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An Abstract Approach to $\mathcal{W}$-Structures Based on Hypersoft Set with Properties

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Abstract. Hypersoft set is an emerging knowledge of study which is projected to address the limitations of soft set for the entitlement of multi-argument approximate function. This function maps sub-parametric tuples to power set of universe. It emphasizes the partitioning of each attribute into its respective attribute-valued set that is missing in existing soft set-like structures. These features make it a completely new mathematical tool for solving problems dealing with uncertainties. In this study, classical concept of weak structures ($\mathcal{W}$-structures) is characterized under hypersoft set environment which will provide a conceptual framework for further characterization of respective topological spaces and other spaces of functional analysis. Some of its important properties and results are investigated. Moreover, new notions of hypersoft weak axioms $\mathcal{W}$-τ_0, $\mathcal{W}$-τ_1 and $\mathcal{W}$-τ_2 are discussed with illustrative examples.

Keywords: Hypersoft set, Hypersoft $\mathcal{W}$-structure, Hypersoft $\mathcal{W}$-τ_0, Hypersoft $\mathcal{W}$-τ_1, Hypersoft $\mathcal{W}$-τ_2.

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1. Introduction

Molodtsov [1] characterized soft set (SST) as a new parametrization tool to address the inadequacy of fuzzy-like structures. Later Maji et al. [2] and Pei et al. [3] extended the work and discussed some of its fundamentals and set-theoretic operations. Shabir et al. [4] applied soft set theory in topological spaces and introduced new notions of soft set topology, later modified by Min [5]. Zorlutuna et al [6], Cagman et al. [7], Roy et al. [8] discussed the properties of soft topology and proposed some modifications. Zakari et al. [9], Min et al. [11] developed a soft weak structure in support of the generalized soft topology. Al-Saadi et al. [10] investigated closed sets for soft weak structure. In many real life situations, distinct attributes are further partitioned in disjoint attribute-valued sets but existing SST is insufficient for dealing with such kind of attribute-valued sets. Hypersoft set (HS-set) [13] is developed to make the SST in line with attribute-valued sets to tackle real life scenarios. HS-set is an extension of SS-Theory as it transforms the single argument function into a multi-argument function. Certain elementary properties, aggregation operations, laws, relations and functions of HS-set, are investigated by [14, 15] for proper understanding and further utilization in different fields. Saeed et al. [16–21] discussed decision-making applications based on complex multi-fuzzy HS-set, mapping on Hs-calsses, neutrosophic HS-graphs and neutrosophic HS-mapping to medical diagnosis and other optimal selections. Rahman et al. [22] developed hybrids of HS-set with complex fuzzy set, complex intuitionistic fuzzy set and complex neutrosophic set. They [23] introduced the notions of convex and concave HS-sets with some properties. Decision-making applications for optimal object selection have been discussed by them under the environments of parameterization of HS-sets in fuzzy set-like structures, bijective HS-sets and complex fuzzy hypersoft in [24–27]. Saqlain et al. [28] investigated single and multi-valued neutrosophic HS-sets and discussed tangent similarity measure of single valued neutrosophic HS-sets. Zulqarnain et al. [29] characterized generalized aggregate operators on neutrosophic HS-sets and discussed their essential properties. Ihsan et al. [30,31] employed the concept of HS-sets in expert system and developed HS expert set and fuzzy HS expert set with application in decision-making. Kamacı et al. [32] extended this work to n-ary fuzzy expert set and discussed its properties. Ajay et al. [33] developed the notions of Alpha Open HS-sets and applied them in MCDM. Musa et al. [34] developed bipolar HS-set and discussed its properties and operations.

1.1. *Motivation*

In many daily-life decision-making problems, we encounter with some scenarios where each attribute is required to be further classified into its respective attribute-valued set. In order to tackle such scenarios, HS-set is projected which employs the cartesian product of disjoint attribute-valued sets as domain of approximate function (i.e. multi-argument approximate function). The existing models [9–12] are insufficient to deal uncertainties with such kind of approximate function. Therefore, the main aim of this study is to generalize these models by developing HS-week structures. All the new proposed operations and properties are explained with the support of illustrated examples.

