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Einstein Aggregation Operators of Simplified Neutrosophic Indeterminate Elements and Their Decision-Making Method

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Abstract: Since current decision problems are becoming more and more complex, the decision environment is becoming more and more uncertain. The simplified neutrosophic indeterminate element (SNIE) was defined to adapt to the expression of the indeterminate and inconsistent information in the indeterminate decision-making problems. SNIE consists of the truth, indeterminacy, and falsity neutrosophic numbers and can express a singled value neutrosophic element or an interval value neutrosophic element depending on the value/range of indeterminacy. In this article, we first define some operational rules of SNIEs based on the Einstein T-norm and T-conorm. Next, SNIE Einstein weighted averaging (SNIEEWA) and SNIE Einstein weighted geometric (SNIEEWG) operators are proposed to aggregate SNIEs. In view of the SNIEEWA and SNIEEWG operators, a multi-attribute decision-making (MADM) method is proposed in the case of SNIEs. Finally, the proposed MADM method is applied to solve indeterminate MADM problems in the case of SNIEs. Furthermore, the validity and effectiveness of the proposed method are verified through an illustrative example and comparative analysis.

Keywords: neutrosophic number; simplified neutrosophic indeterminate element; Einstein weighted averaging operator; Einstein weighted geometric operator

1. Introduction

The fuzzy set (FS) [1] can express a degree of truth membership, but does not express a degree of falsity membership. Therefore, an intuitionistic fuzzy set (IFS) was defined by Atanassov [2, 3], it can express the degrees of truth and falsity memberships simultaneously. Then, Atanassov and Gargov [4] introduced interval-valued IFSs (IvIFS) corresponding to the truth and falsity interval membership degrees.

Since FS, IFS, and IvIFS cannot describe inconsistent, incomplete, and indeterminate information, Smarandache [5] proposed a neutrosophic set (NS), where the truth, indeterminacy, and falsity membership degrees were described independently. Then, the three membership degrees belong to the standard interval [0, 1]/nonstandard interval $]^{-}0, 1^{+}[$. Further, Wang et al. presented a single-valued NS (SvNS) [6] and an interval-valued NS (IvNS) [7]. Next, a simplified NS (SNS) implying SvNS and IvNS introduced by Ye [8] can better apply it in real life because the truth, indeterminacy, and falsity membership degrees in SNS are described in the real unit interval [0, 1].

Since then, researches proposed various aggregation operators and decision-making (DM) methods in the cases of SvNSs and IvNSs. Meanwhile, SNSs can be used in multi-attribute decision-making (MADM) problems with SvNSs and IvNSs in the indeterminate and inconsistent situations. By combining SNS with a hesitant fuzzy notion, Liu and Shi [9] proposed single- and interval-valued neutrosophic hesitant fuzzy sets. Then, Ali et al. [10] introduced a neutrosophic cubic set by the combination of both SvNS and IvNS. The neutrosophic cubic information can represent the single- and interval-valued assessment information of decision makers, then the neutrosophic cubic MADM method [10] solved its DM problems with neutrosophic cubic information.

As another branch of neutrosophic theory, NN was first proposed by Smarandache in 1998 [11] and defined as $\delta = t + \varphi\lambda$ for indeterminacy $\lambda \in [\lambda^-, \lambda^+]$ and t, $\varphi \in \Re$, where t and $\varphi\lambda$ indicate the certain and uncertain terms of NN. Then, the NN δ is a changeable interval number $\delta = [t + \varphi\lambda^-, t + \varphi\lambda^+]$ when λ changes in the range of $\lambda \in [\lambda^-, \lambda^+]$. Therefore, NNs have been widely applied to many fields under indeterminate environment, such as optimization programming [12], mechanical fault diagnosis [13] and various DM problems [14].

With the complexity and variability of real DM problems, there may be the indeterminacy of truth, falsity, and indeterminacy degrees in indeterminate DM problems. Since SNS cannot express the indeterminacy of the three membership degrees, Du et al. [15] defined a simplified neutrosophic indeterminate set/element (SNIS/SNIE) by combining the concept of SNS with NNs, which consists of truth, indeterminacy, and falsity NNs to flexibly express the truth, falsity, and indeterminacy degrees. According to different values or ranges of $\lambda \in [\lambda^-, \lambda^+]$, SNIS can express different SvNSs or IvNSs. In [15], Du et al. proposed two weighted aggregation operators of simplified neutrosophic elements (SNEs) and established a MADM method using the SNIE weighted averaging (SNIEWA) and SNIE weighted geometric (SNIEWG) operators. Then, the Einstein T-norm and T-conorm functions [16] have been widely applied to deal with various fuzzy information [17-22], but the Einstein T-norm and T-conorm functions are not applied in the information aggregation of SNIEs. In this study, therefore, we propose the Einstein T-norm and T-conorm aggregation operators and their MADM method.

The main organization of this article is as the following. The concepts of SNS, NN, and SNIS are briefly reviewed, then the score, accuracy, and certainty functions are introduced to rank SNIEs in Section 2. The SNIE Einstein weighted averaging (SNIEEWA) and SNIE Einstein weighted geometric (SNIEEWG) operators are proposed in Section 3. Then, we put forward a MADM approach corresponding to the SNIEEWA and SNIEEWG operators in Section 4. In Section 5, the proposed MADM approach is applied to an investment selection problem of metal mines, and then its validity and flexibility are indicated by the illustrative example and comparative analysis. The last section draws conclusions and indicates future research.

2. Concepts of SNS, NN, and SNIE

Definition 1 [5, 8]. In a universe set $\tau = \{\tau_1, \tau_2, \ldots, \tau_n\}$, $N = \{<\tau_s, T(\tau_s), D(\tau_s), F(\tau_s)> \mid \tau_s \in \tau\}$ is defined as SNS, where $T(\tau_s)$, $D(\tau_s)$, $F(\tau_s) \in [0, 1]$ or $T(\tau_s)$, $D(\tau_s)$, $F(\tau_s) \subseteq [0, 1]$ ($s = 1, 2, \ldots, n$) are the truth, indeterminacy and falsity membership functions of τ_s to N. The component $<\tau_s, T(\tau_s), D(\tau_s), F(\tau_s)>$ in N is called SNE and is simply denoted as $N_s = <T_s, D_s, F_s>$, which includes the SvNE $N_s = <T_s, D_s, F_s>$ for $T_s, D_s, F_s \in [0, 1]$ and the IvNE $N_s = <[T_s^L, T_s^U], [D_s^L, D_s^U], [F_s^L, F_s^U]>$ for $[T_s^L, T_s^U]$, $[D_s^L, D_s^U]$, $[D_s^L, D_s^U] \subseteq [0, 1]$.

Definition 2 [11]. NN is described as $\delta = t + \varphi\lambda$ for indeterminacy $\lambda \in [\lambda^-, \lambda^+]$ and t, $\varphi \in \Re$, where t and $\varphi\lambda$ describe the certain term and uncertain term of NN, respectively.

Clearly, $\delta = t + \varphi\lambda$ is a changeable interval number $\delta = [t + \varphi\lambda^-, t + \varphi\lambda^+]$ when λ changes in range of $\lambda \in [\lambda^-, \lambda^+]$. It indicates that NN can flexibly express a single value or an indeterminate interval value according to the value/range of $\lambda \in [\lambda^-, \lambda^+]$.

Definition 3 [15]. For a universe set $\tau = \{\tau_1, \tau_2, \ldots, \tau_n\}$, SNIS is defined as $\chi = \{<\tau_s, T(\tau_s, \lambda), D(\tau_s, \lambda), F(\tau_s, \lambda)> \mid \tau_s \in \tau\}$, where $T(\tau_s, \lambda) = t_s + \alpha_s\lambda \subseteq [0, 1]$, $D(\tau_s, \lambda) = d_s + \beta_s\lambda \subseteq [0, 1]$, and $F(\tau_s, \lambda) = f_s + \gamma_s\lambda \subseteq [0,$

1] (s = 1, 2, …, n) for $\lambda \in [\lambda^-, \lambda^+]$ are the truth, indeterminacy, and falsity membership functions in χ. Then, χ_s = <$T(\tau_s, \lambda)$, $D(\tau_s, \lambda)$, $F(\tau_s, \lambda)$> = <$t_s + \alpha_s\lambda$, $d_s + \beta_s\lambda$, $f_s + \gamma_s\lambda$> is called SNIE and is simply denoted as χ_s = <$T_s(\lambda)$, $D_s(\lambda)$, $F_s(\lambda)$> = <$t_s + \alpha_s\lambda$, $d_s + \beta_s\lambda$, $f_s + \gamma_s\lambda$> for $\lambda \in [\lambda^-, \lambda^+]$.

In order to rank different SNIEs, Du et al. [15] defined three functions to compare two SNIEs.

Let χ_s = <$T_s(\lambda)$, $D_s(\lambda)$, $F_s(\lambda)$> = <$t_s + \alpha_s\lambda$, $d_s + \beta_s\lambda$, $f_s + \gamma_s\lambda$> for $\lambda \in [\lambda^-, \lambda^+]$ be any SNIE and $\lambda^* = \lambda^- + \lambda^+$. Thus, Du et al. [15] defined its score, accuracy, and certainty functions:

$$\begin{aligned} S(\chi_s,\lambda) &= \left\{4+T_s(\lambda^-)+T_s(\lambda^+)-D_s(\lambda^-)-D_s(\lambda^+)-F_s(\lambda^-)-F_s(\lambda^+)\right\}/6 \\ &= \left\{4+2t_s-2d_s-2f_s+(\alpha_s-\beta_s-\gamma_s)\lambda^*)\right\}/6 \end{aligned}, \quad S(\chi_s,\lambda)\in[0,1]; \tag{1}$$

$$\begin{aligned} L(\chi_s,\lambda) &= \left\{T_s(\lambda^-)+T_s(\lambda^+)-F_s(\lambda^-)-F_s(\lambda^+)\right\}/2 \\ &= \left\{2t_s-2f_s+(\alpha_s-\gamma_s)\lambda^*\right\}/2 \end{aligned}, \quad L(\chi_s,\lambda)\in[-1,1]; \tag{2}$$

$$C(\chi_s,\lambda) = \left\{T_s(\lambda^-)+T_s(\lambda^+)\right\}/2 = (2t_s+\lambda^*)/2, \quad C(\chi_s,\lambda)\in[0,1]. \tag{3}$$

Suppose that χ_1 = <$T_1(\lambda)$, $D_1(\lambda)$, $F_1(\lambda)$> = <$t_1 + \alpha_1\lambda$, $d_1 + \beta_1\lambda$, $f_1 + \gamma_1\lambda$> and χ_2 = <$T_2(\lambda)$, $D_2(\lambda)$, $F_2(\lambda)$> = <$t_2 + \alpha_2\lambda$, $d_2 + \beta_2\lambda$, $f_2 + \gamma_2\lambda$> for $\lambda \in [\lambda^-, \lambda^+]$ are two SNIEs. We can rank them by the defined three functions. According to their priority, the ranking method is given as follows:

1) If $S(\chi_1, \lambda) > S(\chi_2, \lambda)$, then $\chi_1 > \chi_2$;
2) If $S(\chi_1, \lambda) = S(\chi_2, \lambda)$ and $L(\chi_1, \lambda) > L(\chi_2, \lambda)$, then $\chi_1 > \chi_2$;
3) If $S(\chi_1, \lambda) = S(\chi_2, \lambda)$, $L(\chi_1, \lambda) = L(\chi_2,\lambda)$ and $C(\chi_1, \lambda) > C(\chi_2, \lambda)$, then $\chi_1 > \chi_2$;
4) If $S(\chi_1, \lambda) = S(\chi_2, \lambda)$, $L(\chi_1, \lambda) = L(\chi_2, \lambda)$ and $C(\chi_1, \lambda) = C(\chi_2, \lambda)$, then $\chi_1 = \chi_2$.

