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LR-Type Fully Single-Valued Neutrosophic Linear Programming Problems

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Abstract: A single-valued neutrosophic set, an generalization of intuitionistic fuzzy set, is a powerful model to deal with uncertainty. In this study we present a method to solve LR-type single-valued neutrosophic linear programming problems by using unrestricted LR-type single-valued neutrosophic numbers. We propose the ranking function to transform LR-type single-valued neutrosophic problems into crisp problems. The arithmetic operations for unrestricted LR-type single-valued neutrosophic numbers are introduced. We propose a method to solve the fully single-valued neutrosophic linear programming problems with equality constraints having LR-type single-valued neutrosophic numbers as right hand sides, parameters and variables. We describe our proposed method by solving real life examples.

Keywords: LR-type single-valued neutrosophic numbers, Ranking function, Arithmetic operations.

1 Introduction

First time in the history, Zadeh [53] introduced the theory of fuzzy sets to handel vagueness. Fuzzy logic and fuzzy sets have been applied to many real life applications. Atanassove [8] gave the concept of intuitionistic fuzzy sets which is an extension of fuzzy set. In intuitionistic fuzzy set, we deal with non-membership function as well as membership function. Intuitionistic fuzzy sets are fail to deal with complete information. Intuitionistic fuzzy sets are not able to handle inconsistent information and indeterminate information which exist commonly in the belief system. Smarandache [46] introduced the concept of neutrosophic set theory. Neutrosophic set is an extension of Intuitionistic fuzzy set, there are three independent membership functions namely truth membership, falsity membership and hesitancy membership function to deal with vague information.
Linear programming is a quantitative tool to allocate optimal allocation available sources between competing procedures. It is among the popular techniques applied to several areas like marketing, production, advertising, finance and distribution and so forth. Many problems of science and engineering are modeled in such a way that information about the situation is vague, imprecise or incomplete. Many scientists have been worked on linear programming (LP) and fuzzy linear programming (FLP). First time Bellman and Zadeh [10] introduced the idea of decision making in fuzzy environment. By using multi-objective function Zimmerman [54] gave a technique to solve LP problem. Behera et al. [9] proposed two new methods to solve FLP problems. They solved two types of problems with two different methods. Kaur and Kumar [24] gave an introduction to fuzzy linear programming problems. Kumar et al. [28] presented a method to solve fully fuzzy linear programming (FFLP) problems. Kaur and Kumar [25] presented a method to find exact fuzzy optimal solution of FFLP problems by using unrestricted fuzzy variables. Najafi and Edalatpanah [38] proposed a better technique to solve FFLP problem than Kumar et al. [28]. Kaur and Kumar [26] proposed Mehar's method for solving FFLP problems with LR fuzzy parameters. Najafi et al. [39] solved a nonlinear model for FFLP by using unrestricted fuzzy numbers. Based on multi objective LP problems and lexicographic method Das et al. [17] proposed a new technique to solve FFLP problem with trapezoidal fuzzy numbers. Allahviranloo et al. [6] solved FFLP problem by using a kind of defuzzification approach. Lotfi et al. [29] considered FFLP problems in which all parameters and variable are

triangular fuzzy numbers. They used the concept of the symmetric triangular fuzzy number and popularized a method to defuzzify a general fuzzy quantity. For solving FFLP problems with inequality constraints Prez-Caedo et al. [43] suggested a revised version of a lexicographical-based method.
Later on, in 1983 Atanassove [8] introduced the concept of intuitionistic fuzzy set which is an extension of fuzzy set. In intuitionistic fuzzy set there is a non-membership function along with membership function. Many researchers have worked at certain techniques to solve LP problems in an intuitionistic fuzzy environment by using intuitionistic fuzzy numbers (IFNs) or LR-type IFNs. In an intuitionistic fuzzy environment Angelov [7] has introduced a new technique to the optimization problem. Singh and Yadav [44] introduced the product of LR-type IFNs and solved LR-type intuitionistic fuzzy linear programming (IFLP) problems. Abhishekh and Nishad [2] proposed a new ranking function to obtain an optimal solution of fully LR-intuitionistic fuzzy transportation problem by using LR-type IFNs. Dubey and Mehra [18] solved LP problems with triangular intuitionistic fuzzy numbers (TIFNs). Nagoorgani and Ponnalagu [36] introduced division of TIFN by using accuracy function, score function, α-cut and β-cut. Edalatpanah [19] designed a model of data envelopment analysis with TIFNs and established a strategy to solve it. Kabiraj et al. [23] solved IFLP problems by using a method based on a method suggested by Zimmermann [54]. Malathi and Umadevi [30] IFLP problems in an intuitionistic fuzzy environment. Prez-Caedo and Concepcin-Morales [42] proposed a method to solve LR-type fully intuitionistic fuzzy linear programming (FIFLP) having inequality constraints in which variables and constrains are unrestricted LR-type IFNs. Pythagorean fuzzy linear programming is an extension of intuitionistic fuzzy linear programming. Akram et al. [4, 5] proposed a method to solve pythagorean fuzzy linear programming problems by using pythagorean fuzzy numbers and LR-type pythagorean fuzzy numbers.
Neutrosophic set is an extension of intuitionistic fuzzy set. In neutrosophic set there are three independent membership functions namely truth membership, falsity membership and hesitancy membership function. Smarandache [46] introduced the concept of neutrosophic set theory. Abdel-Basset et al. [1] suggested a technique to solve the fully neutrosophic linear programming (FNLP) problems. Bera and Mahapatra [12] developed the Big-M simplex method to solve neutrosophic linear programming (NLP) problem. Das and Chakraborty [15] considered a pentagonal NLP problem to solve it. Das and Dash [16] solved NLP problems with mixed constraints. Edalatpanah [20] presented a direct algorithm to solve the linear programming problems. Khalifa et al. [27] solved NLP problem with single-valued trapezoidal neutrosophic numbers. Recently, Ahmad et al. [3] have presented a new method to solve LPP using bipolar single-valued neutrosophic sets.
The main contribution of this article is as follows.

1. We present the concept of LR-type SNN and arithmetic operations of LR-type SNNs by using α-cut, β-cut and γ-cut.
2. We propose the idea of ranking function for LR-type SNNs.
3. We promote a technique to solve FSNLPP with equality constraints in which all the parameters and variables are unrestricted LR-type SNNs.
4. We apply proposed method for solving real life problems.

This paper is arranged as follows: In Section 2, basic preliminaries and arithmetic operations are discussed. In Sections 3, methodology for solving problems are explained. In Section 4, numerical problems are solved. In Section 5, conclusion is given.
For more information, the readers are referred to [11, 13, 14, 21, 22, 31, 32, 33, 34, 35, 40, 41, 45, 46, 47, 48, 49, 50, 51, 52].

2 Preliminaries

Definition 1. [46] Let X be a nonempty set. A SNS $\tilde{B}$ in X is an object having the form

$$\tilde{B} = \{x, T_{\tilde{B}(x)}, I_{\tilde{B}(x)}, F_{\tilde{B}(x)} : x \in X\},$$

where the truth membership function $T_{\tilde{B}(x)} : X \to [0,1]$, indeterminacy membership function $I_{\tilde{B}(x)} : X \to [0,1]$ and the falsity membership function $F_{\tilde{B}(x)} : X \to [0,1]$.

Definition 2. [11] Let $\tilde{B}$ be a SNS in X, then its α-cut, β-cut and γ-cut are defined as $\tilde{B}^{\alpha} = \{x \in X : T(x) \geq \alpha\}$, $\tilde{B}^{\beta} = \{x \in X : I(x) \leq \beta\}$ and $\tilde{B}^{\gamma} = \{x \in X : F(x) \leq \gamma\}$ with $\alpha, \beta, \gamma \in [0,1]$.

Definition 3. *A SNN $\tilde{B} = ([b; l, r; l', r'; l'', r'']; \chi, \eta, \zeta)_{LR}$ is defined as an LR-type SNN, if its truth membership $(T_{\tilde{B}}(x))$, Indeterminacy membership $(I_{\tilde{B}}(x))$ and falsity membership $(F_{\tilde{B}}(x))$ functions are defined as:*

$$T_{\tilde{B}}(x) = \begin{cases} L(\frac{b-x}{l}), & x \leq b, l > 0, \\ R(\frac{x-b}{r}), & x \geq b, r > 0, \end{cases}$$

$$I_{\tilde{B}}(x) = \begin{cases} L'(\frac{b-x}{l'}), & x \leq b, l' > 0, \\ R'(\frac{x-b}{r'}), & x \geq b, r' > 0, \end{cases}$$

and

$$F_{\tilde{B}}(x) = \begin{cases} L''(\frac{b-x}{l''}), & x \leq b, l'' > 0, \\ R''(\frac{x-b}{r''}), & x \geq b, r'' > 0, \end{cases}$$

where $l \leq l' \leq l'', r \leq r' \leq r''$, L and R are continues, non-increasing functions on $[0, \infty)$ and $L', R', L'' and R''$ are continuous and non-decreasing functions on $[0, \infty)$ such that

1. $L(0) = R(0) = \chi$,
2. $lim_{x\to\infty} R(x) = lim_{x\to\infty} L(x) = 0$,
3. $L'(0) = R'(0) = \eta$,
4. $lim_{x\to\infty} R'(x) = lim_{x\to\infty} L'(x) = 1$,
5. $L''(0) = R''(0) = \zeta$,
6. $lim_{x\to\infty} R''(x) = lim_{x\to\infty} L''(x) = 1$,

b is called the mean value of $\tilde{B}$, l and r are the left and right spreads of $(T_{\tilde{B}}(x))$, l' and r' are the left and right spreads of $(I_{\tilde{B}}(x))$ and l'' and r'' are the left and right spreads of $(F_{\tilde{B}}(x))$, respectively.

Remark

If we set $L(x) = R(x) = max\{0, \chi - x\}, L'(x) = R'(x) = min\{1, \eta + x\}$ and $L''(x) = R''(x) = min\{1, \zeta + x\}$ then $\tilde{B} = ([b; l, r; l', r'; l'', r'']; \chi, \eta, \zeta)_{LR}$ becomes LR-type triangular single-valued neutrosophic number.

$$L(x) = R(x) = \begin{cases} \chi - x, & 0 \leq x \leq \chi, \\ 0, & otherwise, \end{cases}$$

$$L'(x) = R'(x) = \begin{cases} \eta + x, & \eta \leq x \leq 1, \\ 1, & otherwise, \end{cases}$$

$$L''(x) = R''(x) = \begin{cases} \zeta + x, & \zeta \leq x \leq 1, \\ 1, & otherwise, \end{cases}$$

$\chi, \eta, \zeta \in [0,1]$.

Definition 4. Based on [44], An LR-type SNN $\tilde{B} = ([b; l, r; l', r'; l'', r'']; \chi, \eta, \zeta)_{LR}$ is non-negative, if $b - l'' \geq 0$ and denoted as $\tilde{B} \geq 0$.

Definition 5. Based on [44], An LR-type SNN $\tilde{B} = ([b; l, r; l', r'; l'', r'']; \chi, \eta, \zeta)_{LR}$ is non-positive, if $b + r'' < 0$.

Definition 6. Based on [44], An LR-type SNN $\tilde{B} = ([b; l, r; l', r'; l'', r'']; \chi, \eta, \zeta)_{LR}$ is unrestricted, if b is any real number.

Theorem 7. *Let $\tilde{B} = ([b; l, r; l', r'; l'', r'']; \chi, \eta, \zeta)_{LR}$ be an LR-type SNN, then its α-cut, β-cut and γ-cut are $\tilde{B}^{\alpha} = [b - lL^{-1}(\alpha), b + rR^{-1}(\alpha)]$, $\tilde{B}^{\beta} = [b - l'L'^{-1}(\beta), b + r'R'^{-1}(\beta)]$ and $\tilde{B}^{\gamma} = [b - l''L''^{-1}(\gamma), b + r''R''^{-1}(\gamma)]$, with $\alpha, \beta, \gamma \in [0, 1]$.*
Proof. By using the Definition 2, the theorem can be proved easily.

Definition 8. *Let $\tilde{B} = ([b; l, r; l', r'; l'', r'']; \chi, \eta, \zeta)_{LR}$ be an LR-type SNN, then ranking of $\tilde{B}$, denoted $\Re(\tilde{B})$, can be defined as*
$$\Re(\tilde{B}) = \frac{1}{6}[(\int_0^{\chi} b - lL^{-1}(\alpha))d\alpha + (\int_0^{\chi} b + rR^{-1}(\alpha))d\alpha + (\int_{\eta}^{1} b - l'L'^{-1}(\beta))d\beta + (\int_{\eta}^{1} b + r'R'^{-1}(\beta))d\beta + (\int_{\zeta}^{1} b - l''L''^{-1}(\gamma))d\gamma + (\int_{\zeta}^{1} b + r''R''^{-1}(\gamma))d\gamma].$$
Let $\tilde{B}_1$ and $\tilde{B}_2$ be two LR-type SNNs,

- $\tilde{B}_1 \prec \tilde{B}_2$ *if* $\Re(\tilde{B}_1) < \Re(\tilde{B}_1)$,
- $\tilde{B}_1 \succ \tilde{B}_2$ *if* $\Re(\tilde{B}_1) > \Re(\tilde{B}_1)$,
- $\tilde{B}_1 \approx \tilde{B}_2$ *if* $\Re(\tilde{B}_1) = \Re(\tilde{B}_1)$.

2.1 Arithmetic Operations

Theorem 9. *Let $\tilde{B}_1 = ([b_1; l_1, r_1; l'_1, r'_1; l''_1, r''_1]; \chi_1, \eta_1, \zeta_1)_{LR}$ and $\tilde{B}_2 = ([b_2; l_2, r_2; l'_2, r'_2; l''_2, r''_2]; \chi_2, \eta_2, \zeta_2)_{LR}$ be two LR-type SNNs, then $\tilde{B}_1 \oplus \tilde{B}_2 = ([b_1 + b_2; l_1 + l_2, r_1 + r_2; l'_1 + l'_2, r'_1 + r'_2; l''_1 + l''_2, r''_1 + r''_2]; \chi_1 \wedge \chi_2, \eta_1 \vee \eta_2, \zeta_1 \vee \zeta_2)_{LR}$*
proof. Let $\tilde{B}_1 = ([b_1; l_1, r_1; l'_1, r'_1; l''_1, r''_1]; \chi_1, \eta_1, \zeta_1)_{LR}$ and $\tilde{B}_2 = ([b_2; l_2, r_2; l'_2, r'_2; l''_2, r''_2]; \chi_2, \eta_2, \zeta_2)_{LR}$ be two LR-type SNNs, then their α-cut, β-cut and γ-cut are given as;
$\tilde{B}_1^{\alpha} = [b_1 - l_1L^{-1}(\alpha), b_1 + r_1R^{-1}(\alpha)]$, $\tilde{B}_2^{\alpha} = [b_2 - l_2L^{-1}(\alpha), b_2 + r_2R^{-1}(\alpha)]$,
$\tilde{B}_1^{\beta} = [b_1 - l'_1L'^{-1}(\beta), b_1 + r'_1R'^{-1}(\beta)]$, $\tilde{B}_2^{\beta} = [b_2 - l'_2L'^{-1}(\beta), b_2 + r'_2R'^{-1}(\beta)]$,
$\tilde{B}_1^{\gamma} = [b_1 - l''_1L''^{-1}(\gamma), b_1 + r''_1R''^{-1}(\gamma)]$, $\tilde{B}_2^{\gamma} = [b_2 - l''_2L''^{-1}(\gamma), b_2 + r''_2R''^{-1}(\gamma)]$.
Thus,
$$\tilde{B}_1^{\alpha} + \tilde{B}_2^{\alpha} = [b_1 - l_1L^{-1}(\alpha) + b_2 - l_2L^{-1}(\alpha), b_1 + r_1R^{-1}(\alpha) + b_2 + r_2R^{-1}(\alpha)]. \tag{1}$$
By taking $\alpha = \chi$ in equation (1), we have
$$(\tilde{B}_1 + \tilde{B}_2)^{\alpha=\chi} = b_1 + b_2. \tag{2}$$
By taking $\alpha = 0$ in equation (1), we have
$$(\tilde{B}_1 + \tilde{B}_2)^{\alpha=0} = [b_1 + b_2 - l_1 - l_2, b_1 + b_2 + r_1 + r_2]. \tag{3}$$
Now
$$\tilde{B}_1^{\beta} + \tilde{B}_2^{\beta} = [b_1 - l'_1L'^{-1}(\beta) + b_2 - l'_2L'^{-1}(\beta), b_1 + r'_1R'^{-1}(\beta) + b_2 + r'_2R'^{-1}(\beta)]. \tag{4}$$
By taking $\beta = \eta$ in equation (4), we have
$$(\tilde{B}_1 + \tilde{B}_2)^{\beta=\eta} = b_1 + b_2. \tag{5}$$

By taking $\beta = 1$ in equation (4), we have

$$(\tilde{B}_1 + \tilde{B}_2)^{\beta=1} = [b_1 + b_2 - l_1' - l_2', b_1 + b_2 + r_1' + r_2']. \quad (6)$$

Further,

$$\tilde{B}_1^\gamma + \tilde{B}_2^\gamma = [b_1 - l_1''L^{''-1}(\gamma) + b_2 - l_2''L^{''-1}(\gamma), b_1 + r_1''R^{''-1}(\gamma) + b_2 + r_2''R^{''-1}(\gamma)]. \quad (7)$$

By taking $\gamma = \zeta$ in equation (7), we have

$$(\tilde{B}_1 + \tilde{B}_2)^{\gamma=\zeta} = b_1 + b_2. \quad (8)$$

By taking $\gamma = 1$ in equation (7), we have

$$(\tilde{B}_1 + \tilde{B}_2)^{\gamma=1} = [b_1 + b_2 - l_1'' - l_2'', b_1 + b_2 + r_1'' + r_2'']. \quad (9)$$

By combining the equations (2),(3),(5),(6),(8), and (9), the result follows.