1.2. *Paper Layout*

The rest of paper is organized as:

Section 2: reviews some basic definitions to support the main results.

Section 3: characterizes HS $\mathcal{W}$-structures along with their important properties and results.

Section 4: summarizes the paper with future directions.

2. Preliminaries

In this section, definitions of soft sets , hypersoft sets and soft weak structures are reviewed.

Definition 2.1. [1]
A pair (ψ, R) is called soft set over $\mathcal{U}$, where $\psi : R \to \mathbb{P}(\mathcal{U})$ and R be a subset of a set of attributes $\mathfrak{E}$.

Definition 2.2. [13]
Suppose $b_1, b_2, ..., b_n$, for $b \geq 1$, be n distinct traits, whose corresponding trait values are respectively the sets $\mathcal{Q}_1, \mathcal{Q}_2, ..., \mathcal{Q}_n$, with $\mathcal{Q}_r \bigcap \mathcal{Q}_s = \phi$, $i \neq j$, and r,s $\in \{1, 2, ..., n\}$. Then the pair $(\Psi, \mathcal{Q}_1 \times \mathcal{Q}_2 \times ..... \times \mathcal{Q}_n)$, where $\Psi : \mathcal{Q}_1 \times \mathcal{Q}_2 \times ..... \times \mathcal{Q}_n \to P(\mathcal{U})$ is called a Hypersoft Set over $\mathcal{U}$.

Definition 2.3. [12]
$s\mathcal{W}$ is collection of (ψ, R) over $\mathcal{X}$. if

(i) $\phi, \mathcal{X} \ \in \ s\mathcal{W}$
(ii) $(\psi_a, R_1) \bigcap (\psi_b, R_2)$ in $s\mathcal{W} \ \in \ s\mathcal{W}$.

then $s\mathcal{W}$ is weak structure. $\mathcal{W}$-space is denoted by ($\mathcal{X}$, $s\mathcal{W}$, E). Elements of $s\mathcal{W}$ are $\mathcal{W}$-open and (ψ, R) is soft $\mathcal{W}$-closed if $(\psi, R)^r \ \in s\mathcal{W}$.

3. Hypersoft $\mathcal{W}$-Structures

In this section, hypersoft $\mathcal{W}$-structures are characterized and some of their important properties and results are discussed.

Definition 3.1. Hypersoft $\mathcal{W}$-Structure
Suppose $\mathcal{P}_1, \mathcal{P}_2, \mathcal{P}_3, ..., \mathcal{P}_m$ be disjoint attribute-valued sets corresponding to m distinct attributes $p_1, p_2, p_3, ..., p_m$ respectively and $\mathcal{P} = \mathcal{P}_1 \times \mathcal{P}_2 \times \mathcal{P}_3 \times .... \times \mathcal{P}_m$. A collection $\Omega_{\mathcal{W}}$ of HS-sets defined over $\mathcal{U}$ w.r.t $\mathcal{P}$ is called HS w-Structure if

(i) $\emptyset_{HS}, \mathcal{U}$ belong to $\Omega_{\mathcal{W}}$
(ii) $(\Psi_i, \mathcal{P}) \bigcap (\Psi_j, \mathcal{P})$ in $\Omega_{\mathcal{W}}$ belongs to $\Omega_{\mathcal{W}}$ $\forall\ i \neq j$

A HS set is said to be HS $\mathcal{W}$-open if it belongs to collection $\Omega_{\mathcal{W}}$ and if $(\Psi, \mathcal{P})^r \in \Omega_{\mathcal{W}}$ then HS $\mathcal{W}$-closed.

Example 3.2. Suppose $\mathcal{U} = \{u_1, u_2, u_3, u_4, u_5, u_6, u_7, u_8\}$ and $\mathcal{P} = \{\mathcal{P}_1, \mathcal{P}_2, \mathcal{P}_3, \mathcal{P}_4\}$ such that $\mathcal{P}_1 = \{p_{11}, p_{12}\}, \mathcal{P}_2 = \{p_{21}, p_{22}\}, \mathcal{P}_3 = \{p_{31}, p_{32}\}$.