3.The Einstein Aggregation Operations of SNIEs

3.1. Einstein T-norm and T-conorm operations of SNIEs

Definition 4 [16]. If θ and ϕ are real numbers, the Einstein T-norm function $T(\theta, \phi)$ and T-conorm $T^C(\theta, \phi)$ for $(\theta, \phi) \in [0,1]\times[0,1]$ are defined as the following formulae:

$$T(\theta,\phi)=\frac{\theta\phi}{1+(1-\theta)(1-\phi)} \tag{4}$$

$$T^C(\theta,\phi)=\frac{\theta+\phi}{1+\theta\phi} \tag{5}$$

The above functions are increasing strictly and satisfy $T(\theta, \phi)$, $T^C(\theta, \phi) \in [0, 1]$.

According to Eqs. (4) and (5), some operations of SNIEs are defined as follows.

Definition 5. Suppose that χ_1 = <$T_1(\lambda)$, $D_1(\lambda)$, $F_1(\lambda)$> = <$t_1 + \alpha_1\lambda$, $d_1 + \beta_1\lambda$, $f_1 + \gamma_1\lambda$> and χ_2 = <$T_2(\lambda)$, $D_2(\lambda)$, $F_2(\lambda)$> = <$t_2 + \alpha_2\lambda$, $d_2 + \beta_2\lambda$, $f_2 + \gamma_2\lambda$> are two SNIEs for $t_1 + \alpha_1\lambda$, $d_1 + \beta_1\lambda$, $f_1 + \gamma_1\lambda$, $t_2 + \alpha_2\lambda$, $d_2 + \beta_2\lambda$, $f_2 + \gamma_2\lambda \subseteq [0, 1]$ and $\lambda \in [\lambda^-, \lambda^+]$.

$$\chi_1\oplus\chi_2=\left\langle \begin{array}{l} \left[\dfrac{(t_1+\alpha_1\lambda^-)+(t_2+a_2\lambda^-)}{1+(t_1+\alpha_1\lambda^-)(t_2+\alpha_2\lambda^-)},\dfrac{(t_1+\alpha_1\lambda^+)+(t_2+a_2\lambda^+)}{1+(t_1+\alpha_1\lambda^+)(t_2+\alpha_2\lambda^+)}\right],\left[\dfrac{(d_1+\beta_1\lambda^-)(d_2+\beta_2\lambda^-)}{1+(1-(d_1+\beta_1\lambda^-))(1-(d_2+\beta_2\lambda^-))},\right. \\ \left.\dfrac{(d_1+\beta_1\lambda^+)(d_2+\beta_2\lambda^+)}{1+(1-(d_1+\beta_1\lambda^+))(1-(d_2+\beta_2\lambda^+))}\right],\left[\dfrac{(f_1+\gamma_1\lambda^-)(f_2+\gamma_2\lambda^-)}{1+(1-(f_1+\gamma_1\lambda^-))(1-(f_2+\gamma_2\lambda^-))},\dfrac{(f_1+\gamma_1\lambda^+)(f_2+\gamma_2\lambda^+)}{1+(1-(f_1+\gamma_1\lambda^+))(1-(f_2+\gamma_2\lambda^+))}\right] \end{array}\right\rangle; \tag{6}$$

$$\chi_1\otimes\chi_2=\left\langle \begin{array}{l} \left[\dfrac{(t_1+\alpha_1\lambda^-)(t_2+\alpha_2\lambda^-)}{1+(1-(t_1+\alpha_1\lambda^-))(1-(t_2+\alpha_2\lambda^-))},\dfrac{(t_1+\alpha_1\lambda^+)(t_2+\alpha_2\lambda^+)}{1+(1-(t_1+\alpha_1\lambda^+))(1-(t_2+\alpha_2\lambda^+))}\right],\left[\dfrac{(d_1+\beta_1\lambda^-)+(d_2+\beta_2\lambda^-)}{1+(d_1+\beta_1\lambda^-)(d_2+\beta_2\lambda^-)},\right. \\ \left.\dfrac{(d_1+\beta_1\lambda^+)+(d_2+\beta_2\lambda^+)}{1+(d_1+\beta_1\lambda^+)(d_2+\beta_2\lambda^+)}\right],\left[\dfrac{(f_1+\gamma_1\lambda^-)+(f_2+\gamma_2\lambda^-)}{1+(f_1+\gamma_1\lambda^-)(f_2+\gamma_2\lambda^-)},\dfrac{(f_1+\gamma_1\lambda^+)+(f_2+\gamma_2\lambda^+)}{1+(f_1+\gamma_1\lambda^+)(f_2+\gamma_2\lambda^+)}\right] \end{array}\right\rangle; \tag{7}$$

$$\rho\chi_1=\left\langle\begin{array}{l}\left[\frac{(1+(t_1+\alpha_1\lambda^-))^\rho-(1-(t_1+\alpha_1\lambda^-))^\rho}{(1+(t_1+\alpha_1\lambda^-))^\rho+(1-(t_1+\alpha_1\lambda^-))^\rho},\frac{(1+(t_1+\alpha_1\lambda^+))^\rho-(1-(t_1+\alpha_1\lambda^+))^\rho}{(1+(t_1+\alpha_1\lambda^+))^\rho+(1-(t_1+\alpha_1\lambda^+))^\rho}\right],\left[\frac{2(d_1+\beta_1\lambda^-)^\rho}{(2-(d_1+\beta_1\lambda^-))^\rho+(d_1+\beta_1\lambda^-)^\rho},\right.\\ \left.\frac{2(d_1+\beta_1\lambda^+)^\rho}{(2-(d_1+\beta_1\lambda^+))^\rho+(d_1+\beta_1\lambda^+)^\rho}\right],\left[\frac{2(f_1+\gamma_1\lambda^-)^\rho}{(2-(f_1+\gamma_1\lambda^-))^\rho+(f_1+\gamma_1\lambda^-)^\rho},\frac{2(f_1+\gamma_1\lambda^+)^\rho}{(2-(f_1+\gamma_1\lambda^+))^\rho+(f_1+\gamma_1\lambda^+)^\rho}\right]\end{array}\right\rangle; \quad (8)$$

$$\chi_1^{\rho}=\left\langle\begin{array}{l}\left[\frac{2(t_1+a_1\lambda^-)^\rho}{(2-(t_1+a_1\lambda^-))^\rho+(t_1+a_1\lambda^-)^\rho},\frac{2(t_1+a_1\lambda^+)^\rho}{(2-(t_1+a_1\lambda^+))^\rho+(t_1+a_1\lambda^+)^\rho}\right],\\ \left[\frac{(1+(d_1+\beta_1\lambda^-))^\rho-(1-(d_1+\beta_1\lambda^-))^\rho}{(1+(d_1+\beta_1\lambda^-))^\rho+(1-(d_1+\beta_1\lambda^-))^\rho},\frac{(1+(d_1+\beta_1\lambda^+))^\rho-(1-(d_1+\beta_1\lambda^+))^\rho}{(1+(d_1+\beta_1\lambda^+))^\rho+(1-(d_1+\beta_1\lambda^+))^\rho}\right],\\ \left[\frac{(1+(f_1+\gamma_1\lambda^-))^\rho-(1-(f_1+\gamma_1\lambda^-))^\rho}{(1+(f_1+\gamma_1\lambda^-))^\rho+(1-(f_1+\gamma_1\lambda^-))^\rho},\frac{(1+(f_1+\gamma_1\lambda^+))^\rho-(1-(f_1+\gamma_1\lambda^+))^\rho}{(1+(f_1+\gamma_1\lambda^+))^\rho+(1-(f_1+\gamma_1\lambda^+))^\rho}\right]\end{array}\right\rangle. \quad (9)$$

3.2. Einstein Weighted Arithmetic Average Operator of SNIEs

Definition 6. Let $\chi = \{\chi_1, \chi_2, \ldots, \chi_n\}$ be SNIS, we can define the SNIEEWA operator:

$$SNIEEWA\left(\chi_1,\chi_2,\ldots,\chi_n\right)=\overset{n}{\underset{k=1}{\oplus}}\rho_k\chi_k, \quad (10)$$

where $\rho_k \in [0, 1]$ is the weight of χ_k for $\sum_{k=1}^{n}\rho_k = 1$.

