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Structures on Doubt Neutrosophic Ideals of BCK/BCI-Algebras under (S, T)-Norms

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Abstract. Smarandache implemented the idea of neutrosophic set theory as a method for dealing undetermined data. Neutrosophic set theory is commonly used in various algebric structures, such as groups, rings and BCK/BCI-algebras. At present, there exist no results on doubt neutrosophic ideals of BCK/BCI-algebras using t-conorm and t-norm. First, the notions of (S, T)- normed doubt neutrosophic subalgebras and ideals of BCK/BCI-algebras are introduced and the characteristic properties are described. Then, images and preimages of (S, T)- normed doubt neutrosophic ideals under homomorphism are considered. Moreover, the direct product and (S, T)- product of (S, T)- normed doubt neutrosophic ideals of BCK/BCI-algebras are also discussed.

Keywords: BCK/BCI-algebra; doubt neutrosophic subalgebra (ideal); (S, T)-normed doubt neutrosophic subalgebra (ideal).

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1. Introduction

BCK-algebras entered into mathematics in 1966 through the work of Imai and Iséki [1], and were applied to various mathematical fields, such as group theory, topology, functional analysis and probability theory, etc. In the same way, the concept of a BCI-algebra, which is a generalization of a BCK-algebra, was proposed by Iséki [2]. Zadeh [3] introduced the idea of fuzzy set theory in 1965, where the degree of membership is discussed, and Xi [4] introduced fuzzy subalgebras and ideals in BCK/BCI-algebras in 1991. Later on, fuzzy sets have been generalized to intuitionistic fuzzy sets [5] by adding a non-membership function by Atanassov in 1986 and this concept has been applied to BCK/BCI-algebras by Jun and Kim [6].

As anew idea and based on the concept defined by Xi [4], Jun [7] in 1994 introduced the notions of doubt fuzzy subalgebras and ideals in BCK/BCI-algebras. Bej and Pal [8] introduced the concepts of doubt intuitionistic fuzzy subalgebras and ideals in BCK/BCI-algebras.

Al-Masarwah and Ahmad [9] introduced the concepts of doubt bipolar fuzzy subalgebras and ideals in BCK/BCI-algebras. After that, many other researchers used these ideas and published numerous articles in different branches of algebraic structures [10–14].

Triangular norms were formulated by Schweizer and Sklar [15] to model the distances in probabilistic metric spaces. Triangular norms play an important role in many fields of mathematics, statistics, cooperative games, decision making and artificial intelligence [16]. In particular, in fuzzy set theory, t-conorm (S) and t-norm (T) have been widely used for fuzzy logic, fuzzy relation equations and fuzzy operations. In algebraic structures, Senapati [17] proposed the idea of (imaginable) T-fuzzy subalgebras and (imaginable) T-fuzzy closed ideals of BG-algebras. Kim [18] presented the intuitionistic (S, T)-normed fuzzy subalgebras in BCK/BCI-algebras using triangular norms. Also, Kutukcu and Tuna [19], presented a new classification of intuitionistic fuzzy subalgebras, ideals and implicative ideals in BCK/BCI-algebras.

Neutrosophy, [20, 21] a new branch of science that deals with indeterminacy, was launched by Smarandache in 1998. This concept is a generalization of the classical set, fuzzy set and intuitionistic fuzzy set. Neutrosophic set theory has been applied to several fields of mathematics including decision making [22–24], pattern recognition and medical diagnosis [25] and others [26–31]. In the aspect of algebraic structures, the papers [32–38] address neutrosophic algebraic structures in BCK/BCI-algebras.

As no studies have been reported so far to generalize the above mentioned concepts, so the aim of this present article is:

(1) To propose the concept of (S, T)-normed doubt neutrosophic subalgebras and (S, T)-normed doubt neutrosophic ideals of BCK/BCI-algebras as a generalization of (S, T)-normed intutionistic fuzzy subalgebras and ideals of BCK/BCI-algebras.

(2) To consider images and preimages of (S, T)- normed doubt neutrosophic ideals under homomorphism.

(3) To define and discuss the direct product and (S, T)- product of (S, T)- normed doubt neutrosophic ideals of BCK/BCI-algebras.

