Neutrosophic Stable Random Variables

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Abstract: In this paper, the concept of a neutrosophic stable random variable is introduced. Two definitions of a neutrosophic random variable are presented. We introduced both the neutrosophic probability distribution function and the neutrosophic probability density function, and the convolution with the neutrosophic concept. In addition, we proved some properties of a neutrosophic stable random variable, and three examples are discussed.

Keywords: Random Variables; Stable Distributions; Gaussian Distribution; Cauchy Distribution; Lévy Distribution.

1. Introduction

The term stability in probability theory refers to a property of some probability distributions, which is that the random variable indicative of a sum of independent and identically distributed random variables has the same probability distribution for each of these variables. This property is true for a finite or infinite sum of random variables. Variables that achieve this specificity are called stable random variables. Stability in this concept is called classical stability, and stable distributions represent a large part of the family of all probability distributions. Regarding the tail of the distribution, all stable distributions are heavy-tailed except for the normal distribution, which is light-tailed.

In 1925, Paul Lévy [1] presented stable distributions as a generalization of the normal distribution in several ways. The theory of stable distributions was developed in the messages exchanged between Lévy (1937) [2] and Khintchine (1938) [3], and work on these results was expanded by Gnedenko and Kolmogorov (1949) [4] and then Feller (1970) [5]. Paul Lévy defined a stable distribution by defining its characteristic function and used a Lévy- Khintchine representation for the infinitely divisible distributions. The second definition is the definition related to the stability property of independent and identically distributed random variables, and the third is the generalized central limit theorem, in which the stable distributions appear as the end of a set of independent and identically distributed random variables without imposing the condition contained in the central limit theorem [4], which revolves around the limitation of variance. A recent and condensed overview of the theory of stable distributions can be found in [6–12].

Fuzzy logic can be generalized to Neutrosophic logic by adding the component of indeterminacy.

In probability theory, F. Smarandache defined the neutrosophic probability measure and the probability function. Some researchers introduced many other concepts through the neutrosophic concept such as queuing theory, time series prediction, and modeling in many cases such as linear models, moving averages, and logarithmic models, more information can be founded at [13–23].

In this paper, depending on the geometric isometry (AH-Isometry) [20] (Under publication in Neutrosophic Sets and Systems), the concept of a stable neutrosophic random variable is introduced and provides two definitions of a neutrosophic stable random variable. We also presented some basic properties and present several well-known examples.

2. Preliminaries

2.1. α -Stable distributions

Definition 2.1.1. A random variable X (which is non-degenerate) is said to have a stable distribution if for any positive numbers A and B, there is a positive number C and a real number D such that

$$AX_1 + BX_2 \overset{d}{=} CX + D,$$

where X_1, X_2 are independent copies of X, and where $\overset{d}{=}$ denotes equality in distribution. That X is called strictly stable if the relation $AX_1 + BX_2 \overset{d}{=} CX + D$ hold with $D = 0$.

Dedinition 2.1.2. (*equivalent to definition 2.1*). A random variable X (which is non-degenerate) is said to have a stable distribution if for any $n \geq 0$, there is a positive number C_n and a real number D_n such that

$$X_1 + X_2 + ... + X_n \overset{d}{=} C_n X + D_n,$$

where $X_1, X_2, ..., X_n$ are independent copies of X.

And X is called strictly stable if $X_1 + X_2 + ... + X_n \overset{d}{=} C_n X + D_n$ hold with $D_n = 0$.

Theorem 2.1.3. If $X_1 + X_2 + ... + X_n \overset{d}{=} C_n X$, C_n has the form $C_n = n^{1/\alpha}$. See [5,9] for a proof.

Theorem 2.1.4. If G is strictly stable with characteristic parameter α, then

$$A^{1/\alpha} X_1 + B^{1/\alpha} X_2 \overset{d}{=} \left(A + B\right)^{1/\alpha} X,$$

holds for all $A > 0, B > 0$. See [5] for a proof.

2.2. Neutrosophic Functions on $R(I)$

Depending on information in [20], here are some interesting facts :

Definition 2.2.1.

