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Eigenspace of a Circulant Fuzzy Neutrosophic Soft Matrix

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Abstract. The eigen problem of a Circulant Fuzzy Neutrosophic Soft Matrix (CFNSM) in max-min Fuzzy Neutrosophic Soft Algebra (FNSA) is analyzed. By describing all possible types of Fuzzy Neutrosophic Soft Eigenvectors (FNSEvs), eigen space structures of CFNSM is characterized.

Keywords: Fuzzy Neutrosophic Soft Set (FNSS), Fuzzy Neutrosophic Soft Matrices(FNSMs), Circulant Fuzzy Neutrosophic Soft Matrix (CFNSM), Fuzzy Neutrosophic Soft Eigenvectors (FNSEvs).

1. **Introduction**

In real world, we face so many uncertainties in all walks of life. However most of the existing mathematical tools for formal modeling, reasoning and computing are crisp and precise in character. There are theories viz,theory of probability, evidence, fuzzy set [31], intuitionistic fuzzy set [3], neutrosophic set [26], vague set, interval mathematics, rough set for dealing with uncertainties. These theories have their own difficulties as pointed out by Molodtsov [21]. In 1999, Molodtsov [21] initiated a novel concept of soft set theory, which is completely a new approach for modeling vagueness and uncertainties. Soft set theory has a rich potential for application in solving practical problems in economics, social science, medical science etc.. Later on Maji et al. [22] have proposed the theory of fuzzy soft set. Maji et al. [18,19] extended soft sets to intuitionistic fuzzy soft sets and neutrosophic soft sets.

Eigenvectors of a max-min matrix characterize stable state of the corresponding discrete-events system. Investigation of the max-min eigenvectors of a given matrix is therefore of a great practical importance. The eigenproblem in max-min algebra has been studied by many authors. Interesting results were found in describing the structure of the eigenspace, and algorithms for computing the maximal eigenvector of a given matrix were sugested, see e.g. [5, 6, 23, 24, 31, 32]. The structure of the eigenspace as a union of intervals of increasing eigenvectors is described in [7].

Fuzzy matrices defined first time by Thomason in 1977 [25] and he discussed about the convergence of the powers of a fuzzy matrix. The theory of fuzzy matrices were developed by Kim and Roush [16] as an extension of Boolean matrices. Manoj Bora *et al.* [20] have applied intuitionistic fuzzy soft matrices in the medical diagnosis problem. Arockiarani and Sumathi [1, 2] introduced Fuzzy Neutrosophic Soft Matrix (FNSM) and used them in decision making problems. Broumi *et al.* [4] proposed the concept of generalized interval neutrosophic soft set and studied their operations. Also, they presented an application of it in decision making problem. First time Kavitha *et al.* [10–13, 15] introduced the concept of unique solvability of max-min operation through FNSM equation $Ax = b$ and explained strong regularity of FNSMs over fuzzy neutrosophic soft algebra and computing the greatest X-eigenvector of fuzzy neutrosophic soft matrix. They also introduced the power of FNSM and Periodicity of Interval Fuzzy Neutrosophic Soft Matrices. Murugadas *et al.* proposed the ideas of the Monotone interval fuzzy neutrosophic soft eigenproblem and Solveability of System of Netrosophic Soft Linear Equations in [17]. In [30], Uma et. al, introduced the concept of FNSMs of Type-1 and Type-2.

By max-min FNSA we understand a triplet $(\mathcal{N}, \oplus, \otimes)$, where $\mathcal{N}$ is a linearly ordered FNSS, and $\oplus = \max$, $\otimes = \min$ are binary operations on $\mathcal{N}$. The notation $\mathcal{N}_{(n,n)}, \mathcal{N}_{(n)}$ denotes the set of all Fuzzy Neutrosophic Soft Square Matrices(FNSSMs) (all FNSVs) of given dimension n over $\mathcal{N}$. Operations $\oplus, \otimes$ are extended to FNSMs and FNSVs in formal way.

The eigenproblem for a given FNSM $A \in \mathcal{N}_{(n,n)}$ in max-min FNSA consists of finding a FNSV $\langle x^T, x^I, x^F\rangle \in \mathcal{N}_{(n)}$ (FNSEv) such that the equation
$A \otimes \langle x^T, x^I, x^F\rangle = \langle x^T, x^I, x^F\rangle$ holds true. By the eigenspace of a given FNSM we mean the set of all its FNSEvs.

In this paper the eigenspace structure for a special case of so-called CFNSMs is studied. The paper presents a detailed description of all possible types of FNSEvs of any given CFNSM.

2. Preliminaries

In this section, some basic notions related to this topics are recalled.

Definition 2.1. [26] A neutrosophic set A on the universe of discourse X is defined as $A = \{\langle x,\ T_A(x),\ I_A(x),\ F_A(x)\rangle, x \in X\}$, where $T, I, F : X \to\ ]^-0, 1^+[$ and
$^-0 \leq T_A(x) + I_A(x) + F_A(x) \leq 3^+$. (1)]

From philosophical point of view the NS set takes the value from real standard or non-standard subsets of $]^-0, 1^+[$. But in real life application especially in Scientific and Engineering problems it is difficult to use NS with value from real standard or non-standard subset of $]^-0, 1^+[$. Hence we consider the NS which takes the value from the subset of $[0, 1]$. Therefore we can rewrite equation (1) as $0 \leq T_A(x) + I_A(x) + F_A(x) \leq 3$. In short an element $\widetilde{a}$ in the NS A, can be written as $\widetilde{a} = \langle a^T, a^I, a^F \rangle$, where a^T denotes degree of truth, a^I denotes degree of indeterminacy, a^F denotes degree of falsity such that $0 \leq a^T + a^I + a^F \leq 3$.

Definition 2.2. [1] A NS A on the universe of discourse X is defined as $A = \{x, \langle T_A(x), I_A(x), F_A(x) \rangle, x \in X\}$, where $T, I, F : X \to [0, 1]$ and $0 \leq T_A(x)+I_A(x)+F_A(x) \leq 3$.

Definition 2.3. [21] Let U be the initial universe set and E be a set of parameter. Consider a non-empty set $A, A \subset E$. Let $P(U)$ denotes the set of all NSs of U. The collection (F, A) is termed to be the NSS over U, where F is a mapping given by $F : A \to P(U)$. Here after we simply consider A as NSS over U instead of (F, A).

