NSS

Neutrosophic Sets and Systems, Vol. 48, 2022

University of New Mexico

Bipolar neutrosophic graded soft sets and their topological spaces

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Abstract. Bipolar neutrosophic soft sets and their properties were discussed in several articles by many researchers. Soft sets are parametrized sets and bipolar neutrosophic soft sets are the fusion of bipolar neutrosophic sets and soft sets and there will be a number of decision variables or parameters. In many circumstances, the significance of each parameter is not equal. So the selection at the end might be unfit to the scenario. In this paper, we proposed Bipolar neutrosophic graded soft sets and their topological spaces. The proposed method fills that gap among the selection.

Keywords: Neutrosophy; Bipolar neutrosophic set; Soft set; Bipolar neutrosophic graded soft topology; Topology.

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1. **Introduction**

Exact solution not always exist for real life problems. Most of the scenarios interfere with some unwanted information called uncertainties. Due to uncertainty, one cannot conclude the problem with exact solution. In such cases, conventional methods are not efficient to deal with indeterminate. Fortunately, in recent years, there are many concepts were defined to deal such uncertainties. Neutrosophy is one of the technique which is suitable for problems with uncertainties. Neutrosophy is the extension of Intiutionistic fuzzy theory (originated from fuzzy theory). Neutrosophic sets are derived from Neutrosophy which is used in many decision making problems. Florentin Smarandache [6,8] was introduced this Neutrosophy concept. Neutrosophic set is a set of three memberships namely, Truth, Indeterminacy and False membership range in the non-standard interval $]^{-}0, 1^{+}[$. The non-standard intervals are only for theoretical purpose, but we prefer specific solution for real life problems; Single valued neutrosophic set (SVNS) is the set which is defined by Wang et al. [7] having variables ranges in the standard interval [0,1] instead of non-standard interval.

Majundar et al. [4, 5] proposed some notions on neutrosophic sets and single valued neutrosophic sets. Bipolar neutrosophic sets (BNS), extension of neutrosophic sets were defined by Deli et al. [2] in 2015 and similarity measures of bipolar neutrosophic sets were proposed by Uluay et al. [3] in 2016. In 2012, A.A.Salama et al. [25] extended fuzzy topology and intuitionistic fuzzy topology to neutrosophic topology. In 2017, Francisco Gal. [23] proposed the difference between intiutionistic fuzzy topology and neutrosophic topology and proved that they're not the same. In 2017, Tuhin Bera et al. [22] extended the neutrosophic topology concept to neutrosophic soft topology and proposed some of their properties. Syeda Tayyba et al. [24] proposed a decision making technique using bipolar neutrosophic soft topology. D. Molodtsov [12] introduced soft set theory in 1999. In 2014, Ridvan Sahin et al. [1] proposed some notions on neutrosophic soft sets. Ali et al. [11] introduced the concepts of bipolar neutrosophic soft sets in 2017. In 2019, Arulpandy P et al. [18] and Taha Yasin Ozturk et al. [21] proposed the new approaches on bipolar neutrosophic soft sets and some of their similarity and entropy measurements. Neutrosophic sets are widely used in decision making scenarios. In 2019, Arulpandy et al. [17] were proposed the representation of grayscale images and reduction of indeterminacy in bipolar neutrosophic domain which is very useful for image processing tasks. Also many articles were published in recent years about the applications of neutrsophy in engineering and medical fields [14–20].

In our study, the novel set and topology namely, Bipolar neutrosophic graded soft set (BNGS) and Bipolar neutrosophic soft topological space were proposed. This paper was organized as follows: Section 1 consists introduction and literature survey about the main topic. Section 2 consists some of the preliminaries required for the main topic. Section 3 deals with the proposed set namely, Bipolar neutrosophic graded soft sets and their properties with numerical examples. Section 4 deals with the proposed topology namely, Bipolar neutrosophic graded soft topological spaces with their properties and some propositions about the proposed topology. Finally, Section 5 concludes our study with future research goals.

2. Preliminaries

Definition 2.1. [8] Let X be a universal set and $x \in X$. A Neutrosophic set N is defined by

$$N = \{\langle x, T_N(x), I_N(x), F_N(x)\rangle : x \in X\}$$

where $T_N(x), I_N(x), F_N(x)$ known as truth, indeterminacy and falsify membership values respectively.

Also, $T_N(x), I_N(x), F_N(x) : X \to]^-0, 1^+[$ and it satisfies $^-0 \leq T_N(x) + I_N(x) + F_N(x) \leq 3^+$.

Example 2.2. Let $X = \{x_1, x_2, x_3\}$ be the universe set consists attributes of machines. Also, x_1, x_2 and x_3 denotes reliability, performance and cost of a machine, respectively; $T_N(x), I_N(x)$ and $F_N(x)$ denotes the degree of good service, indeterminacy, degree of poor service respectively. The neutrosophic set N is defined by

$N = \left\{ \langle x_1, 0.7, 0.2, 0.3 \rangle, \langle x_2, 0.2, 0.4, 0.8 \rangle, \langle x_3, 0.4, 0.4, 0.6 \rangle \right\}$

where $^-0 \leq T_N(x) + I_N(x) + F_N(x) \leq 3^+$

Definition 2.3. [7] Single valued neutrosophic set is the neutrosophic set with the membership range of standard interval [0,1]. It is very convenient while solving real life problems.

A single valued neutrosophic set N is defined by

$$N = \{\langle x, T_N(x), I_N(x), F_N(x) \rangle : x \in X\}$$

where $T_N(x), I_N(x), F_N(x) : X \to [0, 1]$ such that $0 \leq T_N(x) + I_N(x) + F_N(x) \leq 3$.

Definition 2.4. [9, 12] Let X be a universe set. A Soft set is a pair (f, E) such that

$$f : E \to P(x)$$

Where $P(x)$ is a power set of X. Soft set is a parameterized family of subsets of the universe set X.

Example 2.5. Let $X = \{x_1, x_2, x_3, x_4\}$ be a set of computer systems and let $E = \{e_1, e_2, e_3\}$ be set of parameters. where e_1=Processor, e_2=Graphics and e_3=Storage.

suppose that

$$f(e_1) = \{x_2, x_3\}$$
$$f(e_2) = \{x_2, x_4\}$$
$$f(e_3) = \{x_1, x_3\}.$$

Then, $f(E) = \{f(e_1), f(e_2), f(e_3)\}$.

The set $f(E)$ is a soft set (parameterized family of subsets of X).

Definition 2.6. [10] A neutrosophic soft set (f_N, E) over X is defined by the set

$$(f_N, E) = \left\{\langle e, f_N(e) \rangle : e \in E, f_N(e) \in NS(X)\right\}$$

where $f_N : E \longrightarrow NS(x)$ such that $f_N(e) = \varphi$ if $e \notin A$.

Also, since $f_N(e)$ is a neutrosophic set over X is defined by

$$f_N(e) = \left\{\left\langle x, T_{f_N(e)}(x), I_{f_N(e)}(x), F_{f_N(e)}(x) \right\rangle : x \in X\right\}$$

where $T_{f_N(e)}(x), I_{f_N(e)}(x), F_{f_N(e)}(x)$ represents truth value of x for the parameter e, indeterminate value of x for the parameter e and false value of x for the parameter e.

Example 2.7. Let $X = \{x_1, x_2, x_3, x_4\}$ be a set of computer systems under consideration. Let $E = \{e_1, e_2, e_3\}$ be set of parameters where e_1, e_2, e_3 represents Processor speed, Graphics index and Storage capacity, respectively.
Then we define

$$(f_N, E) = \{\langle e_1, f_N(e_1)\rangle, \langle e_2, f_N(e_2)\rangle, \langle e_3, f_N(e_3)\rangle\}$$

Here

$$f_N(e_1) = \left\{\langle x_1, 0.1, 0.4, 0.2\rangle, \langle x_2, 0.3, 0.5, 0.3\rangle, \langle x_3, 0.9, 0.2, 0.1\rangle, \langle x_4, 0.4, 0.5, 0.9\rangle\right\}$$

$$f_N(e_2) = \left\{\langle x_1, 0.2, 0.3, 0.4\rangle, \langle x_2, 0.3, 0.5, 0.6\rangle, \langle x_3, 0.4, 0.1, 0.7\rangle, \langle x_4, 0.9, 0.5, 0.6\rangle\right\}$$

$$f_N(e_3) = \left\{\langle x_1, 0.1, 0.5, 0.8\rangle, \langle x_2, 0.7, 0.5, 0.3\rangle, \langle x_3, 0.5, 0.7, 0.2\rangle, \langle x_4, 0.3, 0.5, 0.2\rangle\right\}$$

So that (f_A, E) is a Neutrosophic soft set.

