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Neutrosophic Cubic Hamacher Aggregation Operators and Their Applications in Decision Making

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Abstract. In this paper, firstly, novel approaches of score function and accuracy function are introduced to achieve more practical and convincing comparison results of two neutrosophic cubic values. Furthermore, the neutrosophic cubic Hamacher weighted averaging operator and the neutrosophic cubic Hamacher weighted geometric operator are developed to aggregate neutrosophic cubic values. Some desirable properties of these operators such as idempotency, monotonicity and boundedness are discussed. To deal with the multi-criteria decision making problems in which attribute values take the form of the neutrosophic cubic elements, the decision making algorithms based on some Hamacher aggregation operators, which are extensions of the algebraic aggregation operators and Einstein aggregation operators, are constructed. Finally, the illustrative examples and comparisons are given to verify the proposed algorithms and to demonstrate their practicality and effectiveness.

Keywords: Neutrosophic set; Neutrosophic cubic set; Score function; Accuracy function; Hamacher operations; Decision making

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1. Introduction

In real life, there are many problems with inconsistent, indeterminate and incomplete information which cannot be described by crisp numbers. Under these circumstances, Zadeh [34] proposed the fuzzy set, which is an effective method to deal with such problems. To express uncertainty, Sambuc [26] extended the fuzzy set and initiated the interval valued fuzzy theory. In [33], the researchers discussed the multipolar types of fuzzy sets. In 2012, Jun combined the idea of fuzzy sets and interval valued fuzzy sets to form cubic sets. Some researchers used the cubic sets in different directions to have more applications [23, 24]. In some situations, hesitancy may exist when ones determine the membership degree of an object. Torra [29] improved the hesitant fuzzy set to depict this hesitant information. Moreover, Smarandache [27] introduced the neutrosophic set to reflect the truth, indeterminate and false information simultaneously. In addition, Wang et al. pointed out that the neutrosophic set is difficult to truly apply to practical problems in real world scenarios. To overcome this flaw, they proposed single valued neutrosophic sets [32]. In addition, they put forward that in many real life problems, the degrees of truth, indeterminacy and falsity of a certain statement may be adaptly preferred by interval forms, instead of real numbers [31]. Moreover, many papers were published on the neutrosophic set's case studies [1, 2, 19, 20, 30], their some extensions [3–5, 12, 16], and combining with other theories, like graph theory [11, 18], soft set theory [8, 15, 28], rough set theory [6].

By combining the single valued neutrosophic set and interval neutrosophic set, Jun et al. [14], and Ali et al. [7] introduced the notion of neutrosophic cubic set. These sets enable us to choose both interval values and single values for the membership, indeterminacy and non-membership. This characteristic of neutrosophic cubic sets enables us to deal with ambiguous and uncertain data more efficiently. In addition, the application of sundry extensions of neutrosophic cubic sets studied by researchers in a variety of fields, like decision-making, supplier selection and similarity measure [9, 21, 22, 35].

The aggregation operators are an indispensable part of decision making in neutrosophic cubic environments. In 2019, Khan et al. [17] developed the neutrosophic cubic Einstein weighted geometric operator, and also defined the score and accuracy functions to reveal the superiority among the neutrosophic cubic numbers. It is known that Einstein t-norm and Einstein t-conorm are special forms of Hamacher t-norm and Hamacher t-conorm respectively, that is, Hamacher t-norm and Hamacher t-conorm are the more general version. This paper aims to introduce the neutrosophic cubic Hamacher weighted averaging operator and neutrosophic cubic Hamacher weighted geometric operator, which generalize the aggregation operators proposed by Khan et al. [17]. Furthermore, it proposes new score function and accuracy function, which provide more efficient outputs than Khan et al.'s functions. By using these emerging operators and functions, the phenomenal algorithms are elaborated to solve multi-criteria decision making problems. The contributions of this study can be summarized as follows. The models are proposed to compare neutrosophic cubic numbers, and the operators which are more efficient than some existing netrosophic cubic aggregation operators are developed. In addition to these, it is instilled that these concepts can be used to handle the problems with neutrosophic cubic information.

This paper is arranged as follows. Section 2 presents some fundamental concepts of fuzzy set, neutrosophic set, interval neutrosophic set, cubic set and neutrosophic cubic set. Section 3 presents comparison strategy of two neutrosophic cubic elements. Section 4 is devoted to improve the Hamacher operations of neutrosophic cubic elements. Section 5 introduces neutrosophic cubic Hamacher weighted aggregation operators and their basic properties. Section 6 is devoted to proposing the neutrosophic cubic decision making algorithms with possible applications and analyzing the ranking order with different reducing factors. Section 7 is the conclusion and the future scope of research.

2. Preliminaries

In this part, we briefly remind the definitions of fuzzy set, neutrosophic set, interval neutrosophic set, cubic set, neutrosophic cubic set and neutrosophic cubic element.

Definition 2.1. ([34]) Let $\mathcal{O}$ be a universal set. Then, a fuzzy set (FS) Ψ in $\mathcal{O}$ is defined by

$$\Psi = \{\mu_\Psi(o)/o : o \in \mathcal{O}\}$$

where $\mu_\Psi : \mathcal{O} \to [0,1]$ is said to be the membership function and $\mu_\Upsilon(o)$ denotes the degrees of membership of $o \in \mathcal{O}$ to the set Ψ.

Definition 2.2. ([26]) Let $\mathcal{O}$ be a universal set and $D[0,1]$ be the set of all closed subintervals of the interval [0,1]. Then, an interval-valued fuzzy set (IFS) $\widetilde{\Psi}$ in $\mathcal{O}$ is characterized by

$$\widetilde{\Psi} = \{\widetilde{\mu}_{\widetilde{\Psi}}(o)/o : o \in \mathcal{O}\}$$

where $\widetilde{\mu}_{\widetilde{\Psi}} = [\widetilde{\mu}^L_{\widetilde{\Psi}}, \widetilde{\mu}^U_{\widetilde{\Psi}}] : \mathcal{O} \to D[0,1]$ is said to be the membership function, and $\widetilde{\mu}^L_{\widetilde{\Psi}}(o)$ and $\widetilde{\mu}^U_{\widetilde{\Psi}}(o)$ (where $\widetilde{\mu}^L_{\widetilde{\Psi}}(o) \le \widetilde{\mu}^U_{\widetilde{\Psi}}(o)$) denote the lower degree and upper degree of membership of $o \in \mathcal{O}$ to the set $\widetilde{\Psi}$, respectively.

Definition 2.3. ([27]) Let $\mathcal{O}$ be a universal set. Then, a nuetrosophic set (NS) Υ in $\mathcal{O}$ is described in the following form

$$\Upsilon = \{(\mu_\Upsilon, \iota_\Upsilon, \eta_\Upsilon)/o : o \in \mathcal{O}\}$$

where $\mu_\Upsilon, \iota_\Upsilon, \eta_\Upsilon : \mathcal{O} \to]0^-, 1^+[$ are said to be the functions of membership, indeterminacy and non-membership, respectively. Also, $\mu_\Upsilon(o)$, $\iota_\Upsilon(o)$ and $\eta_\Upsilon(o)$ denote the degrees of membership, indeterminacy and non-membership of $o \in \mathcal{O}$ to the set Υ respectively.

Definition 2.4. ([32]) Let $\mathcal{O}$ be a universal set. Then, a single nuetrosophic set Γ in $\mathcal{O}$ is described in the following form

$$\Upsilon = \{(\mu_\Upsilon, \iota_\Upsilon, \eta_\Upsilon)/o : o \in \mathcal{O}\}$$

where $\mu_\Upsilon, \iota_\Upsilon, \eta_\Upsilon : \mathcal{O} \to [0,1]$ are called the functions of membership, indeterminacy and non-membership, respectively. Also, $\mu_\Upsilon(o)$, $\iota_\Upsilon(o)$ and $\eta_\Upsilon(o)$ denote the degrees of membership, indeterminacy and non-membership of $o \in \mathcal{O}$ to the set Υ respectively.

Remark. Throughout the paper, Υ means the single valued neutrosophic set.

Definition 2.5. ([31]) Let $\mathcal{O}$ be a universal set and $D[0,1]$ be the set of all closed subintervals of the interval [0,1]. Then, an interval neutrosophic set (INS) Υ in $\mathcal{O}$ is characterized by

$$\widetilde{\Upsilon} = \{(\widetilde{\mu}_{\widetilde{\Upsilon}}, \widetilde{\iota}_{\widetilde{\Upsilon}}, \widetilde{\eta}_{\widetilde{\Upsilon}})/o : o \in \mathcal{O}\}$$

where $\widetilde{\mu}_{\widetilde{\Upsilon}}, \widetilde{\iota}_{\widetilde{\Upsilon}}, \widetilde{\eta}_{\widetilde{\Upsilon}} : \mathcal{O} \to D[0,1]$ are termed to be the functions of membership, indeterminacy and non-membership, respectively. Also, $\widetilde{\mu}^L_{\widetilde{\Upsilon}}(o)$, $\widetilde{\mu}^U_{\widetilde{\Upsilon}}(o)$ denote the lower and upper degrees of membership, $\widetilde{\iota}^L_{\widetilde{\Upsilon}}(o)$, $\widetilde{\iota}^U_{\widetilde{\Upsilon}}(o)$ denote the lower and upper degrees of indeterminacy and $\widetilde{\eta}^L_{\widetilde{\Upsilon}}(o)$, $\widetilde{\eta}^U_{\widetilde{\Upsilon}}(o)$ denote the lower and upper degrees of non-membership, respectively.

Definition 2.6. ([13]) Let $\mathcal{O}$ be a universal set. Then, a cubic set (CS) Δ in $\mathcal{O}$ is a structure in the following form

$$\Delta = \{(\widetilde{\Psi}(o), \Psi(o))/o : o \in \mathcal{O}\}$$

where $\widetilde{\Psi}$ is an IFS in $\mathcal{O}$ and Ψ is an FS in $\mathcal{O}$

Definition 2.7. ([7,14]) Let $\mathcal{O}$ be a universal set. Then, a neutrosophic cubic set (NCS) Λ in $\mathcal{O}$ is a structure in the following form

$$\Lambda = \{(\widetilde{\Upsilon}(o), \Upsilon(o))/o : o \in \mathcal{O}\}$$

where $\widetilde{\Upsilon}$ is an INS in $\mathcal{O}$ and Υ is an NS in $\mathcal{O}$

Simply, the structure of neutrosophic cubic set can be considered as follows

$$\widetilde{\Upsilon} = \{((\widetilde{\mu}_{\widetilde{\Upsilon}}(o), \widetilde{\iota}_{\widetilde{\Upsilon}}(o), \widetilde{\eta}_{\widetilde{\Upsilon}}(o)), (\mu_{\Upsilon}(o), \iota_{\Upsilon}(o), \eta_{\Upsilon}(o)))/o : o \in \mathcal{O}\}$$

Furthermore, $((\widetilde{\mu}_{\widetilde{\Upsilon}}(o), \widetilde{\iota}_{\widetilde{\Upsilon}}(o), \widetilde{\eta}_{\widetilde{\Upsilon}}(o)), (\mu_{\Upsilon}(o), \iota_{\Upsilon}(o), \eta_{\Upsilon}(o)))$, which is an element in Λ, is called a neutrsophic cubic element (NCE). For simplicity, an NCE is denoted by $\upsilon_k = (\widetilde{\mu}_k, \widetilde{\iota}_k, \widetilde{\eta}_k, \mu_k, \iota_k, \eta_k)$.

Example 2.8. Suppose that $\mathcal{O} = \{o_1, o_2, o_3, o_4\}$ be a universal set. Then,

(i): a fuzzy set Ψ in $\mathcal{O}$ can be exemplified as follows.

$$\Psi = \{0.3/o_1, 0.7/o_2, 1/o_3, 0.1/o_4\}.$$

(ii): an interval-valued fuzzy set $\widetilde{\Psi}$ in $\mathcal{O}$ can be illustrated as follows.

$$\widetilde{\Psi} = \{[0.3, 0.4]/o_1, [0.4, 0.7]/o_2, [0, 1]/o_3, [0.1, 0.1]/o_4\}$$

(iii): As a sample of a neutrosophic set Υ in $\mathcal{O}$, the following set can be given.

$$\Upsilon = \{(0.3, 0.7, 0.2)/o_1, (0.1, 0.1, 0.1)/o_2, (1, 0.7, 0.3)/o_3, (0, 0, 0.9)/o_4\}$$

(iv): an interval neutrosophic set $\widetilde{\Upsilon}$ in $\mathcal{O}$ can be shown in the following form.

$$\widetilde{\Upsilon} = \left\{ \begin{array}{l} ([0.2, 0.6], [0.4, 0.4], [0.1, 0.8])/o_1, ([0.5, 1], [0.3, 0.4], [0.6, 0.7])/o_2, \\ ([0, 0], [0.1, 0.8], [0.2, 0.4])/o_3, ([0.1, 0.4], [0.3, 0.5], [0.2, 0.2])/o_4 \end{array} \right\}.$$

(v): a cubic set Δ in $\mathcal{O}$ is can be exemplified as follows.

$$\Delta = \{([0.2, 0.6], 0.5)/o_1, ([0.1, 0.5], 0.2)/o_2, ([0.5, 0.7], 1)/o_3, ([0.1, 1], 0.4)/o_4\}.$$

(vi): a neutrosophic cubic set Λ in $\mathcal{O}$ is an object having the following form

$$\Lambda = \left\{ \begin{array}{l} (([0.1, 0.4], [0.1, 0.4], [0.3, 0.6]), (0.5, 0.3, 0.8))/o_1, \\ (([0.8, 0.9], [0.1, 0.7], [0.2, 0.7]), (0.6, 1, 0.7))/o_2, \\ (([0.3, 1], [0, 0.5], [0.4, 0.6]), (0, 0.3, 0.7))/o_3, \\ (([0.4, 0.9], [0.2, 0.2], [0.6, 0.8]), (0.1, 0.1, 0.1))/o_4 \end{array} \right\}.$$

3. **Score and Accuracy Functions of Neutrosophic Cubic Element**

We can develop the score and accuracy functions to compare two NCEs. For comparison of two NCEs, firstly, we use the score functions, sometimes the score values of two NCEs can be equal although they have different components of membership, indeterminacy and non-membership functions. In such cases, it is aimed to achieve a ranking priority between the NCEs using the accuracy function.