To do so, the rest of the article is structured as follows: In Section 2, we review some basic notions. In Section 3, we introduce the notions of (S, T)- normed doubt neutrosophic subalgebras and ideals of BCK/BCI-algebras and then describe some of the characteristic properties. Furthermore, we consider images and preimages of (S, T)- normed doubt neutrosophic ideals under homomorphism. In Section 4, we discuss the direct product and (S, T)- product of (S, T)- normed doubt neutrosophic ideals of BCK/BCI-algebras. Finally, in Section 5, we present the conclusion and future works of the study.

2. Preliminaries

In the current section, we remember some of the basic notions of BCK/BCI-algebras which will be very helpful in further study of the paper. Let X be a BCK/BCI-algebra in what follows, unless otherwise stated.

By a BCI-algebra, we mean a set X with a special element 0 and a binary operation $*$, for all $p, q, s \in X$, that satisfies the following axioms:

(I) $[(p * q) * (p * s)] * (s * q) = 0$,

(II) $[p * (p * q)] * q = 0$,

(III) $p * p = 0$,

(IV) $p * q = 0$ and $q * p = 0$ imply $p = q$.

If a BCI-algebra X satisfies $0 * p = 0$, then X is called a BCK-algebra. In a BCK/BCI-algebra, $p * 0 = p$ holds. A partial ordering $\leq$ on a BCK/BCI-algebra X can be defined by $p \leq q$ if and only if $p * q = 0$. A non-empty subset K of a BCK/BCI-algebra X is called a subalgebra of X if $p * q \in K, \forall p, q \in X$, and an ideal of X if $\forall p, q \in X$,

(1) $0 \in K$,

(2) $p * q \in K$ and $q \in K$ imply $p \in K$.

Definition 2.1. A neutrosophic set in a non-empty set X (see [14]) is a structure of the form:

$$B = \{\langle p; B_T(p), B_I(p), B_F(p)\rangle | p \in X\},$$

where $B_T, B_I, B_F : X \to [0, 1]$. We shall use the symbol $B = (B_T, B_I, B_F)$, for the neutrosophic set $B = \{\langle p; B_T(p), B_I(p), B_F(p)\rangle | p \in X\}$.

If $B = (B_T, B_I, B_F)$ is a neutrosophic set in X, then $\Box B = (B_T, B_I, B_T^c)$ and $\Diamond B = (B_F^c, B_I, B_F)$ are also neutrosophic sets in X.

Definition 2.2 ([15])**.** A function $T : [0, 1] \times [0, 1] \to [0, 1]$ is called a triangular norm, if it satisfies the following conditions: $\forall p, q.s \in [0, 1]$,

(1) $T(0, 0) = 0, T(1, 1) = 1$,

(2) $T(p, T(q, s)) = T(T(p, q), s)$,

(3) $T(p, q) = T(q, p)$,

(4) $T(p, q) \leq T(p, s)$ if $q \leq s$.

If $T(p, 0) = p$ and $T(p, 1) = p$ for all $p \in [0, 1]$, then T is called a t-conorm and a t-norm, respectively. Throughout this paper, denote S and T as a t-conorm and a t-norm, respectively.

Some examples of t-conorms and t-norms are:

(1) $S_M(p, q) = \max\{p, q\}$ and $T_M(p, q) = \min\{p, q\}$.

(2) $S_L(p, q) = \min\{p + q, 1\}$ and $T_L(p, q) = \max\{p + q - 1, 0\}$.

(3) $S_P(p,q) = p + q - pq$ and $T_P(p,q) = pq$.

A t-conorm S and a t-norm T are associated [39], i.e., $S(p,q) = 1-T(1-p,1-q), \forall p,q \in [0,1]$.

Lemma 2.3 ([40]). *For any $p,q \in [0,1]$, we have $0 \leq \max\{p,q\} \leq S(p,q) \leq 1$ and $0 \leq T(p,q) \leq \min\{p,q\} \leq 1$.*

Definition 2.4 ([7]). A fuzzy set λ of X is called a doubt fuzzy subalgebra of X if $\lambda(p * q) \leq \max\{\lambda(p), \lambda(q)\}$ $\forall p,q \in X$, and a doubt fuzzy ideal of X if $\lambda(0) \leq \lambda(p) \leq \max\{\lambda(p*q), \lambda(q)\}$ $\forall p,q \in X$.

Definition 2.5 ([41]). A neutrosophic set $B = (B_T, B_I, B_F)$ of X is called a neutrosophic subalgebra of X if for all $p,q \in X$,

(1) $B_T(p*q) \geq \min\{B_T(p), B_T(q)\}$,
(2) $B_I(p*q) \geq \min\{B_I(p), B_I(q)\}$,
(3) $B_F(p*q) \leq \max\{B_F(p), B_F(q)\}$.

