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Wiener index and applications in the Neutrosophic graphs

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Abstract: In this article, we have examined the Wiener index in neutrosophic graphs. Wiener index is one of the most important topological indices. This index is a distance-based index that is calculated based on the geodesic distance between two vertices. Here, after defining the Wiener index in neutrosophic graphs, we calculated this index for some special modes such as the complete neutrosophic graph, cycle, and tree. In the following, by presenting a several theorems, we compared this index with the connectivity index, which is one of the most important degree-based indicators.

Keywords: Wiener index; partial Wiener index; totally Wiener index; neutrosophic graph; neutrosophic tree; strong spanning tree; connectivity index

1. Introduction

The theory of fuzzy sets was first proposed by Zadeh [20] in 1965, and the concept of fuzzy graph was first introduced by Rosenfeld [13] in 1975. Since then, much research has been done on fuzzy graphs, their properties, and applications. One of these problems was the calculation of degree-based topological indices and distance-based indices in fuzzy graphs. These indicators help by providing a numerical value for each graph so that we can have a good criterion for comparing graphs with the same number of vertices.

After that, Atanassov [6] proposed the theory of intuitionistic fuzzy set. Finally, with the generalization of fuzzy theory by Smarandache [15] in 1995, new sets called neutrosophic sets were born. By presenting this theory, researchers tried to introduce other mathematical concepts in this field. Among them was the concept of graphs, which led to the new concept of neutrosophic graphs.

In recent years, many features and applications of neutrosophic graphs have been proposed by theorists in this field. One of them is the problem of the Decision-Making [1], Solving the supply chain problem [2], application in the NeutroHyperAlgebra and AntiHyper Algebra [16], and Energy and Spectrum [7]. One of these topics is the study of topological indices and its applications in neutrosophic graphs. In [8-10], we examined some of these indicators and their applications.

In this paper, we try to define the Wiener index, which is one of the most important topological indices based on distance, in neutrosophic graphs, and then calculate this index for certain conditions. The calculation of this index in neutrosophic graphs is done for the first time in this paper. Finally, we compare the connectivity index, which is one of the most important degree-based indices, with the Wiener index and present the results.

2. Preliminaries

This section, provides some definitions and theorems needed.

Definition 1. [5] Let $G = (N, M)$ be a single-valued Neutrosophic graph, where N is a Neutrosophic set on V and, M is a Neutrosophic set on E, which satisfy the following

$$T_M(u,v) \le min\big(T_N(u), T_N(v)\big),$$
$$I_M(u,v) \ge max\big(I_N(u), I_N(v)\big),$$
$$F_M(u,v) \ge max\big(F_N(u), F_N(v)\big),$$

where u and v are two vertices of G, and $(u,v) \in E$ is an edge of G.

Definition 2. [5] Let $G = (N, M)$ be a Single-Valued Neutrosophic Graph and P is a path in G. P is a collection of different vertices, $v_0, v_1, v_2, \dots, v_n$ such that $\big(T_M(v_{i-1}, v_i), I_M(v_{i-1}, v_i), F_M(v_{i-1}, v_i)\big) > 0$ for $0 \le i \le n$. P is a Neutrosophic cycle if $v_0 = v_n$ and $n \ge 3$.

Definition 3. [5] Suppose $G = (N, M)$ a single-valued Neutrosophic graph. G is a connected Single-Valued Neutrosophic Graph if there exists no isolated vertex in G. ($v \in V_G$ is the isolated vertex, if there exists no incident edge to the vertex v.)

Definition 4. [9] Let $G = (N, M)$ be the connected Neutrosophic Graph. The partial connectivity index of G is defined as

$$PCI_T(G) = \sum_{u,v\, \in N} T_N(u) T_N(v) CONN_{T_G}(u,v),$$
$$PCI_I(G) = \sum_{u,v\, \in N} I_N(u) I_N(v) CONN_{I_G}(u,v),$$
$$PCI_F(G) = \sum_{u,v\, \in N} F_N(u) F_N(v) CONN_{F_G}(u,v),$$

where $CONN_{T_G}(u,v)$ is the strength of truth, $CONN_{I_G}(u,v)$ the strength of indeterminacy and $CONN_{F_G}(u,v)$ the strength of falsity between two vertices u and v. We have

$$CONN_{T_G}(u,v) = \max\{\min T_M(e) \mid e \in P \; and \; P \; is \; a \; path \; between \; u \; and \; v\},$$
$$CONN_{I_G}(u,v) = \min\{\max I_M(e) \mid e \in P \; and \; P \; is \; a \; path \; between \; u \; and \; v\},$$
$$CONN_{F_G}(u,v) = \min\{\max F_M(e) \mid e \in P \; and \; P \; is \; a \; path \; between \; u \; and \; v\}.$$

Also, the totally connectivity index of G is defined as

$$TCI(G) = \frac{4 + 2PCI_T(G) - 2PCI_F(G) - PCI_I(G)}{6}.$$

Theorem 1. [9] Let $G = (N, M)$ be a complete neutrosophic graph whit $V = \{v_1, v_2, \dots, v_n\}$ such that $t_1 \le t_2 \le \cdots \le t_n$, $i_1 \le i_2 \le \cdots \le i_n$ and $f_1 \ge f_2 \ge \cdots \ge f_n$ where $t_j = T_N(v_j)$, $i_j = I_N(v_j)$ and $f_j = F_N(v_j)$ for $j = 1, 2, \dots, n$. Then

$$PCI_T(G) = \sum_{j=1}^{n-1} t_j^2 \sum_{k=j+1}^{n} t_k, \qquad PCI_I(G) = \sum_{j=1}^{n-1} i_j^2 \sum_{k=j+1}^{n} i_k, \qquad PCI_F(G) = \sum_{j=1}^{n-1} f_j^2 \sum_{k=j+1}^{n} f_k.$$

3. Wiener Index in Neutrosophic Graphs

In this section, which is the main part of the article, we will introduce the Wiener index in neutrosophic graphs. The Wiener index is a distance-based index that is widely used in symmetric graphs.

Like the connectivity index, we divide the Wiener index into a Totally and Partial Wiener index and define it as follows

Definition 5. Let $G = (N, M)$ be the Neutrosophic Graph and $v_1, v_2 \in V$. A strong path P from v_1 to v_2 is called **a neutrosophic geodesic** if there is no strong shorter path between v_1 and v_2.

Note that in the above definition, the shortest strong path must be calculated separately for each of truth (T), indeterminacy (I), and falsity (F) states.