Definition 2.8. [2, 3] Let X be the universe set and $\forall x \in X$. A bipolar neutrosophic set (BNS) BN is defined by

$$BN = \left\{\langle x, T^+(x), I^+(x), F^+(x), T^-(x), I^-(x), F^-(x)\rangle : x \in X\right\}$$

where

$$\text{positive membership-degrees} : T^+, I^+, F^+ : E \to [0, 1]$$
$$\text{negative membership-degrees} : T^-, I^-, F^- : E \to [-1, 0]$$

such that

$$0 \le T^+(x) + I^+(x) + F^+(x) \le 3 \text{ and } -3 \le T^-(x) + I^-(x) + F^-(x) \le 0.$$

Example 2.9. Let $X = \{x_1, x_2, x_3\}$ be the universe set. A bipolar neutrosophic set (BNS) is defined by

$$\begin{aligned} BN = \Big\{&\langle x_1, 0.1, 0.3, 0.4, -0.5, -0.3, -0.7\rangle, \\ &\langle x_2, 0.3, 0.5, 0.8, -0.7, -0.2, -0.7\rangle, \\ &\langle x_3, 0.4, 0.1, 0.7, -0.7, -0.2, -0.9\rangle\Big\} \end{aligned}$$

where $0 \le T^+(x) + I^+(x) + F^+(x) \le 3$; $-3 \le T^-(x) + I^-(x) + F^-(x) \le 0$.
Also $T^+(x), I^+(x), F^+(x) \to [0, 1]$ and $T^-(x), I^-(x), F^-(x) \to [-1, 0]$.

Definition 2.10. [Mumtaz Ali et al. version] [11] Let X be a universe set and E be set of parameters that are describing the elements of X. A bipolar neutrosophic soft set B in X is defined as:

$$B = \{(e, \{(x, T^+(x), I^+(x), F^+(x), T^-(x), I^-(x), F^-(x) : x \in X\} : e \in E\}$$

where T^+, I^+, F^+ ranges from $[0,1]$ and T^-, I^-, F^- ranges from $[-1,0]$. The positive degrees $T^+(x), I^+(x), F^+(x)$, denotes the truth, indeterminate and false values of an element in the BNS set $\mathbb{B}$ and the negative degrees $T^-(x), I^-(x), F^-(x)$ denotes the truth, indeterminate and false values of an element in the BNS set $\mathbb{B}$.

Definition 2.11. [Arulpandy et al. version] [18] Let X be the universe and E be the parameter set. We define a set A be a subset of E.

A bipolar neutrosophic soft set B over X is defined by

$$B{=}(f_A, E) = \left\{ \left\langle e, f_A(e) \right\rangle : e \in E, f_A(e) \in BNS(X) \right\}$$

Here

$$f_A(e) = \left\{ \left\langle x, u^+_{f_A(e)}(x), v^+_{f_A(e)}(x), w^+_{f_A(e)}(x), u^-_{f_A(e)}(x), v^-_{f_A(e)}(x), w^-_{f_A(e)}(x) \right\rangle : x \in X \right\}.$$

where $u^+_{f_A(e)}(x), v^+_{f_A(e)}(x), w^+_{f_A(e)}(x)$ denoted positive truth, indeterminate and false-membership values of x for the parameter e, and similarly $u^-_{f_A(e)}(x), v^-_{f_A(e)}(x), w^-_{f_A(e)}(x)$ denoted positive truth, indeterminate and false-membership values of x for the parameter e.

Example 2.12. Let $X = \{x_1, x_2, x_3, x_4\}$ be universe set and let $E = \{e_1, e_2, e_3\}$ be the parameter set.
Now, let $A = \{e_1, e_2\} \subseteq E$ and $B = \{e_3\} \subseteq E$ be two subsets of E.

Then we define

$$B_1 = (f_A, E) = \{\langle e, f_A(e)\rangle : e \in E, f_A(e) \in BNS(X)\}$$
$$B_2 = (g_B, E) = \{\langle e, g_B(e)\rangle : e \in E, g_B(e) \in BNS(X)\}$$

where,

$$f_A(e_1) = \Big\{ \langle x_1,\ 0.3, 0.5, 0.7, -0.6, -0.5, -0.3\rangle, \langle x_2,\ 0.6, 0.2, 0.7, -0.3, -0.4, -0.6\rangle,$$
$$\langle x_3,\ 0.5, 0.6, 0.3, -0.3, -0.5, -0.3\rangle, \langle x_4,\ 0.4, 0.5, 0.2, -0.7, -0.3, -0.4\rangle \Big\}$$

$$f_A(e_2) = \Big\{ \langle x_1,\ 0.5, 0.4, 0.3, -0.5, -0.6, -0.5\rangle, \langle x_2,\ 0.6, 0.3, 0.4, -0.2, -0.4, -0.7\rangle,$$
$$\langle x_3,\ 0.4, 0.5, 0.3, -0.4, -0.5, -0.8\rangle, \langle x_4,\ 0.7, 0.3, 0.2, -0.5, -0.6, -0.2\rangle \Big\}$$

$$g_B(e_3) = \Big\{ \langle x_1,\ 0.7, 0.4, 0.5, -0.5, -0.6, -0.4 \rangle, \langle x_2,\ 0.5, 0.6, 0.2, -0.3, -0.5, -0.5 \rangle, \langle x_3,\ 0.3, 0.4, 0.2, -0.5, -0.5, -0.3 \rangle, \langle x_4,\ 0.4, 0.5, 0.5, -0.6, -0.4, -0.3 \rangle \Big\}$$

Then B_1 and B_2 are the bipolar neutrosophic soft sets. (parameterized bipolar neutrosophic sets over X).

3. Bipolar neutrosophic graded soft sets

In this section, we have extended the Bipolar neutrosophic soft set (BNSS), namely Bipolar neutrosophic graded soft sets (BNGS) by categorizing parameters as below.

Definition 3.1. Let X be the universe set and E be the parameter set which consists at least two parameters. We define the graded set $G = \{\mathcal{L}, \mathcal{M}, \mathcal{H}\}$ be subsets of the parameter set E in such a way that low priority, medium priority and high priority parameters respectively .

A bipolar neutrosophic graded soft set $\mathbb{BNGS}$ over X is defined by

$$\mathbb{BNGS}=(f_G, E) = \Big\{ \langle e, f_{\mathcal{L}}(e) \rangle, \langle e, f_{\mathcal{M}}(e) \rangle, \langle e, f_{\mathcal{H}}(e) \rangle : e \in E, f(e) \in BNS(X) \Big\}$$

Here

$$f_{\mathcal{L}}(e) = \Big\{ \Big\langle x,\ u^+_{f_{\mathcal{L}(e)}}(x), v^+_{f_{\mathcal{L}(e)}}(x), w^+_{f_{\mathcal{L}(e)}}(x), u^-_{f_{\mathcal{L}(e)}}(x), v^-_{f_{\mathcal{L}(e)}}(x), w^-_{f_{\mathcal{L}(e)}}(x) \Big\rangle : x \in X \Big\}$$

$$f_{\mathcal{M}}(e) = \Big\{ \Big\langle x,\ u^+_{f_{\mathcal{M}(e)}}(x), v^+_{f_{\mathcal{M}(e)}}(x), w^+_{f_{\mathcal{M}(e)}}(x), u^-_{f_{\mathcal{M}(e)}}(x), v^-_{f_{\mathcal{M}(e)}}(x), w^-_{f_{\mathcal{M}(e)}}(x) \Big\rangle : x \in X \Big\}$$

$$f_{\mathcal{H}}(e) = \Big\{ \Big\langle x,\ u^+_{f_{\mathcal{H}(e)}}(x), v^+_{f_{\mathcal{H}(e)}}(x), w^+_{f_{\mathcal{H}(e)}}(x), u^-_{f_{\mathcal{H}(e)}}(x), v^-_{f_{\mathcal{H}(e)}}(x), w^-_{f_{\mathcal{H}(e)}}(x) \Big\rangle : x \in X \Big\}.$$

where $u^+_{f_{\mathcal{L}(e)}}(x), v^+_{f_{\mathcal{L}(e)}}(x), w^+_{f_{\mathcal{L}(e)}}(x)$ represents positive truth, indeterminate and false values of x and similarly $u^-_{f_{\mathcal{L}(e)}}(x), v^-_{f_{\mathcal{L}(e)}}(x), w^-_{f_{\mathcal{L}(e)}}$ represents negative truth, indeterminate and false values of x for the graded parameter e and so on.