Theorem 10. *Let $\tilde{B}_1 = ([b_1; l_1, r_1; l_1', r_1'; l_1'', r_1'']; \chi_1, \eta_1, \zeta_1)_{LR}$ and $\tilde{B}_2 = ([b_2; l_2, r_2; l_2', r_2'; l_2'', r_2'']; \chi_2, \eta_2, \zeta_2)_{LR}$ be two LR-type SNNs, then $\tilde{B}_1 \ominus \tilde{B}_2 = ([b_1 - b_2; l_1 - r_2, r_1 - l_2; l_1' - r_2', r_1' - l_2'; l_1'' - r_2'', r_1'' - l_2'']; \chi_1 \wedge \chi_2, \eta_1 \vee \eta_2, \zeta_1 \vee \zeta_2)_{LR}$*
proof. Let $\tilde{B}_1 = ([b_1; l_1, r_1; l_1', r_1'; l_1'', r_1'']; \chi_1, \eta_1, \zeta_1)_{LR}$ and $\tilde{B}_2 = ([b_2; l_2, r_2; l_2', r_2'; l_2'', r_2'']; \chi_2, \eta_2, \zeta_2)_{LR}$ be two LR-type SNNs, then their α-cut, β-cut and γ-cut are given as;
$\tilde{B}_1^\alpha = [b_1 - l_1L^{-1}(\alpha), b_1 + r_1R^{-1}(\alpha)]$, $\tilde{B}_2^\alpha = [b_2 - l_2L^{-1}(\alpha), b_2 + r_2R^{-1}(\alpha)]$,
$\tilde{B}_1^\beta = [b_1 - l_1'L^{'-1}(\beta), b_1 + r_1'R^{'-1}(\beta)]$, $\tilde{B}_2^\beta = [b_2 - l_2'L^{'-1}(\beta), b_2 + r_2'R^{'-1}(\beta)]$,
$\tilde{B}_1^\gamma = [b_1 - l_1''L^{''-1}(\gamma), b_1 + r_1''R^{''-1}(\gamma)]$, $\tilde{B}_2^\gamma = [b_2 - l_2''L^{''-1}(\gamma), b_2 + r_2''R^{''-1}(\gamma)]$.
Thus,

$$\tilde{B}_1^\alpha - \tilde{B}_2^\alpha = [b_1 - l_1L^{-1}(\alpha) - b_2 - r_2R^{-1}(\alpha), b_1 + r_1R^{-1}(\alpha) - b_2 + l_2L^{-1}(\alpha)]. \quad (10)$$

By taking $\alpha = \chi$ in equation (10), we have

$$(\tilde{B}_1 - \tilde{B}_2)^{\alpha=\chi} = b_1 - b_2. \quad (11)$$

By taking $\alpha = 0$ in equation (10), we have

$$(\tilde{B}_1 - \tilde{B}_2)^{\alpha=0} = [b_1 - b_2 - l_1 - r_2, b_1 - b_2 + r_1 + l_2]. \quad (12)$$

Now

$$\tilde{B}_1^\beta - \tilde{B}_2^\beta = [b_1 - l_1'L^{'-1}(\beta) - b_2 - r_2'R^{'-1}(\beta), b_1 + r_1'R^{'-1}(\beta) - b_2 + l_2'L^{'-1}(\beta)]. \quad (13)$$

By taking $\beta = \eta$ in equation (13), we have

$$(\tilde{B}_1 - \tilde{B}_2)^{\beta=\eta} = b_1 - b_2. \quad (14)$$

By taking $\beta = 1$ in equation (13), we have

$$(\tilde{B}_1 - \tilde{B}_2)^{\beta=1} = [b_1 - b_2 - l_1' - r_2', b_1 - b_2 + r_1' + l_2']. \quad (15)$$

Further,

$$\tilde{B}_1^\gamma - \tilde{B}_2^\gamma = [b_1 - l_1''L^{''-1}(\gamma) - b_2 - r_2''L^{''-1}(\gamma), b_1 + r_1''R^{''-1}(\gamma) - b_2 + l_2''R^{''-1}(\gamma)]. \quad (16)$$

By taking $\gamma = \zeta$ in equation (16), we have

$$(\tilde{B}_1 - \tilde{B}_2)^{\gamma=\zeta} = b_1 - b_2. \tag{17}$$

By taking $\gamma = 1$ in equation (16), we have

$$(\tilde{B}_1 - \tilde{B}_2)^{\gamma=1} = [b_1 - b_2 - l''_1 - r''_2, b_1 - b_2 + r''_1 + l''_2]. \tag{18}$$

By combining the equations (11),(12),(14),(15),(17), and (18), the result follows.

Theorem 11. *Let $\tilde{B} = ([b; l, r; l', r'1; l'', r'']; \chi, \eta, \zeta)_{LR}$ be an LR-type SNN and c be any arbitrary real number, then*

$$c\tilde{B} = \begin{cases} (cb; cl, cr; cl', cr'; cl'', cr''), & c \geq 0, \\ (cb; -cr, -cl; -cr', -cl'; -cr'', -cl''), & c < 0. \end{cases}$$

Proof. *Let $\tilde{B} = ([b; l, r; l', r'1; l'', r'']; \chi, \eta, \zeta)_{LR}$ be an LR-type SNN and c be any arbitrary real number, then $\tilde{B}^{\alpha} = [b - lL^{-1}(\alpha), b + rR^{-1}(\alpha)]$, $\tilde{B}^{\beta} = [b - l'L'^{-1}(\beta), b + r'R'^{-1}(\beta)]$, $\tilde{B}^{\gamma} = [b - l''L''^{-1}(\gamma), b + r''R''^{-1}(\gamma)]$. Now, if $c \geq 0$, then*

$$c\tilde{B}^{\alpha} = [cb - clL^{-1}(\alpha), cb + crR^{-1}(\alpha)]. \tag{19}$$

By taking $\alpha = \chi$ in equation (19), we have

$$c\tilde{B}^{\alpha=\chi} = cb. \tag{20}$$

By taking $\alpha = 0$ in equation (19), we have

$$c\tilde{B}^{\alpha=0} = [cb - cl, cb + cr]. \tag{21}$$

Also,

$$c\tilde{B}^{\beta} = [cb - cl'L'^{-1}(\beta), cb + cr'R'^{-1}(\beta)]. \tag{22}$$

By taking $\beta = \eta$ in equation (22), we have

$$c\tilde{B}^{\beta=\eta} = cb. \tag{23}$$

By taking $\beta = 1$ in equation (22), we have

$$c\tilde{B}^{\beta=1} = [cb - cl', cb + cr']. \tag{24}$$

Further,

$$c\tilde{B}^{\gamma} = [cb - cl''L''^{-1}(\gamma), cb + cr''R''^{-1}(\gamma)]. \tag{25}$$

By taking $\gamma = \zeta$ in equation (25), we have

$$c\tilde{B}^{\gamma=\zeta} = cb. \tag{26}$$

By taking $\gamma = 1$ in equation (25), we have

$$c\tilde{B}^{\gamma=1} = [cb - cl'', cb + cr'']. \tag{27}$$

By combining the equations (20),(21),(23),(24),(26), and (27), the case $c \geq 0$ follows.
If c¡0, then

$$c\tilde{B}^{\alpha} = [cb + crR^{-1}(\alpha), cb - clL^{-1}(\alpha)]. \tag{28}$$

By taking $\alpha = \chi$ in equation (28), we have

$$c\tilde{B}^{\alpha=\chi} = cb. \tag{29}$$

By taking $\alpha = 0$ in equation (28), we have

$$c\tilde{B}^{\alpha=0} = [cb + cr, cb - cl]. \tag{30}$$

Also,

$$c\tilde{B}^{\beta} = [cb + cr'R'^{-1}(\beta), cb - cl'L'^{-1}(\beta)]. \tag{31}$$

By taking $\beta = \eta$ in equation (31), we have

$$c\tilde{B}^{\beta=\eta} = cb. \tag{32}$$

By taking $\beta = 1$ in equation (31), we have

$$c\tilde{B}^{\beta=1} = [cb + cr', cb - cl']. \tag{33}$$

Further,

$$c\tilde{B}^{\gamma} = [cb + cr''R''^{-1}(\gamma), cb - cl''L''^{-1}(\gamma)]. \tag{34}$$

By taking $\gamma = \zeta$ in equation (34), we have

$$c\tilde{B}^{\gamma=\zeta} = cb. \tag{35}$$

By taking $\gamma = 1$ in equation (34), we have

$$c\tilde{B}^{\gamma=1} = [cb + cr'', cb - cl'']. \tag{36}$$

On combining the equations (29),(30),(32),(33),(35), and (36), the case c¡0 follows. thus proof completed.

Theorem 12. *Let $\tilde{B}_1 = ([b_1; l_1, r_1; l_1', r_1'; l_1'', r_1'']; \chi_1, \eta_1, \zeta_1)_{LR}$ and $\tilde{B}_2 = ([b_2; l_2, r_2; l_2', r_2'; l_2'', r_2'']; \chi_2, \eta_2, \zeta_2)_{LR}$ be two non-negative LR-type SNNs, then*
$\tilde{B}_1 \otimes \tilde{B}_2 = ([b_1 b_2; b_1 l_2 + b_2 l_1 - l_1 l_2, b_1 r_2 + b_2 r_1 + r_1 r_2; b_1 l_2' + b_2 l_1' - l_1' l_2', b_1 r_2' + b_2 r_1' + r_1' r_2'; b_1 l_2'' + b_2 l_1'' - l_1'' l_2'', b_1 r_2'' + b_2 r_1'' + r_1'' r_2'']; \chi_1 \wedge \chi_2, \eta_1 \vee \eta_2, \zeta_1 \vee \zeta_2)$.
Proof. *Let $\tilde{B}_1 = ([b_1; l_1, r_1; l_1', r_1'; l_1'', r_1'']; \chi_1, \eta_1, \zeta_1)_{LR}$ and $\tilde{B}_2 = ([b_2; l_2, r_2; l_2', r_2'; l_2'', r_2'']; \chi_2, \eta_2, \zeta_2)_{LR}$ be two non-negative LR-type SNNs, their $\alpha-$cut, $\beta-$cut and $\gamma-$cut are given as:*
$\tilde{B}_1^{\alpha} = [b_1 - l_1 L^{-1}(\alpha), b_1 + r_1 R^{-1}(\alpha)]$, $\tilde{B}_2^{\alpha} = [b_2 - l_2 L^{-1}(\alpha), b_2 + r_2 R^{-1}(\alpha)]$,
$\tilde{B}_1^{\beta} = [b_1 - l_1' L'^{-1}(\beta), b_1 + r_1' R'^{-1}(\beta)]$, $\tilde{B}_2^{\beta} = [b_2 - l_2' L'^{-1}(\beta), b_2 + r_2' R'^{-1}(\beta)]$,
$\tilde{B}_1^{\gamma} = [b_1 - l_1'' L''^{-1}(\gamma), b_1 + r_1'' R''^{-1}(\gamma)]$, $\tilde{B}_2^{\gamma} = [b_2 - l_2'' L''^{-1}(\gamma), b_2 + r_2'' R''^{-1}(\gamma)]$.
Thus,

$$\tilde{B}_1^{\alpha} \tilde{B}_2^{\alpha} = [(b_1 - l_1 L^{-1}(\alpha))(b_2 - l_2 L^{-1}(\alpha)), (b_1 + r_1 R^{-1}(\alpha))(b_2 + r_2 R^{-1}(\alpha))]. \tag{37}$$

By taking $\alpha = \chi$ in equation (37), we have

$$(\tilde{B}_1\tilde{B}_2)^{\alpha=\chi} = b_1b_2. \tag{38}$$

By taking $\alpha = 0$ in equation (37), we have

$$(\tilde{B}_1\tilde{B}_2)^{\alpha=0} = [b_1b_2 - b_1l_2 - b_2l_1 + l_1l_2, b_1b_2 + b_1r_2 + b_2r_1 + r_1r_2]. \tag{39}$$

Now

$$\tilde{B}_1^{\beta}\tilde{B}_2^{\beta} = [(b_1 - l_1'L^{'-1}(\beta))(b_2 - l_2'L^{'-1}(\beta)), (b_1 + r_1'R^{'-1}(\beta))(b_2 + r_2'R^{'-1}(\beta))]. \tag{40}$$

By taking $\beta = \eta$ in equation (40), we have

$$(\tilde{B}_1\tilde{B}_2)^{\beta=\eta} = b_1b_2. \tag{41}$$

By taking $\beta = 1$ in equation (40), we have

$$(\tilde{B}_1\tilde{B}_2)^{\beta=1} = [b_1b_2 - b_1l_2' - b_2l_1' + l_1'l_2', b_1b_2 + b_1r_2' + b_2r_1' + r_1'r_2']. \tag{42}$$

Also

$$\tilde{B}_1^{\gamma}\tilde{B}_2^{\gamma} = [(b_1 - l_1''L^{''-1}(\gamma))(b_2 - l_2''L^{''-1}(\gamma)), (b_1 + r_1''R^{''-1}(\gamma))(b_2 + r_2''R^{''-1}(\gamma))]. \tag{43}$$

By taking $\gamma = \zeta$ in equation (43), we have

$$(\tilde{B}_1\tilde{B}_2)^{\gamma=\zeta} = b_1b_2. \tag{44}$$

By taking $\gamma = 1$ in equation (43), we have

$$(\tilde{B}_1\tilde{B}_2)^{\gamma=1} = [b_1b_2 - b_1l_2'' - b_2l_1'' + l_1''l_2'', b_1b_2 + b_1r_2'' + b_2r_1'' + r_1''r_2'']. \tag{45}$$

By combining the equations (38),(39),(41),(42),(44), and (45), the result follows.

Theorem 13. *Let $\tilde{B}_1 = ([b_1; l_1, r_1; l_1', r_1'; l_1'', r_1'']; \chi_1, \eta_1, \zeta_1)_{LR}$ be non-negative LR-type SNN, and $\tilde{B}_2 = ([b_2; l_2, r_2; l_2', r_2'; l_2'', r_2'']; \chi_2, \eta_2, \zeta_2)_{LR}$ be non-positive LR-type SNN, then $\tilde{B}_1 \otimes \tilde{B}_2 = ([b_1b_2; b_1l_2 - b_2r_1 + l_2r_1, b_1r_2 - b_2l_1 - l_1r_2; b_1l_2' - b_2r_1' + l_2'r_1', b_1r_2' - b_2l_1' - l_1'r_2'; b_1l_2'' - b_2r_1'' + l_2''r_1'', b_1r_2'' - b_2l_1'' - l_1''r_2'']; \chi_1 \wedge \chi_2, \eta_1 \vee \eta_2, \zeta_1 \vee \zeta_2)_{LR}$.*

Proof. *Let $\tilde{B}_1 = ([b_1; l_1, r_1; l_1', r_1'; l_1'', r_1'']; \chi_1, \eta_1, \zeta_1)_{LR}$ be non-negative LR-type SNN, and $\tilde{B}_2 = ([b_2; l_2, r_2; l_2', r_2'; l_2'', r_2'']; \chi_2, \eta_2, \zeta_2)_{LR}$ be non-positive LR-type SNN, their $\alpha-$cut, $\beta-$cut and $\gamma-$cut are given as:*
$\tilde{B}_1^{\alpha} = [b_1 - l_1L^{-1}(\alpha), b_1 + r_1R^{-1}(\alpha)]$, $\tilde{B}_2^{\alpha} = [b_2 - l_2L^{-1}(\alpha), b_2 + r_2R^{-1}(\alpha)]$,
$\tilde{B}_1^{\beta} = [b_1 - l_1'L^{'-1}(\beta), b_1 + r_1'R^{'-1}(\beta)]$, $\tilde{B}_2^{\beta} = [b_2 - l_2'L^{'-1}(\beta), b_2 + r_2'R^{'-1}(\beta)]$,
$\tilde{B}_1^{\gamma} = [b_1 - l_1''L^{''-1}(\gamma), b_1 + r_1''R^{''-1}(\gamma)]$, $\tilde{B}_2^{\gamma} = [b_2 - l_2''L^{''-1}(\gamma), b_2 + r_2''R^{''-1}(\gamma)]$.
Thus,

$$\tilde{B}_1^{\alpha}\tilde{B}_2^{\alpha} = [(b_1 + r_1R^{-1}(\alpha))(b_2 - l_2L^{-1}(\alpha)), (b_1 - l_1L^{-1}(\alpha))(b_2 + r_2R^{-1}(\alpha))]. \tag{46}$$

By taking $\alpha = \chi$ in equation (46), we have

$$(\tilde{B}_1\tilde{B}_2)^{\alpha=\chi} = b_1b_2. \tag{47}$$

By taking $\alpha = 0$ in equation (46), we have

$$(\tilde{B}_1\tilde{B}_2)^{\alpha=0} = [b_1b_2 - b_1l_2 + b_2r_1 - l_2r_1, b_1b_2 + b_1r_2 - b_2l_1 - l_1r_2]. \tag{48}$$

Now

$$\tilde{B}_1^{\beta}\tilde{B}_2^{\beta} = [(b_1 + r'_1 R'^{-1}(\beta))(b_2 - l'_2 L'^{-1}(\beta)), (b_1 - l'_1 L'^{-1}(\beta))(b_2 + r'_2 R'^{-1}(\beta))]. \quad (49)$$

By taking $\beta = \eta$ in equation (49), we have

$$(\tilde{B}_1\tilde{B}_2)^{\beta=\eta} = b_1 b_2. \quad (50)$$

By taking $\beta = 1$ in equation (49), we have

$$(\tilde{B}_1\tilde{B}_2)^{\beta=1} = [b_1b_2 - b_1l'_2 + b_2r'_1 - l'_2r'_1, b_1b_2 + b_1r'_2 - b_2l'_1 - l'_1r'_2]. \quad (51)$$

Also

$$\tilde{B}_1^{\gamma}\tilde{B}_2^{\gamma} = [(b_1 + r''_1 R''^{-1}(\gamma))(b_2 - l''_2 L''^{-1}(\gamma)), (b_1 - l''_1 L''^{-1}(\gamma))(b_2 + r''_2 R''^{-1}(\gamma))]. \quad (52)$$

By taking $\gamma = \zeta$ in equation (52), we have

$$(\tilde{B}_1\tilde{B}_2)^{\gamma=\zeta} = b_1 b_2. \quad (53)$$

By taking $\gamma = 1$ in equation (52), we have

$$(\tilde{B}_1\tilde{B}_2)^{\gamma=1} = [b_1b_2 - b_1l''_2 + b_2r''_1 - l''_2r''_1, b_1b_2 + b_1r''_2 - b_2l''_1 - l''_1r''_2]. \quad (54)$$

By combining the equations (47),(48),(50),(51),(53), and (54), the result follows.