Definition 6. Let $G = (N, M)$ be the Neutrosophic Graph. The **Partial Wiener Index** $(\boldsymbol{PWI})$ of G is defined as

$$PWI_T(G) = \sum_{u,v \in N} T_N(u)T_N(v)d_{s_T}(u,v),$$

$$PWI_I(G) = \sum_{u,v \in N} I_N(u)I_N(v)d_{s_I}(u,v),$$

$$PWI_F(G) = \sum_{u,v \in N} F_N(u)F_N(v)d_{s_F}(u,v),$$

when $d_s(u,v)$ is the minimum, the sum of the weights of the edges in geodesic between u and v.

Also, the **Totally Wiener Index** $(\boldsymbol{TWI})$ of G is defined by

$$TWI(G) = \frac{4 + 2PCWI_T(G) - 2PWI_F(G) - PWI_I(G)}{6}.$$

Example 1. Consider the Neutrosophic Graph $G = (N, M)$ as shown in figure 1, with the vertex set $V = \{a, b, c, d\}$ where $(T_N, I_N, F_N)(a) = (0.4, 0.3, 0.2)$, $(T_N, I_N, F_N)(b) = (0.6, 0.5, 0.2)$, $(T_N, I_N, F_N)(c) = (0.7, 0.2, 0.2)$, and $(T_N, I_N, F_N)(d) = (0.4, 0.2, 0.3)$, whit the edge set $(T_M, I_M, F_M)(a, b) = (0.3, 0.3, 0.3)$, $(T_M, I_M, F_M)(a, c) = (0.4, 0.3, 0.2)$, $(T_M, I_M, F_M)(a, \text{d}) = (0.3, 0.3, 0.2)$, $(T_M, I_M, F_M)(b, \text{d}) = (0.4, 0.4, 0.3)$, $(T_M, I_M, F_M)(c, \text{d}) = (0.4, 0.2, 0.2)$, We have,

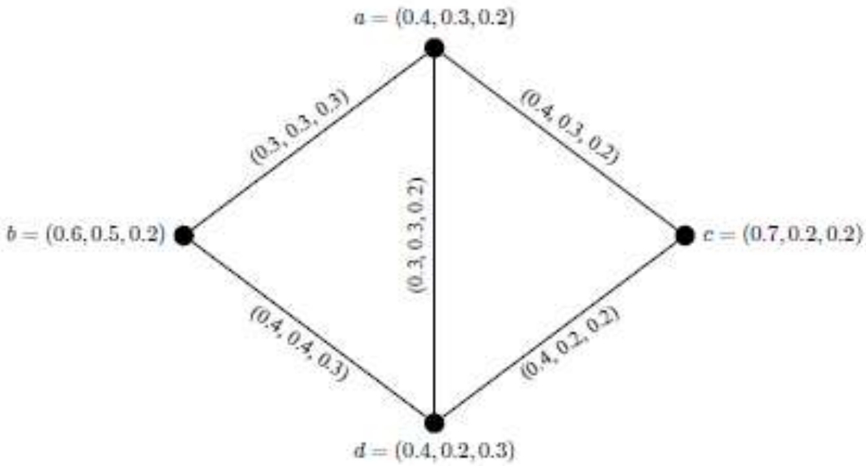

Figure 1. A neutrosophic graph G

Table 1. The sum of the weights of the edges in geodesic between each pair of vertices u and v.

	$d_{s_T}(u,v)$	$d_{s_I}(u,v)$	$d_{s_F}(u,v)$
a,b	$0.4+0.4+0.4=1.2$	0.3	0.3
a,c	0.4	0.3	0.2
a,d	$0.4+0.4=0.8$	0.3	0.2
b,c	$0.4+0.4=0.8$	$0.3+0.3=0.6$	$0.2+0.3=0.5$
b,d	0.4	$0.3+0.3=0.6$	0.3
c,d	0.4	0.2	0.2

$$PWI_T(G)=\sum_{u,v\in N} T_N(u)T_N(v)d_{s_T}(u,v)$$
$$=(0.4)(0.6)(1.2)+(0.4)(0.7)(0.4)+(0.4)(0.4)(0.8)+(0.6)(0.7)(0.8)$$
$$+(0.6)(0.4)(0.4)+(0.7)(0.4)(0.4)=0.288+0.112+0.128+0.336+0.096+0.112$$
$$=1.072,$$

$$PWI_I(G)=\sum_{u,v\in N} I_N(u)I_N(v)d_{s_I}(u,v)$$
$$=(0.3)(0.5)(0.3)+(0.3)(0.2)(0.3)+(0.3)(0.2)(0.3)+(0.5)(0.2)(0.6)$$
$$+(0.5)(0.2)(0.6)+(0.2)(0.2)(0.2)=0.045+0.018+0.018+0.060+0.060+0.008$$
$$=0.209,$$

$$PWI_F(G)=\sum_{u,v\in N} F_N(u)F_N(v)d_{s_F}(u,v)$$
$$=(0.2)(0.2)(0.3)+(0.2)(0.2)(0.2)+(0.2)(0.3)(0.2)+(0.2)(0.2)(0.5)$$
$$+(0.2)(0.3)(0.3)+(0.2)(0.3)(0.2)=0.012+0.008+0.012+0.020+0.018+0.012$$
$$=0.082.$$

$$TWI(G)=\frac{4+2PCWI_T(G)-2PWI_F(G)-PWI_I(G)}{6}=\frac{4+2(1.072)-2(0.209)-(0.082)}{6}=\frac{5.644}{6}$$
$$=0.941.$$

Theorem 2. Let $G=(N,M)$ be a complete neutrosophic graph whit $V=\{v_1,v_2,\dots,v_n\}$ such that $t_1\le t_2\le\cdots\le t_n$, $i_1\le i_2\le\cdots\le i_n$ and $f_1\ge f_2\ge\cdots\ge f_n$ where $t_j=T_N(v_j)$, $i_j=I_N(v_j)$ and $f_j=F_N(v_j)$ for $j=1,2,\dots,n$. Then

$$PWI_T(G)=\sum_{j=1}^{n-1} t_j^2\sum_{k=j+1}^{n} t_k\,,\qquad PWI_I(G)=\sum_{j=1}^{n-1} i_j^2\sum_{k=j+1}^{n} i_k\,,\qquad PWI_F(G)=\sum_{j=1}^{n-1} f_j^2\sum_{k=j+1}^{n} f_k.$$