Definition 2.4. [2] Let $U = \{c_1, c_2, ..., c_m\}$ be the universal set and E be the set of parameters given by $E = \{e_1, e_2, ..., e_m\}$. Let $A \subset E$. A pair (F, A) be a NSS over U. Then the subset of $U \times E$ is defined by $R_A = \{(u, e);\ e \in A,\ u \in F_A(e)\}$
which is called a relation form of $(F_A,\ E)$. The membership function, indeterminacy membership function and non membership function are written by
$T_{R_A} : U \times E \to [0, 1],\ I_{R_A} : U \times E \to [0, 1]$ and $F_{R_A} : U \times E \to [0, 1]$ where $T_{R_A}(u, e) \in [0, 1],\ I_{R_A}(u, e) \in [0, 1]$ and $F_{R_A}(u, e) \in [0, 1]$ are the membership value, indeterminacy value and non membership value respectively of $u \in U$ for each $e \in E$.
If $[(T_{ij},\ I_{ij},\ F_{ij})] = [T_{ij}(u_i, e_j),\ I_{ij}(u_i, e_j)\ , F_{ij}(u_i, e_j)]$ we define a matrix

$$[\langle T_{ij}\ , I_{ij},\ F_{ij} \rangle]_{m \times n} = \begin{bmatrix} \langle T_{11},\ I_{11},\ F_{11} \rangle & \cdots & \langle T_{1n},\ I_{1n},\ F_{1n} \rangle \\ \langle T_{21},\ I_{21},\ F_{21} \rangle & \cdots & \langle T_{2n},\ I_{2n},\ F_{2n} \rangle \\ \vdots & \vdots & \vdots \\ \langle T_{m1},\ I_{m1},\ F_{m1} \rangle & \cdots & \langle T_{mn},\ I_{mn},\ F_{mn} \rangle \end{bmatrix}.$$

Which is called an $m \times n$ $FNSM$ of the NSS (F_A, E) over U.

Definition 2.5. [30] Let $A = (\langle a^T_{ij},\ a^I_{ij},\ a^F_{ij} \rangle),\ B = \langle (b^T_{ij},\ b^I_{ij},\ b^F_{ij} \rangle) \in \mathcal{N}_{(m,n)}$, NSM of order $m \times n$) and $\mathcal{N}_{(n)}$-denotes a square NSM of order n. The component wise addition and component wise multiplication is defined as
$A \oplus B = (sup\{a^T_{ij},\ b^T_{ij}\},\ sup\{a^I_{ij},\ b^I_{ij}\},\ inf\{a^F_{ij},\ b^F_{ij}\})$
$A \otimes B = (inf\{a^T_{ij},\ b^T_{ij}\},\ inf\{a^I_{ij},\ b^I_{ij}\},\ sup\{a^F_{ij},\ b^F_{ij}\})$

Definition 2.6. Let $A \in \mathcal{N}_{(m,n)},\ B \in \mathcal{N}_{(n,p}$, the composition of A and B is defined as

$$A \circ B = \left(\sum_{k=1}^{n} (a_{ik}^{T} \wedge\ b_{kj}^{T}),\ \sum_{k=1}^{n} (a_{ik}^{I} \wedge\ b_{kj}^{I}),\ \prod_{k=1}^{n} (a_{ik}^{F} \vee\ b_{kj}^{F}) \right)$$

equivalently we can write the same as

$$= \left(\bigvee_{k=1}^{n} (a_{ik}^{T} \wedge b_{kj}^{T}),\ \bigvee_{k=1}^{n} (a_{ik}^{I} \wedge b_{kj}^{I}),\ \bigwedge_{k=1}^{n} (a_{ik}^{F} \vee b_{kj}^{F}) \right).$$

The product $A \circ B$ is defined if and only if the number of columns of A is same as the number of rows of B. Then A and B are said to be conformable for multiplication. We shall use AB instead of $A \circ B$.

Where $\sum (a_{ik}^{T} \wedge\ b_{kj}^{T})$ means max-min operation and

$\prod_{k=1}^{n} (a_{ik}^{F} \vee\ b_{kj}^{F})$ means min-max operation.

3. **Eigenvectors of CFNSM**

The characterization of the eigenspace structure for a CFNSM is discussed in this section.

Circulancy of FNSM is analogous to circulancy of classical matrix. Formally, FNSM $A \in \mathcal{N}_{(n,n)}$ is circulant if

$$\langle a_{ij}^{T}, a_{ij}^{I}, a_{ij}^{F} \rangle = \langle a_{i^{'}j^{'}}^{T}, a_{i^{'}j^{'}}^{I}, a_{i^{'}j^{'}}^{F} \rangle \text{ whenever } i - i^{'} \equiv j - j^{'}\ (mod\ n).$$

Hence, CFNSM A is totally determined by its inputs $\langle a_0^T, a_0^I, a_0^F \rangle, \langle a_1^T, a_1^I, a_1^F \rangle, ..., \langle a_{n-1}^T, a_{n-1}^I, a_{n-1}^F \rangle$ in the first row. $\langle a_0^T, a_0^I, a_0^F \rangle$ is the common in all diagonal , and similarly each $\langle a_i^T, a_i^I, a_i^F \rangle$ is common in a line parallel to the FNSM diagonal,

$A(\langle a_0^T, a_0^I, a_0^F \rangle, \langle a_1^T, a_1^I, a_1^F \rangle, ..., \langle a_{n-1}^T, a_{n-1}^I, a_{n-1}^F \rangle) =$

$$\begin{bmatrix} \langle a_0^T\ a_0^I\ a_0^F \rangle & \langle a_1^T\ a_1^I\ a_1^F \rangle & \langle a_2^T\ a_2^I\ a_2^F \rangle & \cdots & \langle a_{n-1}^T\ a_{n-1}^I\ a_{n-1}^F \rangle \\ \langle a_{n-1}^T\ a_{n-1}^I\ a_{n-1}^F \rangle & \langle a_0^T\ a_0^I\ a_0^F \rangle & \langle a_1^T\ a_1^I\ a_1^F \rangle & \cdots & \langle a_{n-2}^T\ a_{n-2}^I\ a_{n-2}^F \rangle \\ \langle a_{n-2}^T\ a_{n-2}^I\ a_{n-2}^F \rangle & \langle a_{n-1}^T\ a_{n-1}^I\ a_{n-1}^F \rangle & \langle a_0^T\ a_0^I\ a_0^F \rangle & \cdots & \langle a_{n-3}^T\ a_{n-3}^I\ a_{n-3}^F \rangle \\ \vdots & \vdots & \vdots & \cdots & \vdots \\ \langle a_1^T\ a_1^I\ a_1^F \rangle & \langle a_2^T\ a_2^I\ a_2^F \rangle & \langle a_3^T\ a_3^I\ a_3^F \rangle & \cdots & \langle a_0^T\ a_0^I\ a_0^F \rangle \end{bmatrix}.$$