Definition 3.1. Let $\upsilon_k = (\widetilde{\mu}_k, \widetilde{\iota}_k, \widetilde{\eta}_k, \mu_k, \iota_k, \eta_k)$ be an NCE, where $\widetilde{\mu}_k = [\widetilde{\mu}_k^L, \widetilde{\mu}_k^U]$, $\widetilde{\iota}_k = [\widetilde{\iota}_k^L, \widetilde{\iota}_k^U]$ and $\widetilde{\eta}_k = [\widetilde{\eta}_k^L, \widetilde{\eta}_k^U]$. Then, the score function f_{scr} is defined by

$$f_{scr} = \frac{\frac{1}{2}\big(6 + (\widetilde{\mu}_k^L + \widetilde{\mu}_k^U) - 2(\widetilde{\iota}_k^L + \widetilde{\iota}_k^U) - (\widetilde{\eta}_k^L + \widetilde{\eta}_k^U)\big) + \big(3 + \mu_k - 2\iota_k - \eta_k\big)}{8}. \tag{1}$$

Proposition 3.2. *The score function of any NCE lies between 0 to 1, i.e., $f_{scr}(\upsilon_k) \in [0,1]$ for any υ_k.*

Proof. Consider $\upsilon_k = (\widetilde{\mu}_k, \widetilde{\iota}_k, \widetilde{\eta}_k, \mu_k, \iota_k, \eta_k)$. By using the definitions of INS and NS, we have all $\widetilde{\mu}_k^L$, $\widetilde{\mu}_k^U$, $\widetilde{\iota}_k^L$, $\widetilde{\iota}_k^U$, $\widetilde{\eta}_k^L$,$\widetilde{\eta}_k^U$, μ_k, ι_k, $\eta_k \in [0,1]$.
Then, it is easily seen that

$$0 \leq \widetilde{\mu}_k^L \leq 1, \quad 0 \leq \widetilde{\mu}_k^U \leq 1 \Rightarrow 0 \leq \widetilde{\mu}_k^L + \widetilde{\mu}_k^U \leq 2, \tag{2}$$

$$0 \leq \widetilde{\iota}_k^L \leq 1, \quad 0 \leq \widetilde{\iota}_k^U \leq 1 \Rightarrow 0 \leq \widetilde{\iota}_k^L + \widetilde{\iota}_k^U \leq 2 \Rightarrow -4 \leq -2(\widetilde{\iota}_k^L + \widetilde{\iota}_k^U) \leq 0, \tag{3}$$

and

$$0 \leq \widetilde{\eta}_k^L \leq 1, \quad 0 \leq \widetilde{\eta}_k^U \leq 1 \Rightarrow 0 \leq \widetilde{\eta}_k^L + \widetilde{\eta}_k^U \leq 2 \Rightarrow -2 \leq -\widetilde{\eta}_k^L - \widetilde{\eta}_k^U \leq 0. \tag{4}$$

By adding Eqs. (2), (3) and (4), we obtain

$$\begin{aligned} &-6 \leq (\widetilde{\mu}_k^L + \widetilde{\mu}_k^U) - 2(\widetilde{\iota}_k^L + \widetilde{\iota}_k^U) - (\widetilde{\eta}_k^L + \widetilde{\eta}_k^U) \leq 2 \\ &\Rightarrow 0 \leq \frac{1}{2}\big(6 + (\widetilde{\mu}_k^L + \widetilde{\mu}_k^U) - 2(\widetilde{\iota}_k^L + \widetilde{\iota}_k^U) - (\widetilde{\eta}_k^L + \widetilde{\eta}_k^U) \leq 4 \end{aligned} \tag{5}$$

In addition, we obtain

$$\begin{aligned} 0 \leq \mu_k \leq 1, \quad -2 \leq -2\iota_k \leq 0, \quad -1 \leq -\eta_k \leq 0 \Rightarrow -3 \leq \mu_k - 2\iota_k - \eta_k \leq 1 \\ \Rightarrow 0 \leq 3 + \mu_k - 2\iota_k - \eta_k \leq 4. \end{aligned} \tag{6}$$

By adding Eqs. (5) and (6) and then dividing by 8, we have

$$0 \leq \frac{\frac{1}{2}\big(6 + (\widetilde{\mu}_k^L + \widetilde{\mu}_k^U) - 2(\widetilde{\iota}_k^L + \widetilde{\iota}_k^U) - (\widetilde{\eta}_k^L + \widetilde{\eta}_k^U)\big) + \big(3 + \mu_k - 2\iota_k - \eta_k\big)}{8} \leq 1. \tag{7}$$

This result completes the proof. □

Definition 3.3. Let $\upsilon_k = (\widetilde{\mu}_k, \widetilde{\iota}_k, \widetilde{\eta}_k, \mu_k, \iota_k, \eta_k)$ be an NCE, where $\widetilde{\mu}_k = [\widetilde{\mu}_k^L, \widetilde{\mu}_k^U]$, $\widetilde{\iota}_k = [\widetilde{\iota}_k^L, \widetilde{\iota}_k^U]$ and $\widetilde{\eta}_k = [\widetilde{\eta}_k^L, \widetilde{\eta}_k^U]$. Then, the accuracy function f_{acr} is defined by

$$f_{acr} = \frac{\frac{1}{2}\big(\widetilde{\mu}_k^L + \widetilde{\mu}_k^U + \widetilde{\iota}_k^L + \widetilde{\iota}_k^U + \widetilde{\eta}_k^L + \widetilde{\eta}_k^U\big) + \mu_k + \iota_k + \eta_k}{6}. \tag{8}$$

Proposition 3.4. *The accuracy function of any NCE lies between 0 to 1, i.e., $f_{acr}(\upsilon_k) \in [0,1]$ for any υ_k.*

Proof. Consider $\upsilon_k = (\widetilde{\mu}_k, \widetilde{\iota}_k, \widetilde{\eta}_k, \mu_k, \iota_k, \eta_k)$. Since $\widetilde{\mu}_k^L$, $\widetilde{\mu}_k^U$, $\widetilde{\iota}_k^L$, $\widetilde{\iota}_k^U$, $\widetilde{\eta}_k^L$,$\widetilde{\eta}_k^U$, μ_k, ι_k, $\eta_k \in [0,1]$ from the definitions of INS and NS, it is obvious that

$$0 \le \frac{1}{2}\big(\widetilde{\mu}_k^L + \widetilde{\mu}_k^U + \widetilde{\iota}_k^L + \widetilde{\iota}_k^U + \widetilde{\eta}_k^L + \widetilde{\eta}_k^U\big) + \mu_k + \iota_k + \eta_k \le 6. \tag{9}$$

Dividing by 6, we have

$$0 \le \frac{\frac{1}{2}\big(\widetilde{\mu}_k^L + \widetilde{\mu}_k^U + \widetilde{\iota}_k^L + \widetilde{\iota}_k^U + \widetilde{\eta}_k^L + \widetilde{\eta}_k^U\big) + \mu_k + \iota_k + \eta_k}{6} \le 1. \tag{10}$$

Thus, the proof is complete. □

The following definition is proposed to compare two NCEs, thereby ensuring the order priority between the NCEs.

Definition 3.5. Let υ_1 and υ_2 be two NCEs. The comparison method for any two NCEs υ_1 and υ_2 is defined as follows:

(1) If $f_{scr}(\upsilon_1) < f_{scr}(\upsilon_1)$ then $\upsilon_1 \prec \upsilon_2$
(2) $f_{scr}(\upsilon_1) > f_{scr}(\upsilon_1)$ then $\upsilon_1 \succ \upsilon_2$
(3) $f_{scr}(\upsilon_1) = f_{scr}(\upsilon_1)$ then
- when $f_{acr}(\upsilon_1) < f_{acr}(\upsilon_1)$, $\upsilon_1 \prec \upsilon_2$
- when $f_{acr}(\upsilon_1) > f_{acr}(\upsilon_1)$, $\upsilon_1 \succ \upsilon_2$
- when $f_{acr}(\upsilon_1) = f_{acr}(\upsilon_1)$, $\upsilon_1 = \upsilon_2$

Example 3.6. We consider any two NCEs as $\upsilon_1 = ([0.4, 0.6], [0.3, 0.4], [0.4, 0.5], 0.8, 0.6, 0.5)$ and $\upsilon_2 = ([0.5, 0.7], [0.2, 0.5], [0.5, 0.6], 0.5, 0.6, 0.2)$. Then, it is obtain $f_{scr}(\upsilon_1) = f_{scr}(\upsilon_2) = 0.5968$. If we compare this two NCEs by using the accuracy functions, then we have $\upsilon_1 \succ \upsilon_2$ since $f_{acr}(\upsilon_1) = 0.5333 > 0.4666 = f_{acr}(\upsilon_2)$.

4. **Hamacher Operations of Neutrosophic Cubic Elements**

The concepts of t-norm and t-conorm, which are useful notions in fuzzy set theory and neutrosophic set theory, are proposed by Roychowdhury and Wang [25]. In 1978, Hamacher [10] defined Hamacher sum ($\oplus_\hbar$) and Hamacher product ($\otimes_\hbar$), which are samples of t-conorm and t-norm, respectively. Hamacher t-norm and Hamacher t-conorm are given as follows.
For all $\hat{a}, \hat{b} \in [0,1]$,

$$\hat{a} \oplus_{\hbar} \hat{b} = \frac{\hat{a}+\hat{b}-\hat{a}\hat{b}-(1-\xi)\hat{a}\hat{b}}{1-(1-\xi)\hat{a}\hat{b}},$$
$$\hat{a} \otimes_{\hbar} \hat{b} = \frac{\hat{a}\hat{b}}{\xi+(1-\xi)(\hat{a}+\hat{b}-\hat{a}\hat{b})}$$

where $\xi > 0$.

Especially, if it is taken $\xi = 1$, then Hamacher t-norm and Hamacher t-conorm will reduce to the form

$$\hat{a} \oplus_{\hbar} \hat{b} = \hat{a} + \hat{b} - \hat{a}\hat{b},$$
$$\hat{a} \otimes_{\hbar} \hat{b} = \hat{a}\hat{b}$$

which represent algebraic t-norm and t-conorm, respectively.

If it is taken $\xi = 2$, then Hamacher t-norm and Hamacher t-conorm will conclude to the form

$$\hat{a} \oplus_{\hbar} \hat{b} = \frac{\hat{a}+\hat{b}}{1-\hat{a}\hat{b}},$$
$$\hat{a} \otimes_{\hbar} \hat{b} = \frac{\hat{a}\hat{b}}{1+(1-\hat{a})(1-\hat{b})}$$

which are called Einstein t-norm and Einstein t-conorm, respectively.

By using the Hamacher t-norm and Hamacher t-conorm, we can create the Hamacher sum and Hamacher product of two NCEs.

Definition 4.1. Let $\upsilon_1 = (\widetilde{\mu}_1, \widetilde{\iota}_1, \widetilde{\eta}_1, \mu_1, \iota_1, \eta_1)$ and $\upsilon_2 = (\widetilde{\mu}_2, \widetilde{\iota}_2, \widetilde{\eta}_2, \mu_2, \iota_2, \eta_2)$ be two CNEs and $\xi > 0$, then the operational rules based on the Hamacher t-norm and Hamacher t-conorm are established as follows:

(a):

$$\upsilon_1 \oplus_{\hbar} \upsilon_2 = \begin{pmatrix} \Big[\frac{\widetilde{\mu}_1^L+\widetilde{\mu}_2^L-\widetilde{\mu}_1^L\widetilde{\mu}_2^L-(1-\xi)\widetilde{\mu}_1^L\widetilde{\mu}_2^L}{1-(1-\xi)\widetilde{\mu}_1^L\widetilde{\mu}_2^L}, \frac{\widetilde{\mu}_1^U+\widetilde{\mu}_2^U-\widetilde{\mu}_1^U\widetilde{\mu}_2^U-(1-\xi)\widetilde{\mu}_1^U\widetilde{\mu}_2^U}{1-(1-\xi)\widetilde{\mu}_1^U\widetilde{\mu}_2^U}\Big], \\ \Big[\frac{\widetilde{\iota}_1^L\widetilde{\iota}_2^L}{\xi+(1-\xi)(\widetilde{\iota}_1^L+\widetilde{\iota}_2^L-\widetilde{\iota}_1^L\widetilde{\iota}_2^L)}, \frac{\widetilde{\iota}_1^U\widetilde{\iota}_2^U}{\xi+(1-\xi)(\widetilde{\iota}_1^U+\widetilde{\iota}_2^U-\widetilde{\iota}_1^U\widetilde{\iota}_2^U)}\Big], \\ \Big[\frac{\widetilde{\eta}_1^L\widetilde{\eta}_2^L}{\xi+(1-\xi)(\widetilde{\eta}_1^L+\widetilde{\eta}_2^L-\widetilde{\eta}_1^L\widetilde{\eta}_2^L)}, \frac{\widetilde{\eta}_1^U\widetilde{\eta}_2^U}{\xi+(1-\xi)(\widetilde{\eta}_1^U+\widetilde{\eta}_2^U-\widetilde{\eta}_1^U\widetilde{\eta}_2^U)}\Big], \\ \frac{\mu_1\mu_2}{\xi+(1-\xi)(\mu_1+\mu_2-\mu_1\mu_2)}, \frac{\iota_1+\iota_2-\iota_1\iota_2-(1-\xi)\iota_1\iota_2}{1-(1-\xi)\iota_1\iota_2}, \frac{\eta_1+\eta_2-\eta_1\eta_2-(1-\xi)\eta_1\eta_2}{1-(1-\xi)\eta_1\eta_2} \end{pmatrix}. \tag{11}$$

