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Neutrosophic Cubic $\beta-$subalgebra

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Abstract. The main objective of this paper is to extend the notion of neutrosophic cubic sets to $\beta-$subalgebra. Some captivating results based on the P-union, P-intersection, R-union, R-intersection of neutrosophic cubic $\beta-$subalgebra have been explored. Further, the engrossing properties of the lower, upper level sets and homomorphism of neutrosophic cubic $\beta-$subalgebras were discussed.

Keywords: cubic set; Neutrosophic set; Neutrosophic cubic set; $\beta-$subalgebra; cubic $\beta-$subalgebra; neutrosophic cubic $\beta-$subalgebra.

1. Introduction

In 1965, Zadeh [30] initiated the concept of fuzzy sets which is a generalisation of the classical notion of a set. The notion of intuitionistic fuzzy set was proposed by Atanassov [4] whose elements have both membership and non-membership degrees. Biswas [5] introduced Rosenfeld's fuzzy subgroups with interval valued membership functions and studied some interesting properties. The idea of $\beta-$algebras has been presented by Neggers and Kim [23] which is a generlization of $BCK/BCI-$algebras where two operations have been used. Samarandache [27] proposed a generlization of intuitionistic fuzzy sets, known as neutrosophic set in which the distinction between the neutrosophic set and intuitionistic fuzzy set are emphasised. The notion of cubic sets introduced by Jun et al. [10, 11] and investigated the characteristics of cubic subgroups. Maji [16] applied the idea of soft set into neutrosophic sets and studied some compelling results.

The thought of fuzzy $\beta-$subalgebras originated by Ansari et al. [3] and relevant results have been examined. The attributes on intuitionistic fuzzy $\beta-$subalgebras were presented by Sujatha et al. [28]. Iqbal et al. [9] developed the idea of neutrosophic cubic subalgebras and ideals of B-algebras. The concept of neutrosophic cubic sets initiated by Jun et al. [12], [13], [14] and they have extended notion of neutrosophic subalgebras set to several types of $BCK/BCI-$algebras. Moreover, the applications of cubic interval valued intuitionistic fuzzy sets in $BCK/BCI-$algebras were provided. Hemavathi et al. [8] expressed the characteristics on interval valued intuitionistic fuzzy $\beta-$subalgebras. Made an approach on normed linear space using neutrosophic sets by Muralikrishna et al. [20] and examined the fascinating results.

The notion of $BMBJ$- neutrosophic aubalgebra in $BCK/BCI-$algebras presented by Bordbar et al. [6] and provided some engrossing results. Ajay et al. [1] discussed about neutrosophic cubic fuzzy dombi hamy mean operators with application to multi-criteria decision making. Akbar Razaei et al. [2] initiated the thought of neutrosophic triplet of $BI-$algebras and relevant results have been studied. Neutrosophic logic theory and applications were developed by Eman AboElHamd et al. [7]. Some aspects on cubic fuzzy $\beta-$subalgebra of $\beta-$ algebra were discussed by Muralikrishna et al. [21]. Mohsin Khalid et al [17], [18], [19] interpreted the concept of translation and multiplication of neutrosophic cubic set and also introduced the notion of $T-MBJ$neutrosophic set under $M-$subalgebra. Moreover, the authors described the properties of $T-$neutrosophic cubic set on $BF-$algebra. Some special characteristics of neutrosophic vague binary $BCK/BCI-$algebra were discussed by Remya et al. [26]. Nanthini et al. [22] initiated the idea of interval valued neutrosophic topological spaces and relevant results have been examined.Diagnosing psychiatric disorder using neutrosophic soft set and its application presented deliberately by Veerappan Chinnadurai et al. [29]. Rajab Ali Borsooei et al. [25] intended to develop the polarity of generalized neutrosophic subalgebras in $BCK/BCI-$algebras. Johnson Awolola [15] introduced the concept of $\alpha-$level sets of neutrosophic set and investigated few of its associated properties. Prakasam Muralikrishna et al. [24] applied the concept of $\beta-$ideal into $MBJ-$neutrosophic set and investigated some engrossing results. With all these inspiration, this paper provides the study of neutrosophic cubic $\beta-$subalgebra. This work is organized into the following sections: Section 1 provides the introduction and section 2 presents the existing definitions required for this study. Section 3 deals the concept of neutrosophic cubic $\beta-$subalgebra, section 4 describes the characteristics on homomorphism of neutrosophic cubic $\beta-$subalgebra and section 5 gives the conclusion and future scope of the work.

2. Preliminaries

This section provides the necessary definitions and examples required for the work.

Definition 2.1. A fuzzy set in a universal set X is defined as $\zeta : X \rightarrow [0,1]$. For each element $x \in X$, $\zeta(x)$ is called the membership value of x.

Definition 2.2. If ζ_1 and ζ_2 are fuzzy sets in X, then the union of ζ_1 and ζ_2, denoted by $\zeta_1 \cup \zeta_2$ is defined by, $(\zeta_1 \cup \zeta_2)(x) = max\{\zeta_1(x), \zeta_2(x)\}\ \forall\ x \in X$.

Definition 2.3. If ζ_1 and ζ_2 are fuzzy sets in X, then the intersection of ζ_1 and ζ_2, denoted by $\zeta_1 \cap \zeta_2$ is defined by, $(\zeta_1 \cap \zeta_2)(x) = min\{\zeta_1(x), \zeta_2(x)\}\ \forall\ x \in X$.

Definition 2.4. Let ζ be a fuzzy set of X. for any $\delta \in [0,1]$, the set $\zeta^\delta = \{x \in X/\zeta(x) \geq \delta\}$ is called an upper level subset of ζ. The level subset ζ^δ of a fuzzy set ζ is a crisp subset of the set X.

Definition 2.5. Let ζ be a fuzzy set of X. For $\delta \in [0,1]$, the set $\zeta_\delta = \{x \in X : \zeta(x) \leq \delta\}$ is called a lower level subsets of ζ.

Definition 2.6. The supremum property of the fuzzy set ζ for the subset A in X is defined as $\zeta(a_0) = \underset{a \in A}{Sup}\ \zeta(a)$ if there exist $a,\ a_0 \in A$.

Definition 2.7. Let $D[0,1]$ denote the family of all closed sub intervals of $[0,1]$. Consider two elements $D_1,D_2 \in D[0,1]$. If $D_1 = [a_1,b_1]$ and $D_2 = [a_2,b_2]$, then $rmax(D_1,D_2) = [max(a_1,a_2), max(b_1,b_2)]$ which is denoted by $D_1 \bigvee^r D_2$ and $rmin(D_1,D_2) = [min(a_1,a_2), min(b_1,b_2)]$ which is denoted by $D_1 \bigwedge^r D_2$.
Thus if $D_i = [a_i,b_i] \in D[0,1]$ for i=1,2,3.... $rsup_i(D_i) = [sup_i(a_i), sup_i(b_i)]$, i.e. $\bigvee_i^r D_i = [\bigvee_i a_i, \bigvee_i b_i]$. Similarly $rinf_i(D_i) = [inf_i(a_i), inf_i(b_i)]$ i.e $\bigwedge_i^r D_i = [\bigwedge_i a_i, \bigwedge_i b_i]$. Now $D_1 \geq D_2$ iff $a_1 \geq a_2$ and $b_1 \geq b_2$. Similarly the relations $D_1 \leq D_2$ and $D_1 = D_2$ are defined.