Proof. Consider neutrosophic graph $G=(N,M)$ with the conditions given in the theorem. According to the definition of the Wiener index

$$PWI_T(G) = \sum_{u,v \in N} T_N(u)T_N(v)d_{s_T}(u,v), \qquad (1)$$

Since G is a complete neutrosophic graph, there is a path of length one between the two vertices. We show that the path is geodesic. Let $u = v_1$. Then for any $2 \le i \le n$, we have $t_1 \le t_i$, it is easy to see that

$$d_{s_T}(v_1,\, v_i) = t_1, \qquad 2 \le i \le n,$$

now, we have for v_2,

$$d_{s_T}(v_2,\, v_i) = t_2, \qquad 3 \le i \le n,$$

for v_k,

$$d_{s_T}(v_k,\, v_i) = t_k, \qquad k+1 \le i \le n,$$

and, we have for v_{n-1},

$$d_{s_T}(v_{n-1},\, v_n) = t_{n-1},$$

now, we get by placing the above relation in (1)

$$\begin{aligned} PWI_T(G) = {} & T_N(v_1)T_N(v_2)t_1 + \cdots + T_N(v_1)T_N(v_n)t_1 + T_N(v_2)T_N(v_3)t_2 + \cdots + T_N(v_2)T_N(v_n)t_2 + \cdots \\ & + T_N(v_k)T_N(v_{k+1})t_k + \cdots + T_N(v_k)T_N(v_n)t_k + \cdots + T_N(v_{n-1})T_N(v_n)t_{n-1} \\ & = t_1t_2t_1 + \cdots + t_1t_nt_1 + t_2t_3t_2 + \cdots + t_2t_nt_2 + \cdots + t_kt_{k+1}t_k + \cdots + t_kt_nt_k + \cdots \\ & + t_{n-1}t_nt_{n-1} \\ & = t_1^2(t_2 + \cdots + t_n) + t_2^2(t_3 + \cdots + t_n) + \cdots + t_k^2(t_{k+1} + \cdots + t_n) + \cdots + t_{n-1}^2 t_n \\ & = \sum_{j=1}^{n-1} t_j^2 \sum_{k=j+1}^{n} t_k. \end{aligned}$$

Similarly, $PWI_I(G)$ and $PWI_F(G)$ can be proved.

□

Corollary 1. Consider the complete neutrosophic graph $G = (N, M)$ with the above theorem conditions, then

$$PWI_T(G) = PCI_T(G),$$
$$PWI_I(G) = PCI_I(G),$$
$$PWI_F(G) = PCI_F(G).$$

Also, $TWI(G) = TCI(G)$.

Proof. According to theorem 1, and the above theorem is clear.

□

Theorem 3. Let $G = (N, M)$ be a neutrosophic graph with $|N^*| = n$, such that G^* is a tree. If for each $uv \in M$, $G - uv$ has two connecting components w_1 and w_2, it has l and k vertices, respectively such that $l + k = n$. Then

$$PWI_T(G) = \sum_{uv \in G} T_M(uv) \sum_{i=1}^{l} T_N(u_i) \sum_{j=1}^{k} T_N(v_j),$$
$$PWI_I(G) = \sum_{uv \in G} I_M(uv) \sum_{i=1}^{l} I_N(u_i) \sum_{j=1}^{k} I_N(v_j),$$

$$PWI_F(G) = \sum_{uv \in G} F_M(uv) \sum_{i=1}^{l} F_N(u_i) \sum_{j=1}^{k} F_N(v_j).$$

Proof. Let $G = (N, M)$ be a neutrosophic graph with $|N^*| = n$, and G^* is a tree. Now suppose we remove the desired edge uv, $uv \in M$, from G. Graph G is divided into two connecting components w_1 and w_2, so that w_1 will contain l vertices and w_2 will contain $k = n - l$ vertices. If $l = 1$ and $k = n - 1$, and $v_1 \in w_1$ then

$$\begin{aligned} PWI_T(G) = & \sum_{u,v \in N} T_N(u) T_N(v) d_{s_T}(u,v) \\ & = T_N(v_1)T_N(v_2)T_M(uv) + T_N(v_1)T_N(v_3)(T_M(uv) + e_1) + \cdots \\ & + T_N(v_1)T_N(v_n)(T_M(uv) + \cdots + e_m) + \sum_{u,v \in N - v_1} T_N(u)T_N(v) d_{s_T}(u,v). \end{aligned}$$

where $e_i \in M$, and $e_i \neq uv$. Repeat the same process for $\sum_{u,v \in N-v_1} T_N(u)T_N(v)d_{s_T}(u,v)$. We continue this until only one vertex remains in w_2. Then, by factoring and summing the number of vertices of the two components, we reach the desired result. Similarly, $PWI_I(G)$ and $PWI_F(G)$ can be proved.
□

Theorem 4. Let $G = (N, M)$ be a connected neutrosophic graph with the unique strong spanning tree T. then

$$PWI_T(G) = PWI_T(T), \qquad PWI_I(G) = PWI_I(T), \qquad PWI_F(G) = PWI_F(T).$$

Hence $TWI(G) = TWI(T)$.

Proof. Let G be a connected neutrosophic graph and T is the unique strong spanning tree of G. By definition of strong spanning tree, if u and v are two vertices of G, we have

$$d_{s_T}(u,v)(G) = d_{s_T}(u,v)(T), \qquad d_{s_I}(u,v)(G) = d_{s_I}(u,v)(T), \qquad d_{s_F}(u,v)(G) = d_{s_F}(u,v)(T).$$

Since, it is clear from the above relation that

$$PWI_T(G) = PWI_T(T), \qquad PWI_I(G) = PWI_I(T), \qquad PWI_F(G) = PWI_F(T).$$

Therefore $TWI(G) = TWI(T)$.
□

Theorem 5. Let $G = (N, M)$ be a neutrosophic graph with $G^* = C_n$. Let M be a constant function. Then

1. For $n = 2m, m \in \mathbb{N}$

$$PWI_T(G) = \sum_{k=1}^{\frac{n}{2}-1} kt \left(\sum_{j=1}^{n} T_N(u_j) T_N(u_{j+k}) \right) + \frac{n}{2} t \sum_{l=1}^{\frac{n}{2}} T_N(u_l) T_N(u_{l+\frac{n}{2}}),$$

$$PWI_I(G) = \sum_{k=1}^{\frac{n}{2}-1} ki \left(\sum_{j=1}^{n} I_N(u_j) I_N(u_{j+k}) \right) + \frac{n}{2} i \sum_{l=1}^{\frac{n}{2}} I_N(u_l) I_N(u_{l+\frac{n}{2}}),$$

$$PWI_F(G) = \sum_{k=1}^{\frac{n}{2}-1} kf\left(\sum_{j=1}^{n} F_N(u_j)F_N(u_{j+k})\right) + \frac{n}{2}f\sum_{l=1}^{\frac{n}{2}} F_N(u_l)F_N(u_{l+\frac{n}{2}}),$$