Now $\mathcal{P} = \mathcal{P}_1 \times \mathcal{P}_2 \times \mathcal{P}_3$

$$\mathcal{P} = \left\{ \begin{array}{ll} q_1 = (p_{11}, p_{21}, p_{31}), & q_2 = (p_{11}, p_{21}, p_{32}), \\ q_3 = (p_{11}, p_{22}, p_{31}), & q_4 = (p_{11}, p_{22}, p_{32}), \\ q_5 = (p_{12}, p_{21}, p_{31}), & q_6 = (p_{11}, p_{21}, p_{32}), \\ q_7 = (p_{11}, p_{22}, p_{31}), & q_8 = (p_{11}, p_{22}, p_{32}) \end{array} \right\}$$

and

$$\Omega_{\mathcal{W}} = \{\emptyset_{HS}, \mathcal{U}, (\Psi_1, \mathcal{P}), (\Psi_2, \mathcal{P}), (\Psi_3, \mathcal{P})\},$$

$$(\Psi_1, \mathcal{P}) = \left\{ \begin{array}{ll} \Psi_1(q_1) = \{u_1, u_2, u_7, u_8\}, & \Psi_1(q_2) = \{u_1, u_3, u_6, u_8\}, \\ \Psi_1(q_4) = \{u_2, u_5, u_7, u_8\}, & \Psi_1(q_6) = \{u_1, u_3, u_5, u_7\}, \\ \Psi_1(q_7) = \{u_4, u_5, u_6, u_8\} & \end{array} \right\},$$

$$(\Psi_2, \mathcal{P}) = \left\{ \begin{array}{ll} \Psi_2(q_1) = \{u_1, u_2, u_3, u_7\}, & \Psi_2(q_3) = \{u_2, u_4, u_5, u_7\}, \\ \Psi_2(q_4) = \{u_1, u_5, u_7, u_8\}, & \Psi_2(q_7) = \{u_4, u_5, u_7, u_8\}, \\ \Psi_2(q_8) = \{u_2, u_5, u_4, u_8\} & \end{array} \right\},$$

$$(\Psi_3, \mathcal{P}) = \left\{ \Psi_3(q_1) = \{u_1, u_2, u_7\}, \quad \Psi_3(q_4) = \{u_5, u_7, u_8\}, \quad \Psi_3(q_7) = \{u_4, u_5, u_8\} \right\}.$$

$\Omega_{\mathcal{W}}$ is a HS $\mathcal{W}$-structure.

Definition 3.3. Hypersoft $\mathcal{W}$-Interior

The HS $\mathcal{W}$-$\mathcal{W}$-interior of $(\Psi, \mathcal{P})$, denoted by $(\Psi, \mathcal{P})^{\circ}$, is defined as

$$(\Psi, \mathcal{P})^{\circ} = \cup \{(\Psi_i, \mathcal{P}) : (\Psi_i, \mathcal{P}) \subseteq (\Psi, \mathcal{P}), (\Psi_i, \mathcal{P}) \in \Omega_{\mathcal{W}}\}.$$

Remark 3.4. If there exists a HS $\mathcal{W}$-open set $(\Psi_2, \mathcal{P})$ s.t $q \in (\Psi_2, \mathcal{P})$ is subset of $(\Psi_1, \mathcal{P})$, then q belongs to $(\Psi_1, \mathcal{P})^{\circ}$.

