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Cosine and Set-Theoretic Similarity Measures for Generalized Multi-Polar Neutrosophic Soft Set with Their Application in Decision Making

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Abstract:

A similarity measure and correlation coefficients are used to tackle many issues that include indistinct and blurred information, excluding is not in a position to deal with the general fuzziness and obscurity of the various problems. In this paper, we study some basic concepts which are helpful to build the structure of the article, such as soft set, neutrosophic soft set, and generalized m-polar neutrosophic soft set. The main objective of this paper is to develop the cosine and set-theoretic similarity measures for the generalized multipolar neutrosophic soft set (GmPNSS). We discuss some basic operations with their properties for GmPNSS. A decision-making approach has been established by using cosine and set-theoretic similarity measures. Also, we introduce the multipolar neutrosophic soft weighted average (mPNSWA) operator and develop a decision-making approach based on mPNSWA. Furthermore, we use to develop techniques to solve multi-criteria decision-making problems. Finally, the advantages, effectiveness, flexibility, and comparative analysis of the algorithms are given with prevailing methods.

Keywords: Neutrosophic set; multipolar neutrosophic set; neutrosophic soft set; multipolar neutrosophic soft set

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1. Introduction

Uncertainty plays a dynamic role in many areas of life (such as modeling, medicine, engineering, etc.). However, people have raised a general question: how do we express and use the concept of uncertainty in mathematical modeling. Many researchers have proposed and recommended different methods of using uncertainty theory. First of all, Zadeh proposed the concept of fuzzy sets [1] to solve those problems that contain uncertainty and ambiguity. It can be seen that in some cases, fuzzy sets cannot handle the situation. To overcome such situations, Turksen [2] proposed the idea of interval-valued fuzzy sets (IVFS). In some cases, we must consider the appropriate representation of the object under the conditions of uncertainty and uncertainty and regard its unbiased value as the fair value of the proper representation of the object, which these fuzzy sets or IVFS cannot process. To overcome these difficulties, Atanassov proposed the Intuitionistic Fuzzy Set (IFS) [3]. The theory proposed by Atanassov only deals with insufficient data considerations and membership and non-member values. However, the IFS theory cannot deal with overall incompatibility and imprecise information. To solve such incompatible and inaccurate records, Smarandache [4] proposed the idea of the neutrosophic set (NS).

A general mathematical tool was proposed by Molodtsov [5] to deal with indeterminate, fuzzy, and not clearly defined substances known as a soft set (SS). Maji et al. [6] extended the work on SS and described some operations and properties. They also used the SS theory for decision-making [7]. Ali et al. [8] revised the Maji approach to SS and developed new operations with their properties. De Morgan's Law on SS theory was proved [9] by using different operators. Cagman and Enginoglu [10] developed the concept of soft matrices with operations and discussed their properties. They also introduced a decision-making method to resolve those problems which contain uncertainty and revised the operations proposed by Molodtsov's SS [11]. In [12], the author's planned some new operations on soft matrices like soft difference product, soft restricted difference product, soft extended difference product, and soft weak-extended difference product with their properties.

Maji [13] offered the idea of a neutrosophic soft set (NSS) with necessary operations and properties. The concept of the possibility NSS was developed by Karaaslan [14] and introduced a neutrosophic soft decision-making method to solve those problems that contain uncertainty based on And-product. Broumi [15] developed the generalized NSS with some operations and properties and used the proposed decision-making concept. To solve MCDM problems with single-valued Neutrosophic numbers (SVNNs) presented by Deli and Subas in [16], they constructed the concept of cut sets of SVNNs. Based on the correlation of IFS, the term CC of SVNSs [17] was introduced. In [18], simplified NSs introduced with some operational laws and aggregation operators such as weighted arithmetic and weighted geometric average operators. They constructed an MCDM method on the base of proposed aggregation operators. Masooma et al. [19] progressed a new concept by combining the multipolar fuzzy set and neutrosophic set, known as the multipolar neutrosophic set. They also established various characterization and operations with examples.

Zulqarnain et al. [20-21] proposed the Einstein weighted ordered average and geometric operators for PFSSs. Zulqarnain et al. [22] introduced operational laws for Pythagorean fuzzy soft numbers (PFSNs) and developed AOs utilizing defined operational laws for PFSNs. They also planned a DM approach to solve MADM problems with the help of presented operators. Riaz et al. [23] prolonged the idea of PFSSs and developed the m polar PFSSs. They also established the TOPSIS method under the considered hybrid structure and proposed a DM methodology to solve the MCGDM problem. Siddique et al. [24] introduced the score matrix for PFSS and established a DM approach using their developed concept. Zulqarnain et al. [25-27] planned the TOPSIS methodology in the PFSS environment based on the correlation coefficient. They also proposed some AOs and interaction AOs for PFSS. Basset et al. [28] applied TODIM and TOPSIS methods under the best-worst approach to raising the overall efficiency of rating beneath uncertainty according to the NS. They also utilized plithogenic set theory to resolve the unsure info and assess the overall commercial enterprise world premiere of manufacturing industries. They utilized the AHP approach to come across the weight vector of your business enterprise concentrations to gain that destination afterward; they had to use VIKOR and TOPSIS methods to utilize the firm's ranking [29].

The authors established the probability multi-valued neutrosophic set by combining the multi-valued neutrosophic set and probability distribution to solve decision-making issues [30]. Kamal et al. [31] proposed the idea of mPNSS with some significant operations and properties. They also used the developed technique for decision-making. Saeed et al. [32] established the concept of mPNSS with its properties and operators. They also developed the distance-based similarity measures and used the proposed similarity measures for decision-making and medical diagnoses. In [33], the authors established the concept of mPNSS with its properties and operators. They also developed the distance-based similarity measures and used the proposed similarity measures for decision-making and medical diagnoses. Zulqarnain et al. [34-35] offered the generalized neutrosophic TOPSIS and an integrated model for neutrosophic TOPSIS. They used their developed techniques for supplier selection and MCDM problems.

In this epoch, experts consider that real life will be moving toward multi-polarization. Thus, there is no doubt that the multi-polarization of information must have managed to succeed in the prosperity of many science and engineering science fields. In information technology, multi-polar technology can be utilized to manipulate several structures. The motivation for expanding and mixing this research work is gradually given in the entire manuscript. We demonstrate that under any appropriate circumstances, different hybrid structures containing fuzzy sets will be converted into the unique privilege of GmPNSS. The multipolar neutrosophic environment is novel to the concept of neutrosophic soft sets of multipolar values. We discuss the effectiveness, flexibility, quality, and advantages of planning work and algorithms. This research will be the most versatile form and will integrate data with appropriate medicine, engineering, artificial intelligence, agriculture, and other daily complications. Current work may apply to other methods and different types of hybrid structures in the future.

The following research is organized: In section 2, we recollected some basic definitions used in the subsequent sequel, such as NS, SS, NSS, and multipolar neutrosophic set. In section 3, we propose the GmPNSS with its properties and operations. Section 4 establishes two different types of similarity measures such as cosine and set-theoretic similarity with their decision-making approaches and graphically representation. We also introduce some operational laws and mPNSWA operators with its decision-making technique based on GmPNSS. Section 5 uses the developed similarity measures and mPNSWA operator for decision-making. A brief comparative analysis has been conducted between proposed methods with existing methodologies in section 6. Finally, the conclusion and future directions are presented in section 7.

2. Preliminaries

In this section, we recollect some basic concepts such as the neutrosophic set, soft set, neutrosophic soft set, and m-polar neutrosophic soft set used in the following sequel.

Definition 2.1 [4]

Let $\mathcal{U}$ be a universe, and $\mathcal{A}$ be an NS on $\mathcal{U}$ is defined as $\mathcal{A} = \{< u, u_{\mathcal{A}}(u), v_{\mathcal{A}}(u), w_{\mathcal{A}}(u) > : u \in \mathcal{U}\}$, where u, v, w: $\mathcal{U} \rightarrow]0^-, 1^+[$ and $0^- \leq u_{\mathcal{A}}(u) + v_{\mathcal{A}}(u) + w_{\mathcal{A}}(u) \leq 3^+$.

Definition 2.2 [5]

Let $\mathcal{U}$ be the universal set and $\mathcal{E}$ be the set of attributes concerning $\mathcal{U}$. Let $\mathcal{P}(\mathcal{U})$ be the power set of $\mathcal{U}$ and $\mathcal{A} \subseteq \mathcal{E}$. A pair $(\mathcal{F}, \mathcal{A})$ is called a soft set over $\mathcal{U}$, and its mapping is given as

$$\mathcal{F}: \mathcal{A} \rightarrow \mathcal{P}(\mathcal{U})$$

It is also defined as:

$$(\mathcal{F}, \mathcal{A}) = \{\mathcal{F}(e) \in \mathcal{P}(\mathcal{U}): e \in \mathcal{E}, \mathcal{F}(e) = \emptyset \ if \ e \neq \mathcal{A}\}$$

Definition 2.3 [13]

Let $\mathcal{U}$ be the universal set and $\mathcal{E}$ be the set of attributes concerning $\mathcal{U}$. Let $\mathcal{P}(\mathcal{U})$ be the Neutrosophic values of $\mathcal{U}$ and $\mathcal{A} \subseteq \mathcal{E}$. A pair $(\mathcal{F}, \mathcal{A})$ is called a Neutrosophic soft set over $\mathcal{U}$ and its mapping is given as

$$\mathcal{F}: \mathcal{A} \rightarrow \mathcal{P}(\mathcal{U})$$

Definition 2.4 [32]

Let $\mathcal{U}$ be a universe of discourse and $\mathcal{E}$ be a set of attributes, and m-polar neutrosophic soft set (mPNSS) $\wp_{\mathfrak{R}}$ over $\mathcal{U}$ defined as
$\wp_{\mathfrak{R}} = \{(< e, \{(u, u_\alpha(u), v_\alpha(u), w_\alpha(u)): u \in \mathcal{U}, \alpha = 1,2,3,\ldots,m\}) > : e \in \mathcal{E}\}$,
where $u_\alpha(u)$, $v_\alpha(u)$, and $w_\alpha(u)$ represents the truthiness, indeterminacy, and falsity, respectively, $u_\alpha(u), v_\alpha(u), w_\alpha(u) \subseteq [0,1]$ and $0 \leq u_\alpha(u) + v_\alpha(u) + w_\alpha(u) \leq 3$, for all $\alpha = 1, 2, 3,\ldots, m$; $e \in \mathcal{E}$ and $u \in \mathcal{U}$. Simply an m-polar neutrosophic number (mPNSN) can be expressed as $\wp = \{< u_\alpha, v_\alpha, w_\alpha >\}$, where $0 \leq u_\alpha + v_\alpha + w_\alpha \leq 3$ and $\alpha = 1, 2, 3,\ldots, m$.

3. Generalized Multi-Polar Neutrosophic soft Set (GmPNSS) with Operators and Properties

In this section, we study the concept of GmPNSS and introduce some basic operations and their properties on GmPNSS.

Definition 3.1

Let $\mathcal{U}$ and E are universal and set of attributes respectively, and $\mathcal{A} \subseteq$ E, if there exists a mapping Φ such as

$\Phi: \mathcal{A} \rightarrow GmPNSS^{\mathcal{U}}$

Then (Φ, $\mathcal{A}$) is called GmPNSS over $\mathcal{U}$ defined as follows

$\Upsilon_K = (\Phi, \mathcal{A}) = \{(u, \Phi_{\mathcal{A}(e)}(u)): e \in E, u \in \mathcal{U}\}$, where

$\Phi_{\mathcal{A}}(e) = \{e, < u, u^\alpha_{\mathcal{A}(e)}(u), v^\alpha_{\mathcal{A}(e)}(u), w^\alpha_{\mathcal{A}(e)}(u) > : u \in \mathcal{U}, e \in E;\ \alpha \in 1,2,3,\ldots,m\}$, and

$0 \leq u^\alpha_{\mathcal{A}(e)}(u) + v^\alpha_{\mathcal{A}(e)}(u) + w^\alpha_{\mathcal{A}(e)}(u) \leq 3$ for all $\alpha \in 1, 2, 3,\ldots, m$; $e \in E$ and $u \in \mathcal{U}$.

Definition 3.2

Let $\Upsilon_{\mathcal{A}}$ and Υ_B are two GmPNSS over $\mathcal{U}$, then $\Upsilon_{\mathcal{A}}$ is called a multi-polar neutrosophic soft subset of Υ_B. If

$u^\alpha_{\mathcal{A}(e)}(u) \leq u^\alpha_{B(e)}(u)$, $v^\alpha_{\mathcal{A}(e)}(u) \leq v^\alpha_{B(e)}(u)$ and $w^\alpha_{\mathcal{A}(e)}(u) \geq w^\alpha_{B(e)}(u)$

for all $\alpha \in 1, 2, 3,\ldots, m$; $e \in E$ and $u \in \mathcal{U}$.

Example 1 Assume $\mathcal{U} = \{u_1, u_2\}$ be a universe of discourse and $E = \{x_1, x_2, x_3, x_4\}$ be a set of attribuites and $\mathcal{A} = B = \{x_1, x_2\} \subseteq E$. Consider $\mathcal{F}_{\mathcal{A}}$ and $\mathcal{G}_B \in$ G3-PNSS over $\mathcal{U}$ can be represented as follows

$$\mathcal{F}_{\mathcal{A}} = \begin{Bmatrix} (x_1, \{\langle u_1, (.5,.2,.1), (.3,.1,.3), (.4,.3,.8), (u_2, (.2,.3,.2), (.2,.1,.3), (.3,.4,.6))\rangle, \\ (x_2, \{\langle u_1, (.3,.1,.4), (0,.1,.5), (.3,.1,.5)), (u_2, (.2,.2,.5), (.3,.1,.5), (.4,.3,.6))\rangle \end{Bmatrix}$$

and

$$\mathcal{G}_B = \begin{Bmatrix} (x_1, \{\langle u_1, (.6,.4,.1), (.4,.3,.2), (.5,.4,.5), (u_2, (.3,.5,.1), (.3,.2,.1), (.4,.5,.4))\rangle, \\ (x_2, \{\langle u_1, (.4,.3,.3), (0,.2,.3), (.4,.2,.5)), (u_2, (.2,.1,.3), (.6,.3,.1), (.5,.3,.1))\rangle \end{Bmatrix}$$

Thus

$\mathcal{F}_{\mathcal{A}} \subseteq \mathcal{G}_B$.