2. For $n = 2m + 1, m \in \mathbb{N}$

$$PWI_T(G) = \sum_{k=1}^{\frac{n-1}{2}} kt\left(\sum_{j=1}^{n} T_N(u_j)T_N(u_{j+k})\right),$$

$$PWI_I(G) = \sum_{k=1}^{\frac{n-1}{2}} ki\left(\sum_{j=1}^{n} I_N(u_j)I_N(u_{j+k})\right),$$

$$PWI_F(G) = \sum_{k=1}^{\frac{n-1}{2}} kf\left(\sum_{j=1}^{n} F_N(u_j)F_N(u_{j+k})\right).$$

Note that for $j + k > n$, $u_{j+k} = u_d$, this is, $j + k \equiv d$ (mode n).

Also for $G - uv$, we have

$$PWI_T(G - uv) = \sum_{k=1}^{n-1} kt\left(\sum_{j=1}^{n-k} T_N(u_j)T_N(u_{j+k})\right),$$

$$PWI_I(G - uv) = \sum_{k=1}^{n-1} ki\left(\sum_{j=1}^{n-k} I_N(u_j)I_N(u_{j+k})\right),$$

$$PWI_F(G - uv) = \sum_{k=1}^{n-1} kf\left(\sum_{j=1}^{n-k} F_N(u_j)F_N(u_{j+k})\right).$$

Where $M = (t, i, f)$ is a constant function.

Proof. First, we assume that G^* is a cycle of even length, and $M = (t, i, f)$ is a constant function. Hence each edge of G is a neutral edge. Then, the maximum length of a neutrosophic geodesic in G is $\frac{n}{2}$. Now consider a case where the distance between two vertices is less than $\frac{n}{2}$. Suppose the distance between u and v is equal to k, where k is less than $\frac{n}{2}$. In that case, we define the geodesic length between the two vertices u and v as follows

$$P_k = \{(u, v) \in N^* \times N^*, \text{k is equal to the geodetic length between u and v}\},$$

On the other hand, we know that there are $\frac{n}{2}$ pairs of vertices (u, v) such that the geodesic length between them is exactly equal to $\frac{n}{2}(t, i, f)$, for these $\frac{n}{2}$ pairs of vertices, it is sufficient to obtain a product of $T_N(u)$ in $T_N(v)$ [Similarly, $I_N(u)$ in $I_N(v)$, and $F_N(u)$ in $F_N(v)$]. And then sum on u and v. Then we get

$$\frac{n}{2}t\sum_{l=1}^{\frac{n}{2}} T_N(u_l)T_N(u_{l+\frac{n}{2}}), \tag{1}$$

[Similarly for I and F]. Now back to the state that $1 \le k < \frac{n}{2}$. For each vertex such as u on the cycle C_n, there is a vertex with distance kt from it. Suppose $k = 1$, so we have

$$T_N(u_1)T_N(u_2) + T_N(u_2)T_N(u_3) + \cdots + T_N(u_j)T_N(u_{j+1}) + \cdots + T_N(u_n)T_N(u_{n+1}),$$

since $n + 1 \equiv 1$ (mode n), hence $T_N(u_n)T_N(u_{n+1}) = T_N(u_n)T_N(u_1)$. Then

for $k = 1$, we have

$$1 \times t \times \sum_{j=1}^{n} T_N(u_j)T_N(u_{j+1}),$$

for $k = 2$,

$$2 \times t \times \sum_{j=1}^{n} T_N(u_j)T_N(u_{j+2}),$$

for $k = m$, $m < \frac{n}{2}$,

$$m \times t \times \sum_{j=1}^{n} T_N(u_j)T_N(u_{j+m}),$$

by continuing this process and summing on k, we get

$$\sum_{k=1}^{\frac{n}{2}-1} kt \left(\sum_{j=1}^{n} T_N(u_j)T_N(u_{j+k}) \right), \qquad (2)$$

use from (1) and (2),

$$PWI_T(G) = (1) + (2) = \sum_{k=1}^{\frac{n}{2}-1} kt \left(\sum_{j=1}^{n} T_N(u_j)T_N(u_{j+k}) \right) + \frac{n}{2} t \sum_{l=1}^{\frac{n}{2}} T_N(u_l)T_N(u_{l+\frac{n}{2}}).$$

To prove that n is odd, note that the maximum distance between the vertices u, and v is $\frac{n-1}{2}$. The continuation of the proof is similar to the case where n is even.

□

Theorem 6. Let $G = (N, M)$ be a neutrosophic tree $|N^*| \ge 3$. Then

$$PCI_T(G) < PWI_T(G), \qquad PCI_I(G) < PWI_I(G), \qquad PCI_F(G) < PWI_F(G).$$

But, $TCI(G)$ need not be less than or equal to $TWI(G)$.

Proof. Let $G = (N, M)$ be a neutrosophic tree and $|N^*| \ge 3$. Since in the neutrosophic tree, there is a unique strong path between vertices u and v, for any u and v. hence this path is the unique strongest path from u to v. then, $d_{s_T}(u, v)$, for each u and v, is equal the sum of the truth-membership values of edges where those edges belong to the strong path from u to v. In other hands, $CONN_{TG}(u, v)$ is truth-membership values of the weakest edge of the $(u - v)-$path. It follows that

$$CONN_{TG}(u, v) \le d_{s_T}(u, v),$$

In the above relation, equality occurs when uv is a strong edge. Otherwise

$$CONN_{TG}(u,v) < d_{s_T}(u,v),$$

hhen, we have

$$PCI_T(G) < PWI_T(G).$$

Similarly, $PWI_I(G)$ and $PWI_F(G)$ can be proved.

□

Here we show with an example that $TCI(G)$ dose not always have to be less than $TWI(G)$.