Definition 2.6 ([41]). A neutrosophic set $B = (B_T, B_I, B_F)$ of X is called a neutrosophic ideal of X if for all $p,q \in X$,

(1) $B_T(0) \geq B_T(p) \geq \min\{B_T(p*q), B_T(q)\}$,
(2) $B_I(0) \geq B_I(p) \geq \min\{B_I(p*q), B_I(q)\}$,
(3) $B_F(0) \leq B_F(p) \leq \max\{B_F(p*q), B_F(q)\}$.

3. (S,T)-Normed doubt neutrosophic ideals

Definition 3.1. A neutrosophic set $B = (B_T, B_I, B_F)$ of X is called a doubt neutrosophic subalgebra of X if for all $p,q \in X$,

(1) $B_T(p*q) \leq \max\{B_T(p), B_T(q)\}$,
(2) $B_I(p*q) \leq \max\{B_I(p), B_I(q)\}$,
(3) $B_F(p*q) \geq \min\{B_F(p), B_F(q)\}$.

Definition 3.2. A neutrosophic set $B = (B_T, B_I, B_F)$ of X is called a doubt neutrosophic subalgebra of X with respect to a t-conorm S and a t-norm T (or simply, an (S,T)-normed doubt neutrosophic subalgebra of X) if for all $p,q \in X$,

(1) $B_T(p*q) \leq S(B_T(p), B_T(q))$,
(2) $B_I(p*q) \leq S(B_I(p), B_I(q))$,
(3) $B_F(p*q) \geq T(B_F(p), B_F(q))$.

Definition 3.3. A neutrosophic set $B = (B_T, B_I, B_F)$ of X is called a doubt neutrosophic ideal of X if for all $p,q \in X$,

(1) $B_T(0) \leq B_T(p) \leq \max\{B_T(p*q), B_T(q)\}$,

(2) $B_I(0) \leq B_I(p) \leq \max\{B_I(p * q), B_I(q)\}$,

(3) $B_F(0) \geq B_F(p) \geq \min\{B_F(p * q), B_F(q)\}$.

Definition 3.4. A neutrosophic set $B = (B_T, B_I, B_F)$ of X is called a doubt neutrosophic ideal of X with respect to a t-conorm S and a t-norm T (or simply, an (S, T)-normed doubt neutrosophic ideal of X) if for all $p, q \in X$,

(1) $B_T(0) \leq B_T(p) \leq S(B_T(p * q), B_T(q))$,

(2) $B_I(0) \leq B_I(p) \leq S(B_I(p * q), B_I(q))$,

(3) $B_F(0) \geq B_F(p) \geq T(B_F(p * q), B_F(q))$.

Example 3.5. Consider a given BCK-algebra $X = \{0, k, l, m\}$ in Table 1:

Table 1. Tabular representation of a BCK-algebra $X = \{0, k, l, m\}$.

$*$	0	k	l	m
0	0	0	0	0
k	k	0	0	k
l	l	k	0	l
m	m	m	m	0

Define a neutrosophic set $B = (B_T, B_I, B_F)$ of X by Table 2:

Table 2. Neutrosophic set $B = (B_T, B_I, B_F)$.

X	$B_T(p)$	$B_I(p)$	$B_F(p)$
0	0	0	1
k	0.50	0.40	0.33
l	0.50	0.40	0.33
m	1	0.90	0

Clearly, $B_T(0) \leq B_T(p) \leq S_M(B_T(p * q), B_T(q))$, $B_I(0) \leq B_I(p) \leq S_M(B_I(p * q), B_I(q))$ and $B_F(0) \geq B_F(p) \geq T_L(B_F(p * q), B_F(q))$ for all $p, q \in X$. Hence, $B = (B_T, B_I, B_F)$ is an (S_M, T_L)-normed doubt neutrosophic ideal of X. Also, note that a t-conorm S_M and a t-norm T_L are not associated.

Remark 3.6. Example 3.5 holds even with the t-conorm S_M and t-norm T_M. Hence, $B = (B_T, B_I, B_F)$ is an (S_M, T_M)-normed doubt neutrosophic ideal of X.

Remark 3.7. Every doubt neutrosophic ideal of X is an (S, T)-normed doubt neutrosophic ideal of X, but the converse is not true.