Definition 3.3

Let $\Upsilon_{\mathcal{A}}$ and Υ_B are two GmPNSS over $\mathcal{U}$, then $\Upsilon_{\mathcal{A}} = \Upsilon_B$, if

$u^\alpha_{\mathcal{A}(e)}(u) \leq u^\alpha_{B(e)}(u)$, $u^\alpha_{B(e)}(u) \leq u^\alpha_{\mathcal{A}(e)}(u)$

$v^\alpha_{\mathcal{A}(e)}(u) \leq v^\alpha_{B(e)}(u)$, $v^\alpha_{B(e)}(u) \leq v^\alpha_{\mathcal{A}(e)}(u)$

$w^\alpha_{\mathcal{A}(e)}(u) \geq w^\alpha_{B(e)}(u)$, $w^\alpha_{B(e)}(u) \geq w^\alpha_{\mathcal{A}(e)}(u)$

for all $i \in 1, 2, 3,\ldots, m$; $e \in E$ and $u \in \mathcal{U}$.

Definition 3.4

Let $\mathcal{F}_{\mathcal{A}}$ be a GmPNSS over $\mathcal{U}$; then empty GmPNSS can be represented as $\mathcal{F}_{\breve{0}}$And defined as follows

$\mathcal{F}_{\breve{0}} = \{e, < u, (0,1,1),\ (0,1,1), \dots, (0,1,1) > : e \in E, u \in \mathcal{U}\}$.

Definition 3.5

Let $\mathcal{F}_{\mathcal{A}}$ be a GmPNSS over $\mathcal{U}$; then universal GmPNSS can be represented as $\mathcal{F}_{\breve{E}}$And defined as follows

$\mathcal{F}_{\breve{E}} = \{e, < u, (1,1,0),\ (1,1,0), \dots, (1,1,0) > : e \in E, u \in \mathcal{U}\}$.

Example 2 Assume $\mathcal{U} = \{u_1, u_2\}$ be a universe of discourse and $E = \{x_1, x_2, x_3, x_4\}$ be a set of attributes. The tabular representation of $\mathcal{F}_{\breve{0}}$ and $\mathcal{F}_{\breve{E}}$ given as follows in table 1 and table 2, respectively.

Table 1. Tablur representation of GmPNSS $\mathcal{F}_{\breve{0}}$

$\mathcal{U}$	$\mathrm{u_1}$	$\mathrm{u_2}$	$\cdots$	$\mathrm{u_n}$
x_1	$(0,1,1)$	$(0,1,1)$	$\cdots$	$(0,1,1)$
x_2	$(0,1,1)$	$(0,1,1)$	$\cdots$	$(0,1,1)$
$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$
x_n	$(0,1,1)$	$(0,1,1)$	$\cdots$	$(0,1,1)$

Table 2. Tablur representation of GmPNSS $\mathcal{F}_{\breve{E}}$

$\mathcal{U}$	$\mathrm{u_1}$	$\mathrm{u_2}$	$\cdots$	$\mathrm{u_n}$
x_1	$(1,1,0)$	$(1,1,0)$	$\cdots$	$(1,1,0)$
x_2	$(1,1,0)$	$(1,1,0)$	$\cdots$	$(1,1,0)$
$\vdots$	$\vdots$	$\vdots$	$\ddots$	$\vdots$
x_n	$(1,1,0)$	$(1,1,0)$	$\cdots$	$(1,1,0)$

Definition 3.6

Let $\mathcal{F}_{\mathcal{A}}$ be a GmPNSS over $\mathcal{U}$, then the complement of GmPNSS is defined as follows

$\mathcal{F}^c_{\mathcal{A}}(e) = \{< u, w^{\alpha}_{\mathcal{A}(e)}(u),\ (1,1,\dots,1) - v^{\alpha}_{\mathcal{A}(e)}(u), u^{\alpha}_{\mathcal{A}(e)}(u) > : u \in \mathcal{U}\}$, for all $\alpha \in 1, 2, 3, \dots, m$; $e \in E$ and $u \in \mathcal{U}$.

Example 3. Assume $\mathcal{U} = \{u_1, u_2\}$ be a universe of discourse and $E = \{x_1, x_2, x_3, x_4\}$ be a set of attributes and $\mathcal{A} = \{x_1, x_2\} \subseteq E$. Consider $\mathcal{F}_{\mathcal{A}} \in$ G3-PNSS over $\mathcal{U}$ can be represented as follows

$$\mathcal{F}_{\mathcal{A}} = \begin{Bmatrix} (x_1, \{\langle u_1, (.6,.4,.1), (.4,.3,.2), (.5,.6,1), \big(u_2, (.3,.5,.1), (.3,.2,.1), (.4,.5,.4)\big)\rangle, \\ \big(x_2, \{\langle u_1, (.4,.3,.3), (0,.2,.3), (.4,.2,.5)\big), \big(u_2, (.2,.1,.7), (.6,.3,1), (.5,.3,.1)\big)\rangle \end{Bmatrix}$$

Then,

$$\mathcal{F}^c_{\mathcal{A}}(x) = \begin{Bmatrix} (x_1, \{\langle u_1, (.1,.6,.6),\ (.2,.7,.4), (.1,.4,.5), \big(u_2, (.1,.5,.3), (.1,.8,.3), (.4,.5,.4)\big)\rangle, \\ \big(x_2, \{\langle u_1, (.3,.7,.4), (.3,.8,0), (.5,.8,.4)\big), \big(u_2, (.7,.9,.2), (1,.7,.6), (.1,.7,.5)\big)\rangle \end{Bmatrix}$$

Proposition 3.7

If $\mathcal{F}_{\mathcal{A}}$ be a GmPNSS, then

1. $(\mathcal{F}_{\mathcal{A}}^{c})^{c} = \mathcal{F}_{\mathcal{A}}$
2. $(\mathcal{F}_{\breve{0}})^{c} = \mathcal{F}_{\breve{E}}$
3. $(\mathcal{F}_{\breve{E}})^{c} = \mathcal{F}_{\breve{0}}$

Proof 1 Let $\mathcal{F}_{\mathcal{A}}(e) = \{e, < u, u_{\mathcal{A}(e)}^{\alpha}(u), v_{\mathcal{A}(e)}^{\alpha}(u), w_{\mathcal{A}(e)}^{\alpha}(u) > : u \in \mathcal{U}, e \in E;\ \alpha \in 1,2,3,\dots,m\}$. Then by using definition 3.6, we get

$$\mathcal{F}_{\mathcal{A}}^{c}(e) = \left\{e, < u, \left(w_{\mathcal{A}(e)}^{\alpha}(u), (1,1,\dots,1) - v_{\mathcal{A}(e)}^{\alpha}(u), u_{\mathcal{A}(e)}^{\alpha}(u)\right), > : u \in \mathcal{U}, e \in E;\ \alpha \in 1,2,3,\dots,m\right\}$$

Again, by using definition 3.6

$(\mathcal{F}_{\mathcal{A}}^{c}(e))^{c}=$

$$\left\{< u, \left(w_{\mathcal{A}(e)}^{\alpha}(u),\ (1,1,\dots,1) - \left((1,1,\dots,1) - v_{\mathcal{A}(e)}^{\alpha}(u)\right), u_{\mathcal{A}(e)}^{\alpha}(u)\right) > : u \in \mathcal{U}, e \in E;\ \alpha \in 1,2,3,\dots,m\right\}$$

.$(\mathcal{F}_{\mathcal{A}}^{c}(e))^{c}= \left\{< u, \left(u_{\mathcal{A}(e)}^{\alpha}(u),\ v_{\mathcal{A}(e)}^{\alpha}(u), w_{\mathcal{A}(e)}^{\alpha}(u)\right) > : u \in \mathcal{U}, e \in E;\ \alpha \in 1,2,3,\dots,m\right\}$.

$(\mathcal{F}_{\mathcal{A}}^{c}(e))^{c}= \mathcal{F}_{\mathcal{A}}(e)$.

Similarly, we can prove 2 and 3.

Definition 3.8

Let $\mathcal{F}_{\mathcal{A}(e)}$ and $\mathcal{G}_{B(e)}$ are two GmPNSS over $\mathcal{U}$, then

$$\mathcal{F}_{\mathcal{A}(e)} \cup \mathcal{G}_{B(e)}= \left\{e, < u, \begin{pmatrix} max\{u_{\mathcal{A}(e)}^{\alpha}(u), u_{B(e)}^{\alpha}(u)\}, \\ min\{v_{\mathcal{A}(e)}^{\alpha}(u), v_{B(e)}^{\alpha}(u)\}, \\ min\{w_{\mathcal{A}(e)}^{\alpha}(u), w_{B(e)}^{\alpha}(u)\} \end{pmatrix} > : u \in \mathcal{U}, e \in E;\ \alpha \in 1,2,3,\dots,m\right\}$$

Example 4. Assume $\mathcal{U} = \{u_1, u_2\}$ be a universe of discourse and $E = \{x_1, x_2, x_3, x_4\}$ be a set of attribuites and $\mathcal{A} = B = \{x_1, x_2\} \subseteq E$. Consider $\mathcal{F}_{\mathcal{A}(e)}$ and $\mathcal{G}_{B(e)} \in$ G3-PNSS over $\mathcal{U}$ can be represented as follows

$$\mathcal{F}_{\mathcal{A}(e)} = \left\{\begin{matrix} (x_1, \{\langle u_1, (.5,.2,.1), (.3,.1,.3), (.4,.3,.8), (u_2, (.2,.3,.2), (.2,.1,.3), (.3,.4,.6))\rangle, \\ (x_2, \{\langle u_1, (.3,.1,.4), (0,.1,.5), (.3,.1,.5)), (u_2, (.2,.2,.5), (.3,.1,.5), (.4,.3,.6))\rangle \end{matrix}\right\}$$

and

$$\mathcal{G}_{B(e)} = \left\{\begin{matrix} (x_1, \{\langle u_1, (.6,.4,.1), (.4,.3,.2), (.5,.4,.5), (u_2, (.3,.5,.1), (.3,.2,.1), (.4,.5,.4))\rangle, \\ (x_2, \{\langle u_1, (.4,.3,.3), (0,.2,.3), (.4,.2,.5)), (u_2, (.2,.1,.3), (.6,.3,.1), (.5,.3,.1))\rangle \end{matrix}\right\}$$

Then

$$\mathcal{F}_{\mathcal{A}(e)} \cup \mathcal{G}_{B(e)}= \left\{\begin{matrix} (x_1, \{\langle u_1, (.6,.2,.1), (.4,.1,.2), (.5,.3,.5)), (u_2, (.3,.3,.1), (.3,.1,.1), (.4,.4,.4))\rangle, \\ (x_2, \{\langle u_1, (.4,.1,.3), (0,.1,.3), (.4,.1,.5)), (u_2, (.2,.1,.3), (.6,.1,.1), (.5,.3,.1))\rangle \end{matrix}\right\}$$

Proposition 3.9

Let $\mathcal{F}_{\breve{A}}$, $\mathcal{G}_{\breve{B}}$, $\mathcal{H}_{\breve{C}}$ are GmPNSS over $\mathcal{U}$. Then

1. $\mathcal{F}_{\breve{A}} \cup \mathcal{F}_{\breve{A}} = \mathcal{F}_{\breve{A}}$
2. $\mathcal{F}_{\breve{A}} \cup \mathcal{G}_{\breve{B}} = \mathcal{G}_{\breve{B}} \cup \mathcal{F}_{\breve{A}}$
3. $(\mathcal{F}_{\breve{A}} \cup \mathcal{G}_{\breve{B}}) \cup \mathcal{H}_{\breve{C}} = \mathcal{F}_{\breve{A}} \cup (\mathcal{G}_{\breve{B}} \cup \mathcal{H}_{\breve{C}})$

Proof 1. As we know that

$\mathcal{F}_{\breve{A}}(e) = \left\{e, < u, u_{\breve{A}(e)}^{\alpha}(u), v_{\breve{A}(e)}^{\alpha}(u), w_{\breve{A}(e)}^{\alpha}(u) > : u \in \mathcal{U}, e \in E;\ \alpha \in 1,2,3,\dots,m\right\}$ be a GmPNSS over $\mathcal{U}$. Then by using definition 3.8

$$\mathcal{F}_{\breve{A}} \cup \mathcal{F}_{\breve{A}} = \left\{ e, < u, \begin{pmatrix} max\left\{ u^{\alpha}_{\breve{A}(e)}(u), u^{\alpha}_{\breve{A}(e)}(u) \right\}, \\ min\left\{ v^{\alpha}_{\breve{A}(e)}(u), v^{\alpha}_{\breve{A}(e)}(u) \right\}, \\ min\left\{ w^{\alpha}_{\breve{A}(e)}(u), w^{\alpha}_{\breve{A}(e)}(u) \right\} \end{pmatrix} > : u \in \mathcal{U}, e \in E;\ \alpha \in 1,2,3,\dots,m \right\}$$

$$\mathcal{F}_{\breve{A}} \cup \mathcal{F}_{\breve{A}} = \left\{ e, < u, u^{\alpha}_{\breve{A}(e)}(u), v^{\alpha}_{\breve{A}(e)}(u),\ w^{\alpha}_{\breve{A}(e)}(u) > : u \in \mathcal{U}, e \in E;\ \alpha \in 1,2,3,\dots,m \right\}$$

$$\mathcal{F}_{\breve{A}} \cup \mathcal{F}_{\breve{A}} = \mathcal{F}_{\breve{A}}.$$