Example 3.2. Let $X = \{x_1, x_2, x_3\}$ be a set variety computers (alternatives) and let $E = \{e_1, e_2, e_3, e_4, e_5\}$ be set of parameters which represents 'Brand', 'Power consumption', 'Processor', 'Price' and 'Modern look', respectively.

People may have different opinions about their priorities. For example, we listed the possible preferences of peoples for the above case.

Old fashioned peoples/ Elders prefers only quality and durability and they won't bother about trendy look.

For them, the choices are

$\mathcal{L} = \{$Brand, Modern look$\}$, $\mathcal{M} = \{$Power consumption$\}$ and $\mathcal{H} = \{$Processor, Price$\}$

i.e. $\mathcal{L} = \{e_1, e_5\}$, $\mathcal{M} = \{e_2\}$ and $\mathcal{H} = \{e_3, e_4\}$

Modern peoples/ Students prefers good looking and latest technology with affordable price.

For them, the choices are

$\mathcal{L} = \{$Brand, Power consumption$\}$, $\mathcal{M} = \{$Price$\}$ and $\mathcal{H} = \{$Processor, Modern look$\}$
i.e. $\mathcal{L} = \{e_1, e_2\}$, $\mathcal{M} = \{e_4\}$ and $\mathcal{H} = \{e_3, e_5\}$
Professionals/ Office workers prefers quality best in class and they won't worry about budget. For them, the choices are
$\mathcal{L} = \{$Power consumption, Price$\}$, $\mathcal{M} = \{$Brand$\}$ and $\mathcal{H} = \{$Processor, Modern look$\}$
i.e. $\mathcal{L} = \{e_2, e_4\}$, $\mathcal{M} = \{e_1\}$ and $\mathcal{H} = \{e_3, e_5\}$

Example 3.3. Let $X = \{x_1, x_2, x_3\}$ be a set of alternatives and $E = \{e_1, e_2, e_3, e_4\}$ be the parameter set for X. The graded set G be $G = \{\mathcal{L}, \mathcal{M}, \mathcal{H}\}$.
In this problem, the graded parameters are $\mathcal{L} = \{e_4\}, \mathcal{M} = \{e_1, e_2\}, \mathcal{H} = \{e_3\}$. Here we define two $\mathbb{BNGS}$s $\mathbb{B}_1$ and $\mathbb{B}_2$ as follows.

$$\begin{aligned}\mathbb{B}_1 = (f_G, E) &= \left\{ \langle e, f_{\mathcal{L}}(e)\rangle, \langle e, f_{\mathcal{M}}(e)\rangle, \langle e, f_{\mathcal{H}}(e)\rangle : e \in E, f(e) \in BNS(X) \right\} \\ &= \left\{ \langle e_4, f_{\mathcal{L}}(e_4)\rangle, \langle e_1, f_{\mathcal{M}}(e_1)\rangle, \langle e_2, f_{\mathcal{M}}(e_2)\rangle \langle e_3, f_{\mathcal{H}}(e_3)\rangle \right\}\end{aligned}$$

Here,

$f_{\mathcal{L}}(e_4) =$
$\left\{\langle x_1,\ 0.3, 0.5, 0.4, -0.2, -0.5, -0.7\rangle, \langle x_2,\ 0.5, 0.4, 0.1, -0.3, -0.5, -0.2\rangle, \langle x_3,\ 0.1, 0.7, 0.3, -0.2, -0.4, -0.1\rangle\right\}$
$f_{\mathcal{M}}(e_1) =$
$\left\{\langle x_1,\ 0.1, 0.3, 0.2, -0.5, -0.2, -0.4\rangle, \langle x_2,\ 0.3, 0.7, 0.3, -0.2, -0.7, -0.5\rangle, \langle x_3,\ 0.7, 0.2, 0.4, -0.3, -0.4, -0.5\rangle\right\}$
$f_{\mathcal{M}}(e_2) =$
$\left\{\langle x_1,\ 0.4, 0.3, 0.1, -0.6, -0.7, -0.3\rangle, \langle x_2,\ 0.2, 0.5, 0.6, -0.3, -0.4, -0.7\rangle, \langle x_3,\ 0.4, 0.1, 0.7, -0.7, -0.1, -0.4\rangle\right\}$
$f_{\mathcal{H}}(e_3) =$
$\left\{\langle x_1,\ 0.3, 0.5, 0.7, -0.2, -0.1, -0.8\rangle, \langle x_2,\ 0.7, 0.2, 0.3, -0.1, -0.5, -0.7\rangle, \langle x_3,\ 0.2, 0.1, 0.8, -0.4, -0.5, -0.6\rangle\right\}$

Also,

$$\begin{aligned}\mathbb{B}_2 = (f_G, E) &= \left\{ \langle e, f_{\mathcal{L}}(e)\rangle, \langle e, f_{\mathcal{M}}(e)\rangle, \langle e, f_{\mathcal{H}}(e)\rangle : e \in E, f(e) \in BNS(X) \right\} \\ &= \left\{ \langle e_4, f_{\mathcal{L}}(e_4)\rangle, \langle e_1, f_{\mathcal{M}}(e_1)\rangle, \langle e_2, f_{\mathcal{M}}(e_2)\rangle \langle e_3, f_{\mathcal{H}}(e_3)\rangle \right\}\end{aligned}$$

Here,

$f_{\mathcal{L}}(e_4) =$
$\left\{\langle x_1,\ 0.2, 0.4, 0.3, -0.3, -0.6, -0.8\rangle, \langle x_2,\ 0.6, 0.5, 0.2, -0.2, -0.4, -0.1\rangle, \langle x_3,\ 0.2, 0.6, 0.4, -0.1, -0.3, -0.2\rangle\right\}$
$f_{\mathcal{M}}(e_1) =$
$\left\{\langle x_1,\ 0.2, 0.4, 0.3, -0.4, -0.1, -0.3\rangle, \langle x_2,\ 0.4, 0.8, 0.4, -0.3, -0.6, -0.4\rangle, \langle x_3,\ 0.6, 0.3, 0.5, -0.2, -0.3, -0.4\rangle\right\}$
$f_{\mathcal{M}}(e_2) =$
$\left\{\langle x_1,\ 0.3, 0.2, 0.3, -0.5, -0.6, -0.2\rangle, \langle x_2,\ 0.3, 0.6, 0.7, -0.2, -0.3, -0.8\rangle, \langle x_3,\ 0.5, 0.2, 0.6, -0.6, -0.2, -0.5\rangle\right\}$

$f_{\mathcal{H}}(e_3) =$
$\{\langle x_1,\ 0.2, 0.5, 0.6, -0.3, -0.1, -0.7\rangle, \langle x_2,\ 0.6, 0.2, 0.7, -0.3, -0.2, -0.4\rangle, \langle x_3,\ 0.4, 0.7, 0.8, -0.3, -0.2, -0.5\rangle\}$

3.1. *Properties of BNGS*

Let $\{\mathbb{B}_i : i = 1, 2...n\}$ be set of all Bipolar neutrosophic graded soft sets defined as below.
$\mathbb{B}_i = \{\langle e, f_{L_i}(e)\rangle, \langle e, f_{M_i}(e)\rangle, \langle e, f_{H_i}(e)\rangle : e \in E, f(e) \in \mathbb{BNS}\}$.
For any $i = 1, 2$, $L \cup M \cup H = E$ and $L \cap M \cap H = \phi$