Example 2. Consider the Neutrosophic tree $G = (N, M)$ as shown in figure 2,

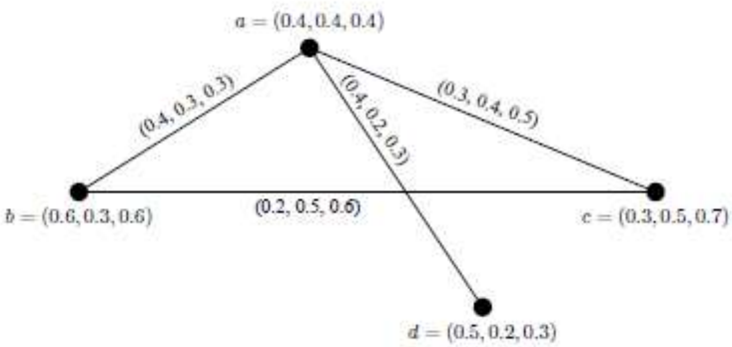

Figure 2. A neutrosophic tree with $V = \{a, b, c, d\}$

Note that here bc is a weak edge.

Table 2. The strength of connectedness and the geodesic between each pair of vertices u and v.

	$CONN_{TG}(u,v)$	$CONN_{IG}(u,v)$	$CONN_{FG}(u,v)$	$d_{s_T}(u,v)$	$d_{s_I}(u,v)$	$d_{s_F}(u,v)$
a,b	0.4	0.3	0.3	0.4	0.3	0.3
a,c	0.3	0.4	0.5	0.3	0.4	0.5
a,d	0.4	0.2	0.3	0.4	0.2	0.3
b,c	0.3	0.4	0.5	0.7	0.7	0.8
b,d	0.4	0.3	0.3	0.8	0.5	0.6
c,d	0.3	0.4	0.5	0.7	0.6	0.8

By direct calculations, we have

$$PCI_T(G) = \sum_{u,v \in N} T_N(u)T_N(v)CONN_{T_G}(u,v)$$

$$= 0.4 * 0.6 * 0.4 + 0.4 * 0.3 * 0.3 + 0.4 * 0.5 * 0.4 + 0.6 * 0.3 * 0.3 + 0.6 * 0.5 * 0.4$$
$$+ 0.3 * 0.5 * 0.3 = 0.096 + 0.036 + 0.080 + 0.054 + 0.120 + 0.045 = 0.431,$$

$$PCI_I(G) = \sum_{u,v \in N} I_N(u)I_N(v)CONN_{I_G}(u,v) = 0.036 + 0.080 + 0.016 + 0.060 + 0.018 + 0.040 = 0.25,$$

$$PCI_F(G) = \sum_{u,v \in N} F_N(u)F_N(v)CONN_{F_G}(u,v) = 0.072 + 0.14 + 0.036 + 0.21 + 0.054 + 0.105 = 0.617,$$

$$TCI(G) = \frac{4 + 2PCI_T(G) - 2PCI_F(G) - PCI_I(G)}{6} = \frac{3.378}{6} = 0.563.$$

$$PWI_T(G) = \sum_{u,v \in N} T_N(u)T_N(v)d_{s_T}(u,v) = 0.096 + 0.036 + 0.08 + 0.126 + 0.24 + 0.105 = 0.683,$$

$$PWI_I(G) = \sum_{u,v \in N} I_N(u)I_N(v)d_{s_I}(u,v) = 0.036 + 0.08 + 0.016 + 0.105 + 0.03 + 0.060 = 0.327,$$

$$PWI_F(G) = \sum_{u,v \in N} F_N(u)F_N(v)d_{s_F}(u,v) = 0.072 + 0.14 + 0.036 + 0.336 + 0.108 + 0.168 = 0.86,$$

$$TWI(G) = \frac{4 + 2PCWI_T(G) - 2PWI_F(G) - PWI_I(G)}{6} = \frac{3.319}{6} = 0.553.$$

As seen in this example

$$PCI_T(G) = 0.431 < PWI_T(G) = 0.683,$$
$$PCI_I(G) = 0.25 < PWI_I(G) = 0.327,$$
$$PCI_F(G) = 0.617 < PWI_F(G) = 0.86.$$

But, we have $TCI(G) = 0.563 > TWI(G) = 0.553$.

The neutrosophic graph shown in the figure below is also a tree in which $PCI_T(G) < PWI_T(G)$, $PCI_I(G) < PWI_I(G)$, $PCI_F(G) < PWI_F(G)$. And, $TCI(G) < TWI(G)$.

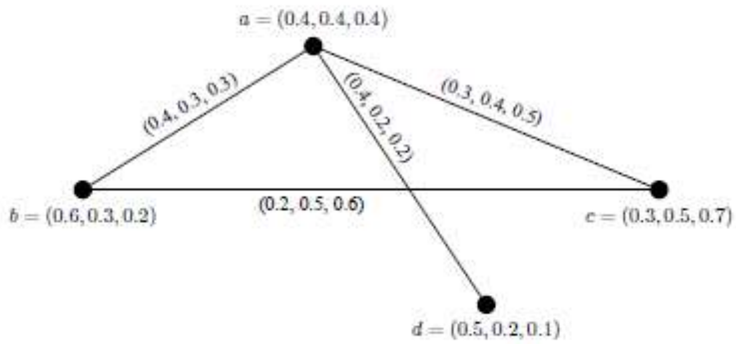

Figure 3. A neutrosophic tree with $V = \{a, b, c, d\}$

Theorem 7. Let $G = (N, M)$ be a neutrosophic tree $|N^*| \geq 3$, With G^* is a star. Let M be a constant function. if v_1 is the center vertex and $v_2, v_3, \ldots, v_n$ are the vertices adjacent to vertex v_1, then

$$PWI_T(G) = 2t \sum_{j=1}^{n-1} T_N(v_j) \sum_{k=j+1}^{n} T_N(v_k) - tT_N(v_1) \sum_{j=2}^{n} T(v_j),$$

$$PWI_I(G) = 2i \sum_{j=1}^{n-1} I_N(v_j) \sum_{k=j+1}^{n} I_N(v_k) - iI_N(v_1) \sum_{j=2}^{n} I(v_j),$$

$$PWI_F(G) = 2f \sum_{j=1}^{n-1} F_N(v_j) \sum_{k=j+1}^{n} F_N(v_k) - fF_N(v_1) \sum_{j=2}^{n} F(v_j),$$

where $M = (t, i, f)$.