Proof 2. As we know that

$\mathcal{F}_{\breve{A}}(e) = \left\{ e, < u, u^{\alpha}_{\breve{A}(e)}(u), v^{\alpha}_{\breve{A}(e)}(u),\ w^{\alpha}_{\breve{A}(e)}(u) > : u \in \mathcal{U}, e \in E;\ \alpha \in 1,2,3,\dots,m \right\}$ and $\mathcal{G}_{\breve{B}}(e) = \left\{ e, < u, u^{\alpha}_{\breve{B}(e)}(u), v^{\alpha}_{\breve{B}(e)}(u),\ w^{\alpha}_{\breve{B}(e)}(u) > : u \in \mathcal{U}, e \in E;\ \alpha \in 1,2,3,\dots,m \right\}$ are two GmPNSS over $\mathcal{U}$. Then

$$\mathcal{F}_{\breve{A}} \cup \mathcal{G}_{\breve{B}} = \left\{ e, < u, \begin{pmatrix} max\left\{ u^{\alpha}_{\breve{A}(e)}(u), u^{\alpha}_{\breve{B}(e)}(u) \right\}, \\ min\left\{ v^{\alpha}_{\breve{A}(e)}(u), v^{\alpha}_{\breve{B}(e)}(u) \right\}, \\ min\left\{ w^{\alpha}_{\breve{A}(e)}(u), w^{\alpha}_{\breve{B}(e)}(u) \right\} \end{pmatrix} > : u \in \mathcal{U}, e \in E;\ \alpha \in 1,2,3,\dots,m \right\}$$

$$\mathcal{F}_{\breve{A}} \cup \mathcal{G}_{\breve{B}} = \left\{ e, < u, \begin{pmatrix} max\left\{ u^{\alpha}_{\breve{B}(e)}(u), u^{\alpha}_{\breve{A}(e)}(u) \right\}, \\ min\left\{ v^{\alpha}_{\breve{B}(e)}(u), v^{\alpha}_{\breve{A}(e)}(u) \right\}, \\ min\left\{ w^{\alpha}_{\breve{B}(e)}(u), w^{\alpha}_{\breve{A}(e)}(u) \right\} \end{pmatrix} > : u \in \mathcal{U}, e \in E;\ \alpha \in 1,2,3,\dots,m \right\}$$

$$\mathcal{F}_{\breve{A}} \cup \mathcal{G}_{\breve{B}} = \mathcal{G}_{\breve{B}} \cup \mathcal{F}_{\breve{A}}$$

Similarly, we can prove 3.

Definition 3.10

Let $\mathcal{F}_{\mathcal{A}(e)}$ and $\mathcal{G}_{B(e)}$ are GmPNSS over $\mathcal{U}$, then

$$\mathcal{F}_{\mathcal{A}(e)} \cap \mathcal{G}_{B(e)} = \left\{ e, < u, \begin{pmatrix} min\{ u^{\alpha}_{\mathcal{A}(e)}(u), u^{\alpha}_{B(e)}(u) \}, \\ max\{ v^{\alpha}_{\mathcal{A}(e)}(u), v^{\alpha}_{B(e)}(u) \}, \\ max\{ w^{\alpha}_{\mathcal{A}(e)}(u), w^{\alpha}_{B(e)}(u) \} \end{pmatrix} > : u \in \mathcal{U}, e \in E;\ \alpha \in 1,2,3,\dots,m \right\}$$

Proposition 3.11

Let $\mathcal{F}_{\breve{A}}$, $\mathcal{G}_{\breve{B}}$, $\mathcal{H}_{\breve{C}}$ are GmPNSS over $\mathcal{U}$. Then

1. $\mathcal{F}_{\breve{A}} \cap \mathcal{F}_{\breve{A}} = \mathcal{F}_{\breve{A}}$
2. $\mathcal{F}_{\breve{A}} \cap \mathcal{G}_{\breve{B}} = \mathcal{G}_{\breve{B}} \cap \mathcal{F}_{\breve{A}}$
3. $(\mathcal{F}_{\breve{A}} \cap \mathcal{G}_{\breve{B}}) \cap \mathcal{H}_{\breve{C}} = \mathcal{F}_{\breve{A}} \cap (\mathcal{G}_{\breve{B}} \cap \mathcal{H}_{\breve{C}})$

Proof Similar to proposition 3.9, by using definition 3.11, we can prove easily.

Remark 3.12 Generally, if $\mathcal{F}_{\breve{A}} \neq \mathcal{F}_{\breve{0}}$ and $\mathcal{F}_{\breve{A}} \neq \mathcal{F}_{E}$, then the law of contradiction $\mathcal{F}_{\breve{A}} \cap {\mathcal{F}_{\breve{A}}}^{C} = \mathcal{F}_{\breve{0}}$ and the law of the excluded middle $\mathcal{F}_{\breve{A}} \cap {\mathcal{F}_{\breve{A}}}^{C} = \mathcal{F}_{E}$ does not hold in mPIVNSS. But in classical set theory law of contradiction and excluded middle always hold.

Proposition 3.13

Let $\mathcal{F}_{\breve{A}}$ and $\mathcal{G}_{\breve{B}}$ are GmPNSS over $\mathcal{U}$, then

1. $(\mathcal{F}_{\breve{A}(e)} \cup \mathcal{G}_{\breve{B}(e)})^{C} = {\mathcal{F}_{\breve{A}(e)}}^{C} \cap {\mathcal{G}_{\breve{B}(e)}}^{C}$

2. $(\mathcal{F}_{\breve{A}(e)} \cap G_{B(e)})^C = \mathcal{F}_{\breve{A}(e)}{}^C \cup \mathcal{G}_{\breve{B}(e)}{}^C$

Proof 1 As we know that

$\mathcal{F}_{\breve{A}}(e) = \left\{e, < u, u^{\alpha}_{\breve{A}(e)}(u), v^{\alpha}_{\breve{A}(e)}(u), w^{\alpha}_{\breve{A}(e)}(u) > : u \in \mathcal{U}, e \in E;\ \alpha \in 1,2,3,\dots,m\right\}$ and $\mathcal{G}_{\breve{B}}(e) = \left\{e, < u, u^{\alpha}_{\breve{B}(e)}(u), v^{\alpha}_{\breve{B}(e)}(u), w^{\alpha}_{\breve{B}(e)}(u) > : u \in \mathcal{U}, e \in E;\ \alpha \in 1,2,3,\dots,m\right\}$ are two GmPNSS.

By using definition 3.8, we get

$$\mathcal{F}_{\breve{A}}(e) \cup \mathcal{G}_{\breve{B}(e)} = \left\{e, < u, \begin{pmatrix} max\left\{u^{\alpha}_{\breve{A}(e)}(u), u^{\alpha}_{\breve{B}(e)}(u)\right\}, \\ min\left\{v^{\alpha}_{\breve{A}(e)}(u), v^{\alpha}_{\breve{B}(e)}(u)\right\}, \\ min\left\{w^{\alpha}_{\breve{A}(e)}(u), w^{\alpha}_{\breve{B}(e)}(u)\right\} \end{pmatrix} > : u \in \mathcal{U}, e \in E;\ \alpha \in 1,2,3,\dots,m \right\}$$

Now by using definition 3.6, we get

$$\left(\mathcal{F}_{\breve{A}}(e) \cup \mathcal{G}_{\breve{B}(e)}\right)^c = \left\{(e, (< u, \begin{pmatrix} min\left\{w^{\alpha}_{\breve{A}(e)}(u), w^{\alpha}_{\breve{B}(e)}(u)\right\}, \\ (1,1,\dots,1) - min\left\{v^{\alpha}_{\breve{A}(e)}(u), v^{\alpha}_{\breve{B}(e)}(u)\right\}, \\ max\left\{u^{\alpha}_{\breve{A}(e)}(u), u^{\alpha}_{\breve{B}(e)}(u)\right\} \end{pmatrix} > : u \in \mathcal{U}, e \in E;\ \alpha \in 1,2,3,\dots,m \right\}$$

Now

$$\mathcal{F}_{\breve{A}}(e)^C = \left\{< u, \left(w^{\alpha}_{\breve{A}(e)}(u), (1,1,\dots,1) - v^{\alpha}_{\breve{A}(e)}(u), u^{\alpha}_{\breve{A}(e)}(u)\right) > : u \in \mathcal{U}, e \in E;\ \alpha \in 1,2,3,\dots,m\right\}$$

$$\mathcal{G}_{\breve{B}(e)}{}^C = \left\{< u, \left(w^{\alpha}_{\breve{B}(e)}(u), (1,1,\dots,1) - v^{\alpha}_{\breve{B}(e)}(u), u^{\alpha}_{\breve{B}(e)}(u)\right) > : u \in \mathcal{U}, e \in E;\ \alpha \in 1,2,3,\dots,m\right\}$$

By using definition 3.10

$$\mathcal{F}_{\breve{A}}(e)^C \cap \mathcal{G}_{\breve{B}(e)}{}^C = \left\{(e, < u, \begin{pmatrix} min\left\{w^{\alpha}_{\breve{A}(e)}(u), w^{\alpha}_{\breve{B}(e)}(u)\right\}, \\ \max\left\{(1,1,\dots,1) - v^{\alpha}_{\breve{A}(e)}(u), (1,1,\dots,1) - v^{\alpha}_{\breve{B}(e)}(u)\right\}, \\ max\left\{u^{\alpha}_{\breve{A}(e)}(u), u^{\alpha}_{\breve{B}(e)}(u)\right\} \end{pmatrix} > : u \in \mathcal{U}, e \in E;\ \alpha \in 1,2,3,\dots,m \right\}$$

$$\mathcal{F}_{\breve{A}}(e)^C \cap \mathcal{G}_{\breve{B}(e)}{}^C = \left\{(e, (< u, \begin{pmatrix} min\left\{w^{\alpha}_{\breve{A}(e)}(u), w^{\alpha}_{\breve{B}(e)}(u)\right\}, \\ (1,1,\dots,1) - min\left\{v^{\alpha}_{\breve{A}(e)}(u), v^{\alpha}_{\breve{B}(e)}(u)\right\}, \\ max\left\{u^{\alpha}_{\breve{A}(e)}(u), u^{\alpha}_{\breve{B}(e)}(u)\right\} \end{pmatrix} > : u \in \mathcal{U}, e \in E; \alpha \in 1,2,3,\dots,m \right\}$$

Hence

$(\mathcal{F}_{\breve{A}}(e) \cup \mathcal{G}_{\breve{B}(e)})^C = \mathcal{F}_{\breve{A}}(e)^C \cap \mathcal{G}_{\breve{B}(e)}{}^C$.

Proof 2 By using definition 3.10, we have

$$\mathcal{F}_{\breve{A}}(e) \cap \mathcal{G}_{\breve{B}(e)} = \left\{e, < u, \begin{pmatrix} min\left\{u^{\alpha}_{\breve{A}(e)}(u), u^{\alpha}_{\breve{B}(e)}(u)\right\}, \\ max\left\{v^{\alpha}_{\breve{A}(e)}(u), v^{\alpha}_{\breve{B}(e)}(u)\right\}, \\ max\left\{w^{\alpha}_{\breve{A}(e)}(u), w^{\alpha}_{\breve{B}(e)}(u)\right\} \end{pmatrix} > : u \in \mathcal{U}, e \in E;\ \alpha \in 1,2,3,\dots,m \right\}$$

Now by using definition 3.6, we get

$\left(\mathcal{F}_{\check{A}}(e) \cap \mathcal{G}_{\check{B}(e)}\right)^{c}$ =

$$\left\{ (e, (< u, \begin{pmatrix} max\left\{w^{\alpha}_{\check{A}(e)}(u), w^{\alpha}_{\check{B}(e)}(u)\right\}, \\ (1,1,\dots,1) - max\left\{v^{\alpha}_{\check{A}(e)}(u), v^{\alpha}_{\check{B}(e)}(u)\right\}, \\ min\left\{u^{\alpha}_{\check{A}(e)}(u), u^{\alpha}_{\check{B}(e)}(u)\right\} \end{pmatrix} > : u \in \mathcal{U}, e \in E;\ \alpha \in 1, 2, 3, \dots, m \right\}$$

Now

$$\mathcal{F}_{\check{A}}(e)^{C} = \left\{ < u, \left(w^{\alpha}_{\check{A}(e)}(u), (1,1,\dots,1) - v^{\alpha}_{\check{A}(e)}(u), u^{\alpha}_{\check{A}(e)}(u) \right) > : u \in \mathcal{U}, e \in E;\ \alpha \in 1, 2, 3, \dots, m \right\}$$

$$\mathcal{G}_{\check{B}(e)}{}^{C} = \left\{ < u, \left(w^{\alpha}_{\check{B}(e)}(u), (1,1,\dots,1) - v^{\alpha}_{\check{B}(e)}(u), u^{\alpha}_{\check{B}(e)}(u) \right) > : u \in \mathcal{U}, e \in E;\ \alpha \in 1, 2, 3, \dots, m \right\}$$

By using definition 3.8

$\mathcal{F}_{\check{A}}(e)^{C} \cup \mathcal{G}_{\check{B}(e)}{}^{C}$=

$$\left\{ (e, < u, \begin{pmatrix} max\left\{w^{\alpha}_{\check{A}(e)}(u), w^{\alpha}_{\check{B}(e)}(u)\right\}, \\ min\left\{(1,1,\dots,1) - v^{\alpha}_{\check{A}(e)}(u), (1,1,\dots,1) - v^{\alpha}_{\check{B}(e)}(u)\right\}, \\ min\left\{u^{\alpha}_{\check{A}(e)}(u), u^{\alpha}_{\check{B}(e)}(u)\right\} \end{pmatrix} > : u \in \mathcal{U}, e \in E;\ \alpha \in 1, 2, 3, \dots, m \right\}$$

$\mathcal{F}_{\check{A}}(e)^{C} \cup \mathcal{G}_{\check{B}(e)}{}^{C}$=

$$\left\{ (e, (< u, \begin{pmatrix} max\left\{w^{\alpha}_{\check{A}(e)}(u), w^{\alpha}_{\check{B}(e)}(u)\right\}, \\ (1,1,\dots,1) - max\left\{v^{\alpha}_{\check{A}(e)}(u), v^{\alpha}_{\check{B}(e)}(u)\right\}, \\ min\left\{u^{\alpha}_{\check{A}(e)}(u), u^{\alpha}_{\check{B}(e)}(u)\right\} \end{pmatrix} > : u \in \mathcal{U}, e \in E;\ \alpha \in 1, 2, 3, \dots, m \right\}$$

Hence

$(\mathcal{F}_{\check{A}(e)} \cap G_{B(e)})^{C} = \mathcal{F}_{\check{A}(e)}{}^{C} \cup \mathcal{G}_{\check{B}(e)}{}^{C}$.