Theorem 14. *Let $\tilde{B}_1 = ([b_1; l_1, r_1; l'_1, r'_1; l''_1, r''_1]; \chi_1, \eta_1, \zeta_1)_{LR}$ be non-positive LR-type SNN, and $\tilde{B}_2 = ([b_2; l_2, r_2; l'_2, r'_2; l''_2, r''_2]; \chi_2, \eta_2, \zeta_2)_{LR}$ be non-negative LR-type SNN, then $\tilde{B}_1 \otimes \tilde{B}_2 = ([b_1b_2; b_2l_1 - b_1r_2 + l_1r_2, -b_1l_2 + b_2r_1 - l_2r_1; b_2l'_1 - b_1r'_2 + l'_1r'_2, -b_1l'_2 + b_2r'_1 - l'_2r'_1; b_2l''_1 - b_1r''_2 + l''_1r''_2, -b_1l''_2 + b_2r''_1 - l''_2r''_1]; \chi_1 \wedge \chi_2, \eta_1 \vee \eta_2, \zeta_1 \vee \zeta_2)_{LR}$.*

Proof. *Let $\tilde{B}_1 = ([b_1; l_1, r_1; l'_1, r'_1; l''_1, r''_1]; \chi_1, \eta_1, \zeta_1)_{LR}$ be non-positive LR-type SNN, and $\tilde{B}_2 = ([b_2; l_2, r_2; l'_2, r'_2; l''_2, r''_2]; \chi_2, \eta_2, \zeta_2)_{LR}$ be non-negative LR-type SNN, their $\alpha-$cut, $\beta-$cut and $\gamma-$cut are given as:*
$\tilde{B}_1^{\alpha} = [b_1 - l_1L^{-1}(\alpha), b_1 + r_1R^{-1}(\alpha)]$, $\tilde{B}_2^{\alpha} = [b_2 - l_2L^{-1}(\alpha), b_2 + r_2R^{-1}(\alpha)]$,
$\tilde{B}_1^{\beta} = [b_1 - l'_1L'^{-1}(\beta), b_1 + r'_1R'^{-1}(\beta)]$, $\tilde{B}_2^{\beta} = [b_2 - l'_2L'^{-1}(\beta), b_2 + r'_2R'^{-1}(\beta)]$,
$\tilde{B}_1^{\gamma} = [b_1 - l''_1L''^{-1}(\gamma), b_1 + r''_1R''^{-1}(\gamma)]$, $\tilde{B}_2^{\gamma} = [b_2 - l''_2L''^{-1}(\gamma), b_2 + r''_2R''^{-1}(\gamma)]$.
Thus,

$$\tilde{B}_1^{\alpha}\tilde{B}_2^{\alpha} = [(b_1 - l_1L^{-1}(\alpha))(b_2 + r_2R^{-1}(\alpha)), (b_1 + r_1R^{-1}(\alpha))(b_2 - l_2L^{-1}(\alpha))]. \quad (55)$$

By taking $\alpha = \chi$ in equation (55), we have

$$(\tilde{B}_1\tilde{B}_2)^{\alpha=\chi} = b_1 b_2. \quad (56)$$

By taking $\alpha = 0$ in equation (55), we have

$$(\tilde{B}_1\tilde{B}_2)^{\alpha=0} = [b_1b_2 - b_2l_1 + b_1r_2 - l_1r_2, b_1b_2 - b_1l_2 + b_2r_1 - l_2r_1]. \quad (57)$$

Now

$$\tilde{B}_1^{\beta}\tilde{B}_2^{\beta} = [(b_1 - l'_1 L'^{-1}(\beta))(b_2 + r'_2 R'^{-1}(\beta)), (b_1 + r'_1 R'^{-1}(\beta))(b_2 - l'_2 L'^{-1}(\beta))]. \quad (58)$$

By taking $\beta = \eta$ in equation (58), we have

$$(\tilde{B}_1\tilde{B}_2)^{\beta=\eta} = b_1 b_2. \quad (59)$$

By taking $\beta = 1$ in equation (58), we have

$$(\tilde{B}_1\tilde{B}_2)^{\beta=1} = [b_1b_2 - b_2l'_1 + b_1r'_2 - l'_1r'_2, b_1b_2 - b_1l'_2 + b_2r'_1 - l'_2r'_1]. \tag{60}$$

Also

$$\tilde{B}_1^{\gamma}\tilde{B}_2^{\gamma} = [(b_1 - l''_1L''^{-1}(\gamma))(b_2 + r''_2R''^{-1}(\gamma)), (b_1 + r''_1R''^{-1}(\gamma))(b_2 - l''_2L''^{-1}(\gamma))]. \tag{61}$$

By taking $\gamma = \zeta$ in equation (61), we have

$$(\tilde{B}_1\tilde{B}_2)^{\gamma=\zeta} = b_1b_2. \tag{62}$$

By taking $\gamma = 1$ in equation (61), we have

$$(\tilde{B}_1\tilde{B}_2)^{\gamma=1} = [b_1b_2 - b_2l''_1 + b_1r''_2 - l''_1r''_2, b_1b_2 - b_1l''_2 + b_2r''_1 - l''_2r''_1]. \tag{63}$$

By combining the equations (56),(57),(59),(60),(62), and (63), the result follows.

Theorem 15. *Let $\tilde{B}_1 = ([b_1; l_1, r_1; l'_1, r'_1; l''_1, r''_1]; \chi_1, \eta_1, \zeta_1)_{LR}$ and $\tilde{B}_2 = ([b_2; l_2, r_2; l'_2, r'_2; l''_2, r''_2]; \chi_2, \eta_2, \zeta_2)_{LR}$ be two non-positive LR-type SNNs, then $\tilde{B}_1 \otimes \tilde{B}_2 = ([b_1b_2; -b_1r_2 - b_2r_1 - r_1r_2, -b_1l_2 - b_2l_1 + l_1l_2; -b_1r'_2 - b_2r'_1 - r'_1r'_2, -b_1l'_2 - b_2l'_1 + l'_1l'_2; -b_1r''_2 - b_2r''_1 - r''_1r''_2, -b_1l''_2 - b_2l''_1 + l''_1l''_2]; \chi_1 \wedge \chi_2, \eta_1 \vee \eta_2, \zeta_1 \vee \zeta_2)$.*

Proof. *Let $\tilde{B}_1 = ([b_1; l_1, r_1; l'_1, r'_1; l''_1, r''_1]; \chi_1, \eta_1, \zeta_1)_{LR}$ and $\tilde{B}_2 = ([b_2; l_2, r_2; l'_2, r'_2; l''_2, r''_2]; \chi_2, \eta_2, \zeta_2)_{LR}$ be two non-negative LR-type SNNs, their $\alpha-$cut, $\beta-$cut and $\gamma-$cut are given as:*
$\tilde{B}_1^{\alpha} = [b_1 - l_1L^{-1}(\alpha), b_1 + r_1R^{-1}(\alpha)]$, $\tilde{B}_2^{\alpha} = [b_2 - l_2L^{-1}(\alpha), b_2 + r_2R^{-1}(\alpha)]$,
$\tilde{B}_1^{\beta} = [b_1 - l'_1L'^{-1}(\beta), b_1 + r'_1R'^{-1}(\beta)]$, $\tilde{B}_2^{\beta} = [b_2 - l'_2L'^{-1}(\beta), b_2 + r'_2R'^{-1}(\beta)]$,
$\tilde{B}_1^{\gamma} = [b_1 - l''_1L''^{-1}(\gamma), b_1 + r''_1R''^{-1}(\gamma)]$, $\tilde{B}_2^{\gamma} = [b_2 - l''_2L''^{-1}(\gamma), b_2 + r''_2R''^{-1}(\gamma)]$.
Thus,

$$\tilde{B}_1^{\alpha}\tilde{B}_2^{\alpha} = [(b_1 + r_1R^{-1}(\alpha))(b_2 + r_2R^{-1}(\alpha)), (b_1 - l_1L^{-1}(\alpha))(b_2 - l_2L^{-1}(\alpha))]. \tag{64}$$

By taking $\alpha = \chi$ in equation (64), we have

$$(\tilde{B}_1\tilde{B}_2)^{\alpha=\chi} = b_1b_2. \tag{65}$$

By taking $\alpha = 0$ in equation (64), we have

$$(\tilde{B}_1\tilde{B}_2)^{\alpha=0} = [b_1b_2 + b_1r_2 + b_2r_1 + r_1r_2, b_1b_2 - b_1l_2 - b_2l_1 + l_1l_2]. \tag{66}$$

Now

$$\tilde{B}_1^{\beta}\tilde{B}_2^{\beta} = [(b_1 + r'_1R'^{-1}(\beta))(b_2 + r'_2R'^{-1}(\beta)), (b_1 - l'_1L'^{-1}(\beta))(b_2 - l'_2L'^{-1}(\beta))]. \tag{67}$$

By taking $\beta = \eta$ in equation (67), we have

$$(\tilde{B}_1\tilde{B}_2)^{\beta=\eta} = b_1b_2. \tag{68}$$

By taking $\beta = 1$ in equation (67), we have

$$(\tilde{B}_1\tilde{B}_2)^{\beta=1} = [b_1b_2 + b_1r'_2 + b_2r'_1 + r'_1r'_2, b_1b_2 - b_1l'_2 - b_2l'_1 + l'_1l'_2]. \tag{69}$$

Also

$$\tilde{B}_1^{\gamma}\tilde{B}_2^{\gamma} = [(b_1 + r''_1R''^{-1}(\gamma))(b_2 + r''_2R''^{-1}(\gamma)), (b_1 - l''_1L''^{-1}(\gamma))(b_2 - l''_2L''^{-1}(\gamma))]. \tag{70}$$

By taking $\gamma = \zeta$ in equation (70), we have

$$(\tilde{B}_1\tilde{B}_2)^{\gamma=\zeta} = b_1b_2. \tag{71}$$

By taking $\gamma = 1$ in equation (70), we have

$$(\tilde{B}_1\tilde{B}_2)^{\gamma=1} = [b_1b_2 + b_1r''_2 + b_2r''_1 + r''_1r''_2, b_1b_2 - b_1l''_2 - b_2l''_1 + l''_1l''_2]. \tag{72}$$

By combining the equations (65),(66),(68),(69),(71), and (72), the result follows.

Theorem 16. *Let $\tilde{B}_1 = ([b_1; l_1, r_1; l'_1, r'_1; l''_1, r''_1]; \chi_1, \eta_1, \zeta_1)_{LR}$ be an LR-type SNN in which $b_1 - l''_1 < 0$, $b_1 - l'_1 \geq 0$, and $\tilde{B}_2 = ([b_2; l_2, r_2; l'_2, r'_2; l''_2, r''_2]; \chi_2, \eta_2, \zeta_2)_{LR}$ be an unrestricted LR-type SNN, then $\tilde{B}_1 \otimes \tilde{B}_2 = ([b; l, r; l'r'; l'', r'']; \chi, \eta, \zeta)_{LR}$, where $b = b_1b_2, l = b_1b_2 - min\{b_1b_2 - l_2b_1 - l_1b_2 + l_1l_2, b_1b_2 - l_2b_1 + r_1b_2 - l_2r_1\}, r = max\{b_1b_2 + r_2b_1 + r_1b_2 + r_1r_2, b_1b_2 + r_2b_1 - l_1b_2 - l_1r_2\} - b_1b_2, l' = b_1b_2 - min\{b_1b_2 - l'_2b_1 - l'_1b_2 + l'_1l'_2, b_1b_2 - l'_2b_1 + r'_1b_2 - l'_2r'_1\}, r' = max\{b_1b_2 + r'_2b_1 + r'_1b_2 + r'_1r'_2, b_1b_2 + r'_2b_1 - l'_1b_2 - l'_1r'_2\} - b_1b_2, l'' = b_1b_2 - min\{b_1b_2 - l''_1b_2 + r''_2b_1 - l''_1r''_2, b_1b_2 + r''_1b_2 - l''_2b_1 - l''_2r''_1\}$ and $r'' = max\{b_1b_2 - l''_1b_2 - l''_2b_1 + l''_1l''_2, b_1b_2 + r''_1b_2 + r''_2b_1 + r''_1r''_2\} - b_1b_2$.*
here $\chi = \chi_1 \wedge \chi_2, \eta = \eta_1 \vee \eta_2, \zeta = \zeta_1 \vee \zeta_2$.
Proof. Let $\tilde{B}_1 = ([b_1; l_1, r_1; l'_1, r'_1; l''_1, r''_1]; \chi_1, \eta_1, \zeta_1)_{LR}$ be an LR-type SNN such that $b_1 - l''_1 < 0$, $b_1 - l'_1 \geq 0$, $b_1 - l_1 \geq 0$, and $\tilde{B}_2 = ([b_2; l_2, r_2; l'_2, r'_2; l''_2, r''_2]; \chi_2, \eta_2, \zeta_2)_{LR}$ be an unrestricted LR-type SNN, and their $\alpha-$cut, $\beta-$cut and $\gamma-$cut are given as:
$\tilde{B}_1^{\alpha} = [b_1 - l_1L^{-1}(\alpha), b_1 + r_1R^{-1}(\alpha)]$, $\tilde{B}_2^{\alpha} = [b_2 - l_2L^{-1}(\alpha), b_2 + r_2R^{-1}(\alpha)]$,
$\tilde{B}_1^{\beta} = [b_1 - l'_1L'^{-1}(\beta), b_1 + r'_1R'^{-1}(\beta)]$, $\tilde{B}_2^{\beta} = [b_2 - l'_2L'^{-1}(\beta), b_2 + r'_2R'^{-1}(\beta)]$,
$\tilde{B}_1^{\gamma} = [b_1 - l''_1L''^{-1}(\gamma), b_1 + r''_1R''^{-1}(\gamma)]$, $\tilde{B}_2^{\gamma} = [b_2 - l''_2L''^{-1}(\gamma), b_2 + r''_2R''^{-1}(\gamma)]$.
Thus,

$$\tilde{B}_1^{\alpha}\tilde{B}_2^{\alpha} = [min\{(b_1 - l_1L^{-1}(\alpha))(b_2 - l_2L^{-1}(\alpha)), (b_1 + r_1R^{-1}(\alpha))(b_2 + r_2R^{-1}(\alpha))\},$$

$$max\{(b_1 + r_1R^{-1}(\alpha))(b_2 + r_2R^{-1}(\alpha)), (b_1 - l_1L^{-1}(\alpha))(b_2 - l_2L^{-1}(\alpha))\}]. \tag{73}$$

By taking $\alpha = \chi$ in equation (73), we have

$$(\tilde{B}_1\tilde{B}_2)^{\alpha=\chi} = b_1b_2. \tag{74}$$

By taking $\alpha = 0$ in equation (73), we have

$$(\tilde{B}_1\tilde{B}_2)^{\alpha=0} = [min\{b_1b_2 - b_1l_2 - b_2l_1 + l_1l_2, b_1b_2 - b_1l_2 + b_2r_1 - l_2r_1\},$$

$$max\{b_1b_2 + b_1r_2 + b_2r_1 + r_1r_2, b_1b_2 + b_1r_2 - b_2l_1 - l_1r_2\}]. \tag{75}$$

