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Introduction to the IndetermSoft Set and IndetermHyperSoft Set

Florentin Smarandache[1]

[1] University of New Mexico, Mathematics, Physics, and Natural Science Division, 705 Gurley Ave., Gallup, NM 87301, USA; smarand@unm.edu

Abstract: In this paper one introduces for the first time the *IndetermSoft Set*, as extension of the classical (determinate) Soft Set, that deals with indeterminate data, and similarly the HyperSoft Set extended to *IndetermHyperSoft Set*, where 'Indeterm' stands for 'Indeterminate' (uncertain, conflicting, not unique outcome). They are built on an *IndetermSoft Algebra* that is an algebra dealing with *IndetermSoft Operators* resulted from our real world. Afterwards, the corresponding Fuzzy / Intuitionistic Fuzzy / Neutrosophic / and other fuzzy-extension IndetermSoft Set & IndetermHyperSoft Set are presented together with their applications.

Keywords: Soft Set; HyperSoft Set; IndetermSoft Set; IndetermHyperSoft Set; IndetermSoft Operators; IndetermSoft Algebra.

1. Introduction

The classical Soft Set is based on a determinate function (whose values are certain, and unique), but in our world there are many sources that, because of lack of information or ignorance, provide indeterminate (uncertain, and not unique – but hesitant or alternative) information.

They can be modelled by operators having some degree of indeterminacy due to the imprecision of our world.

The paper recalls the definitions of the classical Soft Set and HyperSoft Set, then shows the distinction between determinate and indeterminate soft functions.

The neutrosophic triplets <*Function, NeutroFunction, AntiFunction*> and <*Operator, NeutroOperator, AntiOperator*> are brought into discussion, as parts of the <*Algebra, NeutroAlgebra, AntiAlgebra*> (Smarandache, 2019).

Similarly, distinctions between determinate and indeterminate operators are taken into consideration.

Afterwards, an IndetermSoft Algebra is built, using a determinate soft operator (*joinAND*), and three indeterminate soft operators *(disjoinOR, exclussiveOR, NOT)*, whose properties are further on studied.

IndetermSoft Algebra and IndetermHyperSoft Algebra are subclasses of the IndetermAlgebra.

The IndetermAlgebra is introduced as an algebra whose space or operators have some degree of indeterminacy ($I > 0$), and it is a subclass of the NeutroAlgebra.

It was proved that the IndetermSoft Algebra and IndetermHyperSoft Algebra are non-Boolean Algebras, since many Boolean Laws fail.

2. Definition of Classical Soft Set

Let U be a universe of discourse, H a non-empty subset of U, with $P(H)$ the powerset of H, and a an attribute, with its set of attribute values denoted by A. Then the pair (F, H), where $F: A \to P(H)$, is called a Classical Soft Set over H.

Molodtsov [1] has defined in 1999 the Soft Set, and Maji [2] the Neutrosophic Soft Set in 2013.

3. Definition of the Determinate (Classical) Soft Function

The above function $F: A \to P(H)$, where for each $x \in A$, $f(x) \in P(H)$, and f(x) is certain and unique, is called a Determinate (Classical) Function.

4. First Definition of the IndetermSoft Set

One introduces it for the first time. Let U be a universe of discourse, H a non-empty subset of U, and $P(H)$ the powerset of H. Let a be an attribute, and A be a set of this attribute values.

A function $F: A \to P(H)$ is called an *IndetermSoft Set (or Function)* if:

i. the set A has some indeterminacy;
ii. or *P(H)* has some indeterminacy;
iii. or there exist at least an attribute value $v \in A$, such that *F(v)* = indeterminate (unclear, uncertain, or not unique);
iv. or any two or all three of the above situations.

The IndetermSoft Set (or Function F) has some degree of indeterminacy, and as such it is a particular case of the NeutroFunction [6, 7], defined in 2014 – 2015, that one recalls below.

5. <Function, NeutroFunction, AntiFunction>

We have formed the above neutrosophic triplet [10, 11].

i. **(Classical) Function**, which is a function well-defined (inner-defined) for all elements in its domain of definition, or (T, I, F) = (1,0,0).
ii. **NeutroFunction** (or Neutrosophic Function), which is a function partially well-defined (degree of truth T), partially indeterminate (degree of indeterminacy I), and partially outer-defined (degree of falsehood F) on its domain of definition, where $(T, I, F) \notin \{(1,0,0), (0,0,1)\}$.
iii. **AntiFunction**, which is a function outer-defined for all the elements in its domain of definition, or (T, I, F) = (0, 0, 1).

6. Applications of the Soft Set

A detective must find the criminal(s) out of a crowd of suspects. He uses the testimonies of several witnesses.