Proposition 3.14

Let $\mathcal{F}_{\widetilde{A(e)}}$, $\mathcal{G}_{\widetilde{B(e)}}$, $\mathcal{H}_{\widetilde{C(e)}}$ are GmPNSS over $\mathcal{U}$. Then

1. $\mathcal{F}_{\widetilde{A(e)}} \cup (\mathcal{G}_{\widetilde{B(e)}} \cap \mathcal{H}_{\widetilde{C(e)}}) = (\mathcal{F}_{\widetilde{A(e)}} \cup \mathcal{G}_{\widetilde{B(e)}}) \cap (\mathcal{F}_{\widetilde{A(e)}} \cup \mathcal{H}_{\widetilde{C(e)}})$
2. $\mathcal{F}_{\widetilde{A(e)}} \cap (\mathcal{G}_{\widetilde{B(e)}} \cup \mathcal{H}_{\widetilde{C(e)}}) = (\mathcal{F}_{\widetilde{A(e)}} \cap \mathcal{G}_{\widetilde{B(e)}}) \cup (\mathcal{F}_{\widetilde{A(e)}} \cap \mathcal{H}_{\check{C}(e)})$
3. $\mathcal{F}_{\widetilde{A(e)}} \cup (\mathcal{F}_{\check{A}(e)} \cap \mathcal{G}_{\widetilde{B(e)}}) = \mathcal{F}_{\widetilde{A(e)}}$
4. $\mathcal{F}_{\widetilde{A(e)}} \cap (\mathcal{F}_{\widetilde{A(e)}} \cup \mathcal{G}_{\widetilde{B(e)}}) = \mathcal{F}_{\widetilde{A(e)}}$

Proof 1 As we know that

$$\mathcal{G}_{\widetilde{B(e)}} \cap \mathcal{H}_{\widetilde{C(e)}} = \left\{ e, < u, \begin{pmatrix} min\left\{u^{\alpha}_{\widetilde{B(e)}}(u), u^{\alpha}_{\widetilde{C(e)}}(u)\right\}, \\ max\left\{v^{\alpha}_{\widetilde{B(e)}}(u), v^{\alpha}_{\widetilde{C(e)}}(u)\right\}, \\ max\left\{w^{\alpha}_{\widetilde{B(e)}}(u), w^{\alpha}_{\widetilde{C(e)}}(u)\right\} \end{pmatrix} > : u \in \mathcal{U}, e \in E;\ \alpha \in 1, 2, 3, \dots, m \right\}$$

$\mathcal{F}_{\widetilde{A(e)}} \cup (\mathcal{G}_{\widetilde{B(e)}} \cap \mathcal{H}_{\widetilde{C(e)}})$ =

$$\left\{e, < u, \begin{pmatrix} max\left\{u^{\alpha}_{\overline{A(e)}}(u), min\left\{u^{\alpha}_{\overline{B(e)}}(u), u^{\alpha}_{\overline{C(e)}}(u)\right\}\right\}, \\ min\left\{v^{\alpha}_{\overline{A(e)}}(u), max\left\{v^{\alpha}_{\overline{B(e)}}(u), v^{\alpha}_{\overline{C(e)}}(u)\right\}\right\} \\ min\left\{w^{\alpha}_{\overline{B(e)}}(u), max\left\{w^{\alpha}_{\overline{B(e)}}(u), w^{\alpha}_{\overline{C(e)}}(u)\right\}\right\} \end{pmatrix} > : u \in \mathcal{U}, e \in E;\ \alpha \in 1, 2, 3, \ldots, m\right\}$$

$$\mathcal{F}_{\overline{A(e)}} \cup \mathcal{G}_{\overline{B(e)}} = \left\{e, < u, \begin{pmatrix} max\left\{u^{\alpha}_{\overline{A(e)}}(u), u^{\alpha}_{\overline{B(e)}}(u)\right\}, \\ min\left\{v^{\alpha}_{\overline{A(e)}}(u), v^{\alpha}_{\overline{B(e)}}(u)\right\}, \\ min\left\{w^{\alpha}_{\overline{A(e)}}(u), w^{\alpha}_{\overline{B(e)}}(u)\right\} \end{pmatrix} > : u \in \mathcal{U}, e \in E;\ \alpha \in 1, 2, 3, \ldots, m\right\}$$

$$\mathcal{F}_{\overline{A(e)}} \cup \mathcal{H}_{\overline{C(e)}} = \left\{e, < u, \begin{pmatrix} max\left\{u^{\alpha}_{\overline{A(e)}}(u), u^{\alpha}_{\overline{C(e)}}(u)\right\}, \\ min\left\{v^{\alpha}_{\overline{A(e)}}(u), v^{\alpha}_{\overline{C(e)}}(u)\right\}, \\ min\left\{w^{\alpha}_{\overline{A(e)}}(u), w^{\alpha}_{\overline{C(e)}}(u)\right\} \end{pmatrix} > : u \in \mathcal{U}, e \in E;\ \alpha \in 1, 2, 3, \ldots, m\right\}$$

$(\mathcal{F}_{\overline{A(e)}} \cup \mathcal{G}_{\overline{B(e)}}) \cap (\mathcal{F}_{\overline{A(e)}} \cup \mathcal{H}_{\overline{C(e)}}) =$

$$\left\{e, < u, \begin{pmatrix} min\left\{max\left\{u^{\alpha}_{\overline{A(e)}}(u), u^{\alpha}_{\overline{B(e)}}(u)\right\}, max\left\{u^{\alpha}_{\overline{A(e)}}(u), u^{\alpha}_{\overline{C(e)}}(u)\right\}\right\}, \\ max\left\{min\left\{v^{\alpha}_{\overline{A(e)}}(u), v^{\alpha}_{\overline{B(e)}}(u)\right\}, min\left\{v^{\alpha}_{\overline{A(e)}}(u), v^{\alpha}_{\overline{C(e)}}(u)\right\}\right\}, \\ max\left\{min\left\{w^{\alpha}_{\overline{A(e)}}(u), w^{\alpha}_{\overline{B(e)}}(u)\right\}, min\left\{w^{\alpha}_{\overline{A(e)}}(u), w^{\alpha}_{\overline{C(e)}}(u)\right\}\right\} \end{pmatrix} > : u \in \mathcal{U}, e \in E; \alpha \in 1, 2, 3, \ldots, m\right\}$$

$(\mathcal{F}_{\overline{A(e)}} \cup \mathcal{G}_{\overline{B(e)}}) \cap (\mathcal{F}_{\overline{A(e)}} \cup \mathcal{H}_{\overline{C(e)}}) =$

$$\left\{e, < u, \begin{pmatrix} min\left\{u^{\alpha}_{\overline{A(e)}}(u), max\left\{u^{\alpha}_{\overline{B(e)}}(u), u^{\alpha}_{\overline{C(e)}}(u)\right\}\right\}, \\ max\left\{v^{\alpha}_{\overline{A(e)}}(u), min\left\{v^{\alpha}_{\overline{B(e)}}(u), v^{\alpha}_{\overline{C(e)}}(u)\right\}\right\}, \\ max\left\{w^{\alpha}_{\overline{A(e)}}(u), min\left\{w^{\alpha}_{\overline{B(e)}}(u), w^{\alpha}_{\overline{C(e)}}(u)\right\}\right\} \end{pmatrix} > : u \in \mathcal{U}, e \in E; \alpha \in 1, 2, 3, \ldots, m\right\}$$

Hence

$\mathcal{F}_{\overline{A(e)}} \cup (\mathcal{G}_{\overline{B(e)}} \cap \mathcal{H}_{\overline{C(e)}}) = (\mathcal{F}_{\overline{A(e)}} \cup \mathcal{G}_{\overline{B(e)}}) \cap (\mathcal{F}_{\overline{A(e)}} \cup \mathcal{H}_{\overline{C(e)}})$.

Proof 2. As we know that

$$\mathcal{G}_{\overline{B(e)}} \cup \mathcal{H}_{\overline{C(e)}} = \left\{e, < u, \begin{pmatrix} max\left\{u^{\alpha}_{\overline{B(e)}}(u), u^{\alpha}_{\overline{C(e)}}(u)\right\}, \\ min\left\{v^{\alpha}_{\overline{B(e)}}(u), v^{\alpha}_{\overline{C(e)}}(u)\right\}, \\ min\left\{w^{\alpha}_{\overline{B(e)}}(u), w^{\alpha}_{\overline{C(e)}}(u)\right\} \end{pmatrix} > : u \in \mathcal{U}, e \in E;\ \alpha \in 1, 2, 3, \ldots, m\right\}$$

$\mathcal{F}_{\overline{A(e)}} \cap (\mathcal{G}_{\overline{B(e)}} \cup \mathcal{H}_{\overline{C(e)}}) =$

$$\left\{e, < u, \begin{pmatrix} min\left\{u^{\alpha}_{\overline{A(e)}}(u), max\left\{u^{\alpha}_{\overline{B(e)}}(u), u^{\alpha}_{\overline{C(e)}}(u)\right\}\right\}, \\ max\left\{v^{\alpha}_{\overline{A(e)}}(u), min\left\{v^{\alpha}_{\overline{B(e)}}(u), v^{\alpha}_{\overline{C(e)}}(u)\right\}\right\}, \\ max\left\{w^{\alpha}_{\overline{B(e)}}(u), min\left\{w^{\alpha}_{\overline{B(e)}}(u), w^{\alpha}_{\overline{C(e)}}(u)\right\}\right\} \end{pmatrix} >: u \in \mathcal{U}, e \in E; \alpha \in 1, 2, 3, \ldots, m\right\}$$

$$\mathcal{F}_{\overline{A(e)}} \cap \mathcal{G}_{\overline{B(e)}} = \left\{e, < u, \begin{pmatrix} min\left\{u^{\alpha}_{\overline{A(e)}}(u), u^{\alpha}_{\overline{B(e)}}(u)\right\}, \\ max\left\{v^{\alpha}_{\overline{A(e)}}(u), v^{\alpha}_{\overline{B(e)}}(u)\right\}, \\ max\left\{w^{\alpha}_{\overline{A(e)}}(u), w^{\alpha}_{\overline{B(e)}}(u)\right\} \end{pmatrix} > : u \in \mathcal{U}, e \in E;\ \alpha \in 1, 2, 3, \ldots, m\right\}$$

$$\mathcal{F}_{\overline{A(e)}} \cap \mathcal{H}_{\overline{C(e)}} = \left\{e, < u, \begin{pmatrix} min\left\{u^{\alpha}_{\overline{A(e)}}(u), u^{\alpha}_{\overline{C(e)}}(u)\right\}, \\ max\left\{v^{\alpha}_{\overline{A(e)}}(u), v^{\alpha}_{\overline{C(e)}}(u)\right\}, \\ max\left\{w^{\alpha}_{\overline{A(e)}}(u), w^{\alpha}_{\overline{C(e)}}(u)\right\} \end{pmatrix} > : u \in \mathcal{U}, e \in E;\ \alpha \in 1, 2, 3, \ldots, m\right\}$$

$(\mathcal{F}_{\widetilde{A(e)}} \cap \mathcal{G}_{\widetilde{B(e)}}) \cup (\mathcal{F}_{\widetilde{A(e)}} \cap \mathcal{H}_{\widetilde{C(e)}}) =$

$$\left\{ e, < u, \begin{pmatrix} max\left\{min\left\{u^{\alpha}_{\widetilde{A(e)}}(u), u^{\alpha}_{\widetilde{B(e)}}(u)\right\}, min\left\{u^{\alpha}_{\widetilde{A(e)}}(u), u^{\alpha}_{\widetilde{C(e)}}(u)\right\}\right\}, \\ min\left\{max\left\{v^{\alpha}_{\widetilde{A(e)}}(u), v^{\alpha}_{\widetilde{B(e)}}(u)\right\}, max\left\{v^{\alpha}_{\widetilde{A(e)}}(u), v^{\alpha}_{\widetilde{C(e)}}(u)\right\}\right\}, \\ min\left\{max\left\{w^{\alpha}_{\widetilde{A(e)}}(u), w^{\alpha}_{\widetilde{B(e)}}(u)\right\}, max\left\{w^{\alpha}_{\widetilde{A(e)}}(u), w^{\alpha}_{\widetilde{C(e)}}(u)\right\}\right\} \end{pmatrix} > : u \in \mathcal{U}, e \in E;\ \alpha \in 1,2,3,\dots,m \right\}$$

$(\mathcal{F}_{\widetilde{A(e)}} \cap \mathcal{G}_{\widetilde{B(e)}}) \cup (\mathcal{F}_{\widetilde{A(e)}} \cap \mathcal{H}_{\widetilde{C(e)}}) =$

$$\left\{ e, < u, \begin{pmatrix} max\left\{u^{\alpha}_{\widetilde{A(e)}}(u), min\left\{u^{\alpha}_{\widetilde{B(e)}}(u), u^{\alpha}_{\widetilde{C(e)}}(u)\right\}\right\}, \\ min\left\{v^{\alpha}_{\widetilde{A(e)}}(u), max\left\{v^{\alpha}_{\widetilde{B(e)}}(u), v^{\alpha}_{\widetilde{C(e)}}(u)\right\}\right\}, \\ min\left\{w^{\alpha}_{\widetilde{A(e)}}(u), max\left\{w^{\alpha}_{\widetilde{B(e)}}(u), w^{\alpha}_{\widetilde{C(e)}}(u)\right\}\right\} \end{pmatrix} > : u \in \mathcal{U}, e \in E;\ \alpha \in 1,2,3,\dots,m \right\}$$

Hence

$\mathcal{F}_{\widetilde{A(e)}} \cap (\mathcal{G}_{\widetilde{B(e)}} \cup \mathcal{H}_{\widetilde{C(e)}}) = (\mathcal{F}_{\widetilde{A(e)}} \cap \mathcal{G}_{\widetilde{B(e)}}) \cup (\mathcal{F}_{\widetilde{A(e)}} \cap \mathcal{H}_{\check{C}(e)})$.