Proof. Let $G = (N, M)$ be a neutrosophic tree $|N^*| \geq 3$, With G^* is a star. Since $= (t, i, f)$ is a constant function and v_i is the center vertex, for each $v_i,\ 1 < i \leq n$, we have

$$d_{s_T}(v_1, v_i) = t, \qquad d_{s_I}(v_1, v_i) = i, \qquad d_{s_F}(v_1, v_i) = f.$$

Also, for v_i and $v_j,\ i, j \neq 1$, then

$$d_{s_T}(v_j, v_i) = 2t, \qquad d_{s_I}(v_j, v_i) = 2i, \qquad d_{s_F}(v_j, v_i) = 2f.$$

Then

$$\begin{aligned}
PWI_T(G) &= \sum_{v_i, v_j \in N} T_N(v_i) T_N(v_j) d_{s_T}(v_i,\ v_j) \\
&= \sum_{v_j \in N} T_N(v_1) T_N(v_j) d_{s_T}(v_1,\ v_j) + \sum_{\substack{v_i, v_j \in N \\ i \neq 1}} T_N(v_i) T_N(v_j) d_{s_T}(v_i,\ v_j) \\
&= tT_N(v_1) \sum_{j=2}^{n} T(v_j) + 2t \sum_{\substack{v_i, v_j \in N \\ i \neq 1}} T_N(v_i) T_N(v_j) d_{s_T}(v_i,\ v_j) \\
&= tT_N(v_1) \sum_{j=2}^{n} T(v_j) + \left[2t \sum_{j=1}^{n-1} T_N(v_j) \sum_{k=j+1}^{n} T_N(v_k) - 2tT_N(v_1) \sum_{j=2}^{n} T(v_j) \right] \\
&= 2t \sum_{j=1}^{n-1} T_N(v_j) \sum_{k=j+1}^{n} T_N(v_k) - tT_N(v_1) \sum_{j=2}^{n} T(v_j).
\end{aligned}$$

Similarly, $PWI_I(G)$ and $PWI_F(G)$ can be proved.
□

4. Applications

One of the most important topics is the use of neutrosophic sets in other sciences and also the use of these assemblies to model various problems. Many applications have been discussed by experts so far. Which can be referred to as application of neutrosophic in graphs [12, 17-19], application in algebraic topics [11, 14], application in intelligent systems and optimization [3, 4].

Here the Wiener index is calculated for a neutrosophic graph associated with a real-time example. You can see this issue and its explanation on the website www.pantechsolutions.net. The neutrosophic graph of this issue is also given in [5]. There, the author examines energy, Laplacian energy, and signless Laplacian energy. We also use the modeling used in [5] here. This neutrosophic graph is intended for four different time periods. According to each time period, we define a neutrosophic graph in the following order:

- G_1 from 16 January 2018 to 15 February 2018 (figure 3);
- G_2 from 16 February 2018 to 15 March 2018 (figure 4);
- G_3 from 16 March 2018 to 15 April 2018 (figure 5);
- G_4 from 16 April 2018 to 15 May 2018 (figure 6);

We now calculate the Wiener index (partial Wiener index and totally Wiener index) for each of the above time periods.

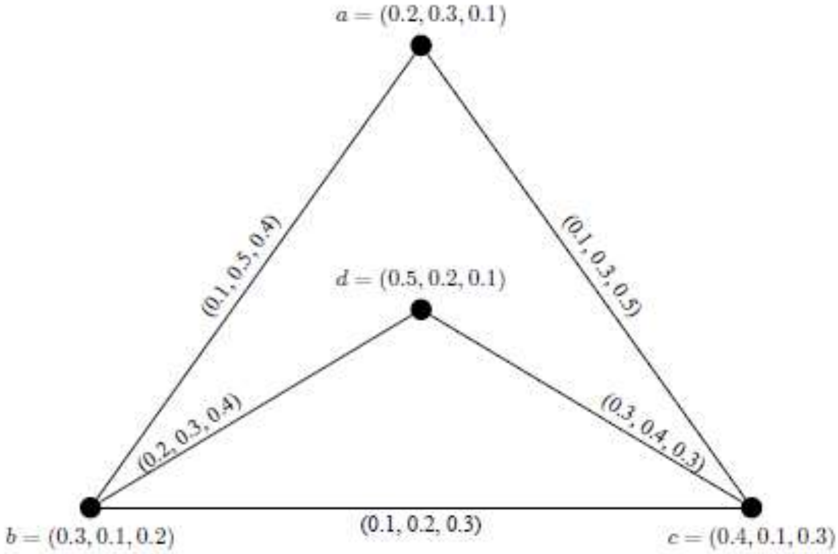

Figure 4. Neutrosophic graph G_1

Table 3. The sum of the weights of the edges in geodesic between each pair of vertices u and v.

	$d_{s_T}(u,v)$	$d_{s_I}(u,v)$	$d_{s_F}(u,v)$
a,b	0.1	$0.3+0.2=0.5$	0.4
a,c	0.1	0.3	$0.4+0.3=0.7$
a,d	$0.1+0.2=0.3$	$0.3+0.4=0.7$	$0.4+0.4=0.8$
b,c	$0.2+0.3=0.5$	0.2	0.3
b,d	0.2	0.3	$0.3+0.3=0.6$
c,d	0.3	$0.2+0.3=0.5$	0.3

$$PWI_T(G_1)=\sum_{u,v\in N}T_N(u)T_N(v)d_{s_T}(u,v)$$
$$=(0.2)(0.3)(0.1)+(0.2)(0.4)(0.1)+(0.2)(0.5)(0.3)+(0.3)(0.4)(0.5)+(0.3)(0.5)(0.2)+(0.4)(0.5)(0.3)=0.194,$$

$$PWI_I(G_1)=\sum_{u,v\in N}I_N(u)I_N(v)d_{s_I}(u,v)$$
$$=(0.3)(0.1)(0.5)+(0.3)(0.1)(0.3)+(0.3)(0.2)(0.7)+(0.1)(0.1)(0.2)+(0.1)(0.2)(0.3)+(0.1)(0.2)(0.5)=0.084,$$

$$PWI_F(G_1)=\sum_{u,v\in N}F_N(u)F_N(v)d_{s_F}(u,v)$$
$$=(0.1)(0.2)(0.4)+(0.1)(0.3)(0.7)+(0.1)(0.1)(0.8)+(0.2)(0.3)(0.3)+(0.2)(0.1)(0.6)+(0.3)(0.1)(0.3)=0.076,$$

$$TWI(G_1)=\frac{4+2PCWI_T(G_1)-2PWI_F(G_1)-PWI_I(G_1)}{6}=\frac{4+2(0.194)-2(0.076)-(0.084)}{6}=\frac{4.152}{6}$$
$$=0.692.$$