Set $N = \{1, 2, ..., n\}$ and $N_0 = \{0, 1, ..., n-1\}$. Further for a given CFNSM $A = A(\langle a_0^T, a_0^I, a_0^F \rangle, \langle a_1^T, a_1^I, a_1^F \rangle, ..., \langle a_{n-1}^T, a_{n-1}^I, a_{n-1}^F \rangle)$, a strictly non increasing sequence $M(A) = (s_1, s_2, ...)$ of the length $l(A)$ by repetition

$$s_r = \begin{cases} \max\{\langle a_i^T, a_i^I, a_i^F \rangle;\ i \in N_0\}\ for\ r = 1 \\ \max\{\langle a_i^T, a_i^I, a_i^F \rangle < s_{r-1};\ i \in N_0\}\ for\ r > 1 \end{cases}$$

Here $s_r = \langle s_r^T, s_r^I, s_r^F \rangle$. Henceforth $s_1 > s_2 > ...$ and $l(A)$ the length of the sequence $M(A)$ is the first l satisfying $\{\langle a_i^T, a_i^I, a_i^F \rangle;\ i \in N_0\} = \{s_r; 1 \leq r \leq l\}$. Use the notation $L(A) = \{1, 2, ..., l(A)\}$. Denote P_r as the set of all positions of the value s_r in the first row of the FNSM A, for any $r \in L(A)$ i.e.

$$P_r = \{i \in N_0;\ \langle a_i^T, a_i^I, a_i^F \rangle = s_r\}$$

and we set the highest common factors(HCF) d_r, e_r as follows

$$d_r = HCF(P_r \cup \{n\}),\ e_r = HCF(d_1, d_2, ..., d_r) = HCF(e_{r-1}, d_r).$$

Remark 3.1. The indices of FNSM values $\langle a_i^T, a_i^I, a_i^F \rangle$, and their placements, are numbers in $N_0 = \{0, 1, ..., n-1\}$, while the row and columns of the FNSM are indexed between 1 and n. Thus, for all $k \in N$, the kth row of A will be like this

$$A_k = (..., \langle a_{kk}^T, a_{kk}^I, a_{kk}^F \rangle, \langle a_{kk+1}^T, a_{kk+1}^I, a_{kk+1}^F \rangle, \langle a_{kk+2}^T, a_{kk+2}^I, a_{kk+2}^F \rangle, ...)$$

and for any position $p \in P_r$, we have $\langle a_{kk+p}^T, a_{kk+p}^I, a_{kk+p}^F \rangle = s_r$ (here the column index is computed modulo $k + p\ n$).

The next two lemmas are vital in this work.

Lemma 3.2. Let CFNSM $A = A(\langle a_0^T, a_0^I, a_0^F \rangle, \langle a_1^T, a_1^I, a_1^F \rangle, ..., \langle a_{n-1}^T, a_{n-1}^I, a_{n-1}^F \rangle)$ be given, let $(\langle x^T, x^I, x^F \rangle) = (\langle x_1^T, x_1^I, x_1^F \rangle, \langle x_2^T, x_2^I, x_2^F \rangle, ..., \langle x_n^T, x_n^I, x_n^F \rangle)$ be FNSEv of A, let $k \in N, r \in L(A)$ and $p \in P_r(A)$. If $\langle x_k^T, x_k^I, x_k^F \rangle < s_r$, then $\langle x_k^T, x_k^I, x_k^F \rangle = \langle x_{k+p}^T, x_{k+p}^I, x_{k+p}^F \rangle$.

Proof. Assume that $\langle x_k^T, x_k^I, x_k^F \rangle < \langle x_{k+p}^T, x_{k+p}^I, x_{k+p}^F \rangle$. Then by Remark 3.1

$$\langle x_k^T, x_k^I, x_k^F \rangle < s_r \otimes \langle x_{k+p}^T, x_{k+p}^I, x_{k+p}^F \rangle = \langle a_{kk+p}^T, a_{kk+p}^I, a_{kk+p}^F \rangle \otimes \langle x_{k+p}^T, x_{k+p}^I, x_{k+p}^F \rangle \leq A_k \otimes (\langle x^T, x^I, x^F \rangle),$$

i.e $(\langle x^T, x^I, x^F \rangle)$ cannot be eigenvector of A, a contradiction. Then $\langle x_k^T, x_k^I, x_k^F \rangle \geq \langle x_{k+p}^T, x_{k+p}^I, x_{k+p}^F \rangle$. Repeating like this we get, due to the cyclicity of A,

$$\langle x_k^T, x_k^I, x_k^F \rangle \geq \langle x_{k+p}^T, x_{k+p}^I, x_{k+p}^F \rangle \geq \langle x_{k+2p}^T, x_{k+2p}^I, x_{k+2p}^F \rangle \geq ... \geq \langle x_k^T, x_k^I, x_k^F \rangle$$

hence, $\langle x_k^T, x_k^I, x_k^F \rangle = \langle x_{k+p}^T, x_{k+p}^I, x_{k+p}^F \rangle$ must be hold true.