Theorem 1. Let $\chi_k = \langle T_k(\lambda), D_k(\lambda), F_k(\lambda)\rangle = \langle t_k + \alpha_k\lambda, d_k + \beta_k\lambda, f_k + \gamma_k\lambda\rangle$ ($k = 1, 2, \ldots, n$) for $\lambda \in [\lambda^-, \lambda^+]$ be a group of SNIEs with the related weights $\rho_k \in [0, 1]$ for $\sum_{k=1}^{n}\rho_k = 1$. Thus, the Eq. (10) can be calculated by the following equation:

$$SNIEEWA(\chi_1,\chi_2,\ldots,\chi_n)=\left\langle\begin{array}{l}\left[\frac{\prod_{k=1}^n(1+t_k+\alpha_k\lambda^-)^{\rho_k}-\prod_{k=1}^n(1-t_k-\alpha_k\lambda^-)^{\rho_k}}{\prod_{k=1}^n(1+t_k+\alpha_k\lambda^-)^{\rho_k}+\prod_{k=1}^n(1-t_k-\alpha_k\lambda^-)^{\rho_k}},\frac{\prod_{k=1}^n(1+t_k+\alpha_k\lambda^+)^{\rho_k}-\prod_{k=1}^n(1-t_k-\alpha_k\lambda^+)^{\rho_k}}{\prod_{k=1}^n(1+t_k+\alpha_k\lambda^+)^{\rho_k}+\prod_{k=1}^n(1-t_k-\alpha_k\lambda^+)^{\rho_k}}\right],\\ \left[\frac{2\prod_{k=1}^n(d_k+\beta_k\lambda^-)^{\rho_k}}{\prod_{k=1}^n(2-d_k-\beta_k\lambda^-)^{\rho_k}+\prod_{k=1}^n(d_k+\beta_k\lambda^-)^{\rho_k}},\frac{2\prod_{k=1}^n(d_k+\beta_k\lambda^+)^{\rho_k}}{\prod_{k=1}^n(2-d_k-\beta_k\lambda^+)^{\rho_k}+\prod_{k=1}^n(d_k+\beta_k\lambda^+)^{\rho_k}}\right],\\ \left[\frac{2\prod_{k=1}^n(f_k+\gamma_k\lambda^-)^{\rho_k}}{\prod_{k=1}^n(2-f_k-\gamma_k\lambda^-)^{\rho_k}+\prod_{k=1}^n(f_k+\gamma_k\lambda^-)^{\rho_k}},\frac{2\prod_{k=1}^n(f_k+\gamma_k\lambda^+)^{\rho_k}}{\prod_{k=1}^n(2-f_k-\gamma_k\lambda^+)^{\rho_k}+\prod_{k=1}^n(f_k+\gamma_k\lambda^+)^{\rho_k}}\right]\end{array}\right\rangle. \quad (11)$$

Proof:

(1) If $n = 2$, by Eqs. (6) and (8), we can get the result:

$SNIEEWA(\chi_1,\chi_2)=\rho_1\chi_1\oplus\rho_2\chi_2$

$$=\left\langle\begin{array}{l}\left[\frac{(1+T_1(\lambda^-))^{\rho_1}-(1-T_1(\lambda^-))^{\rho_1}}{(1+T_1(\lambda^-))^{\rho_1}+(1-T_1(\lambda^-))^{\rho_1}},\frac{(1+T_1(\lambda^+))^{\rho_1}-(1-T_1(\lambda^+))^{\rho_1}}{(1+T_1(\lambda^+))^{\rho_1}+(1-T_1(\lambda^+))^{\rho_1}}\right],\left[\frac{2D_1(\lambda^-)^{\rho_1}}{(2-D_1(\lambda^-))^{\rho_1}+D_1(\lambda^-)^{\rho_1}},\right.\\ \left.\frac{2D_1(\lambda^+)^{\rho_1}}{(2-D_1(\lambda^+))^{\rho_1}+D_1(\lambda^+)^{\rho_1}}\right],\left[\frac{2F_1(\lambda^-)^{\rho_1}}{(2-F_1(\lambda^-))^{\rho_1}+F_1(\lambda^-)^{\rho_1}},\frac{2F_1(\lambda^+)^{\rho_1}}{(2-F_1(\lambda^+))^{\rho_1}+F_1(\lambda^+)^{\rho_1}}\right]\end{array}\right\rangle$$

$$\oplus\left\langle\begin{array}{l}\left[\dfrac{(1+T_2(\lambda^-))^{\rho_2}-(1-T_2(\lambda^-))^{\rho_2}}{(1+T_2(\lambda^-))^{\rho_2}+(1-T_2(\lambda^-))^{\rho_2}},\dfrac{(1+T_2(\lambda^+))^{\rho_2}-(1-T_2(\lambda^+))^{\rho_2}}{(1+T_2(\lambda^+))^{\rho_2}+(1-T_2(\lambda^+))^{\rho_2}}\right],\left[\dfrac{2D_2(\lambda^-)^{\rho_2}}{(2-D_2(\lambda^-))^{\rho_2}+D_2(\lambda^-)^{\rho_2}},\right.\\ \left.\dfrac{2D_2(\lambda^+)^{\rho_2}}{(2-D_2(\lambda^+))^{\rho_2}+D_2(\lambda^+)^{\rho_2}}\right],\left[\dfrac{2F_2(\lambda^-)^{\rho_2}}{(2-F_2(\lambda^-))^{\rho_2}+F_2(\lambda^-)^{\rho_2}},\dfrac{2F_2(\lambda^+)^{\rho_2}}{(2-F_2(\lambda^+))^{\rho_2}+F_1(\lambda^+)^{\rho_2}}\right]\end{array}\right\rangle$$

$$=\left\langle\begin{array}{l}\left[\dfrac{(1+T_1(\lambda^-))^{\rho_1}(1+T_2(\lambda^-))^{\rho_2}-(1-T_1(\lambda^-))^{\rho_1}(1-T_2(\lambda^-))^{\rho_2}}{(1+T_1(\lambda^-))^{\rho_1}(1+T_2(\lambda^-))^{\rho_2}+(1-T_1(\lambda^-))^{\rho_1}(1-T_2(\lambda^-))^{\rho_2}},\dfrac{(1+T_1(\lambda^+))^{\rho_1}(1+T_2(\lambda^+))^{\rho_2}-(1-T_1(\lambda^+))^{\rho_1}(1-T_2(\lambda^+))^{\rho_2}}{(1+T_1(\lambda^+))^{\rho_1}(1+T_2(\lambda^+))^{\rho_2}+(1-T_1(\lambda^+))^{\rho_1}(1-T_2(\lambda^+))^{\rho_2}}\right],\\ \left[\dfrac{2D_1(\lambda^-)^{\rho_1}D_2(\lambda^-)^{\rho_2}}{(2-D_1(\lambda^-))^{\rho_1}(2-D_2(\lambda^-))^{\rho_2}+D_1(\lambda^-)^{\rho_1}D_2(\lambda^-)^{\rho_2}},\dfrac{2D_1(\lambda^+)^{\rho_1}D_2(\lambda^+)^{\rho_2}}{(2-D_1(\lambda^+))^{\rho_1}(2-D_2(\lambda^+))^{\rho_2}+D_1(\lambda^+)^{\rho_1}D_2(\lambda^+)^{\rho_2}}\right],\\ \left[\dfrac{2F_1(\lambda^-)^{\rho_1}F_2(\lambda^-)^{\rho_2}}{(2-F_1(\lambda^-))^{\rho_1}((2-F_2(\lambda^-))^{\rho_2}+F_1(\lambda^-)^{\rho_1}F_2(\lambda^-)^{\rho_2}},\dfrac{2F_1(\lambda^+)^{\rho_1}F_2(\lambda^+)^{\rho_2}}{(2-F_1(\lambda^+))^{\rho_1}((2-F_2(\lambda^+))^{\rho_2}+F_1(\lambda^+)^{\rho_1}F_2(\lambda^+)^{\rho_2}}\right]\end{array}\right\rangle$$

$$=\left\langle\begin{array}{l}\left[\dfrac{\prod_{k=1}^{2}(1+T_k(\lambda^-))^{\rho_k}-\prod_{k=1}^{2}(1-T_k(\lambda^-))^{\rho_k}}{\prod_{k=1}^{2}(1+T_k(\lambda^-))^{\rho_k}+\prod_{k=1}^{2}(1-T_k(\lambda^-))^{\rho_k}},\dfrac{\prod_{k=1}^{2}(1+T_k(\lambda^+))^{\rho_k}-\prod_{k=1}^{2}(1-T_k(\lambda^+))^{\rho_k}}{\prod_{k=1}^{2}(1+T_k(\lambda^+))^{\rho_k}+\prod_{k=1}^{2}(1-T_k(\lambda^+))^{\rho_k}}\right],\left[\dfrac{2\prod_{k=1}^{2}D_k(\lambda^-)^{\rho_k}}{\prod_{k=1}^{2}(2-D_k(\lambda^-))^{\rho_k}+\prod_{k=1}^{2}D_k(\lambda^-)^{\rho_k}},\right.\\ \left.\dfrac{2\prod_{k=1}^{2}D_k(\lambda^+)^{\rho_k}}{\prod_{k=1}^{2}(2-D_k(\lambda^+))^{\rho_k}+\prod_{k=1}^{2}D_k(\lambda^+)^{\rho_k}}\right],\left[\dfrac{2\prod_{k=1}^{2}F_k(\lambda^-)^{\rho_k}}{\prod_{k=1}^{2}(2-F_k(\lambda^-))^{\rho_k}+\prod_{k=1}^{2}F_k(\lambda^-)^{\rho_k}},\dfrac{2\prod_{k=1}^{2}F_k(\lambda^+)^{\rho_k}}{\prod_{k=1}^{2}(2-F_k(\lambda^+))^{\rho_k}+\prod_{k=1}^{2}F_k(\lambda^+)^{\rho_k}}\right]\end{array}\right\rangle$$

$$=\left\langle\begin{array}{l}\left[\dfrac{\prod_{k=1}^{2}(1+t_k+\alpha_k\lambda^-)^{\rho_k}-\prod_{k=1}^{2}(1-t_k-\alpha_k\lambda^-)^{\rho_k}}{\prod_{k=1}^{2}(1+t_k+\alpha_k\lambda^-)^{\rho_k}+\prod_{k=1}^{2}(1-t_k-\alpha_k\lambda^-)^{\rho_k}},\dfrac{\prod_{k=1}^{2}(1+t_k+\alpha_k\lambda^+)^{\rho_k}-\prod_{k=1}^{2}(1-t_k-\alpha_k\lambda^+)^{\rho_k}}{\prod_{k=1}^{2}(1+t_k+\alpha_k\lambda^+)^{\rho_k}+\prod_{k=1}^{2}(1-t_k-\alpha_k\lambda^+)^{\rho_k}}\right],\\ \left[\dfrac{2\prod_{k=1}^{2}(d_k+\beta_k\lambda^-)^{\rho_k}}{\prod_{k=1}^{2}(2-d_k-\beta_k\lambda^-)^{\rho_k}+\prod_{k=1}^{2}(d_k+\beta_k\lambda^-)^{\rho_k}},\dfrac{2\prod_{k=1}^{2}(d_k+\beta_k\lambda^+)^{\rho_k}}{\prod_{k=1}^{2}(2-d_k-\beta_k\lambda^+)^{\rho_k}+\prod_{k=1}^{2}(d_k+\beta_k\lambda^+)^{\rho_k}}\right],\\ \left[\dfrac{2\prod_{k=1}^{2}(f_k+\gamma_k\lambda^-)^{\rho_k}}{\prod_{k=1}^{2}(2-f_k-\gamma_k\lambda^-)^{\rho_k}+\prod_{k=1}^{2}(f_k+\gamma_k\lambda^-)^{\rho_k}},\dfrac{2\prod_{k=1}^{2}(f_k+\gamma_k\lambda^+)^{\rho_k}}{\prod_{k=1}^{2}(2-f_k-\gamma_k\lambda^+)^{\rho_k}+\prod_{k=1}^{2}(f_k+\gamma_k\lambda^+)^{\rho_k}}\right]\end{array}\right\rangle.$$