Definition 2.8. An interval valued fuzzy set A defined on X is given by A=$\{(x,[\zeta_A^L(x), \zeta_A^U(x)])\}\ \forall\ x \in X$ (briefly denoted by $A = [\zeta_A^L, \zeta_A^U]$), where ζ_A^L and ζ_A^U are two fuzzy sets in X such that $\zeta_A^L(x) \leq \sigma_A^U(x)\ \forall\ x \in X$. Let $\overline{\zeta}_A(x) = [\zeta_A^L(x), \zeta_A^U(x)]\ \forall\ x \in X$ and let $D[0,1]$ denotes the family of all closed sub intervals of $[0,1]$. If $\zeta_A^L(x) = \zeta_A^U(x) = c$, say, where $0 \leq c \leq 1$, then $\overline{\zeta}_A(x) = [c,c]$ also for the sake of convenience, to belong to $D[0,1]$. Thus $\overline{\zeta}_A(x) \in D[0,1]\ \forall\ x \in X$, and therefore the interval valued fuzzy set A is given by
$A = \{(x, \overline{\zeta}_A(x))\}\ \forall\ x \in X$, where $\overline{\zeta}_A : X \rightarrow D[0,1]$.
Now let us define what is known as *refined mimimum*$(rmin)$ of two elements in $D[0,1]$. Let us define the symbols " $\geq$ " , " $\leq$ ", and " $=$ " in case of two elements in $D[0,1]$. Consider two elements $D_1 := [a_1,b_1]$ and $D_2 := [a_2,b_2] \in D[0,1]$. Then $rmin(D_1,D_2) =$

$[min\{a_1, a_2\}, min\{b_1, b_2\}]$; $D_1 \geq D_2$ if and only if $a_1 \geq a_2,\ b_1 \geq b_2$; Similarly , $D_1 \leq D_2$ and $D_1 = D_2$.

Definition 2.9. An Intuitionistic fuzzy set (IFS) in a nonempty set X is defined by A= $\{\langle x, \zeta_A(x), \eta_A(x)\rangle / x \in X\}$ where $\zeta_A : X \rightarrow [0,1]$ is a membership function of A and $\eta_A : X \rightarrow [0,1]$ is a non membership function of A satisfying $0 \leq \zeta_A(x) + \eta_A(x) \leq 1\ \forall\ x \in X$.

Definition 2.10. An intuitionistic fuzzy set A is said to have sup-inf property if for any subset T of X there exists $x_0 \in T$ such that $\zeta_A(x_0) = \underset{x \in T}{Sup}\ \zeta_A(x)$ and $\eta_A(x_0) = \underset{x \in T}{inf}\ \eta_A(x)$.

Definition 2.11. A $\beta-$ algebra is a non-empty set X with a constant 0 and two binary operations $+$ and $-$ satisfying the following axioms:

(i) $x - 0 = x$

(ii) $(0 - x) + x = 0$

(iii) $(x - y) - z = x - (z + y)\ \forall\ x, y, z \in X$.

Example 2.12. The following Cayley table shows $(X = \{0, 1, 2, 3\}, +, -, 0)$ is a $\beta-$algebra.

Table 1. β-algebra

+	0	1	2	3
0	0	1	2	3
1	1	3	0	2
2	2	0	3	1
3	3	2	1	0

−	0	1	2	3
0	0	2	1	3
1	1	0	3	2
2	2	3	0	1
3	3	1	2	0

Definition 2.13. A non empty subset A of a $\beta-$algebra $(X, +, -, 0)$ is called a β-subalgebra of X, if

(i) $x + y \in A$ and

(ii) $x - y \in A\ \ \forall\ x, y \in A$.

Definition 2.14. Let X be a non empty set. By a cubic set in X we mean a structure

$$C = \{\langle x, \overline{\zeta}_C(x), \eta_C(x)\rangle :\ x \in\ X\}$$

in which $\overline{\zeta}_C$ is an interval valued fuzzy set in X and η_C is a fuzzy set in X.

Definition 2.15. Let $A = \{\langle x, \overline{\zeta}_A(x), \eta_A(x)\rangle :\ x \in\ X\}$ and $B = \{\langle x, \overline{\zeta}_B(x), \eta_B(x)\rangle :\ x \in\ X\}$ be two cubic sets on X, then the intersection of A and B denoted by $A \cap B$ is defined by $A \cap B = \{\langle x, \overline{\zeta}_{A\cap B}(x), \eta_{A\cap B}(x)\rangle\} = \{\langle x, rmin\{\zeta_A(x), \zeta_B(x)\}, max(\eta_A(x), \eta_B(x)), \rangle : x \in X\}$.

Definition 2.16. A cubic set $C = \{\langle x, \overline{\zeta}_C(x), \eta_C(x)\rangle : x \in X\}$ is said to have rsup-inf property if for any subset T of X there exists $x_0 \in T$ such that $\overline{\zeta}_C(x_0) = \underset{x \in T}{rsup}\ \overline{\zeta}_C(x)$ and $\eta_C(x_0) = \underset{x \in T}{inf}\ \eta_C(x)$.

Definition 2.17. Let $C = \{\langle x, \overline{\zeta}_C(x), \eta_C(x)\rangle :\ x \in\ X\}$ be a cubic set in X. Then the set C is a cubic fuzzy $\beta-$ subalgebra if it satisfies the following conditions.
(i) $\overline{\zeta}_C(x+y) \geq rmin\{\overline{\zeta}_C(x), \overline{\zeta}_C(y)\}$ & $\overline{\zeta}_C(x-y) \geq rmin\{\overline{\zeta}_C(x), \overline{\zeta}_C(y)\}$
(ii) $\eta_C(x+y) \leq max\{\eta_C(x), \eta_C(y)\}$ & $\eta_C(x-y) \leq max\{\eta_C(x), \eta_C(y)\}$ $\forall\ x, y \in X$.

Definition 2.18. A neutrosophic set in X is a structre of the form $\Omega = \{\langle x : \omega_T(x), \omega_I(x), \omega_F(x)\rangle / x \in X\}$. Where $\omega_T : X \to [0,1]$ is a truth membership function, $\omega_I : X \to [0,1]$ is a indeterminate membership function and $\omega_F : X \to [0,1]$ is a false membership function.

Definition 2.19. An interval neutrosophic set in X is a structre of the form $\Delta = \{\langle x : \delta_T(x), \delta_I(x), \delta_F(x)\rangle / x \in X\}$ where $\delta_T, \delta_I, \delta_F$ are interval valued fuzzy sets in X, which are called an interval truth membership function, an interval indeterminate membership function and an interval false membership function respectively.

Definition 2.20. Let X be a non-empty set. A neutrosophic cubic set is a pair $C = (\Delta, \omega)$ where $\Delta = \{x : \delta_T(x), \delta_I(x), \delta_F(x)/x \in X\}$ is interval valued neutrosophic set $\Omega = \{\langle x : \omega_T(x), \Omega_I(x), \omega_F(x)/x \in X\rangle\}$ is neutrosophic set.For our convenience, the neutrosophic cubic set will be denoted as $C = (\delta_{T,I,F}, \omega_{T,I,F}) = \{\langle x, \delta_{T,I,F}(x), \omega_{T,I,F}(x)\rangle\}$.