Let $R(I) = \{a + bI;\ a, b \in R\}$ where $I^2 = I$ be the neutrosophic field of reals. The one-dimensional isometry (AH-Isometry) is defined as follows: [19]

$$T : R(I) \rightarrow R \times R$$
$$T(a + bI) = (a, a + b).$$

Some properties of an algebraic isomorphism T :

1. T is bijective.
2. T is invertible by

$$T^{-1} : R \times R \to R(I)$$

$$T^{-1}(a,b) = a + (b-a)I$$

3.

$$T\left[\left(a+bI\right)+\left(c+dI\right)\right] = T\left(a+bI\right)+T\left(c+dI\right)$$

And

$$T\left[\left(a+bI\right).\left(c+dI\right)\right] = T\left(a+bI\right).T\left(c+dI\right).$$

And more can be found in [20].

3. Neutrosophic Stable Random Variables

Definition 3.1. Aneutrosophic random variable $X_N = X + YI$ is said to have a neutrosophic stable distribution if for any positive numbers $A_N = A_1 + A_2 I$ and $B_N = B_1 + B_2 I$, there is a positive number $C_N = C_1 + C_2 I$ and a number $D_N = D_1 + D_2 I$ such that

$$A_N X_N^{(1)} + B_N X_N^{(2)} \overset{d}{=} C_N X_N + D_N, \tag{1}$$

where $X_N^{(1)} = X_1 + Y_1 I$ and $X_N^{(2)} = X_2 + Y_2 I$ are independent copies of X_N, and where $"\overset{d}{=}"$ denotes equality in distribution.

Remark 3.1. The right hand side of (1) takes the form

$$C_N X_N + D_N = C_1 X + I[L\zeta - C_1 X] + D_N,$$

where $C_1 + C_2 = L,\ X + Y = \zeta$.

Proof By taking T for the left hand side of (1) we obtain

$$T\left[A_N X_N^{(1)} + B_N X_N^{(2)}\right] \overset{d}{=} (A_1, A_1 + A_2)(X_1, X_1 + Y_1) + (B_1, B_1 + B_2)(X_2, X_2 + Y_2),$$

$$\overset{d}{=} (A_1 X_1, (A_1 + A_2)(X_1 + Y_1)) + (B_1 X_2, (B_1 + B_2)(X_2 + Y_2)),$$

$$\overset{d}{=} (A_1 X_1 + B_1 X_2, (A_1 + A_2)(X_1 + Y_1) + (B_1 + B_2)(X_2 + Y_2)).$$

By taking T^{-1} for both sides we obtain

$$A_N X_N^{(1)} + B_N X_N^{(2)} \overset{d}{=} A_1 X_1 + B_1 X_2 + I[(A_1 + A_2)(X_1 + Y_1) + (B_1 + B_2)(X_2 + Y_2) - A_1 X_1 + B_1 X_2].$$

Since $A_1 X_1 + B_1 X_2 \overset{d}{=} C_1 X + D_1$, $(A_1 + A_2)(X_1 + Y_1) + (B_1 + B_2)(X_2 + Y_2) \overset{d}{=} (C_1 + C_2)(X + Y) + (D_1 + D_2)$ then

$$A_N X_N^{(1)} + B_N X_N^{(2)} \overset{d}{=} C_1 X + D_1 + I[(C_1 + C_2)(X + Y) + (D_1 + D_2) - (C_1 X + D_1)],$$

and $$A_N X_N^{(1)} + B_N X_N^{(2)} \overset{d}{=} C_1 X + I[(C_1 + C_2)(X+Y) - C_1 X] + D_1 + D_2 I \,.$$

Finally

$$C_N X_N + D_N = C_1 X + I[L\zeta - C_1 X] + D_N \,. \tag{2}$$

Definition 3.2. A neutrosophic stable random variable is called neutrosophic strictly stable if (1) holds with $D_N = 0_N = 0 + 0I$.