(2) Set $n = m$. Then, the following formula can hold:

$$SNIEEWA(\chi_1,\chi_2,...,\chi_m)=\left\langle\begin{array}{l}\left[\dfrac{\prod_{k=1}^{m}(1+t_k+\alpha_k\lambda^-)^{\rho_k}-\prod_{k=1}^{m}(1-t_k-\alpha_k\lambda^-)^{\rho_k}}{\prod_{k=1}^{m}(1+t_k+\alpha_k\lambda^-)^{\rho_k}+\prod_{k=1}^{m}(1-t_k-\alpha_k\lambda^-)^{\rho_k}},\dfrac{\prod_{k=1}^{m}(1+t_k+\alpha_k\lambda^+)^{\rho_k}-\prod_{k=1}^{m}(1-t_k-\alpha_k\lambda^+)^{\rho_k}}{\prod_{k=1}^{m}(1+t_k+\alpha_k\lambda^+)^{\rho_k}+\prod_{k=1}^{m}(1-t_k-\alpha_k\lambda^+)^{\rho_k}}\right],\\ \left[\dfrac{2\prod_{k=1}^{m}(d_k+\beta_k\lambda^-)^{\rho_k}}{\prod_{k=1}^{m}(2-d_k-\beta_k\lambda^-)^{\rho_k}+\prod_{k=1}^{m}(d_k+\beta_k\lambda^-)^{\rho_k}},\dfrac{2\prod_{k=1}^{m}(d_k+\beta_k\lambda^+)^{\rho_k}}{\prod_{k=1}^{m}(2-d_k-\beta_k\lambda^+)^{\rho_k}+\prod_{k=1}^{m}(d_k+\beta_k\lambda^+)^{\rho_k}}\right],\\ \left[\dfrac{2\prod_{k=1}^{m}(f_k+\gamma_k\lambda^-)^{\rho_k}}{\prod_{k=1}^{m}(2-f_k-\gamma_k\lambda^-)^{\rho_k}+\prod_{k=1}^{m}(f_k+\gamma_k\lambda^-)^{\rho_k}},\dfrac{2\prod_{k=1}^{m}(f_k+\gamma_k\lambda^+)^{\rho_k}}{\prod_{k=1}^{m}(2-f_k-\gamma_k\lambda^+)^{\rho_k}+\prod_{k=1}^{m}(f_k+\gamma_k\lambda^+)^{\rho_k}}\right]\end{array}\right\rangle. \quad (12)$$

(3) If $n = m + 1$, according to the formulae (6), (8) and (12), we can get

$$SNIEEWA(\chi_1,\chi_2,...,\chi_{m+1})=SNIEEWA(\chi_1,\chi_2,...,\chi_m)\oplus\rho_{m+1}\chi_{m+1}$$

$$=\left\langle\begin{array}{l}\left[\frac{\prod_{k=1}^{m}(1+t_k+\alpha_k\lambda^-)^{\rho_k}-\prod_{k=1}^{m}(1-t_k-\alpha_k\lambda^-)^{\rho_k}}{\prod_{k=1}^{m}(1+t_k+\alpha_k\lambda^-)^{\rho_k}+\prod_{k=1}^{m}(1-t_k-\alpha_k\lambda^-)^{\rho_k}},\frac{\prod_{k=1}^{m}(1+t_k+\alpha_k\lambda^+)^{\rho_k}-\prod_{k=1}^{m}(1-t_k-\alpha_k\lambda^+)^{\rho_k}}{\prod_{k=1}^{m}(1+t_k+\alpha_k\lambda^+)^{\rho_k}+\prod_{k=1}^{m}(1-t_k-\alpha_k\lambda^+)^{\rho_k}}\right],\\ \left[\frac{2\prod_{k=1}^{m}(d_k+\beta_k\lambda^-)^{\rho_k}}{\prod_{k=1}^{m}(2-d_k-\beta_k\lambda^-)^{\rho_k}+\prod_{k=1}^{m}(d_k+\beta_k\lambda^-)^{\rho_k}},\frac{2\prod_{k=1}^{m}(d_k+\beta_k\lambda^+)^{\rho_k}}{\prod_{k=1}^{m}(2-d_k-\beta_k\lambda^+)^{\rho_k}+\prod_{k=1}^{m}(d_k+\beta_k\lambda^+)^{\rho_k}}\right],\\ \left[\frac{2\prod_{k=1}^{m}(f_k+\gamma_k\lambda^-)^{\rho_k}}{\prod_{k=1}^{m}(2-f_k-\gamma_k\lambda^-)^{\rho_k}+\prod_{k=1}^{m}(f_k+\gamma_k\lambda^-)^{\rho_k}},\frac{2\prod_{k=1}^{m}(f_k+\gamma_k\lambda^+)^{\rho_k}}{\prod_{k=1}^{m}(2-f_k-\gamma_k\lambda^+)^{\rho_k}+\prod_{k=1}^{m}(f_k+\gamma_k\lambda^+)^{\rho_k}}\right]\end{array}\right\rangle$$

$$\oplus\left\langle\begin{array}{l}\left[\frac{(1+(t_{m+1}+\alpha_{m+1}\lambda^-))^{\rho_{m+1}}-(1-(t_{m+1}+\alpha_{m+1}\lambda^-))^{\rho_{m+1}}}{(1+(t_{m+1}+\alpha_{m+1}\lambda^-))^{\rho_{m+1}}+(1-(t_{m+1}+\alpha_{m+1}\lambda^-))^{\rho_{m+1}}},\frac{(1+(t_{m+1}+\alpha_{m+1}\lambda^+))^{\rho_{m+1}}-(1-(t_{m+1}+\alpha_{m+1}\lambda^+))^{\rho_{m+1}}}{(1+(t_{m+1}+\alpha_{m+1}\lambda^+))^{\rho_{m+1}}+(1-(t_{m+1}+\alpha_{m+1}\lambda^+))^{\rho_{m+1}}}\right],\\ \left[\frac{2(d_{m+1}+\beta_{m+1}\lambda^-)^{\rho_{m+1}}}{(2-(d_{m+1}+\beta_{m+1}\lambda^-))^{\rho_{m+1}}+(d_{m+1}+\beta_{m+1}\lambda^-)^{\rho_{m+1}}},\frac{2(d_{m+1}+\beta_{m+1}\lambda^+)^{\rho_{m+1}}}{(2-(d_{m+1}+\beta_{m+1}\lambda^+))^{\rho_{m+1}}+(d_{m+1}+\beta_{m+1}\lambda^+)^{\rho_{m+1}}}\right],\\ \left[\frac{2(f_{m+1}+\gamma_{m+1}\lambda^-)^{\rho_{m+1}}}{(2-(f_{m+1}+\gamma_{m+1}\lambda^-))^{\rho_{m+1}}+(f_{m+1}+\gamma_{m+1}\lambda^-)^{\rho_{m+1}}},\frac{2(f_{m+1}+\gamma_{m+1}\lambda^+)^{\rho_{m+1}}}{(2-(f_{m+1}+\gamma_{m+1}\lambda^+))^{\rho_{m+1}}+(f_{m+1}+\gamma_{m+1}\lambda^+)^{\rho_{m+1}}}\right]\end{array}\right\rangle$$

$$=\left\langle\begin{array}{l}\left[\frac{\prod_{k=1}^{m+1}(1+t_k+\alpha_k\lambda^-)^{\rho_k}-\prod_{k=1}^{m+1}(1-t_k-\alpha_k\lambda^-)^{\rho_k}}{\prod_{k=1}^{m+1}(1+t_k+\alpha_k\lambda^-)^{\rho_k}+\prod_{k=1}^{m+1}(1-t_k-\alpha_k\lambda^-)^{\rho_k}},\frac{\prod_{k=1}^{m+1}(1+t_k+\alpha_k\lambda^+)^{\rho_k}-\prod_{k=1}^{m+1}(1-t_k-\alpha_k\lambda^+)^{\rho_k}}{\prod_{k=1}^{m+1}(1+t_k+\alpha_k\lambda^+)^{\rho_k}+\prod_{k=1}^{m+1}(1-t_k-\alpha_k\lambda^+)^{\rho_k}}\right],\\ \left[\frac{2\prod_{k=1}^{m+1}(d_k+\beta_k\lambda^-)^{\rho_k}}{\prod_{k=1}^{m+1}(2-d_k-\beta_k\lambda^-)^{\rho_k}+\prod_{k=1}^{m+1}(d_k+\beta_k\lambda^-)^{\rho_k}},\frac{2\prod_{k=1}^{m+1}(d_k+\beta_k\lambda^+)^{\rho_k}}{\prod_{k=1}^{m+1}(2-d_k-\beta_k\lambda^+)^{\rho_k}+\prod_{k=1}^{m+1}(d_k+\beta_k\lambda^+)^{\rho_k}}\right],\\ \left[\frac{2\prod_{k=1}^{m+1}(f_k+\gamma_k\lambda^-)^{\rho_k}}{\prod_{k=1}^{m+1}(2-f_k-\gamma_k\lambda^-)^{\rho_k}+\prod_{k=1}^{m+1}(f_k+\gamma_k\lambda^-)^{\rho_k}},\frac{2\prod_{k=1}^{m+1}(f_k+\gamma_k\lambda^+)^{\rho_k}}{\prod_{k=1}^{m+1}(2-f_k-\gamma_k\lambda^+)^{\rho_k}+\prod_{k=1}^{m+1}(f_k+\gamma_k\lambda^+)^{\rho_k}}\right]\end{array}\right\rangle.$$

Thus, we have proved that the Eq. (11) can hold for any k.

The SNIEEWA operator implies the following properties.

(P1) Idempotency: Set χ_k = <$T_k(\lambda)$, $D_k(\lambda)$, $F_k(\lambda)$> = <$t_k + \alpha_k\lambda$, $d_k + \beta_k\lambda$, $f_k + \gamma_k\lambda$> as a group of SNIEs for $\lambda\in[\lambda^-, \lambda^+]$ and k = 1, 2, …, n. If $\chi_k = \chi$, then $SNIEEWA(\chi_1, \chi_2, \ldots, \chi_n) = \chi$.

(P2) Boundedness: Set χ_k = <$T_k(\lambda)$, $D_k(\lambda)$, $F_k(\lambda)$> = <$t_k + \alpha_k\lambda$, $d_k + \beta_k\lambda$, $f_k + \gamma_k\lambda$> as a group of SNIEs for $\lambda\in[\lambda^-, \lambda^+]$ and k = 1, 2, …, n. Let the minimum and maximum SNIEs be

$$\chi_{\min}=\left\langle\left[\min_k(t_k+\alpha_k\lambda^-),\min_k(t_k+\alpha_k\lambda^+)\right],\left[\max_k(d_k+\beta_k\lambda^-),\max_k(d_k+\beta_k\lambda^+)\right],\left[\max_k(f_k+\gamma_k\lambda^-),\max_k(f_k+\gamma_k\lambda^+)\right]\right\rangle,$$

$$\chi_{\max}=\left\langle\left[\max_k(t_k+\alpha_k\lambda^-),\max_k(t_k+\alpha_k\lambda^+)\right],\left[\min_k(d_k+\beta_k\lambda^-),\min_k(d_k+\beta_k\lambda^+)\right],\left[\min_k(f_k+\gamma_k\lambda^-),\min_k(f_k+\gamma_k\lambda^+)\right]\right\rangle.$$

Then, there is $\chi_{\min} \le SNIEEWA(\chi_1, \chi_2, \ldots, \chi_n) \le \chi_{\max}$.