Definition 2.21. Let f be a mapping from X to Y. If $C = (\delta_{T,I,F}, \omega_{T,I,F})$ is neutrosophic cubic set of X. Then the image of C under f is denoted by $f(C)$ and is defined as $f(C) = \{\langle x, f_{rsup}(\delta_{T,I,F}), f_{inf}(\omega_{T,I,F})\rangle / x \in X\}$, where

$$f_{rsup}(\delta_{T,I,F})(y) = \begin{cases} \underset{x\in f^{-1}(y)}{rsup} (\delta_{T,I,F})(x) & : if\ f^{-1}(y) \neq \phi \\ [0,0] & : Otherwise \end{cases}$$

$$f_{inf}(\omega_{T,I,F})(y) = \begin{cases} \underset{x\in f^{-1}(y)}{inf} (\omega_{T,I,F})(x) & : if\ f^{-1}(y) \neq \phi \\ 1 & : Otherwise \end{cases}$$

Definition 2.22. Let f be a mapping from X to Y. If $C = (\delta_{T,I,F}, \omega_{T,I,F})$ is neutrosophic cubic set of X. Then the inverse image of C is defined as $f^{-1}(C) = \{\langle x, f_{rsup}(\delta_{T,I,F}), f_{inf}(\omega_{T,I,F})\rangle / x \in X\}$, with $f^{-1}(\delta_{T,I,F}(x) = (\delta_{T,I,F}(f(x))$ and $f^{-1}(\omega_{T,I,F}(x) = (\omega_{T,I,F}(f(x))$.

Definition 2.23. For any $C_i = (\Delta_i, \Omega_i)$,where $\Delta_i = \{\langle x, \delta_{iT}(x), \delta_{iI}(x), \delta_{iF}(x)\rangle : x \in X\}$ and $\Omega_i = \{\langle x, \omega_{iT}(x), \omega_{iI}(x), \omega_{iF}(x)\rangle : x \in X\}$ for $i \in k$. $P-$ union, $P-$intersection & $R-$union,$R-$intersection is defined respectively by
$P-$union: $\bigcup_{i\in k} C_i = \left(\bigcup_{i\in k} \Delta_i, \bigvee_{i\in k} \Omega_i\right)$

$P-$intersection: $\bigcap_{i\in k} C_i = \left(\bigcap_{i\in k} \Delta_i, \bigwedge_{i\in k} \Omega_i\right)$
$R-$union: $\bigcup_{i\in k} C_i = \left(\bigcup_{i\in k} \Delta_i, \bigwedge_{i\in k} \Omega_i\right)$
$R-$intersection: $\bigcap_{i\in k} C_i = \left(\bigcap_{i\in k} \Delta_i, \bigvee_{i\in k} \Omega_i\right)$
where

$$\bigcup_{i\in k} \Delta_i = \left\{\langle x; \left(\bigcup_{i\in k} \delta_{iT}\right)(x), \left(\bigcup_{i\in k} \delta_{iI}\right)(x), \left(\bigcup_{i\in k} \delta_{iF}\right)(x)/x \in X\right\}$$
$$\bigvee_{i\in k} \Omega_i = \left\{\langle x; \left(\bigvee_{i\in k} \omega_{iT}\right)(x), \left(\bigvee_{i\in k} \omega_{iI}\right)(x), \left(\bigvee_{i\in k} \omega_{iF}\right)(x)/x \in X\right\}$$
$$\bigcap_{i\in k} \Delta_i = \left\{\langle x; \left(\bigcap_{i\in k} \delta_{iT}\right)(x), \left(\bigcap_{i\in k} \delta_{iI}\right)(x), \left(\bigcap_{i\in k} \delta_{iF}\right)(x)/x \in X\right\}$$
$$\bigwedge_{i\in k} \Omega_i = \left\{\langle x; \left(\bigwedge_{i\in k} \omega_{iT}\right)(x), \left(\bigwedge_{i\in k} \omega_{iI}\right)(x), \left(\bigwedge_{i\in k} \omega_{iF}\right)(x)/x \in X\right\}.$$

Definition 2.24. Let C be a neutrosophic cubic set of X where $C = (\delta_{T,I,F}, \omega_{T,I,F})$. For $[s_{T_1}, s_{T_2}], [s_{I_1}, s_{I_2}], [s_{F_1}, s_{F_2}] \in D[0,1]$ and $t_{T_1}, t_{T_1}, t_{F_1} \in [0,1]$,
the set $U(\delta_{T,I,F}/[s_{T_1}, s_{T_2}], [s_{I_1}, s_{I_2}], [s_{F_1}, s_{F_2}] = \{x \in X/\delta_T(x) \geq [s_{T_1}, s_{T_2}], \delta_T(x) \geq [s_{I_1}, s_{I_2}], \delta_T(x) \geq [s_{F_1}, s_{F_2}]\}$ is called upper $([s_{T_1}, s_{T_2}], [s_{I_1}, s_{I_2}], [s_{F_1}, s_{F_2}])$-level of C and $L(\omega_{T,I,F}/(t_{T_1}, t_{T_1}, t_{F_1})) = \{x \in X/\omega_T(x) \leq t_{T_1}, \omega_I(x) \leq t_{I_1}, \omega_F(x) \leq t_{F_1}$ is called lower $(t_{T_1}, t_{T_1}, t_{F_1})$-level set of A.
For our convenience, we are introducing the new notion as
$U(\delta_{T,I,F}/[S_{T,I,F_1}, S_{T,I,F_2}] = \{x \in X/\delta_{T,I,F}(x) \geq [S_{T,I,F_1}, S_{T,I,F_2}]\}$ is called upper $[s_{T,I,F_1}, s_{T,I,F_2}]$-level set of C and $L(\omega_{T,I,F}/[t_{T,I,F_1}, t_{T,I,F_2}] = \{x \in X/\omega_{T,I,F}(x) \leq [t_{T,I,F_1}, t_{T,I,F_2}]\}$ is called lower t_{T,I,F_1}-level set of C.

3. Neutrosophic Cubic $\beta-$ Subalgebra

This section introduces the notion of neutrosophic cubic $\beta-$ subalgebra and discusses some engrossing results.

Definition 3.1. $C = \{x, \Delta(x), \Omega(x)/x \in X\})$ be a neutrosophic cubic set in X. Then the set C is a neutrosophic cubic $\beta-$subalgebra if it satisfies the following conditions:
$NS1:$
$\delta_T(x+y) \geq rmin\{\delta_T(x), \delta_T(y)\}$ & $\delta_T(x-y) \geq rmin\{\delta_T(x), \delta_T(y)\}$
$\delta_I(x+y) \geq rmin\{\delta_I(x), \delta_I(y)\}$ & $\delta_I(x-y) \geq rmin\{\delta_I(x), \delta_I(y)\}$
$\delta_F(x+y) \geq rmin\{\delta_F(x), \delta_F(y)\}$ & $\delta_F(x-y) \geq rmin\{\delta_F(x), \delta_F(y)\}$
$NS2$:
$\omega_T(x+y) \leq max\{\omega_T(x), \omega_T(y)\}$ & $\omega_T(x-y) \leq max\{\omega_T(x), \omega_T(y)\}$
$\omega_I(x+y) \leq max\{\omega_I(x), \omega_I(y)\}$ & $\omega_I(x-y) \leq max\{w_I(x), \omega_I(y)\}$
$\omega_F(x+y) \leq max\{\omega_F(x), \omega_F(y)\}$ & $\omega_F(x-y) \leq max\{\omega_F(x), \omega_F(y)\}$

For our convenience the neutrosophic cubic set will be denoted as
$C = (\delta_{T,I,F}, \omega_{T,I,F}) = \{\langle x, \delta_{T,I,F}(x), \omega_{T,I,F}(x)\rangle\}$ with conditions
$(i)\delta_{T,I,F}(x+y) \geq rmin\{\delta_{T,I,F}(x), \delta_{T,I,F}(y)\}$ & $\delta_{T,I,F}(x-y) \geq rmin\{\delta_{T,I,F}(x), \delta_{T,I,F}(y)\}$
$(ii)\omega_{T,I,F}(x+y) \leq max\{\omega_{T,I,F}(x), \omega_{T,I,F}(y)\}$ & $\omega_{T,I,F}(x-y) \leq max\{\omega_{T,I,F}(x), \omega_{T,I,F}(y)\}$.