Definition 3.3. Aneutrosophic random variable $X_N = X + YI$ is referred to as neutrosophic stable if there exist constants $0 < A_N^{(n)} = A_1^{(n)} + A_2^{(n)} I$ and $B_N^{(n)} = B_1^{(n)} + B_2^{(n)} I$ such that

$$\sum_{i=1}^{n} X_N^{(n)} \overset{d}{=} B_N^{(n)} + A_N^{(n)} X_N \,, \tag{3}$$

where $X_N^{(1)}, X_N^{(2)}, \ldots$ are independent neutrosophic random variables each having the same distribution as X_N .

Again, if $B_N^{(n)} = 0_N$, then X_N in (3) is called neutrosophic strictly stable, i.e.

$$\sum_{i=1}^{n} X_N^{(n)} \overset{d}{=} A_N^{(n)} X_N \,, \tag{4}$$

Theorem 3.1. In relation (4), the constant $A_N^{(n)}$ has the form

$$A_N^{(n)} = n^{1_N/\alpha_N}, \quad n^{1_N/\alpha_N} = n^{1/\alpha} + I\left(n^{1/\alpha} - n^{1/\alpha}\right) = n^{1/\alpha} + 0I, \quad \alpha_N = \alpha + \alpha I.$$

Proof Rewriting (4) as the sequence of sums

$$\begin{aligned}
&X_N^{(1)} + X_N^{(2)} \overset{d}{=} A_N^{(2)} X_N \\
&X_N^{(1)} + X_N^{(2)} + X_N^{(3)} \overset{d}{=} A_N^{(3)} X_N \\
&X_N^{(1)} + X_N^{(2)} + X_N^{(3)} + X_N^{(4)} \overset{d}{=} A_N^{(4)} X_N \\
&\ldots
\end{aligned} \tag{5}$$

We cosider only those sums which contain 2^k terms, $k = 1, 2, \ldots$:

$$\begin{aligned}
&X_N^{(1)} + X_N^{(2)} \overset{d}{=} A_N^{(2)} X_N \\
&X_N^{(1)} + X_N^{(2)} + X_N^{(3)} + X_N^{(4)} \overset{d}{=} A_N^{(4)} X_N \\
&X_N^{(1)} + X_N^{(2)} + X_N^{(3)} + X_N^{(4)} + X_N^{(5)} + X_N^{(6)} + X_N^{(7)} + X_N^{(8)} \overset{d}{=} A_N^{(8)} X_N \\
&\ldots \\
&X_N^{(1)} + X_N^{(2)} + \ldots + X_N^{(2^k - 1)} + X_N^{(2^k)} \overset{d}{=} A_N^{(2^k)} X_N \\
&\ldots
\end{aligned}$$

Making use the first formula, we transform the second one as follows:

$$S_N^{(4)} = \left(X_N^{(1)} + X_N^{(2)}\right) + \left(X_N^{(3)} + X_N^{(4)}\right) \overset{d}{=} A_N^{(2)}\left(X_N^{(1)} + X_N^{(2)}\right) \overset{d}{=} \left(A_N^{(2)}\right)^2 X_N .$$

Here we keep in mind that $X_N^{(1)} + X_N^{(2)} \overset{d}{=} X_N^{(3)} + X_N^{(4)}$. Applying this reasoning to the third formula, we obtain

$$\begin{aligned} S_N^{(8)} &= \left(X_N^{(1)} + X_N^{(2)}\right) + \left(X_N^{(3)} + X_N^{(4)}\right) + \left(X_N^{(5)} + X_N^{(6)}\right) + \left(X_N^{(7)} + X_N^{(8)}\right) \\ &\overset{d}{=} A_N^{(2)}\left(X_N^{(1)} + X_N^{(2)}\right) + A_N^{(2)}\left(X_N^{(5)} + X_N^{(6)}\right) \\ &\overset{d}{=} \left(A_N^{(2)}\right)^2 X_N^{(1)} + \left(A_N^{(2)}\right)^2 X_N^{(5)} \\ &\overset{d}{=} \left(A_N^{(2)}\right)^2 \left(X_N^{(1)} + X_N^{(5)}\right) = \left(A_N^{(2)}\right)^3 X_N . \end{aligned}$$