Example 3.5. Considering example 3.2, we have

$$(\Psi_1, \mathcal{P})^{\circ} = \{(\Psi_3, \mathcal{P})\}.$$

Theorem 3.6. *If $(\Psi_1, \mathcal{P})$ and $(\Psi_2, \mathcal{P})$ belongs to $\Omega_{\mathcal{W}}$, then*

(i) $(\Psi, \mathcal{P})^{\circ}$ *is subset of* $(\Psi, \mathcal{P})$

(ii) *If* $(\Psi_1, \mathcal{P})$ *is subset of* $(\Psi_2, \mathcal{P})$ *then* $(\Psi_1, \mathcal{P})^{\circ}$ *is subset of* $(\Psi_2, \mathcal{P})^{\circ}$

(iii) *HS $\mathcal{W}$-interior of intersection of* $(\Psi_1, \mathcal{P})$ *and* $(\Psi_2, \mathcal{P})$ *is equal to intersection of HS $\mathcal{W}$-interior of* $(\Psi_1, \mathcal{P})$ *and HS $\mathcal{W}$-interior of* $(\Psi_2, \mathcal{P})$

(iv) $((\Psi, \mathcal{P})^{\circ})^{\circ}$ *is equal to* $(\Psi, \mathcal{P})^{\circ}$

Proof. (i) is obvious.

(ii) Given $(\Psi_1, \mathcal{P})$ is subset of $(\Psi_2, \mathcal{P})$

From (i) $(\Psi_1, \mathcal{P})^{\circ}$ is subset of $(\Psi_1, \mathcal{P})$ and $(\Psi_2, \mathcal{P})^{\circ}$ is subset of $(\Psi_2, \mathcal{P})$.

implies $(\Psi_1, \mathcal{P})^{\circ}$ is subset of $(\Psi_2, \mathcal{P})$

but $(\Psi_2, \mathcal{P})^{\circ}$ is subset of $(\Psi_2, \mathcal{P})$.

Hence $(\Psi_1, \mathcal{P})^\circ$ is subset of $(\Psi_2, \mathcal{P})^\circ$

(iii) Since intersection of $(\Psi_1, \mathcal{P})$ and $(\Psi_2, \mathcal{P})$ is subset of $(\Psi_1, \mathcal{P})$, Intersection of $(\Psi_1, \mathcal{P})$ and $(\Psi_2, \mathcal{P})$ is subset of $(\Psi_2, \mathcal{P})$.

from (i) $(\Psi, \mathcal{P})^\circ$ is subset of $(\Psi, \mathcal{P})$ implies

HS $\mathcal{W}$-interior of intersection of $(\Psi_1, \mathcal{P})$ and $(\Psi_2, \mathcal{P})$ is subset of $(\Psi_1, \mathcal{P})^\circ$ and HS $\mathcal{W}$-interior of intersection of $(\Psi_1, \mathcal{P})$ and $(\Psi_2, \mathcal{P})$ is subset of $(\Psi_2, \mathcal{P})^\circ$.

So HS $\mathcal{W}$-interior of intersection of $(\Psi_1, \mathcal{P})$ and $(\Psi_2, \mathcal{P})$ is subset of intersection of HS $\mathcal{W}$-interior of $(\Psi_1, \mathcal{P})$ and HS $\mathcal{W}$-interior of $(\Psi_2, \mathcal{P})$.

Also intersection of HS $\mathcal{W}$-interior of $(\Psi_1, \mathcal{P})$ and HS $\mathcal{W}$-interior of $(\Psi_2, \mathcal{P})$ is subset of intersection of $(\Psi_1, \mathcal{P})$ and $(\Psi_2, \mathcal{P})$.

Therefore intersection of HS $\mathcal{W}$-interior of $(\Psi_1, \mathcal{P})$ and HS $\mathcal{W}$-interior of $(\Psi_2, \mathcal{P})$ is open subset of intersection of $(\Psi_1, \mathcal{P})$ and $(\Psi_2, \mathcal{P})$.

Hence intersection of HS $\mathcal{W}$-interior of $(\Psi_1, \mathcal{P})$ and HS $\mathcal{W}$-interior of $(\Psi_2, \mathcal{P})$ is subset of HS $\mathcal{W}$-interior of intersection of $(\Psi_1, \mathcal{P})$ and $(\Psi_2, \mathcal{P})$.

HS $\mathcal{W}$-interior of intersection of $(\Psi_1, \mathcal{P})$ and $(\Psi_2, \mathcal{P})$ is equal to intersection of HS $\mathcal{W}$-interior of $(\Psi_1, \mathcal{P})$ and HS $\mathcal{W}$-interior of $(\Psi_2, \mathcal{P})$.