Definition 3.4. Let $\mathbb{B}_1$ and $\mathbb{B}_2$ be two $\mathbb{BNGS}$s. Then their union $\mathbb{B}_1 \cup \mathbb{B}_2$ is defined as

$$\mathbb{B}_1 \cup \mathbb{B}_2 = \left\{\left\langle e, \cup_i f^{(i)}_{\mathcal{L}}(e)\right\rangle, \left\langle e, \cup_i f^{(i)}_{\mathcal{M}}(e)\right\rangle, \left\langle e, \cup_i f^{(i)}_{\mathcal{H}}(e)\right\rangle\right\}.$$

Here,

$$\bigcup_i f^{(i)}_{\mathcal{L}}(e) = \Big\{\langle x, max\left[u^+_{f_{L_i}(e)}(x)\right], min\left[v^+_{f_{L_i}(e)}(x)\right], min\left[w^+_{f_{L_i}(e)}(x)\right], min\left[u^-_{f_{L_i}(e)}(x)\right], max\left[v^-_{f_{L_i}(e)}(x)\right], max\left[w^-_{f_{L_i}(e)}(x)\right]\rangle\Big\}$$

$$\bigcup_i f^{(i)}_{\mathcal{M}}(e) = \Big\{\langle x, max\left[u^+_{M_{L_i}(e)}(x)\right], min\left[v^+_{M_{L_i}(e)}(x)\right], min\left[w^+_{M_{L_i}(e)}(x)\right], min\left[u^-_{M_{L_i}(e)}(x)\right], max\left[v^-_{M_{L_i}(e)}(x)\right], max\left[w^-_{M_{L_i}(e)}(x)\right]\rangle\Big\}$$

$$\bigcup_i f^{(i)}_{\mathcal{H}}(e) = \Big\{\langle x, max\left[u^+_{H_{L_i}(e)}(x)\right], min\left[v^+_{H_{L_i}(e)}(x)\right], min\left[w^+_{H_{L_i}(e)}(x)\right], min\left[u^-_{H_{L_i}(e)}(x)\right], max\left[v^-_{H_{L_i}(e)}(x)\right], max\left[w^-_{H_{L_i}(e)}(x)\right]\rangle\Big\}$$

Example 3.5. Consider the $\mathbb{BNGS}$ sets $\mathbb{B}_1$ and $\mathbb{B}_2$ defined in Example 3.3. Then their union is defined by
$\mathbb{B}_1 \cup \mathbb{B}_2 = \left\{\left\langle e_4, \cup_i f^{(i)}_{\mathcal{L}}(e_4)\right\rangle, \left\langle e_1, \cup_i f^{(i)}_{\mathcal{M}}(e_1)\right\rangle, \left\langle e_2, \cup_i f^{(i)}_{\mathcal{M}}(e_2)\right\rangle, \left\langle e_3, \cup_i f^{(i)}_{\mathcal{H}}(e_3)\right\rangle\right\}$ Here
$\cup_i f^{(i)}_{\mathcal{L}}(e_4) =$
$\{\langle x_1,\ 0.3, 0.4, 0.3, -0.3, -0.5, -0.7\rangle, \langle x_2,\ 0.6, 0.4, 0.1, -0.3, -0.4, -0.1\rangle, \langle x_3,\ 0.2, 0.6, 0.3, -0.2, -0.3, -0.1\rangle\}$
$\cup_i f^{(i)}_{\mathcal{M}}(e_1) =$
$\{\langle x_1,\ 0.2, 0.3, 0.2, -0.5, -0.1, -0.3\rangle, \langle x_2,\ 0.4, 0.7, 0.3, -0.3, -0.6, -0.4\rangle, \langle x_3,\ 0.7, 0.2, 0.4, -0.3, -0.3, -0.4\rangle\}$
$\cup_i f^{(i)}_{\mathcal{M}}(e_2) =$
$\{\langle x_1,\ 0.4, 0.2, 0.1, -0.6, -0.6, -0.2\rangle, \langle x_2,\ 0.3, 0.5, 0.6, -0.3, -0.3, -0.7\rangle, \langle x_3,\ 0.5, 0.1, 0.6, -0.7, -0.1, -0.4\rangle\}$
$\cup_i f^{(i)}_{\mathcal{H}}(e_3) =$
$\{\langle x_1,\ 0.3, 0.5, 0.6, -0.3, -0.1, -0.7\rangle, \langle x_2,\ 0.7, 0.2, 0.3, -0.3, -0.1, -0.7\rangle, \langle x_3,\ 0.4, 0.1, 0.8, -0.4, -0.2, -0.5\rangle\}$

Definition 3.6. Let $\mathbb{B}_1$ and $\mathbb{B}_2$ be two $\mathbb{BNGS}$s. Then their intersection $\mathbb{B}_1 \cap \mathbb{B}_2$ is defined as

$$\mathbb{B}_1 \cap \mathbb{B}_2 = \left\{ \left\langle e, \cap_i f_{\mathcal{L}}^{(i)}(e) \right\rangle, \left\langle e, \cap_i f_{\mathcal{M}}^{(i)}(e) \right\rangle, \left\langle e, \cap_i f_{\mathcal{H}}^{(i)}(e) \right\rangle \right\}.$$

Here,

$$\bigcap_i f_{\mathcal{L}}^{(i)}(e) = \Big\{ \langle x, min\left[u^+_{f_{L_i}(e)}(x)\right], max\left[v^+_{f_{L_i}(e)}(x)\right], max\left[w^+_{f_{L_i}(e)}(x)\right], max\left[u^-_{f_{L_i}(e)}(x)\right], min\left[v^-_{f_{L_i}(e)}(x)\right], min\left[w^-_{f_{L_i}(e)}(x)\right] \rangle \Big\}$$

$$\bigcap_i f_{\mathcal{M}}^{(i)}(e) = \Big\{ \langle x, min\left[u^+_{M_{L_i}(e)}(x)\right], max\left[v^+_{M_{L_i}(e)}(x)\right], max\left[w^+_{M_{L_i}(e)}(x)\right], max\left[u^-_{M_{L_i}(e)}(x)\right], min\left[v^-_{M_{L_i}(e)}(x)\right], min\left[w^-_{M_{L_i}(e)}(x)\right] \rangle \Big\}$$

$$\bigcap_i f_{\mathcal{H}}^{(i)}(e) = \Big\{ \langle x, min\left[u^+_{H_{L_i}(e)}(x)\right], max\left[v^+_{H_{L_i}(e)}(x)\right], max\left[w^+_{H_{L_i}(e)}(x)\right], max\left[u^-_{H_{L_i}(e)}(x)\right], min\left[v^-_{H_{L_i}(e)}(x)\right], min\left[w^-_{H_{L_i}(e)}(x)\right] \rangle \Big\}$$

Example 3.7. Consider the $\mathbb{BNGS}$ sets $\mathbb{B}_1$ and $\mathbb{B}_2$ defined in Example 3.3. Then their intersection is defined by
$\mathbb{B}_1 \cap \mathbb{B}_2 = \left\{ \left\langle e_4, \cap_i f_{\mathcal{L}}^{(i)}(e_4) \right\rangle, \left\langle e_1, \cap_i f_{\mathcal{M}}^{(i)}(e_1) \right\rangle, \left\langle e_2, \cap_i f_{\mathcal{M}}^{(i)}(e_2) \right\rangle, \left\langle e_3, \cap_i f_{\mathcal{H}}^{(i)}(e_3) \right\rangle \right\}$ Here
$\cap_i f_{\mathcal{L}}^{(i)}(e_4) =$
$\{\langle x_1,\ 0.2, 0.5, 0.4, -0.2, -0.6, -0.8\rangle, \langle x_2,\ 0.5, 0.5, 0.2, -0.2, -0.5, -0.2\rangle, \langle x_3,\ 0.1, 0.7, 0.4, -0.1, -0.4, -0.2\rangle\}$
$\cap_i f_{\mathcal{M}}^{(i)}(e_1) =$
$\{\langle x_1,\ 0.1, 0.4, 0.3, -0.4, -0.2, -0.4\rangle, \langle x_2,\ 0.3, 0.8, 0.4, -0.2, -0.7, -0.5\rangle, \langle x_3,\ 0.6, 0.3, 0.5, -0.2, -0.4, -0.5\rangle\}$
$\cap_i f_{\mathcal{M}}^{(i)}(e_2) =$
$\{\langle x_1,\ 0.3, 0.3, 0.3, -0.5, -0.7, -0.3\rangle, \langle x_2,\ 0.2, 0.6, 0.7, -0.2, -0.4, -0.8\rangle, \langle x_3,\ 0.4, 0.2, 0.7, -0.6, -0.2, -0.5\rangle\}$
$\cap_i f_{\mathcal{H}}^{(i)}(e_3) =$
$\{\langle x_1,\ 0.2, 0.5, 0.7, -0.2, -0.3, -0.8\rangle, \langle x_2,\ 0.6, 0.2, 0.7, -0.1, -0.5, -0.7\rangle, \langle x_3,\ 0.2, 0.7, 0.8, -0.3, -0.5, -0.6\rangle\}$

Remark 3.8. Suppose $\mathbb{B}_1$ and $\mathbb{B}_2$ are two $\mathbb{BNGS}$s with unequal number of graded parameters (i.e. the cardinality of L_1 and L_2 are not equal and so on.).
Let the universal parameter set $E = \{e_1, e_2, e_3, e_4\}$. We define two $\mathbb{BNGS}$s as follows.