For the sum of 2^k terms, we similarly obtain

$$S_N^{(2^k)} \overset{d}{=} A_N^{(2^k)} X_N \overset{d}{=} A_N^{(k)} X_N .$$

Comparing this with (4), with $n = 2^k$, we obtain:

$$A_N^{(n)} = \left(A_N^{(2)}\right)^k = \left(A_N^{(2)}\right)^{(\log n)/\log 2} ;$$

hence

$$\log A_N^{(n)} = [(\log n) / \log 2] \log A_N^{(2)} = \log n^{(\log A_N^{(2)})/\log 2} .$$

Thus, for the sequence of sums we obtain

$$A_N^{(n)} = n^{1_N/(\alpha_N^{(2)})}, \quad \alpha_N^{(2)} = \log 2 / \log A_N^{(2)}, \quad n=2^k, \quad k = 1, 2, \ldots . \tag{6}$$

Choosing now from (5) those sums which contain 3^k terms, and repeating the above reasoning, we arrive at

$$A_N^{(n)} = n^{1_N/(\alpha_N^{(3)})}, \quad \alpha_N^{(3)} = \log 3 / \log A_N^{(3)}, \quad n = 3^k, \quad k = 1, 2, \ldots \tag{7}$$

In the general case,

$$A_N^{(n)} = n^{1_N/(\alpha_N^{(m)})}, \quad \alpha_N^{(m)} = \log m / \log A_N^{(m)}, \quad n=m^k, \quad k = 1, 2, \ldots \tag{8}$$

We set $m = 4$. By virtue of (8),

$$\alpha_N^{(4)} = \log 4 / \log A_N^{(4)} ,$$

whereas (6) with $k = 2$ yields

$$\log A_N^{(4)} = \left(1_N / \alpha_N^{(2)}\right) \log 4.$$

Comparing the two last formulae, we conclude that

$$\alpha_N^{(2)} = \alpha_N^{(4)}.$$

By induction, we come to the conlusion that all $\alpha_N^{(m)}$ are equal to each other:

$$\alpha_N^{(m)} = \alpha_N.$$

The following expression hence holds for the scale factors $A_N^{(n)}$:

$$A_N^{(n)} = n^{1_N/\alpha_N}, \qquad n=1,2,3,... \tag{9}$$

whereas (4) takes the form

$$S_N^{(n)} = \sum_{i=1}^{n} X_N^{(n)} \overset{d}{=} n^{1_N/\alpha_N} X_N, \tag{10}$$

$$T\left(n^{1_N/\alpha_N}\right) = T\left(n^{\frac{1+I}{(\alpha+\alpha I)}}\right) = T\left(n+0I\right)^{T\left(\frac{1+I}{\alpha+\alpha I}\right)} = (n,n)^{\frac{(1,2)}{(\alpha,2\alpha)}}$$

$$=(n,n)^{(1/\alpha,2/(2\alpha))} = \left(n^{1/\alpha}, n^{1/\alpha}\right).$$

By taking T^{-1} for both sides of the last relation, then

$$n^{1_N/\alpha_N} = n^{1/\alpha} + I\left(n^{1/\alpha} - n^{1/\alpha}\right). \qquad \square$$

Remark 3.2. The right hand side of the relation (4) takes the form

$$n^{1_N/\alpha_N} X_N = n^{1/\alpha} X + I\left(n^{1/\alpha}(X+Y) - n^{1/\alpha} X\right).$$

In fact

$$T\left(n^{1_N/\alpha_N} X_N\right) = T\left(n^{(\alpha+\alpha I)^{-1}(1_N)}\right) T\left(X+YI\right) = T\left(n+0I\right)^{T(\alpha+\alpha I)^{T(-1)} T(1_N)} T\left(X+YI\right)$$

$$= (n,n)^{(\alpha,2\alpha)^{(-1,-1)}(1,2)}\left(X, X+Y\right) = (n,n)^{(1/\alpha,1/2\alpha))(1,2)}\left(X, X+Y\right)$$

$$= \left(n^{1/\alpha}, n^{2/2\alpha}\right)\left(X, X+Y\right) = \left(n^{1/\alpha} X, n^{1/\alpha}(X+Y)\right).$$

Note that $1_N = 1 + I$, and in the neutrosophic field: $\dfrac{1_N}{\alpha_N} = 1_N.\alpha_N^{-1}$.