Now

$$\tilde{B}_1^{\beta}\tilde{B}_2^{\beta} = [min\{(b_1 - l'_1L'^{-1}(\beta))(b_2 - l'_2L'^{-1}(\beta)), (b_1 + r'_1R'^{-1}(\beta))(b_2 + r'_2R'^{-1}(\beta))\},$$

$$max\{(b_1 + r'_1R'^{-1}(\beta))(b_2 + r'_2R'^{-1}(\beta)), \{(b_1 - l'_1L'^{-1}(\beta))(b_2 - l'_2L'^{-1}(\beta))\}]. \tag{76}$$

By taking $\beta = \eta$ in equation (76), we have

$$(\tilde{B}_1\tilde{B}_2)^{\beta=\eta} = b_1b_2. \tag{77}$$

By taking $\beta = 1$ in equation (76), we have

$$(\tilde{B}_1\tilde{B}_2)^{\beta=1} = [min\{b_1b_2 - b_1l'_2 - b_2l'_1 + l'_1l'_2, b_1b_2 - b_1l'_2 + b_2r'_1 - l'_2r'_1\},$$

$$max\{b_1b_2 + b_1r'_2 + b_2r'_1 + r'_1r'_2, b_1b_2 + b_1r'_2 - b_2l'_1 - l'_1r'_2\}]. \quad (78)$$

Also

$$\tilde{B}_1^{\alpha}\tilde{B}_2^{\alpha} = [min\{(b_1 - l''_1L''^{-1}(\gamma))(b_2 + r''_2R''^{-1}(\gamma)), (b_1 + r''_1R''^{-1}(\gamma))(b_2 - l''_2L''^{-1}(\gamma))\},$$

$$max\{\{(b_1 - l''_1L''^{-1}(\gamma))(b_2 - l''_2L''^{-1}(\gamma)), (b_1 + r''_1R''^{-1}(\gamma))(b_2 + r''_2R''^{-1}(\gamma))\}]. \quad (79)$$

By taking $\gamma = \zeta$ in equation (79), we have

$$(\tilde{B}_1\tilde{B}_2)^{\gamma=\zeta} = b_1b_2. \quad (80)$$

By taking $\gamma = 1$ in equation (79), we have

$$(\tilde{B}_1\tilde{B}_2)^{\gamma=1} = [min\{b_1b_2 + b_1r''_2 - b_2l''_1 - l''_1r''_2, b_1b_2 - b_1l''_2 + b_2r''_1 - l''_2r''_1\},$$

$$max\{b_1b_2 - b_1l''_2 - b_2l''_1 + l''_1l''_2, b_1b_2 + b_1r''_2 + b_2r''_1 + r''_1r''_2\}]. \quad (81)$$

By combining the equations (74),(75),(77),(78),(80), and (81), the result follows.
By using similar method as used in the above theorem the following theorems can be proved easily.

Theorem 17. *Let $\tilde{B}_1 = ([b_1; l_1, r_1; l'_1, r'_1; l''_1, r''_1]; \chi_1, \eta_1, \zeta_1)_{LR}$ be an LR-type SNN in which $b_1 - l'_1 < 0$, $b_1 - l_1 \geq 0$, and $\tilde{B}_2 = ([b_2; l_2, r_2; l'_2, r'_2; l''_2, r''_2]; \chi_2, \eta_2, \zeta_2)_{LR}$ be an unrestricted LR-type SNN, then $\tilde{B}_1 \otimes \tilde{B}_2 = ([b; l, r; l'r'; l'', r'']; \chi, \eta, \zeta)_{LR}$, where $b = b_1b_2, l = b_1b_2 - min\{b_1b_2 - l_2b_1 - l_1b_2 + l_1l_2, b_1b_2 - l_2b_1 + r_1b_2 - l_2r_1\}, r = max\{b_1b_2 + r_2b_1 + r_1b_2 + r_1r_2, b_1b_2 + r_2b_1 - l_1b_2 - l_1r_2\} - b_1b_2, l' = b_1b_2 - min\{b_1b_2 - l'_1b_2 + r'_2b_1 - l'_1r'_2, b_1b_2 + r'_1b_2 - l'_2b_1 - l'_2r'_1\}, r' = max\{b_1b_2 - l'_1b_2 - l'_2b_1 + l'_1l'_2, b_1b_2 + r'_1b_2 + r'_2b_1 + r'_1r'_2\} - b_1b_2, l'' = b_1b_2 - min\{b_1b_2 - l''_1b_2 + r''_2b_1 - l''_1r''_2, b_1b_2 + r''_1b_2 - l''_2b_1 - l''_2r''_1\}$ and $r'' = max\{b_1b_2 - l''_1b_2 - l''_2b_1 + l''_1l''_2, b_1b_2 + r''_1b_2 + r''_2b_1 + r''_1r''_2\} - b_1b_2$.*
here $\chi = \chi_1 \wedge \chi_2, \eta = \eta_1 \vee \eta_2, \zeta = \zeta_1 \vee \zeta_2$.

Theorem 18. *Let $\tilde{B}_1 = ([b_1; l_1, r_1; l'_1, r'_1; l''_1, r''_1]; \chi_1, \eta_1, \zeta_1)_{LR}$ be an LR-type SNN in which $b_1 - l_1 < 0$, $b_1 \geq 0$, and $\tilde{B}_2 = ([b_2; l_2, r_2; l'_2, r'_2; l''_2, r''_2]; \chi_2, \eta_2, \zeta_2)_{LR}$ be an unrestricted LR-type SNN, then $\tilde{B}_1 \otimes \tilde{B}_2 = ([b; l, r; l'r'; l'', r'']; \chi, \eta, \zeta)_{LR}$, where $b = b_1b_2, l = b_1b_2 - min\{b_1b_2 - l_1b_2 + r_2b_1 - l_1r_2, b_1b_2 + r_1b_2 - l_2b_1 - l_2r_1\}, r = max\{b_1b_2 - l_1b_2 - l_2b_1 + l_1l_2, b_1b_2 + r_1b_2 + r_2b_1 + r_1r_2\} - b_1b_2, l' = b_1b_2 - min\{b_1b_2 - l'_1b_2 + r'_2b_1 - l'_1r'_2, b_1b_2 + r'_1b_2 - l'_2b_1 - l'_2r'_1\}, r' = max\{b_1b_2 - l'_1b_2 - l'_2b_1 + l'_1l'_2, b_1b_2 + r'_1b_2 + r'_2b_1 + r'_1r'_2\} - b_1b_2, l'' = b_1b_2 - min\{b_1b_2 - l''_1b_2 + r''_2b_1 - l''_1r''_2, b_1b_2 + r''_1b_2 - l''_2b_1 - l''_2r''_1\}$ and $r'' = max\{b_1b_2 - l''_1b_2 - l''_2b_1 + l''_1l''_2, b_1b_2 + r''_1b_2 + r''_2b_1 + r''_1r''_2\} - b_1b_2$.*
here $\chi = \chi_1 \wedge \chi_2, \eta = \eta_1 \vee \eta_2, \zeta = \zeta_1 \vee \zeta_2$.

Theorem 19. *Let $\tilde{B}_1 = ([b_1; l_1, r_1; l'_1, r'_1; l''_1, r''_1]; \chi_1, \eta_1, \zeta_1)_{LR}$ be an LR-type SNN in which $b_1 < 0$, $b_1 + r_1 \geq 0$, and $\tilde{B}_2 = ([b_2; l_2, r_2; l'_2, r'_2; l''_2, r''_2]; \chi_2, \eta_2, \zeta_2)_{LR}$ be an unrestricted LR-type SNN, then $\tilde{B}_1 \otimes \tilde{B}_2 = ([b; l, r; l'r'; l'', r'']; \chi, \eta, \zeta)_{LR}$, where $b = b_1b_2, l = b_1b_2 - min\{b_1b_2 - l_1b_2 + r_2b_1 - l_1r_2, b_1b_2 + r_1b_2 - l_2b_1 - l_2r_1\}, r = max\{b_1b_2 - l_1b_2 - l_2b_1 + l_1l_2, b_1b_2 + r_1b_2 + r_2b_1 + r_1r_2\} - b_1b_2, l' = b_1b_2 - min\{b_1b_2 - l'_1b_2 + r'_2b_1 - l'_1r'_2, b_1b_2 + r'_1b_2 - l'_2b_1 - l'_2r'_1\}, r' = max\{b_1b_2 - l'_1b_2 - l'_2b_1 + l'_1l'_2, b_1b_2 + r'_1b_2 + r'_2b_1 + r'_1r'_2\} - b_1b_2, l'' = b_1b_2 - min\{b_1b_2 - l''_1b_2 + r''_2b_1 - l''_1r''_2, b_1b_2 + r''_1b_2 - l''_2b_1 - l''_2r''_1\}$ and $r'' = max\{b_1b_2 - l''_1b_2 - l''_2b_1 + l''_1l''_2, b_1b_2 + r''_1b_2 + r''_2b_1 + r''_1r''_2\} - b_1b_2$.*
here $\chi = \chi_1 \wedge \chi_2, \eta = \eta_1 \vee \eta_2, \zeta = \zeta_1 \vee \zeta_2$.

Theorem 20. *Let $\tilde{B}_1 = ([b_1; l_1, r_1; l'_1, r'_1; l''_1, r''_1]; \chi_1, \eta_1, \zeta_1)_{LR}$ be an LR-type SNN in which $b_1 + r_1 < 0$, $b_1 + r'_1 \geq 0$, and $\tilde{B}_2 = ([b_2; l_2, r_2; l'_2, r'_2; l''_2, r''_2]; \chi_2, \eta_2, \zeta_2)_{LR}$ be an unrestricted LR-type SNN, then $\tilde{B}_1 \otimes \tilde{B}_2 = ([b; l, r; l'r'; l'', r'']; \chi, \eta, \zeta)_{LR}$,*

where $b = b_1b_2, l = b_1b_2 - min\{b_1b_2 - l_1b_2 + r_2b_1 - l_1r_2, b_1b_2 + r_2b_1 + r_1b_2 + r_1r_2\}, r = max\{b_1b_2 + r_1b_2 - l_2b_1 - l_2r_1, b_1b_2 - l_2b_1 - l_1b_2 + l_1l_2\} - b_1b_2, l' = b_1b_2 - min\{b_1b_2 - l'_1b_2 + r'_2b_1 - l'_1r'_2, b_1b_2 + r'_1b_2 - l'_2b_1 - l'_2r'_1\}, r' = max\{b_1b_2 - l'_1b_2 - l'_2b_1 + l'_1l'_2, b_1b_2 + r'_1b_2 + r'_2b_1 + r'_1r'_2\} - b_1b_2, l'' = b_1b_2 - min\{b_1b_2 - l''_1b_2 + r''_2b_1 - l''_1r''_2, b_1b_2 + r''_1b_2 - l''_2b_1 - l''_2r''_1\}$ *and* $r'' = max\{b_1b_2 - l''_1b_2 - l''_2b_1 + l''_1l''_2, b_1b_2 + r''_1b_2 + r''_2b_1 + r''_1r''_2\} - b_1b_2.$
here $\chi = \chi_1 \wedge \chi_2, \eta = \eta_1 \vee \eta_2, \zeta = \zeta_1 \vee \zeta_2.$

Theorem 21. *Let* $\tilde{B}_1 = ([b_1; l_1, r_1; l'_1, r'_1; l''_1, r''_1]; \chi_1, \eta_1, \zeta_1)_{LR}$ *be an* LR*-type SNN in which* $b_1 + r'_1 < 0$, $b_1 + r''_1 \geq 0$, *and* $\tilde{B}_2 = ([b_2; l_2, r_2; l'_2, r'_2; l''_2, r''_2]; \chi_2, \eta_2, \zeta_2)_{LR}$ *be an unrestricted* LR*-type SNN, then* $\tilde{B}_1 \otimes \tilde{B}_2 = ([b; l, r; l'r'; l'', r'']; \chi, \eta, \zeta)_{LR}$, *where* $b = b_1b_2, l = b_1b_2 - min\{b_1b_2 - l_1b_2 + r_2b_1 - l_1r_2, b_1b_2 + r_2b_1 + r_1b_2 + r_1r_2\}, r = max\{b_1b_2 + r_1b_2 - l_2b_1 - l_2r_1, b_1b_2 - l_2b_1 - l_1b_2 + l_1l_2\} - b_1b_2, l' = b_1b_2 - min\{b_1b_2 - l'_1b_2 + r'_2b_1 - l'_1r'_2, b_1b_2 + r'_2b_1 + r'_1b_2 + r'_1r'_2\}, r' = max\{b_1b_2 + r'_1b_2 - l'_2b_1 - l'_2r'_1, b_1b_2 - l_2b_1 - l'_1b_2 + l'_1l'_2\} - b_1b_2, l'' = b_1b_2 - min\{b_1b_2 - l''_1b_2 + r''_2b_1 - l''_1r''_2, b_1b_2 + r''_1b_2 - l''_2b_1 - l''_2r''_1\}$ *and* $r'' = max\{b_1b_2 - l''_1b_2 - l''_2b_1 + l''_1l''_2, b_1b_2 + r''_1b_2 + r''_2b_1 + r''_1r''_2\} - b_1b_2.$
here $\chi = \chi_1 \wedge \chi_2, \eta = \eta_1 \vee \eta_2, \zeta = \zeta_1 \vee \zeta_2.$

Theorem 22. *Let* $\tilde{B}_1 = ([b_1; l_1, r_1; l'_1, r'_1; l''_1, r''_1]; \chi_1, \eta_1, \zeta_1)_{LR}$ *be an* LR*-type SNN in which* $b_1 + r''_1 < 0$, *and* $\tilde{B}_2 = ([b_2; l_2, r_2; l'_2, r'_2; l''_2, r''_2]; \chi_2, \eta_2, \zeta_2)_{LR}$ *be an unrestricted* LR*-type SNN, then* $\tilde{B}_1 \otimes \tilde{B}_2 = ([b; l, r; l'r'; l'', r'']; \chi, \eta, \zeta)_{LR}$, *where* $b = b_1b_2, l = b_1b_2 - min\{b_1b_2 - l_1b_2 + r_2b_1 - l_1r_2, b_1b_2 + r_2b_1 + r_1b_2 + r_1r_2\}, r = max\{b_1b_2 + r_1b_2 - l_2b_1 - l_2r_1, b_1b_2 - l_2b_1 - l_1b_2 + l_1l_2\} - b_1b_2, l' = b_1b_2 - min\{b_1b_2 - l'_1b_2 + r'_2b_1 - l'_1r'_2, b_1b_2 + r'_2b_1 + r'_1b_2 + r'_1r'_2\}, r' = max\{b_1b_2 + r'_1b_2 - l'_2b_1 - l'_2r'_1, b_1b_2 - l_2b_1 - l'_1b_2 + l'_1l'_2\} - b_1b_2, l'' = b_1b_2 - min\{b_1b_2 - l''_1b_2 + r''_2b_1 - l''_1r''_2, b_1b_2 + r''_2b_1 + r''_1b_2 + r''_1r''_2\}$ *and* $r'' = max\{b_1b_2 + r''_1b_2 - l''_2b_1 - l''_2r''_1, b_1b_2 - l_2b_1 - l''_1b_2 + l''_1l''_2\} - b_1b_2.$
here $\chi = \chi_1 \wedge \chi_2, \eta = \eta_1 \vee \eta_2, \zeta = \zeta_1 \vee \zeta_2.$

Theorem 23. *Let* $\tilde{B}_1 = ([b_1; l_1, r_1; l'_1, r'_1; l''_1, r''_1]; \chi_1, \eta_1, \zeta_1)_{LR}$ *be an* LR*-type SNN in which* $b_1 - l''_1 \geq 0$, *and* $\tilde{B}_2 = ([b_2; l_2, r_2; l'_2, r'_2; l''_2, r''_2]; \chi_2, \eta_2, \zeta_2)_{LR}$ *be an unrestricted* LR*-type SNN, then* $\tilde{B}_1 \otimes \tilde{B}_2 = ([b; l, r; l'r'; l'', r'']; \chi, \eta, \zeta)_{LR}$, *where* $b = b_1b_2, l = b_1b_2 - min\{b_1b_2 - l_2b_1 - l_1b_2 + l_1l_2, b_1b_2 - l_2b_1 + r_1b_2 - l_2r_1\}, r = max\{b_1b_2 + r_2b_1 + r_1b_2 + r_1r_2, b_1b_2 + r_2b_1 - l_1b_2 - l_1r_2\} - b_1b_2, l' = b_1b_2 - min\{b_1b_2 - l'_2b_1 - l'_1b_2 + l'_1l'_2, b_1b_2 - l'_2b_1 + r'_1b_2 - l'_2r'_1\}, r' = max\{b_1b_2 + r'_2b_1 + r'_1b_2 + r'_1r'_2, b_1b_2 + r'_2b_1 - l'_1b_2 - l'_1r'_2\} - b_1b_2, l'' = b_1b_2 - min\{b_1b_2 - l''_2b_1 - l''_1b_2 + l''_1l''_2, b_1b_2 - l''_2b_1 + r''_1b_2 - l''_2r''_1\}$ *and* $r'' = max\{b_1b_2 + r''_2b_1 + r''_1b_2 + r''_1r''_2, b_1b_2 + r''_2b_1 - l''_1b_2 - l''_1r''_2\} - b_1b_2.$
here $\chi = \chi_1 \wedge \chi_2, \eta = \eta_1 \vee \eta_2, \zeta = \zeta_1 \vee \zeta_2.$