(b):

$$\upsilon_1 \otimes_{\hbar} \upsilon_2 = \begin{pmatrix} \Big[\frac{\widetilde{\mu}_1^L\widetilde{\mu}_2^L}{\xi+(1-\xi)(\widetilde{\mu}_1^L+\widetilde{\mu}_2^L-\widetilde{\mu}_1^L\widetilde{\mu}_2^L)}, \frac{\widetilde{\mu}_1^U\widetilde{\mu}_2^U}{\xi+(1-\xi)(\widetilde{\mu}_1^U+\widetilde{\mu}_2^U-\widetilde{\mu}_1^U\widetilde{\mu}_2^U)}\Big], \\ \Big[\frac{\widetilde{\iota}_1^L+\widetilde{\iota}_2^L-\widetilde{\iota}_1^L\widetilde{\iota}_2^L-(1-\xi)\widetilde{\iota}_1^L\widetilde{\iota}_2^L}{1-(1-\xi)\widetilde{\iota}_1^L\widetilde{\iota}_2^L}, \frac{\widetilde{\iota}_1^U+\widetilde{\iota}_2^U-\widetilde{\iota}_1^U\widetilde{\iota}_2^U-(1-\xi)\widetilde{\iota}_1^U\widetilde{\iota}_2^U}{1-(1-\xi)\widetilde{\iota}_1^U\widetilde{\iota}_2^U}\Big], \\ \Big[\frac{\widetilde{\eta}_1^L+\widetilde{\eta}_2^L-\widetilde{\eta}_1^L\widetilde{\eta}_2^L-(1-\xi)\widetilde{\eta}_1^L\widetilde{\eta}_2^L}{1-(1-\xi)\widetilde{\eta}_1^L\widetilde{\eta}_2^L}, \frac{\widetilde{\eta}_1^U+\widetilde{\eta}_2^U-\widetilde{\eta}_1^U\widetilde{\eta}_2^U-(1-\xi)\widetilde{\eta}_1^U\widetilde{\eta}_2^U}{1-(1-\xi)\widetilde{\eta}_1^U\widetilde{\eta}_2^U}\Big], \\ \frac{\mu 1+\mu_2-\mu_1\mu_2-(1-\xi)\mu_1\mu_2}{1-(1-\xi)\mu_1\mu_2}, \frac{\iota_1\iota_2}{\xi+(1-\xi)(\iota_1+\iota_2-\iota_1\iota_2)}, \frac{\eta_1\eta_2}{\xi+(1-\xi)(\eta_1+\eta_2-\eta_1\eta_2)} \end{pmatrix}. \tag{12}$$

(c):

$$q\upsilon_1 = \begin{pmatrix} \Big[\frac{(1+(\xi-1)\widetilde{\mu}_1^L)^q-(1-\widetilde{\mu}_1^L)^q}{(1+(\xi-1)\widetilde{\mu}_1^L)^q+(\xi-1)(1-\widetilde{\mu}_1^L)^q}, \frac{(1+(\xi-1)\widetilde{\mu}_1^U)^q-(1-\widetilde{\mu}_1^U)^q}{(1+(\xi-1)\widetilde{\mu}_1^U)^q+(\xi-1)(1-\widetilde{\mu}_1^U)^q}\Big], \\ \Big[\frac{\xi(\widetilde{\iota}_1^L)^q}{(1+(\xi-1)(1-\widetilde{\iota}_1^L))^q+(\xi-1)(\widetilde{\iota}_1^L)^q}, \frac{\xi(\widetilde{\iota}_1^U)^q}{(1+(\xi-1)(1-\widetilde{\iota}_1^U))^q+(\xi-1)(\widetilde{\iota}_1^U)^q},\Big], \\ \Big[\frac{\xi(\widetilde{\eta}_1^L)^q}{(1+(\xi-1)(1-\widetilde{\eta}_1^L))^q+(\xi-1)(\widetilde{\eta}_1^L)^q}, \frac{\xi(\widetilde{\eta}_1^U)^q}{(1+(\xi-1)(1-\widetilde{\eta}_1^U))^q+(\xi-1)(\widetilde{\eta}_1^U)^q},\Big], \\ \frac{\xi\mu_1^q}{(1+(\xi-1)(1-\mu_1))^q+(\xi-1)\mu_1^q}, \frac{(1+(\xi-1)\iota_1)^q-(1-\iota_1)^q}{(1+(\xi-1)\iota_1)^q+(\xi-1)(1-\iota_1)^q}, \frac{(1+(\xi-1)\eta_1)^q-(1-\eta_1)^q}{(1+(\xi-1)\eta_1)^q+(\xi-1)(1-\eta_1)^q} \end{pmatrix} \tag{13}$$

where $q > 0$.

(d):

$$\upsilon_1^q = \begin{pmatrix} \Big[\frac{\xi(\widetilde{\mu}_1^L)^q}{(1+(\xi-1)(1-\widetilde{\mu}_1^L))^q+(\xi-1)(\widetilde{\mu}_1^L)^q}, \frac{\xi(\widetilde{\mu}_1^U)^q}{(1+(\xi-1)(1-\widetilde{\mu}_1^U))^q+(\xi-1)(\widetilde{\mu}_1^U)^q}, \Big], \\ \Big[\frac{(1+(\xi-1)\widetilde{\iota}_1^L)^q-(1-\widetilde{\iota}_1^L)^q}{(1+(\xi-1)\widetilde{\iota}_1^L)^q+(\xi-1)(1-\widetilde{\iota}_1^L)^q}, \frac{(1+(\xi-1)\widetilde{\iota}_1^U)^q-(1-\widetilde{\iota}_1^U)^q}{(1+(\xi-1)\widetilde{\iota}_1^U)^q+(\xi-1)(1-\widetilde{\iota}_1^U)^q}\Big], \\ \Big[\frac{(1+(\xi-1)\widetilde{\eta}_1^L)^q-(1-\widetilde{\eta}_1^L)^q}{(1+(\xi-1)\widetilde{\eta}_1^L)^q+(\xi-1)(1-\widetilde{\eta}_1^L)^q}, \frac{(1+(\xi-1)\widetilde{\eta}_1^U)^q-(1-\widetilde{\eta}_1^U)^q}{(1+(\xi-1)\widetilde{\eta}_1^U)^q+(\xi-1)(1-\widetilde{\eta}_1^U)^q}\Big], \\ \frac{(1+(\xi-1)\mu_1)^q-(1-\mu_1)^q}{(1+(\xi-1)\mu_1)^q+(\xi-1)(1-\mu_1)^q}, \frac{\xi\iota_1^q}{(1+(\xi-1)(1-\iota_1))^q+(\xi-1)\iota_1^q}, \frac{\xi\eta_1^q}{(1+(\xi-1)(1-\eta_1))^q+(\xi-1)\eta_1^q} \end{pmatrix} \tag{14}$$

where $q > 0$.

Example 4.2. Assume that two NCEs are $\upsilon_1 = ([0.4, 1], [0.7, 0.8], [0, 0.2], 0.5, 0, 0.7)$ and $\upsilon_1 = ([0.2, 0.4], [0.5, 0.6], [0.5, 0.6], 0.1, 1, 0.4)$ and $q = 2$. Then, for $\xi = 3$

$\upsilon_1 \oplus_\hbar \upsilon_2 = ([0.8095, 1], [0.2692, 0.4137], [0, 0.0731], 0.0263, 1, 0.8846)$,

$\upsilon_1 \otimes_\hbar \upsilon_2 = ([0.0408, 0.4], [0.9117, 0.9591], [0.5, 0.7419], 0.5909, 0, 0.2058)$,

$q\upsilon_2 = ([0.4074, 0.7272], [0.1666, 0.2727], [0.1666, 0.2727], 0.0038, 1, 0.7272)$,

$\upsilon_1^q = ([0.1237, 1], [0.9545, 0.9824], [0, 0.4074], 0.8333, 0, 0.4152)$.

Proposition 4.3. *Let υ_1 and υ_2 be two NCEs and $q, q' > 0$.*

(1): $\upsilon_1 \oplus_\hbar \upsilon_2 = \upsilon_2 \oplus_\hbar \upsilon_1$.

(2): $\upsilon_1 \otimes_\hbar \upsilon_2 = \upsilon_2 \otimes_\hbar \upsilon_1$.

(3): $q(\upsilon_1 \oplus_\hbar \upsilon_2) = q\upsilon_1 \oplus_\hbar q\upsilon_2$.

(4): $q\upsilon_1 \oplus_\hbar q'\upsilon_1) = (q + q')\upsilon_1$.

(5): $(\upsilon_1 \otimes_\hbar \upsilon_2)^q = \upsilon_1^q \otimes_\hbar \upsilon_2^q$.

(6): $\upsilon_1^q \otimes_\hbar \upsilon_1^{q'} = \upsilon_1^{q+q'}$.

Proof. They are easily seen from the formulas in Definition 4.1, hence omitted. □

5. Neutrosophic Cubic Hamacher Weighted Aggregation Operators

In this section, we will introduce the neutrodophic cubic Hamacher weighted averaging operator and neutrosophic cubic Hamacher weighted geometric operator.

Definition 5.1. Let υ_k $(k = 1, 2, ..., r)$ be a collection of the CNEs. Then, neutrosophic cubic Hamacher weighted averaging (NCHWA) operator is defined as the mapping $NCHWA_\varpi : \mathcal{N}^r \to \mathcal{N}$ such that

$$NCHWA_\varpi(\upsilon_1, \upsilon_2, ..., \upsilon_r) = \bigoplus_{k=1}^{r}{}_\hbar (\varpi_k \upsilon_k) \tag{15}$$

where $\mathcal{N}$ is the set of all NCEs and $\varpi = (\varpi_1, \varpi_2, ..., \varpi_r)^T$ is weight vector of $(\upsilon_1, \upsilon_2, ..., \upsilon_r)$ such that $\varpi_k \in [0, 1]$ and $\sum_{k=1}^{r} \varpi_k = 1$.

Theorem 5.2. *The aggregation value of NCEs by using the NCHWA operator is still an NCE, and even*

$$NCHWA_{\varpi}(\upsilon_1,\upsilon_2,...,\upsilon_r)=\bigoplus_{\hbar\,k=1}^{r}(\varpi_k\upsilon_k)=$$
$$\left(\begin{array}{l}\Big[\frac{\prod_{k=1}^{r}(1+(\xi-1)\widetilde{\mu}_k^L)^{\varpi_k}-\prod_{k=1}^{r}(1-\widetilde{\mu}_k^L)^{\varpi_k}}{\prod_{k=1}^{r}(1+(\xi-1)\widetilde{\mu}_k^L)^{\varpi_k}+(\xi-1)\prod_{k=1}^{r}(1-\widetilde{\mu}_k^L)^{\varpi_k}},\frac{\prod_{k=1}^{r}(1+(\xi-1)\widetilde{\mu}_k^U)^{\varpi_k}-\prod_{k=1}^{r}(1-\widetilde{\mu}_k^U)^{\varpi_k}}{\prod_{k=1}^{r}(1+(\xi-1)\widetilde{\mu}_k^U)^{\varpi_k}+(\xi-1)\prod_{k=1}^{r}(1-\widetilde{\mu}_k^U)^{\varpi_k}}\Big],\\ \Big[\frac{\xi\prod_{k=1}^{r}(\widetilde{\iota}_k^L)^{\varpi_k}}{\prod_{k=1}^{r}(1+(\xi-1)(1-\widetilde{\iota}_k^L))^{\varpi_k}+(\xi-1)\prod_{k=1}^{r}(\widetilde{\iota}_k^L)^{\varpi_k}},\frac{\xi\prod_{k=1}^{r}(\widetilde{\iota}_k^U)^{\varpi_k}}{\prod_{k=1}^{r}(1+(\xi-1)(1-\widetilde{\iota}_k^U))^{\varpi_k}+(\xi-1)\prod_{k=1}^{r}(\widetilde{\iota}_k^U)^{\varpi_k}}\Big],\\ \Big[\frac{\xi\prod_{k=1}^{r}(\widetilde{\eta}_k^L)^{\varpi_k}}{\prod_{k=1}^{r}(1+(\xi-1)(1-\widetilde{\eta}_k^L))^{\varpi_k}+(\xi-1)\prod_{k=1}^{r}(\widetilde{\eta}_k^L)^{\varpi_k}},\frac{\xi\prod_{k=1}^{r}(\widetilde{\eta}_k^U)^{\varpi_k}}{\prod_{k=1}^{r}(1+(\xi-1)(1-\widetilde{\eta}_k^U))^{\varpi_k}+(\xi-1)\prod_{k=1}^{r}(\widetilde{\eta}_k^U)^{\varpi_k}}\Big],\\ \frac{\xi\prod_{k=1}^{r}(\mu_k)^{\varpi_k}}{\prod_{k=1}^{r}(1+(\xi-1)(1-\mu_k))^{\varpi_k}+(\xi-1)\prod_{k=1}^{r}(\mu_k)^{\varpi_k}},\frac{\prod_{k=1}^{r}(1+(\xi-1)\iota_k)^{\varpi_k}-\prod_{k=1}^{r}(1-\iota_k)^{\varpi_k}}{\prod_{k=1}^{r}(1+(\xi-1)\iota_k)^{\varpi_k}+(\xi-1)\prod_{k=1}^{r}(1-\iota_k)^{\varpi_k}},\frac{\prod_{k=1}^{r}(1+(\xi-1)\eta_k)^{\varpi_k}-\prod_{k=1}^{r}(1-\eta_k)^{\varpi_k}}{\prod_{k=1}^{r}(1+(\xi-1)\eta_k)^{\varpi_k}+(\xi-1)\prod_{k=1}^{r}(1-\eta_k)^{\varpi_k}}\end{array}\right) \tag{16}$$