Proof 3. As

$$\mathcal{F}_{\widetilde{A(e)}} \cap \mathcal{G}_{\widetilde{B(e)}} = \left\{ e, < u, \begin{pmatrix} min\left\{u^{\alpha}_{\widetilde{A(e)}}(u), u^{\alpha}_{\widetilde{B(e)}}(u)\right\}, \\ max\left\{v^{\alpha}_{\widetilde{A(e)}}(u), v^{\alpha}_{\widetilde{B(e)}}(u)\right\}, \\ max\left\{w^{\alpha}_{\widetilde{A(e)}}(u), w^{\alpha}_{\widetilde{B(e)}}(u)\right\} \end{pmatrix} > : u \in \mathcal{U}, e \in E;\ \alpha \in 1,2,3,\dots,m \right\}$$

$\mathcal{F}_{\widetilde{A(e)}} \cup (\mathcal{F}_{\check{A}(e)} \cap \mathcal{G}_{\widetilde{B(e)}}) =$

$$\left\{ e, < u, \begin{pmatrix} max\left\{u^{\alpha}_{\widetilde{A(e)}}(u), min\left\{u^{\alpha}_{\widetilde{A(e)}}(u), u^{\alpha}_{\widetilde{B(e)}}(u)\right\}\right\}, \\ min\left\{v^{\alpha}_{\widetilde{A(e)}}(u), max\left\{v^{\alpha}_{\widetilde{A(e)}}(u), v^{\alpha}_{\widetilde{B(e)}}(u)\right\}\right\}, \\ min\left\{w^{\alpha}_{\widetilde{A(e)}}(u), max\left\{w^{\alpha}_{\widetilde{A(e)}}(u), w^{\alpha}_{\widetilde{B(e)}}(u)\right\}\right\} \end{pmatrix} > : u \in \mathcal{U}, e \in E;\ \alpha \in 1,2,3,\dots,m \right\}$$

$$\mathcal{F}_{\widetilde{A(e)}} \cup (\mathcal{F}_{\check{A}(e)} \cap \mathcal{G}_{\widetilde{B(e)}}) = \left\{ e, < u, u^{\alpha}_{\check{A}(e)}(u), v^{\alpha}_{\check{A}(e)}(u),\ w^{\alpha}_{\check{A}(e)}(u) > : u \in \mathcal{U}, e \in E;\ \alpha \in 1,2,3,\dots,m \right\}$$

Hence

$\mathcal{F}_{\widetilde{A(e)}} \cup (\mathcal{F}_{\check{A}(e)} \cap \mathcal{G}_{\widetilde{B(e)}}) = \mathcal{F}_{\widetilde{A(e)}}$.

Proof 4. $\mathcal{F}_{\widetilde{A(e)}} \cap (\mathcal{F}_{\widetilde{A(e)}} \cup \mathcal{G}_{\widetilde{B(e)}}) = \mathcal{F}_{\widetilde{A(e)}}$

As

$$\mathcal{F}_{\widetilde{A(e)}} \cup \mathcal{G}_{\widetilde{B(e)}} = \left\{ e, < u, \begin{pmatrix} max\left\{u^{\alpha}_{\widetilde{A(e)}}(u), u^{\alpha}_{\widetilde{B(e)}}(u)\right\}, \\ min\left\{v^{\alpha}_{\widetilde{A(e)}}(u), v^{\alpha}_{\widetilde{B(e)}}(u)\right\}, \\ min\left\{w^{\alpha}_{\widetilde{A(e)}}(u), w^{\alpha}_{\widetilde{B(e)}}(u)\right\} \end{pmatrix} > : u \in \mathcal{U}, e \in E;\ \alpha \in 1,2,3,\dots,m \right\}$$

$\mathcal{F}_{\widetilde{A(e)}} \cap (\mathcal{F}_{\check{A}(e)} \cup \mathcal{G}_{\widetilde{B(e)}}) =$

$$\left\{ e, < u, \begin{pmatrix} min\left\{u^{\alpha}_{\widetilde{A(e)}}(u), max\left\{u^{\alpha}_{\widetilde{A(e)}}(u), u^{\alpha}_{\widetilde{B(e)}}(u)\right\}\right\}, \\ max\left\{v^{\alpha}_{\widetilde{A(e)}}(u), min\left\{v^{\alpha}_{\widetilde{A(e)}}(u), v^{\alpha}_{\widetilde{B(e)}}(u)\right\}\right\}, \\ max\left\{w^{\alpha}_{\widetilde{A(e)}}(u), min\left\{w^{\alpha}_{\widetilde{A(e)}}(u), w^{\alpha}_{\widetilde{B(e)}}(u)\right\}\right\} \end{pmatrix} > : u \in \mathcal{U}, e \in E;\ \alpha \in 1,2,3,\dots,m \right\}$$

$$\mathcal{F}_{\widetilde{A(e)}} \cap (\mathcal{F}_{\check{A}(e)} \cup \mathcal{G}_{\widetilde{B(e)}}) = \left\{ e, < u, u^{\alpha}_{\check{A}(e)}(u), v^{\alpha}_{\check{A}(e)}(u),\ w^{\alpha}_{\check{A}(e)}(u) > : u \in \mathcal{U}, e \in E;\ \alpha \in 1,2,3,\dots,m \right\}$$

Hence

$\mathcal{F}_{\widetilde{A(e)}} \cap (\mathcal{F}_{\check{A}(e)} \cup \mathcal{G}_{\widetilde{B(e)}}) = \mathcal{F}_{\widetilde{A(e)}}$.

Definition 3.15

Let $\mathcal{F}_{\breve{A}}$, $\mathcal{G}_{\breve{B}}$ are GmPNSS, then their difference defined as follows

$$\mathcal{F}_{\breve{A}} \setminus \mathcal{G}_{\breve{B}} = \begin{Bmatrix} (< u, min\left\{u^{\alpha}_{\breve{A}(e)}(u), u^{\alpha}_{\breve{B}(e)}(u)\right\}, max\left\{v^{\alpha}_{\breve{A}(e)}(u),\ (1,1,\ldots,1) - v^{\alpha}_{\breve{B}(e)}(u)\right\}, \\ max\left\{w^{\alpha}_{\breve{A}(e)}(u), w^{\alpha}_{\breve{B}(e)}(u)\right\} >: u \in \mathcal{U}; \alpha \in 1,2,3,\ldots,m) \end{Bmatrix}$$

Definition 3.16

Let $\mathcal{F}_{\breve{A}}$, $\mathcal{G}_{\breve{B}}$ are GmPNSS, then their addition is defined as follows

$$\mathcal{F}_{\breve{A}} + \mathcal{G}_{\breve{B}} = \begin{Bmatrix} (< u, min\left\{u^{\alpha}_{\breve{A}(e)}(u) + u^{\alpha}_{\breve{B}(e)}(u), (1,1,\ldots,1)\right\}, min\left\{v^{\alpha}_{\breve{A}(e)}(u) + v^{\alpha}_{\breve{B}(e)}(u), (1,1,\ldots,1)\right\}, \\ min\left\{w^{\alpha}_{\breve{A}(e)}(u) + w^{\alpha}_{\breve{B}(e)}(u), (1,1,\ldots,1)\right\} >: u \in \mathcal{U};\ i\ \in\ 1,2,3,\ldots,m) \end{Bmatrix}$$

Definition 3.17

Let $\mathcal{F}_{\breve{A}}$ be a GmPNSS; then its scalar multiplication is represented as $\mathcal{F}_{\breve{A}}(e).\breve{a}$, where $\breve{a} \in [0, 1]$ and defined as follows

$$\mathcal{F}_{\breve{A}}.\breve{a} =$$

$$\left\{(< u, min\left\{u^{\alpha}_{\breve{A}(e)}(u).\breve{a}, (1,1,\ldots,1)\right\}, min\left\{v^{\alpha}_{\breve{A}(e)}(u).\breve{a}, (1,1,\ldots,1)\right\}, min\left\{w^{\alpha}_{\breve{A}(e)}(u).\breve{a}, (1,1,\ldots,1)\right\} >: u \in \mathcal{U})\right\}$$

Definition 3.18

Let $\mathcal{F}_{\breve{A}}$ be a GmPNSS; then its scalar division is represented as $\mathcal{F}_{\breve{A}}/\breve{a}$, where $\breve{a} \in [0, 1]$ and defined as follows

$$\mathcal{F}_{\breve{A}}/\breve{a} =$$

$$\left\{(< u, min\left\{u^{\alpha}_{\breve{A}(e)}(u)/\breve{a}, (1,1,\ldots,1)\right\}, min\left\{v^{\alpha}_{\breve{A}(e)}(u)/\breve{a}, (1,1,\ldots,1)\right\},\ min\left\{w^{\alpha}_{\breve{A}(e)}(u)/\breve{a}, (1,1,\ldots,1)\right\} >: u \in \mathcal{U})\right\}$$

4. Similarity Measures and Their Decision-Making Approaches

Many mathematicians developed various methodologies to solve MCDM problems in the past few years, such as aggregation operators for different hybrid structures, CC, similarity measures, and decision-making applications. Some operational laws and mPNSWA operator with its decision-making approach are established for GmPNSS.

Definition 4.1

Let $\boldsymbol{\mathcal{F}_{\breve{A}}}$, $\boldsymbol{\mathcal{G}_{\breve{B}}}$ are two GmPNSS over the universe of discourse $\boldsymbol{\mathcal{U}} = \{\boldsymbol{u_1}, \boldsymbol{u_2}, \ldots, \boldsymbol{u_j}\}$, then cosine similarity measure between $\boldsymbol{\mathcal{F}_{\breve{A}}}$ and $\boldsymbol{\mathcal{G}_{\breve{B}}}$ defined as

$\mathcal{F}_{\breve{A}}(e) = \left\{e, < u, u^{\alpha}_{\breve{A}(e)}(u), v^{\alpha}_{\breve{A}(e)}(u),\ w^{\alpha}_{\breve{A}(e)}(u) >: u \in \mathcal{U}, e \in E;\ \alpha \in 1,2,3,\ldots,m\right\}$ and $\mathcal{G}_{\breve{B}}(e) = \left\{e, < u, u^{\alpha}_{\breve{B}(e)}(u), v^{\alpha}_{\breve{B}(e)}(u),\ w^{\alpha}_{\breve{B}(e)}(u) >: u \in \mathcal{U}, e \in E;\ \alpha \in 1,2,3,\ldots,m\right\}$

$$\mathcal{S}^1_{GmPNSS}(\mathcal{F}_{\breve{A}}, \mathcal{G}_{\breve{B}}) =$$

$$\frac{1}{mn}\sum_{j=1}^{n}\sum_{\alpha=1}^{m}\frac{\left(\left(u^{\alpha}_{\breve{A}(e)}(u)\right)\left(u^{\alpha}_{\breve{B}(e)}(u)\right)+\left(v^{\alpha}_{\breve{A}(e)}(u)\right)\left(v^{\alpha}_{\breve{B}(e)}(u)\right)+\left(w^{\alpha}_{\breve{A}(e)}(u)\right)\left(w^{\alpha}_{\breve{B}(e)}(u)\right)\right)}{\left(\sqrt{\left(\left(u^{\alpha}_{\breve{A}(e)}(u)\right)^2+\left(v^{\alpha}_{\breve{A}(e)}(u)\right)^2+\left(w^{\alpha}_{\breve{A}(e)}(u)\right)^2\right)}\sqrt{\left(\left(u^{\alpha}_{\breve{B}(e)}(u)\right)^2+\left(v^{\alpha}_{\breve{B}(e)}(u)\right)^2+\left(w^{\alpha}_{\breve{B}(e)}(u)\right)^2\right)}\right)}$$

Proposition 4.2

Let $\mathcal{F}_{\breve{A}}$, $\mathcal{G}_{\breve{B}}$, and $\mathcal{H}_{\breve{C}} \in$ GmPNSS, then the following properties hold

1. $0 \leq \mathcal{S}^1_{GmPNSS}(\mathcal{F}_{\breve{A}}, \mathcal{G}_{\breve{B}}) \leq 1$
2. $\mathcal{S}^1_{GmPNSS}(\mathcal{F}_{\breve{A}}, \mathcal{G}_{\breve{B}}) = \mathcal{S}^1_{GmPNSS}(\mathcal{G}_{\breve{B}}, \mathcal{F}_{\breve{A}})$

3. If $\mathcal{F}_{\breve{A}} \subseteq \mathcal{G}_{\breve{B}} \subseteq \mathcal{H}_{\breve{C}}$, then $\mathcal{S}^1_{GmPNSS}(\mathcal{F}_{\breve{A}}, \mathcal{H}_{\breve{C}}) \leq \mathcal{S}^1_{mPNSS}(\mathcal{F}_{\breve{A}}, \mathcal{G}_{\breve{B}})$ and $\mathcal{S}^1_{mPNSS}(\mathcal{F}_{\breve{A}}, \mathcal{H}_{\breve{C}}) \leq \mathcal{S}^1_{mPNSS}(\mathcal{G}_{\breve{B}}, \mathcal{H}_{\breve{C}})$.