By taking T^{-1} for both sides of the last relation, the proof will be completed.

Let us prove the relation $\dfrac{1_N}{\alpha_N} = 1_N.\alpha_N^{-1}$ in the general case where $\alpha_N = \alpha_1 + \alpha_2 I$:

$$\frac{1_N}{\alpha_N}=\frac{1+I}{\alpha_1+\alpha_2 I}\Rightarrow T\left(\frac{1_N}{\alpha_N}\right)=T\left(\frac{1+I}{\alpha_1+\alpha_2 I}\right)=\frac{(1,2)}{(\alpha_1,\alpha_1+\alpha_2)}=\left(\frac{1}{\alpha_1},\frac{2}{\alpha_1+\alpha_2}\right),$$

$$1_N.\alpha_N^{-1}=(1+I)(\alpha_1+\alpha_2 I)^{-1}\Rightarrow T\left[(1+I)(\alpha_1+\alpha_2 I)^{-1}\right]=T(1+I)T(\alpha_1+\alpha_2 I)^{T(-1)}\cdot$$

$$=(1,2)(\alpha_1,\alpha_1+\alpha_2)^{(-1,-1)}=(1,2)\left(\frac{1}{\alpha_1},\frac{1}{\alpha_1+\alpha_2}\right)=\left(\frac{1}{\alpha_1},\frac{2}{\alpha_1+\alpha_2}\right).$$

Theorem 3.2. If G_N is a neutrosophic strictly stable distribution with characteristic parameter $\alpha_N=\alpha+\alpha I$ then

$$(A)^{1_N/\alpha_N}X_N^{(1)}+(B)^{1_N/\alpha_N}X_N^{(2)}\overset{d}{=}(A+B)^{1/\alpha}X_N,$$

for $A,\ B>0$.

Proof By recognizing the relation (10), for any positive numbers $A,\ B,$ let $X_N^{(i)},\ i=\overbrace{1,2,\ldots,A}^{A\text{ number}},\overbrace{A+1,\ldots,n}^{\text{B number}}.$ be neutrosophic strictly stable random variables.

Then, we have $S_N^{(A)}=\sum_{i=1}^{A}X_N^{(i)}$, $S_N^{(B)}=\sum_{i=A+1}^{n}X_N^{(i)}$, and $S_N^{(A+B)}=\sum_{i=1}^{A+B}X_N^{(i)}$, hence

$$S_N^{(A)}=\sum_{i=1}^{A}X_N^{(i)}\overset{d}{=}A^{1/\alpha}X_N^{(1)},\ S_N^{(B)}=\sum_{i=A+1}^{n}X_N^{(i)}\overset{d}{=}B^{1/\alpha}X_N^{(2)},\text{ and }S_N^{(A+B)}=\sum_{i=A+1}^{A+B}X_N^{(i)}\overset{d}{=}(A+B)^{1/\alpha}X_N.$$

Since $S_N^{(A)}+S_N^{(B)}=S_N^{(A+B)}$, then $(A)^{1_N/\alpha_N}X_N^{(1)}+(B)^{1_N/\alpha_N}X_N^{(2)}\overset{d}{=}(A+B)^{1/\alpha}X_N$. □

The neutrosophic convolution

Let X_N be a neutrosophic random variable, its neutrosophic density function is $f_{X_N}(x_N)$. We stand for the neutrosophic probability distribution function by $F_{X_N}(x_N)$ and we define it as

$$F_{X_N}(x_N)=\mathrm{P}(X_N\le x_N)=\int_{-\infty_N}^{x_N}f_N(t_N)\,dt_N.$$

What the right hand side form is?