Proof. This can be proved by mathematical induction.
When $r=1$, for the left side of Eq. (16), $NCHWA_{\varpi}(\upsilon_1)=\varpi_1\upsilon_1=\upsilon_1$ and for the right side of Eq. (16) we have

$$\left(\begin{array}{l}\Big[\frac{1+(\xi-1)\widetilde{\mu}_1^L-(1-\widetilde{\mu}_1^L)}{1+(\xi-1)\widetilde{\mu}_1^L+(\xi-1)(1-\widetilde{\mu}_1^L)},\frac{1+(\xi-1)\widetilde{\mu}_1^U-(1-\widetilde{\mu}_1^U)}{1+(\xi-1)\widetilde{\mu}_1^U+(\xi-1)(1-\widetilde{\mu}_1^U)}\Big],\\ \Big[\frac{\xi\widetilde{\iota}_1^L}{1+(\xi-1)(1-\widetilde{\iota}_1^L)+(\xi-1)\widetilde{\iota}_1^L},\frac{\xi\widetilde{\iota}_1^U}{1+(\xi-1)(1-\widetilde{\iota}_1^U)+(\xi-1)\widetilde{\iota}_1^U}\Big],\\ \Big[\frac{\xi\widetilde{\eta}_1^L}{1+(\xi-1)(1-\widetilde{\eta}_1^L)+(\xi-1)\widetilde{\eta}_1^L},\frac{\xi\widetilde{\eta}_1^U}{1+(\xi-1)(1-\widetilde{\eta}_1^U)+(\xi-1)\widetilde{\eta}_1^U}\Big],\\ \frac{\xi\mu_1}{1+(\xi-1)(1-\mu_1)+(\xi-1)\mu_1},\frac{1+(\xi-1)\iota_1-(1-\iota_1)}{1+(\xi-1)\iota_1+(\xi-1)(1-\iota_1)},\frac{1+(\xi-1)\eta_1-(1-\eta_1)}{1+(\xi-1)\eta_1+(\xi-1)(1-\eta_1)}\end{array}\right).$$

Suppose that Eq. (16) holds for $r=t$, i.e., we have

$$NCHWA_{\varpi}(\upsilon_1,\upsilon_2,...,\upsilon_t)=\bigoplus_{\hbar\,k=1}^{t}(\varpi_k\upsilon_k)=$$
$$\left(\begin{array}{l}\Big[\frac{\prod_{k=1}^{t}(1+(\xi-1)\widetilde{\mu}_k^L)^{\varpi_k}-\prod_{k=1}^{t}(1-\widetilde{\mu}_k^L)^{\varpi_k}}{\prod_{k=1}^{t}(1+(\xi-1)\widetilde{\mu}_k^L)^{\varpi_k}+(\xi-1)\prod_{k=1}^{t}(1-\widetilde{\mu}_k^L)^{\varpi_k}},\frac{\prod_{k=1}^{t}(1+(\xi-1)\widetilde{\mu}_k^U)^{\varpi_k}-\prod_{k=1}^{t}(1-\widetilde{\mu}_k^U)^{\varpi_k}}{\prod_{k=1}^{t}(1+(\xi-1)\widetilde{\mu}_k^U)^{\varpi_k}+(\xi-1)\prod_{k=1}^{t}(1-\widetilde{\mu}_k^U)^{\varpi_k}}\Big],\\ \Big[\frac{\xi\prod_{k=1}^{t}(\widetilde{\iota}_k^L)^{\varpi_k}}{\prod_{k=1}^{t}(1+(\xi-1)(1-\widetilde{\iota}_k^L))^{\varpi_k}+(\xi-1)\prod_{k=1}^{t}(\widetilde{\iota}_k^L)^{\varpi_k}},\frac{\xi\prod_{k=1}^{t}(\widetilde{\iota}_k^U)^{\varpi_k}}{\prod_{k=1}^{t}(1+(\xi-1)(1-\widetilde{\iota}_k^U))^{\varpi_k}+(\xi-1)\prod_{k=1}^{t}(\widetilde{\iota}_k^U)^{\varpi_k}}\Big],\\ \Big[\frac{\xi\prod_{k=1}^{t}(\widetilde{\eta}_k^L)^{\varpi_k}}{\prod_{k=1}^{t}(1+(\xi-1)(1-\widetilde{\eta}_k^L))^{\varpi_k}+(\xi-1)\prod_{k=1}^{t}(\widetilde{\eta}_k^L)^{\varpi_k}},\frac{\xi\prod_{k=1}^{t}(\widetilde{\eta}_k^U)^{\varpi_k}}{\prod_{k=1}^{t}(1+(\xi-1)(1-\widetilde{\eta}_k^U))^{\varpi_k}+(\xi-1)\prod_{k=1}^{t}(\widetilde{\eta}_k^U)^{\varpi_k}}\Big],\\ \frac{\xi\prod_{k=1}^{t}(\mu_k)^{\varpi_k}}{\prod_{k=1}^{t}(1+(\xi-1)(1-\mu_k))^{\varpi_k}+(\xi-1)\prod_{k=1}^{t}(\mu_k)^{\varpi_k}},\frac{\prod_{k=1}^{t}(1+(\xi-1)\iota_k)^{\varpi_k}-\prod_{k=1}^{t}(1-\iota_k)^{\varpi_k}}{\prod_{k=1}^{t}(1+(\xi-1)\iota_k)^{\varpi_k}+(\xi-1)\prod_{k=1}^{t}(1-\iota_k)^{\varpi_k}},\frac{\prod_{k=1}^{t}(1+(\xi-1)\eta_k)^{\varpi_k}-\prod_{k=1}^{t}(1-\eta_k)^{\varpi_k}}{\prod_{k=1}^{t}(1+(\xi-1)\eta_k)^{\varpi_k}+(\xi-1)\prod_{k=1}^{t}(1-\eta_k)^{\varpi_k}}\end{array}\right).$$

When $r = t+1$,

$NCHWA_{\varpi}(\upsilon_1, \upsilon_2, ..., \upsilon_{t+1}) = NCHWA_{\varpi}(\upsilon_1, \upsilon_2, ..., \upsilon_t) \oplus_{\hbar} (\varpi_{t+1}\upsilon_{t+1}) =$

$$\begin{pmatrix} \Big[\frac{\prod_{k=1}^{t}(1+(\xi-1)\widetilde{\mu}_k^L)^{\varpi_k}-\prod_{k=1}^{t}(1-\widetilde{\mu}_k^L)^{\varpi_k}}{\prod_{k=1}^{t}(1+(\xi-1)\widetilde{\mu}_k^L)^{\varpi_k}+(\xi-1)\prod_{k=1}^{t}(1-\widetilde{\mu}_k^L)^{\varpi_k}}, \frac{\prod_{k=1}^{t}(1+(\xi-1)\widetilde{\mu}_k^U)^{\varpi_k}-\prod_{k=1}^{t}(1-\widetilde{\mu}_k^U)^{\varpi_k}}{\prod_{k=1}^{t}(1+(\xi-1)\widetilde{\mu}_k^U)^{\varpi_k}+(\xi-1)\prod_{k=1}^{t}(1-\widetilde{\mu}_k^U)^{\varpi_k}}\Big], \\ \Big[\frac{\xi\prod_{k=1}^{t}(\widetilde{\iota}_k^L)^{\varpi_k}}{\prod_{k=1}^{t}(1+(\xi-1)(1-\widetilde{\iota}_k^L))^{\varpi_k}+(\xi-1)\prod_{k=1}^{t}(\widetilde{\iota}_k^L)^{\varpi_k}}, \frac{\xi\prod_{k=1}^{t}(\widetilde{\iota}_k^U)^{\varpi_k}}{\prod_{k=1}^{t}(1+(\xi-1)(1-\widetilde{\iota}_k^U))^{\varpi_k}+(\xi-1)\prod_{k=1}^{t}(\widetilde{\iota}_k^U)^{\varpi_k}}\Big], \\ \Big[\frac{\xi\prod_{k=1}^{t}(\widetilde{\eta}_k^L)^{\varpi_k}}{\prod_{k=1}^{t}(1+(\xi-1)(1-\widetilde{\eta}_k^L))^{\varpi_k}+(\xi-1)\prod_{k=1}^{t}(\widetilde{\eta}_k^L)^{\varpi_k}}, \frac{\xi\prod_{k=1}^{t}(\widetilde{\eta}_k^U)^{\varpi_k}}{\prod_{k=1}^{t}(1+(\xi-1)(1-\widetilde{\eta}_k^U))^{\varpi_k}+(\xi-1)\prod_{k=1}^{t}(\widetilde{\eta}_k^U)^{\varpi_k}}\Big], \\ \frac{\xi\prod_{k=1}^{t}(\mu_k)^{\varpi_k}}{\prod_{k=1}^{t}(1+(\xi-1)(1-\mu_k))^{\varpi_k}+(\xi-1)\prod_{k=1}^{t}(\mu_k)^{\varpi_k}}, \frac{\prod_{k=1}^{t}(1+(\xi-1)\iota_k)^{\varpi_k}-\prod_{k=1}^{t}(1-\iota_k)^{\varpi_k}}{\prod_{k=1}^{t}(1+(\xi-1)\iota_k)^{\varpi_k}+(\xi-1)\prod_{k=1}^{t}(1-\iota_k)^{\varpi_k}}, \frac{\prod_{k=1}^{t}(1+(\xi-1)\eta_k)^{\varpi_k}-\prod_{k=1}^{t}(1-\eta_k)^{\varpi_k}}{\prod_{k=1}^{t}(1+(\xi-1)\eta_k)^{\varpi_k}+(\xi-1)\prod_{k=1}^{t}(1-\eta_k)^{\varpi_k}} \end{pmatrix}$$

$$\oplus_{\hbar} \begin{pmatrix} \Big[\frac{1+(\xi-1)\widetilde{\mu}_{t+1}^L-(1-\widetilde{\mu}_{t+1}^L)}{1+(\xi-1)\widetilde{\mu}_{t+1}^L+(\xi-1)(1-\widetilde{\mu}_{t+1}^L)}, \frac{1+(\xi-1)\widetilde{\mu}_{t+1}^U-(1-\widetilde{\mu}_{t+1}^U)}{1+(\xi-1)\widetilde{\mu}_{t+1}^U+(\xi-1)(1-\widetilde{\mu}_{t+1}^U)}\Big], \\ \Big[\frac{\xi\widetilde{\iota}_1^L}{1+(\xi-1)(1-\widetilde{\iota}_{t+1}^L)+(\xi-1)\widetilde{\iota}_{t+1}^L}, \frac{\xi\widetilde{\iota}_{t+1}^U}{1+(\xi-1)(1-\widetilde{\iota}_{t+1}^U)+(\xi-1)\widetilde{\iota}_{t+1}^U}\Big], \\ \Big[\frac{\xi\widetilde{\eta}_{t+1}^L}{1+(\xi-1)(1-\widetilde{\eta}_{t+1}^L)+(\xi-1)\widetilde{\eta}_{t+1}^L}, \frac{\xi\widetilde{\eta}_{t+1}^U}{1+(\xi-1)(1-\widetilde{\eta}_{t+1}^U)+(\xi-1)\widetilde{\eta}_{t+1}^U}\Big], \\ \frac{\xi\mu_{t+1}}{1+(\xi-1)(1-\mu_{t+1})+(\xi-1)\mu_{t+1}}, \frac{1+(\xi-1)\iota_{t+1}-(1-\iota_{t+1})}{1+(\xi-1)\iota_{t+1}+(\xi-1)(1-\iota_{t+1})}, \frac{1+(\xi-1)\eta_{t+1}-(1-\eta_{t+1})}{1+(\xi-1)\eta_{t+1}+(\xi-1)(1-\eta_{t+1})} \end{pmatrix}.$$