(P3) Monotonicity: Set χ_k = <$T_k(\lambda)$, $D_k(\lambda)$, $F_k(\lambda)$> and χ_k^*= <$T_k^*(\lambda)$, $D_k^*(\lambda)$, $F_k^*(\lambda)$> as two groups of SNIEs for $\lambda\in[\lambda^-, \lambda^+]$ and k = 1, 2, …, n. If $\chi_k \subseteq \chi_k^*$, then $SNIEEWA(\chi_1, \chi_2, \ldots, \chi_n) \subseteq SNIEEWA(\chi_1^*, \chi_2^*, \ldots, \chi_n^*)$.

Next, we give proofs of the three properties.

Proof：

(P1) Let χ_k = <$t_k + \alpha_k\lambda$, $d_k + \beta_k\lambda$, $f_k + \gamma_k\lambda$> = χ = <$t + \alpha\lambda$, $d + \beta\lambda$, $f + \gamma\lambda$> for k = 1, 2, …, n be SNIE with the related weight $\rho_k \in [0, 1]$ for $\sum_{k=1}^{n}\rho_k = 1$. Then, we can get the result:

$$SNIEEWA(\chi_1,\chi_2,...,\chi_n)$$
$$=\left\langle \begin{array}{l} \left[\frac{\prod_{k=1}^{n}(1+t_k+\alpha_k\lambda^-)^{\rho_k}-\prod_{k=1}^{n}(1-t_k-\alpha_k\lambda^-)^{\rho_k}}{\prod_{k=1}^{n}(1+t_k+\alpha_k\lambda^-)^{\rho_k}+\prod_{k=1}^{n}(1-t_k-\alpha_k\lambda^-)^{\rho_k}},\frac{\prod_{k=1}^{n}(1+t_k+\alpha_k\lambda^+)^{\rho_k}-\prod_{k=1}^{n}(1-t_k-\alpha_k\lambda^+)^{\rho_k}}{\prod_{k=1}^{n}(1+t_k+\alpha_k\lambda^+)^{\rho_k}+\prod_{k=1}^{n}(1-t_k-\alpha_k\lambda^+)^{\rho_k}}\right], \\ \left[\frac{2\prod_{k=1}^{n}(d_k+\beta_k\lambda^-)^{\rho_k}}{\prod_{k=1}^{n}(2-d_k-\beta_k\lambda^-)^{\rho_k}+\prod_{k=1}^{n}(d_k+\beta_k\lambda^-)^{\rho_k}},\frac{2\prod_{k=1}^{n}(d_k+\beta_k\lambda^+)^{\rho_k}}{\prod_{k=1}^{n}(2-d_k-\beta_k\lambda^+)^{\rho_k}+\prod_{k=1}^{n}(d_k+\beta_k\lambda^+)^{\rho_k}}\right], \\ \left[\frac{2\prod_{k=1}^{n}(f_k+\gamma_k\lambda^-)^{\rho_k}}{\prod_{k=1}^{n}(2-f_k-\gamma_k\lambda^-)^{\rho_k}+\prod_{k=1}^{n}(f_k+\gamma_k\lambda^-)^{\rho_k}},\frac{2\prod_{k=1}^{n}(f_k+\gamma_k\lambda^+)^{\rho_k}}{\prod_{k=1}^{n}(2-f_k-\gamma_k\lambda^+)^{\rho_k}+\prod_{k=1}^{n}(f_k+\gamma_k\lambda^+)^{\rho_k}}\right] \end{array}\right\rangle$$
$$=\left\langle \begin{array}{l} \left[\frac{(1+t+\alpha\lambda^-)^{\sum_{k=1}^{n}\rho_k}-(1-t-\alpha\lambda^-)^{\sum_{k=1}^{n}\rho_k}}{(1+t+\alpha\lambda^-)^{\sum_{k=1}^{n}\rho_k}+(1-t-\alpha\lambda^-)^{\sum_{k=1}^{n}\rho_k}},\frac{(1+t+\alpha\lambda^+)^{\sum_{k=1}^{n}\rho_k}-(1-t-\alpha\lambda^+)^{\sum_{k=1}^{n}\rho_k}}{(1+t+\alpha\lambda^+)^{\sum_{k=1}^{n}\rho_k}+(1-t-\alpha\lambda^+)^{\sum_{k=1}^{n}\rho_k}}\right], \\ \left[\frac{2(d+\beta\lambda^-)^{\sum_{k=1}^{n}\rho_k}}{(2-d-\beta\lambda^-)^{\sum_{k=1}^{n}\rho_k}+(d+\beta\lambda^-)^{\sum_{k=1}^{n}\rho_k}},\frac{2(d+\beta\lambda^+)^{\sum_{k=1}^{n}\rho_k}}{(2-d-\beta\lambda^+)^{\sum_{k=1}^{n}\rho_k}+(d+\beta\lambda^+)^{\sum_{k=1}^{n}\rho_k}}\right], \\ \left[\frac{2(f+\gamma\lambda^-)^{\sum_{k=1}^{n}\rho_k}}{(2-f-\gamma\lambda^-)^{\sum_{k=1}^{n}\rho_k}+(f+\gamma\lambda^-)^{\sum_{k=1}^{n}\rho_k}},\frac{2(f+\gamma\lambda^+)^{\sum_{k=1}^{n}\rho_k}}{(2-f-\gamma\lambda^+)^{\sum_{k=1}^{n}\rho_k}+(f+\gamma\lambda^+)^{\sum_{k=1}^{n}\rho_k}}\right] \end{array}\right\rangle$$
$$=\left\langle\left[t+\alpha\lambda^-,t+\alpha\lambda^+\right],\left[d+\beta\lambda^-,d+\beta\lambda^+\right],\left[f+\gamma\lambda^-,f+\gamma\lambda^+\right]\right\rangle$$
$$=\chi$$

(P2) Assume that there is $SNIEEWA(\chi_1, \chi_2, \ldots, \chi_n) = \langle T(\lambda), D(\lambda), F(\lambda)\rangle$. According to Eq. (11), we know that

$$T(\lambda)=\frac{\prod_{k=1}^{n}(1+t_k+\alpha_k\lambda)^{\rho_k}-\prod_{k=1}^{n}(1-t_k-\alpha_k\lambda)^{\rho_k}}{\prod_{k=1}^{n}(1+t_k+\alpha_k\lambda)^{\rho_k}+\prod_{k=1}^{n}(1-t_k-\alpha_k\lambda)^{\rho_k}},D(\lambda)=\frac{2\prod_{k=1}^{n}(d_k+\beta_k\lambda)^{\rho_k}}{\prod_{k=1}^{n}(2-d_k-\beta_k\lambda)^{\rho_k}+\prod_{k=1}^{n}(d_k+\beta_k\lambda)^{\rho_k}},$$

$F(\lambda)=\frac{2\prod_{k=1}^{n}(f_k+\gamma_k\lambda)^{\rho_k}}{\prod_{k=1}^{n}(2-f_k-\gamma_k\lambda)^{\rho_k}+\prod_{k=1}^{n}(f_k+\gamma_k\lambda)^{\rho_k}}$ for $\lambda\in[\lambda^-, \lambda^+]$ are increasing functions of λ. So, we can get the following inequations:

$$\min_k(t_k+\alpha_k\lambda^-)\le T(\lambda^-)=\frac{\prod_{k=1}^{n}(1+t_k+\alpha_k\lambda^-)^{\rho_k}-\prod_{k=1}^{n}(1-t_k-\alpha_k\lambda^k)^{\rho_k}}{\prod_{k=1}^{n}(1+t_k+\alpha_k\lambda^-)^{\rho_k}+\prod_{k=1}^{n}(1-t_k-\alpha_k\lambda^k)^{\rho_k}}\le\max_k(t_k+\alpha_k\lambda^-),$$

$$\min_k(t_k+\alpha_k\lambda^+)\le T(\lambda^+)=\frac{\prod_{k=1}^{n}(1+t_k+\alpha_k\lambda^+)^{\rho_k}-\prod_{k=1}^{n}(1-t_k-\alpha_k\lambda^+)^{\rho_k}}{\prod_{k=1}^{n}(1+t_k+\alpha_k\lambda^+)^{\rho_k}+\prod_{k=1}^{n}(1-t_k-\alpha_k\lambda^+)^{\rho_k}}\le\min_k(t_k+\alpha_k\lambda^+),$$

$$\min_k(d_k+\beta_k\lambda^-)\le D(\lambda^-)=\frac{2\prod_{k=1}^{n}(d_{i_k}+\beta_k\lambda^-)^{\rho_k}}{\prod_{k=1}^{n}(2-d_k-\beta_k\lambda^-)^{\rho_i}+\prod_{k=1}^{n}(d_k+\beta_k\lambda^-)^{\rho_k}}\le\max_k(d_k+\beta_k\lambda^-),$$

$$\min_k(d_k+\beta_k\lambda^+)\le D(\lambda^+)=\frac{2\prod_{k=1}^{n}(d_k+\beta_k\lambda^+)^{\rho_k}}{\prod_{k=1}^{n}(2-d_k-\beta_k\lambda^+)^{\rho_k}+\prod_{k=1}^{n}(d_k+\beta_k\lambda^+)^{\rho_k}}\le\max_k(d_k+\beta_k\lambda^+),$$

$$\min_k(f_k+\gamma_k\lambda^-)\le F(\lambda^-)=\frac{2\prod_{k=1}^{n}(f_k+\gamma_k\lambda^-)^{\rho_k}}{\prod_{k=1}^{n}(2-f_k-\gamma_k\lambda^-)^{\rho_k}+\prod_{k=1}^{n}(f_k+\gamma_k\lambda^-)^{\rho_k}}\le\max_k(f_k+\gamma_k\lambda^-),$$

$$\min_k(f_k+\gamma_k\lambda^+)\le F(\lambda^+)=\frac{2\prod_{k=1}^{n}(f_k+\gamma_k\lambda^+)^{\rho_k}}{\prod_{k=1}^{n}(2-f_k-\gamma_k\lambda^+)^{\rho_k}+\prod_{k=1}^{n}(f_k+\gamma_k\lambda^+)^{\rho_k}}\le\max_k(f_k+\gamma_k\lambda^+).$$

According to Eq. (1), we get the score values of $SNIEEWA(\chi_1, \chi_2, \ldots, \chi_n)$, $\chi_{\min}$, and $\chi_{\max}$:

$$S(SNIEEWA(\chi_1,\chi_2,\ldots,\chi_n))=(4+T(\lambda^-)+T(\lambda^+)-D(\lambda^-)-D(\lambda^+)-F(\lambda^-)-F(\lambda^+))/6,$$

$$S(\chi_{\min})=\begin{Bmatrix}(4+\min_k(t_k+\alpha_k\lambda^-)+\min_k(t_k+\alpha_k\lambda^+)-\max_k(d_k+\beta_k\lambda^-)\\-\max_k(d_k+\beta_k\lambda^+)-\max_k(f_k+\gamma_k\lambda^-)-\max_k(f_k+\gamma_k\lambda^+))/6\end{Bmatrix},$$

$$S(\chi_{\max})=\begin{Bmatrix}(4+\max_k(t_k+\alpha_k\lambda^-)+\max_k(t_k+\alpha_k\lambda^+)-\min_k(d_k+\beta_k\lambda^-\\-\min_k(d_k+\beta_k\lambda^+)-\min_k(f_k+\gamma_k\lambda^-)-\min_k(f_k+\gamma_k\lambda^+))/6\end{Bmatrix}.$$

We can get $S(\chi_{\min}) \le S(SNIEEWA(\chi_1, \chi_2,\ldots,\chi_n)) \le S(\chi_{\max})$. Thus $\chi_{\min} \le SNIEEWA(\chi_1, \chi_2, \ldots, \chi_n) \le \chi_{\max}$.