Proof: Using the above definition, the proof of these properties can be done quickly.

Definition 4.3

Let $\mathcal{F}_{\breve{A}}$, $\mathcal{G}_{\breve{B}}$ are two GmPNSS over the universe of discourse $\mathcal{U} = \{u_1, u_2, \ldots, u_j\}$, then set-theoretic similarity measure between $\mathcal{F}_{\breve{A}}$ and $\mathcal{G}_{\breve{B}}$ defined as

$$\mathcal{S}^2_{GmPNSS}(\mathcal{F}_{\breve{A}}, \mathcal{G}_{\breve{B}}) = \frac{1}{mn}\sum_{j=1}^{n}\sum_{\alpha=1}^{m}\frac{\left(\left(u^{\alpha}_{\breve{A}(e)}(u)\right)\left(u^{\alpha}_{\breve{B}(e)}(u)\right)+\left(v^{\alpha}_{\breve{A}(e)}(u)\right)\left(v^{\alpha}_{\breve{B}(e)}(u)\right)+\left(w^{\alpha}_{\breve{A}(e)}(u)\right)\left(w^{\alpha}_{\breve{B}(e)}(u)\right)\right)}{max\left\{\left(\left(u^{\alpha}_{\breve{A}(e)}(u)\right)^2+\left(v^{\alpha}_{\breve{A}(e)}(u)\right)^2+\left(w^{\alpha}_{\breve{A}(e)}(u)\right)^2\right),\left(\left(u^{\alpha}_{\breve{B}(e)}(u)\right)^2+\left(v^{\alpha}_{\breve{B}(e)}(u)\right)^2+\left(w^{\alpha}_{\breve{B}(e)}(u)\right)^2\right)\right\}}$$

Proposition 4.4

Let $\mathcal{F}_{\breve{A}}$, $\mathcal{G}_{\breve{B}}$, and $\mathcal{H}_{\breve{C}} \in$ GmPNSS, then the following properties hold

1. $0 \leq \mathcal{S}^2_{GmPNSS}(\mathcal{F}_{\breve{A}}, \mathcal{G}_{\breve{B}}) \leq 1$
2. $\mathcal{S}^2_{GmPNSS}(\mathcal{F}_{\breve{A}}, \mathcal{G}_{\breve{B}}) = \mathcal{S}^2_{GmPNSS}(\mathcal{G}_{\breve{B}}, \mathcal{F}_{\breve{A}})$
3. If $\mathcal{F}_{\breve{A}} \subseteq \mathcal{G}_{\breve{B}} \subseteq \mathcal{H}_{\breve{C}}$, then $\mathcal{S}^2_{GmPNSS}(\mathcal{F}_{\breve{A}}, \mathcal{H}_{\breve{C}}) \leq \mathcal{S}^2_{mPNSS}(\mathcal{F}_{\breve{A}}, \mathcal{G}_{\breve{B}})$ and $\mathcal{S}^2_{mPNSS}(\mathcal{F}_{\breve{A}}, \mathcal{H}_{\breve{C}}) \leq \mathcal{S}^2_{mPNSS}(\mathcal{G}_{\breve{B}}, \mathcal{H}_{\breve{C}})$.

Proof: Using the above definition, the proof of these properties can be done quickly.

4.5 Algorithm 1 for Similarity Measures of GmPNSS

Step 1. Pick out the set containing parameters.

Step 2. Construct the GmPNSS according to experts.

Step 3. Compute the cosine similarity measure by using definition 4.1.

Step 4. Compute the set-theoretic similarity measure for GmPNSS by utilizing definition 4.3.

Step 5. An alternative with a maximum value with cosine similarity measure has the maximum rank according to considered numerical illustration.

Step 6. An alternative with a maximum value with a set-theoretic similarity measure has the maximum rank according to considered numerical illustration.

Step 7. Analyze the ranking.

A flowchart of the presented algorithm can see in figure 1.

Definition 4.6

Let $\mathcal{F}_{\breve{A}}(e) = \left\{< u^{\alpha}_{\breve{A}(e)}(u), v^{\alpha}_{\breve{A}(e)}(u), w^{\alpha}_{\breve{A}(e)}(u) >\right\}$, $\mathcal{G}_{\breve{B}}(e) = \left\{< u^{\alpha}_{\breve{B}(e)}(u), v^{\alpha}_{\breve{B}(e)}(u), w^{\alpha}_{\breve{B}(e)}(u) >\right\}$, and $\mathcal{H}_{\breve{B}(e)} = \left\{< u^{\alpha}_{\breve{C}(e)}(u), v^{\alpha}_{\breve{C}(e)}(u), w^{\alpha}_{\breve{C}(e)}(u) >\right\}$ are three mPNSNs, the basic operators for mPNSNs are defined as when $\delta > 0$

1. $\mathcal{F}_{\breve{A}}(e) \oplus \mathcal{G}_{\breve{B}}(e) = \left\langle u^{\alpha}_{\breve{A}(e)}(u) + u^{\alpha}_{\breve{B}(e)}(u) - u^{\alpha}_{\breve{A}(e)}(u)u^{\alpha}_{\breve{B}(e)}(u), v^{\alpha}_{\breve{A}(e)}(u) * v^{\alpha}_{\breve{B}(e)}(u), w^{\alpha}_{\breve{A}(e)}(u) * w^{\alpha}_{\breve{B}(e)}(u)\right\rangle$
2. $\mathcal{F}_{\breve{A}}(e) \otimes \mathcal{G}_{\breve{B}}(e) = \left\langle u^{\alpha}_{\breve{A}(e)}(u) * u^{\alpha}_{\breve{B}(e)}(u), v^{\alpha}_{\breve{A}(e)}(u) + v^{\alpha}_{\breve{B}(e)}(u) - v^{\alpha}_{\breve{A}(e)}(u)v^{\alpha}_{\breve{B}(e)}(u), w^{\alpha}_{\breve{A}(e)}(u) + w^{\alpha}_{\breve{B}(e)}(u) - w^{\alpha}_{\breve{A}(e)}(u)w^{\alpha}_{\breve{B}(e)}(u)\right\rangle$
3. $\delta\mathcal{F}_{\breve{A}}(e) = \left\langle 1 - \left(1 - u^{\alpha}_{\breve{A}(e)}(u)\right)^{\delta}, \left(v^{\alpha}_{\breve{A}(e)}(u)\right)^{\delta}, \left(w^{\alpha}_{\breve{A}(e)}(u)\right)^{\delta}\right\rangle$

4. $\left(\mathcal{F}_{\breve{A}}(e)\right)^{\delta}=\left\langle\left(u_{\breve{A}(e)}^{\alpha}(u)\right)^{\delta}, 1-\left(1-v_{\breve{A}(e)}^{\alpha}(u)\right)^{\delta}, 1-\left(1-w_{\breve{A}(e)}^{\alpha}(u)\right)^{\delta}\right\rangle$.

Proposition 4.7

Let $\mathcal{F}_{\breve{A}}$, $\mathcal{G}_{\breve{B}}$, and $\mathcal{H}_{\breve{C}}$ $\in$ mPNSNs and $\delta,\ \delta_1,\ \delta_2 > 0$, then the following laws hold

1. $\mathcal{F}_{\breve{A}} \oplus \mathcal{G}_{\breve{B}} = \mathcal{G}_{\breve{B}} \oplus \mathcal{F}_{\breve{A}}$
2. $\mathcal{F}_{\breve{A}} \otimes \mathcal{G}_{\breve{B}} = \mathcal{G}_{\breve{B}} \otimes \mathcal{F}_{\breve{A}}$
3. $\delta(\mathcal{F}_{\breve{A}} \oplus \mathcal{G}_{\breve{B}})= \delta\mathcal{G}_{\breve{B}} \oplus \delta\mathcal{F}_{\breve{A}}$
4. $\left(\mathcal{F}_{\breve{A}} \otimes \mathcal{G}_{\breve{B}}\right)^{\delta} = \left(\mathcal{F}_{\breve{A}}\right)^{\delta} \otimes \left(\mathcal{G}_{\breve{B}}\right)^{\delta}$
5. $\delta_1\mathcal{F}_{\breve{A}} \oplus \delta_2\mathcal{F}_{\breve{A}}= (\delta_1 \oplus \delta_2)\mathcal{F}_{\breve{A}}$
6. $\left(\mathcal{F}_{\breve{A}}\right)^{\delta_1} \otimes \left(\mathcal{F}_{\breve{A}}\right)^{\delta_2} = \left(\mathcal{F}_{\breve{A}}\right)^{\delta_1+\delta_2}$
7. $\left(\mathcal{F}_{\breve{A}} \oplus \mathcal{G}_{\breve{B}}\right) \oplus \mathcal{H}_{\breve{C}} = \mathcal{F}_{\breve{A}} \oplus \left(\mathcal{G}_{\breve{B}} \oplus \mathcal{H}_{\breve{C}}\right)$
8. $\left(\mathcal{F}_{\breve{A}} \otimes \mathcal{G}_{\breve{B}}\right) \otimes \mathcal{H}_{\breve{C}} = \mathcal{F}_{\breve{A}} \otimes \left(\mathcal{G}_{\breve{B}} \otimes \mathcal{H}_{\breve{C}}\right)$

Proof. The proof of the above laws is straightforward by using definition 4.6.

Definition 4.8

Let $\mathcal{F}_{\breve{A}}(e_{ij}) = \left\langle\left\{u_{\breve{A}(e_{ij})}^{\alpha}(u), v_{\breve{A}(e_{ij})}^{\alpha}(u),\ w_{\breve{A}(e_{ij})}^{\alpha}(u)\right\}\right\rangle$ be a collection of mPNSNs, Ω_i and γ_j are weight vector for expert's and parameters respectively with given conditions $\Omega_i > 0$, $\sum_{i=1}^{n}\Omega_i = 1$, $\gamma_j > 0$, $\sum_{j=1}^{m}\gamma_j = 1$, where $(i = 1,2,\dots,n, and\ j = 1,2,\dots,m)$. Then mPIVNSWA operator defined as mPNSWA: $\Delta^n \rightarrow \Delta$ defined as follows

$mPNSWA\left(\mathcal{F}_{\breve{A}}(e_{11}), \mathcal{F}_{\breve{A}}(e_{12}),\dots,\mathcal{F}_{\breve{A}}(e_{nk})\right) = \oplus_{j=1}^{k}\gamma_j\left(\oplus_{i=1}^{n}\Omega_i\mathcal{F}_{\breve{A}}(e_{ij})\right)$.

Proposition 4.9

Let $\mathcal{F}_{\breve{A}}(e_{ij}) = \left\langle\left\{u_{\breve{A}(e_{ij})}^{\alpha}(u), v_{\breve{A}(e_{ij})}^{\alpha}(u),\ w_{\breve{A}(e_{ij})}^{\alpha}(u)\right\}\right\rangle$ be a collection of mPNSNs, where $(i = 1,2,\dots,n, and\ j = 1,2,\dots,k)$, the aggregated value is also an mPNSNs, such as

$mPNSWA\left(\mathcal{F}_{\breve{A}}(e_{11}), \mathcal{F}_{\breve{A}}(e_{12}),\dots,\mathcal{F}_{\breve{A}}(e_{nk})\right)$

$$= \left\langle 1-\prod_{j=}^{m}\left(\prod_{i=1}^{n}\left(1-u_{\breve{A}(e_{ij})}^{\alpha}(u)\right)^{\Omega_i}\right)^{\gamma_j}, 1-\left(1-\prod_{j=1}^{m}\left(\prod_{i=1}^{n}\left(1-v_{\breve{A}(e_{ij})}^{\alpha}(u)\right)^{\Omega_i}\right)^{\gamma_j}\right),\ 1-\left(1-\prod_{j=1}^{m}\left(\prod_{i=1}^{n}\left(1-w_{\breve{A}(e_{ij})}^{\alpha}(u)\right)^{\Omega_i}\right)^{\gamma_j}\right)\right\rangle$$

Proof. We can prove easily by using IFSWA [32].

Definition 4.10

Let $\mathcal{F}_{\breve{A}}(e) = \left\{< u_{\breve{A}(e)}^{\alpha}(u), v_{\breve{A}(e)}^{\alpha}(u),\ w_{\breve{A}(e)}^{\alpha}(u) >\right\}$ be an mPNSN, then the score, accuracy, and certainty functions for GmPNSN respectively defined as follows

$$\mathbb{S}(\mathcal{F}_{\breve{A}}) = \frac{1}{6m}\sum_{\alpha=1}^{m}\left(6+u_{\breve{A}(e)}^{\alpha}(u)-v_{\breve{A}(e)}^{\alpha}(u)-w_{\breve{A}(e)}^{\alpha}(u)\right)$$

$$\mathbb{A}(\mathcal{F}_{\breve{A}}) = \frac{1}{4m}\left(4+u_{\breve{A}(e)}^{\alpha}(u)-w_{\breve{A}(e)}^{\alpha}(u)\right)$$

$$\mathbb{C}(\mathcal{F}_{\breve{A}}) = \frac{1}{2m}\left(2+u_{\breve{A}(e)}^{\alpha}(u)\right)$$

where $\alpha = 1, 2,\cdots,\ m$.