Example 3.8. Consider a given BCK-algebra $X = \{0, 1, 2, 3, 4\}$ in Table 3:

TABLE 3. Tabular representation of a BCK-algebra $X = \{0, 1, 2, 3, 4\}$.

$*$	0	1	2	3	4
0	0	0	0	0	0
1	1	0	1	0	0
2	2	2	0	0	0
3	3	2	1	0	0
4	4	4	4	4	0

Define a neutrosophic set $B = (B_T, B_I, B_F)$ of X by Table 4:

TABLE 4. Neutrosophic set $B = (B_T, B_I, B_F)$.

X	$B_T(p)$	$B_I(p)$	$B_F(p)$
0	0.50	0.50	0.33
1	0.50	0.50	0.33
2	0.50	0.50	0.33
3	0.75	0.75	0.25
4	0.75	0.75	0.25

Clearly, $B_T(0) \leq B_T(p) \leq S_L(B_T(p*q), B_T(q))$, $B_I(0) \leq B_I(p) \leq S_L(B_I(p*q), B_I(q))$ and $B_F(0) \geq B_F(p) \geq T_P(B_F(p*q), B_F(q))$ for all $p, q \in X$. Hence, $B = (B_T, B_I, B_F)$ is an (S_L, T_P)-normed doubt neutrosophic ideal of X, but it is not a doubt neutrosophic ideal of X.

Lemma 3.9. *If $B = (B_T, B_I, B_F)$ is an (S, T)-normed doubt neutrosophic ideal of X, then so is $\Box B = (B_T, B_I, B_T^c)$, where a t-conorm S and a t-norm T are associated.*

Proof. Let $B = (B_T, B_I, B_F)$ be an (S, T)-normed doubt neutrosophic ideal of X. Then, $B_T(0) \leq B_T(p) \forall p \in X$ and so $1 - B_T^c(0) \leq 1 - B_T^c(p)$. Hence, $B_T^c(0) \geq B_T^c(p)$. Also, for all $p, q \in X$, we have $B_T(p) \leq S(B_T(p*q), B_T(q))$ and so $1 - B_T^c(p) \leq S(1 - B_T^c(p*q), 1 - B_T^c(q))$ which implies $B_T^c(p) \geq 1 - S(1 - B_T^c(p*q), 1 - B_T^c(q))$. Since S and T are associated, we have $B_T^c(p) \geq T(B_T^c(p*q), B_T^c(q))$. Thus, $\Box B = (B_T, B_I, B_T^c)$ is an (S, T)-normed doubt neutrosophic ideal of X. □

Lemma 3.10. *If $B = (B_T, B_I, B_F)$ is an (S, T)-normed doubt neutrosophic ideal of X, then so is $\Diamond B = (B_F^c, B_I, B_F)$, where a t-conorm S and a t-norm T are associated.*

Proof. The proof is similar to the proof of Lemma 3.9. □

Combining Lemmas 3.9 and 3.10, we deduce that:

Theorem 3.11. *A neutrosophic set $B = (B_T, B_I, B_F)$ is an (S,T)-normed doubt neutrosophic ideal of X if and only if $\Box B$ and $\Diamond B$ are (S,T)-normed doubt neutrosophic ideals of X, where a t-conorm S and a t-norm T are associated.*

Lemma 3.12. *Every (S,T)-normed doubt neutrosophic ideal $B = (B_T, B_I, B_F)$ of X satisfies: for all $p, q \in X$,*

$$p \leq q \Rightarrow B_T(p) \leq B_T(q), B_I(p) \leq B_I(q) \text{ and } B_F(q) \geq B_F(p).$$

Proof. Let $p, q \in X$ be such that $p \leq q$. Then, $p * q = 0$ and so

$$B_T(p) \leq S(B_T(p * q), B_T(q)) = S(B_T(0), B_T(q)) \leq B_T(q),$$
$$B_I(p) \leq S(B_I(p * q), B_I(q)) = S(B_I(0), B_I(q)) \leq B_I(q),$$

and

$$B_F(p) \geq T(B_F(p * q), B_F(q)) = T(B_F(0), B_F(q)) \geq B_F(q).$$

This completes the proof. □

Theorem 3.13. *Every (S,T)-normed doubt neutrosophic ideal of X is an (S,T)-normed doubt neutrosophic subalgebra of X.*