$$= \begin{pmatrix} \Big[\frac{\prod_{k=1}^{t+1}(1+(\xi-1)\widetilde{\mu}_k^L)^{\varpi_k}-\prod_{k=1}^{t+1}(1-\widetilde{\mu}_k^L)^{\varpi_k}}{\prod_{k=1}^{t+1}(1+(\xi-1)\widetilde{\mu}_k^L)^{\varpi_k}+(\xi-1)\prod_{k=1}^{t+1}(1-\widetilde{\mu}_k^L)^{\varpi_k}}, \frac{\prod_{k=1}^{t+1}(1+(\xi-1)\widetilde{\mu}_k^U)^{\varpi_k}-\prod_{k=1}^{t+1}(1-\widetilde{\mu}_k^U)^{\varpi_k}}{\prod_{k=1}^{t+1}(1+(\xi-1)\widetilde{\mu}_k^U)^{\varpi_k}+(\xi-1)\prod_{k=1}^{t+1}(1-\widetilde{\mu}_k^U)^{\varpi_k}}\Big], \\ \Big[\frac{\xi\prod_{k=1}^{t+1}(\widetilde{\iota}_k^L)^{\varpi_k}}{\prod_{k=1}^{t+1}(1+(\xi-1)(1-\widetilde{\iota}_k^L))^{\varpi_k}+(\xi-1)\prod_{k=1}^{t+1}(\widetilde{\iota}_k^L)^{\varpi_k}}, \frac{\xi\prod_{k=1}^{t+1}(\widetilde{\iota}_k^U)^{\varpi_k}}{\prod_{k=1}^{t+1}(1+(\xi-1)(1-\widetilde{\iota}_k^U))^{\varpi_k}+(\xi-1)\prod_{k=1}^{t+1}(\widetilde{\iota}_k^U)^{\varpi_k}}\Big], \\ \Big[\frac{\xi\prod_{k=1}^{t+1}(\widetilde{\eta}_k^L)^{\varpi_k}}{\prod_{k=1}^{t+1}(1+(\xi-1)(1-\widetilde{\eta}_k^L))^{\varpi_k}+(\xi-1)\prod_{k=1}^{t+1}(\widetilde{\eta}_k^L)^{\varpi_k}}, \frac{\xi\prod_{k=1}^{t+1}(\widetilde{\eta}_k^U)^{\varpi_k}}{\prod_{k=1}^{t+1}(1+(\xi-1)(1-\widetilde{\eta}_k^U))^{\varpi_k}+(\xi-1)\prod_{k=1}^{t+1}(\widetilde{\eta}_k^U)^{\varpi_k}}\Big], \\ \frac{\xi\prod_{k=1}^{t+1}(\mu_k)^{\varpi_k}}{\prod_{k=1}^{t+1}(1+(\xi-1)(1-\mu_k))^{\varpi_k}+(\xi-1)\prod_{k=1}^{t+1}(\mu_k)^{\varpi_k}}, \frac{\prod_{k=1}^{t+1}(1+(\xi-1)\iota_k)^{\varpi_k}-\prod_{k=1}^{t+1}(1-\iota_k)^{\varpi_k}}{\prod_{k=1}^{t+1}(1+(\xi-1)\iota_k)^{\varpi_k}+(\xi-1)\prod_{k=1}^{t+1}(1-\iota_k)^{\varpi_k}}, \frac{\prod_{k=1}^{t+1}(1+(\xi-1)\eta_k)^{\varpi_k}-\prod_{k=1}^{t+1}(1-\eta_k)^{\varpi_k}}{\prod_{k=1}^{t+1}(1+(\xi-1)\eta_k)^{\varpi_k}+(\xi-1)\prod_{k=1}^{t+1}(1-\eta_k)^{\varpi_k}}, \end{pmatrix}.$$

So, Eq. (16) holds for $r = t+1$. Thus, the proof is complete. □

Definition 5.3. Let υ_k $(k = 1, 2, ..., r)$ be a collection of the CNEs. Then, neutrosophic cubic Hamacher weighted geometric (NCHWG) operator is defined as the mapping $NCHWG_{\varpi} : \mathcal{N}^r \to \mathcal{N}$ such that

$$NCHWG_{\varpi}(\upsilon_1, \upsilon_2, ..., \upsilon_r) = \bigotimes_{k=1}^{r}{}_{\hbar}\, \upsilon_k^{\varpi_k} \tag{17}$$

where $\mathcal{N}$ is the set of all NCEs and $\varpi = (\varpi_1, \varpi_2, ..., \varpi_r)^T$ is weight vector of $(\upsilon_1, \upsilon_2, ..., \upsilon_r)$ such that $\varpi_k \in [0, 1]$ and $\sum_{k=1}^{r} \varpi_k = 1$.

Theorem 5.4. *The aggregation value of NCEs by using the NCHWG operator is still an NCE, and even*

$$NCHWG_{\varpi}(\upsilon_1,\upsilon_2,...,\upsilon_r)=\bigotimes_{k=1}^{r}{}_{\hbar}\,\upsilon_k^{\varpi_k}=$$

$$\left(\begin{array}{l}\Big[\frac{\xi\prod_{k=1}^{r}(\widetilde{\mu}_k^L)^{\varpi_k}}{\prod_{k=1}^{r}(1+(\xi-1)(1-\widetilde{\mu}_k^L))^{\varpi_k}+(\xi-1)\prod_{k=1}^{r}(\widetilde{\mu}_k^L)^{\varpi_k}},\frac{\xi\prod_{k=1}^{r}(\widetilde{\mu}_k^U)^{\varpi_k}}{\prod_{k=1}^{r}(1+(\xi-1)(1-\widetilde{\mu}_k^U))^{\varpi_k}+(\xi-1)\prod_{k=1}^{r}(\widetilde{\mu}_k^U)^{\varpi_k}}\Big],\\ \Big[\frac{\prod_{k=1}^{r}(1+(\xi-1)\widetilde{\iota}_k^L)^{\varpi_k}-\prod_{k=1}^{r}(1-\widetilde{\iota}_k^L)^{\varpi_k}}{\prod_{k=1}^{r}(1+(\xi-1)\widetilde{\iota}_k^L)^{\varpi_k}+(\xi-1)\prod_{k=1}^{r}(1-\widetilde{\iota}_k^L)^{\varpi_k}},\frac{\prod_{k=1}^{r}(1+(\xi-1)\widetilde{\iota}_k^U)^{\varpi_k}-\prod_{k=1}^{r}(1-\widetilde{\iota}_k^U)^{\varpi_k}}{\prod_{k=1}^{r}(1+(\xi-1)\widetilde{\iota}_k^U)^{\varpi_k}+(\xi-1)\prod_{k=1}^{r}(1-\widetilde{\iota}_k^U)^{\varpi_k}}\Big],\\ \Big[\frac{\prod_{k=1}^{r}(1+(\xi-1)\widetilde{\eta}_k^L)^{\varpi_k}-\prod_{k=1}^{r}(1-\widetilde{\eta}_k^L)^{\varpi_k}}{\prod_{k=1}^{r}(1+(\xi-1)\widetilde{\eta}_k^L)^{\varpi_k}+(\xi-1)\prod_{k=1}^{r}(1-\widetilde{\eta}_k^L)^{\varpi_k}},\frac{\prod_{k=1}^{r}(1+(\xi-1)\widetilde{\eta}_k^U)^{\varpi_k}-\prod_{k=1}^{r}(1-\widetilde{\eta}_k^U)^{\varpi_k}}{\prod_{k=1}^{r}(1+(\xi-1)\widetilde{\eta}_k^U)^{\varpi_k}+(\xi-1)\prod_{k=1}^{r}(1-\widetilde{\eta}_k^U)^{\varpi_k}}\Big],\\ \frac{\prod_{k=1}^{r}(1+(\xi-1)\mu_k)^{\varpi_k}-\prod_{k=1}^{r}(1-\mu_k)^{\varpi_k}}{\prod_{k=1}^{r}(1+(\xi-1)\mu_k)^{\varpi_k}+(\xi-1)\prod_{k=1}^{r}(1-\mu_k)^{\varpi_k}},\frac{\xi\prod_{k=1}^{r}(\iota_k)^{\varpi_k}}{\prod_{k=1}^{r}(1+(\xi-1)(1-\iota_k))^{\varpi_k}+(\xi-1)\prod_{k=1}^{r}(\iota_k)^{\varpi_k}},\frac{\xi\prod_{k=1}^{r}(\eta_k)^{\varpi_k}}{\prod_{k=1}^{r}(1+(\xi-1)(1-\eta_k))^{\varpi_k}+(\xi-1)\prod_{k=1}^{r}(\eta_k)^{\varpi_k}}\end{array}\right) \tag{18}$$

Proof. It can be demonstrated similar to the proof of Theorem 5.2. □

Theorem 5.5. *(Idempotency)*
Let υ_k $(k=1,2,...,r)$ be a collection of the NCEs. If $\upsilon_k=\upsilon$ for all $k=1,2,...,r$ then

(1): $NCHWA_{\varpi}(\upsilon_1,\upsilon_2,...,\upsilon_r)=\upsilon$.
(2): $NCHWG_{\varpi}(\upsilon_1,\upsilon_2,...,\upsilon_r)=\upsilon$.

Proof. Let's prove (2), the other can be proved similar to this. Assume $\upsilon_k=\upsilon$ for all $k=1,2,...,r$. By Theorem 5.4, we obtain that

$$NCHWG_{\Omega}(\upsilon_1,\upsilon_2,...,\upsilon_r)=\bigotimes_{k=1}^{r}{}_{\hbar}\,\upsilon_k^{\varpi_j}=\bigotimes_{k=1}^{r}{}_{\hbar}\,\upsilon^{\varpi_j}=$$

$$\left(\begin{array}{l}\Big[\frac{\xi\prod_{k=1}^{r}(\widetilde{\mu}^L)^{\varpi_k}}{\prod_{k=1}^{r}(1+(\xi-1)(1-\widetilde{\mu}^L))^{\varpi_k}+(\xi-1)\prod_{k=1}^{r}(\widetilde{\mu}^L)^{\varpi_k}},\frac{\xi\prod_{k=1}^{r}(\widetilde{\mu}^U)^{\varpi_k}}{\prod_{k=1}^{r}(1+(\xi-1)(1-\widetilde{\mu}^U))^{\varpi_k}+(\xi-1)\prod_{k=1}^{r}(\widetilde{\mu}^U)^{\varpi_k}}\Big],\\ \Big[\frac{\prod_{k=1}^{r}(1+(\xi-1)\widetilde{\iota}^L)^{\varpi_k}-\prod_{k=1}^{r}(1-\widetilde{\iota}^L)^{\varpi_k}}{\prod_{k=1}^{r}(1+(\xi-1)\widetilde{\iota}^L)^{\varpi_k}+(\xi-1)\prod_{k=1}^{r}(1-\widetilde{\iota}^L)^{\varpi_k}},\frac{\prod_{k=1}^{r}(1+(\xi-1)\widetilde{\iota}^U)^{\varpi_k}-\prod_{k=1}^{r}(1-\widetilde{\iota}^U)^{\varpi_k}}{\prod_{k=1}^{r}(1+(\xi-1)\widetilde{\iota}^U)^{\varpi_k}+(\xi-1)\prod_{k=1}^{r}(1-\widetilde{\iota}^U)^{\varpi_k}}\Big],\\ \Big[\frac{\prod_{k=1}^{r}(1+(\xi-1)\widetilde{\eta}^L)^{\varpi_k}-\prod_{k=1}^{r}(1-\widetilde{\eta}^L)^{\varpi_k}}{\prod_{k=1}^{r}(1+(\xi-1)\widetilde{\eta}^L)^{\varpi_k}+(\xi-1)\prod_{k=1}^{r}(1-\widetilde{\eta}_k^L)^{\varpi_k}},\frac{\prod_{k=1}^{r}(1+(\xi-1)\widetilde{\eta}_k^U)^{\varpi_k}-\prod_{k=1}^{r}(1-\widetilde{\eta}^U)^{\varpi_k}}{\prod_{k=1}^{r}(1+(\xi-1)\widetilde{\eta}^U)^{\varpi_k}+(\xi-1)\prod_{k=1}^{r}(1-\widetilde{\eta}^U)^{\varpi_k}}\Big],\\ \frac{\prod_{k=1}^{r}(1+(\xi-1)\mu)^{\varpi_k}-\prod_{k=1}^{r}(1-\mu)^{\varpi_k}}{\prod_{k=1}^{r}(1+(\xi-1)\mu)^{\varpi_k}+(\xi-1)\prod_{k=1}^{r}(1-\mu)^{\varpi_k}},\frac{\xi\prod_{k=1}^{r}(\iota)^{\varpi_k}}{\prod_{k=1}^{r}(1+(\xi-1)(1-\iota))^{\varpi_k}+(\xi-1)\prod_{k=1}^{r}(\iota)^{\varpi_k}},\frac{\xi\prod_{k=1}^{r}(\eta)^{\varpi_k}}{\prod_{k=1}^{r}(1+(\xi-1)(1-\eta))^{\varpi_k}+(\xi-1)\prod_{k=1}^{r}(\eta)^{\varpi_k}}\end{array}\right)=\upsilon.$$

□

Theorem 5.6. *(Monotonicity)*
Let υ_k and ν_k $(k=1,2,...,r)$ be two collections of the NCEs. If $\upsilon_k\leq\upsilon_k'$ for all $k=1,2,...,r$ then

(1): $NCHWA_{\varpi}(\upsilon_1,\upsilon_2,...,\upsilon_r)\leq NCHWA_{\varpi}(\upsilon_1',\upsilon_2',...,\upsilon_r')$.
(2): $NCHWG_{\varpi}(\upsilon_1,\upsilon_2,...,\upsilon_r)\leq NCHWG_{\varpi}(\upsilon_1',\upsilon_2',...,\upsilon_r')$.