(P3) If $\chi_k = <T_k(\lambda), D_k(\lambda), F_k(\lambda)>$ and $\chi_k^*=<T_k^*(\lambda), D_k^*(\lambda), F_k^*(\lambda)>$ for $\lambda\in [\lambda^-, \lambda^+]$, $k = 1, 2, \ldots, n$, and $\chi_k \subseteq \chi_*$, then they satisfy the following constraints: $T_k(\lambda^-) \le T_k^*(\lambda^-)$, $T_k(\lambda^+) \le T_k^*(\lambda^+)$, $D_k(\lambda^-) \ge D_k^*(\lambda^-)$, $D_k(\lambda^+) \ge D_k^*(\lambda^+)$, $F_k(\lambda^-) \ge F_k^*(\lambda^-)$, and $F_k(\lambda^+) \ge F_k^*(\lambda^+)$. We use Eq. (11) to calculate $SNIEEWA(\chi_1, \chi_2, \ldots, \chi_n)$ and $SNIEEWA(\chi_1^*, \chi_2^*, \ldots, \chi_n^*)$: $SNIEEWA(\chi_1, \chi_2, \ldots, \chi_n) = <[T(\lambda^-), T(\lambda^+)], [D(\lambda^-), D(\lambda^+)], [F(\lambda^-)\ F(\lambda^+)]>$ and $SNIEEWA(\chi_1^*, \chi_2^*, \ldots, \chi_n^*) = <[T^*(\lambda^-), T^*(\lambda^+)], [D^*(\lambda^-), D^*(\lambda^+)], [F^*(\lambda^-), F^*(\lambda^+)]>$. Obviously, we can get $T(\lambda^-) \le T^*(\lambda^-)$, $T(\lambda^+) \le T^*(\lambda^+)$, $D(\lambda^-) \ge D^*(\lambda^-)$, $D(\lambda^+) \ge D^*(\lambda^+)$, $F(\lambda^-) \ge F^*(\lambda^-)$, and $F(\lambda^+) \ge F^*(\lambda^+)$. Hence $SNIEEWA(\chi_1, \chi_2, \ldots, \chi_n) \subseteq SNIEEWA(\chi_1^*, \chi_2^*, \ldots, \chi_n^*)$ holds.

3.3. Einstein Weighted Geometric Average Operator of SNIEs

Definition 7. Let $\chi = \{\chi_1, \chi_2, \ldots, \chi_n\}$ be SNIS, we can define the SNIEEWG Operator of SNIEs:

$$SNIEEWG(\chi_1,\chi_2,\ldots,\chi_n)=\otimes_{k=1}^{n}\chi_k^{\rho_k}, \tag{13}$$

where $\rho_k \in [0, 1]$ are weights for $\sum_{k=1}^{n}\rho_k = 1$.

Theorem 2. Let $\chi_k = <T_k(\lambda), D_k(\lambda), F_k(\lambda)> = < t_k + \alpha_k\lambda, d_k + \beta_k\lambda, f_k + \gamma_k\lambda >$ for $k = 1, 2, \ldots, n$ and $\lambda \in [\lambda^-, \lambda^+]$ be SNIEs with the related weights $\rho_k \in [0,1]$ for $\sum_{k=1}^{n}\rho_k = 1$. Then according to the operational rules (7) and (9), Eq. (11) can be calculated by

$$SNIEEWG(\chi_1,\chi_2,...,\chi_n)=\left\langle\begin{array}{l}\left[\dfrac{2\prod_{k=1}^{n}(t_k+\alpha_k\lambda^-)^{\rho_k}}{\prod_{k=1}^{n}(2-t_k-\alpha_k\lambda^-)^{\rho_k}+\prod_{k=1}^{n}(t_k+\alpha_k\lambda^-)^{\rho_k}},\dfrac{2\prod_{k=1}^{n}(t_k+\alpha_k\lambda^+)^{\rho_k}}{\prod_{k=1}^{n}(2-t_k-\alpha_k\lambda^+)^{\rho_k}+\prod_{k=1}^{n}(t_k+\alpha_k\lambda^+)^{\rho_k}}\right],\\ \left[\dfrac{\prod_{k=1}^{n}(1+d_k+\beta_k\lambda^-)^{\rho_k}-\prod_{k=1}^{n}(1-d_k-\beta_k\lambda^-)^{\rho_k}}{\prod_{k=1}^{n}(1+d_k+\beta_k\lambda^-)^{\rho_k}+\prod_{k=1}^{n}(1-d_k-\beta_k\lambda^-)^{\rho_k}},\dfrac{\prod_{k=1}^{n}(1+d_k+\beta_k\lambda^+)^{\rho_k}-\prod_{k=1}^{n}(1-d_k-\beta_k\lambda^+)^{\rho_k}}{\prod_{k=1}^{n}(1+d_k+\beta_k\lambda^+)^{\rho_k}+\prod_{k=1}^{n}(1-d_k-\beta_k\lambda^+)^{\rho_k}}\right],\\ \left[\dfrac{\prod_{k=1}^{n}(1+f_k+\gamma_k\lambda^-)^{\rho_k}-\prod_{k=1}^{n}(1-f_k-\gamma_k\lambda^-)^{\rho_k}}{\prod_{k=1}^{n}(1+f_k+\gamma_k\lambda^-)^{\rho_k}+\prod_{k=1}^{n}(1-f_k-\gamma_k\lambda^-)^{\rho_k}},\dfrac{\prod_{k=1}^{n}(1+f_k+\gamma_k\lambda^+)^{\rho_k}-\prod_{k=1}^{n}(1-f_k-\gamma_k\lambda^+)^{\rho_k}}{\prod_{k=1}^{n}(1+f_k+\gamma_k\lambda^+)^{\rho_k}+\prod_{k=1}^{n}(1-f_k-\gamma_k\lambda^+)^{\rho_k}}\right]\end{array}\right\rangle. \quad (14)$$

In view of the same proof of Theorem 1, we can proof Theorem 2, which is omitted here.

4. MADM Method with the SNIEEWA or SNIEEWG Operator

For a MADM problem, there are an alternative set $\eta = \{\eta_1, \eta_2, \ldots, \eta_m\}$ and an attribute set $C = \{C_1, C_2, \ldots, C_n\}$ with the weight vector $\rho = \{\rho_1, \rho_2, \ldots, \rho_n\}$. The assessment values given by the decision makers are in the SNIE form. Thus, the evaluation value of the attribute C_j for the alternative η_i is specified as $\chi_{ij} = \langle T_{ij}(\lambda), D_{ij}(\lambda), F_{ij}(\lambda)\rangle = \langle t_{ij} + \alpha_{ij}\lambda, d_{ij} + \beta_{ij}\lambda, f_{ij} + \gamma_{ij}\lambda\rangle$ for $t_{ij} + \alpha_{ij}\lambda, d_{ij} + \beta_{ij}\lambda, f_{ij} + \gamma_{ij}\lambda \in [0, 1]$ and $\lambda \in [\lambda^-, \lambda^+]$, ($i = 1, 2, \ldots, m; j = 1, 2, \ldots, n$). Hence, all the assessed SNIEs constitute the SNIE decision matrix $\chi = (\chi_{ij})_{m\times n}$. Then, the MADM method is shown as the following steps.

Step1: Calculate the aggregation value of SNIEs χ_{ij} for η_i and some ranges of λ by the following aggregation formula:

$$\chi_i = SNIEEWA(\chi_{i1},\chi_{i2},...,\chi_{in}) = \mathop{\oplus}_{j=1}^{n}\rho_j\chi_{ij}$$
$$=\left\langle\begin{array}{l}\left[\dfrac{\prod_{j=1}^{n}(1+t_{ij}+\alpha_{ij}\lambda^-)^{\rho_j}-\prod_{j=1}^{n}(1-t_{ij}-a_{ij}\lambda^-)^{\rho_j}}{\prod_{j=1}^{n}(1+t_{ij}+\alpha_{ij}\lambda^-)^{\rho_j}+\prod_{j=1}^{n}(1-t_{ij}-a_{ij}\lambda^-)^{\rho_j}},\dfrac{\prod_{j=1}^{n}(1+t_{ij}+\alpha_{ij}\lambda^+)^{\rho_j}-\prod_{j=1}^{n}(1-t_{ij}-a_{ij}\lambda^+)^{\rho_j}}{\prod_{j=1}^{n}(1+t_{ij}+\alpha_{ij}\lambda^+)^{\rho_j}+\prod_{j=1}^{n}(1-t_{ij}-a_{ij}\lambda^+)^{\rho_j}}\right],\\ \left[\dfrac{2\prod_{j=1}^{n}(d_{ij}+\beta_{ij}\lambda^-)^{\rho_j}}{\prod_{j=1}^{n}(2-d_{ij}-\beta_{ij}\lambda^-)^{\rho_j}+\prod_{j=1}^{n}(d_{ij}+\beta_{ij}\lambda^-)^{\rho_j}},\dfrac{2\prod_{j=1}^{n}(d_{ij}+\beta_{ij}\lambda^+)^{\rho_j}}{\prod_{j=1}^{n}(2-d_{ij}-\beta_{ij}\lambda^+)^{\rho_j}+\prod_{j=1}^{n}(d_{ij}+\beta_{ij}\lambda^+)^{\rho_j}}\right],\\ \left[\dfrac{2\prod_{j=1}^{n}(f_{ij}+\gamma_{ij}\lambda^-)^{\rho_j}}{\prod_{j=1}^{n}(2-f_{ij}-\gamma_{ij}\lambda^-)^{\rho_j}+\prod_{j=1}^{n}(f_{ij}+\gamma_{ij}\lambda^-)^{\rho_j}},\dfrac{2\prod_{j=1}^{n}(f_{ij}+\gamma_{ij}\lambda^+)^{\rho_j}}{\prod_{j=1}^{n}(2-f_{ij}-\gamma_{ij}\lambda^+)^{\rho_j}+\prod_{j=1}^{n}(f_{ij}+\gamma_{ij}\lambda^+)^{\rho_j}}\right]\end{array}\right\rangle \quad (15)$$