$\mathbb{B}_1 = \{\langle e, f_{L_1}(e)\rangle, \langle e, f_{M_1}(e)\rangle, \langle e, f_{H_1}(e)\rangle : e \in E\}$
where $L_1 = \{e_1\}, M_1 = \{e_2\}, H_1 = \{e_3, e_4\}$.
$\mathbb{B}_2 = \{\langle e, f_{L_2}(e)\rangle, \langle e, f_{M_2}(e)\rangle, \langle e, f_{H_2}(e)\rangle : e \in E\}$
where $L_2 = \{e_1, e_2\}, H_2 = \{e_3, e_4\}$.

Then their union $\mathbb{B}_1 \cup \mathbb{B}_2$ is defined as

$$\mathbb{B}_1 \cup \mathbb{B}_2 = \left\{\langle e, f_{L_1 \cup L_2}(e)\rangle, \langle e, f_{M_1 \cup \phi}(e)\rangle, \langle e, f_{H_1 \cup H_2}(e)\rangle\right\}.$$

Also, the intersection $\mathbb{B}_1 \cap \mathbb{B}_2$ is defined as

$$\mathbb{B}_1 \cap \mathbb{B}_2 = \left\{\langle e, f_{L_1 \cap L_2}(e)\rangle, \langle e, f_{M_1 \cap \phi}(e)\rangle, \langle e, f_{H_1 \cap H_2}(e)\rangle\right\}.$$

Definition 3.9. Let $\mathbb{B}$ be a $\mathbb{BNGS}$. Then the complement of $\mathbb{B}$ is defined as

$$\mathbb{B}^c = \left\{\langle e, f_L^c(e)\rangle, \langle e, f_M^c(e)\rangle, \langle e, f_H^c(e)\rangle\right\}.$$

Here,

$$\begin{aligned} f_L^c(e) =& \left\{w^+_{f_L}(e), 1 - v^+_{f_L}(e), u^+_{f_L}(e), w^-_{f_L}(e), -1 - v^-_{f_L}(e), u^-_{f_L}(e)\right\} \\ f_M^c(e) =& \left\{w^+_{f_M}(e), 1 - v^+_{f_M}(e), u^+_{f_M}(e), w^-_{f_M}(e), -1 - v^-_{f_M}(e), u^-_{f_M}(e)\right\} \\ f_H^c(e) =& \left\{w^+_{f_H}(e), 1 - v^+_{f_H}(e), u^+_{f_H}(e), w^-_{f_H}(e), -1 - v^-_{f_H}(e), u^-_{f_H}(e)\right\} \end{aligned}$$

Example 3.10. Consider the $\mathbb{BNGS}$ set $\mathbb{B}_1$ in Example 3.3. Then the complement is defined by

$\mathbb{B}^c = \left\{\langle e, f^c_{\mathcal{L}}(e_4)\rangle, \langle e, f^c_{\mathcal{M}}(e_1)\rangle, \langle e, f^c_{\mathcal{M}}(e_2)\rangle, \langle e, f^c_{\mathcal{H}}(e_3)\rangle\right\}$. Here

$f^c_{\mathcal{L}}(e_4) =$

$\left\{\langle x_1,\ 0.4, 0.5, 0.3, -0.7, -0.5, -0.2\rangle, \langle x_2,\ 0.1, 0.6, 0.5, -0.2, -0.5, -0.3\rangle, \langle x_3,\ 0.3, 0.3, 0.1, -0.1, -0.6, -0.2\rangle\right\}$

$f^c_{\mathcal{M}}(e_1) =$

$\left\{\langle x_1,\ 0.2, 0.7, 0.1, -0.4, -0.8, -0.5\rangle, \langle x_2,\ 0.3, 0.3, 0.3, -0.5, -0.3, -0.2\rangle, \langle x_3,\ 0.4, 0.8, 0.7, -0.5, -0.6, -0.3\rangle\right\}$

$f^c_{\mathcal{M}}(e_2) =$

$\left\{\langle x_1,\ 0.1, 0.7, 0.4, -0.3, -0.3, -0.6\rangle, \langle x_2,\ 0.6, 0.5, 0.2, -0.7, -0.6, -0.3\rangle, \langle x_3,\ 0.7, 0.9, 0.4, -0.4, -0.9, -0.7\rangle\right\}$

$f^c_{\mathcal{H}}(e_3) =$

$\left\{\langle x_1,\ 0.7, 0.5, 0.3, -0.8, -0.9, -0.2\rangle, \langle x_2,\ 0.3, 0.8, 0.7, -0.7, -0.5, -0.1\rangle, \langle x_3,\ 0.8, 0.9, 0.2, -0.6, -0.5, -0.4\rangle\right\}$

Definition 3.11. Let $\phi_{\mathbb{B}}$ be a null $\mathbb{BNGS}$ and is defined as

$$\phi_{\mathbb{B}} = \left\{\ \langle e_i, f_\phi(e_i)\rangle : e \in E\right\}$$

Here $f_\phi(e_i) = \left\{\ \langle x_i, 0, 1, 1, 0, -1, -1\rangle : x \in X\right\}$

Definition 3.12. Let $1_{\mathbb{B}}$ be a complete $\mathbb{BNGS}$ and is defined as

$$1_{\mathbb{B}} = \left\{\ \langle e_i, f_C(e_i)\rangle : e \in E\right\}$$

Here $f_C(e_i) = \left\{\ \langle x_i, 1, 0, 0, -1, 0, 0\rangle : x \in X\right\}$

Proposition 3.13. *For any* $\mathbb{BNGS}$ *set,*

(i). $\mathbb{B} \cup \phi_{\mathbb{B}} = \mathbb{B}$

(ii). $\mathbb{B} \cup 1_{\mathbb{B}} = 1_{\mathbb{B}}$

(iii). $\mathbb{B} \cap \phi_{\mathbb{B}} = \phi_{\mathbb{B}}$

(iv). $\mathbb{B} \cap 1_{\mathbb{B}} = \mathbb{B}$

Proof. By the definition of union and intersection of $\mathbb{BNGS}$s, results are obvious. □

Proposition 3.14. *For any three $\mathbb{BNGS}$ sets $\mathbb{B}_1, \mathbb{B}_2$ and $\mathbb{B}_3$, the following relations are hold.*

(i). $\mathbb{B}_1 \cup \mathbb{B}_2 = \mathbb{B}_2 \cup \mathbb{B}_1$

(ii). $\mathbb{B}_1 \cap \mathbb{B}_2 = \mathbb{B}_2 \cap \mathbb{B}_1$

(iii). $\mathbb{B}_1 \cup (\mathbb{B}_2 \cup \mathbb{B}_3) = (\mathbb{B}_1 \cup \mathbb{B}_2) \cup \mathbb{B}_3$

(iv). $\mathbb{B}_1 \cap (\mathbb{B}_2 \cap \mathbb{B}_3) = (\mathbb{B}_1 \cap \mathbb{B}_2) \cap \mathbb{B}_3$

Proof. It is obvious. □

Proposition 3.15. *For any two $\mathbb{BNGS}$ sets $\mathbb{B}_1$ and $\mathbb{B}_2$, the following conditions are hold.[De'Morgans law]*

(i). $(\mathbb{B}_1 \cup \mathbb{B}_2)^c = (\mathbb{B}_1)^c \cap (\mathbb{B}_2)^c$