Suppose that $X_N=X+YI$, and the probability density functions of $X,\ Y$ are $f,\ g$ respectively. By taking T for both sides, we obtain

$$T\left(F_{X_N}(x_N)\right)=T\left(\int_{-\infty_N}^{x_N}f_N(t_N)\,dt_N\right). \tag{11}$$

$$T\left(\int_{-\infty_N}^{x_N}f_N(t_N)\,dt_N\right)\equiv T\left(\int_{-\infty_N}^{x_N}\right)T\left(f_N(t_N)\right)T\left(d(t_N)\right). \tag{12}$$

By taking $T\left(\int_{-\infty_N}^{x_N}\right) \equiv T\left(\int_{-\infty-\infty I}^{x+yI}\right) = \left(\left(\int_{-\infty}^{x}\right),\left(\int_{-\infty}^{x+y}\right)\right)$, $T\left(f_N(t_N)\right) \equiv \left(f_X(t_1),(f*g)_{X+Y}(t_1+t_2)\right)$,

and $T\left(d(t_N)\right) = T\left(d\left(t_1+t_2I\right)\right) = T\left(dt_1+Idt_2\right) = \left(dt_1,dt_1+dt_2\right) = \left(dt_1,d(t_1+t_2)\right)$.

Hence, the right hand side of (12) becomes

$$T\left(\int_{-\infty_N}^{x_N}\right)T\left(f_N(t_N)\right)T\left(d(t_N)\right) = \left(\left(\int_{-\infty}^{x}\right),\left(\int_{-\infty}^{x+y}\right)\right)\left(f_X(t_1),(f*g)_{X+Y}(t_1+t_2)\right)\left(dt_1,d(t_1+t_2)\right)$$

$$= \left(\left(\int_{-\infty}^{x} f_X(t_1)dt_1\right),\left(\int_{-\infty}^{x+y} (f*g)_{X+Y}(t_1+t_2)d(t_1+t_2)\right)\right).$$

Now, the relation (11) becomes

$$T\left(F_{X_N}(x_N)\right) = \left(\left(\int_{-\infty}^{x} f_X(t_1)dt_1\right),\left(\int_{-\infty}^{x+y} (f*g)_{X+Y}(t_1+t_2)d(t_1+t_2)\right)\right)$$

By taking T^{-1} for both sides, we obtain

$$F_{X_N}(x_N) = \int_{-\infty}^{x} f_X(t_1)dt_1 + I\left(\int_{-\infty}^{x+y} (f_*g)_{X+Y}(t_1+t_2)d(t_1+t_2) - \int_{-\infty}^{x} f_X(t_1)dt_1\right). \tag{13}$$

Definition 3.4. Suppose that X_N, Y_N are two independent neutrosophic random variables. $F_{X_N}(x_N)$, $G_{Y_N}(y_N)$ and $f_N = f_{X_N}(x_N)$, $g_N = g_{Y_N}(y_N)$ are their neutrosophic probability distribution functions and neutrosophic probability density functions respectively. The neutrosophic convolution of $F_N = F_{X_N}(x_N)$ and $G_N = G_{Y_N}(y_N)$ can be defined as

$$F_N *_N G_N = \int_{-\infty_N}^{x_N} \left(f_N *_N g_N\right) d\left(t_N\right), \tag{14}$$

where

$$f_N *_N g_N = \int_{-\infty_N}^{\infty_N} f_N\left(t_N - y_N\right) g_N\left(y_N\right) dy_N. \quad (15$$

Theorem 3.3. According to the above hypotheses, the relations (14) and (15) hold, and (15) takes the form

$$f_{N*_N} g_N = (f_{X_1} * g_{Y_1})(t_1) + I\left[(f_{X_1+X_2} * g_{Y_1+Y_2})(t_1+t_2) - (f_{X_1} * g_{Y_1})(t_1)\right], \tag{16}$$

where $(f_{X_1+X_2} * g_{Y_1+Y_2})(t_1+t_2)$ is the convolution of the variables $X = X_1 + X_2$ and $Y = Y_1 + Y_2$.