3 Methodology

In this section, a new method is presented to find the single-valued neutrosophic optimal solution of FSNLP problems with equality constraints, in which all the parameters are represented by LR-type SNNs.

$$\text{Maximize/ Minimize } \sum_{j=1}^{n} C_j \otimes X_j; \tag{82}$$

subject to

$$\sum_{j=1}^{n} A_{ij} \otimes X_j = B_i, \forall i = 1, 2, 3, \cdots, m.$$

where C_j, A_{ij}, B_i and X_j are LR-type SNNs.
Step 1. Assuming $C_j = ([c_j; p_j, q_j; p'_j, q'_j; p''_j, q''_j]; \chi_j, \eta_j, \zeta_j)_{LR}, X_j = ([x_j; y_j, z_j; y'_j, z'_j; y''_j, z''_j]; \phi_j, \theta_j, \kappa_j)_{LR}, A_{ij} = ([a_{ij}; l_j, r_j; l'_j, r'_j; l''_j, r''_j]; \xi_{ij}, \psi_{ij}, \varGamma_{ij})_{LR}$, and $B_i = ([b_i; s_j, t_j; s'_j, t'_j; s''_j, t''_j]; \epsilon_j, \varepsilon_j, \phi_j)$, the FSNLP problem can be trans-

formed as follows;

$$\text{Maximize/ Minimize}\ \left(\ \sum_{j=1}^{n}([c_j;p_j,q_j;p'_j,q'_j;p''_j,q''_j];\chi_j,\eta_j,\zeta_j)_{LR}\otimes([x_j;y_j,z_j;y'_j,z'_j;y''_j,z''_j];\phi_j,\theta_j,\kappa_j)_{LR}\ \right);\quad(83)$$

subject to

$$\sum_{j=1}^{n}([a_{ij};l_j,r_j;l'_j,r'_j;l''_j,r''_j];\xi_{ij},\psi_{ij},\Gamma_{ij})_{LR}\otimes([x_j;y_j,z_j;y'_j,z'_j;y''_j,z''_j];\phi_j,\theta_j,\kappa_j)_{LR}$$
$$=([b_i;s_j,t_j;s'_j,t'_j;s''_j,t''_j];\epsilon_j,\varepsilon_j,\phi_j)_{LR},\forall i=1,2,3,\cdots,m.$$

where $([x_j;y_j,z_j;y'_j,z'_j;y''_j,z''_j];\phi_j,\theta_j,\kappa_j)_{LR}$ are LR-type SNN, $\forall j=1,2,3\cdots,n$.
Step 2. Using product of LR-type SNNs defined in Section (2.1) and assuming

$$([a_{ij};l_j,r_j;l'_j,r'_j;l''_j,r''_j];\xi_{ij},\psi_{ij},\Gamma_{ij})_{LR}\otimes([x_j;y_j,z_j;y'_j,z'_j;y''_j,z''_j];\phi_j,\theta_j,\kappa_j)_{LR}$$
$$=([a^*_{ij};l^*_j,r^*_j;l^{*'}_j,r^{*'}_j;l^{*''}_j,r^{*''}_j];\xi^*_{ij},\psi^*_{ij},\Gamma^*_{ij})_{LR}.$$
$$Here$$
$$\xi_{ij}\wedge\phi_j=\xi^*_{ij},\psi_{ij}\vee\theta_j=\psi^*_{ij},\Gamma_{ij}\vee\kappa_j=\Gamma^*_{ij}.$$

The FSNLP problem (83) can be transformed as follows;

$$\text{Maximize/ Minimize}\ \left(\ \sum_{j=1}^{n}([c_j;p_j,q_j;p'_j,q'_j;p''_j,q''_j];\chi_j,\eta_j,\zeta_j)_{LR}\otimes([x_j;y_j,z_j;y'_j,z'_j;y''_j,z''_j];\phi_j,\theta_j,\kappa_j)_{LR}\ \right);\quad(84)$$

subject to

$$([a^*_{ij};l^*_j,r^*_j;l^{*'}_j,r^{*'}_j;l^{*''}_j,r^{*''}_j];\xi^*_{ij},\psi^*_{ij},\Gamma^*_{ij})_{LR}=([b_i;s_j,t_j;s'_j,t'_j;s''_j,t''_j];\epsilon_j,\varepsilon_j,\phi_j)_{LR},\forall i=1,2,3,\cdots,m.$$

where $([x_j;y_j,z_j;y'_j,z'_j;y''_j,z''_j];\phi_j,\theta_j,\kappa_j)_{LR}$ are LR-type SNN, $\forall j=1,2,3\cdots,n$.
Step 3. Using arithmetic operations defined in Section (2.1), above problem becomes:

$$\text{Maximize/ Minimize}\ \left(\ \sum_{j=1}^{n}([c_j;p_j,q_j;p'_j,q'_j;p''_j,q''_j];\chi_j,\eta_j,\zeta_j)_{LR}\otimes([x_j;y_j,z_j;y'_j,z'_j;y''_j,z''_j];\phi_j,\theta_j,\kappa_j)_{LR}\ \right);\quad(85)$$

subject to

$$\sum_{j=1}^{n}a^*_{ij}=b_i,\sum_{j=1}^{n}l^*_j=s_j,\sum_{j=1}^{n}r^*_j=t_j,\sum_{j=1}^{n}l^{*'}_j=s'_j,\sum_{j=1}^{n}r^{*'}_j=t'_j,\sum_{j=1}^{n}l^{*''}_j=s''_j,\sum_{j=1}^{n}r^{*''}_j=t''_j,$$
$$\wedge[\xi^*_{ij}]=\epsilon_j,\vee[\psi^*_{ij}]=\varepsilon_j,\vee[\Gamma^*_{ij}]=\phi_j,$$

$y_j\geq 0,z_j\geq 0,\quad y'_j-y_j\geq 0,\quad z'_j-z_j\geq 0,\quad y''_j-y'_j\geq 0,\quad z''_j-z'_j\geq 0,\forall j=1,2,3,\cdots,n.$
and $\phi_j,\theta_j,\kappa_j\in[0,1]$.
Step 4. Now we have to find LR-type single-valued neutrosophic feasible solution
$X^k=([x^k_j;y^k_j,z^k_j;y^{k'}_j,z^{k'}_j;y^{k''}_j,z^{k''}_j];\phi_j,\theta_j,\kappa_j)_{LR}$. By applying ranking, the FSNLP problem can be solved

$$\text{Maximize/ Minimize}\ \Re\left(\ \sum_{j=1}^{n}([c_j;p_j,q_j;p'_j,q'_j;p''_j,q''_j];\chi_j,\eta_j,\zeta_j)_{LR}\otimes([x_j;y_j,z_j;y'_j,z'_j;y''_j,z''_j];\phi_j,\theta_j,\kappa_j)_{LR}\ \right);\quad(86)$$

subject to

$$\sum_{j=1}^{n} a^*_{ij} = b_i, \sum_{j=1}^{n} l^*_j = s_j, \sum_{j=1}^{n} r^*_j = t_j, \sum_{j=1}^{n} l^{*'}_j = s'_j, \sum_{j=1}^{n} r^{*'}_j = t'_j, \sum_{j=1}^{n} l^{*''}_j = s''_j, \sum_{j=1}^{n} r^{*''}_j = t''_j,$$
$$\wedge[\xi^*_{ij}] = \epsilon_j, \vee[\psi^*_{ij}] = \varepsilon_j, \vee[\Gamma^*_{ij}] = \phi_j,$$

$y_j \geq 0, z_j \geq 0, \quad y'_j - y_j \geq 0, \quad z'_j - z_j \geq 0, \quad y''_j - y'_j \geq 0, \quad z''_j - z'_j \geq 0, \forall j = 1, 2, 3, \cdots, n.$
and $\phi_j, \theta_j, \kappa_j \in [0, 1]$.
Step 5. Assuming $([c_j; p_j, q_j; p'_j, q'_j; p''_j, q''_j]; \chi_j, \eta_j, \zeta_j)_{LR} \otimes ([x_j; y_j, z_j; y'_j, z'_j; y''_j, z''_j]; \phi_j, \theta_j, \kappa_j)_{LR}$ $= ([x^s_j; y^s_j, z^s_j; ys'_j, zs'_j; ys''_j, zs''_j]; \phi^s_j, \theta^s_j, \kappa^s_j)_{LR}$ the problem (86) can be written as:

$$\text{Maximize/Minimize } \Re\left(\sum_{j=1}^{n}([x^s_j; y^s_j, z^s_j; ys'_j, zs'_j; ys''_j, zs''_j]; \phi^s_j, \theta^s_j, \kappa^s_j)_{LR} \right); \tag{87}$$

subject to

$$\sum_{j=1}^{n} a^*_{ij} = b_i, \sum_{j=1}^{n} l^*_j = s_j, \sum_{j=1}^{n} r^*_j = t_j, \sum_{j=1}^{n} l^{*'}_j = s'_j, \sum_{j=1}^{n} r^{*'}_j = t'_j, \sum_{j=1}^{n} l^{*''}_j = s''_j, \sum_{j=1}^{n} r^{*''}_j = t''_j,$$
$$\wedge[\xi^*_{ij}] = \epsilon_j, \vee[\psi^*_{ij}] = \varepsilon_j, \vee[\Gamma^*_{ij}] = \phi_j,$$

$y_j \geq 0, z_j \geq 0, \quad y'_j - y_j \geq 0, \quad z'_j - z_j \geq 0, \quad y''_j - y'_j \geq 0, \quad z''_j - z'_j \geq 0, \forall j = 1, 2, 3, \cdots, n.$
and $\phi_j, \theta_j, \kappa_j \in [0, 1]$.
Step 6. As ranking function is linear thus the problem (87) can be written as:

$$\text{Maximize/Minimize } \left(\sum_{j=1}^{n} \Re([x^s_j; y^s_j, z^s_j; ys'_j, zs'_j; ys''_j, zs''_j]; \phi^s_j, \theta^s_j, \kappa^s_j)_{LR} \right); \tag{88}$$

subject to

$$\begin{aligned}&\sum_{j=1}^{n} a^*_{ij} = b_i, \sum_{j=1}^{n} l^*_j = s_j, \sum_{j=1}^{n} r^*_j = t_j, \sum_{j=1}^{n} l^{*'}_j = s'_j, \sum_{j=1}^{n} r^{*'}_j = t'_j, \sum_{j=1}^{n} l^{*''}_j = s''_j, \sum_{j=1}^{n} r^{*''}_j = t''_j,\\ &\wedge[\xi^*_{ij}] = \epsilon_j, \vee[\psi^*_{ij}] = \varepsilon_j, \vee[\Gamma^*_{ij}] = \phi_j,\end{aligned} \tag{89}$$

$y_j \geq 0, z_j \geq 0, \quad y'_j - y_j \geq 0, \quad z'_j - z_j \geq 0, \quad y''_j - y'_j \geq 0, \quad z''_j - z'_j \geq 0, \forall j = 1, 2, 3, \cdots, n.$
and $\phi_j, \theta_j, \kappa_j \in [0, 1]$.
Step 7. Using the definition of ranking function defined in Section (2.1), the problem can be converted into:

$$\text{Maximize/Minimize} \begin{pmatrix} \sum_{j=1}^{n}[\frac{1}{6}\{(\int_0^{\chi} x^s_j - y^s_j L^{-1}(\alpha)d\alpha) + (\int_0^{\chi} x^s_j - z^s_j R^{-1}(\alpha)d\alpha) + \\ (\int_{\eta}^{1} x^s_j - y^{s'}_j L'^{-1}(\beta)d\beta) + (\int_{\eta}^{1} x^s_j - z^{s'}_j R'^{-1}(\beta)d\beta) \\ +(\int_{\zeta}^{1} x^s_j - y^{s''}_j L''^{-1}(\gamma)d\gamma) + (\int_{\zeta}^{1} x^s_j - z^{s''}_j R''^{-1}(\gamma)d\gamma)\}] \end{pmatrix}; \tag{90}$$

subject to

$$\sum_{j=1}^{n} a^*_{ij} = b_i, \sum_{j=1}^{n} l^*_j = s_j, \sum_{j=1}^{n} r^*_j = t_j, \sum_{j=1}^{n} l^{*'}_j = s'_j, \sum_{j=1}^{n} r^{*'}_j = t'_j, \sum_{j=1}^{n} l^{*''}_j = s''_j, \sum_{j=1}^{n} r^{*''}_j = t''_j,$$
$$\wedge[\xi^*_{ij}] = \epsilon_j, \vee[\psi^*_{ij}] = \varepsilon_j, \vee[\Gamma^*_{ij}] = \phi_j,$$

$y_j \geq 0, z_j \geq 0, \quad y'_j - y_j \geq 0, \quad z'_j - z_j \geq 0, \quad y''_j - y'_j \geq 0, \quad z''_j - z'_j \geq 0, \forall j = 1, 2, 3, \cdots, n.$
and $\phi_j, \theta_j, \kappa_j \in [0, 1]$.
Step 8 Solve the crisp linear programming problem (90) by proposed method to find the optimal solution $x_j, y_j, z_j, y'_j, z'_j, y''_j, z''_j, \chi_j, \eta_j, \zeta_j$. **Step 9** Find the LR-type single-valued neutrosophic optimal solution X_j of the FSNLP problem by substituting the values of $x_j, y_j, z_j, y'_j, z'_j, y''_j, z''_j, \chi_j, \eta_j$ and ζ_j in $X_j = ([x_j; y_j, z_j; y'_j, z'_j; y''_j, z''_j]; \chi_j, \eta_j, \zeta_j)_{LR}$.
Step 10 Find the LR-type single-valued neutrosophic optimal solution of the FSNLP problem (82) by substituting the

values of X_j in $\sum_{j=1}^{n} C_j \otimes X_j$.

Theorem 24. *The solution of FSNLP problem with LR-type SNNs*

$$\text{Maximize/ Minimize} \sum_{j=1}^{n} C_j \otimes X_j \quad \text{subject to} \quad \sum_{j=1}^{n} A_{ij} \otimes X_j = B_i, \forall i = 1, 2, 3, \cdots, m. \tag{91}$$

where C_j, A_{ij}, B_i and X_j are LR-type SNNs, exists when the solution of the associated crisp LPP

$$\text{Maximize/ Minimize} \, \Re\left(\sum_{j=1}^{n}([x_j^s; y_j^s, z_j^s; ys'_j, zs'_j; ys''_j, zs''_j]; \phi_j^s, \theta_j^s, \kappa_j^s)_{LR} \right);$$

subject to

$$\sum_{j=1}^{n} a_{ij}^* = b_i, \sum_{j=1}^{n} l_j^* = s_j, \sum_{j=1}^{n} r_j^* = t_j, \sum_{j=1}^{n} l_j^{*'} = s'_j, \sum_{j=1}^{n} r_j^{*'} = t'_j, \sum_{j=1}^{n} l_j^{*''} = s''_j, \sum_{j=1}^{n} r_j^{*''} = t''_j,$$
$$\wedge[\xi_{ij}^*] = \epsilon_j, \vee[\psi_{ij}^*] = \varepsilon_j, \vee[\Gamma_{ij}^*] = \phi_j,$$

$y_j \geq 0, z_j \geq 0, \quad y'_j - y_j \geq 0, \quad z'_j - z_j \geq 0, \quad y''_j - y'_j \geq 0, \quad z''_j - z'_j \geq 0, \forall j = 1, 2, 3, \cdots, n.$
and $\phi_j, \theta_j, \kappa_j \in [0, 1]$, exists. Otherwise, there is no guarantee that the LR-type single-valued neutrosophic optimal solution exists.