Proof. (1) If $\upsilon_k \leq \upsilon'_k$ then we have

$$\left\{\begin{array}{l} \widetilde{\mu}_k^L \leq \widetilde{\mu'}_k^L,\ \widetilde{\mu}_k^U \leq \widetilde{\mu'}_k^U \\ \widetilde{\iota}_k^L \geq \widetilde{\iota'}_k^L,\ \widetilde{\iota}_k^U \geq \widetilde{\iota'}_k^U \\ \widetilde{\eta}_k^L \geq \widetilde{\eta'}_k^L,\ \widetilde{\eta}_k^L \geq \widetilde{\eta'}_k^L \\ \mu_k \geq \mu'_k,\ \iota_k \leq \iota'_k,\ \eta_k \leq \eta'_k \end{array}\right\}_{(k=1,2,...,r)}.$$

With these assumptions, we find that

$$\left(\begin{array}{l} \left[\frac{\prod_{k=1}^{r}(1+(\xi-1)\widetilde{\mu}_k^L)^{\varpi_k}-\prod_{k=1}^{r}(1-\widetilde{\mu}_k^L)^{\varpi_k}}{\prod_{k=1}^{r}(1+(\xi-1)\widetilde{\mu}_k^L)^{\varpi_k}+(\xi-1)\prod_{k=1}^{r}(1-\widetilde{\mu}_k^L)^{\varpi_k}}, \frac{\prod_{k=1}^{r}(1+(\xi-1)\widetilde{\mu}_k^U)^{\varpi_k}-\prod_{k=1}^{r}(1-\widetilde{\mu}_k^U)^{\varpi_k}}{\prod_{k=1}^{r}(1+(\xi-1)\widetilde{\mu}_k^U)^{\varpi_k}+(\xi-1)\prod_{k=1}^{r}(1-\widetilde{\mu}_k^U)^{\varpi_k}}\right], \\ \left[\frac{\xi\prod_{k=1}^{r}(\widetilde{\iota}_k^L)^{\varpi_k}}{\prod_{k=1}^{r}(1+(\xi-1)(1-\widetilde{\iota}_k^L))^{\varpi_k}+(\xi-1)\prod_{k=1}^{r}(\widetilde{\iota}_k^L)^{\varpi_k}}, \frac{\xi\prod_{k=1}^{r}(\widetilde{\iota}_k^U)^{\varpi_k}}{\prod_{k=1}^{r}(1+(\xi-1)(1-\widetilde{\iota}_k^U))^{\varpi_k}+(\xi-1)\prod_{k=1}^{r}(\widetilde{\iota}_k^U)^{\varpi_k}}\right], \\ \left[\frac{\xi\prod_{k=1}^{r}(\widetilde{\eta}_k^L)^{\varpi_k}}{\prod_{k=1}^{r}(1+(\xi-1)(1-\widetilde{\eta}_k^L))^{\varpi_k}+(\xi-1)\prod_{k=1}^{r}(\widetilde{\eta}_k^L)^{\varpi_k}}, \frac{\xi\prod_{k=1}^{r}(\widetilde{\eta}_k^U)^{\varpi_k}}{\prod_{k=1}^{r}(1+(\xi-1)(1-\widetilde{\eta}_k^U))^{\varpi_k}+(\xi-1)\prod_{k=1}^{r}(\widetilde{\eta}_k^U)^{\varpi_k}}\right], \\ \frac{\xi\prod_{k=1}^{r}(\mu_k)^{\varpi_k}}{\prod_{k=1}^{r}(1+(\xi-1)(1-\mu_k))^{\varpi_k}+(\xi-1)\prod_{k=1}^{r}(\mu_k)^{\varpi_k}}, \frac{\prod_{k=1}^{r}(1+(\xi-1)\iota_k)^{\varpi_k}-\prod_{k=1}^{r}(1-\iota_k)^{\varpi_k}}{\prod_{k=1}^{r}(1+(\xi-1)\iota_k)^{\varpi_k}+(\xi-1)\prod_{k=1}^{r}(1-\iota_k)^{\varpi_k}}, \frac{\prod_{k=1}^{r}(1+(\xi-1)\eta_k)^{\varpi_k}-\prod_{k=1}^{r}(1-\eta_k)^{\varpi_k}}{\prod_{k=1}^{r}(1+(\xi-1)\eta_k)^{\varpi_k}+(\xi-1)\prod_{k=1}^{r}(1-\eta_k)^{\varpi_k}}, \end{array}\right)$$

$$\leq \left(\begin{array}{l} \left[\frac{\prod_{k=1}^{r}(1+(\xi-1)\widetilde{\mu'}_k^L)^{\varpi_k}-\prod_{k=1}^{r}(1-\widetilde{\mu'}_k^L)^{\varpi_k}}{\prod_{k=1}^{r}(1+(\xi-1)\widetilde{\mu'}_k^L)^{\varpi_k}+(\xi-1)\prod_{k=1}^{r}(1-\widetilde{\mu'}_k^L)^{\varpi_k}}, \frac{\prod_{k=1}^{r}(1+(\xi-1)\widetilde{\mu'}_k^U)^{\varpi_k}-\prod_{k=1}^{r}(1-\widetilde{\mu'}_k^U)^{\varpi_k}}{\prod_{k=1}^{r}(1+(\xi-1)\widetilde{\mu'}_k^U)^{\varpi_k}+(\xi-1)\prod_{k=1}^{r}(1-\widetilde{\mu'}_k^U)^{\varpi_k}}\right], \\ \left[\frac{\xi\prod_{k=1}^{r}(\widetilde{\iota'}_k^L)^{\varpi_k}}{\prod_{k=1}^{r}(1+(\xi-1)(1-\widetilde{\iota'}_k^L))^{\varpi_k}+(\xi-1)\prod_{k=1}^{r}(\widetilde{\iota'}_k^L)^{\varpi_k}}, \frac{\xi\prod_{k=1}^{r}(\widetilde{\iota'}_k^U)^{\varpi_k}}{\prod_{k=1}^{r}(1+(\xi-1)(1-\widetilde{\iota'}_k^U))^{\varpi_k}+(\xi-1)\prod_{k=1}^{r}(\widetilde{\iota'}_k^U)^{\varpi_k}}\right], \\ \left[\frac{\xi\prod_{k=1}^{r}(\widetilde{\eta'}_k^L)^{\varpi_k}}{\prod_{k=1}^{r}(1+(\xi-1)(1-\widetilde{\eta'}_k^L))^{\varpi_k}+(\xi-1)\prod_{k=1}^{r}(\widetilde{\eta'}_k^L)^{\varpi_k}}, \frac{\xi\prod_{k=1}^{r}(\widetilde{\eta'}_k^U)^{\varpi_k}}{\prod_{k=1}^{r}(1+(\xi-1)(1-\widetilde{\eta'}_k^U))^{\varpi_k}+(\xi-1)\prod_{k=1}^{r}(\widetilde{\eta'}_k^U)^{\varpi_k}}\right], \\ \frac{\xi\prod_{k=1}^{r}(\mu'_k)^{\varpi_k}}{\prod_{k=1}^{r}(1+(\xi-1)(1-\mu'_k))^{\varpi_k}+(\xi-1)\prod_{k=1}^{r}(\mu'_k)^{\varpi_k}}, \frac{\prod_{k=1}^{r}(1+(\xi-1)\iota'_k)^{\varpi_k}-\prod_{k=1}^{r}(1-\iota'_k)^{\varpi_k}}{\prod_{k=1}^{r}(1+(\xi-1)\iota'_k)^{\varpi_k}+(\xi-1)\prod_{k=1}^{r}(1-\iota'_k)^{\varpi_k}}, \frac{\prod_{k=1}^{r}(1+(\xi-1)\eta'_k)^{\varpi_k}-\prod_{k=1}^{r}(1-\eta'_k)^{\varpi_k}}{\prod_{k=1}^{r}(1+(\xi-1)\eta'_k)^{\varpi_k}+(\xi-1)\prod_{k=1}^{r}(1-\eta'_k)^{\varpi_k}} \end{array}\right)$$

Then $\bigoplus_{k=1}^{r}{}_{\hbar}(\varpi_k\upsilon_k) \leq \bigoplus_{k=1}^{r}{}_{\hbar}(\varpi_k\upsilon'_k)$, so $NCHWA_{\varpi}(\upsilon_1,\upsilon_2,...,\upsilon_r) \leq NCHWA_{\varpi}(\upsilon'_1,\upsilon'_2,...,\upsilon'_r)$.
(2) It is shown similar to the proof of (1) by using Eq. (18). □

Theorem 5.7. *(Boundedness Property)*
Let υ_k $(k=1,2,...,r)$ be a collection of the NCEs. Then

(1): $\upsilon_{min} \leq NCHWA_{\varpi}(\upsilon_1,\upsilon_2,...,\upsilon_r) \leq \upsilon_{max}$

(2): $\upsilon_{min} \leq NCHWG_{\varpi}(\upsilon_1,\upsilon_2,...,\upsilon_r) \leq \upsilon_{max}$

where

$$\upsilon_{min} = \left\{\begin{array}{l} [min\{\widetilde{\mu}_k^L\}, min\{\widetilde{\mu}_k^U\}], \\ [max\{\widetilde{\iota}_k^L\}, max\{\widetilde{\iota}_k^U\}], \\ [max\{\widetilde{\eta}_k^L\}, max\{\widetilde{\eta}_k^U\}], \\ max\{\mu_k\}, min\{\iota_k\}, min\{\eta_k\} \end{array}\right\}_{(k=1,2,...,r)}$$

and

$$\upsilon_{max} = \left\{ \begin{array}{l} [max\{\widetilde{\mu}_k^L\}, max\{\widetilde{\mu}_k^U\}], \\ [min\{\widetilde{\iota}_k^L\}, min\{\widetilde{\iota}_k^U\}], \\ [min\{\widetilde{\eta}_k^L\}, min\{\widetilde{\eta}_k^U\}], \\ min\{\mu_k\}, max\{\iota_k\}, max\{\eta_k\} \end{array} \right\}_{(k=1,2,...,r)}.$$

Proof. They can be proved using similar techniques, therefore omitted. □

6. The approaches to multiple-criteria decision making under neutrosophic cubic environment

Let o_i $(i = 1, 2, \ldots, p)$ be a fixed of alternatives, e_k $(k = 1, 2, \ldots, r)$ be a criterion and ϖ_k $(k = 1, 2, \ldots, r)$ be the weight of criterion e_k $(k = 1, 2, \ldots, r)$ respectively such that $\varpi_k \in [0, 1]$ and $\sum_{k=1}^{r} \varpi_k = 1$. Let υ_k^i denotes the neutrosophic cubic element (NCE) of the alternative o_i with respect to criterion e_k.

Algorithm 1.

Step 1. Obtain the aggregation value υ^i of neutrosophic cubic elements $\upsilon_1^i, \upsilon_2^i, \ldots, \upsilon_r^i$ by using of neutrosophic cubic Hamacher weighted averaging (NCHWA) operator or neutrosophic cubic Hamacher weighted geometric (NCHWG) operator, i.e., respectively

$$NCHWA_{\varpi}(\upsilon_1^i, \upsilon_2^i, \ldots, \upsilon_r^i) = \bigoplus_{k=1}^{r}{}_{\hbar} (\varpi_k \upsilon_k^i) \ \forall \ i = 1, 2, \ldots, p$$

or

$$NCHWG_{\varpi}(\upsilon_1^i, \upsilon_2^i, \ldots, \upsilon_r^i) = \bigotimes_{k=1}^{r}{}_{\hbar} (\upsilon_k^i)^{\varpi_k} \ \forall \ i = 1, 2, \ldots, p.$$

Step 2. Compute the value of score function $f_{scr}(\upsilon^i)$ $\forall$ $i = 1, 2, \ldots, p$. If $f_{scr}(\upsilon^{p_1}) = f_{scr}(\upsilon^{p_2})$ for any $p_1, p_2 \in \{1, 2, \ldots, p\}$, then compute the values of accuracy function $f_{acr}(\upsilon^{p_1})$ and $f_{scr}(\upsilon^{p_2})$ to compare these alternatives.

Step 3. Find the optimal alternative according to the values obtained in Step 2.

Example 6.1. In order to illustrate the proposed algorithm, the problem for logistic center location selection is described here. Assume that a new modern logistic center is required in a town. There are three locations o_1, o_2 and o_3. A committee of decision makers has been formed to choice the optimal location on the basis of three parameters (namely, cost (e_1), distance to customers (e_2), distance to suppliers (e_3), environmental impact (e_4), quality of service (e_5), transportation (e_6)) with respect to the evaluation of decision committee. As a result of the evaluation, the decision committee gives Table 1 with the neutrosophic cubic values.

Table 1. The collective evaluation values of location with respect to criteria.

$E/\mathcal{O}$	o_1	o_2	o_3
e_1	$\begin{pmatrix}[0.3,0.7],[0.2,1],\\ [0,0.3],0.4,0.1,1\end{pmatrix}$	$\begin{pmatrix}[0.7,0.8],[0.3,0.3],\\ [0.4,0.5],0.5,0.7,0.4\end{pmatrix}$	$\begin{pmatrix}[0.2,0.3],[0.1,0.5],\\ [0.3,0.5],0.2,0,0.6\end{pmatrix}$
e_2	$\begin{pmatrix}[0.5,0.5],[0,0.5],\\ [0.2,0.3],0.4,0.4,0\end{pmatrix}$	$\begin{pmatrix}[0.7,0.9],[0.3,0.3],\\ [0,0.2],0.4,0.4,0.5\end{pmatrix}$	$\begin{pmatrix}[0,0.4],[0.1,0.4],\\ [0.5,1],0.5,0,0.7\end{pmatrix}$
e_3	$\begin{pmatrix}[0.2,0.6],[0.5,0.7],\\ [0,0.1],0.6,0.6,0.6\end{pmatrix}$	$\begin{pmatrix}[0.8,1],[0.4,0.5],\\ [0,0.1],0.5,0.5,0.4\end{pmatrix}$	$\begin{pmatrix}[0.2,0.6],[0.4,0.4],\\ [0.2,0.3],0.4,0.5,0.4\end{pmatrix}$
e_4	$\begin{pmatrix}[0.1,0.2],[0.3,0.6],\\ [0.1,0.4],0.6,0.4,0.1\end{pmatrix}$	$\begin{pmatrix}[0.5,1],[0.4,0.6],\\ [0.5,0.6],0.5,0.5,0.4\end{pmatrix}$	$\begin{pmatrix}[0.3,0.3],[0.1,0.6],\\ [0,1],0.4,0.2,0.4\end{pmatrix}$
e_5	$\begin{pmatrix}[0.4,0.7],[0.4,0.5],\\ [0.1,0.2],0.5,0,0.5\end{pmatrix}$	$\begin{pmatrix}[0.4,1],[0.3,0.8],\\ [0.2,0.6],0.6,0.4,0.3\end{pmatrix}$	$\begin{pmatrix}[0.5,0.5],[0.2,0.7],\\ [0.7,0.8],0.6,0.3,0.4\end{pmatrix}$
e_6	$\begin{pmatrix}[0.1,0.3],[0.4,0.7],\\ [0.4,0.5],0.3,0.3,0.3\end{pmatrix}$	$\begin{pmatrix}[0.5,0.7],[0.4,0.5],\\ [0.4,0.5],0.4,0.6,0.9\end{pmatrix}$	$\begin{pmatrix}[0.3,0.6],[0.2,0.8],\\ [0.5,0.5],0.1,0.4,0.1\end{pmatrix}$

Also, the decision committee determines the weight of criteria as $\varpi = (0.1, 0.2, 0.1, 0.3, 0.2, 0.1)^T$. We are ready to apply the proposed approach to solve this problem based on the neutrosophic cubic information.