Let the crowd of suspects be the set $S = \{s_1, s_2, s_3, s_4, s_5\} \cup \{\phi\}$, where $\{\phi\}$ is the empty (null) element, and the attribute *c* = *criminal*,

which has two attribute-values *C* = *{yes, no}*.

i. Let the function $F_1: C \to P(S)$, where $P(S)$ is the powerset of *S*, represent the information provided by the witness W_1.

For example,

$F_1(yes) = s_3$, which means that, according to the witness W_1, the suspect s_3 is the criminal,

and $F_1(no) = s_4$, which similarly means, according to the witness W_1, that the suspect s_4 is not the criminal.

These are determined (exact) information, provided by witness W_1, therefore this is a classical Soft Set.

ii. Further on, let the function $F_2 : C \to P(S)$, where $P(S)$ is the powerset of *S*, represent the information provided by the witness W_2.

For example,

$F_2(yes) = \{\phi\}$, the null-element, which means that according to the witness W_2, none of the suspects in the set *S* is the criminal. This is also a determinate information as in classical Soft Set.

7. Indeterminate Operator as Extension of the Soft Set

iii. Again, let the function $F_3 : C \to P(S)$, where $P(S)$ is the powerset of *S*, represent the information provided by the witness W_3.

This witness is not able to provide a certain and unique information, but some indeterminate (uncertain, not unique but alternative) information.

For example:

$F_3(yes) = NOT(s_2)$

and $F_3(no) = s_3$ *OR* s_4

The third source (W_3) provides indeterminate (unclear, not unique) information, since $NOT(s_2)$ means that s_2 is not the criminal, then consequently: either one, or two, or more suspects from the remaining set of suspects $\{s_1, s_3, s_4, s_5\}$ may be the criminal(s), or $\{\phi\}$ (none of the remaining suspects is the criminal), whence one has:

$C_4^1 + C_4^2 + C_4^3 + C_4^4 + 1 = 2^4 = 16$ possibilities (alternatives, or outcomes), resulted from a single input, to chose from, where C_n^m means combinations of *n* elements taken into groups of *m* elements, for integers $0 \le m \le n$.

Indeterminate information again, since:

s_3 *OR* s_4 means: either {s_3 *yes*, and s_4 *no*}, or {s_3 *no*, and s_4 *yes*}, or {s_3 *yes*, and s_4 *yes*}, therefore 3 possible (alternatives) outcomes to chose from.

Thus, $F_3 : C \to P(S)$ is an Indeterminate Soft Function (or renamed/contracted as IndetermSoft Function).

8. Indeterminate Attribute-Value Extension of the Soft Set

Let's extend the previous Applications of the Soft Set with the crowd of suspects being the set $S = \{s_1, s_2, s_3, s_4, s_5\} \cup \{\phi\}$, where $\{\phi\}$ is the empty (null) element, and the attribute *c* = *criminal*, but the attribute *c* has this time three attribute-values *K* = *{yes, no, maybe}*, as in the new branch of philosophy, called neutrosophy, where between the opposites <*A*> = *yes*, and <*antiA*> = *no*, there is the indeterminacy (or neutral) <*neutA*> = *maybe*.

And this is provided by witness W_4 and defined as:

$F_4 : K \to P(S)$

For example: $F_4(maybe) = s_5$, which means that the criminal is maybe s_5.

There also is some indeterminacy herein as well because the attribute-value "maybe" means unsure, uncertain.

One can transform this one into a Fuzzy (or Intuitionistic Fuzzy, or Neutrosophic, or other Fuzzy-Extension) Soft Sets in the following ways:

$F_4(maybe) = s_5$ is approximately equivalent to $F_4(yes) = s_5$*(some appurtenance degree)*

or

$F_4(maybe) = s_5$ is approximately equivalent to $F_4(no) = s_5$*(some non-appurtenance degree)*

Let's consider the bellow example.

Fuzzy Soft Set as:

F_4(maybe) = s_5 is approximately equivalent to *F_4(yes) = s_5(0.6)*, or the chance that s_5 be a criminal is *60%;*

Intuitionistic Fuzzy Soft Set as:

F_4(maybe) = s_5 is approximately equivalent to *F_4(yes) = s_5(0.6, 0.3)*, or the chance that *s_5* be a criminal is *60%*, and chance that *s_5* not be a criminal is *30%*;

Neutrosophic Soft Set as:

F_4(maybe) = s_5 is approximately equivalent to *F_4(yes) = s_5(0.6, 0.2, 0.3)*, or the chance that s_5 is a criminal is *60%*, indeterminate-chance of criminal-noncriminal is *20%*, and chance that *s_5* not be a criminal is *30%*.

And similarly for other *Fuzzy-Extension Soft Set*.