Proof. Let $B = (B_T, B_I, B_F)$ be an (S,T)-normed doubt neutrosophic ideal of X. Since $p*q \leq p$ $\forall p, q \in X$, it follows from Lemma 3.12 that $B_T(p * q)) \leq B_T(p), B_I(p * q)) \leq B_I(p)$ and $B_F(p * q)) \geq B_F(p)$. Then,

$$B_T(p * q)) \leq B_T(p) \leq S(B_T(p * q), B_T(q)) \leq S(B_T(p), B_T(q)),$$
$$B_I(p * q)) \leq B_I(p) \leq S(B_I(p * q), B_I(q)) \leq S(B_I(p), B_I(q)),$$

and

$$B_F(p * q)) \geq B_F(p) \geq T(B_F(p * q), B_F(q)) \geq T(B_F(p), B_F(q)).$$

Hence, $B = (B_T, B_I, B_F)$ is an (S,T)-normed doubt neutrosophic subalgebra of X. □

Remark 3.14. The converse of Theorem 3.13 is not hold in general.

Example 3.15. Reconsider the BCK-algebra X given in Example 3.5. Define a neutrosophic set $B = (B_T, B_I, B_F)$ of X by Table 5:

Table 5. Neutrosophic set $B = (B_T, B_I, B_F)$.

X	$B_T(p)$	$B_I(p)$	$B_F(p)$
0	0	0	1
k	0.50	0.50	0.33
l	1	1	0
m	1	1	0

Clearly, $B = (B_T, B_I, B_F)$ is an (S_M, T_M)-normed doubt neutrosophic subalgebra of X, but it is not (S_M, T_M)-normed doubt neutrosophic ideal of X, since

$$B_T(l) = 1 > \max\{B_T(l * k), B_T(k)\},$$
$$B_I(l) = 1 > \max\{B_I(l * k), B_I(k)\},$$

and

$$B_F(l) = 0 < \min\{B_F(l * k), B_F(k)\}.$$

Definition 3.16. A mapping $\theta : X \to Y$ of BCK/BCI-algebras is said to be a homomorphism if $\theta(p * q) = \theta(p) * \theta(q) \forall p, q \in X$. If $\theta : X \to Y$ is a homomorphism, then $\theta(0) = 0$.

Let $\theta : X \to Y$ be a homomorphism of BCK/BCI-algebras. For any neutrosophic set $B = (B_T, B_I, B_F)$ in Y, we define a new neutrosophic set $B[\theta] = (B_T[\theta], B_I[\theta], B_F[\theta])$ such that for all $p \in X$,

$$B_T[\theta] : X \to [0, 1], B_T[\theta](p) = B_T(\theta(p)),$$
$$B_I[\theta] : X \to [0, 1], B_I[\theta](p) = B_I(\theta(p)),$$
$$B_F[\theta] : X \to [0, 1], B_F[\theta](p) = B_F(\theta(p)).$$

Theorem 3.17. *Let $\theta : X \to Y$ be a homomorphism of BCK/BCI-algebras. If $B = (B_T, B_I, B_F)$ is an (S, T)-normed doubt neutrosophic ideal of Y, then $B[\theta] = (B_T[\theta], B_I[\theta], B_F[\theta])$ is an (S, T)-normed doubt neutrosophic ideal of X.*

Proof. We first have

$$B_T[\theta](0) = B_T(\theta(0)) = B_T(0) \leq B_T(\theta(p)) = B_T[\theta](p),$$
$$B_I[\theta](0) = B_I(\theta(0)) = B_I(0) \leq B_I(\theta(p)) = B_I[\theta](p),$$
$$B_F[\theta](0) = B_F(\theta(0)) = B_F(0) \geq B_F(\theta(p)) = B_F[\theta](p)$$

for all $p, q \in X$. Let $p, q \in X$. Then,

$$\begin{aligned} B_T[\theta](p) = B_T(\theta(p)) &\leq S(B_T(\theta(p) * \theta(q)), B_T(\theta(q))) \\ &= S(B_T(\theta(p * q)), B_T(\theta(q))) \\ &= S(B_T[\theta](p * q), B_T[\theta](q)), \end{aligned}$$

$$\begin{aligned} B_I[\theta](p) = B_I(\theta(p)) &\leq S(B_I(\theta(p) * \theta(q)), B_I(\theta(q))) \\ &= S(B_I(\theta(p * q)), B_I(\theta(q))) \\ &= S(B_I[\theta](p * q), B_I[\theta](q)) \end{aligned}$$

and

$$\begin{aligned} B_F[\theta](p) = B_F(\theta(p)) &\geq T(B_F(\theta(p) * \theta(q)), B_F(\theta(q))) \\ &= T(B_F(\theta(p * q)), B_F(\theta(q))) \\ &= T(B_F[\theta](p * q), B_F[\theta](q)). \end{aligned}$$