Step 1. By applying NCHWA operator with $q = 4$, we get the following aggregation values.

$\upsilon^1 = ([0.2666, 0.4733], [0, 0.6218], [0, 0.2913], 0.4846, 0.2941, 1)$,

$\upsilon^2 = ([0.5834, 1], [0.3476, 0.5184], [0, 0.4131], 0.4879, 0.4944, 0.4756)$,

$\upsilon^3 = ([0.2541, 0.4233], [0.1438, 0.5658], [0, 0.7997], 0.3837, 0.1721, 0.4583)$.

Step 2. Using Eq. (1) given in Definition 3.1, the value of score function are obtained as $f_{scr}(\upsilon^1) = 0.5623$, $f_{scr}(\upsilon^2) = 0.5928$ and $f_{scr}(\upsilon^3) = 0.6013$.

Step 3. Then, we obtain the ranking order of three locations as $o_3 \succ o_2 \succ o_1$. Therefore, we suggest o_3 as the optimal choice and so a new logistic center location.

Table 2 presents the ranking order of alternatives for some values of ξ.

Table 2. The ranking order according to NCHWA operator with some values of ξ.

ξ	$f_{scr}(\upsilon^1)$	$f_{scr}(\upsilon^2)$	$f_{scr}(\upsilon^3)$	ranking order
$\xi = 0.1$	0.5584	0.5898	0.5947	$o_3 \succ o_2 \succ o_1$
$\xi = 1$	0.5593	0.5927	0.5987	$o_3 \succ o_2 \succ o_1$
$\xi = 2$	0.5605	0.5929	0.5999	$o_3 \succ o_2 \succ o_1$
$\xi = 4$	0.5623	0.5928	0.6013	$o_3 \succ o_2 \succ o_1$
$\xi = 10$	0.5657	0.5925	0.6033	$o_3 \succ o_2 \succ o_1$
$\xi = 100$	0.5763	0.5922	0.6088	$o_3 \succ o_2 \succ o_1$

Figure 1 gives a graphical representation of score values for some values of ξ.

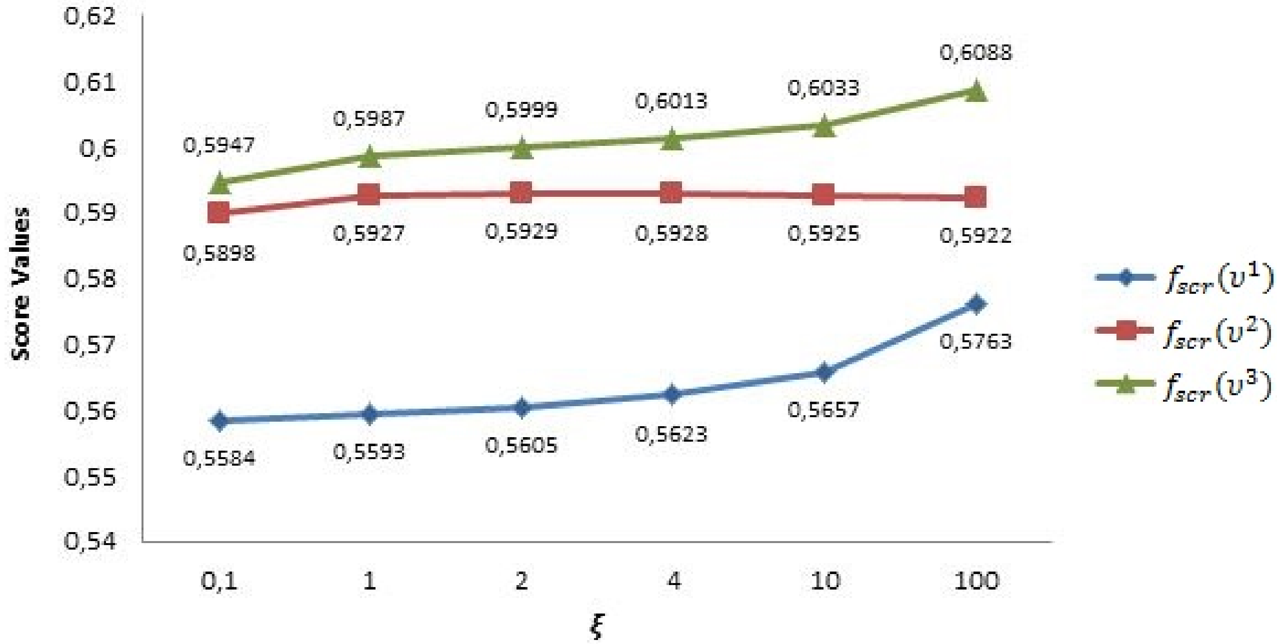

Figure 1. Graphical representation of score values for some values of ξ.

Algorithm 1 is efficient for decision making problems that include the evaluations of a single decision maker, but it cannot be used for decision systems with multiple experts. Now, we create a decision making model based on the neutrosophic cubic Hamacher weighted aggregation operators to deal with multi-criteria group decision making which includes the evaluations of two or more decision makers (experts).

Let o_i $(i = 1, 2, \ldots, p)$ be a fixed of alternatives, e_k $(k = 1, 2, \ldots, r)$ be a criterion and ϖ_k $(k = 1, 2, \ldots, r)$ be the weight of criterion e_k $(k = 1, 2, \ldots, r)$ respectively such that $\varpi_k \in [0, 1]$ and $\sum_{k=1}^{r} \varpi_k = 1$. Also, let D_j $(j = 1, 2, \ldots, v)$ be a fixed of decision makers and Ω_j $(j = 1, 2, \ldots, v)$ be the weight of decision maker D_j $(j = 1, 2, \ldots, v)$ respectively such that $\Omega_j \in [0, 1]$ and $\sum_{j=1}^{v} \Omega_j = 1$. Let $\upsilon_k^{(i,j)}$ denotes the neutrosophic cubic element (NCE) of the alternative o_i with respect to criterion e_k for the decision maker D_j.

Algorithm 2.

Step 1. Obtain the aggregation value $\upsilon^{(i,j)}$ of neutrosophic cubic elements $\upsilon_1^{(i,j)}, \upsilon_2^{(i,j)}, \ldots, \upsilon_r^{(i,j)}$ for each decision maker D_j by using the neutrosophic cubic Hamacher weighted averaging (NCHWA) operator or neutrosophic cubic Hamacher weighted geometric (NCHWG) operator.

For instance, for a decision making problem with two decision makers ($j = 1, 2$), obtain

$$NCHWA_{\varpi}(\upsilon_1^{(i,1)}, \upsilon_2^{(i,1)}, \ldots, \upsilon_r^{(i,1)}) = \bigoplus_{k=1}^{r}{}_{\hbar}(\varpi_k \upsilon_k^{(i,1)}) \ \forall\ i = 1, 2, \ldots, p,$$

$$NCHWA_{\varpi}(\upsilon_1^{(i,2)}, \upsilon_2^{(i,2)}, \ldots, \upsilon_r^{(i,2)}) = \bigoplus_{k=1}^{r}{}_{\hbar}(\varpi_k \upsilon_k^{(i,2)}) \ \forall\ i = 1, 2, \ldots, p$$

or

$$NCHWG_{\varpi}(\upsilon_1^{(i,1)}, \upsilon_2^{(i,1)}, \ldots, \upsilon_r^{(i,1)}) = \bigotimes_{k=1}^{r}{}_{\hbar}(\upsilon_k^{(i,1)})^{\varpi_k} \ \forall\ i = 1, 2, \ldots, p,$$

$$NCHWG_{\varpi}(\upsilon_1^{(i,2)}, \upsilon_2^{(i,2)}, \ldots, \upsilon_r^{(i,2)}) = \bigotimes_{k=1}^{r}{}_{\hbar}(\upsilon_k^{(i,2)})^{\varpi_k} \ \forall\ i = 1, 2, \ldots, p.$$

Step 2. Compute the value of score function $f_{scr}(\upsilon^{(i,j)})$ $\forall\ i = 1, 2, \ldots, p$ and $j = 1, 2, \ldots, v$ for each aggregation value $\upsilon^{(i,j)}$.

Step 3. Calculate the standardized score values for each decision maker by using the following formula:

$$\mathfrak{S}(i,j) = \Omega_j \frac{f_{scr}(\upsilon^{(i,j)})}{\sqrt{(f_{scr}(\upsilon^{(1,j)}))^2 + (f_{scr}(\upsilon^{(2,j)}))^2 + ... + (f_{scr}(\upsilon^{(p,j)}))^2}}.$$

Step 4. Calculate the decision value of each alternative by using the following formula:

$$\mathfrak{D}(i) = \frac{1}{v} \sum_{j=1}^{v} \mathfrak{S}(i,j).$$

If the decision values of any two alternatives are equal then in Step 3, the standardized accuracy values of these two alternatives are calculated (that is, f_{acr} substituted for f_{scr} in the formula $\mathfrak{S}(i,j)$).

Step 5. Find the optimal alternative according to the decision values obtained in Step 4.

Example 6.2. (adapted from [17]) Mobile companies play a major role in Pakistans stock market. The performance of these companies affects capital market resources and have become a common concern of creditors, shareholders, government authorities and other stakeholders. In this example, an investor company wants to invest the capital tax in listed companies. They acquire two types of decision makers (experts): Attorney and market maker.The attorney is acquired to look at the legal matters and the market maker is encouraged to provide his/her expertise in the capital market issues. The data are collected on the basis of stock market analysis and growth in different areas. Let the listed mobile companies be (o_1) Zong, (o_2) Jazz, (o_3) Telenor and (o_4) Ufone, which have higher ratios of earnings than the others available in the market, from the three alternatives of (e_1) stock market trends, (e_2) policy directions and (e_3) the annual performance. The two decision makers (D_j $j = 1, 2$) evaluated the mobile companies (o_i, $i = 1, 2, 3, 4$) with respect to the corresponding attributes (e_k, $k = 1, 2, 3$), and proposed their assessments consisting of neutrosophic cubic values in Table 3 and Table 4.

Assume that the weight of attributes is $\varpi = (0.35, 0.30, 0.35)^T$, and the weight of decision makers is $\Omega = (0.9, 0.1)^T$. Let's provide a solution for this decision making problem using the NCHWG operator on the attributes.

Table 3. The neutrosophic cubic values of attorney's assessment.

$E/\mathcal{O}$	o_1	o_2	o_3	o_4
e_1	$\begin{pmatrix} [0.2,0.6],[0.4,0.6], \\ [0.5,0.8],0.7,0.4,0.3 \end{pmatrix}$	$\begin{pmatrix} [0.3,0.5],[0.6,0.9], \\ [0.3,0.6],0.3,0.6,0.7 \end{pmatrix}$	$\begin{pmatrix} [0.6,0.9],[0.2,0.7], \\ [0.4,0.9],0.5,0.5,0.6 \end{pmatrix}$	$\begin{pmatrix} [0.4,0.8],[0.5,0.9], \\ [0.3,0.8],0.5,0.8,0.5 \end{pmatrix}$
e_2	$\begin{pmatrix} [0.1,0.4],[0.5,0.8], \\ [0.4,0.8],0.6,0.7,0.5 \end{pmatrix}$	$\begin{pmatrix} [0.5,0.9],[0.1,0.3], \\ [0.4,0.8],0.8,0.3,0.6 \end{pmatrix}$	$\begin{pmatrix} [0.2,0.6],[0.3,0.7], \\ [0.3,0.8],0.4,0.6,0.5 \end{pmatrix}$	$\begin{pmatrix} [0.2,0.7],[0.4,0.9], \\ [0.5,0.7],0.6,0.4,0.5 \end{pmatrix}$
e_3	$\begin{pmatrix} [0.4,0.6],[0.2,0.7], \\ [0.5,0.9],0.4,0.5,0.3 \end{pmatrix}$	$\begin{pmatrix} [0.2,0.7],[0.1,0.6], \\ [0.4,0.7],0.5,0.4,0.7 \end{pmatrix}$	$\begin{pmatrix} [0.5,0.9],[0.7,0.9], \\ [0.1,0.5],0.5,0.6,0.4 \end{pmatrix}$	$\begin{pmatrix} [0.3,0.5],[0.5,0.9], \\ [0.3,0.7],0.3,0.3,0.8 \end{pmatrix}$

Table 4. The neutrosophic cubic values of market maker's assessment.