Proof. Straightforward. □

4 Numerical Examples

Example 1. A Company Manufacturing Problem. A company manufactures two types of face mask: cotton face mask and wool face mask. Each face mask has to pass through two different machines: M_1 and M_2. M_1 machine can work for $([50; 26, 62; 38, 101; 49.2, 192]; 0.7, 0.6, 0.5)_{LR}$ minutes per week and M_2 machine can work for $([68; 41, 94; 54, 133; 67.8, 262]; 0.6, 0.5, 0.3)_{LR}$ minutes per week. Ten hundred cotton face masks required $([5; 2, 3; 3, 5; 4.7, 6]; 0.9, 0.6, 0.3)_{LR}$ minutes on M_1 machine and $([6; 1, 2; 2, 3; 5.5, 5]; 0.8, 0.3, 0.2)_{LR}$ minutes on M_2 machine. Ten hundred wool masks required $([7; 3, 4; 4, 5; 6.9, 7]; 0.6, 0.1, 0.2)_{LR}$ minutes and $([8; 3, 4; 4, 5; 7.9, 8]; 0.7, 0.5, 0.2)_{LR}$ minutes on M_1 and M_2, respectively. The profit is Rs. $([12; 5, 6; 7, 8; 10, 11]; 0.6, 0.2, 0.3)_{LR}$ per thousand for cotton face masks and Rs. $([14; 4, 7; 6, 9; 8, 13]; 0.8, 0.5, 0.4)_{LR}$ per thousand for wool face masks. The company wants to maximize the profit.
We apply the method discussed in Section (3).

$$Maximize([12; 5, 6; 7, 8; 10, 11]; 0.6, 0.2, 0.3)_{LR} \otimes X_1 \oplus ([14; 4, 7; 6, 9; 8, 13]; 0.8, 0.5, 0.4)_{LR} \otimes X_2$$

subject to

$$([5; 2, 3; 3, 5; 4.7, 6]; 0.9, 0.6, 0.3)_{LR} \otimes X_1 \oplus ([6; 1, 2; 2, 3; 5.5, 5]; 0.8, 0.3, 0.2)_{LR} \otimes X_2$$
$$= ([50; 26, 62; 38, 101; 49.2, 192]; 0.7, 0.6, 0.5)_{LR}$$
$$([7; 3, 4; 4, 5; 6.9, 7]; 0.6, 0.1, 0.2)_{LR} \otimes X_1 \oplus ([8; 3, 4; 4, 5; 7.9, 8]; 0.7, 0.5, 0.2)_{LR} \otimes X_2$$
$$= ([68; 41, 94; 54, 133; 67.8, 262]; 0.6, 0.5, 0.3)_{LR}$$

where $([x_1; l_1, r_1; l'_1, r'_1; l''_1, r''_1]; \chi_1, \eta_1, \zeta_1)$ and $([x_2; l_2, r_2; l'_2, r'_2; l''_2, r''_2]; \chi_2, \eta_2, \zeta_2)_{LR}$ are LR-type SNNs.
$\chi_1, \eta_1, \zeta_1, \chi_2, \eta_2, \zeta_2 \in [0, 1]$.
Step 1: Let $X_1 = ([x_1; l_1, r_1; l'_1, r'_1; l''_1, r''_1]; \chi_1, \eta_1, \zeta_1)_{LR}$ and $X_2 = ([x_2; l_2, r_2; l'_2, r'_2; l''_2, r''_2]; \chi_2, \eta_2, \zeta_2)_{LR}$, then prob-

lems can be written as

$$Maximize([12;5,6;7,8;10,11];0.6,0.2,0.3)_{LR}\otimes([x_1;l_1,r_1;l'_1,r'_1;l''_1,r''_1];\chi_1,\eta_1,\zeta_1)_{LR}\oplus$$
$$([14;4,7;6,9;8,13];0.8,0.5,0.4)_{LR}\otimes([x_2;l_2,r_2;l'_2,r'_2;l''_2,r''_2];\chi_2,\eta_2,\zeta_2)_{LR}$$

subject to

$$([5;2,3;3,5;4.7,6];0.9,0.6,0.3)_{LR}\otimes([x_1;l_1,r_1;l'_1,r'_1;l''_1,r''_1];\chi_1,\eta_1,\zeta_1)_{LR}\oplus([6;1,2;2,3;5.5,5];0.8,0.3,0.2)_{LR}$$
$$\otimes([x_2;l_2,r_2;l'_2,r'_2;l''_2,r''_2];\chi_2,\eta_2,\zeta_2)_{LR}=([50;26,62;38,101;49.2,192];0.7,0.6,0.5)_{LR}$$
$$([7;3,4;4,5;6.9,7];0.6,0.1,0.2)_{LR}\otimes([x_1;l_1,r_1;l'_1,r'_1;l''_1,r''_1];\chi_1,\eta_1,\zeta_1)_{LR}\oplus([8;3,4;4,5;7.9,8];0.7,0.5,0.2)_{LR}$$
$$\otimes([x_2;l_2,r_2;l'_2,r'_2;l''_2,r''_2];\chi_2,\eta_2,\zeta_2)_{LR}=([68;41,94;54,133;67.8,262];0.6,0.5,0.3)_{LR}$$

where $([x_1;l_1,r_1;l'_1,r'_1;l''_1,r''_1];\chi_1,\eta_1,\zeta_1)$ and $([x_2;l_2,r_2;l'_2,r'_2;l''_2,r''_2];\chi_2,\eta_2,\zeta_2)_{LR}$ are LR-type SNNs. $\chi_1,\eta_1,\zeta_1,\chi_2,\eta_2,\zeta_2\in[0,1]$.

Step 2: Using product defined in Section (2.1), the FSNLP, problem obtained in step 1, can be written as:

$$\text{Maximize}([12x_1;12x_1-min\{7x_1-7l_1,18x_1-18l_1\},max\{18x_1+18r_1,7x_1+7r_1\}-12x_1;12x_1$$
$$-min\{5x_1-5l'_1,20x_1-20l'_1\},max\{20x_1+20r'_1,5x_1+5r'_1\}-12x_1;12x_1-min$$
$$\{2x_1-2l''_1,23x_1-23l''_1\},max\{23x_1+23r''_1,2x_1+2r''_1\}-12x_1];0.6\wedge\chi_1,0.2\vee\eta_1,0.3\vee\zeta_1)_{LR}$$
$$\oplus([14x_2;14x_2-min\{10x_2-10l_2,21x_2-21l_2\},max\{21x_2+21r_2,10x_2+10r_2\}-14x_2;14x_2$$
$$-min\{8x_2-8l'_2,23x_2-23l'_2\},max\{23x_2+23r'_2,8x_2+8r'_2\}-14x_2;14x_2-min$$
$$\{6x_2-6l''_2,27x_2-27l''_2\},max\{27x_2+27r''_2,6x_2+6r''_2\}-14x_2];0.8\wedge\chi_2,0.5\vee\eta_2,0.4\vee\zeta_2)_{LR}$$

subject to

$$([5x_1;5x_1-min\{3x_1-3l_1,8x_1-8l_1\},max\{8x_1+8r_1,3x_1+3r_1\}-5x_1;5x_1$$
$$-min\{2x_1-2l'_1,10x_1-10l'_1\},max\{10x_1+10r'_1,2x_1+2r'_1\}-5x_1;5x_1-min$$
$$\{0.3x_1-0.3l''_1,11x_1-11l''_1\},max\{11x_1+11r''_1,x_1+r''_1\}-5x_1];0.9\wedge\chi_1,0.6\vee\eta_1,0.3\vee\zeta_1)_{LR}$$
$$\oplus([6x_2;6x_2-min\{5x_2-5l_2,8x_2-8l_2\},max\{8x_2+8r_2,5x_2+5r_2\}-6x_2;6x_2$$
$$-min\{4x_2-4l'_2,9x_2-9l'_2\},max\{9x_2+9r'_2,4x_2+4r'_2\}-6x_2;6x_2-min$$
$$\{0.5x_2-0.5l''_2,11x_2-11l''_2\},max\{11x_2+11r''_2,2x_2+2r''_2\}-6x_2];0.8\wedge\chi_2,0.3\vee\eta_2,0.2\vee\zeta_2)_{LR}$$
$$=([50;26,62;38,101;49.2,192];0.7,0.6,0.5)_{LR}$$
$$([7x_1;7x_1-min\{4x_1-4l_1,11x_1-11l_1\},max\{11x_1+11r_1,4x_1+4r_1\}-7x_1;7x_1$$
$$-min\{3x_1-3l'_1,12x_1-12l'_1\},max\{12x_1+12r'_1,3x_1+3r'_1\}-7x_1;7x_1-min$$
$$\{0.1x_1-0.1l''_1,14x_1-14l''_1\},max\{14x_1+14r''_1,x_1+r''_1\}-7x_1];0.6\wedge\chi_1,0.1\vee\eta_1,0.2\vee\zeta_1)_{LR}$$
$$\oplus([8x_2;8x_2-min\{5x_2-5l_2,12x_2-12l_2\},max\{12x_2+12r_2,5x_2+5r_2\}-8x_2;8x_2$$
$$-min\{4x_2-4l'_2,13x_2-13l'_2\},max\{13x_2+13r'_2,4x_2+4r'_2\}-8x_2;8x_2-min$$
$$\{0.1x_2-0.1l''_2,16x_2-16l''_2\},max\{16x_2+16r''_2,2x_2+2r''_2\}-8x_2];0.7\wedge\chi_2,0.5\vee\eta_2,0.2\vee\zeta_2)_{LR}$$
$$=([68;41,94;54,133;67.8,262];0.6,0.5,0.3)_{LR}$$

where $([x_1;l_1,r_1;l'_1,r'_1;l''_1,r''_1];\chi_1,\eta_1,\zeta_1)$ and $([x_2;l_2,r_2;l'_2,r'_2;l''_2,r''_2];\chi_2,\eta_2,\zeta_2)_{LR}$ are LR-type SNNs. $\chi_1,\eta_1,\zeta_1,\chi_2,\eta_2,\zeta_2\in[0,1]$.

Step 3: Using the arithmetic operations which are defined in Section (2.1), the FSNLP, problem obtained in step 2, can be rewritten as:

$$\begin{aligned}\text{Maximize}(&[12x_1+14x_2;12x_1-min\{7x_1-7l_1,18x_1-18l_1\}+14x_2-min\{10x_2-10l_2,21x_2-21l_2\},\\&+max\{18x_1+18r_1,7x_1+7r_1\}-12x_1+max\{21x_2+21r_2,10x_2+10r_2\}-14x_2;12x_1\\&-min\{5x_1-5l_1',20x_1-20l_1'\}+14x_2-min\{8x_2-8l_2',23x_2-23l_2'\},max\{20x_1+20r_1',\\&5x_1+5r_1'\}-12x_1+max\{23x_2+23r_2',8x_2+8r_2'\}-14x_2;12x_1-min\{2x_1-2l_1'',23x_1-23l_1''\}\\&+14x_2-min\{6x_2-6l_2'',27x_2-27l_2''\},max\{23x_1+23r_1'',2x_1+2r_1''\}-12x_1+max\{27x_2+27r_2''\\&,6x_2+6r_2''\}-14x_2];\wedge[(0.6\wedge\chi_1)\wedge(0.8\wedge\chi_2)],\vee[(0.2\vee\eta_1)\vee(0.5\vee\eta_2)],\vee[(0.3\vee\zeta_1)\vee(0.4\vee\zeta_2)]_{LR}\end{aligned}$$

subject to

$$\begin{aligned}5x_1+6x_2=50,7x_1+8x_2=68,\\5x_1-min\{3x_1-3l_1,8x_1-8l_1\}+6x_2-min\{5x_2-5l_2,8x_2-8l_2\}=26,\\max\{8x_1+8r_1,3x_1+3r_1\}-5x_1+max\{8x_2+8r_2,5x_2+5r_2\}-6x_2=62,\\5x_1-min\{2x_1-2l_1',10x_1-10l_1'\}+6x_2-min\{4x_2-4l_2',9x_2-9l_2'\}=38,\\max\{10x_1+10r_1',2x_1+2r_1'\}-5x_1+max\{9x_2+9r_2',4x_2+4r_2'\}-6x_2=101,\\5x_1-min\{0.3x_1-0.3l_1'',11x_1-11l_1''\}+6x_2-min\{0.5x_2-0.5l_2'',11x_2-11l_2''\}=49.2,\\max\{11x_1+11r_1'',x_1+r_1''\}-5x_1+max\{11x_2+11r_2'',2x_2+2r_2''\}-6x_2=192,\\\wedge[(0.9\wedge\chi_1)\wedge(0.8\wedge\chi_2)]=0.7,\vee[(0.6\vee\eta_1)\vee(0.3\vee\eta_2)]=0.6,\vee[(0.3\vee\zeta_1)\vee(0.2\vee\zeta_2)]=0.5,\\7x_1-min\{4x_1-4l_1,11x_1-11l_1\}+8x_2-min\{5x_2-5l_2,12x_2-12l_2\}=41,\\max\{11x_1+11r_1,4x_1+4r_1\}-7x_1+max\{12x_2+12r_2,5x_2+5r_2\}-8x_2=94,\\7x_1-min\{3x_1-3l_1',12x_1-12l_1'\}+8x_2-min\{4x_2-4l_2',13x_2-13l_2'\}=54,\\max\{12x_1+12r_1',3x_1+3r_1'\}-7x_1+max\{13x_2+13r_2',4x_2+4r_2'\}-8x_2=133,\\7x_1-min\{0.1x_1-0.1l_1'',14x_1-14l_1''\}+8x_2-min\{0.1x_2-0.1l_2'',16x_2-16l_2''\}=67.8,\\max\{14x_1+14r_1'',x_1+r_1''\}-7x_1+max\{16x_2+16r_2'',2x_2+2r_2''\}-8x_2=262,\\\wedge[(0.6\wedge\chi_1)\wedge(0.7\wedge\chi_2)]=0.6,\vee[(0.1\vee\eta_1)\vee(0.5\vee\eta_2)]=0.5,\vee[(0.2\vee\zeta_1)\vee(0.2\vee\zeta_2)]=0.3,\\l_1\geq0,r_1\geq0,l_1'-l_1\geq0,r_1'-r_1\geq0,l_1''-l_1'\geq0,r_1''-r_1'\geq0,l_2\geq0,r_2\geq0,l_2'-l_2\geq0,r_2'-r_2\geq0,\\l_2''-l_2'\geq0,r_2''-r_2'\geq0,\chi_1,\eta_1,\zeta_1,\chi_2,\eta_2,\zeta_2\in[0,1].\end{aligned}$$

Step 4: Using the ranking function which are defined in Section (3), the FSNLP, problem obtained in step 3, can be rewritten as:

$$\begin{aligned}\text{Maximize}\Re(&[12x_1+14x_2;12x_1-min\{7x_1-7l_1,18x_1-18l_1\}+14x_2-min\{10x_2-10l_2,21x_2-21l_2\},\\&+max\{18x_1+18r_1,7x_1+7r_1\}-12x_1+max\{21x_2+21r_2,10x_2+10r_2\}-14x_2;12x_1\\&-min\{5x_1-5l_1',20x_1-20l_1'\}+14x_2-min\{8x_2-8l_2',23x_2-23l_2'\},max\{20x_1+20r_1',\\&5x_1+5r_1'\}-12x_1+max\{23x_2+23r_2',8x_2+8r_2'\}-14x_2;12x_1-min\{2x_1-2l_1'',23x_1-23l_1''\}\\&+14x_2-min\{6x_2-6l_2'',27x_2-27l_2''\},max\{23x_1+23r_1'',2x_1+2r_1''\}-12x_1+max\{27x_2+27r_2''\\&,6x_2+6r_2''\}-14x_2];\wedge[(0.6\wedge\chi_1)\wedge(0.8\wedge\chi_2)],\vee[(0.2\vee\eta_1)\vee(0.5\vee\eta_2)],\vee[(0.3\vee\zeta_1)\vee(0.4\vee\zeta_2)])_{LR}\end{aligned}$$

subject to

$$5x_1 + 6x_2 = 50, 7x_1 + 8x_2 = 68,$$
$$5x_1 - min\{3x_1 - 3l_1, 8x_1 - 8l_1\} + 6x_2 - min\{5x_2 - 5l_2, 8x_2 - 8l_2\} = 26,$$
$$max\{8x_1 + 8r_1, 3x_1 + 3r_1\} - 5x_1 + max\{8x_2 + 8r_2, 5x_2 + 5r_2\} - 6x_2 = 62,$$
$$5x_1 - min\{2x_1 - 2l'_1, 10x_1 - 10l'_1\} + 6x_2 - min\{4x_2 - 4l'_2, 9x_2 - 9l'_2\} = 38,$$
$$max\{10x_1 + 10r'_1, 2x_1 + 2r'_1\} - 5x_1 + max\{9x_2 + 9r'_2, 4x_2 + 4r'_2\} - 6x_2 = 101,$$
$$5x_1 - min\{0.3x_1 - 0.3l''_1, 11x_1 - 11l''_1\} + 6x_2 - min\{0.5x_2 - 0.5l''_2, 11x_2 - 11l''_2\} = 49.2,$$
$$max\{11x_1 + 11r''_1, x_1 + r''_1\} - 5x_1 + max\{11x_2 + 11r''_2, 2x_2 + 2r''_2\} - 6x_2 = 192,$$
$$\wedge[(0.9 \wedge \chi_1) \wedge (0.8 \wedge \chi_2)] = 0.7, \vee[(0.6 \vee \eta_1) \vee (0.3 \vee \eta_2)] = 0.6, \vee[(0.3 \vee \zeta_1) \vee (0.2 \vee \zeta_2)] = 0.5,$$
$$7x_1 - min\{4x_1 - 4l_1, 11x_1 - 11l_1\} + 8x_2 - min\{5x_2 - 5l_2, 12x_2 - 12l_2\} = 41,$$
$$max\{11x_1 + 11r_1, 4x_1 + 4r_1\} - 7x_1 + max\{12x_2 + 12r_2, 5x_2 + 5r_2\} - 8x_2 = 94,$$
$$7x_1 - min\{3x_1 - 3l'_1, 12x_1 - 12l'_1\} + 8x_2 - min\{4x_2 - 4l'_2, 13x_2 - 13l'_2\} = 54,$$
$$max\{12x_1 + 12r'_1, 3x_1 + 3r'_1\} - 7x_1 + max\{13x_2 + 13r'_2, 4x_2 + 4r'_2\} - 8x_2 = 133,$$
$$7x_1 - min\{0.1x_1 - 0.1l''_1, 14x_1 - 14l''_1\} + 8x_2 - min\{0.1x_2 - 0.1l''_2, 16x_2 - 16l''_2\} = 67.8,$$
$$max\{14x_1 + 14r''_1, x_1 + r''_1\} - 7x_1 + max\{16x_2 + 16r''_2, 2x_2 + 2r''_2\} - 8x_2 = 262,$$
$$\wedge[(0.6 \wedge \chi_1) \wedge (0.7 \wedge \chi_2)] = 0.6, \vee[(0.1 \vee \eta_1) \vee (0.5 \vee \eta_2)] = 0.5, \vee[(0.2 \vee \zeta_1) \vee (0.2 \vee \zeta_2)] = 0.3,$$
$$l_1 \geq 0, r_1 \geq 0, l'_1 - l_1 \geq 0, r'_1 - r_1 \geq 0, l''_1 - l'_1 \geq 0, r''_1 - r'_1 \geq 0, l_2 \geq 0, r_2 \geq 0, l'_2 - l_2 \geq 0, r'_2 - r_2 \geq 0,$$
$$l''_2 - l'_2 \geq 0, r''_2 - r'_2 \geq 0, \chi_1, \eta_1, \zeta_1, \chi_2, \eta_2, \zeta_2 \in [0, 1].$$