(ii). $(\mathbb{B}_1 \cap \mathbb{B}_2)^c = (\mathbb{B}_1)^c \cup (\mathbb{B}_2)^c$

Proof. It is obvious. □

Proposition 3.16. *For any three $\mathbb{BNGS}$ sets $\mathbb{B}_1, \mathbb{B}_2$ and $\mathbb{B}_3$, the following relations are hold.[Distributive law]*

(i). $\mathbb{B}_1 \cap (\mathbb{B}_2 \cup \mathbb{B}_3) = (\mathbb{B}_1 \cap \mathbb{B}_2) \cup (\mathbb{B}_1 \cap \mathbb{B}_3)$

(ii). $\mathbb{B}_1 \cup (\mathbb{B}_2 \cap \mathbb{B}_3) = (\mathbb{B}_1 \cup \mathbb{B}_2) \cap (\mathbb{B}_1 \cup \mathbb{B}_3)$

Proof. It is obvious. □

4. Bipolar neutrosophic graded soft topological space

Let X be a universal set which consists alternatives and $\mathbb{BNGS}(x)$ be the collection of all $\mathbb{BNGS}$s in X. Then the collection $\tau_{\mathbb{B}}$ containing all $\mathbb{BNGS}$s is called $\mathbb{BNGS}$-topology if it holds the following conditions.

(1) $\phi_{\mathbb{B}}, 1_{\mathbb{B}} \in \tau_{\mathbb{B}}$

(2) $\bigcup_{i \in n} \mathbb{B}_i \in \tau_{\mathbb{B}}$ for each $\mathbb{B}_i \in \tau_{\mathbb{B}}$

(3) $\mathbb{B}_i \cap \mathbb{B}_j \in \tau_{\mathbb{B}}$ for any $\mathbb{B}_i, \mathbb{B}_j \in \tau_{\mathbb{B}}$

Then the pair $(X, \tau_{\mathbb{B}})$ is called $\mathbb{BNGS}$-topological space. The members of $\tau_{\mathbb{B}}$ are called open $\mathbb{BNGS}$s and their complements are called closed $\mathbb{BNGS}$s.

Example 4.1. Let $X = x_1, x_2$ be set of alternatives and $E = e_1, e_2, e_3$ be a parameter set. We define the graded parameter set as $G = \mathcal{L} = e_1, \mathcal{M} = e_2, \mathcal{H} = e_3$. Now let us define a topology on (X, E) as follows.

$$\tau_{\mathcal{B}} = \left\{\phi_{\mathbb{B}}, 1_{\mathbb{B}}, \mathbb{B}_1, \mathbb{B}_2, \mathbb{B}_3, \mathbb{B}_4\right\}$$

Here $\phi_{\mathbb{B}}, 1_{\mathbb{B}}$ are null and complete $\mathbb{BNGS}$ respectively. Also,

$$\mathbb{B}_1 = \left\{\left\langle e_1, f^{(1)}_{\mathcal{L}}(e_1)\right\rangle, \left\langle e_2, f^{(1)}_{\mathcal{M}}(e_2)\right\rangle, \left\langle e_3, f^{(1)}_{\mathcal{H}}(e_3)\right\rangle\right\}$$
$$= \left\{\left\langle e_1, \{\langle x_1,\ 1, 0, 1, -1, 0, 0\rangle, \langle x_2,\ 0.5, 0.2, 0.4, -0.5, -0.4, -0.3\rangle\}\right\rangle,\right.$$
$$\left\langle e_2, \{\langle x_1,\ 0.4, 0.6, 0.3, -0.4, -0.7, -0.2\rangle, \langle x_2,\ 0.7, 0.2, 0.1, -0.3, -0.5, -0.7\rangle\}\right\rangle,$$
$$\left.\left\langle e_3, \{\langle x_1,\ 0.5, 0.3, 0.7, -0.2, -0.4, -0.8\rangle, \langle x_2,\ 0.4, 0.3, 0.5, -0.1, -0.4, -0.6\rangle\}\right\rangle\right\}$$
$$\mathbb{B}_2 = \left\{\left\langle e_1, f^{(2)}_{\mathcal{L}}(e_1)\right\rangle, \left\langle e_2, f^{(2)}_{\mathcal{M}}(e_2)\right\rangle, \left\langle e_3, f^{(2)}_{\mathcal{H}}(e_3)\right\rangle\right\}$$
$$= \left\{\left\langle e_1, \{\langle x_1,\ 0.3, 0.1, 0.7, -0.5, -0.6, -0.3\rangle, \langle x_2,\ 0, 1, 1, -0.7, 0, -1\rangle\}\right\rangle,\right.$$
$$\left\langle e_2, \{\langle x_1,\ 0.2, 0.5, 0.7, -1, 0, -0.2\rangle, \langle x_2,\ 0.9, 0.1, 0.3, -0.1, -0.6, -0.3\rangle\}\right\rangle,$$
$$\left.\left\langle e_3, \{\langle x_1,\ 0.3, 0.5, 0.3, -0.2, 0, -0.4\rangle, \langle x_2,\ 0.7, 0.4, 0.1, -0.3, -0.5, -0.1\rangle\}\right\rangle\right\}$$
$$\mathbb{B}_3 = \left\{\left\langle e_1, f^{(3)}_{\mathcal{L}}(e_1)\right\rangle, \left\langle e_2, f^{(3)}_{\mathcal{M}}(e_2)\right\rangle, \left\langle e_3, f^{(3)}_{\mathcal{H}}(e_3)\right\rangle\right\}$$
$$= \left\{\left\langle e_1, \{\langle x_1,\ 1, 0, 0.7, -1, 0, 0\rangle, \langle x_2,\ 0.5, 0.2, 0.4, -0.7, 0, -0.3\rangle\}\right\rangle,\right.$$
$$\left\langle e_2, \{\langle x_1,\ 0.4, 0.5, 0.3, -1, 0, -0.2\rangle, \langle x_2,\ 0.9, 0.1, 0.1, -0.3, -0.5, -0.3\rangle\}\right\rangle,$$
$$\left.\left\langle e_3, \{\langle x_1,\ 0.5, 0.3, 0.3, -0.2, 0, -0.4\rangle, \langle x_2,\ 0.7, 0.3, 0.1, -0.3, -0.4, -0.1\rangle\}\right\rangle\right\}$$
$$\mathbb{B}_4 = \left\{\left\langle e_1, f^{(4)}_{\mathcal{L}}(e_1)\right\rangle, \left\langle e_2, f^{(4)}_{\mathcal{M}}(e_2)\right\rangle, \left\langle e_3, f^{(4)}_{\mathcal{H}}(e_3)\right\rangle\right\}$$
$$= \left\{\left\langle e_1, \{\langle x_1,\ 0.3, 0.1, 1, -0.5, -0.6, -0.3\rangle, \langle x_2,\ 0, 1, 1, -0.5, -0.4, -1\rangle\}\right\rangle,\right.$$
$$\left\langle e_2, \{\langle x_1,\ 0.2, 0.6, 0.7, -0.4, -0.7, -0.2\rangle, \langle x_2,\ 0.7, 0.2, 0.3, -0.1, -0.6, -0.7\rangle\}\right\rangle,$$
$$\left.\left\langle e_3, \{\langle x_1,\ 0.3, 0.5, 0.7, -0.2, -0.4, -0.8\rangle, \langle x_2,\ 0.4, 0.4, 0.5, -0.1, -0.5, -0.6\rangle\}\right\rangle\right\}$$

Here, $\mathbb{B}_1 \cup \mathbb{B}_2 = \mathbb{B}_3, \mathbb{B}_2 \cup \mathbb{B}_3 = \mathbb{B}_3, \mathbb{B}_1 \cup \mathbb{B}_3 = \mathbb{B}_3, \mathbb{B}_3 \cup \mathbb{B}_4 = \mathbb{B}_3$ and so on. Also, $\mathbb{B}_1 \cap \mathbb{B}_2 = \mathbb{B}_4, \mathbb{B}_2 \cap \mathbb{B}_3 = \mathbb{B}_4, \mathbb{B}_1 \cap \mathbb{B}_3 = \mathbb{B}_4, \mathbb{B}_3 \cap \mathbb{B}_4 = \mathbb{B}_4$ and so on.