Example 3.2. For the $\beta-$algebra X in the example 2.6, the Cubic set $C = \{x, \Delta(x), \Omega(x)/x \in X\})$ on X as follows.

	0	1	2	3
δ_T	[0.4,0.6]	[0.3,0.7]	[0.4,0.6]	[0.3,0.7]
δ_I	[0.3,0.5]	[0.2,0.4]	[0.3,0.5]	[0.2,0.4]
δ_F	[0.2,0.3]	[0.1,0.2]	[0.2,0.3]	[0.1,0.2]

	0	1	2	3
ω_T	0.2	0.4	0.2	0.4
ω_I	0.3	0.5	0.3	0.5
ω_F	0.4	0.6	0.4	0.6

is a neutrosophic cubic fuzzy $\beta-$sub algebra of X.

Proposition 3.3. *Let $C = \{\langle x, \delta_{T,I,F}(x), \omega_{T,I,F}(x)\rangle : x \in X\}$ is a neutrosophic cubic $\beta-$subalgebra of X. Then $\delta_{T,I,F}(0) \geq \delta_{T,I,F}(x)$ and $\omega_{T,I,F}(0) \leq \omega_{T,I,F}(x)$ $\forall\ x \in X$. Thus $\delta_{T,I,F}(0)$ & $\omega_{T,I,F}(0)$ are upper bounds and lower bounds of $\delta_{T,I,F}(x)$ & $\omega_{T,I,F}(x)$ respectively.*
proof: (1) For every $x \in X$,

$$\begin{aligned}\delta_{T,I,F}(0) &= \delta_{T,I,F}(x-x)\\ &\geq rmin\{\delta_{T,I,F}(x), \delta_{T,I,F}(x)\}\\ &= \delta_{T,I,F}(x)\end{aligned}$$

$\therefore\ \delta_{T,I,F}(0) \geq \delta_{T,I,F}(x)$ *and*

$$\begin{aligned}\omega_{T,I,F}(0) &= \omega_{T,I,F}(x-x)\\ &\leq max\{\omega_{T,I,F}(x), \omega_{T,I,F}(x)\}\\ &= \omega_{T,I,F}(x)\end{aligned}$$

$\therefore\ \omega_{T,I,F}(0) \leq \omega_{T,I,F}(x)$.

Theorem 3.4. *Let $C = \{\langle x, \delta_{T,I,F}(x), \omega_{T,I,F}(x)\rangle : x \in X\}$ be a neutrosophic cubic $\beta-$subalgebra of X. If there exists a sequence $\{x_n\}$ of X such that $\lim_{n\to\infty} \delta_{T,I,F}(x_n) = [1,1]$ and $\lim_{n\to\infty} \omega_{T,I,F}(x_n) = 0$. Then $\delta_{T,I,F}(x_n) = [1,1]$ and $\omega_{T,I,F}(x_n) = 0$.*
Proof: By using Proposition 3.3, $\delta_{T,I,F}(0) \geq \delta_{T,I,F}(x)$ $\forall x \in X$, then we have $\delta_{T,I,F}(0) \geq \delta_{T,I,F}(x_n)$ $\forall n \in Z^+$. Consider, $[1,1] \geq \delta_{T,I,F}(0) \geq \lim_{n\to\infty} \delta_{T,I,F}(x_n) = [1,1]$ Hence, $\delta_{T,I,F}(0) = [1,1]$. Moreover using proposition 3.3, $\omega_{T,I,F}(0) \leq \omega_{T,I,F}(x)$ $\forall x \in X$, then we have $\omega_{T,I,F}(0) \leq \omega_{T,I,F}(x_n)$ $\forall n \in Z^+$ Consider, $0 \leq \omega_{T,I,F}(0) \leq \lim_{n\to\infty} \omega_{T,I,F}(x_n) = 0$. Hence, $\omega_{T,I,F}(0) = 0$.

Theorem 3.5. *The $R-$intersection of any set of neutrosophic cubic $\beta-$subalgebras of X is also a neutrosophic cubic $\beta-$subalgebra of X.*

Proof: *Let $C_i = \{\langle x, \delta_{iT,I,F}, \omega_{iT,I,F}\rangle / x \in X\}$ where $i \in k$ be a sets of neutrosophic cubic $\beta-$subalgebras of X and $x, y \in X$. Then*

$$\begin{aligned}(\cap\delta_{iT,I,F})(x+y) &= rinf\ \delta_{iT,I,F}(x+y)\\ &\geq rinf\ \{rmin\{\delta_{iT,I,F}(x), \delta_{iT,I,F}(y)\}\}\\ &= rmin\{rinf\ \delta_{iT,I,F}(x), rinf\ \delta_{iT,I,F}(y)\}\\ &= rmin\{\cap\delta_{iT,I,F}(x), \cap\delta_{iT,I,F}(y)\}\end{aligned}$$

$\therefore\ \cap\delta_{iT,I,F}(x+y) \geq rmin\{\cap\delta_{iT,I,F}(x), \cap\delta_{iT,I,F}(y)\}$

Similarly,

$\delta_{iT,I,F}(x-y) \geq rmin\{\cap\delta_{iT,I,F}(x), \cap\delta_{iT,I,F}(y)\}$ *and*

$$\begin{aligned}(\vee\omega_{iT,I,F})(x+y) &= sup\ \omega_{iT,I,F}(x+y)\\ &\leq sup\ \{max\{\omega_{iT,I,F}(x), \omega_{iT,I,F}(y)\}\}\\ &= max\{sup\ \omega_{iT,I,F}(x), sup\ \omega_{iT,I,F}(y)\}\\ &= max\{\vee\omega_{iT,I,F}(x), \vee\omega_{iT,I,F}(y)\}\end{aligned}$$

$\therefore\ \vee\omega_{iT,I,F}(x+y) \leq max\{\vee\omega_{iT,I,F}(x), \vee\omega_{iT,I,F}(y)\}$

In the same way, $\omega_{iT,I,F}(x-y) \leq max\{\vee\delta_{iT,I,F}(x), \vee\omega_{iT,I,F}(y)\}$. Hence $R-$intersection of C_i is a neutrosophic cubic $\beta-$subalgebra of X.

Theorem 3.6. *The $C_i = \{\langle x, \delta_{iT,I,F}, \omega_{iT,I,F}\rangle\}/x \in X$ where $i \in k$ be a sets of neutrosophic cubic $\beta-$subalgebras of X. If $inf\{max\{\omega_{iT,I,F}(x), \omega_{iT,I,F}(y) = max\{inf\omega_{iT,I,F}(x), inf\omega_{iT,I,F}(y)\}\forall x \in X$. Then the $P-$intersection of C_i is also a neutrosophic cubic $\beta-$subalgebra of X.*

Proof: *Let $C_i = \{\langle x, \delta_{iT,I,F}, \omega_{iT,I,F}\rangle / x \in X\}$ where $i \in k$ be a sets of neutrosophic cubic $\beta-$subalgebras of X and $x, y \in X$. Then*

$$\begin{aligned}(\cap\delta_{iT,I,F})(x+y) &= rinf\ \delta_{iT,I,F}(x+y)\\ &\geq rinf\ \{rmin\{\delta_{iT,I,F}(x), \delta_{iT,I,F}(y)\}\}\\ &= rmin\{rinf\ \delta_{iT,I,F}(x), rinf\ \delta_{iT,I,F}(y)\}\\ &= rmin\{\cap\delta_{iT,I,F}(x), \cap\delta_{iT,I,F}(y)\}\end{aligned}$$

$\therefore\ \cap\delta_{iT,I,F}(x+y) \geq rmin\{\cap\delta_{iT,I,F}(x), \cap\delta_{iT,I,F}(y)\}$

In the same manner, $\delta_{iT,I,F}(x-y) \geq rmin\{\cap\delta_{iT,I,F}(x), \cap\delta_{iT,I,F}(y)\}$ *and*

$$\begin{aligned}(\wedge\omega_{iT,I,F})(x+y) &= inf\ \omega_{iT,I,F}(x+y)\\ &\leq inf\ \{max\{\omega_{iT,I,F}(x), \omega_{iT,I,F}(y)\}\}\\ &= max\{inf\omega_{iT,I,F}(x), inf\ \omega_{iT,I,F}(y)\}\\ &= max\{\wedge\omega_{iT,I,F}(x), \vee\omega_{iT,I,F}(y)\}\end{aligned}$$

$\therefore\ \wedge\omega_{iT,I,F}(x+y) \leq max\{\wedge\omega_{iT,I,F}(x), \wedge\omega_{iT,I,F}(y)\}$

Similarly, $\wedge\omega_{iT,I,F}(x-y) \leq max\{\wedge\omega_{iT,I,F}(x), \wedge\omega_{iT,I,F}(y)\}$. *Hence* $P-$*intersection of* C_i *is a neutrosophic cubic* $\beta-$*subalgebra of* X.