Lemma 3.3. Let CFNSM $A = A(\langle a_0^T, a_0^I, a_0^F \rangle, \langle a_1^T, a_1^I, a_1^F \rangle, ..., \langle a_{n-1}^T, a_{n-1}^I, a_{n-1}^F \rangle)$ be given. Let $(\langle x^T, x^I, x^F \rangle)$ be FNSEv of A, let $k, l \in N, r \in L(A)$. If $\langle x_k^T, x_k^I, x_k^F \rangle < s_r$, then the following result hold

(i) if $k \equiv l\ mod\ d_r$ then $\langle x_k^T, x_k^I, x_k^F \rangle = \langle x_l^T, x_l^I, x_l^F \rangle$,

(ii)if $k \equiv l\ mod\ e_r$ then $\langle x_k^T, x_k^I, x_k^F \rangle = \langle x_l^T, x_l^I, x_l^F \rangle$.

Proof. (i) Clearly d_r can be expressed as a linear combination of values in $P_r \cup \{n\}$ with non-negative coefficients from number theory. By repeated use of Lemma 3.2 (i) is obtained. (ii) follows directly from the definition of e_r and (i).

Theorem 3.4. Let CFNSM $A = A(\langle a_0^T, a_0^I, a_0^F\rangle, \langle a_1^T, a_1^I, a_1^F\rangle, ..., \langle a_{n-1}^T, a_{n-1}^I, a_{n-1}^F\rangle)$ be given, let $\langle x^T, x^I, x^F\rangle$ be FNSEv of A. Then $\langle x_k^T, x_k^I, x_k^F\rangle < s_1$, holds true for every $k \in N$.

Proof: By contradiction, that if$\langle x_k^T, x_k^I, x_k^F\rangle > s_1$ for some $k \in N$. Then, the inequality $\langle x_k^T, x_k^I, x_k^F\rangle > \langle a_i^T, a_i^I, a_i^F\rangle$ holds for every $i \in N_0$, by definition of s_1, which gives $\langle x_k^T, x_k^I, x_k^F\rangle > \langle a_{kj}^T, a_{kj}^I, a_{kj}^F\rangle$ for every $j \in N$. Hence

$$\langle x_k^T, x_k^I, x_k^F\rangle > \bigoplus_{j\in N}(\langle a_{kj}^T, a_{kj}^I, a_{kj}^F\rangle \otimes \langle x_j^T, x_j^I, x_j^F\rangle) = A_k \otimes (\langle x^T, x^I, x^F\rangle),$$

i.e. $\langle x_k^T, x_k^I, x_k^F\rangle \neq A_k \otimes (\langle x^T, x^I, x^F\rangle)$ and, thus, $(\langle x^T, x^I, x^F\rangle)$ is not a eigenvector of A.

Theorem 3.5. Let CFNSM $A = A(\langle a_0^T, a_0^I, a_0^F\rangle, \langle a_1^T, a_1^I, a_1^F\rangle, ..., \langle a_{n-1}^T, a_{n-1}^I, a_{n-1}^F\rangle)$ be given, such that the diagonal input $\langle a_0^T, a_0^I, a_0^F\rangle$ is greater than all other inputs of the FNSM. If a FNSV $(\langle x^T, x^I, x^F\rangle) \in \mathcal{N}_{(n)}$ has inputs fulfilling the inequalities $s_2 \leq \langle x_k^T, x_k^I, x_k^F\rangle \leq s_1$ for every $k \in N$, then $(\langle x^T, x^I, x^F\rangle)$ is FNSEv of A.

Proof: By definition of P_r, the hypothesis of the theorem gives $P_1 = \{0\}$ and thus

$$A_k \otimes (\langle x^T, x^I, x^F\rangle) = \bigoplus_{j\in N}(\langle a_{kj}^T, a_{kj}^I, a_{kj}^F\rangle \otimes \langle x_j^T, x_j^I, x_j^F\rangle) =$$
$$(\langle a_{kk}^T, a_{kk}^I, a_{kk}^F\rangle \otimes \langle x_k^T, x_k^I, x_k^F\rangle) \oplus \bigoplus_{j\in N\setminus\{k\}}(\langle a_{kj}^T, a_{kj}^I, a_{kj}^F\rangle \otimes \langle x_j^T, x_j^I, x_j^F\rangle).$$

Further, we have $\langle a_{kk}^T, a_{kk}^I, a_{kk}^F\rangle \otimes \langle x_k^T, x_k^I, x_k^F\rangle = s_1 \otimes \langle x_k^T, x_k^I, x_k^F\rangle = \langle x_k^T, x_k^I, x_k^F\rangle$,

$\bigoplus_{j\in N\setminus\{k\}}(\langle a_{kj}^T, a_{kj}^I, a_{kj}^F\rangle \otimes \langle x_j^T, x_j^I, x_j^F\rangle \leq \bigoplus_{j\in N\setminus\{k\}}(s_2 \otimes \langle x_j^T, x_j^I, x_j^F\rangle) = s_2$,

hence $\langle x_k^T, x_k^I, x_k^F\rangle = \langle a_{kk}^T, a_{kk}^I, a_{kk}^F\rangle \otimes \langle x_k^T, x_k^I, x_k^F\rangle \leq A_k \otimes \langle x^T, x^I, x^F\rangle \leq \langle x_k^T, x_k^I, x_k^F\rangle \oplus s_2 = \langle x_k^T, x_k^I, x_k^F\rangle$.

for every $k \in N$, i.e. $A \otimes (\langle x^T, x^I, x^F\rangle) = (\langle x^T, x^I, x^F\rangle)$.

Remark 3.6. Theorem 3.5 is a special case of the sufficient part of Theorem 3.8. The assertions of Lemma 3.3 are fulfilled, as in Theorem 3.5 we have $P_1 = \{0\}$ and $d_1 = e_1 = n$, hence, the equivalence relation modulo n is the identity relation on N_0.

Remark 3.7. If the maximal input of the CFNSM is not unique, or if it is placed on other position than the diagonal one, then $0 < e_1 < n$ and the equivalence modulo e_1 differs from the identity relation on N_0. Hence, the inputs of any FNSEv cannot be arbitrary value in the interval $\langle s_2, s_1\rangle$ but according to Lemma 3.3, some repetitions must occur, see Example 4.2.