Therefore, $B[\theta] = (B_T[\theta], B_I[\theta], B_F[\theta])$ is an (S, T)-normed doubt neutrosophic ideal of X. □

Theorem 3.18. *Let $\theta : X \to Y$ be an onto homomorphism of BCK/BCI-algebras and let $B = (B_T, B_I, B_F)$ be a neutrosophic set of Y. If $B[\theta] = (B_T[\theta], B_I[\theta], B_F[\theta])$ is an (S, T)-normed doubt neutrosophic ideal of X, then $B = (B_T, B_I, B_F)$ is an (S, T)-normed doubt neutrosophic ideal of Y.*

Proof. For any $b \in Y$, there exists $a \in X$ such that $\theta(a) = b$. Then,

$$\begin{aligned} B_T(0) = B_T(\theta(0)) = B_T[\theta](0) \leq B_T[\theta](a) = B_T(\theta(a)) = B_T(b), \\ B_I(0) = B_I(\theta(0)) = B_I[\theta](0) \leq B_I[\theta](a) = B_I(\theta(a)) = B_I(b), \\ B_F(0) = B_F(\theta(0)) = B_F[\theta](0) \geq B_F[\theta](a) = B_F(\theta(a)) = B_F(b). \end{aligned}$$

Let $p, q \in Y$. Then, $\theta(a) = p$ and $\theta(b) = q$ for some $a, b \in X$. It follows that

$$\begin{aligned} B_T(p) = B_T(\theta(a)) &= B_T[\theta](a) \\ &\leq S(B_T[\theta](a * b), B_T[\theta](b)) \\ &= S(B_T(\theta(a * b)), B_T(\theta(b))) \\ &= S(B_T(\theta(a) * \theta(b)), B_T(\theta(b))) \\ &= S(B_T(p * q), B_T(q)), \end{aligned}$$

$$\begin{aligned} B_I(p) = B_I(\theta(a)) &= B_I[\theta](a) \\ &\leq S(B_I[\theta](a * b), B_I[\theta](b)) \\ &= S(B_I(\theta(a * b)), B_I(\theta(b))) \\ &= S(B_I(\theta(a) * \theta(b)), B_I(\theta(b))) \\ &= S(B_I(p * q), B_I(q)) \end{aligned}$$

and

$$\begin{aligned} B_F(p) = B_F(\theta(a)) &= B_F[\theta](a) \\ &\geq T(B_F[\theta](a * b), B_F[\theta](b)) \\ &= T(B_F(\theta(a * b)), B_F(\theta(b))) \\ &= T(B_F(\theta(a) * \theta(b)), B_F(\theta(b))) \\ &= T(B_F(p * q), B_F(q)), \end{aligned}$$

Therefore, $B = (B_T, B_I, B_F)$ is an (S, T)-normed doubt neutrosophic ideal of Y. □

4. **Product of (S, T)-normed doubt neutrosophic ideals**

In this section, we discuss the direct product and (S, T)- product of (S, T)- normed doubt neutrosophic ideals of BCK/BCI-algebras.

Lemma 4.1 ([16]). *Let S and T be a t-conorm and t-norm, respectively. Then,*

$$S(S(p, q), S(a, b)) = S(S(p, a), S(q, b)),$$
$$T(T(p, q), T(a, b)) = T(T(p, a), T(q, b))$$

for all $p, q, a, b \in [0, 1]$.

Theorem 4.2. *Let $X = P_1 \times P_2$ be the direct product BCK/BCI-algebra of two BCK/BCI-algebras P_1 and P_2. If $B = (B_T, B_I, B_F)$ (resp., $C = (C_T, C_I, C_F)$) is an (S, T)-normed doubt neutrosophic ideal of P_1 (resp., P_2), then $D = (D_T, D_I, D_F)$ is an (S, T)-normed doubt neutrosophic ideal of X defined by $D_T = B_T \times C_T, D_I = B_I \times C_I$, and $D_F = B_F \times C_F$ such that*

$$D_T(p_1, p_2) = (B_T \times C_T)(p_1, p_2) = S(B_T(p_1), C_T(p_2)),$$
$$D_I(p_1, p_2) = (B_I \times C_I)(p_1, p_2) = S(B_I(p_1), C_I(p_2)),$$
$$D_F(p_1, p_2) = (B_F \times C_F)(p_1, p_2) = T(B_F(p_1), C_F(p_2))$$

for all $(p_1, p_2) \in X$.