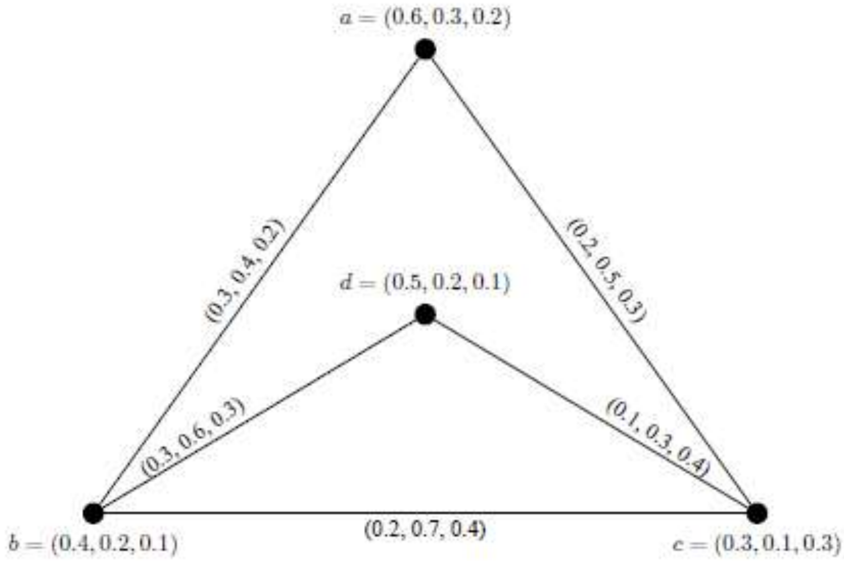

Figure 5. Neutrosophic graph G_2

Table 4. The sum of the weights of the edges in geodesic between each pair of vertices u and v.

	$d_{s_T}(u,v)$	$d_{s_I}(u,v)$	$d_{s_F}(u,v)$
a,b	0.3	0.4	0.2
a,c	0.2	0.5	0.3
a,d	0.3	$0.3+0.5=0.8$	$0.2+0.3=0.5$
b,c	0.2	$0.4+0.5=0.9$	$0.2+0.3=0.5$
b,d	0.3	$0.4+0.5+0.3=1.2$	0.3
c,d	$0.2+0.3=0.5$	0.3	$0.3+0.2+0.3=0.8$

$$PWI_T(G_2) = \sum_{u,v\in N} T_N(u)T_N(v)d_{s_T}(u,v) = 0.072 + 0.036 + 0.090 + 0.024 + 0.060 + 0.075 = 0.357,$$

$$PWI_I(G_2) = \sum_{u,v\in N} I_N(u)I_N(v)d_{s_I}(u,v) = 0.024 + 0.015 + 0.048 + 0.018 + 0.048 + 0.006 = 0.159,$$

$$PWI_F(G_2) = \sum_{u,v\in N} F_N(u)F_N(v)d_{s_F}(u,v) = 0.004 + 0.018 + 0.010 + 0.015 + 0.003 + 0.024 = 0.074,$$

$$TWI(G_2) = \frac{4 + 2PCWI_T(G_2) - 2PWI_F(G_2) - PWI_I(G_2)}{6} = \frac{4 + 2(0.357) - 2(0.074) - (0.159)}{6} = \frac{4.307}{6}$$
$$= 0.718.$$

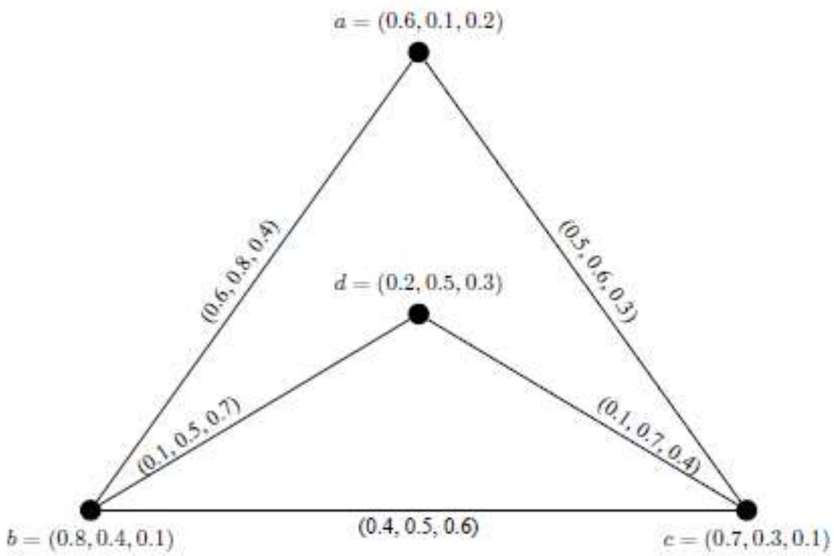

Figure 6. Neutrosophic graph G_3

Table 5. The sum of the weights of the edges in geodesic between each pair of vertices u and v.

	$d_{s_T}(u,v)$	$d_{s_I}(u,v)$	$d_{s_F}(u,v)$
a,b	0.6	$0.5+0.6=1.1$	0.4
a,c	0.5	0.6	0.3
a,d	$0.5+0.1=0.6$	$0.6+0.5+0.5=1.6$	$0.3+0.4=0.7$
b,c	$0.6+0.5=1.1$	0.5	$0.4+0.3=0.7$
b,d	0.1	0.5	$0.4+0.3+0.4=1.1$
c,d	0.1	$0.5+0.5=1$	0.4

$$PWI_T(G_3) = \sum_{u,v\in N} T_N(u)T_N(v)d_{s_T}(u,v) = 0.288+0.210+0.072+0.616+0.016+0.014 = 1.216,$$

$$PWI_I(G_3) = \sum_{u,v\in N} I_N(u)I_N(v)d_{s_I}(u,v) = 0.044+0.018+0.080+0.060+0.1+0.15 = 0.452,$$

$$PWI_F(G_3) = \sum_{u,v\in N} F_N(u)F_N(v)d_{s_F}(u,v) = 0.008+0.006+0.042+0.007+0.033+0.012 = 0.108$$

$$TWI(G_3) = \frac{4+2PCWI_T(G_3)-2PWI_F(G_3)-PWI_I(G_3)}{6} = \frac{4+2(1.216)-2(0.108)-(0.452)}{6} = \frac{5.548}{6}$$