The $\tau_{\mathbb{B}}$ satisfies all three conditions of topology. So $\tau_{\mathbb{B}}$ is a $\mathbb{BNGS}$-topology.

Proposition 4.2. *Let $(X, \tau_{\mathbb{B}})$ be an $\mathbb{BNGS}$. Then the following conditions hold.*

- *$\phi_{\mathbb{B}}$ and $1_{\mathbb{B}}$ are open $\mathbb{BNGS}$s.*

- *Union of any number of open* $\mathbb{BNGS}$*s is open.*
- *Intersection of finite number of closed* $\mathbb{BNGS}$*s is closed.*

Definition 4.3. Let $(X, \tau_{\mathbb{B}})$ and $(X, \tau'_{\mathbb{B}})$ be two $\mathbb{BNGS}$ in X. Two $\mathbb{BNGS}$'s are said to be Comparable if $\tau_{\mathbb{B}} \subseteq \tau'_{\mathbb{B}}$ or $\tau'_{\mathbb{B}} \subseteq \tau_{\mathbb{B}}$.
If $\tau_{\mathbb{B}} \subseteq \tau'_{\mathbb{B}}$, then $\tau_{\mathbb{B}}$ is courser or weaker than $\tau'_{\mathbb{B}}$. In other words, $\tau'_{\mathbb{B}}$ is stronger or finer than $\tau_{\mathbb{B}}$ and vice versa.

4.1. *Example*

Proposition 4.4. *Let* $(X, \tau_{\mathbb{B}_1})$ *and* $(X, \tau_{\mathbb{B}_2})$ *be two* $\mathbb{BNGS}$*-topological spaces over* (X, E)*. Suppose* $\tau_{\mathbb{B}_1} \cap \tau_{\mathbb{B}_2} = \{B : B \in P(X)\}$*. Then* $\tau_{\mathbb{B}_1} \cap \tau_{\mathbb{B}_2}$ *is also a* $\mathbb{BNGS}$*-topology.*

Proof. $B_1 \cap B_2$ must satisfy topology conditions in order to be a $\mathbb{BNGS}$-topology.

i). Clearly $\phi_{\mathbb{B}}, 1_{\mathbb{B}} \in \tau_{\mathbb{B}_1} \cap \tau_{\mathbb{B}_2}$.

ii). Let $B_1, B_2 \in \tau_{\mathbb{B}_1} \cap \tau_{\mathbb{B}_2}$
$\Rightarrow B_1, B_2 \in \mathbb{B}_1 and B_1, B_2 \in \mathbb{B}_2$
$\Rightarrow B_1 \cap B_2 \in \mathbb{B}_1 and B_1 \cap B_2 \in \mathbb{B}_2$
$\Rightarrow B_1 \cap B_2 \in \tau_{\mathbb{B}_1} \cap \tau_{\mathbb{B}_2}$

iii). Let $\{B_i\} \in \tau_{\mathbb{B}_1} \cap \tau_{\mathbb{B}_2}$
$\Rightarrow \{B_i\} \in \tau_{\mathbb{B}_1} and \{B_i\} \in \tau_{\mathbb{B}_2}$
$\Rightarrow \cup_i B_i \in \tau_{\mathbb{B}_1} and \cup_i B_i \in \tau_{\mathbb{B}_2}$
$\Rightarrow \cup_i B_i \in \tau_{\mathbb{B}_1} \cap \tau_{\mathbb{B}_2}$.

Hence $\tau_{\mathbb{B}_1} \cap \tau_{\mathbb{B}_2}$ in a $\mathbb{BNGS}$-topology. □

Remark 4.5. The union of any two $\mathbb{BNGS}$-topologies may or may not be a topology.
Since it may or may not satisfy the topology conditions.

i). $\phi_{\mathbb{B}}, 1_{\mathbb{B}} \in \tau_{\mathbb{B}_1} \cup \tau_{\mathbb{B}_2}$ always holds.

ii). For $B_1, B_2 \in \tau_{\mathbb{B}_1} \cup \tau_{\mathbb{B}_2}$, both B_1 and B_2 may or may not be in both $\tau_{\mathbb{B}_1}$ and $\tau_{\mathbb{B}_2}$.

Definition 4.6. Let $(X, \tau_{\mathbb{B}})$ be a $\mathbb{BNGS}$-topological space over (X, E). Let $B \in \mathbb{BNGS}(X, E)$.
Then the interior of B is defined by
$B^o = \bigcup \{N : N$is a bipolar neutrosophic graded soft open set and $N \in B\}$.
i.e. It is the union of all open $\mathbb{BNGS}$ open subsets of B.

Proposition 4.7. *Let* $(X, \tau_{\mathbb{B}})$ *be a* $\mathbb{BNGS}$*-topological space over* (X, E) *and* $B_1, B_2 \in \mathbb{BNGS}(X, E)$*. Then*

(i). $B_1^o \in B_1$ *and* B_1^o *is the largest open set.*

(ii). $B_1 \in B_2 \Rightarrow B_1^o \in B_2^o$.

(iii). B_1^o *is an open* $\mathbb{BNGS}$*. i.e.* $B_1^o \in \tau_{\mathbb{B}}$.

(iv). B_1 *is* $\mathbb{BNGS}$ *soft open set if and only if* $B_1^o = B_1$.

(v). $(B_1^o)^o = B_1^o$.

(vi). $(\phi_{\mathbb{B}\not{k}})^o = \phi_{\mathbb{B}}$ *and* $(1_{\mathbb{B}})^o = 1_{\mathbb{B}}$.

(vii). $(B_1 \cap B_2)^o = B_1^o \cap B_2^o$.

(viii). $B_1^o \cup B_2^o \subset (B_1 \cup B_2)^o$.

Proof. (i) Since B_1^o is the union of all open sets in B_1, B_1^o is the largest open set which contained in B_1.

(ii) Let $B_1 \in B_2 \Rightarrow B_1^o \subset B_1 \subset B_2 \Rightarrow B_1^o \subset B_2$and also $B_2^o \subset B_2$.
But B_2^o is the largest open set in B_2. Hence $B_1^o \subset B_2^o$.

(iii) By definition of $\mathbb{BNGS}$-toppology $\tau_{\mathbb{B}}$, it is obvious.

(iv) $B_1^o \subset B_1$ and let B_1 be bipolar neutrosophic graded soft open set.
$B_1 \subset B_1 \Rightarrow B_1 \subset \cap\{B_2 \in \tau_{\mathbb{B}} : B_2 \subset B_1\} = B_1^o$
$\Rightarrow B_1 \subset B_1^o \Rightarrow B_1 = B_1^o$. Conversely, let $B_1 = B_1^o$. Then $B_1 = B_1^o \in \tau_{\mathbb{B}} \Rightarrow B_1$ is open bipolar neutrosophic graded soft open set.

(v) If B_1 is an open $\mathbb{BNGS}$, then $B_1^o = B_1$. Clearly B_1^o is an open $\mathbb{BNGS}$. Hence $(B_1^o)^o = B_1^o$.

(vi) Since $\phi_{\mathbb{B}}, 1_{\mathbb{B}} \in \tau_{\mathbb{B}}$. So they are open $\mathbb{BNGS}$. Hence it is obvious from (iv).

(vii) $B_1 \cap B_2 \subset B_1$ and $B_1 \cap B_2 \subset B_2 \Rightarrow (B_1 \cap B_2)^o \subset B_1^o$ and $(B_1 \cap B_2)^o \subset B_2^o$
$\Rightarrow (B_1 \cap B_2)^o \subset B_1^o \cap B_2^o$.
Further, $B_1^o \subset B_1$ and $B_2^o \subset B_2$. Then $B_1^o \cap B_2^o \subset B_1 \cap B_2$. But $(B_1 \cap B_2)^o \subset B_1 \cap B_2$ and it is the largest open set. So $B_1^o \cap B_2^o \subset (B_1 \cap B_2)^o$.
Hence $(B_1 \cap B_2)^o = B_1^o \cap B_2^o$.

(viii) $B_1 \subset B_1 \cup B_2$ and $B_2 \subset B_1 \cup B_2$. $\Rightarrow B_1^o \subset (B_1 \cup B_2)^o$and $B_1^o \subset (B_1 \cup B_2)^o$
$\Rightarrow B_1^o \cup B_2^o \subset (B_1 \cup B_2)^o$.