Theorem 3.8. Let CFNSM $A = A(\langle a_0^T, a_0^I, a_0^F\rangle, \langle a_1^T, a_1^I, a_1^F\rangle, ..., \langle a_{n-1}^T, a_{n-1}^I, a_{n-1}^F\rangle)$ be given. A FNSV $(\langle x^T, x^I, x^F\rangle) \in \mathcal{N}_{(n)}$ is FNSEv of A if and only if there is a partition $\mathcal{T}$, on N, such that for every class $t \in \mathcal{T}$ there exist $(\langle x^T(t), x^I(t), x^F(t)\rangle) \in \mathcal{N}$ and $r(t) \in L(A)$, satisfying the following conditions

(i) $\langle x_k^T, x_k^I, x_k^F\rangle = \langle x^T(t), x^I(t), x^F(t)\rangle \leq s_1$ for every $k \in t$,

(ii)$r(t) = \max\{r \in S(A); x(t) < s_r\}$,

(iii) t is an equivalence class in $\mathcal{N}$ modulo $e_r(t)$.

Proof:($\Rightarrow$) The conditions (i)-(iii) follow from Lemma 3.3 and Theorem 3.4.

($\Leftarrow$) Let (i)-(iii) be satisfied. If $(\langle x^T(t), x^I(t), x^F(t)\rangle) = s_1$, then according to (ii), $r(t)$ is the maximum of the $\emptyset$, which is the least element in $S(A)$, i.e. $r(t) = 1$ in this case.

For arbitrary, but fixed $k \in N$, there is $t \in \mathcal{T}$ with $k \in t$ and $P_1 \neq \emptyset$ by definition, hence there is $p \in P_1$, and $\langle a_p^T, a_p^I, a_p^F\rangle = s_1$. Therefore, $k \equiv k + p \ mod\ e_{r(t)}$ and by conditions (i),(iii), we have

$\langle x_k^T, x_k^I, x_k^F\rangle = \langle x_{k+p}^T, x_{k+p}^I, x_{k+p}^F\rangle = s_1 \otimes \langle x_{k+p}^T, x_{k+p}^I, x_{k+p}^F\rangle = \langle a_{kk+p}^T, a_{kk+p}^I, a_{kk+p}^F\rangle \otimes \langle x_{k+p}^T, x_{k+p}^I, x_{k+p}^F\rangle \leq \bigoplus_{j\in N}(\langle a_{kj}^T, a_{kj}^I, a_{kj}^F\rangle \otimes \langle x_j^T, x_j^I, x_j^F\rangle) = A_k \otimes \langle x^T, x^I, x^F\rangle$.

To prove the otherside, consider any $j \in N$. If $j \in t$, then $\langle x_j^T, x_j^I, x_j^F\rangle = \langle x_k^T, x_k^I, x_k^F\rangle$, by (i). Thus,

$\bigoplus_{j\in t}(\langle a_{kj}^T, a_{kj}^I, a_{kj}^F\rangle \otimes \langle x_j^T, x_j^I, x_j^F\rangle) = \bigoplus_{j\in t}(\langle a_{kj}^T, a_{kj}^I, a_{kj}^F\rangle \otimes \langle x_k^T, x_k^I, x_k^F\rangle) \leq \langle x_k^T, x_k^I, x_k^F\rangle$.

If $j \notin t$, then $j, k \not\cong mode_{r(t)}$. Therefore, $p = j - k$ is not a multiple of the HCF $e_{r(T)}$, and so, the difference p cannot be expressed as a linear combination with integer coefficients, of the values in $P_1 \cup P_2 \cup ... \cup P_{r(t)} \cup \{n\}$, from definition of $e_{r(t)}$. As a result we have $\langle a_p^T, a_p^I, a_p^F\rangle = s_q$ for some $q > r(t)$, which implies $s_q \leq \langle x^T(t), x^I(t), x^F(t)\rangle$, by assumption (ii). Therefore $\langle a_{kj}^T, a_{kj}^I, a_{kj}^F\rangle = \langle a_{kk+p}^T, a_{kk+p}^I, a_{kk+p}^F\rangle = s_q \leq \langle x_k^T, x_k^I, x_k^F\rangle$. Thus we have

$\bigotimes_{j\in N\setminus t}(\langle a_{kj}^T, a_{kj}^I, a_{kj}^F\rangle \otimes \langle x_j^T, x_j^I, x_j^F\rangle) \leq \bigotimes_{j\in N\setminus t}(\langle a_{kj}^T, a_{kj}^I, a_{kj}^F\rangle \leq \langle x_k^T, x_k^I, x_k^F\rangle$.

Summarizing we get

$\langle x_k^T, x_k^I, x_k^F\rangle \leq A_k \otimes (\langle x^T, x^I, x^F\rangle) = \bigotimes_{j\in t}(\langle a_{kj}^T, a_{kj}^I, a_{kj}^F\rangle \otimes \langle x_j^T, x_j^I, x_j^F\rangle) \oplus \bigotimes_{j\in N\setminus t}(\langle a_{kj}^T, a_{kj}^I, a_{kj}^F\rangle \otimes \langle x_j^T, x_j^I, x_j^F\rangle$.

As $k \in N$ is arbitrary, we have

$A \otimes (\langle x^T, x^I, x^F\rangle) = (\langle x^T, x^I, x^F\rangle)$.

4. **Examples of FNSEvs**

Examples of FNSEvs of CFNSM are illustrated here.

Example 4.1. Let $n = 6$ and let

$A = A(\langle 1, 1, 0\rangle, \langle 0.1, 0.1, 0.9\rangle, \langle 0.3, 0.2, 0.7\rangle, \langle 0.7, 0.6, 0.3\rangle, \langle 0.3, 0.2, 0.7\rangle, \langle 0, 0, 1\rangle)$ be a CFNSM generated by inputs on positions $(0, 1, ..., 5)$ in the first row. Then $M(A) = (s_1, s_2, ..., s_5) = (\langle 1, 1, 0\rangle, \langle 0.7, 0.6, 0.3\rangle, \langle 0.3, 0.2, 0.7\rangle, \langle 0.1, 0.1, 0.9\rangle, \langle 0, 0, 1\rangle)$. The maximal input $s_1 = \langle 1, 1, 0\rangle$ is on the diagonal, i.e. on position 0 and nowhere else, the second largest input has value $s_2 = \langle 0.7, 0.6, 0.3\rangle$. Hence, in view of Theorem 3.5, any FNSV with arbitrary inputs from interval $[\langle 0.7, 0.6, 0.3\rangle, \langle 1, 1, 0\rangle]$, e.g.