Proof Because of the independence of X_N, Y_N:

$$F_N *_N G_N = \int_{-\infty_N}^{\infty_N} \int_{-\infty_N}^{\infty_N} f_N(x_N) g_N(y_N)\, d(x_N) d(y_N)$$

$$= \int_{-\infty_N}^{\infty_N} \left[\int_{-\infty_N}^{t_N - y_N} f_N(x_N)\, d(x_N) \right] g_N(y_N) d(y_N)$$

$$= \int_{-\infty_N}^{\infty_N} \left[\int_{-\infty_N}^{x_N} f_N(t_N - y_N)\, d(t_N) \right] g_N(y_N) d(y_N)$$

$$= \int_{-\infty_N}^{x_N} \left[\int_{-\infty_N}^{\infty_N} f_N(t_N - y_N)\, g_N(y_N) d(y_N) \right] d(t_N).$$

Prove the relation (16) is similar to prove (13). □

Based on the previous facts, the convolution can be generalized for n.

4. Applications

There are three fundamental and well-known examples of stable laws, let $q(x)$ is the probability density function of stable random variable X :

4.1. Gaussian Distribution

In (16), two classical convolutions are well-known for the gaussian distribution. Because of the independence and identically in distribution for stable random variables, $(f_{X_1+X_2} * g_{Y_1+Y_2})(t_1+t_2)$ becomes the convolution of four gaussian random variables with one dimensional. The same applies to the rest of the examples.

We have

$$q\left(x; a, \sigma^2\right) = \frac{1}{\sqrt{2\pi}\sigma} \exp\left\{ -\frac{(x-a)^2}{2\sigma^2} \right\}, \quad -\infty < x < \infty, \quad \sigma > 0.$$

Since (See [9])

$$q(x; a_1, \sigma_1) * q(x; a_2, \sigma_2) = q(x; a_1 + a_2, \sqrt{\sigma_1^2 + \sigma_2^2}),$$

$$q(x; a_1, \sigma_1) * q(x; a_2, \sigma_2) * q(x; a_3, \sigma_3) * q(x; a_4, \sigma_4) = q(x; a_1 + a_2 + a_3 + a_4, \sqrt{\sigma_1^2 + \sigma_2^2 + \sigma_3^2 + \sigma_4^2}),$$

then

$$q_{N *_N} q_N = q(x; a_1 + a_2, \sqrt{\sigma_1^2 + \sigma_2^2}) + I\left[q(x; a_1 + a_2 + a_3 + a_4, \sqrt{\sigma_1^2 + \sigma_2^2 + \sigma_3^2 + \sigma_4^2}) - q(x; a_1 + a_2, \sqrt{\sigma_1^2 + \sigma_2^2}) \right].$$

4.2. Cauchy Distribution

Without losing generality, it is known that the convolution of a Cauchy probability density function with a scale parameter equal to one is

$$q(x) * q(x) = \frac{1}{2} q_{X_1 + X_2}\left(\frac{x}{2}\right).$$

And

$$q(x)*q(x)*q(x)*q(x)=\frac{1}{4}q_{X_1+X_2+X_3+X_4}\left(\frac{x}{4}\right).$$

Then

$$q_{N*_N}q_N=\frac{1}{2}q_{X_1+X_2}\left(\frac{x}{2}\right)+I\left[\frac{1}{4}q_{X_1+X_2+X_3+X_4}\left(\frac{x}{4}\right)-\frac{1}{2}q_{X_1+X_2}\left(\frac{x}{2}\right)\right].$$

4.3. Lévy Distribution

We have for Lévy Distribution that

$$q(x)*q(x)=(1/4)q(x/4).$$

And

$$q(x)*q(x)=(1/4)q(x/4).$$
$$q(x)*q(x)*q(x)*q(x)=(1/16)q(x/16).$$

Then

$$q_{N*_N}q_N=(1/4)q(x/4)+I\left[(1/16)q(x/16)-(1/4)q(x/4)\right].$$

5. Conclusions

In this paper, we suggested some basic definitions of the neutrosophic stable random variable and generalize some of the main properties of the classical stable distributions to the neutrosophic field. We also defined both the neutrosophic probability distribution function and the neutrosophic probability density function, then we defined the convolution with the neutrosophic concept. Finally, we supported the article with three examples of stable distributions with the neutrosophical concept, which are famous distributions in classical stability. Later, we will extend the work in the field of neutrosophic stability and work to generalize and prove more profound facts.

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