$E/\mathcal{O}$	o_1	o_2	o_3	o_4
e_1	$\begin{pmatrix} [0.3,0.6],[0.2,0.6], \\ [0.2,0.6],0.8,0.7,0.2 \end{pmatrix}$	$\begin{pmatrix} [0.2,0.5],[0.6,0.9], \\ [0.3,0.7],0.4,0.8,0.7 \end{pmatrix}$	$\begin{pmatrix} [0.5,0.9],[0.2,0.6], \\ [0.3,0.8],0.7,0.7,0.8 \end{pmatrix}$	$\begin{pmatrix} [0.3,0.5],[0.3,0.9], \\ [0.2,0.5],0.6,0.5,0.4 \end{pmatrix}$
e_2	$\begin{pmatrix} [0.3,0.8],[0.4,0.8], \\ [0.3,0.8],0.6,0.7,0.4 \end{pmatrix}$	$\begin{pmatrix} [0.4,0.9],[0.1,0.4], \\ [0.5,0.8],0.6,0.5,0.7 \end{pmatrix}$	$\begin{pmatrix} [0.2,0.5],[0.2,0.7], \\ [0.5,0.8],0.6,0.7,0.2 \end{pmatrix}$	$\begin{pmatrix} [0.4,0.7],[0.2,0.8], \\ [0.3,0.7],0.6,0.7,0.7 \end{pmatrix}$
e_3	$\begin{pmatrix} [0.2,0.7],[0.2,0.6], \\ [0.3,0.8],0.5,0.3,0.5 \end{pmatrix}$	$\begin{pmatrix} [0.4,0.9],[0.1,0.4], \\ [0.5,0.8],0.6,0.5,0.7 \end{pmatrix}$	$\begin{pmatrix} [0.3,0.5],[0.3,0.9], \\ [0.2,0.5],0.6,0.5,0.4 \end{pmatrix}$	$\begin{pmatrix} [0.2,0.6],[0.5,0.9], \\ [0.2,0.8],0.4,0.4,0.8 \end{pmatrix}$

Steps 1-3. By using NCHWG operator with $q = 100$, the aggregation values, the score values and the standardized score values of alternatives are obtained as in Table 5.

Table 5. The aggregation values, score values and standardized score values.

j	$\upsilon^{(i,j)}$	$f_{scr}(\upsilon^{(i,j)})$	$\mathfrak{S}(i,j)$
$j=1$	$\upsilon^{(1,1)} = \begin{pmatrix} [0.2163,0.5402],\ [0.3849,0.7015], \\ [0.4696,0.8416],\ 0.5685,\ 0.5276,\ 0.3558 \end{pmatrix}$	0.4787	0.4558
	$\upsilon^{(2,1)} = \begin{pmatrix} [0.3137,0.7198],\ [0.2205,0.6595], \\ [0.3636,0.7015],\ 0.5306,\ 0.4365,\ 0.6713 \end{pmatrix}$	0.5113	0.4869
	$\upsilon^{(3,1)} = \begin{pmatrix} [0.4314,0.8382],\ [0.3926,0.7892], \\ [0.2397,0.7661],\ 0.4996,\ 0.5654,\ 0.4998 \end{pmatrix}$	0.4736	0.4509
	$\upsilon^{(4,1)} = \begin{pmatrix} [0.2984,0.6761],\ [0.4696,0.9], \\ [0.3562,0.738],\ 0.4568,\ 0.5156,\ 0.6179 \end{pmatrix}$	0.4223	0.4021
$j=2$	$\upsilon^{(1,1)} = \begin{pmatrix} [0.2671,0.5957],\ [0.3535,0.8279], \\ [0.3146,0.6665],\ 0.5778,\ 0.4689,\ 0.498 \end{pmatrix}$	0.4877	0.0515
	$\upsilon^{(2,1)} = \begin{pmatrix} [0.3209,0.8035],\ [0.2205,0.6247], \\ [0.4266,0.7681],\ 0.5305,\ 0.6179,\ 0.7 \end{pmatrix}$	0.4642	0.0491
	$\upsilon^{(3,1)} = \begin{pmatrix} [0.3288,0.6799],\ [0.232,0.7624], \\ [0.3146,0.7113],\ 0.6365,\ 0.634,\ 0.4799 \end{pmatrix}$	0.4856	0.0513
	$\upsilon^{(4,1)} = \begin{pmatrix} [0.2882,0.5975],\ [0.3297,0.8758], \\ [0.2272,0.677],\ 0.5305,\ 0.5276,\ 0.6439 \end{pmatrix}$	0.452	0.0478

Step 4-5. Consequently, we obtain the decision values of alternatives as $\mathfrak{D}(1) = 0.2536$, $\mathfrak{D}(2) = 0.268$, $\mathfrak{D}(3) = 0.2511$, $\mathfrak{D}(4) = 0.2249$. Then, the ranking order of alternatives is $o_2 \succ o_1 \succ o_3 \succ o_4$, and so the optimal choice is o_2.

In Table 6, we discuss the ranking order of alternatives for some values of ξ. Thus, we exhibit that the standardized score values and decision values show slight changes synchronous to the range of ξ.

TABLE 6. The ranking order according to NCHWG operator with some values of ξ.

ξ	i	$\mathfrak{S}(i,1)$	$\mathfrak{S}(i,2)$	$\mathfrak{D}(i)$	ranking order
$\xi = 0.1$	$i=1$	0.4636	0.0541	0.2588	$o_2 \succ o_1 \succ o_3 \succ o_4$
	$i=2$	0.4727	0.0456	0.2591	
	$i=3$	0.4313	0.0514	0.2413	
	$i=4$	0.4306	0.0484	0.2395	
$\xi = 1$	$i=1$	0.4606	0.0532	0.2569	$o_2 \succ o_1 \succ o_3 \succ o_4$
	$i=2$	0.4766	0.0463	0.2614	
	$i=3$	0.4402	0.0516	0.2459	
	$i=4$	0.4205	0.0484	0.2344	
$\xi = 2$	$i=1$	0.4593	0.0525	0.2559	$o_2 \succ o_1 \succ o_3 \succ o_4$
	$i=2$	0.4789	0.0477	0.2633	
	$i=3$	0.4435	0.0514	0.2474	
	$i=4$	0.4158	0.0481	0.2319	
$\xi = 4$	$i=1$	0.4581	0.0522	0.2551	$o_2 \succ o_1 \succ o_3 \succ o_4$
	$i=2$	0.4813	0.0482	0.2674	
	$i=3$	0.4463	0.0514	0.2488	
	$i=4$	0.4113	0.048	0.2296	
$\xi = 10$	$i=1$	0.4569	0.0518	0.2543	$o_2 \succ o_1 \succ o_3 \succ o_4$
	$i=2$	0.484	0.0486	0.2663	
	$i=3$	0.4487	0.0513	0.25	
	$i=4$	0.4067	0.0479	0.2273	
$\xi = 100$	$i=1$	0.4558	0.0515	0.2536	$o_2 \succ o_1 \succ o_3 \succ o_4$
	$i=2$	0.4869	0.0491	0.268	
	$i=3$	0.4509	0.0513	0.2511	
	$i=4$	0.4021	0.0478	0.2249	

In Figure 2, a figuration of the decision values of alternatives for some values of ξ is presented. Thus, the effect of the range of ξ on the selection priority is illustrated.

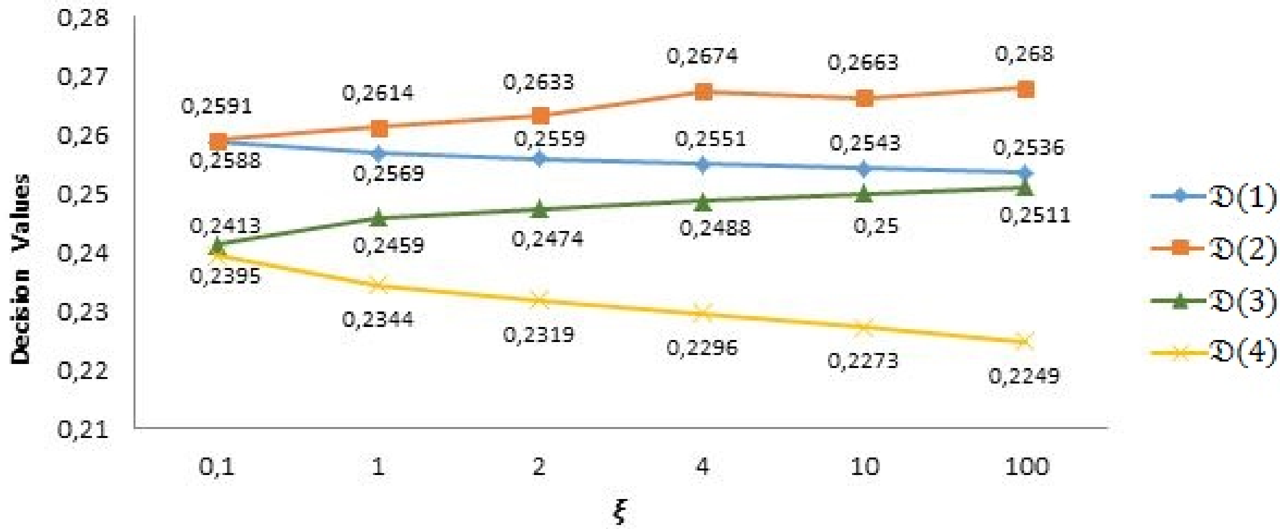

Figure 2. Graphical representation of decision values for some values of ξ.

Discussion and Comparison: If $\Omega = (0.4, 0.6)^T$ in Example 6.2, then this example is the same as the problem in "Application" (see: Section 6 on page 21) of [17]. For $\Omega = (0.4, 0.6)^T$, using NCHWG operator with $q = 100$, we rank the alternatives as $o_1 \succ o_2 \succ o_3 \succ o_4$. But it is proposed as a priority order of alternatives in [17] that $o_3 \succ o_2 \succ o_1 \succ o_4$. We think the reason for this order is the concepts of score function and accuracy function given in Definitions 18 and 19 of [17], and these functions should be improved. Let us demonstrate that the score and accuracy functions in Definitions 18 and 19 of [17] give erroneous outputs for some neutrosophic cubic elements. Let $\upsilon_1 = ([0.5, 0.7], [0.2, 0.5], [0.5, 0.6], 0.5, 0.8, 0.2)$ and $\upsilon_2 = ([0.4, 0.6], [0.3, 0.4], [0.4, 0.5], 0.8, 0.6, 0.5)$ be two NCEs. It is evident that υ_1 and υ_2 do not have identical values, i.e., $\upsilon_1 \neq \upsilon_2$. By Definition 18 in [17], the score values of υ_1 and υ_2 are $S(\upsilon_1) = 0.5 - 0.5 + 0.7 - 0.6 + 0.5 - 0.2 = 0.4$ and $S(\upsilon_2) = 0.4 - 0.4 + 0.6 - 0.5 + 0.8 - 0.5 = 0.4$, respectively. By Definition 19 in [17], the accuracy values of υ_1 and υ_2 are $H(\upsilon_1) = \frac{1}{9}(0.5 + 0.2 + 0.5 + 0.7 + 0.5 + 0.6 + 0.5 + 0.8 + 0.2) = 0.5$ and $H(\upsilon_2) = \frac{1}{9}(0.4 + 0.3 + 0.4 + 0.6 + 0.4 + 0.5 + 0.8 + 0.6 + 0.5) = 0.5$, respectively. By the comparison method in Definition 20 of [17], $\upsilon_1 = \upsilon_2$, which is against our intuition. By using the score function in Definition 3.1, we obtain $f_{scr}(\upsilon_1) = 0.5468$ and $f_{scr}(\upsilon_1) = 0.5968$, so $\upsilon_1 \prec \upsilon_2$. Also if the score function in Definition 3.1 is used for NCGW in Eq. (4) of [17]:

$$NCWG = \begin{pmatrix} \upsilon_1 \begin{pmatrix} [0.2375,0.6195], \\ [0.2885,0.7916], \\ [0.3567,0.8146], \\ 0.6315, 0.5757, 0.2851 \end{pmatrix} \\ \upsilon_2 \begin{pmatrix} [0.4426,0.7657], \\ [0.2165,0.5915], \\ [0.5382,0.7804], \\ 0.4827, 0.5729, 0.5282 \end{pmatrix} \\ \upsilon_3 \begin{pmatrix} [0.3500,0.6616], \\ [0.3335,0.8142], \\ [0.3131,0.7498], \\ 0.5791, 0.6133, 0.4439 \end{pmatrix} \\ \upsilon_4 \begin{pmatrix} [0.3327,0.6774], \\ [0.3630,0.7787], \\ [0.2888,0.7396], \\ 0.4906, 0.5359, 0.5692 \end{pmatrix} \end{pmatrix} \tag{15}$$

then it is calculated as $f_{scr}(\upsilon_1) = 0.4947$, $f_{scr}(\upsilon_2) = 0.4931$, $f_{scr}(\upsilon_3) = 0.4669$ and $f_{scr}(\upsilon_4) = 0.4589$. By these score values, we say that the ranking order of alternatives is $o_1 \succ o_2 \succ o_3 \succ o_4$. This result coincides with the output of Algorithm 2. Thereby, the efficiency of score function, accuracy function and decision making algorithms presented in this study are displayed.

7. Conclusions

In this study, we described a comparison strategy for two neutrosophic cubic elements. Some new aggregation operators for the neutrosophic cubic sets based on Hamacher t-norm and Hamacher t-conorm, which are a generalization of the operators based on algebraic t-norm and t-conorm or Einstein t-norm and Einstein t-conorm, were proposed and their basic properties were investigated. They were applied to solve the MCDM problems in which attribute values take the form of neutrosophic cubic elements. In addition, compared with the existing algorithm based on Einstein geometric aggregations under the neutrosophic cubic environment, the proposed algorithms can give the satisfactory sorting value of each alternative.

In further research, it is necessary and meaningful to give the applications of these aggregation operators to the other domains such as medical diagnosis, pattern recognition and selection of renewable energy.

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