(iv) From (i), it follows $((\Psi, \mathcal{P})^\circ)^\circ$ is subset of $(\Psi, \mathcal{P})^\circ$. For any HS $\mathcal{W}$-open set $(\Psi_1, \mathcal{P})$ s.t $((\Psi_1, \mathcal{P})$ is subset of $(\Psi, \mathcal{P})^\circ$,

$(\Psi_1, \mathcal{P})$ is equal to $(\Psi_1, \mathcal{P})^\circ$ is subset of $((\Psi, \mathcal{P})^\circ)^\circ$, so $(\Psi, \mathcal{P})^\circ \ddot{\subset} ((\Psi, \mathcal{P})^\circ)^\circ$ Consequently, we have

$((\Psi, \mathcal{P})^\circ)^\circ$ is equal to $(\Psi, \mathcal{P})^\circ$ □

Definition 3.7. Hypersoft $\mathcal{W}$-exterior

The HS $\mathcal{W}$-exterior of $(\Psi, \mathcal{P})$, denoted by $(\Psi, \mathcal{P})^\varepsilon$, is defined as

$$(\Psi, \mathcal{P})^\varepsilon = ((\Psi, \mathcal{P})^c)^\circ$$

Example 3.8. Consider the sets given in example 3.2, let we have a hypersoft set

$$(\Psi, \mathcal{P}) = \left\{ \begin{array}{ll} \Psi(q_1) = \{u_1, u_2, u_7, u_8\}, & \Psi(q_2) = \{u_1, u_3, u_6, u_8\}, \\ \Psi(q_4) = \{u_2, u_5, u_7, u_8\}, & \Psi(q_6) = \{u_1, u_3, u_5, u_7\}, \\ \Psi(q_7) = \{u_4, u_5, u_6, u_8\} & \end{array} \right\}$$

.

$$((\Psi, \mathcal{P}))^c = \left\{ \begin{array}{ll} \Psi(q_1) = \{u_3, u_4, u_5, u_6\}, & \Psi(q_2) = \{u_2, u_4, u_5, u_7\}, \\ \Psi(q_4) = \{u_1, u_3, u_4, u_6\}, & \Psi(q_6) = \{u_2, u_4, u_6, u_8\}, \\ \Psi(q_7) = \{u_1, u_2, u_3, u_7\} & \end{array} \right\}$$

.

$$(\Psi_4, \mathcal{P}) = \left\{ \begin{array}{ll} \Psi_4(q_1) = \{u_3, u_5, u_6\}, & \Psi_4(q_2) = \{u_2, u_5, u_7\}, \\ \Psi_4(q_4) = \{u_1, u_3, u_6\}, & \Psi_4(q_6) = \{u_2, u_4, u_6\}, \\ \Psi_4(q_7) = \{u_1, u_3, u_7\} & \end{array} \right\}$$

.

$$(\Psi, \mathcal{P})^{\varepsilon} = (\Psi_4, \mathcal{P})$$

Definition 3.9. Hypersoft $\mathcal{W}$-boundry
The HS $\mathcal{W}$-boundry of $(\Psi, \mathcal{P})$, denoted by $(\Psi, \mathcal{P})^b$, contains those HS sets which do not belongs to HS $\mathcal{W}$-interior and HS exterior.

Example 3.10. in example 3.2, we have

$$(\Psi, \mathcal{P})^b = \{(\Psi_1, \mathcal{P}), (\Psi_2, \mathcal{P})\}$$

Definition 3.11. Hypersoft $\mathcal{W}$-Closure
HS $\mathcal{W}$-closure of $(\Psi, \mathcal{P})$ is denoted by $(\Psi, \mathcal{P})^\bullet$, is defined as

$$(\Psi, \mathcal{P})^\bullet = \bigcap\{(\Psi_1, \mathcal{P}) : (\Psi, \mathcal{P}) \subseteq (\Psi_1, \mathcal{P}), (\Psi_1, \mathcal{P})^c \in \Omega_{\mathcal{W}}\}$$

.

