
Proof: Straightforward.
Remark 3.7: The converse of the proposition is not true
Example 3.14
From Example 3.10, we have
Let $Q = \{< a, 0.8, 0.8, 0.8 >, < b, 0.5, 0.5, 0.5 >\}$ $Q^C = \{< a, 0.8, 0.2, 0.8 >, < b, 0.5, 0.5, 0.5 >\}$
$P \vee Q = \{< a, 0.9, 0.3, 0.2 >, < b, 0.6, 0.2, 0.3 >\}$
Now τ_2-N-Cl(Q)= 1_X and (τ_i, τ_j)-N-Cl(Q) = 1_X
τ_2-N-Cl(Q^C)= $1_X$and (τ_i, τ_j)-N-Cl((Q^C) = 1_X
So, (τ_1, τ_2)-N-Bd(Q)= 1_X
Now τ_2-N-Cl($P \vee Q$)= 1_X and (τ_i, τ_j)-N-Cl($P \vee Q$) = 1_X
τ_2-N-Cl($[P \vee Q]^C$)= $(C∧D)^C$ and (τ_i, τ_j)-N-Cl($[P \vee Q]^C$) = $(A∧B)^C$
So, (τ_1, τ_2)-N-Bd($P \vee Q$)= $(A∧B)^C$
Now (τ_i, τ_j)-N-Bd(P∨Q) =$(A∧B)^C$ and (τ_i, τ_j)-N-Bd(P) ∨ (τ_i, τ_j)-N-Bd(Q)= 1_X
Hence (τ_i, τ_j)-N-Bd(P∨Q) ≠(τ_i, τ_j)-N-Bd(P) ∨ (τ_i, τ_j)-N-Bd(Q).

Proposition 3.9
Let A and B be neutrosophic set in (X, τ_i, τ_j). Then

(τ_i, τ_j)-N-Bd(A∧B) ≤ (τ_i, τ_j)-N-Bd(A) ∨ (τ_i, τ_j)-N-Bd(B)

Proof: Straightforward.

Conclusion: In this work we have redefined the definition of Bitopological space with the help of netrosophic set. Then we have investigated the properties of interior, closure and boundary of neutrosophic bitopological spaces. Hope our work will help in further study of neutrosophic generalized closed sets in neutrosophiv bitopological space. This may lead a new beginning for further research on the study of generalized closed sets in neutrosophic bitopological space associated with digraph and directed graphs. This may also lead to the new properties of separation axioms on neutrosophic bitopological space.

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References

1. L. A. Zadeh , Fuzzy sets, *Information and Control*, 8 (1965), 338-353
2. K. T Atanassov., Intuitionistic fuzzy sets, Fuzzy sets and systems, 20. (1986), 87-96,
3. F. Smarandache, Neutrosophic set - a generalization of the intuitionistic fuzzy set, International Journal of Pure and Applied Mathematics, 24(3) (2005) 287–297.
4. F. Smarandache, Neutrosophy and neutrosophic logic, first international conference on neutrosophy, neutrosophic logic, set, probability, and statis- tics, University of New Mexico, Gallup, NM 87301, USA(2002).
5. C. L. Chang , Fuzzy Topological Space, Journal of Mathematical Analysis and Application 24 (1968), 182-190
6. D. Coker, An introduction to intuitionistic fuzzy topological spaces, Fuzzy Sets and Systems, 88 (1997), 81-89.
7. F. G. Lupi´a˜nez, On neutrosophic topology, The International Journal of Systems and Cybernetics, 37(6) (2008), 797–800.
8. F. G. Lupi´a˜nez, Interval neutrosophic sets and topology, The International Journal of Systems and Cybernetics, 38(3/4) (2009), 621–624.
9. F. G. Lupi´a˜nez, On various neutrosophic topologies, The International Journal of Systems and Cybernetics, 38(6) (2009), 1009–1013.
10. F. G. Lupi´a˜nez, On neutrosophic paraconsistent topology, The International Journal of Systems and Cybernetics, 39(4) (2010), 598–601.
11. A. Salama and S. AL-Blowi, Generalized neutrosophic set and generalized neutrosophic topological spaces, Computer Science and Engineering, 2(7) (2012), 129–132.
12. J. C. Kelly, Bitopoligicaal spaces, Pro. London math soc., 313 (1963), 71-89
13. A. Kandil, Nouth A. A., El-Sheikh S. A., On fuzzy bitopological spaces, Fuzzy sets and system, 74(1995) 353-363
14. S. J. Lee, J. T. Kim, Some properties of Intuitionistic Fuzzy Bitopological Spaces, SCIS-ISIS 2012, Kobe, Japan, Nov. 20-24.
15. P. M. Pu and Y. M. Liu, Fuzzy topology I: neighbourhood structure of a fuzzy point and Moore-Smith convergence, J. Math. Anal. Appl. 76, (1980) 571–599
16. R. H. Warren, Boundary of a fuzzy set, Indiana Univ. Math. J. 26(1977), 191–197
17. M. Abdel-Baset, R.. Mohamed, A. E. N. H. Zaied & F. Samarandache, A Hybride plitogenic decision-making approach with quality function deployment for selecting supply chain sustainability metrics, Symmetry, 11(7)(2019), 903.
18. M. Abdel-Baset, G. Manogaran, A. Gamal & F. Samarandache, A Group decision making framework based on neutrosophic TOPSIS approach for smart medical device selection , Journal of Medical System, 43(2)(2019), 38.
19. M. Abdel-Baset & R. Mohamed, A Novel plithogenic TOPSIS-CRITIC model for sustainable supply chain risk management, Journal of Cleaner Production, 247(2020), 119586.
20. Wang H, Smarandache F, Zhang YQ, Sunderraman R Single valued neutrosophic sets, Multispace and Multistructure, 4(2010): 410-413.

21. T. Nanthini and A. Pushpalatha, Interval Valued Neutrosophic Topological Spaces, Neutrosophic Sets and Systems, vol. 32, (2020), pp. 52-60. DOI: 10.5281/zenodo.3723139
22. R. Dhavaseelan, Md. Hanif PAGE: Neutrosophic Almost Contra α-Continuous Functions, Neutrosophic Sets and Systems, vol. 29, (2019), pp. 71-77, DOI: 10.5281/zenodo.3514403
23. R. Dhavaseelan, S. Jafari, R. Narmada Devi, Md. Hanif Page: Neutrosophic Baire Spaces, Neutrosophic Sets and Systems, Vol. 16 (2017), pp. 20-23. doi.org/10.5281/zenodo.831920.
24. M Abdel-Basset, Mai Mohamed and F. Smarandache, Comment on "A Novel Method for Solving the Fully Neutrosophic Linear Programming Problems: Suggested Modifications", Neutrosophic Sets and Systems, vol. 31, (2020), pp. 305-309. DOI: 10.5281/zenodo.3659265
25. R. K. Al-Hamido, T. Gharibah, S. Jafari, F. Smarandache: On Neutrosophic Crisp Topology via N-Topology, Neutrosophic Sets and Systems, vol. 23, (2018), pp. 96-109. DOI:10.5281/zenodo.2156509.

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