$$\chi_i = SNIEEWG(\chi_{i1},\chi_{i2},...,\chi_{in}) = \mathop{\otimes}_{j=1}^{n}\chi_{ij}^{\rho_j}$$

or

$$=\left\langle\begin{array}{l}\left[\dfrac{2\prod_{j=1}^{n}(t_{ij}+\alpha_{ij}\lambda^-)^{\rho_j}}{\prod_{j=1}^{n}(2-t_{ij}-\alpha_{ij}\lambda^-)^{\rho_j}+\prod_{j=1}^{n}(t_{ij}+\alpha_{ij}\lambda^-)^{\rho_j}},\dfrac{2\prod_{j=1}^{n}(t_{ij}+\alpha_{ij}\lambda^+)^{\rho_j}}{\prod_{j=1}^{n}(2-t_{ij}-\alpha_{ij}\lambda^+)^{\rho_j}+\prod_{j=1}^{n}(t_{ij}+\alpha_{ij}\lambda^+)^{\rho_j}}\right],\\ \left[\dfrac{\prod_{j=1}^{n}(1+d_{ij}+\beta_{ij}\lambda^-)^{\rho_j}-\prod_{j=1}^{n}(1-d_{ij}-\beta_{ij}\lambda^-)^{\rho_j}}{\prod_{j=1}^{n}(1+d_{ij}+\beta_{ij}\lambda^-)^{\rho_j}+\prod_{j=1}^{n}(1+d_{ij}+\beta_{ij}\lambda^-)^{\rho_j}},\dfrac{\prod_{j=1}^{n}(1+d_{ij}+\beta_{ij}\lambda^+)^{\rho_j}-\prod_{j=1}^{n}(1-d_{ij}-\beta_{ij}\lambda^+)^{\rho_j}}{\prod_{j=1}^{n}(1+d_{ij}+\beta_{ij}\lambda^+)^{\rho_j}+\prod_{j=1}^{n}(1+d_{ij}+\beta_{ij}\lambda^+)^{\rho_j}}\right],\\ \left[\dfrac{\prod_{j=1}^{n}(1+f_{ij}+\gamma_{ij}\lambda^-)^{\rho_j}-\prod_{j=1}^{n}(1-f_{ij}-\gamma_{ij}\lambda^-)^{\rho_j}}{\prod_{j=1}^{n}(1+f_{ij}+\gamma_{ij}\lambda^-)^{\rho_j}+\prod_{j=1}^{n}(1-f_{ij}-\gamma_{ij}\lambda^-)^{\rho_j}},\dfrac{\prod_{j=1}^{n}(1+f_{ij}+\gamma_{ij}\lambda^+)^{\rho_j}-\prod_{j=1}^{n}(1-f_{ij}-\gamma_{ij}\lambda^+)^{\rho_j}}{\prod_{j=1}^{n}(1+f_{ij}+\gamma_{ij}\lambda^+)^{\rho_j}+\prod_{j=1}^{n}(1-f_{ij}-\gamma_{ij}\lambda^+)^{\rho_j}}\right]\end{array}\right\rangle. \quad (16)$$

Step2: Calculate the values of the score function $S(\chi_i, \lambda)$ (accuracy function $L(\chi_i,\lambda)$ and certainty function $C(\chi_i,\lambda)$) by Eq. (1) (Eqs. (2) and (3)).

Step3: Rank the alternatives and define the best one.

5. Illustrative Example, Sensitivity Analysis, and Comparison

5.1. Illustrative Example

In an illustrative example, we apply the proposed MADM method to the risk assessment of the investment selection of metallic mines. The mining projects have great uncertainty and a long cycle. Then, there are investment projects of four candidate mines, denoted as a set of four alternatives $\eta = \{\eta_1, \eta_2, \eta_3, \eta_4\}$. The key evaluation factors/attributes of the four candidate mines contain the economic factor (C_1), the safety factor (C_2), and the environmental risk factor (C_3) in the investment evaluation process. The weight vector ρ = (0.3, 0.36, 0.34) addresses the importance of the three attributes. Because of evaluation information uncertainty in the four candidate mines, the decision makers/experts are required to evaluate each candidate mine on the three attributes in the SNIE form. Their evaluation information is provided by the SNIEs $\chi_{ij} = \langle T_{ij}(\lambda), D_{ij}(\lambda), F_{ij}(\lambda)\rangle = \langle t_{ij} + \alpha_{ij}\lambda, d_{ij} + \beta_{ij}\lambda, f_{ij} + \gamma_{ij}\lambda\rangle$ for j = 1, 2, 3; i = 1, 2, 3, 4. Thus, the decision matrix of all SNIEs is shown below:

$$\chi = \begin{bmatrix} \langle 0.7+0.2\lambda, 0.2+0.1\lambda, 0.2+0.2\lambda\rangle & \langle 0.7+0.2\lambda, 0.1+0.3\lambda, 0.1+0.1\lambda\rangle & \langle 0.6+0.2\lambda, 0.2+0.2\lambda, 0.2+0.2\lambda\rangle \\ \langle 0.7+0.2\lambda, 0.2+0.1\lambda, 0.3+0.1\lambda\rangle & \langle 0.8+0.1\lambda, 0.1+0.2\lambda, 0.1+0.3\lambda\rangle & \langle 0.7+0.1\lambda, 0.2+0.2\lambda, 0.1+0.1\lambda\rangle \\ \langle 0.8+0.1\lambda, 0.2+0.1\lambda, 0.1+0.2\lambda\rangle & \langle 0.7+0.1\lambda, 0.2+0.1\lambda, 0.1+0.2\lambda\rangle & \langle 0.7+0.2\lambda, 0.3+0.1\lambda, 0.2+0.1\lambda\rangle \\ \langle 0.7+0.1\lambda, 0.1+0.2\lambda, 0.2+0.1\lambda\rangle & \langle 0.8+0.1\lambda, 0.1+0.2\lambda, 0.2+0.1\lambda\rangle & \langle 0.7+0.1\lambda, 0.2+0.1\lambda, 0.2+0.2\lambda\rangle \end{bmatrix}.$$

According to the evaluation information and the proposed MADM method, the decision steps are shown below.

Step1: Aggregate SNIEs χ_{ij} for η_i (i = 1, 2, 3, 4; j = 1,2, 3) by Eq. (15) or (16). The indeterminate λ is specified as $\lambda = [\lambda^-, \lambda^+]$ = [0, 0], [0, 0.5], [0, 1], [0, 1.5]. The aggregation values of Eq. (15) or (16) are listed in Tables 1 and 2.

Table 1. The aggregation values corresponding to the SNIEEWA operator

$\lambda = [\lambda^-, \lambda^+]$	Aggregation value
λ= [0, 0]	χ_1 = <[0.6685, 0.6685],[0.1565, 0.1565],[0.1565, 0.1565]>, χ_2 = <[0.7400, 0.7400],[0.1565, 0.1565],[0.1407, 0.1407]>, χ_3 = <[0.7337, 0.7337],[0.2301, 0.2301],[0.1271, 0.1271]>, χ_4 = <[0.7400, 0.7400],[0.1271, 0.1271],[0.2000, 0.2000]>
λ= [0, 0.5]	χ_1 = <[0.6685, 0.7699],[0.1565, 0.2661],[0.1565, 0.2354]>, χ_2 = <[0.7400, 0.8050],[0.1565, 0.2460],[0.1407, 0.2343]>, χ_3 = <[0.7337, 0.8007],[0.2301, 0.2809],[0.1270, 0.2159]>, χ_4 = <[0.7400, 0.7913],[0.1271, 0.2159],[0.2000, 0.2661]>
λ= [0, 1]	χ_1 = <[0.6685, 0.8729],[0.1565, 0.3676],[0.1565, 0.3147]>, χ_2 = <[0.7400, 0.8729],[0.1565, 0.3315],[0.1407, 0.3190]>, χ_3 = <[0.7337, 0.8711],[0.2301, 0.3315],[0.1270, 0.3000]>, χ_4 = <[0.7400, 0.8435],[0.1271, 0.3000],[0.2000, 0.3315]>
λ= [0, 1.5]	χ_1 = <[0.6685, 1.0000],[0.1565, 0.4673],[0.1565, 0.3945]>, χ_2 = <[0.7400, 1.0000],[0.1565, 0.4158],[0.1407, 0.4015]>, χ_3 = <[0.7337, 1.0000],[0.2301, 0.3819],[0.1270, 0.3824]>, χ_4 = <[0.7400, 0.8983],[0.1271, 0.3824],[0.2000, 0.3966]>

Table 2. The aggregation values corresponding to the SNIEEWG operator

$\lambda = [\lambda^-, \lambda^+]$	Aggregation value
λ = [0, 0]	χ_1 = <[0.6651, 0.6651],[0.1645, 0.1644],[0.1644, 0.1644]>, χ_2 = <[0.7354, 0.7354],[0.1645, 0.1644],[0.1617, 0.1617]>, χ_3 = <[0.7294, 0.7294],[0.2346, 0.2346],[0.1343, 0.1343]>, χ_4 = <[0.7354, 0.7354],[0.1343, 0.1343],[0.2000, 0.2000]>
λ = [0, 0.5]	χ_1 = <[0.6651, 0.7654],[0.1645, 0.2672],[0.1644, 0.2473]>, χ_2 = <[0.7354, 0.8006],[0.1645, 0.2495],[0.1617, 0.2477]>, χ_3 = <[0.7294, 0.7967],[0.2346, 0.2847],[0.1343, 0.2171]>, χ_4 = <[0.7354, 0.7855],[0.1343, 0.2171],[0.2000, 0.2672]>
λ = [0, 1]	χ_1 = <[0.6651, 0.8657],[0.1645, 0.3709],[0.1644, 0.3311]>, χ_2 = <[0.7354, 0.8657],[0.1645, 0.3349],[0.1617, 0.3351]>, χ_3 = <[0.7294, 0.8637],[0.2346, 0.3349],[0.1343, 0.3000]>, χ_4 = <[0.7354, 0.8356],[0.1343, 0.3000],[0.2000, 0.3349]>
λ = [0, 1.5]	χ_1 = <[0.6651, 0.9659],[0.1645, 0.4769],[0.1644, 0.4165]>, χ_2 = <[0.7354, 0.9307],[0.1645, 0.4210],[0.1617, 0.4259]>, χ_3 = <[0.7294, 0.9307],[0.2346, 0.3851],[0.1343, 0.3832]>, χ_4 = <[0.7354, 0.8858],[0.1343, 0.3832],[0.2000, 0.4036]>

Step 2: Calculate the scores of $S(\chi_i, \lambda)$ by Eq. (1) and show the results in Tables 3 and 4.