Proof. Let $(p_1, p_2), (q_1, q_2) \in P_1 \times P_2$. Since $X = P_1 \times P_2$ is a BCK/BCI-algebra, we have

$$\begin{aligned} D_T(0,0) = (B_T \times C_T)(0,0) &= S(B_T(0), C_T(0)) \\ &\leq S(B_T(p_1), C_T(p_2)) \\ &= S((B_T \times C_T)(p_1, p_2)) \\ &= D_T(p_1, p_2), \end{aligned}$$

$$\begin{aligned} D_I(0,0) = (B_I \times C_I)(0,0) &= S(B_I(0), C_I(0)) \\ &\leq S(B_I(p_1), C_I(p_2)) \\ &= S((B_I \times C_I)(p_1, p_2)) \\ &= D_I(p_1, p_2), \end{aligned}$$

and

$$\begin{aligned} D_F(0,0) = (B_F \times C_F)(0,0) &= T(B_F(0), C_F(0)) \\ &\geq T(B_F(p_1), C_F(p_2)) \\ &= T((B_F \times C_F)(p_1, p_2)) \\ &= D_F(p_1, p_2). \end{aligned}$$

Also,

$$\begin{aligned} D_T(p_1, p_2) = (B_T \times C_T)(p_1, p_2) &= S(B_T(p_1), C_T(p_2)) \\ &\leq S\Big(S(B_T(p_1 * q_1), B_T(q_1)), S(C_T(p_2 * q_2), C_T(q_2))\Big) \\ &= S\Big(S(B_T(p_1 * q_1), C_T(p_2 * q_2)), S(B_T(q_1), C_T(q_2))\Big) \\ &= S\Big((B_T \times C_T)(p_1 * q_1, p_2 * q_2), (B_T \times C_T)(q_1, q_2)\Big) \\ &= S\Big((B_T \times C_T)((p_1, p_2) * (q_1, q_2)), (B_T \times C_T)(q_1, q_2)\Big) \\ &= S\Big(D_T((p_1, p_2) * (q_1, q_2)), D_T(q_1, q_2)\Big), \end{aligned}$$

$$\begin{aligned} D_I(p_1, p_2) = (B_I \times C_I)(p_1, p_2) &= S(B_I(p_1), C_I(p_2)) \\ &\leq S\Big(S(B_I(p_1 * q_1), B_I(q_1)), S(C_I(p_2 * q_2), C_I(q_2))\Big) \\ &= S\Big(S(B_I(p_1 * q_1), C_I(p_2 * q_2)), S(B_I(q_1), C_I(q_2))\Big) \\ &= S\Big((B_I \times C_I)(p_1 * q_1, p_2 * q_2), (B_I \times C_I)(q_1, q_2)\Big) \\ &= S\Big((B_I \times C_I)((p_1, p_2) * (q_1, q_2)), (B_I \times C_I)(q_1, q_2)\Big) \\ &= S\Big(D_I((p_1, p_2) * (q_1, q_2)), D_I(q_1, q_2)\Big), \end{aligned}$$

and

$$\begin{aligned}
D_F(p_1,p_2) = (B_F \times C_F)(p_1,p_2) &= T(B_F(p_1), C_F(p_2)) \\
&\geq T\Big(T(B_F(p_1 * q_1), B_F(q_1)), T(C_F(p_2 * q_2), C_F(q_2))\Big) \\
&= T\Big(T(B_F(p_1 * q_1), C_F(p_2 * q_2)), T(B_F(q_1), C_F(q_2))\Big) \\
&= T\Big((B_F \times C_T)(p_1 * q_1, p_2 * q_2), (B_F \times C_F)(q_1, q_2)\Big) \\
&= T\Big((B_F \times C_F)((p_1,p_2) * (q_1,q_2)), (B_F \times C_F)(q_1, q_2)\Big) \\
&= T\Big(D_F((p_1,p_2) * (q_1,q_2)), D_F(q_1,q_2)\Big).
\end{aligned}$$

This completes the proof. □

Definition 4.3. Let $B = (B_T, B_I, B_F)$ and $C = (C_T, C_I, C_F))$ be two neutrosophic sets of a BCK/BCI-algebra X. Then, (S,T)- product of B and C, written as $[B.C]_{(S,T)}$, are defined by

$$[B.C]_{(S,T)} = ([B_T.C_T]_S, [B_I.C_I]_S, [B_F.C_F]_T),$$

where

$$[B_T.C_T]_S(p) = S(B_T(p), C_T(p)),$$
$$[B_I.C_I]_S(p) = S(B_I(p), C_I(p))$$

and

$$[B_F.C_F]_T(p) = T(B_F(p), C_F(p))$$

for all $p \in X$.