Example 3.12. It is clear from example 3.2

$$(\Psi_3, \mathcal{P})^\bullet = \{(\Psi_1, \mathcal{P})\}$$

Theorem 3.13.
If $q \in (\Psi, \mathcal{P})^\bullet$, then $(\Psi_i, \mathcal{P}) \bigcap (\Psi, \mathcal{P}) \neq \emptyset \ \forall \ (\Psi_i, \mathcal{P}) \in \Omega_{\mathcal{W}}$ s.t $q \in (\Psi_i, \mathcal{P})$.

Proof. Suppose $q \in (\Psi, \mathcal{P})^\bullet$ then there exists $(\Psi_i, \mathcal{P}) \in \Omega_{\mathcal{W}}$ s.t $q \in (\Psi_i, \mathcal{P})$
and $(\Psi_i, \mathcal{P}) \bigcap (\Psi, \mathcal{P}) = \emptyset$
this implies $(\Psi, \mathcal{P}) \ddot{\subset} (\Psi_i, \mathcal{P})^c$ so $(\Psi, \mathcal{P})^\bullet \ddot{\subset} (\Psi_i, \mathcal{P})^c$ and $q \notin (\Psi, \mathcal{P})^\bullet$. So it is a contradiction. □

Theorem 3.14.
If $(\Psi_1, \mathcal{P})$ and $(\Psi_2, \mathcal{P})$ are two HS sets then

(i) $(\Psi, \mathcal{P})$ *is subset of* $(\Psi, \mathcal{P})^\bullet$
(ii) *if* $(\Psi_1, \mathcal{P})$ *is subset of* $(\Psi_2, \mathcal{P})$ *then* $(\Psi_1, \mathcal{P})^\bullet$ *is subset of* $(\Psi_2, \mathcal{P})^\bullet$
(iii) $(\Psi_1, \mathcal{P})^\bullet \cup (\Psi_2, \mathcal{P})^\bullet = ((\Psi_1, \mathcal{P}) \cup (\Psi_2, \mathcal{P}))^\bullet$
(iv) $((\Psi, \mathcal{P})^\bullet)^\bullet = (\Psi, \mathcal{P})^\bullet$

Proof. (i) is obvious.
(ii) Since $(\Psi_1, \mathcal{P})$ is subset of $(\Psi_2, \mathcal{P})$
from (i) $(\Psi_1, \mathcal{P})$ is subset of $(\Psi_1, \mathcal{P})^\bullet$ and $(\Psi_2, \mathcal{P})$ is subset of $(\Psi_2, \mathcal{P})^\bullet$
then $(\Psi_1, \mathcal{P})$ *is subset of* $(\Psi_2, \mathcal{P})^\bullet$
but $(\Psi_1, \mathcal{P})$ *is subset of* $(\Psi_1, \mathcal{P})^\bullet$ implies $(\Psi_1, \mathcal{P})^\bullet$ *is subset of* $(\Psi_2, \mathcal{P})^\bullet$
(iii) Since $(\Psi_1, \mathcal{P})$ *is subset of* $(\Psi_1, \mathcal{P}) \cup (\Psi_2, \mathcal{P})$, $(\Psi_2, \mathcal{P})$ is subset of $(\Psi_1, \mathcal{P}) \cup (\Psi_2, \mathcal{P})$

and $(\Psi,\mathcal{P})$ *is subset of* $(\Psi,\mathcal{P})^{\bullet}$ then $(\Psi_1,\mathcal{P})^{\bullet}$ *is subset of* $((\Psi_1,\mathcal{P})\cup(\Psi_2,\mathcal{P}))^{\bullet}$ and $(\Psi_2,\mathcal{P})^{\bullet}$ *is subset of* $((\Psi_1,\mathcal{P})\cup(\Psi_2,\mathcal{P}))^{\bullet}$,
$(\Psi_1,\mathcal{P})^{\bullet}\cup(\Psi_2,\mathcal{P})^{\bullet}$ *is subset of* $((\Psi_1,\mathcal{P})\cup(\Psi_2,\mathcal{P}))^{\bullet}$
also $(\Psi_1,\mathcal{P})\cup(\Psi_2,\mathcal{P}))^{\bullet}$ *is subset of* $(\Psi_1,\mathcal{P})^{\bullet}\cup(\Psi_2,\mathcal{P})^{\bullet}$ Hence