□

Definition 4.8. Let $(X, \tau_{\mathbb{B}})$ be a $\mathbb{BNGS}$-topological space over (X, E) and $B_1 \in \mathbb{BNGS}(X, E)$. Then the closure of B is defined by
$\overline{B} = \bigcap\{N : N$ is bipolar neutrosophic graded soft closed set and $N \supset B\}$.
i.e. It is the intersection of all bipolar neutrosophic graded soft closed subsets of B.

Proposition 4.9. *Let* $(X, \tau_{\mathbb{B}})$ *be a* $\mathbb{BNGS}$*-topological space over* (X, E) *and* $B_1, B_2 \in \mathbb{BNGS}(X, E)$*. Then*

(i). $B_1 \subset \overline{B_1}$ *and* $\overline{B_1}$ *is the smallest closed set.*

(ii). $B_1 \subset B_2 \Rightarrow \overline{B_1} \subset \overline{B_2}$.

(iii). $\overline{B_1}$ *is closed* $\mathbb{BNGS}$*. i.e.* $\overline{B_1} \in \tau_{\mathbb{B}}^c$.

(iv). B_1 *is* $\mathbb{BNGS}$ *closed set if and only if* $\overline{B_1} = B_1$.

(v). $\overline{\overline{B_1}} = \overline{B_1}$.

(vi). $\overline{\phi_{\mathbb{B}}} = \phi_{\mathbb{B}}$ *and* $\overline{1_{\mathbb{B}}} = 1_{\mathbb{B}}$.

(vii). $\overline{B_1 \cup B_2} = \overline{B_1} \cup \overline{B_2}$.

(viii). $\overline{B_1 \cup B_2} \subset \overline{B_1} \cap \overline{B_2}$.

Proof. (i) Since $\overline{B_1}$ is the intersection of all closed sets in B_1, $\overline{B_1}$ is the smallest closed set which contains B_1.

(ii) Let $B_1 \subset B_2$. Also $B_1 \subset \overline{B_1}$ and $B_2 \subset \overline{B_2} \Rightarrow B_1 \subset B_2 \subset \overline{B_2}$.
But $\overline{B_1}$ is the smallest set containing B_1. So $B_1 \subset \overline{B_1} \subset \overline{B_2}$. Hence $\overline{B_1} \subset \overline{B_2}$.

(iii) By definition of $\mathbb{BNGS}$-toppology $\tau_{\mathbb{B}}$ and $\overline{B_1}$, it is obvious.

(iv) $B_1 \subset \overline{B_1}$ and let B_1 be bipolar neutrosophic graded soft closed set. Then $B_1 \subset B_1$.
$\overline{B_1} = \bigcap \{B_2 \in \tau_{\mathbb{B}}^c : B_2 \supset B_1\} \subset \{B_1 \in \tau_{\mathbb{B}}^c : B_1 \supset B_1\} = B_1$
$\Rightarrow \overline{B_1} \subset B_1$
$\Rightarrow B_1 = \overline{B_1}$. Conversely, let $B_1 = \overline{B_1}$. Then $(\overline{B_1})^c \in \tau_{\mathbb{B}} \Rightarrow B_1^c \in \tau_{\mathbb{B}}$
$\Rightarrow B_1^c$ is open $\Rightarrow B_1$ is closed.

(v) If N is closed $\mathbb{BNGS}$, then $N = \overline{N}$. But $\overline{N}$ is closed by default. Replacing N by $\overline{B_1}$, we get $\overline{\overline{B_1}} = \overline{B_1}$.

(vi) Since $\phi_{\mathbb{B}}, 1_{\mathbb{B}} \in \tau_{\mathbb{B}}$ are both open and closed. So the result is obvious by (iv).

(vii) $B_1 \subset B_1 \cup B_2$ and $B_2 \subset B_1 \cup B_2 \Rightarrow \overline{B_1} \subset \overline{B_1 \cup B_2}$ and $\overline{B_2} \subset \overline{B_1 \cup B_2}$
$\Rightarrow \overline{B_1} \cup \overline{B_2} \subset \overline{B_1 \cup B_2}$.

Also, $B_1 \subset \overline{B_1}$ and $B_2 \subset \overline{B_2}$. Then $B_1 \cup B_2 \subset \overline{B_1} \cup \overline{B_2}$.
But $B_1 \cup B_2 \subset \overline{B_1 \cup B_2} \subset \overline{B_1} \cup \overline{B_2}$.
Hence $\overline{B_1 \cup B_2} = \overline{B_1} \cup \overline{B_2}$.

(viii) $B_1 \cap B_2 \subset B_1$ and $B_1 \cap B_2 \subset B_2$. $\Rightarrow \overline{B_1 \cap B_2} \subset \overline{B_1}$and $\overline{B_1 \cap B_2} \subset \overline{B_2}$
$\Rightarrow \overline{B_1 \cap B_2} \subset \overline{B_1} \cap \overline{B_2}$.

□

Definition 4.10. Let $(X, \tau_{\mathbb{B}})$ be a $\mathbb{BNGS}$-topological space over (X, E) and $B \in \mathbb{BNGS}(X, E)$. Then the boundary of B is denoted by $Bd(B)$ and is defined by $Bd(B) = \overline{B} \cap \overline{B^c}$.

Proposition 4.11. *Let* $(X, \tau_{\mathbb{B}})$ *be a* $\mathbb{BNGS}$*-topological space over* (X, E) *and* $B \in \mathbb{BNGS}(X, E)$*. Then* *(i).* $B^o \cap Bd(B) = \phi_{\mathbb{B}}$.

(ii). $\overline{B} = B^o \cup Bd(B)$.

(iii). $Bd(B) = \phi_{\mathbb{B}}$ *if and only if* B *is both closed and open.*

(iv). $Bd(B) = \overline{B} \cap (B^o)^c$.

Proof. (i) $B^o \cap Bd(B) = B^o \cap (\overline{B} \cap \overline{B^c}) = B^o \cap (\overline{B} \cap (B^o)^c)$
$= B^o \cap (B^o)^c \cap \overline{B} = \phi_{\mathbb{B}} \cap \overline{B} = \phi_{\mathbb{B}}$.

(ii) $B^o \cup Bd(B) = B^o \cup (\overline{B} \cap \overline{B^c}) = B^o \cup (\overline{B} \cap (B^o)^c)$
$= (B^o \cup \overline{B}) \cap (B^o \cup (B^o)^c) = (B^o \cup \overline{B}) \cap 1_{\mathbb{B}}$
$= (B^o \cup \overline{B}) = \overline{B}$. [Since $B^o \subset B \subset \overline{B}$]

(iii) $Bd(B) = \overline{B} \cap \overline{B^c} = \phi_{\mathbb{B}}$
$\Rightarrow \overline{B} \cap (B^o)^c = \phi_{\mathbb{B}} \Rightarrow \overline{B} \cap ((B^o)^c)^c \neq \phi_{\mathbb{B}}$
$\Rightarrow \overline{B} \cap B^o \neq \phi_{\mathbb{B}} \Rightarrow \overline{B} \subset B^o$
$\Rightarrow B \subset \overline{B} \subset B^o \Rightarrow B \subset B^o$.
Also we know that $B^o \subset B$. Hence $B = B^o \Rightarrow B$ is open.
Further $\overline{B} \subset B^o \subset B \Rightarrow \overline{B} \subset B$, but we have $B \subset \overline{B}$
$\Rightarrow B = \overline{B} \Rightarrow B$ is closed.
Conversely, if B is both open and closed, then $B = B^o$ and $B = \overline{B}$.
Now $Bd(B) = \overline{B} \cap \overline{B^c} = \overline{B} \cap (B^o)^c = B \cap B^c = \phi_{\mathbb{B}}$.

(iv) $Bd(B) = \overline{B} \cap \overline{B^c} = \overline{B} \cap (B^o)^c$.

□

5. Conclusion

Bipolar neutrosophic graded soft sets and some of their properties with real life examples were proposed in this paper. BNGS is the extension of bipolar neutrosophic soft set by categorizing parameter set. Further, we proposed bipolar neutrosophic graded soft topological spaces with their properties and some propositions about the BNGS-topology. In future, we will try to explore the real life applications and construct the algorithm based the BNGS set and their topological structure.

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Received: Nov 30, 2021. Accepted: Feb 3, 2022