Theorem 3.7. *The* $C_i = \{\langle x, \delta_{iT,I,F}, \omega_{iT,I,F}\rangle\}/x \in X$ *where* $i \in k$ *be a sets of neutrosophic cubic* $\beta-$*subalgebras of* X. *If* $sup\ \{rmin\{\delta_{iT,I,F}(x), \delta_{iT,I,F}(y) = rmin\{sup\ \delta_{iT,I,F}(x), sup\ \delta_{iT,I,F}(y)\}\forall x \in X$. *Then the* $P-$*union of* C_i *is also a neutrosophic cubic* $\beta-$*subalgebra of* X.

Proof: Let $C_i = \{\langle x, \delta_{iT,I,F}, \omega_{iT,I,F}\rangle/x \in X\}$ *where* $i \in k$ *be a sets of neutrosophic cubic* $\beta-$*subalgebras of* X *and* $x, y \in X$ *such that* $sup\{rmin\{\delta_{iT,I,F}(x), \delta_{iT,I,F}(y) = rmin\{sup\ \delta_{iT,I,F}(x), sup\ \delta_{iT,I,F}(y)\}\forall x \in X$. *Then for* $x, y \in X$,

$$\begin{aligned}(\cup\delta_{iT,I,F})(x+y) &= rsup\ \delta_{iT,I,F}(x+y)\\ &\geq rsup\ \{rmin\{\delta_{iT,I,F}(x), \delta_{iT,I,F}(y)\}\}\\ &= rmin\{rsup\ \delta_{iT,I,F}(x), rsup\ \delta_{iT,I,F}(y)\}\\ &= rmin\{\cup\delta_{iT,I,F}(x), \cup\delta_{iT,I,F}(y)\}\end{aligned}$$

$\therefore\ \cup\delta_{iT,I,F}(x+y) \geq rmin\{\cup\delta_{iT,I,F}(x), \cup\delta_{iT,I,F}(y)\}$

Likewise, $\cup\delta_{iT,I,F}(x-y) \geq rmin\{\cup\delta_{iT,I,F}(x), \cup\delta_{iT,I,F}(y)\}$ *and*

$$\begin{aligned}(\vee\omega_{iT,I,F})(x+y) &= sup\ \omega_{iT,I,F}(x+y)\\ &\leq sup\ \{max\{\omega_{iT,I,F}(x), \omega_{iT,I,F}(y)\}\}\\ &= max\{sup\ \omega_{iT,I,F}(x), sup\ \omega_{iT,I,F}(y)\}\\ &= max\{\vee\omega_{iT,I,F}(x), \vee\omega_{iT,I,F}(y)\}\end{aligned}$$

$\therefore\ \vee\omega_{iT,I,F}(x+y) \leq max\{\vee\omega_{iT,I,F}(x), \vee\omega_{iT,I,F}(y)\}$

Similarly, $\vee\omega_{iT,I,F}(x-y) \leq max\{\vee\omega_{iT,I,F}(x), \vee\omega_{iT,I,F}(y)\}$. *Hence* $P-$*union of* C_i *is a neutrosophic cubic* $\beta-$*subalgebra of* X.

Theorem 3.8. *The* $C_i = \{\langle x, \delta_{iT,I,F}, \omega_{iT,I,F}\rangle\}/x \in X$ *where* $i \in k$ *be a sets of neutrosophic cubic* $\beta-$*subalgebras of* X. *If* $inf\ \{max\{\omega_{iT,I,F}(x), \omega_{iT,I,F}(y)\} =$

$max\{inf\ \omega_{iT,I,F}(x), inf\ \omega_{iT,I,F}(y)\}$ & $sup\ \{rmin\{\omega_{iT,I,F}(x), \omega_{iT,I,F}(y)\} = rmin\{sup\ \omega_{iT,I,F}(x), sup\ \omega_{iT,I,F}(y)\}\forall x \in X$. Then the $R-$union of C_i is also a neutrosophic cubic $\beta-$subalgebra of X.

Proof: Let $C_i = \{\langle x, \delta_{iT,I,F}, \omega_{iT,I,F}\rangle / x \in X\}$ where $i \in k$ be a sets of neutrosophic cubic $\beta-$subalgebras of X such that $inf\{max\{\delta_{iT,I,F}(x), \delta_{iT,I,F}(y)\}$ & $sup\{rmin\{\delta_{iT,I,F}(x), \delta_{iT,I,F}(y)\} = rmin\{sup\ \delta_{iT,I,F}(x), sup\ \delta_{iT,I,F}(y)\}\forall x \in X$. Then for $x, y \in X$,

$$\begin{aligned}(\cup\delta_{iT,I,F})(x+y) &= rsup\ \delta_{iT,I,F}(x+y)\\ &\geq rsup\ \{rmin\{\delta_{iT,I,F}(x), \delta_{iT,I,F}(y)\}\}\\ &= rmin\{rsup\ \delta_{iT,I,F}(x), rsup\ \delta_{iT,I,F}(y)\}\\ &= rmin\{\cup\delta_{iT,I,F}(x), \cup\delta_{iT,I,F}(y)\}\end{aligned}$$

$\therefore\ \cup\delta_{iT,I,F}(x+y) \geq rmin\{\cup\delta_{iT,I,F}(x), \cup\delta_{iT,I,F}(y)\}$

In the same way, $\cup\delta_{iT,I,F}(x-y) \geq rmin\{\cup\delta_{iT,I,F}(x), \cup\delta_{iT,I,F}(y)\}$ and

$$\begin{aligned}(\wedge\omega_{iT,I,F})(x+y) &= inf\ \omega_{iT,I,F}(x+y)\\ &\leq inf\ \{max\{\omega_{iT,I,F}(x), \omega_{iT,I,F}(y)\}\}\\ &= max\{inf\ \omega_{iT,I,F}(x), inf\ \omega_{iT,I,F}(y)\}\\ &= max\{\wedge\omega_{iT,I,F}(x), \wedge\omega_{iT,I,F}(y)\}\end{aligned}$$

$\therefore\ \wedge\omega_{iT,I,F}(x+y) \leq max\{\wedge\omega_{iT,I,F}(x), \wedge\omega_{iT,I,F}(y)\}$

Similarly, $\wedge\omega_{iT,I,F}(x-y) \leq max\{\wedge\omega_{iT,I,F}(x), \wedge\omega_{iT,I,F}(y)\}$. Hence $R-$union of C_i is a neutrosophic cubic $\beta-$subalgebra of X.