$(\langle x^T, x^I, x^F\rangle) = (\langle 0.9, 0.8, 0.1\rangle, \langle 0.8, 0.7, 0.2\rangle, \langle 0.7, 0.6, 0.3\rangle, \langle 0.8, 0.7, 0.2\rangle, \langle 0.8, 0.7, 0.2\rangle,$
$\langle 0.7, 0.6, 0.3\rangle)^t$ is an FNSEv of A.

$$A = \begin{bmatrix} \langle 1,1,0\rangle & \langle 0.1,0.1,0.9\rangle & \langle 0.3,0.2,0.7\rangle & \langle 0.7,0.6,0.3\rangle & \langle 0.3,0.2,0.7\rangle & \langle 0,0,1\rangle \\ \langle 0,0,1\rangle & \langle 1,1,0\rangle & \langle 0.1,0.1,0.9\rangle & \langle 0.3\ 0.2\ 0.7\rangle & \langle 0.7,0.6,0.3\rangle & \langle 0.3,0.2,0.7\rangle \\ \langle 0.3,0.2,0.7\rangle & \langle 0,0,1\rangle & \langle 1,1,0\rangle & \langle 0.1,0.1,0.9\rangle & \langle 0.3,0.2,0.7\rangle & \langle 0.7,0.6,0.3\rangle \\ \langle 0.7,0.6,0.3\rangle & \langle 0.3,0.2,0.7\rangle & \langle 0,0,1\rangle & \langle 1,1,0\rangle & \langle 0.1,0.1,0.9\rangle & \langle 0.3,0.2,0.7\rangle \\ \langle 0.3,0.2,0.7\rangle & \langle 0.7,0.6,0.3\rangle & \langle 0.3,0.2,0.7\rangle & \langle 0,0,1\rangle & \langle 1,1,0\rangle & \langle 0.1,0.1,0.9\rangle \\ \langle 0.1,0.1,0.9\rangle & \langle 0.3,0.2,0.7\rangle & \langle 0.7,0.6,0.3\rangle & \langle 0.3,0.2,0.7\rangle & \langle 0,0,1\rangle & \langle 1,1,0\rangle \end{bmatrix} \otimes$$

$$\begin{bmatrix} \langle 0.9,0.8,0.1\rangle \\ \langle 0.8,0.7,0.2\rangle \\ \langle 0.7,0.6,0.3\rangle \\ \langle 0.8,0.7,0.2\rangle \\ \langle 0.8,0.7,0.2\rangle \\ \langle 0.7,0.6,0.3\rangle \end{bmatrix} = \begin{bmatrix} \langle 0.9,0.8,0.1\rangle \\ \langle 0.8,0.7,0.2\rangle \\ \langle 0.7,0.6,0.3\rangle \\ \langle 0.8,0.7,0.2\rangle \\ \langle 0.8,0.7,0.2\rangle \\ \langle 0.7,0.6,0.3\rangle \end{bmatrix}.$$

Example 4.2. In this example we show further FNSEvs of the FNSM $A = A(\langle 1,1,0\rangle, \langle 0.1,0.1,0.9\rangle, \langle 0.3,0.2,0.7\rangle, \langle 0.7,0.6,0.3\rangle, \langle 0.3,0.2,0.7\rangle, \langle 0,0,1\rangle)$ from the previous example. If an FNSEv should contain inputs not belonging to the interval $\langle s_2, s_1\rangle = \langle\langle 0.7,0.6,0.3\rangle, \langle 1,1,0\rangle\rangle$, then in view of Theorem 3.4, such inputs con not be large then $s_1 = \langle 1,1,0\rangle$. Hence such inputs must be less than the value $s_2 = \langle 0.7,0.6,0.3\rangle$ and some repetitions must occur, by Lemma 3.3.

The position sets for particular inputs are $P_1 = \{0\}$ for $s_1 = \langle 1,1,0\rangle, P_2 = \{3\}$ for $s_2 = \langle 0.7,0.6,0.3\rangle, P_3 = \{2,4\}$ for $s_3 = \langle 0.3,0.2,0.7\rangle, P_4 = \{1\}$ for $s_4 = \langle 0.1,0.1,0.9\rangle, P_5 = \{1\}$ for $s_5 = \langle 0,0,1\rangle$. By definition of the HCF d_r, e_r we get

$$d_1 = HCF(P_1 \cup \{n\}) = HCF(0,6) = 6 \ e_1 = 6$$
$$d_2 = HCF(P_2 \cup \{n\}) = HCF(3,6) = 3 \ e_2 = HCF(d_1,d_2) = HCF(6,3) = 3$$
$$d_3 = HCF(P_3 \cup \{n\}) = HCF(2,4,6) = 2 \ e_3 = HCF(e_2,d_3) = HCF(3,2) = 1$$
$$d_4 = HCF(P_4 \cup \{n\}) = HCF(1,6) = 1 \ e_4 = HCF(e_3,d_4) = HCF(1,1) = 1$$

Further $e_5 = 1$. By Lemma 3.3, any input $\langle x_k^T, x_k^I, x_k^F\rangle < s_r$ must be repeated in $\langle x^T, x^I, x^F\rangle$ after e_r positions. In particular, inputs less than value $s_2 = \langle 0.7,0.6,0.3\rangle$ must be repeated after 3rd positions, inputs less than $s_3 = \langle 0.3,0.2,0.7\rangle$ must be repeated on every second position. However, inputs which are not less than $s_2 = \langle 0.7,0.6,0.3\rangle$ can be arbitrary. The above conditions are satisfied e.g. by FNSV $(\langle x^T, x^I, x^F\rangle) = (\langle 0.4,0.3,0.6\rangle, \langle 0.5,0.4,0.6\rangle, \langle 0.6,0.5,0.4\rangle, \langle 0.4,0.3,0.6\rangle, \langle 0.5,0.4,0.6\rangle, \langle 0.6,0.5,0.4\rangle)^t$ which is therefore an FNSEv of A, in the view of Theorem 3.8