Table 3. Scores and ranking orders corresponding to the SNIEEWA operator

$\lambda = [\lambda^-, \lambda^+]$	Score of $S(\chi_i, \lambda)$	Ranking	The best
λ = [0, 0]	0.7852, 0.8143, 0.7922, 0.8043	$\eta_2 > \eta_4 > \eta_3 > \eta_1$	η_2
λ = [0, 0.5]	0.7706, 0.7946, 0.7801, 0.7870	$\eta_2 > \eta_4 > \eta_3 > \eta_1$	η_2
λ = [0, 1]	0.7577, 0.7775, 0.7694, 0.7708	$\eta_2 > \eta_4 > \eta_3 > \eta_1$	η_2
λ = [0, 1.5]	0.7489, 0.7709, 0.7687, 0.7554	$\eta_2 > \eta_3 > \eta_4 > \eta_1$	η_2

Table 4. Scores and ranking orders corresponding to the SNIEEWG operator

$\lambda = [\lambda^-, \lambda^+]$	Score of $S(\chi_i, \lambda)$	Ranking	The best
λ = [0, 0]	0.7788, 0.8031, 0.7868, 0.8004	$\eta_2 > \eta_4 > \eta_3 > \eta_1$	η_2
λ = [0, 0.5]	0.7645, 0.7854, 0.7759, 0.7837	$\eta_2 > \eta_4 > \eta_3 > \eta_1$	η_2
λ = [*0, 1*]	0.7500, 0.7675, 0.7649, 0.7670	$\eta_2 > \eta_4 > \eta_3 > \eta_1$	η_2
λ = [0, 1.5]	0.7348, 0.7488, 0.7538, 0.7500	$\eta_3 > \eta_4 > \eta_2 > \eta_1$	η_3

Step 3: The ranking orders are listed in Tables 3 and 4. There are exactly consistent ranking results between the SNIEEWA operator and the SNIEEWG operator when the indeterminate ranges of λ are λ = [0, 0], [0, 0.5], [0, 1], then η_2 is the best alternative. While the indeterminate range is λ= [0, 1.5], the ranking orders are very different between two aggregation operators, then the best alternative is η_2 corresponding to the SNIEEWA operator and η_3 corresponding to the SNIEEWG operator.

5.2. Sensitivity Analysis

An SNIE can represent an SvNE or an IvNE regarding the value or range of the indeterminate λ. In the above example, we have specified several indeterminate ranges of λ to make decisions.

The results show that the ranking orders are the same within certain ranges of λ = [0, 0], [0, 0.5], [0, 1] corresponding to two aggregation operators, while the ranking orders are very different in the range of λ= [0, 1.5]. The above example demonstrates that more ranking orders based on the two aggregation operators are nearly consistent. Due to the variability of the indeterminate λ, the proposed MADM approach is valid and flexible. To further analyze the change of decision results with the indeterminate variety of λ, we show the relational graphs corresponding to the indeterminate value of λ and the score of η_i in Figures 1 and 2**.**

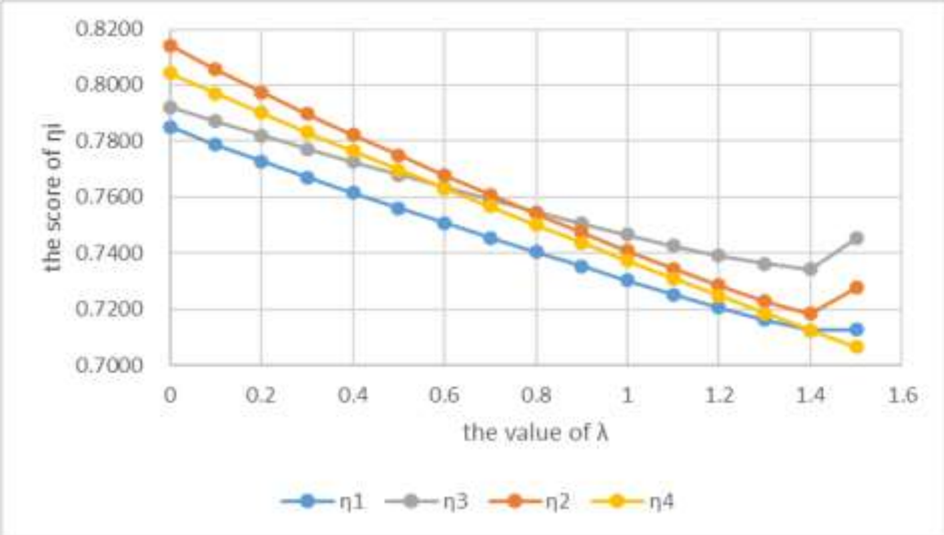

Figure 1. Relationship between the score of η_i and the value of λ corresponding to the SNIEEWA operator

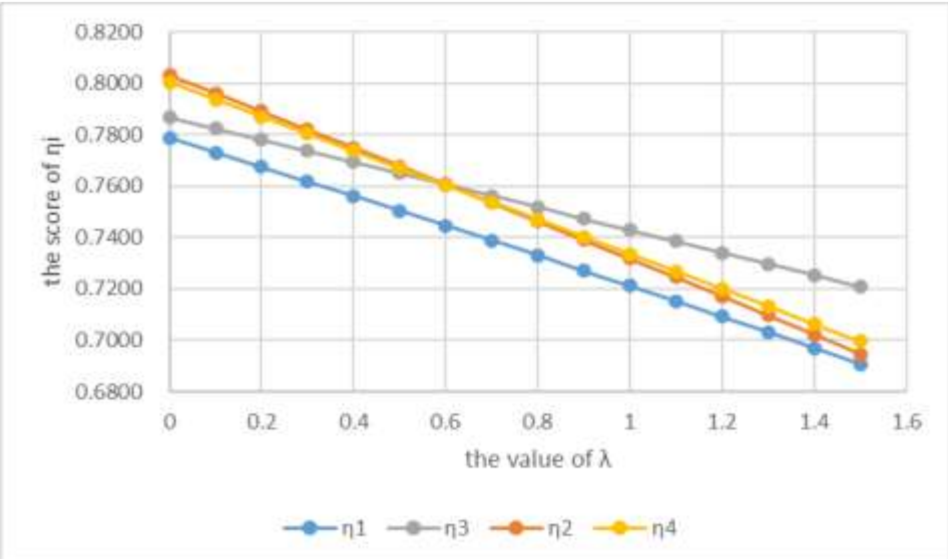

Figure 2. Relationship between the score of η_i and the value of λ corresponding to the SNIEEWG operator

SNIE is reduced to SvNE when the indeterminate λ is a single value. In Figure 1, the ranking order is $\eta_2 > \eta_4 > \eta_3 > \eta_1$ and the best alternative is η_2 when λ is in the range of $\lambda \in [0, 0.8]$ corresponding to the SNIEEWA operator. Then, the best alternative is η_3 in the range of $\lambda \in [0.8, 1.5]$. In Figure 2, the ranking order is $\eta_2 > \eta_4 > \eta_3 > \eta_1$ when the value of λ is less than 0.6 corresponding to the SNIEEWG operator. The ranking order is $\eta_3 > \eta_4 > \eta_2 > \eta_1$ when the value of λ is greater than 0.6. Although the ranking orders are not exactly identical with different aggregation operators, they are some sensitivities to different values/ranges of λ.

5.3. Comparison and Discussion

Du et al. [15] first put forward the concept of SNIE and a MADM approach based on the weighted aggregation operators of SNIEs. To compare the proposed MADM approach with the existing MADM approach [15], the ranking results of the existing MADM approach [15] in specified ranges of λ are indicated corresponding to the SNIEWA and SNIEWG operators and shown in Table 5. The ranking results corresponding to the proposed SNIEEWA operator and the existing SNIEWA operator [15] are identical in all ranges of λ. Corresponding to the proposed SNIEEG operator and the existing SNIEWG operator [15], the ranking results are different only in the range of λ = [0, 1]. In terms of all the results, η_2 is the best investment selection, conversely η_1 is the worst one.

Table 5. Decision results of different methods

	Ranking order in different ranges of λ			
λ	λ = [0, 0]	λ = [0, 0.5]	λ = [0, 1]	λ = [0, 1.5]
SNIEEWA	$\eta_2 > \eta_4 > \eta_3 > \eta_1$	$\eta_2 > \eta_4 > \eta_3 > \eta_1$	$\eta_2 > \eta_4 > \eta_3 > \eta_1$	$\eta_2 > \eta_4 > \eta_3 > \eta_1$
SNIEEWG	$\eta_2 > \eta_4 > \eta_3 > \eta_1$	$\eta_2 > \eta_4 > \eta_3 > \eta_1$	$\eta_2 > \eta_4 > \eta_3 > \eta_1$	$\eta_3 > \eta_4 > \eta_2 > \eta_1$
SNIEWA [15]	$\eta_2 > \eta_4 > \eta_3 > \eta_1$	$\eta_2 > \eta_4 > \eta_3 > \eta_1$	$\eta_2 > \eta_4 > \eta_3 > \eta_1$	$\eta_2 > \eta_4 > \eta_3 > \eta_1$
SNIEWG [15]	$\eta_2 > \eta_4 > \eta_3 > \eta_1$	$\eta_2 > \eta_4 > \eta_3 > \eta_1$	$\eta_4 > \eta_2 > \eta_3 > \eta_1$	$\eta_3 > \eta_4 > \eta_2 > \eta_1$

6. Conclusions

In this article, we presented the SNIEEWA and SNIEEWG operators of SNIEs with respect to the Einstein t-norm and t-conorm operations. On the basis of the SNIEEWA and SNIEEWG operators, the MADM method was developed and applied to the selection problem of mine investments. In the illustrative example, the decision results were analyzed under the single- and interval-valued neutrosophic indeterminate situations, which indicated some sensitivities to different values/ranges of λ. Compared the existing MADM approach [15] in the situation of interval-valued neutrosophic indeterminate information, the ranking results demonstrated that the proposed approach is valid. Since SNIS can flexibly express neutrosophic information according to indetermination ranges of λ, the proposed MADM method reflected its efficiency and flexibility regarding interval indeterminate ranges.

Since SNIS is a flexible form for describing indeterminate and inconsistent assessment information, it can be used in many indeterminate problems. In future research, more aggregation operators, similarity measures, and decision-making methods will be developed and applied to many fields in neutrosophic indeterminate environment.

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