Theorem 3.9. *Neutrosophic cubic set $C_i = \{\Delta_{T,I,F}, \Omega_{T,I,F}\}$of X is a neutrosophic cubic $\beta-$subalgebra of X if and ony if $\delta^L_{T,I,F}, \delta^U_{T,I,F}$ & $\omega_{T,I,F}$ are fuzzy subalgebras of X.*

Proof: Let $\delta^L_{T,I,F}, \delta^U_{T,I,F}$ & $\omega_{T,I,F}$ are fuzzy subalgebras of X and $x, y \in X$.Then

$\delta^L_{T,I,F}(x+y) \geq min\{\delta^L_{T,I,F}(x), \delta^L_{T,I,F}(y)\}$

$\delta^U_{T,I,F}(x+y) \geq min\{\delta^U_{T,I,F}(x), \delta^U_{T,I,F}(y)\}$ *and*

$\omega_{T,I,F}(x+y) \leq max\{\omega_{T,I,F}(x), \omega_{T,I,F}(y)\}$

Now

$$\begin{aligned}\delta_{T,I,F}(x+y) &= [\delta^L_{T,I,F}(x+y), \delta^U_{T,I,F}(x+y)]\\ &\geq [min\{\delta^L_{T,I,F}(x), \delta^L_{T,I,F}(y)\}, min\{\delta^U_{T,I,F}(x), \delta^U_{T,I,F}(y)\}]\\ &\geq rmin\{[\delta^L_{T,I,F}(x), \delta^U_{T,I,F}(x)], [\delta^L_{T,I,F}(y), \delta^U_{T,I,F}(y)]\}\\ &= rmin\{\delta_{T,I,F}(x), \delta_{T,I,F}(y)\}\end{aligned}$$

$\therefore$ *C is neutrosophic cubic $\beta-$subalgebra of X.*

Conversely, assume that C is neutrosophic cubic $\beta-$subalgebra of X. For any $x, y \in X$,

$$\begin{aligned}[\delta^L_{T,I,F}(x+y), \delta^U_{T,I,F}(x+y)] &= \delta_{T,I,F}(x+y) \\ &\geq rmin\{\delta_{T,I,F}(x), \delta_{T,I,F}(y)\} \\ &\geq rmin\{[\delta^L_{T,I,F}(x), \delta^U_{T,I,F}(x)], [\delta^L_{T,I,F}(y), \delta^U_{T,I,F}(y)]\}\end{aligned}$$

Thus, $\delta^L_{T,I,F}(x+y) \geq min\{\delta^L_{T,I,F}(x), \delta^L_{T,I,F}(y)\}$, $\delta^U_{T,I,F}(x+y) \geq min\{\delta^U_{T,I,F}(x), \delta^U_{T,I,F}(y)\}$ and $\omega_{T,I,F}(x+y) \leq max\{\omega_{T,I,F}(x), \omega_{T,I,F}(y)\}$. Hence $\delta^L_{T,I,F}, \delta^U_{T,I,F}$ and $\omega_{T,I,F}$ are fuzzy subalgebra of X.

Remark 3.10. The sets denoted by $I_{\delta_{T,I,F}}$ and $I_{\omega_{T,I,F}}$ are also subalgebra of X which are defined as $I_{\delta_{T,I,F}} = \{x \in X/\delta_{T,I,F}(x) = \delta_{T,I,F}(0)\}$ and $I_{\omega_{T,I,F}} = \{x \in X/\omega_{T,I,F}(x) = \omega_{T,I,F}(0)\}$.

Theorem 3.11. *Let $C = (\delta_{T,I,F}, \omega_{T,I,F})$ be a neutrosophic cubic $\beta-$subalgebra of X. Then the sets $I_{\delta_{T,I,F}}$ and $I_{\omega_{T,I,F}}$ are also subalgebra of X.*

Proof: Let $x, y \in I_{\delta_{T,I,F}}$.

Then $\delta_{T,I,F}(x) = \delta_{T,I,F}(0) = \delta_{T,I,F}(y)$. Consider

$$\begin{aligned}\delta_{T,I,F}(x+y) &\geq rmin\{\delta_{T,I,F}(x), \delta_{T,I,F}(y)\} \\ &\geq rmin\{\delta_{T,I,F}(0), \delta_{T,I,F}(0)\} \\ &= \delta_{T,I,F}(0)\end{aligned}$$

$\therefore \delta_{T,I,F}(x+y) \geq \delta_{T,I,F}(0)$*. By using proposition 3.3, $\delta_{T,I,F}(0) \geq \delta_{T,I,F}(x+y)$*

Then we have $\delta_{T,I,F}(x+y) = \delta_{T,I,F}(0)$ or equivalently, $x+y \in I_{\delta_{T,I,F}}$

Similarly, $x-y \in I_{\delta_{T,I,F}}$.

Now, let $x, y \in I_{\delta_{T,I,F}}$. Then $\omega_{T,I,F}(x) = \omega_{T,I,F}(0) = \omega_{T,I,F}(y)$.

Consider

$$\begin{aligned}\omega_{T,I,F}(x+y) &\leq max\{\delta_{T,I,F}(x), \delta_{T,I,F}(y)\} \\ &= max\{\omega_{T,I,F}(0), \omega_{T,I,F}(0)\} \\ &= \omega_{T,I,F}(0)\end{aligned}$$

$\therefore \omega_{T,I,F}(x+y) \leq \omega_{T,I,F}(0)$*. By using proposition 3.3, $\omega_{T,I,F}(0) \leq \omega_{T,I,F}(x+y)$*

Then we have $\omega_{T,I,F}(x+y) = \omega_{T,I,F}(0)$ or equivalently, $x+y \in I_{\omega_{T,I,F}}$

Similarly,$x-y \in I_{\omega_{T,I,F}}$. Hence the sets $I_{\delta_{T,I,F}}$ and $I_{\omega_{T,I,F}}$ are $\beta-$subalgebras of X.

Theorem 3.12. *Let P be a non empty subset of X and $C = (\delta_{T,I,F}, \omega_{T,I,F})$ be a neutrosophic cubic $\beta-$subalgebra of X defined by*

$$\delta_{T,I,F}(x) = \begin{cases} [\phi_{T,I,F_1}, \phi_{T,I,F_1}] : & if\ x \in P \\ [\psi_{T,I,F_1}, \psi_{T,I,F_1}] : & Otherwise \end{cases} \qquad \omega_{T,I,F}(x) = \begin{cases} \rho_{T,I,F} : & if\ x \in P \\ \epsilon_{T,I,F} : & Otherwise \end{cases}$$

$\forall\ [\phi_{T,I,F_1}, \phi_{T,I,F_2}], [\psi_{T,I,F_1}, \psi_{T,I,F_2}] \in D[0,1]$ *and* $\rho_{T,I,F}, \epsilon_{T,I,F} \in [0,1]$ *with* $[\phi_{T,I,F_1}, \phi_{T,I,F_2}] \geq [\psi_{T,I,F_1}, \psi_{T,I,F_2}]$ *and* $\rho_{T,I,F} \leq \epsilon_{T,I,F}$. *Then* C *is a neutrosophic cubic* $\beta-$*subalgebra of* $X \Leftrightarrow P$ *is a* $\beta-$*subalgebra of* X.

Proof: Let C *be a neutrosophic cubic* $\beta-$*subalgebra of* X.

Let $x, y \in X$ *such tat* $x, y \in P$. *Then*

$$\begin{aligned} \delta_{T,I,F}(x+y) &\geq rmin\{\delta_{T,I,F}(x), \delta_{T,I,F}(y)\} \\ &\geq rmin\{[\phi_{T,I,F_1}, \phi_{T,I,F_2}], [\phi_{T,I,F_1}, \phi_{T,I,F_2}]\} \\ &= [\phi_{T,I,F_1}, \phi_{T,I,F_2}] \end{aligned}$$

and

$$\begin{aligned} \omega_{T,I,F}(x+y) &\leq max\{\omega_{T,I,F}(x), \omega_{T,I,F}(y)\} \\ &\leq max\{\rho_{T,I,F}, \rho_{T,I,F}\} \\ &= \rho_{T,I,F} \end{aligned}$$

Therefore $x + y \in P$. *Similarly, we have* $x - y \in P$.