$$A = \begin{bmatrix} \langle 1,1,0\rangle & \langle 0.1,0.1,0.9\rangle & \langle 0.3,0.2,0.7\rangle & \langle 0.7,0.6,0.3\rangle & \langle 0.3,0.2,0.7\rangle & \langle 0,0,1\rangle \\ \langle 0,0,1\rangle & \langle 1,1,0\rangle & \langle 0.1,0.1,0.9\rangle & \langle 0.3\ 0.2\ 0.7\rangle & \langle 0.7,0.6,0.3\rangle & \langle 0.3,0.2,0.7\rangle \\ \langle 0.3,0.2,0.7\rangle & \langle 0,0,1\rangle & \langle 1,1,0\rangle & \langle 0.1,0.1,0.9\rangle & \langle 0.3,0.2,0.7\rangle & \langle 0.7,0.6,0.3\rangle \\ \langle 0.7,0.6,0.3\rangle & \langle 0.3,0.2,0.7\rangle & \langle 0,0,1\rangle & \langle 1,1,0\rangle & \langle 0.1,0.1,0.9\rangle & \langle 0.3,0.2,0.7\rangle \\ \langle 0.3,0.2,0.7\rangle & \langle 0.7,0.6,0.3\rangle & \langle 0.3,0.2,0.7\rangle & \langle 0,0,1\rangle & \langle 1,1,0\rangle & \langle 0.1,0.1,0.9\rangle \\ \langle 0.1,0.1,0.9\rangle & \langle 0.3,0.2,0.7\rangle & \langle 0.7,0.6,0.3\rangle & \langle 0.3,0.2,0.7\rangle & \langle 0,0,1\rangle & \langle 1,1,0\rangle \end{bmatrix} \otimes \begin{bmatrix} \langle 0.4,0.3,0.6\rangle \\ \langle 0.5,0.4,0.6\rangle \\ \langle 0.6,0.5,0.4\rangle \\ \langle 0.4,0.3,0.6\rangle \\ \langle 0.5,0.4,0.6\rangle \\ \langle 0.6,0.5,0.4\rangle \end{bmatrix} = \begin{bmatrix} \langle 0.4,0.3,0.6\rangle \\ \langle 0.5,0.4,0.6\rangle \\ \langle 0.6,0.5,0.4\rangle \\ \langle 0.4,0.3,0.6\rangle \\ \langle 0.5,0.4,0.6\rangle \\ \langle 0.6,0.5,0.4\rangle \end{bmatrix}.$$

We may note that if an FNSEv $(\langle x^T, x^I, x^F\rangle)$ of A should contain an input $\langle x_k^T, x_k^I, x_k^F\rangle < s_3 = \langle 0.3, 0.2, 0.7\rangle$, then such an input would be repeated after every $e_2 = 1$ position, in other words the FNSEv would have only that single input, i.e. it would be a constant FNSV.

Example 4.3. labelE3 This example illustrates Remark 3.7 by analyzing FNSEvs of the FNSM $B = B(\langle 1,1,0\rangle, \langle 0.1,0.1,0.9\rangle, \langle 1,1,0\rangle, \langle 0.7,0.6,0.3\rangle, \langle 0.3,0.2,0.7\rangle, \langle 0,0,1\rangle)$ which differs from FNSM A in a single input, namely $\langle b_3^T, b_3^I, b_3^F\rangle = \langle 1,1,0\rangle$. Thus, the maximal input of the FNSM B is placed on the diagonal position 0 and also on a non-diagonal position 3. We have $P_1 = \{0,3\}$ for $s_1 = \langle 1,1,0\rangle$ and $e_1 = d_1 = HCF(0,2,6) = 2$. Theorem 3.5 can not be applied, and the input values belonging to the interval $\langle s_2, s_1\rangle = \langle\langle 0.7,0.6,0.3\rangle, \langle 1,1,0\rangle\rangle$ must be repeated after $e_1 = 2$ positions. In fact, the same is true for all input values in the interval $\langle s_3, s_1\rangle$, because it can be easily computed that $e_1 = e_2 = 2$.

$$B = \begin{bmatrix} \langle 1,1,0\rangle & \langle 0.1,0.1,0.9\rangle & \langle 1,1,0\rangle & \langle 0.3.0.2,0.7\rangle & \langle 0.7,0.6,0.3\rangle & \langle 0,0,1\rangle \\ \langle 0,0,1\rangle & \langle 1,1,0\rangle & \langle 0.1,0.1,0.9\rangle & \langle 1,1,0\rangle & \langle 0.3\ 0.2\ 0.7\rangle & \langle 0.7,0.6,0.3\rangle \\ \langle 0.7,0.6,0.3\rangle & \langle 0,0,1\rangle & \langle 1,1,0\rangle & \langle 0.1,0.1,0.9\rangle & \langle 1,1,0\rangle & \langle 0.3,0.2,0.7\rangle \\ \langle 0.3,0.2,0.7\rangle & \langle 0.7,0.6,0.3\rangle & \langle 0,0,1\rangle & \langle 1,1,0\rangle & \langle 0.1,0.1,0.9\rangle & \langle 1,1,0\rangle \\ \langle 1,1,0\rangle & \langle 0.3,0.2,0.7\rangle & \langle 0.7,0.6,0.3\rangle & \langle 0,0,1\rangle & \langle 1,1,0\rangle & \langle 0.1,0.1,0.9\rangle \\ \langle 0.1,0.1,0.9\rangle & \langle 1,1,0\rangle & \langle 0.3,0.2,0.7\rangle & \langle 0.7,0.6,0.3\rangle & \langle 0,0,1\rangle & \langle 1,1,0\rangle \end{bmatrix} \otimes \begin{bmatrix} \langle 0.3,0.2,0.7\rangle \\ \langle 0.4,0.3,0.6\rangle \\ \langle 0.3,0.2,0.7\rangle \\ \langle 0.4,0.3,0.6\rangle \\ \langle 0.3,0.2,0.7\rangle \\ \langle 0.4,0.3,0.6\rangle \end{bmatrix} = \begin{bmatrix} \langle 0.3,0.2,0.7\rangle \\ \langle 0.4,0.3,0.6\rangle \\ \langle 0.3,0.2,0.7\rangle \\ \langle 0.4,0.3,0.6\rangle \\ \langle 0.3,0.2,0.7\rangle \\ \langle 0.4,0.3,0.6\rangle \end{bmatrix}.$$

5. Conclusion

We study the eigenspace of a circulant max-min matrix, and propose the characterization of eigenspace structure for circulant fuzzy neutrosophic soft matrix. Further examples are given for all possible types of fuzzy neutrosophic soft eigenvectors.