$$(\Psi_1,\mathcal{P})^{\bullet}\cup(\Psi_2,\mathcal{P})^{\bullet}=((\Psi_1,\mathcal{P})\cup(\Psi_2,\mathcal{P}))^{\bullet}$$

(iv) From (i), $(\Psi,\mathcal{P})$ is subset of $(\Psi,\mathcal{P})^{\bullet}$ then $(\Psi,\mathcal{P})^{\bullet}$ is subset of $((\Psi,\mathcal{P})^{\bullet})^{\bullet}$,
$((\Psi,\mathcal{P})^{\bullet})^{\bullet}=(\Psi,\mathcal{P})$ is subset of $(\Psi,\mathcal{P})^{\bullet}$, then $((\Psi,\mathcal{P})^{\bullet})^{\bullet}$ is subset of $(\Psi,\mathcal{P})^{\bullet}$
Consequently, we have
$((\Psi,\mathcal{P})^{\bullet})^{\bullet}=(\Psi,\mathcal{P})^{\bullet}$ □

Remark 3.15.

(i) if $(\Psi,\mathcal{P})\in\Omega_{\mathcal{W}}$ then $(\Psi,\mathcal{P})=((\Psi,\mathcal{P}))^{\circ}$
(ii) if $(\Psi,\mathcal{P})^{r}\in\Omega_{\mathcal{W}}$ then $(\Psi,\mathcal{P})=((\Psi,\mathcal{P}))^{\bullet}$

Definition 3.16. Hypersoft $\mathcal{W}$-τ_0
If $u_1,u_2\in\mathcal{U}$ and $u_1\neq u_2$, $\exists$ a HS $\mathcal{W}$-open set $(\Psi,\mathcal{P})$ s.t $u_1\in(\Psi,\mathcal{P})$ and $u_2\notin(\Psi,\mathcal{P})$ or $u_1\notin(\Psi,\mathcal{P})$ and $u_2\in(\Psi,\mathcal{P})$ then $(\mathcal{U},\Omega_{\mathcal{W}},\mathcal{P})$ is called $\mathcal{W}$-τ_0

Example 3.17. Suppose $\mathcal{U}=\{u_1,u_2\}$ then $\Omega_{\mathcal{W}}=\{\emptyset,\mathcal{U},(\Psi,\mathcal{P})\}$ where
$(\Psi,\mathcal{P})=\{\Psi_1(q_1)=\{u_1\}\}$ is $\mathcal{W}$-τ_0.

Theorem 3.18.
If $\mathcal{U}$ is a relative HS $\mathcal{W}$-τ_0 space, then for each $u_1,u_2\in\mathcal{U}$ such that $u_1\neq u_2$, we have $(u_1,\mathcal{P})^{\bullet}\neq(u_2,\mathcal{P})^{\bullet}$.

Proof. For every $u_1,u_2\in\mathcal{U}$ and $u_1\neq u_2$ $\exists$ a HS $(\Psi,\mathcal{P})\in\Omega_{\mathcal{W}}$ s.t $u_1\in(\Psi,\mathcal{P})$ and $u_2\in(\Psi,\mathcal{P})^{c}$.
Therefore $(\Psi,\mathcal{P})^{c}$ is a HS $\mathcal{W}$-closed set s.t $u_1\notin(\Psi,\mathcal{P})^{c}$ and $u_2\in(\Psi,\mathcal{P})^{c}$.
Since $(u_2,\mathcal{P})^{\bullet}\ddot{\subset}(\Psi,\mathcal{P})^{c}$ and $u_1\notin(u_2,\mathcal{P})^{\bullet}$ Thus $(u_1,\mathcal{P})^{\bullet}\neq(u_2,\mathcal{P})^{\bullet}$. □