Definition 4.11

Let $\mathcal{F}_{\breve{A}}$, $\mathcal{G}_{\breve{B}} \in$ mPIVNSS, then comparison approach is present as follows

1. If $\mathbb{S}(\mathcal{F}_{\breve{A}}) > \mathbb{S}(\mathcal{G}_{\breve{B}})$, then $\mathcal{F}_{\breve{A}}$ is superior to $\mathcal{G}_{\breve{B}}$.
2. If $\mathbb{S}(\mathcal{F}_{\breve{A}}) = \mathbb{S}(\mathcal{G}_{\breve{B}})$ and $\mathbb{A}(\mathcal{F}_{\breve{A}}) > \mathbb{A}(\mathcal{G}_{\breve{B}})$, then $\mathcal{F}_{\breve{A}}$ is superior to $\mathcal{G}_{\breve{B}}$.
3. If $\mathbb{S}(\mathcal{F}_{\breve{A}}) = \mathbb{S}(\mathcal{G}_{\breve{B}})$, $\mathbb{A}(\mathcal{F}_{\breve{A}}) = \mathbb{A}(\mathcal{G}_{\breve{B}})$, and $\mathbb{C}(\mathcal{F}_{\breve{A}}) > \mathbb{C}(\wp_{\Re_1})$, then $\mathcal{F}_{\breve{A}}$ is superior to $\mathcal{G}_{\breve{B}}$.
4. If $\mathbb{S}(\mathcal{F}_{\breve{A}}) = \mathbb{S}(\mathcal{G}_{\breve{B}})$, $\mathbb{A}(\mathcal{F}_{\breve{A}}) > \mathbb{A}(\mathcal{G}_{\breve{B}})$, and $\mathbb{C}(\mathcal{F}_{\breve{A}}) = \mathbb{C}(\wp_{\Re_1})$, then $\mathcal{F}_{\breve{A}}$ is indifferent to $\mathcal{G}_{\breve{B}}$, can be denoted as $\mathcal{F}_{\breve{A}} \sim \mathcal{G}_{\breve{B}}$.

4.2 Decision-making approach based mPNSWA for GmPNSS

Assume a set of "s" alternatives such as $\beta = \{\beta^1, \beta^2, \beta^3, \dots, \beta^s\}$ for assessment under the team of experts such as $\mathcal{U} = \{u_1, u_2, u_3, \dots, u_n\}$ with weights $\Omega = (\Omega_1, \Omega_1, \dots, \Omega_n)^T$ such that $\Omega_i > 0$, $\sum_{i=1}^{n} \Omega_i = 1$. Let $\mathcal{E} = \{e_1, e_2, \dots, e_m\}$ be a set of attributes with weights $\gamma = (\gamma_1, \gamma_2, \gamma_3, \dots, \gamma_m)^T$ be a weight vector for parameters such as $\gamma_j > 0$, $\sum_{j=1}^{m} \gamma_j = 1$. The team of experts $\{u_i$: $i = 1, 2, \dots, n\}$ evaluate the alternatives $\{\beta^{(z)}$: $z = 1, 2, \dots, s\}$ under the considered parameters $\{e_j$: $j = 1, 2, \dots, m\}$ given in the form of mPIVNSNs $\mathcal{L}_{ij}^{(z)} = \left(u_{\alpha_{ij}}^{(z)}, v_{\alpha_{ij}}^{(z)}, w_{\alpha_{ij}}^{(z)}\right)$, where $0 \leq u_{\alpha_{ij}}^{(z)}, v_{\alpha_{ij}}^{(z)}, w_{\alpha_{ij}}^{(z)} \leq 1$ and $0 \leq u_{\alpha_{ij}}^{(z)}, + v_{\alpha_{ij}}^{(z)} + w_{\alpha_{ij}}^{(z)} \leq 3$. So $\Delta_k = \left(u_{\alpha_{ij}}^{(z)}, v_{\alpha_{ij}}^{(z)}, w_{\alpha_{ij}}^{(z)}\right)$ for all i, j. Experts give their preferences for each alternative in terms of mPNSNs by using the mPNSWA operator in the form of $\Delta_k = \left(u_{\alpha_{ij}}^{(z)}, v_{\alpha_{ij}}^{(z)}, w_{\alpha_{ij}}^{(z)}\right)$. Compute the score values for each alternative and analyze the ranking of the alternatives, the algorithm of the proposed approach is presented in Figure: 1.

4.2.1 Algorithm 2 for mPNSWA Operator

Step 1. Develop the m-polar neutrosophic soft matrix for each alternative.

Step 2. Aggregate the mPNSNs for each alternative into a collective decision matrix Δ_k by using the mPNSWA operator.

Step 3. Compute the score value for each alternative Δ_k by using equation 14, where $k = 1, 2, \cdots, s$.

Step 4. Rank the alternatives $\beta^{(k)}$ and choose the best alternative.

Step 5. End.

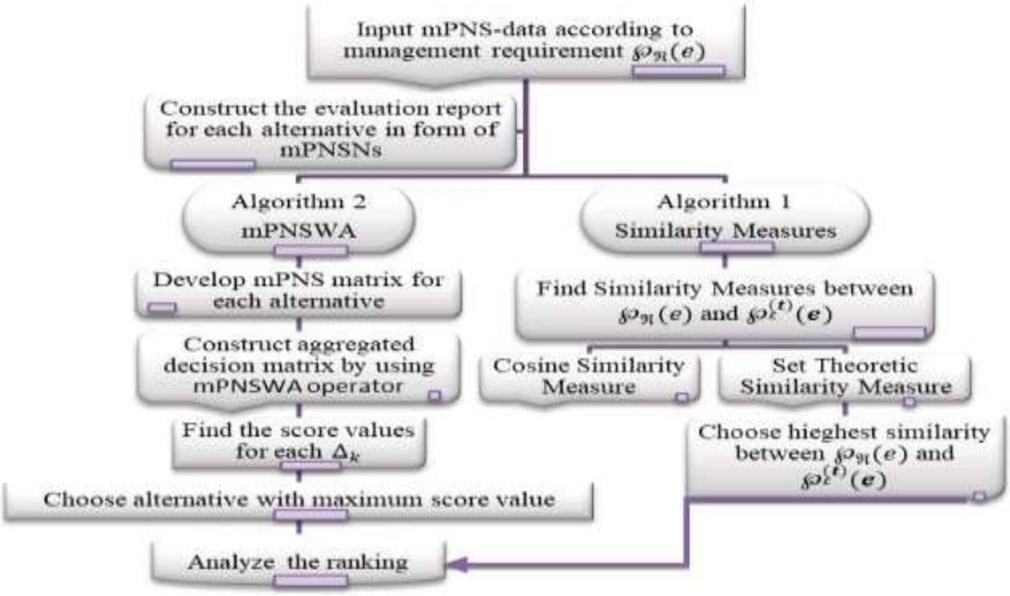


Figure 1: Flowchart of Proposed Algorithm 1 and Algorithm 2

5. Application of Similarity Measures and mPNSWA Operator in Decision Making

In this section, we proposed the algorithm for GmPNSS by using developed similarity measures and the mPNSWA operator. We also used the proposed methods for decision-making in real-life problems.

5.1. Problem Formulation and Application of GmPNSS For Decision Making

A construction company calls for the appointment of a civil engineer to supervise the workers. Several engineers apply for the civil engineer post, simply four engineers call for an interview based on experience for undervaluation such as S = $\{S_1,\ S_2,\ S_3,\ S_4\}$ be a set of selected engineers call for the interview. The managing director of the hires a committee of four experts X = $\{X_1,\ X_2,\ X_3,\ X_4\}$ for the selection of civil engineer. First of all, the committee decides the set of parameters such as E = $\{x_1,\ x_2,\ x_3\}$, where x_1, x_2, and x_3 represents the personality, communication skills, and qualifications for the selection of civil engineer. The experts evaluate the applicants under defined parameters and forward the evaluation performa to the company's managing director. Finally, the director scrutinizes the best applicant based on the expert's evaluation report.

5.1.1. Application of GmPNSS For Decision Making

Assume S = $\{S_1,\ S_2,\ S_3,\ S_4\}$ be a set of civil engineers who are shortlisted for interview and E = $\{x_1$ = personality, x_2 = communication skills, x_3 = qualification$\}$ be a set of parameters for the selection of civil engineer. Let $\mathcal{F}$ and $\mathcal{G} \subseteq E$; then we construct the G3-PNSS $\Phi_{\mathcal{F}}(x)$ according to the requirement of the construction company such as follows

Table 3. Construction of G3-PNSS of all Applicants According to Company Requirement

$\boldsymbol{\Phi_{\mathcal{F}}(x)}$	$\boldsymbol{x_1}$	$\boldsymbol{x_2}$	$\boldsymbol{x_3}$
$\mathbf{X_1}$	(.8,.5,.6),(.5,.4,.2),(.4,.3,.6)	(.4,.8,.6),(.7,.6,.5),(.4,.1,.3)	(.7,.8,.5),(.8,.4,.7),(.6,.5,.2)
$\mathbf{X_2}$	(.5,.6,.5),(.9,.5,.8),(.6,.4,.5)	(.7,.5,.8),(.7,.5,.7),(.3,.5,.9)	(.6,.4,.9),(.2,.5,.2),(.9,.4,.6)
$\mathbf{X_3}$	(.2,.5,.4),(.7,.3,.2),(.6,.4,.5)	(.3,.5,.7),(.4,.6,.2),(.6,.7,.9)	(.5,.2,.4),(.7,.5,.9),(.6,.3,.4)

X_4	(.9,.5,.1),(.3,.4,.6),(.6,.5,.2)	(.9,.5,.6),(.3,.4,.3),(.6,.3,.9)	(.9,.5,.7),(.7,.4,.3),(.4,.7,.6)

Now we will construct the G3-PNSS $\varphi_{\mathcal{G}}^{t}$ according to four experts, where t = 1, 2, 3, 4.

Table 4. G3-PNSS Evaluation Report According to Experts of S_1

$\varphi_{\mathcal{G}}^{1}$	x_1	x_2	x_3
X_1	(.3,.5,.2),(.8,.7,.3),(.7,.2,.9)	(.9,.5,.1),(.3,.4,.6),(.1,.5,.2)	(.9,.5,.1),(.7,.4,.3),(.6,.7,.2)
X_2	(.7,.8,.3),(.6,.1,.2),(.2,.4,.6)	(.9,.5,.6),(.7,.2,.3),(.4,.7,.6)	(.7,.2,.4),(.3,.9,.7),(.5,.9,.1)
X_3	(.7,.3,.2),(.2,.1,.2),(.7,.9,.8)	(.7,.2,.1),(.7,.4,.5),(.1,.7,.9)	(.7,.8,.6),(.7,.2,.5),(.7,.3,.2)
X_4	(.3,.2,.7),(.5,.6,.2),(.4,.6,.8)	(.7,.2,.6),(.7,.4,.9),(.8,.6,.9)	(.2,.9,.6),(.7,.4,.2),(.7,.7,.9)

Table 5. G3-PNSS Evaluation Report According to Experts of S_2

$\varphi_{\mathcal{G}}^{2}$	x_1	x_2	x_3
X_1	(.6,.2,.7),(.8,.7,.9),(.7,.5,.6)	(.1,.5,.6),(.3,.4,.6),(.6,.5,.2)	(.9,.5,.1),(.7,.4,.2),(.6,.3,.9)
X_2	(.1,.2,.4),(.1,.2,.2),(.7,.4,.9)	(.3,.5,.7),(.4,.2,.3),(.4,.7,.6)	(.7,.2,.4),(.3,.9,.7),(.3,.5,.1)
X_3	(.2,.6,.7),(.2,.7,.6),(.4,.5,.2)	(.7,.2,.1),(.6,.3,.5),(.1,.7,.4)	(.7,.5,.6),(.7,.2,.5),(.7,.3,.9)
X_4	(.8,.1,.9),(.4,.2,.6),(.2,.7,.1)	(.4,.2,.6),(.7,.4,.3),(.5,.7,.9)	(.2,.9,.1),(.1,.4,.2),(.4,.7,.9)

Table 6. G3-PNSS Evaluation Report According to Experts of S_3

$\varphi_{\mathcal{G}}^{3}$	x_1	x_2	x_3
X_1	(.7,.4,.1),(.7,.3,.1),(.7,.4,.6)	(.4,.9,.6),(.7,.2,.5),(.7,.3,.2)	(.7,.4,.6),(.9,.4,.3),(.1,.4,.5)
X_2	(.6,.2,.3),(.7,.4,.3),(.6,.2,.5)	(.6,.2,.1),(.5,.4,.7),(.3,.5,.1)	(.6,.2,.7),(.5,.4,.3),(.6,.4,.7)
X_3	(.6,.2,.1),(.6,.3,.5),(.4,.7,.9)	(.2,.7,.4),(.3,.6,.2),(.5,.3,.9)	(.4,.2,.6),(.7,.4,.3),(.5,.4,.9)
X_4	(.4,.2,.3),(.4,.1,.3),(.4,.5,.2)	(.1,.6,.5),(.3,.2,.6),(.1,.5,.2)	(.6,.1,.4),(.3,.7,.4),(.4,.3,.2)

Table 7. G3-PNSS Evaluation Report According to Experts of S_4

$\varphi_{\mathcal{G}}^{4}$	x_1	x_2	x_3
X_1	(.2,.1,.2),(.3,.5,.4),(.9,.2,.7)	(.4,.8,.6),(.4,.7,.5),(.4,.5,.3)	(.2,.5,.6),(.5,.6,.2),(.4,.8,.6)
X_2	(.1,.3,.1),(.9,.4,.6),(.3,.3,.8)	(.7,.2,.6),(.7,.4,.2),(.4,.7,.9)	(.5,.6,.5),(.3,.5,.8),(.6,.4,.5)
X_3	(.7,.2,.1),(.6,.3,.5),(.4,.5,.9)	(.7,.2,.1),(.6,.3,.5),(.1,.7,.4)	(.3,.5,.7),(.4,.5,.2),(.6,.3,.9)
X_4	(.4,.1,.7),(.9,.6,.2),(.4,.8,.1)	(.6,.1,.7),(.2,.4,.7),(.4,.5,.2)	(.2,.6,.4),(.3,.1,.6),(.4,.3,.2)