References

1. Arockiarani.I, Sumathi.I.R., and Martina Jency, *Fuzzy Neutrosophic Soft Topological Spaces,* IJMA,4(10) (2013), 225-238.
2. Arockiarani.I and Sumathi.I.R., *A Fuzzy Neutrosophic Soft Matrix Approach in Decision making,* JGRMA, 2(2) ,(2014), 14-23.
3. Atanassov.K, *Intuitionistic Fuzzy Sets*, Fuzzy Sets and System, 20, (1983), 87-96.
4. Broumi.S, Sahin.R, Smarandache.F., *Generalized Interval Neutrosophic Soft Set and its Decision Making Problem,* Journal of New Results in Science, 7, 2014, 29-47.
5. Cechlarova.K., *Eigenvectors in Bottleneck Algebra*, Linear Algebra and its Application, 175(15), (1992), 63-73.
6. Cuninghame-Green.R.A., *Minimax Algebra, Lecture notes in Economics and Mathematics System 166*, Springer-Verlag, Berlin, (1979).
7. Gavalec.M., *Monotone Eigenspace Structure in Max-Min Algebra*, Linear Algebra and its Application, 345(1-3), (2002), 149-167.
8. Gavalec.M, Plavka.J., *Monotone Interval Eigenproblem in Max-Min Algebra*, Kybernetika, 46(3), (2010), 387-396.
9. Deli.I, Broumi.S., *Neutrosophic Soft Sets and Neutrosophic Soft Matrices Based on Decision Making*, Journal of Intelligent and Fuzzy Systems, 28(5), (2014), 1-28.
10. Kavitha.M, Murugadas.P and Sriram.S., *Minimal Solution of Fuzzy Neutrosophic Soft Matrix*, Journal of Linear and Topological Algebra, 6, (2017), 171-189.
11. Kavitha.M, Murugadas.P and Sriram.S, *On the λ-Robustness of Fuzzy Neutrosophic Soft Matrix*, International Journal of Fuzzy Mathematical Archive, 2, (2017), 267-287.
12. Kavitha.M, Murugadas.P and Sriram.S., *Computing the Greatest X-eigenvector of Fuzzy Neutrosophic Soft Matrix*, Internation Journal of Mathamatics and its Applications, 4, (2017), 893-907.
13. Kavitha.M, Murugadas.P and Sriram.S., On the Power of Fuzzy Neutrosophic Soft Matrix, *Journal of Linear and Topological Algebra*, 7, (2018), 133-147.
14. Murugadas.P, Kavitha.M, Sriram.S., *Monotone interval fuzzy neutrosophic soft eigenproblem*, Malaya Journal of Matematik, Vol. S, No. 1, 342-350, 2019.
15. Kavitha.M, Murugadas.P and Sriram.S, *Priodicity of Interval Fuzzy Neutrosophic Soft Matrices*, Advances in Mathematics: Scientic Journal, 9, (2020), 1661-1670.
16. Kim.K.H, Roush.F.W., *Generalized Fuzzy Matrices,* Fuzzy Sets and System, 4 (3), 1980, 293-315.
17. Murugadas.P, Kavitha.M., *Solvability of System of Neutrosophic soft Linear Equations*, Neutrosophic Sets and System,vol.40,(2021), 254-269.
18. Maji.P.K., *A Neutrosophic Soft Set Approach to a Decision Making Problem*, Annals of Fuzzy Mathematics and Informatics, 3(2), (2012), 313-319.
19. Maji.P.K, Roy.A.R and Biswas.R., *On Intuitionistic Fuzzy Soft Set,* International Journal of Fuzzy Mathematics, 12(3), (2004), 669-684.
20. Manoj Bora, Banashree Bora, Tridiv Sut., *Intuitionistic Fuzzy Soft Matrix Theory and its Application in Medical Diagnosis,* Annals of Fuzzy Mathematics and Informatics, x, 2013, pp. 1-xx.

21. Molodtsov.D., *Soft Set Theory First Results,* Computer & Mathematics with Applications, 37 (4-5), 1999, 19-31.
22. Maji.P.K, Biswas.R and Ray.R., *Fuzzy Soft Set*,J. Fuzzy Math., 9, (2001), 589-602.
23. Plavka.J., *Eigenproblem for Cirulant Matrices in Max-Algebra*, Optimization 50, (2001), 477-483.
24. Plavka.J., *l-parametric Eigenproblem in Max Algebra*, Discrete Applied Mathematics, 150, (2005), 16-28.
25. Thomason M.G., *Convergence of Powers of Fuzzy Matrix,* J. of Math. Anal and Appl, 57, 1977, 476-480.
26. Smarandache.F., *Neutrosophic Set, a Generalization of the Intuitionistic Fuzzy Set,* Inter. J.Pure and Appl Math, 24 (3), 2005, 287-297.
27. Smarandache.F.,*Extension of Soft Set to Hypersoft Set, and then to Plithogenic Hypersoft Set*, Neutrosophic Sets and Systems, vol. 22, (2018), 168-170.
28. Florentin Smarandache.F.,*Extension of Soft Set to Hypersoft Set, and then to Plithogenic Hypersoft Set* (revisited), Octogon Mathematical Magazine, vol. 27(1), April (2019), 413-418.
29. Sumathi.I.R and Arokiarani.I, *New Operations on Fuzzy Neutrosophic Soft Matrices*, International Journal of Innovative Research and Studies, 3(3), (2014), 110-124.
30. Uma.R, Murugadas.P and Sriram.S, *Fuzzy Neutrosophic Soft Matrices of Type-I and Type-II*, Communicated.
31. Zimmermann.K., *Extremal Algebra (in Czech)*, Ekon.ustav CSAV, Praha 1976.
32. Zimmermann.U., *Linear and Combinatorial Optimization in Ordered Algebraic Structure*, (Ann. Discrete Math. 10), North Holland, Amsterdam 1981.
33. Zadeh.L.A, *Fuzzy Sets, Information and Control*, 8, (1965), 338-353.

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