Theorem 4.4. *Let S and T be a t-conorm and s-norm, respectively. Let $B = (B_T, B_I, B_F)$ and $C = (C_T, C_I, C_F))$ be two (S,T)- normed doubt neutrosophic ideals of X. If S_1 is a t-conorm which dominates S, i.e.,*

$$S_1(S(p,q), S(a,b)) \leq S(S_1(p,a), S_1(q,b))$$

and T_1 is a t-norm which dominates T, i.e.,

$$T_1(T(p,q), T(a,b)) \geq T(T_1(p,a), T_1(q,b))$$

for all $p,q,a,b \in [0,1]$, then $([B_T.C_T]_{S_1}, [B_I.C_I]_{S_1}, [B_F.C_F]_{T_1})$ is an (S,T)-normed doubt neutrosophic ideal of X.

Proof. For any $p \in X$, we have

$$[B_T.C_T]_{S_1}(0) = S_1(B_T(0), C_T(0)) \leq S_1(B_T(p), C_T(p)) = [B_T.C_T]_{S_1}(p),$$

$$[B_I.C_I]_{S_1}(0) = S_1(B_I(0), C_I(0)) \leq S_1(B_I(p), C_I(p)) = [B_I.C_I]_{S_1}(p),$$

$$[B_F.C_F]_{T_1}(0) = T_1(B_F(0), C_F(0)) \geq T_1(B_F(p), C_F(p)) = [B_F.C_F]_{T_1}(p).$$

Also, for all $p, q \in X$, we have

$$\begin{aligned}[B_T.C_T]_{S_1}(p) &= S_1(B_T(p), C_T(p)) \\ &\leq S_1\Big(S(B_T(p*q), B_T(q)), S(B_T(p*q), B_T(q))\Big) \\ &\leq S\Big(S_1(B_T(p*q), B_T(p*q)), S_1(B_T(q), B_T(q))\Big) \\ &= S\Big([B_T.C_T]_{S_1}(p*q), [B_T.C_T]_{S_1}(q)\Big),\end{aligned}$$

$$\begin{aligned}[B_I.C_I]_{S_1}(p) &= S_1(B_I(p), C_I(p)) \\ &\leq S_1\Big(S(B_I(p*q), B_I(q)), S(B_I(p*q), B_I(q))\Big) \\ &\leq S\Big(S_1(B_I(p*q), B_I(p*q)), S_1(B_I(q), B_I(q))\Big) \\ &= S\Big([B_I.C_I]_{S_1}(p*q), [B_I.C_I]_{S_1}(q)\Big),\end{aligned}$$

and

$$\begin{aligned}[B_F.C_F]_{T_1}(p) &= T_1(B_F(p), C_F(p)) \\ &\geq T_1\Big(T(B_F(p*q), B_F(q)), T(B_T(p*q), B_F(q))\Big) \\ &\geq T\Big(T_1(B_F(p*q), B_F(p*q)), T_1(B_F(q), B_F(q))\Big) \\ &= T\Big([B_F.C_F]_{T_1}(p*q), [B_F.C_F]_{T_1}(q)\Big).\end{aligned}$$

This completes the proof. □

5. **Conclusions**

In this paper, we have introduced the notions of (S, T)- normed doubt neutrosophic subalgebras and ideals of BCK/BCI-algebras and described the characteristic properties. Then, we have considered images and preimages of (S, T)- normed doubt neutrosophic ideals under homomorphism. Moreover, we have discussed the direct product and (S, T)- product of (S, T)-normed doubt neutrosophic ideals of BCK/BCI-algebras. We aim to extend our notions to

(1) (S, T)- normed doubt generalized neutrosophic positive implicative ideals of BCK-algebras.

(2) (S, T)- normed doubt generalized neutrosophic ideals of BCK/BCI-algebras.

(3) (S,T)- normed doubt cubic neutrosophic ideals of BCK/BCI-algebras.

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