Hence P *is a* $\beta-$*subalgebra of* X.

Conversely, suppose that P *is a* $\beta-$*subalgebra of* X. *Let* $x, y \in X$.

Case(i): If $x, y \in P$ *then* $x + y \in P$ *&* $x - y \in P$

Thus $\delta_{T,I,F}(x+y) = [\phi_{T,I,F_1}, \phi_{T,I,F_2}] = rmin\{\delta_{T,I,F}(x), \delta_{T,I,F}(y)\}$

Similarly, $\delta_{T,I,F}(x-y) = rmin\{\delta_{T,I,F}(x), \delta_{T,I,F}(y)\}$ *and*

$\omega_{T,I,F}(x+y) = \rho_{T,I,F} = max\{\omega_{T,I,F}(x), \omega_{T,I,F}(y)\}$.

In the same way, $\omega_{T,I,F}(x-y) = max\{\omega_{T,I,F}(x), \delta_{T,I,F}(y)\}$

Case (ii): if $x, y \notin B$,*then*

$\delta_{T,I,F}(x+y) = [\psi_{T,I,F_1}, \psi_{T,I,F_2}] = rmin\{\delta_{T,I,F}(x), \delta_{T,I,F}(y)\}$

Similarly, $\delta_{T,I,F}(x-y) = rmin\{\delta_{T,I,F}(x), \delta_{T,I,F}(y)\}$ *and*

$\omega_{T,I,F}(x+y) = \epsilon_{T,I,F} = max\{\omega_{T,I,F}(x), \omega_{T,I,F}(y)\}$.

In the same way, $\omega_{T,I,F}(x-y) = max\{\omega_{T,I,F}(x), \delta_{T,I,F}(y)\}$

Hence C *is a neutrosophic cubic* $\beta-$*subalgebra of* X.

Now,

$$\begin{aligned} I_{\delta_{T,I,F}} &= \{x \in X, \delta_{T,I,F}(x) = \delta_{T,I,F}(0)\} \\ &= \{x \in X, \delta_{T,I,F}(x) = [\phi_{T,I,F_1}, \phi_{T,I,F_2}]\} \\ &= P \end{aligned}$$

$$\begin{aligned} I_{\omega_{T,I,F}} &= \{x \in X, \omega_{T,I,F}(x) = \omega_{T,I,F}(0)\} \\ &= \{x \in X, \omega_{T,I,F}(x) = \rho_{T,I,F}\} \\ &= P. \end{aligned}$$

Theorem 3.13. *If $C = (\delta_{T,I,F}, \omega_{T,I,F})$ be a neutrosophic cubic $\beta-$subalgebra of X then the upper $[s_{T,I,F_1}, s_{T,I,F_2}]$-level and lower t_{T,I,F_1}-level set of C are $\beta-$subalgebra of X.*
proof: Let $x, y \in U(\delta_{T,I,F}/[s_{T,I,F_1}, s_{T,I,F_2}])$, then $\delta_{T,I,F}(x) \geq [s_{T,I,F_1}, s_{T,I,F_2}]$ and $\delta_{T,I,F}(y) \geq [s_{T,I,F_1}, s_{T,I,F_2}]$. It follows that $\delta_{T,I,F}(x+y) \geq rmin\{\delta_{T,I,F}(x), \delta_{T,I,F}(y) \geq [s_{T,I,F_1}, s_{T,I,F_2}]$
$\Rightarrow x + y \in U(\delta_{T,I,F}/[s_{T,I,F_1}, s_{T,I,F_2}])$. Similarly, $x - y \in U(\delta_{T,I,F}/[s_{T,I,F_1}, s_{T,I,F_2}]$.
Hence $U(\delta_{T,I,F}/[s_{T,I,F_1}, s_{T,I,F_2}]$ is a $\beta-$subalgebra of X.
Let $x, y \in L(\omega_{T,I,F}/t_{T,I,F_1})$ then $\omega_{T,I,F}(x) \leq t_{T,I,F_1}$ and $\omega_{T,I,F}(y) \leq t_{T,I,F_1}$.
It follows that $\omega_{T,I,F}(x+y) \leq max\{\omega_{T,I,F}(x), \omega_{T,I,F}(y) \leq t_{T,I,F_1}\}$
$\Rightarrow x + y \in L(\omega_{T,I,F}/t_{T,I,F_1})$. Similarly, $x - y \in L(\omega_{T,I,F}/t_{T,I,F_1})$.
Hence $L(\omega_{T,I,F}/t_{T,I,F_1})$ is a $\beta-$subalgebra of X.

Theorem 3.14. *Let $C = (\delta_{T,I,F}, \omega_{T,I,F})$ be a neutrosophic cubic set of X, such that the sets $U(\delta_{T,I,F}/[s_{T,I,F_1}, s_{T,I,F_2}])$ and $L(\omega_{T,I,F}/t_{T,I,F_1})$ are $\beta-$subalgebra of X for every $[s_{T,I,F_1}, s_{T,I,F_2}] \in D[0,1]$ and $t_{T,I,F_1} \in [0,1]$. Then $C = (\delta_{T,I,F}, \omega_{T,I,F})$ is neutrosophic cubic $\beta-$subalgebra of X.*
proof: Let $U(\delta_{T,I,F}/[s_{T,I,F_1}, s_{T,I,F_2}])$ and $L(\omega_{T,I,F}/t_{T,I,F_1})$ are $\beta-$subalgebra of X for every $[s_{T,I,F_1}, s_{T,I,F_2}] \in D[0,1]$ and $t_{T,I,F_1} \in [0,1]$.
On the contrary, let $x_0, y_0 \in X$ be such that $\delta_{T,I,F}(x_0 + y_0) < rmin\{\delta_{T,I,F}(x_0), \delta_{T,I,F}(y_0)\}$
Let$\delta_{T,I,F}(x_0) = [\theta_1, \theta_2], \delta_{T,I,F}(y_0) = [\theta_3, \theta_4]$ and $\delta_{T,I,F}(x_0 + y_0) = [s_{T,I,F_1}, s_{T,I,F_2}]$. Then
$[s_{T,I,F_1}, s_{T,I,F_2}] < rmin\{[\theta_1, \theta_2], [\theta_3, \theta_4]\} = [min\{\theta_1, \theta_2\}, min\{\theta_3, \theta_4\}]$
So, $\delta_{T,I,F_1} < min\{[\theta_1, \theta_3]\}$ and $\delta_{T,I,F_2} < min\{[\theta_2, \theta_4]\}$
Let us consider,

$$\begin{aligned} [\gamma_1, \gamma_2] &= (1/2)[\delta_{T,I,F}(x_0 + y_0) + rmin\{\delta_{T,I,F}(x_0), \delta_{T,I,F}(y_0)\}] \\ &= (1/2)[s_{T,I,F_1}, s_{T,I,F_2}] + min\{\theta_1, \theta_3\}, min\{\theta_3, \theta_4\} \\ &= (1/2)(s_{T,I,F_1} + min\{\theta_1, \theta_3\}), (1/2)(s_{T,I,F_1} + min\{\theta_2, \theta_4\}) \end{aligned}$$