$$= 0.925.$$

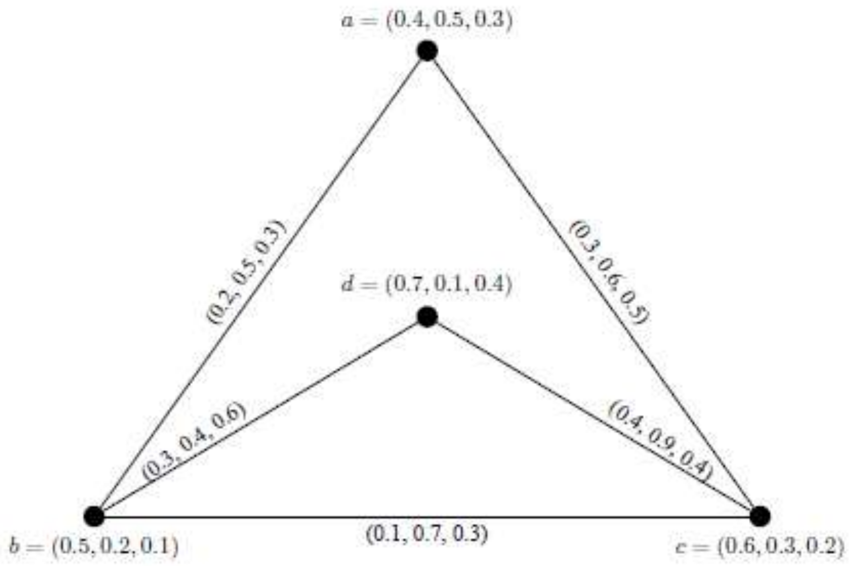

Figure 7. Neutrosophic graph G_4

Table 6. The sum of the weights of the edges in geodesic between each pair of vertices u and v.

	$d_{s_T}(u,v)$	$d_{s_I}(u,v)$	$d_{s_F}(u,v)$
a,b	$0.3+0.4+0.3=1$	0.5	0.3
a,c	0.3	0.6	$0.3+0.3=0.6$
a,d	$0.3+0.4=0.7$	$0.5+0.4=0.9$	$0.5+0.4=0.9$
b,c	$0.3+0.4=0.7$	$0.5+0.6=1.1$	0.3
b,d	0.3	0.4	$0.3+0.4=0.7$
c,d	0.4	$0.6+0.5=1.1$	0.4

$$PWI_T(G_4) = \sum_{u,v\in N} T_N(u)T_N(v)d_{s_T}(u,v) = 0.20 + 0.072 + 0.196 + 0.210 + 0.105 + 0.168 = 0.951,$$

$$PWI_I(G_4) = \sum_{u,v\in N} I_N(u)I_N(v)d_{s_I}(u,v) = 0.050 + 0.180 + 0.045 + 0.066 + 0.008 + 0.033 = 0.382,$$

$$PWI_F(G_4) = \sum_{u,v\in N} F_N(u)F_N(v)d_{s_F}(u,v) = 0.009 + 0.036 + 0.108 + 0.006 + 0.028 + 0.032 = 0.219,$$

$$TWI(G_4) = \frac{4 + 2PCWI_T(G_4) - 2PWI_F(G_4) - PWI_I(G_4)}{6} = \frac{4 + 2(0.951) - 2(0.219) - (0.382)}{6} = \frac{5.082}{6}$$
$$= 0.847.$$

Now, using the Wiener index obtained for each of the neutrosophic graphs G_1, G_2, G_3, and G_4, we can compare these four components in the time intervals given in the problem.

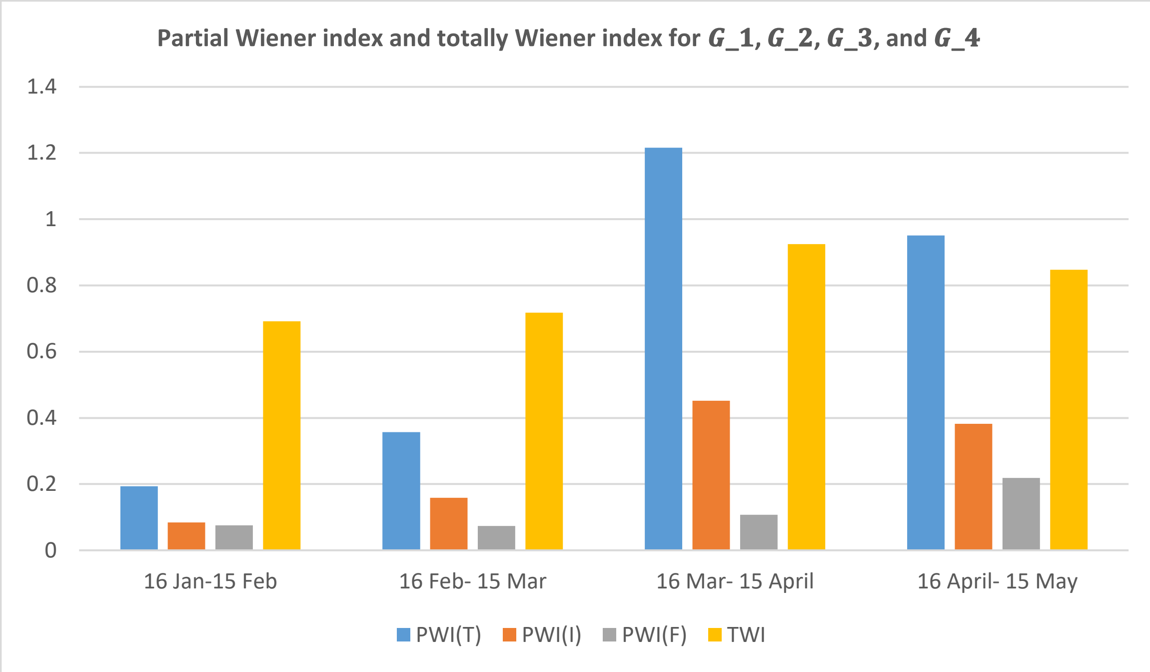

Figure 8. Wiener index comparison chart in G_1, G_2, G_3, and G_4

As shown in Figure 7, they can be easily studied using the Wiener index and assigning a logical value to each of the neutrosophic graphs.

Conclusion

In this article, we examine the Wiener index in neutrosophic graphs. First, this index was defined for this group of graphs and then it was calculated for certain modes of neutrosophic graphs. In the following, we provide an example of the application of this index in real problems. As you can see here, this index, which is one of the most important topological indices based on distance, can be a good criterion for comparing neutrosophic graphs under the same conditions. This index can also be studied and used for bipolar and interval valued neutrosophic graphs.

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