Definition 3.19. Hypersoft $\mathcal{W}$-τ_1
If for each $u_1,u_2\in\mathcal{U}$ s.t $u_1\neq u_2$, $\exists$ HS $\mathcal{W}$-open sets $(\Psi_1,\mathcal{P})$ and $(\Psi_2,\mathcal{P})$ s.t $u_1\in(\Psi_1,\mathcal{P})$ and $u_2\notin(\Psi_1,\mathcal{P})$ and $u_1\notin(\Psi_2,\mathcal{P})$ and $u_2\in(\Psi_2,\mathcal{P})$ then HS $\Omega_{\mathcal{W}}$ space is known as $\mathcal{W}$-τ_1 .

Example 3.20. Suppose $\mathcal{U}=\{u_1,u_2\}$ then $\Omega_{\mathcal{W}}=\{\emptyset,\mathcal{U},(\Psi_1,\mathcal{P}),(\Psi_2,\mathcal{P})\}$ where
$(\Psi_1,\mathcal{P})=\{\Psi_1(q_1)=\{u_1\}\}$ and $(\Psi_2,\mathcal{P})=\{\Psi_2(q_1)=\{u_2\}\}$ is $\mathcal{W}$-τ_1.

Theorem 3.21.
A HS $\mathcal{W}$-space $(\mathcal{U},\Omega_{\mathcal{W}},\mathcal{P})$ is HS $\mathcal{W}$-τ_1 if $(u,\mathcal{P})$ is HS $\mathcal{W}$-closed set for all $u\in\mathcal{U}$.

Proof. suppose $u_1, u_2 \in \mathcal{U}$ and $u_1 \neq u_2$ $\exists$ HS $\mathcal{W}$-open sets $(u_1, \mathcal{P})^c$ and $(u_2, \mathcal{P})^c$ s.t $u_1 \in (u_1, \mathcal{P})^c$, $u_2 \in (u_1, \mathcal{P})^c$ and $u_2 \notin (u_2, \mathcal{P})^c$, $u_1 \in (u_2, \mathcal{P})^c$, It prove that $\mathcal{U}$ is HS $\mathcal{W}$-τ_1. □

Definition 3.22. Hypersoft $\mathcal{W}$-τ_2
$\mathcal{W}$-τ_2 if for each $u_1, u_2 \in \mathcal{U}$ s.t $u_1 \neq u_2$, $\exists$ HS $\mathcal{W}$-open sets $(\Psi_1, \mathcal{P})$ and $(\Psi_2, \mathcal{P})$ then each $u_1 \in (\Psi_1, \mathcal{P})$, $u_2 \in (\Psi_2, \mathcal{P})$ and $(\Psi_1, \mathcal{P}) \bigcap (\Psi_2, \mathcal{P}) = \emptyset$

Example 3.23. Suppose $\mathcal{U} = \{u_1, u_2\}$ then $\Omega_{\mathcal{W}} = \{\emptyset, \mathcal{U}, (\Psi_1, \mathcal{P}), (\Psi_2, \mathcal{P})\}$ where $(\Psi_1, \mathcal{P}) = \{\Psi_1(q_1) = \{u_1\}\}$ and $(\Psi_2, \mathcal{P}) = \{\Psi_2(q_1) = \{u_2\}\}$ is $\mathcal{W}$-τ_2.

4. Conclusions

In this study, weak structures are characterized under hypersoft set environment, and some of its essential properties and results are discussed. Moreover, some separation axioms like τ_0, τ_1, and τ_2 are introduced with the help of weak structures on hypersoft set. Further study may include the development of :

(1) HS-compact spaces
(2) HS-connected spaces
(3) HS-normed spaces
(4) HS-Hilbert spaces
(5) HS-inner product spaces
(6) HS-metric spaces

with their applications in decision-making by using certain techniques like TOPSIS, MCDM etc.

Conflicts of Interest:
The authors declare no conflict of interest.

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