$\therefore, min\{\theta_1, \theta_3\} > \gamma_1 = (1/2)(s_{T,I,F_1} + min\{\theta_1, \theta_3\}) > s_{T,I,F_1}$
and $\therefore, min\{\theta_2, \theta_4\} > \gamma_2 = (1/2)(s_{T,I,F_2} + min\{\theta_2, \theta_4\}) > s_{T,I,F_2}$

Hence $[min\{\theta_1,\theta_3\}, min\{\theta_2,\theta_4\}] > [\gamma_1,\gamma_2] > [s_{T,I,F_1}, s_{T,I,F_2}]$,*so that* $x_0 + y_0 \notin U(\delta_{T,I,F}/[s_{T,I,F_1}, s_{T,I,F_2}])$ *which is a contradiction. Since* $\delta_{T,I,F}(x_0) = [\theta_1,\theta_2] \geq [min\{\theta_1,\theta_3\}, min\{\theta_2,\theta_4\}] > [\gamma_1,\gamma_2]$ *and* $\delta_{T,I,F}(y_0) = [\theta_3,\theta_4] \geq [min\{\theta_1,\theta_3\}, min\{\theta_2,\theta_4\}] > [\gamma_1,\gamma_2]$. $\Rightarrow x_0 + y_0 \in U(\delta_{T,I,F}/[s_{T,I,F_1}, s_{T,I,F_2}])$. *Thus* $\delta_{T,I,F}(x + y) \geq rmin\{\delta_{T,I,F}(x), \delta_{T,I,F}(y)\}\ \forall\ x, y \in X$.

Similarly,$\delta_{T,I,F}(x-y) \geq rmin\{\delta_{T,I,F}(x), \delta_{T,I,F}(y)\}\ \forall\ x, y \in X$. *In the same way, we can prove* $\omega_{T,I,F}(x+y) = \omega_{T,I,F}(x+y) \leq max\{\omega_{T,I,F}(x), \omega_{T,I,F}(y)\}\ \forall\ x, y \in X$.

4. Homomorphism of Neutrosophic Cubic $\beta-$subalgebras

In this section, some of the interesting results on homomorphism of neutrosophic cubic $\beta-$subalgebra is being investigated.

Theorem 4.1. *Suppose that* $f : X \to Y$ *be a homomorphism from a* $\beta-$*algebra* X *to* Y. *If* $C = (\delta_{T,I,F}\omega_{T,I,F})$ *is a neutrosophic cubic* $\beta-$*subalgebra of* X, *then the image* $f(C) = \{\langle x, f_{rsup}(\delta_{T,I,F}), f_{inf}(\omega_{T,I,F})\rangle / x \in X\}$ *of* C *under* f *is a neutrosophic cubic* $\beta-$*subalgebra of* Y.

Proof: Let $C = (\delta_{T,I,F}, \omega_{T,I,F})$ *be a neutrosophic cubic* $\beta-$*subalgebra of* X *and let* $y_1, y_2 \in Y$. *We know that* $\{x_1 + x_2/x_1 \in f^{-1}(y_1)\ \&\ x_2 \in f^{-1}(y_2)\} \subseteq \{x \in X/x \in f^{-1}(y_1 + y_2)\}$.*Now*

$$\begin{aligned} f_{rsup}(\delta_{T,I,F})(y_1+y_2) &= rsup\{\delta_{T,I,F}(x)/x \in f^{-1}(y_1+y_2)\} \\ &= rsup\{\delta_{T,I,F}(x_1+x_2)/x_1 \in f^{-1}(y_1)\ \&\ x_2 \in f^{-1}(y_2)\} \\ &\geq rsup\{rmin\{\delta_{T,I,F}(x_1), \delta_{T,I,F}(x_2)/x_1 \in f^{-1}(y_1)\ \&\ x_2 \in f^{-1}(y_2)\} \\ &= rmin\{rsup\{\delta_{T,I,F}(x_1)/x_1 \in f^{-1}(y_1), \delta_{T,I,F}(x_2)/x_2 \in f^{-1}(y_2)\} \end{aligned}$$

In the same manner, we have

$f_{rsup}(\delta_{T,I,F})(y_1 - y_2) \geq rmin\{rsup\{\delta_{T,I,F}(x_1)/x_1 \in f^{-1}(y_1), \delta_{T,I,F}(x_2)/x_2 \in f^{-1}(y_2)\}$. *Also,*

$$\begin{aligned} f_{inf}(\delta_{T,I,F})(y_1+y_2) &= inf\{\omega_{T,I,F}(x)/x \in f^{-1}(y_1+y_2)\} \\ &= inf\{\omega_{T,I,F}(x_1+x_2)/x_1 \in f^{-1}(y_1)\ \&\ x_2 \in f^{-1}(y_2)\} \\ &\leq inf\{max\{\omega_{T,I,F}(x_1), \omega_{T,I,F}(x_2)/x_1 \in f^{-1}(y_1)\ \&\ x_2 \in f^{-1}(y_2)\} \\ &= max\{inf\{\omega_{T,I,F}(x_1)/x_1 \in f^{-1}(y_1), \omega_{T,I,F}(x_2)/x_2 \in f^{-1}(y_2)\} \end{aligned}$$

In the same way, we have

$f_{inf}(\omega_{T,I,F})(y_1 - y_2) \leq max\{inf\{\omega_{T,I,F}(x_1)/x_1 \in f^{-1}(y_1), \omega_{T,I,F}(x_2)/x_2 \in f^{-1}(y_2)\}$.

Theorem 4.2. *Suppose that* $f : X \to Y$ *be a homomorphism of* $\beta-$*algebra. If* $C = (\delta_{T,I,F}, \omega_{T,I,F})$ *is a neutrosophic cubic* $\beta-$*subalgebra of* Y, *then the pre-image* $f^{-1}(C) = \{\langle x, f^{-1}(\delta_{T,I,F}), f^{-1}(\omega_{T,I,F})\rangle / x \in X\}$ *of* C *under* f *is a neutrosophic cubic* $\beta-$*subalgebra of* X.

proof: Assume that $C = (\delta_{T,I,F}\omega_{T,I,F})$ *is a neutrosophic cubic* $\beta-$*subalgebra of* Y *and let* $x, y \in X$. *Then*

$$\begin{aligned} f^{-1}(\delta_{T,I,F})(x+y) &= \delta_{T,I,F}(f(x+y)) \\ &= \delta_{T,I,F}(f(x)+f(y)) \\ &\geq rmin\{\delta_{T,I,F}(f(x)), \delta_{T,I,F}(f(y))\} \\ &= rmin\{f^{-1}(\delta_{T,I,F})(x), f^{-1}(\delta_{T,I,F})(y)\} \end{aligned}$$

Similarly, $f^{-1}(\delta_{T,I,F})(x-y) \geq rmin\{f^{-1}(\delta_{T,I,F})(x), f^{-1}(\delta_{T,I,F})(y)\}$

$$\begin{aligned} f^{-1}(\omega_{T,I,F})(x+y) &= \omega_{T,I,F}(f(x+y)) \\ &= \omega_{T,I,F}(f(x)+f(y)) \\ &\leq max\{\omega_{T,I,F}(f(x)), \omega_{T,I,F}(f(y))\} \\ &= max\{f^{-1}(\omega_{T,I,F})(x), f^{-1}(\omega_{T,I,F})(y)\} \end{aligned}$$

Similarly, $f^{-1}(\omega_{T,I,F})(x-y) \leq rmin\{f^{-1}(\omega_{T,I,F})(x), f^{-1}(\omega_{T,I,F})(y)\}$
$\therefore f^{-1}(C) = \{\langle x, f^{-1}(\delta_{T,I,F}), f^{-1}(\omega_{T,I,F})\rangle / x \in X\}$ *of* C *under* f *is a neutrosophic cubic* $\beta-$*subalgebra of* X.

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