Step 5: Using $\min\{a, b\} = \frac{a+b}{2} - |\frac{a-b}{2}|$, $\max\{a, b\} = \frac{a+b}{2} + |\frac{a-b}{2}|$, the FSNLP, problem obtained in step 4, can be rewritten as:

$$\text{Maximize}([\frac{(96 + 48\chi - 48\eta - 48\zeta + \chi^2 + (\eta - 1)^2 + (\zeta - 1)^2)}{12}x_1 +$$
$$\frac{(112 + 56\chi - 56\eta - 56\zeta + 3\chi^2 + 3(\eta - 1)^2 + 5(\zeta - 1)^2)}{12}x_2 - \frac{25}{24}\chi^2 l_1 - \frac{11}{24}\chi^2|x_1 - l_1| - \frac{31}{24}\chi^2 l_2$$
$$- \frac{11}{24}\chi^2|x_2 - l_2| + \frac{25}{24}\chi^2 r_1 + \frac{11}{24}\chi^2|x_1 + r_1| + \frac{31}{24}\chi^2 r_2 + \frac{11}{24}\chi^2|x_2 + r_2| - \frac{25}{24}(\eta - 1)^2 l'_1$$
$$- \frac{15}{24}(\eta - 1)^2|x_1 - l'_1| - \frac{31}{24}(\eta - 1)^2 l'_2 - \frac{15}{24}(\eta - 1)^2|x_2 - l'_2| + \frac{25}{24}(\eta - 1)^2 r'_1 + \frac{15}{24}(\eta - 1)^2|x_1 + r'_1|$$
$$+ \frac{31}{24}(\eta - 1)^2 r'_2 + \frac{15}{24}(\eta - 1)^2|x_2 + r'_2| - \frac{25}{24}(\zeta - 1)^2 l''_1 - \frac{21}{24}(\zeta - 1)^2|x_1 - l''_1| - \frac{33}{24}(\zeta - 1)^2 l''_2$$
$$- \frac{21}{24}(\zeta - 1)^2|x_2 - l''_2| + \frac{25}{24}(\zeta - 1)^2 r''_1 + \frac{21}{24}(\zeta - 1)^2|x_1 + r''_1| + \frac{33}{24}(\zeta - 1)^2 r''_2 + \frac{21}{24}(\zeta - 1)^2|x_2 + r''_2|])$$

subject to

$$5x_1+6x_2=50, 7x_1+8x_2=68,$$
$$-\frac{1}{2}x_1+\frac{11}{2}l_1+\frac{5}{2}|x_1-l_1|-\frac{x_2}{2}+\frac{13}{2}l_2+\frac{3}{2}|x_2-l_2|=26,$$
$$\frac{1}{2}x_1+\frac{11}{2}r_1+\frac{5}{2}|x_1+r_1|+\frac{x_2}{2}+\frac{13}{2}r_2+\frac{3}{2}|x_2+r_2|=62,$$
$$-x_1+6l_1'+4|x_1-l_1'|-\frac{x_2}{2}+\frac{13}{2}l_2'+\frac{5}{2}|x_2-l_2'|=38,$$
$$x_1+6r_1'+4|x_1+r_1'|+\frac{x_2}{2}+\frac{13}{2}r_2'+\frac{5}{2}|x_2+r_2'|=101,$$
$$-\frac{1.3}{2}x_1+\frac{11.3}{2}l_1''+\frac{10.7}{2}|x_1-l_1''|+\frac{0.5}{2}x_2+\frac{11.5}{2}l_2''+\frac{10.5}{2}|x_2-l_2''|=49.2,$$
$$x_1+6r_1''+5|x_1+r_1''|+\frac{x_2}{2}+\frac{13}{2}r_2''+\frac{9}{2}|x_2+r_2''|=192,$$
$$\wedge[(0.9\wedge\chi_1)\wedge(0.8\wedge\chi_2)]=0.7, \vee[(0.6\vee\eta_1)\vee(0.3\vee\eta_2)]=0.6, \vee[(0.3\vee\zeta_1)\vee(0.2\vee\zeta_2)]=0.5,$$
$$-\frac{1}{2}x_1+\frac{15}{2}l_1+\frac{7}{2}|x_1-l_1|-\frac{x_2}{2}+\frac{17}{2}l_2+\frac{7}{2}|x_2-l_2|=41,$$
$$\frac{1}{2}x_1+\frac{15}{2}r_1+\frac{7}{2}|x_1+r_1|+\frac{x_2}{2}+\frac{17}{2}r_2+\frac{7}{2}|x_2+r_2|=94,$$
$$-\frac{1}{2}x_1+\frac{15}{2}l_1'+\frac{9}{2}|x_1-l_1'|-\frac{x_2}{2}+\frac{17}{2}l_2'+\frac{9}{2}|x_2-l_2'|=54,$$
$$\frac{1}{2}x_1+\frac{15}{2}r_1'+\frac{9}{2}|x_1+r_1'|+\frac{x_2}{2}+\frac{17}{2}r_2'+\frac{9}{2}|x_2+r_2'|=133,$$
$$-\frac{0.1}{2}x_1+\frac{14.1}{2}l_1''+\frac{13.9}{2}|x_1-l_1''|-\frac{0.1}{2}x_2+\frac{16.1}{2}l_2''+\frac{15.9}{2}|x_2-l_2''|=67.8,$$
$$\frac{1}{2}x_1+\frac{15}{2}r_1''+\frac{13}{2}|x_1+r_1''|+x_2+9r_2''+7|x_2+r_2''|=262,$$
$$\wedge[(0.6\wedge\chi_1)\wedge(0.7\wedge\chi_2)]=0.6, \vee[(0.1\vee\eta_1)\vee(0.5\vee\eta_2)]=0.5, \vee[(0.2\vee\zeta_1)\vee(0.2\vee\zeta_2)]=0.3,$$
$$l_1\geq 0, r_1\geq 0, l_1'-l_1\geq 0, r_1'-r_1\geq 0, l_1''-l_1'\geq 0, r_1''-r_1'\geq 0, l_2\geq 0, r_2\geq 0, l_2'-l_2\geq 0, r_2'-r_2\geq 0,$$
$$l_2''-l_2'\geq 0, r_2''-r_2'\geq 0, \chi_1,\eta_1,\zeta_1,\chi_2,\eta_2,\zeta_2\in[0,1].$$

Step 6: By solving the crisp mathematical problem obtained in step 5, we get the optimal solution $x_1=4, l_1=1, r_1=2, l_1'=2, r_1'=3, l_1''=3, r_1''=7, x_2=5, l_2=2, r_2=3, l_2'=3, r_2'=4, l_2''=4, r_2''=6, \chi_1=0.7, \eta_1=0.5, \zeta_1=0.4, \chi_2=0.9, \eta_2=0.4, \zeta_2=0.5$.
Step 7: Substituting the values of $x_1, l_1, r_1, l_1', r_1', l_1'', r_1'', x_2, l_2, r_2, l_2', r_2', l_2'', r_2'', \chi_1, \eta_1, \zeta_1, \chi_2, \eta_2$ and ζ_2 in $X_1=([x_1; l_1, r_1; l_1', r_1'; l_1'', r_1'']; \chi_1, \eta_1, \zeta_1)_{LR}$ and $X_2=([x_2; l_2, r_2; l_2', r_2'; l_2'', r_2'']; \chi_2, \eta_2, \zeta_2)_{LR}$ the exact LR-type single-valued neutrosophic optimal solution is $X_1=([4; 1, 2; 2, 3; 3, 7]; 0.7, 0.5, 0.4)_{LR}, X_2=([5; 2, 3; 3, 4; 4, 6]; 0.9, 0.4, 0.5)_{LR}$.
Step 8: By substituting the values of X_1 and X_2, obtained in Step 7, into the objective function, the LR-type single-valued neutrosophic optimal value is $([118; 67, 158; 92, 229; 110, 432]; 0.6, 0.5, 0.5)_{LR}$.

Example 2.

$$Minimize([10; 3, 5; 4, 6; 7, 8]; 0.8, 0.4, 0.5)_{LR}\otimes X_1\oplus([16; 4, 6; 8, 10; 12, 14]; 0.7, 0.3, 0.2)_{LR}\otimes X_2$$

subject to

$$([7; 3, 4; 4, 6; 5, 7]; 0.6, 0.5, 0.4)_{LR} \otimes X_1 \oplus ([9; 4, 5; 6, 7; 8, 9]; 0.7, 0.1, 0.3)_{LR} \otimes X_2$$
$$= ([87; 56, 149; 75, 216; 84, 297]; 0.6, 0.5, 0.4)_{LR}$$
$$([10; 4, 6; 8, 9; 9, 10]; 0.9, 0.2, 0.1)_{LR} \otimes X_1 \oplus ([11; 4, 6; 6, 8; 9, 10]; 0.8, 0.3, 0.4)_{LR} \otimes X_2$$
$$= ([115; 70, 198; 101, 284; 112, 377]; 0.7, 0.5, 0.4)_{LR}$$

Step 1: *Let* $X_1 = ([x_1; l_1, r_1; l_1', r_1'; l_1'', r_1'']; \chi_1, \eta_1, \zeta_1)_{LR}$ *and* $X_2 = ([x_2; l_2, r_2; l_2', r_2'; l_2'', r_2'']; \chi_2, \eta_2, \zeta_2)_{LR}$, *then problems can be written as*

$$Minimize([10; 3, 5; 4, 6; 7, 8]; 0.8, 0.4, 0.5)_{LR} \otimes ([x_1; l_1, r_1; l_1', r_1'; l_1'', r_1'']; \chi_1, \eta_1, \zeta_1)_{LR} \oplus$$
$$([16; 4, 6; 8, 10; 12, 14]; 0.7, 0.3, 0.2)_{LR} \otimes ([x_2; l_2, r_2; l_2', r_2'; l_2'', r_2'']; \chi_2, \eta_2, \zeta_2)_{LR}$$

subject to

$$([7; 3, 4; 4, 6; 5, 7]; 0.6, 0.5, 0.4)_{LR} \otimes ([x_1; l_1, r_1; l_1', r_1'; l_1'', r_1'']; \chi_1, \eta_1, \zeta_1)_{LR} \oplus ([9; 4, 5; 6, 7; 8, 9]; 0.7, 0.1, 0.3)_{LR} \otimes$$
$$([x_2; l_2, r_2; l_2', r_2'; l_2'', r_2'']; \chi_2, \eta_2, \zeta_2)_{LR} = ([87; 56, 149; 75, 216; 84, 297]; 0.6, 0.5, 0.4)_{LR}$$
$$([10; 4, 6; 8, 9; 9, 10]; 0.9, 0.2, 0.1)_{LR} \otimes ([x_1; l_1, r_1; l_1', r_1'; l_1'', r_1'']; \chi_1, \eta_1, \zeta_1)_{LR} \oplus ([11; 4, 6; 6, 8; 9, 10]; 0.8, 0.3, 0.4)_{LR} \otimes$$
$$([x_2; l_2, r_2; l_2', r_2'; l_2'', r_2'']; \chi_2, \eta_2, \zeta_2)_{LR} = ([115; 70, 198; 101, 284; 112, 377]; 0.7, 0.5, 0.4)_{LR}$$

where $([x_1; l_1, r_1; l_1', r_1'; l_1'', r_1'']; \chi_1, \eta_1, \zeta_1)$ *and* $([x_2; l_2, r_2; l_2', r_2'; l_2'', r_2'']; \chi_2, \eta_2, \zeta_2)_{LR}$ *are* LR*-type SNNs.*
$\chi_1, \eta_1, \zeta_1, \chi_2, \eta_2, \zeta_2 \in [0, 1]$.
Step 2: *Using product defined in Section (2.1), the FSNLP, problem obtained in step 1, can be written as:*

$$\textit{Minimize}([10x_1; 10x_1 - min\{7x_1 - 7l_1, 15x_1 - 15l_1\}, max\{15x_1 + 15r_1, 7x_1 + 7r_1\} - 10x_1; 10x_1$$
$$- min\{6x_1 - 6l_1', 16x_1 - 16l_1'\}, max\{16x_1 + 16r_1', 6x_1 + 6r_1'\} - 10x_1; 10x_1 - min$$
$$\{3x_1 - 3l_1'', 18x_1 - 18l_1''\}, max\{18x_1 + 18r_1'', 3x_1 + 3r_1''\} - 10x_1]; 0.8 \wedge \chi_1, 0.4 \vee \eta_1, 0.5 \vee \zeta_1)_{LR}$$
$$\oplus([16x_2; 16x_2 - min\{12x_2 - 12l_2, 22x_2 - 22l_2\}, max\{22x_2 + 22r_2, 12x_2 + 12r_2\} - 16x_2; 16x_2$$
$$- min\{8x_2 - 8l_2', 26x_2 - 26l_2'\}, max\{26x_2 + 26r_2', 8x_2 + 8r_2'\} - 16x_2; 16x_2 - min$$
$$\{4x_2 - 4l_2'', 30x_2 - 30l_2''\}, max\{30x_2 + 30r_2'', 4x_2 + 4r_2''\} - 16x_2]; 0.7 \wedge \chi_2, 0.3 \vee \eta_2, 0.2 \vee \zeta_2)_{LR}$$

subject to

$$
\begin{aligned}
&([7x_1;7x_1-min\{4x_1-4l_1,11x_1-11l_1\},max\{11x_1+11r_1,4x_1+4r_1\}-7x_1;7x_1\\
&-min\{3x_1-3l_1',13x_1-13l_1'\},max\{13x_1+13r_1',3x_1+3r_1'\}-7x_1;7x_1-min\\
&\{2x_1-2l_1'',14x_1-14l_1''\},max\{14x_1+14r_1'',2x_1+2r_1''\}-7x_1];0.6\wedge\chi_1,0.5\vee\eta_1,0.4\vee\zeta_1)_{LR}\\
&\oplus([9x_2;9x_2-min\{5x_2-5l_2,14x_2-14l_2\},max\{14x_2+14r_2,5x_2+5r_2\}-9x_2;9x_2\\
&-min\{3x_2-3l_2',16x_2-16l_2'\},max\{16x_2+16r_2',3x_2+3r_2'\}-9x_2;9x_2-min\\
&\{x_2-l_2'',18x_2-18l_2''\},max\{18x_2+18r_2'',x_2+r_2''\}-9x_2];0.7\wedge\chi_2,0.1\vee\eta_2,0.3\vee\zeta_2)_{LR}\\
&=([87;56,149;75,216;84,297];0.6,0.5,0.4)_{LR}\\
&([10x_1;10x_1-min\{6x_1-6l_1,16x_1-16l_1\},max\{16x_1+16r_1,6x_1+6r_1\}-10x_1;10x_1\\
&-min\{2x_1-2l_1',19x_1-19l_1'\},max\{19x_1+19r_1',2x_1+2r_1'\}-10x_1;10x_1-min\\
&\{x_1-l_1'',20x_1-20l_1''\},max\{20x_1+20r_1'',x_1+r_1''\}-10x_1];0.9\wedge\chi_1,0.2\vee\eta_1,0.1\vee\zeta_1)_{LR}\\
&\oplus([11x_2;11x_2-min\{7x_2-7l_2,17x_2-17l_2\},max\{17x_2+17r_2,7x_2+7r_2\}-11x_2;11x_2\\
&-min\{5x_2-5l_2',19x_2-19l_2'\},max\{19x_2+19r_2',5x_2+5r_2'\}-11x_2;11x_2-min\\
&\{2x_2-2l_2'',21x_2-21l_2''\},max\{21x_2+21r_2'',2x_2+2r_2''\}-11x_2];0.8\wedge\chi_2,0.3\vee\eta_2,0.4\vee\zeta_2)_{LR}\\
&=([115;70,198;101,284;112,377];0.7,0.5,0.4)_{LR}
\end{aligned}
$$

where $([x_1;l_1,r_1;l_1',r_1';l_1'',r_1''];\chi_1,\eta_1,\zeta_1)$ *and* $([x_2;l_2,r_2;l_2',r_2';l_2'',r_2''];\chi_2,\eta_2,\zeta_2)_{LR}$ *are* LR*-type SNNs.* $\chi_1,\eta_1,\zeta_1,\chi_2,\eta_2,\zeta_2\in[0,1]$.