5.1.2 Solution by using Algorithm 1

By using Tables 3-7, compute the cosine similarity measure between $\mathcal{S}_{GmPNSS}^{1}(\Phi_{\mathcal{F}}(x),\varphi_{\mathcal{G}}^{1}(x))$, $\mathcal{S}_{GmPNSS}^{1}(\Phi_{\mathcal{F}}(x),\varphi_{\mathcal{G}}^{2}(x))$, $\mathcal{S}_{GmPNSS}^{1}(\Phi_{\mathcal{F}}(x),\varphi_{\mathcal{G}}^{3}(x))$, and $\mathcal{S}_{GmPNSS}^{1}(\Phi_{\mathcal{F}}(x),\varphi_{\mathcal{G}}^{1}(x))$ by using equation 4.1, such as

$\mathcal{S}^{1}_{GmPNSS}(\Phi_{\mathcal{F}}(x), \varphi^{1}_{\mathcal{G}}(x)) = \frac{1}{3\times4}\left\{\frac{(.8)(.3)+(.5)(.5)+(.6)(.2)}{\sqrt{(.8)^2+(.5)^2+(.6)^2}\sqrt{(.3)^2+(.5)^2+(.2)^2}} + \frac{(.5)(.8)+(.4)(.7)+(.2)(.3)}{\sqrt{(.5)^2+(.4)^2+(.2)^2}\sqrt{(.8)^2+(.7)^2+(.3)^2}} + \cdots + \frac{(.4)(.7)+(.7)(.7)+(.6)(.9)}{\sqrt{(.4)^2+(.7)^2+(.6)^2}\sqrt{(.7)^2+(.7)^2+(.9)^2}}\right\} = \frac{1}{12}\left(\frac{28.99}{34.4799}\right) = 0.07007.$

Similarly, we can find the cosine similarity measure between $\mathcal{S}^{1}_{GmPNSS}(\Phi_{\mathcal{F}}(x), \varphi^{2}_{\mathcal{G}}(x))$, $\mathcal{S}^{1}_{GmPNSS}(\Phi_{\mathcal{F}}(x),\varphi^{3}_{\mathcal{G}}(x))$, and $\mathcal{S}^{1}_{GmPNSS}(\Phi_{\mathcal{F}}(x),\varphi^{4}_{\mathcal{G}}(x))$ given as $\mathcal{S}^{1}_{GmPNSS}(\Phi_{\mathcal{F}}(x),\varphi^{2}_{\mathcal{G}}(x)) = \frac{1}{12}\left(\frac{26.32}{32.3767}\right) = 0.06771$, $\mathcal{S}^{1}_{GmPNSS}(\Phi_{\mathcal{F}}(x),\varphi^{3}_{\mathcal{G}}(x)) = \frac{1}{12}\left(\frac{25.4}{29.4056}\right) = 0.06943$, and $\mathcal{S}^{1}_{GmPNSS}(\Phi_{\mathcal{F}}(x), \varphi^{4}_{\mathcal{G}}(x)) = \frac{1}{12}\left(\frac{25.48}{30.88764}\right) = 0.06874$. This shows that $\mathcal{S}^{1}_{GmPNSS}(\Phi_{\mathcal{F}}(x), \varphi^{1}_{\mathcal{G}}(x)) > \mathcal{S}^{1}_{GmPNSS}(\Phi_{\mathcal{F}}(x), \varphi^{3}_{\mathcal{G}}(x)) > \mathcal{S}^{1}_{GmPNSS}(\Phi_{\mathcal{F}}(x), \varphi^{4}_{\mathcal{G}}(x)) > \mathcal{S}^{1}_{GmPNSS}(\Phi_{\mathcal{F}}(x),\varphi^{2}_{\mathcal{G}}(x))$. It can be seen from this ranking alternative $\beta^{(1)}$ is most relevant and similar to $\Phi_{\mathcal{F}}(x)$. Therefore $\beta^{(1)}$ is the best alternative for the vacant position of associate professor, the ranking of other alternatives given as $\beta^{(1)} > \beta^{(3)} > \beta^{(4)} > \beta^{(2)}$.

Now we compute the set-theoretic similarity measure by using Definition 4.3 between $\mathcal{S}^{2}_{GmPNSS}(\Phi_{\mathcal{F}}(x),\varphi^{1}_{\mathcal{G}}(x))$, $\mathcal{S}^{2}_{GmPNSS}(\Phi_{\mathcal{F}}(x),\varphi^{2}_{\mathcal{G}}(x))$, $\mathcal{S}^{2}_{GmPNSS}(\Phi_{\mathcal{F}}(x),\varphi^{3}_{\mathcal{G}}(x))$, and $\mathcal{S}^{4}_{GmPNSS}(\Phi_{\mathcal{F}}(x),\varphi^{1}_{\mathcal{G}}(x))$ From Tables 1-5, we can find the set-theoretic similarity measure for each alternative by using definition 4.3 given as $\mathcal{S}^{2}_{GmPNSS}(\Phi_{\mathcal{F}}(x), \varphi^{1}_{\mathcal{G}}(x)) = 0.06986$, $\mathcal{S}^{2}_{GmPNSS}(\Phi_{\mathcal{F}}(x), \varphi^{2}_{\mathcal{G}}(x)) = 0.06379$, $\mathcal{S}^{2}_{GmPNSS}(\Phi_{\mathcal{F}}(x), \varphi^{3}_{\mathcal{G}}(x)) = 0.06157$, and $\mathcal{S}^{2}_{GmPNSS}(\Phi_{\mathcal{F}}(x), \varphi^{4}_{\mathcal{G}}(x)) = 0.06176$. This shows that $\mathcal{S}^{2}_{mPNSS}(\wp_{\Re}(e), \wp^{(1)}_{\mathcal{L}}(e)) > \mathcal{S}^{2}_{GmPNSS}(\Phi_{\mathcal{F}}(x), \varphi^{2}_{\mathcal{G}}(x)) > \mathcal{S}^{2}_{GmPNSS}(\Phi_{\mathcal{F}}(x), \varphi^{4}_{\mathcal{G}}(x)) > \mathcal{S}^{2}_{GmPNSS}(\Phi_{\mathcal{F}}(x), \varphi^{3}_{\mathcal{G}}(x))$. Therefore $\beta^{(1)}$ is the best alternative for the vacant position of associate professor by using set-theoretic similarity measure, the ranking of other alternatives given as $\beta^{(1)} > \beta^{(2)} > \beta^{(4)} > \beta^{(3)}$. Graphically representation of results can be seen in Fig. 2.

5.1.3 Solution by using Algorithm 2

Step 1. The experts will evaluate the condition in the case of mPNSNs, and there are just four alternatives; parameters and a summary of their scores given in Tables 4, 5, 6, and 7.

Step 2. Experts' opinions on each alternative are summarized by using proposition 4.9. Therefore, we have

$\Delta_1 = \langle(.3144, .5379, .4259), (.1819, .3711, .4126), (.2129, .3421, .1328)\rangle$,

$\Delta_2 = \langle(.1815, .5420, .3844), (.3546, .5937, .2725), (.4526, .5031, .3725)\rangle$,

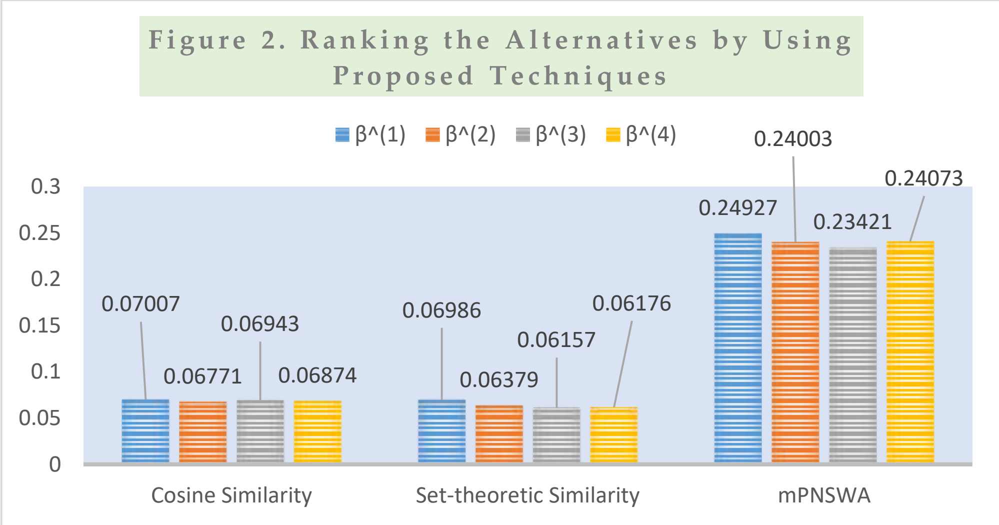
$\Delta_3 = \langle(.2904, .4223, .3755), (.3761, .5547, .4136), (.2516, .4732, .4631)\rangle$, and

$\Delta_4 = \langle(.2713, .5445, .1756), (.3530, .5201, .5641), (.4547, .4153, .5263)\rangle$.

Step 3. Compute the Score values by using definition 4.10.

$\mathbb{S}(\Delta_1) = .24927$, $\mathbb{S}(\Delta_2) = .24003$, $\mathbb{S}(\Delta_3) = .23421$, and $\mathbb{S}(\Delta_4) = .24073$

Step 4. Therefore, the ranking of the alternatives is as follows $\mathbb{S}(\Delta_1) > \mathbb{S}(\Delta_4) > \mathbb{S}(\Delta_2) > \mathbb{S}(\Delta_3)$. So, $\beta^{(1)} > \beta^{(4)} > \beta^{(2)} > \beta^{(3)}$, hence, the alternative $\beta^{(1)}$ is the most suitable alternative for the company.



6. Discussion and Comparative Analysis

The following section will discuss the effectiveness, naivety, flexibility, and advantages of the proposed methods and algorithms. We also conducted a brief comparative analysis of suggested strategies and existing methods.

6.1 Advantages, flexibility, and Superiority of Proposed Approach

The recommended technique is practical and applicable to all forms of input data. We introduce two novel algorithms based on GmPNSS, and one is similarity measures, the other is mPNSWA. This manuscript has established two different types of similarity measures, such as cosine and set-theoretic similarity measures. Both algorithms are practical and can provide the best results in MCDM problems. The recommended algorithms are simple and easy to understand, can deepen their understanding, and apply to many choices and metrics. All algorithms are flexible and easy to change to adapt to different situations, inputs, and outputs. There are subtle differences between the rankings of the suggested methods because different techniques have different ranking methods, so that they can be affordable according to their considerations.

6.2. Results and Discussion

Through this research and comparative analysis, we have concluded that the results obtained by the proposed method are more general than the existing methods. However, in the decision-making process, compared with the current decision-making methods, it contains more information to deal with the uncertainty in the data. Moreover, the hybrid structure of many FSs becomes a particular case of mPNSS, add some suitable conditions. Among them, the information related to the object can be expressed more accurately and empirically, so it is a convenient tool for combining inaccurate and uncertain information in the decision-making process. Therefore, our proposed method is effective, flexible, simple, and superior to other hybrid structures of fuzzy sets.

Table 8: Comparative analysis between some existing techniques and the proposed approach

	Set	Truthiness	Indeterminacy	Falsity	Multi-polarity	Loss of information
Chen et al. [38]	mPFS	✓	×	×	✓	×

Xu et al. [36]	IFS	✓	×	✓	×	×
Zhang et al. [40]	IFS	✓	×	✓	×	✓
Yager [43, 44]	PFS	✓	×	✓	×	×
Naeem et al. [37]	mPFS	✓	×	✓	✓	×
Zhang et al. [31]	INSs	✓	✓	✓	×	×
Ali et al. [39]	BPNSS	✓	✓	✓	×	×
Saeed et al. [45]	mPIVNS	✓	✓	✓	✓	✓
Saqlain et al. [32]	mPNSS	✓	✓	✓	✓	×
Proposed approach	GmPNSS	✓	✓	✓	✓	×

It turns out that this is a contemporary issue. Why do we have to embody novel algorithms based on the proposed novel structure? Many indications compared with other existing methods; the recommended method may be an exception. We remember the following fact: the mixed form limits IFS, picture fuzzy sets, FS, fuzzy hesitation sets, NS, and other fuzzy sets and cannot provide complete information about the situation. But our m-polar model GmPNSS can deal with truthiness, indeterminacy, and falsity, so it is most suitable for MCDM. Due to the exaggerated multipolar neutrosophy, these three degrees are independent and provide a lot of information about alternative norms. Other similarity measures of available hybrid structures are converted into exceptional cases of GmPNSS. A comparative analysis of some already existing techniques is listed in Table 8. Therefore, this model has more versatility and can efficiently resolve complications than intuitionistic, neutrosophic, hesitant, image, and ambiguity substitution. The similarity measure established for GmPNSS becomes better than the existing similarity measure for MCDM.

7. Conclusion

This paper studies some basic concepts such as soft set, NSS, mPNSS, and GmPNSS. We discussed various operations with their properties and numerical examples for GmPNSS. We developed the idea of cosine similarity measure and set-theoretic similarity measure for GmPNSS with some properties in this research. We also presented the introduced multipolar neutrosophic weighted average operator for GmPNSS and established some operational laws for GmPNSS. The concept of score function, accuracy function, and certainty function is developed to compare m-polar neutrosophic numbers. Furthermore, decision-making approaches have been developed for GmPNSS based on proposed techniques. To verify the effectiveness of our developed techniques, we presented an illustration to solve MCDM problems. We gave a comprehensive comparative analysis of proposed techniques with existing methods. In the future, the concept of mPNSS will be extended to interval-valued mPNSS. It will solve real-life problems such as medical diagnoses, decision-making, etc.

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