Step 3: *Using the arithmetic operations which are defined in Section (2.1), the FSNLP, problem obtained in step 2, can be rewritten as:*

$$
\begin{aligned}
&\textit{Minimize}([10x_1+16x_2;10x_1-min\{7x_1-7l_1,15x_1-15l_1\}+16x_2-min\{12x_2-12l_2,22x_2-22l_2\},\\
&max\{15x_1+15r_1,7x_1+7r_1\}-10x_1+max\{22x_2+22r_2,12x_2+12r_2\}-16x_2;10x_1\\
&-min\{6x_1-6l_1',16x_1-16l_1'\}+16x_2-min\{8x_2-8l_2',26x_2-26l_2'\},max\{16x_1+16r_1',6x_1+6r_1'\}-10x_1\\
&+max\{26x_2+26r_2',8x_2+8r_2'\}-16x_2;10x_1-min\{3x_1-3l_1'',18x_1-18l_1''\}+16x_2-min\{4x_2-4l_2'',\\
&30x_2-30l_2''\},max\{18x_1+18r_1'',3x_1+3r_1''\}-10x_1+max\{30x_2+30r_2'',4x_2+4r_2''\}-16x_2];\wedge[(0.8\wedge\chi_1)\\
&\wedge(0.7\wedge\chi_2)],\vee[(0.4\vee\eta_1)\vee(0.3\vee\eta_2)],\vee[(0.5\vee\zeta_1)\vee(0.2\vee\zeta_2]))_{LR}
\end{aligned}
$$

subject to

$$
\begin{aligned}
&7x_1+9x_2=87,10x_1+11x_2=115,\\
&7x_1-min\{4x_1-4l_1,11x_1-11l_1\}+9x_2-min\{5x_2-5l_2,14x_2-14l_2\}=56,\\
&max\{11x_1+11r_1,4x_1+4r_1\}-7x_1+max\{14x_2+14r_2,5x_2+5r_2\}-9x_2=149,\\
&7x_1-min\{3x_1-3l'_1,13x_1-13l'_1\}+9x_2-min\{3x_2-3l'_2,16x_2-16l'_2\}=75\\
&max\{13x_1+13r'_1,3x_1+3r'_1\}-7x_1+max\{16x_2+16r'_2,3x_2+3r'_2\}-9x_2=216,\\
&7x_1-min\{2x_1-2l''_1,14x_1-14l''_1\}+9x_2-min\{x_2-l''_2,18x_2-18l''_2\}=84,\\
&max\{14x_1+14r''_1,2x_1+2r''_1\}-7x_1)+max\{18x_2+18r''_2,x_2+r''_2\}-9x_2)=297,\\
&\wedge[((0.6\wedge\chi_1)\wedge(0.7\wedge\chi_2)]=0.6,\vee[(0.5\vee\eta_1)\vee(0.1\vee\eta_2)]=0.5,\vee[(0.4\vee\zeta_1)\vee(0.3\vee\zeta_2)]=0.4,\\
&10x_1-min\{6x_1-6l_1,16x_1-16l_1\}+11x_2-min\{7x_2-7l_2,17x_2-17l_2\}=70,\\
&max\{16x_1+16r_1,6x_1+6r_1\}-10x_1+max\{17x_2+17r_2,7x_2+7r_2\}-11x_2=198,\\
&10x_1-min\{2x_1-2l'_1,19x_1-19l'_1\}+11x_2-min\{5x_2-5l'_2,19x_2-19l'_2\}=101,\\
&max\{19x_1+19r'_1,2x_1+2r'_1\}-10x_1+max\{19x_2+19r'_2,5x_2+5r'_2\}-11x_2=284,\\
&10x_1-min\{x_1-l''_1,20x_1-20l''_1\}+11x_2-min\{2x_2-2l''_2,21x_2-21l''_2\}=112,\\
&max\{20x_1+20r''_1,x_1+r''_1\}-10x_1+max\{21x_2+21r''_2,2x_2+2r''_2\}-11x_2=377,\\
&\wedge[(0.9\wedge\chi_1)\wedge(0.8\wedge\chi_2)]=0.7,\vee[(0.2\vee\eta_1)\vee(0.3\vee\eta_2)]=0.5,\vee[(0.1\vee\zeta_1)\vee(0.4\vee\zeta_2)]=0.4,\\
&l_1\geq 0,r_1\geq 0,l'_1-l_1\geq 0,r'_1-r_1\geq 0,l''_1-l'_1\geq 0,r''_1-r'_1\geq 0,l_2\geq 0,r_2\geq 0,l'_2-l_2\geq 0,r'_2-r_2\geq 0,\\
&l''_2-l'_2\geq 0,r''_2-r'_2\geq 0,\chi_1,\eta_1,\zeta_1,\chi_2,\eta_2,\zeta_2\in[0,1].
\end{aligned}
$$

Step 4: *Using the ranking function which are defined in Section (3), the FSNLP, problem obtained in step 3, can be rewritten as:*

$$
\begin{aligned}
&\textit{Minimize}\,\Re([10x_1+16x_2;10x_1-min\{7x_1-7l_1,15x_1-15l_1\}+16x_2-min\{12x_2-12l_2,22x_2-22l_2\},\\
&max\{15x_1+15r_1,7x_1+7r_1\}-10x_1+max\{22x_2+22r_2,12x_2+12r_2\}-16x_2;10x_1\\
&-min\{6x_1-6l'_1,16x_1-16l'_1\}+16x_2-min\{8x_2-8l'_2,26x_2-26l'_2\},max\{16x_1+16r'_1,6x_1+6r'_1\}-10x_1\\
&+max\{26x_2+26r'_2,8x_2+8r'_2\}-16x_2;10x_1-min\{3x_1-3l''_1,18x_1-18l''_1\}+16x_2-min\{4x_2-4l''_2,\\
&30x_2-30l''_2\},max\{18x_1+18r''_1,3x_1+3r''_1\}-10x_1+max\{30x_2+30r''_2,4x_2+4r''_2\}-16x_2];\wedge[(0.8\wedge\chi_1)\\
&\wedge(0.7\wedge\chi_2)],\vee[(0.4\vee\eta_1)\vee(0.3\vee\eta_2)],\vee[(0.5\vee\zeta_1)\vee(0.2\vee\zeta_2]])_{LR}
\end{aligned}
$$

subject to

$$\begin{aligned}
7x_1+9x_2=87, 10x_1+11x_2=115,\\
7x_1-min\{4x_1-4l_1, 11x_1-11l_1\}+9x_2-min\{5x_2-5l_2, 14x_2-14l_2\}=56,\\
max\{11x_1+11r_1, 4x_1+4r_1\}-7x_1+max\{14x_2+14r_2, 5x_2+5r_2\}-9x_2=149,\\
7x_1-min\{3x_1-3l_1', 13x_1-13l_1'\}+9x_2-min\{3x_2-3l_2', 16x_2-16l_2'\}=75\\
max\{13x_1+13r_1', 3x_1+3r_1'\}-7x_1+max\{16x_2+16r_2', 3x_2+3r_2'\}-9x_2=216,\\
7x_1-min\{2x_1-2l_1'', 14x_1-14l_1''\}+9x_2-min\{x_2-l_2'', 18x_2-18l_2''\}=84,\\
max\{14x_1+14r_1'', 2x_1+2r_1''\}-7x_1)+max\{18x_2+18r_2'', x_2+r_2''\}-9x_2)=297,\\
\wedge[((0.6\wedge\chi_1)\wedge(0.7\wedge\chi_2)]=0.6, \vee[(0.5\vee\eta_1)\vee(0.1\vee\eta_2)]=0.5, \vee[(0.4\vee\zeta_1)\vee(0.3\vee\zeta_2)]=0.4,\\
10x_1-min\{6x_1-6l_1, 16x_1-16l_1\}+11x_2-min\{7x_2-7l_2, 17x_2-17l_2\}=70,\\
max\{16x_1+16r_1, 6x_1+6r_1\}-10x_1+max\{17x_2+17r_2, 7x_2+7r_2\}-11x_2=198,\\
10x_1-min\{2x_1-2l_1', 19x_1-19l_1'\}+11x_2-min\{5x_2-5l_2', 19x_2-19l_2'\}=101,\\
max\{19x_1+19r_1', 2x_1+2r_1'\}-10x_1+max\{19x_2+19r_2', 5x_2+5r_2'\}-11x_2=284,\\
10x_1-min\{x_1-l_1'', 20x_1-20l_1''\}+11x_2-min\{2x_2-2l_2'', 21x_2-21l_2''\}=112,\\
max\{20x_1+20r_1'', x_1+r_1''\}-10x_1+max\{21x_2+21r_2'', 2x_2+2r_2''\}-11x_2=377,\\
\wedge[(0.9\wedge\chi_1)\wedge(0.8\wedge\chi_2)]=0.7, \vee[(0.2\vee\eta_1)\vee(0.3\vee\eta_2)]=0.5, \vee[(0.1\vee\zeta_1)\vee(0.4\vee\zeta_2)]=0.4,\\
l_1\geq 0, r_1\geq 0, l_1'-l_1\geq 0, r_1'-r_1\geq 0, l_1''-l_1'\geq 0, r_1''-r_1'\geq 0, l_2\geq 0, r_2\geq 0, l_2'-l_2\geq 0, r_2'-r_2\geq 0,\\
l_2''-l_2'\geq 0, r_2''-r_2'\geq 0, \chi_1, \eta_1, \zeta_1, \chi_2, \eta_2, \zeta_2\in[0,1].
\end{aligned}$$

Step 5: *Using min*$\{a,b\}=\frac{a+b}{2}-|\frac{a-b}{2}|$, *max*$\{a,b\}=\frac{a+b}{2}+|\frac{a-b}{2}|$, *the FSNLP, problem obtained in step 4, can be rewritten as:*

$$\begin{aligned}
&\textit{Minimize}([\frac{(80+40\chi-40\eta-40\zeta+2\chi^2+2(\eta-1)^2+(\zeta-1)^2)}{12}x_1+\\
&\frac{(64+32\chi-32\eta-32\zeta+\chi^2+(\eta-1)^2+(\zeta-1)^2)}{6}x_2-\frac{11}{12}\chi^2 l_1-\frac{1}{3}\chi^2|x_1-l_1|-\frac{17}{12}\chi^2 l_2\\
&-\frac{5}{12}\chi^2|x_2-l_2|+\frac{11}{12}\chi^2 r_1+\frac{1}{3}\chi^2|x_1+r_1|+\frac{17}{12}\chi^2 r_2+\frac{5}{12}\chi^2|x_2+r_2|-\frac{11}{12}(\eta-1)^2 l_1'\\
&-\frac{5}{12}(\eta-1)^2|x_1-l_1'|-\frac{17}{12}(\eta-1)^2 l_2'-\frac{9}{12}(\eta-1)^2|x_2-l_2'|+\frac{11}{12}(\eta-1)^2 r_1'+\frac{5}{12}(\eta-1)^2|x_1+r_1'|\\
&+\frac{17}{12}(\eta-1)^2 r_2'+\frac{9}{12}(\eta-1)^2|x_2+r_2'|-\frac{21}{24}(\zeta-1)^2 l_1''-\frac{15}{24}(\zeta-1)^2|x_1-l_1''|-\frac{17}{12}(\zeta-1)^2 l_2''\\
&-\frac{13}{12}(\zeta-1)^2|x_2-l_2''|+\frac{21}{24}(\zeta-1)^2 r_1''+\frac{15}{24}(\zeta-1)^2|x_1+r_1''|+\frac{17}{12}(\zeta-1)^2 r_2''+\frac{13}{124}(\zeta-1)^2|x_2+r_2''|])
\end{aligned}$$

subject to

$$\begin{aligned}
7x_1+9x_2=87, 10x_1+11x_2=115,\\
-\frac{1}{2}x_1+\frac{15}{2}l_1+\frac{7}{2}|x_1-l_1|-\frac{x_2}{2}+\frac{19}{2}l_2+\frac{9}{2}|x_2-l_2|=56,\\
\frac{1}{2}x_1+\frac{15}{2}r_1+\frac{7}{2}|x_1+r_1|+\frac{x_2}{2}+\frac{19}{2}r_2+\frac{9}{2}|x_2+r_2|=149,\\
-x_1+8l'_1+5|x_1-l'_1|-\frac{x_2}{2}+\frac{19}{2}l'_2+\frac{13}{2}|x_2-l'_2|=75\\
x_1+8r'_1+5|x_1+r'_1|+\frac{x_2}{2}+\frac{19}{2}r'_2+\frac{13}{2}|x_2+r'_2|=216,\\
-x_1+8l''_1+6|x_1-l''_1|-\frac{x_2}{2}+\frac{19}{2}l''_2+\frac{17}{2}|x_2-l''_2|=84,\\
x_1+8r''_1+6|x_1+r''_1|+\frac{x_2}{2}+\frac{19}{2}r''_2+\frac{17}{2}|x_2+r''_2|=297,\\
\wedge[((0.6\wedge\chi_1)\wedge(0.7\wedge\chi_2)]=0.6, \vee[(0.5\vee\eta_1)\vee(0.1\vee\eta_2)]=0.5, \vee[(0.4\vee\zeta_1)\vee(0.3\vee\zeta_2)]=0.4,\\
-x_1+11l_1+5|x_1-l_1|-x_2+12l_2+5|x_2-l_2|=70,\\
x_1+11r_1+5|x_1+r_1|+x_2+12r_2+5|x_2+r_2|=198,\\
-\frac{1}{2}x_1+\frac{21}{2}l'_1+\frac{17}{2}|x_1-l'_1|-x_2+12l'_2+7|x_2-l'_2|=101,\\
\frac{1}{2}x_1+\frac{21}{2}r'_1+\frac{17}{2}|x_1+r'_1|+x_2+12r'_2+7|x_2+r'_2|=284,\\
-\frac{1}{2}x_1+\frac{21}{2}l''_1+\frac{19}{2}|x_1-l''_1|-\frac{x_2}{2}+\frac{23}{2}l''_2+\frac{19}{2}|x_2-l''_2|=112,\\
\frac{1}{2}x_1+\frac{21}{2}r''_1+\frac{19}{2}|x_1+r''_1|+\frac{x_2}{2}+\frac{23}{2}r''_2+\frac{19}{2}|x_2+r''_2|=377,\\
\wedge[(0.9\wedge\chi_1)\wedge(0.8\wedge\chi_2)]=0.7, \vee[(0.2\vee\eta_1)\vee(0.3\vee\eta_2)]=0.5, \vee[(0.1\vee\zeta_1)\vee(0.4\vee\zeta_2)]=0.4,\\
l_1\geq 0, r_1\geq 0, l'_1-l_1\geq 0, r'_1-r_1\geq 0, l''_1-l'_1\geq 0, r''_1-r'_1\geq 0, l_2\geq 0, r_2\geq 0, l'_2-l_2\geq 0, r'_2-r_2\geq 0,\\
l''_2-l'_2\geq 0, r''_2-r'_2\geq 0, \chi_1,\eta_1,\zeta_1,\chi_2,\eta_2,\zeta_2\in[0,1].
\end{aligned}$$

Step 6:*By solving the crisp mathematical problem obtained in step 5, we get the optimal solution* $x_1=6, l_1=2, r_1=4, l'_1=4, r'_1=5, l''_1=5, r''_1=6, x_2=5, l_2=2, r_2=4, l'_2=3, r'_2=5, l''_2=4, r''_2=7, \chi_1=0.7, \eta_1=0.5, \zeta_1=0.3, \chi_2=0.9, \eta_2=0.3, \zeta_2=0.4$.

Step 7: *Substituting the values of* $x_1, l_1, r_1, l'_1, r'_1, l''_1, r''_1, x_2, l_2, r_2, l'_2, r'_2, l''_2, r''_2, \chi_1, \eta_1, \zeta_1, \chi_2, \eta_2$ *and* ζ_2 *in* $X_1=([x_1; l_1, r_1; l'_1, r'_1; l''_1, r''_1]; \chi_1, \eta_1, \zeta_1)_{LR}$ *and* $X_2=([x_2; l_2, r_2; l'_2, r'_2; l''_2, r''_2]; \chi_2, \eta_2, \zeta_2)_{LR}$ *the exact* LR*-type single-valued neutrosophic optimal solution is* $X_1=([6; 2, 4; 4, 5; 5, 6]; 0.7, 0.5, 0.3)_{LR}, X_2=([5; 2, 4; 3, 5; 4, 7]; 0.9, 0.3, 0.4)_{LR}$.

Step 8: *By substituting the values of* X_1 *and* X_2*, obtained in Step 7, into the objective function, the* LR*-type single-valued neutrosophic optimal value is* $([140; 76, 208; 112, 296; 133, 436]; 0.7, 0.5, 0.5)_{LR}$.

5 Conclusion

In this paper, we have applied the concept of neutrosophic sets to the LPPs. We have defined unrestricted LR-type SNNs and their arithmetic operations. We have developed ranking function of the LR-type SNN. We have proposed a method to solve the FSNLP problems with equality constraints having unrestricted LR-type SNNs as right hand side, parameters and variables. We have solved numerical examples to explain it which satisfies the given constraints.

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