Or, equivalently, employing the attribute-value "no", one may consider:

Fuzzy Soft Set as:

F_4(maybe) = s_5 is approximately equivalent to *F_4(no) = s_5(0.4)*, or the chance that s_5 is not a criminal is *40%;*

Intuitionistic Fuzzy Soft Set as:

F_4(maybe) = s_5 is approximately equivalent to *F_4(no) = s_5(0.3, 0.6)*, or the chance that s_5 is not a criminal is *30%*, and chance that *s_5* is a criminal is *60%*;

Neutrosophic Soft Set as:

F_4(maybe) = s_5 is approximately equivalent to *F_4(no) = s_5(0.3, 0.2, 0.6)*, or the chance that s_5 is not a criminal is *30%*, indeterminate-chance of criminal-noncriminal is *20%*, and chance that *s_5* is a criminal is *60%*.

And similarly for other *Fuzzy-Extension Soft Set*.

9. HyperSoft Set

Smarandache has extended in 2018 the Soft Set to the HyperSoft Set [3, 4] by transforming the function *F* from a uni-attribute function into a multi-attribute function.

9.1. Definition of HyperSoft Set

Let $\mathcal{U}$ be a universe of discourse, H a non-empty set included in U, and $P(H)$ the powerset of H. Let $a_1, a_2, \dots a_n$, where $n \geq 1$, be n distinct attributes, whose corresponding attribute values are respectively the sets $A_1, A_2, \dots, A_n$, with $A_i \cap A_j = \emptyset$ for $i \neq j$, and $i, j \in \{1, 2, \dots, n\}$. Then the pair $(F, A_1 \times A_2 \times \dots \times A_n)$, where $A_1 \times A_2 \times \dots \times A_n$ represents a Cartesian product, with

$F: A_1 \times A_2 \times \dots \times A_n \to P(H)$

is called a HyperSoft Set.

For example,

let

$$(e_1, e_2, ..., e_n) \in A_1 \times A_2 \times ... \times A_n$$

then

$$F(e_1, e_2, ..., e_n) = G \in P(H).$$

9.2. Classification of HyperSoft Sets

With respect to the types of sets, such as: classical, fuzzy, intuitionistic fuzzy, neutrosophic, plithogenic, and all other fuzzy-extension sets, one respectively gets: the Crisp HyperSoft Set, Fuzzy HyperSoft Set, Intuitionistic Fuzzy HyperSoft Set, Neutrosophic HyperSoft Set, Plithogenic HyperSoft Set, and all other fuzzy-extension HyperSoft Sets [3, 5-9].

The HyperSoft degrees of T = truth, I = indeterminacy, F = falsehood, H = hesitancy, N = neutral etc. assigned to these Crisp HyperSoft Set, Fuzzy HyperSoft Set, Intuitionistic Fuzzy HyperSoft Set, Neutrosophic HyperSoft Set, Plithogenic HyperSoft Set, and all other fuzzy-extension HyperSoft Sets verify the same conditions of inclusion and inequalities as in their corresponding fuzzy and fuzzy-extension sets.

9.3. Applications of HyperSoft Set and its corresponding Fuzzy / Intuitionistic Fuzzy / Neutrosophic HyperSoft Set

Let $H = \{h_1, h_2, h_3, h_4\}$ be a set of four houses, and two attributes:
s = size, whose attribute values are *S = {small, medium, big}*,
and *l = location*, whose attribute values are *L = {central, peripherical}.*
Then $F: S \times L \to P(H)$ is a HyperSoft Set.

i. For example, *F(small, peripherical) =* $\{h_2, h_3\}$, which means that the houses that are *small and peripherical* are h_2 and h_3.

ii. A Fuzzy HyperSoft Set may assign some fuzzy degrees, for example:

F(small, peripherical) = $\{h_2(0.7), h_3(0.2)\}$, which means that with respect to the attributes' values *small and peripherical* all together, h_2 meets the requirements of being both small and peripherical in a fuzzy degree of *70%*, while h_3 in a fuzzy degree of *20%*.

iii. Further on, a Intuitionistic Fuzzy HyperSoft Set may assign some intuitionistic fuzzy degrees, for example:

F(small, peripherical) = $\{h_2(0.7, 0.1), h_3(0.2, 0.6)\}$, which means that with respect to the attributes' values *small and peripherical* all together, h_2 meets the requirements of being both small and peripherical in a intuitionistic fuzzy degree of *70%*, and does not meet it in a intuitionistic fuzzy degree of *10%*; and similarly for h_3.

iv. Further on, a Neutrosophic HyperSoft Set may assign some neutrosophic degrees, for example:

F(small, peripherical) = $\{h_2(0.7, 0.5, 0.1), h_3(0.2, 0.3, 0.6)\}$, which means that with respect to the attributes' values *small and peripherical* all together, h_2 meets the requirements of being both small and peripherical in a neutrosophic degree of *70%*, the indeterminate-requirement in a neutrosophic degree of *50%*, and does not meet the requirement in a neutrosophic degree of *10%*. And similarly, for h_3.

v. In the same fashion for other fuzzy-extension HyperSoft Sets.

10. Operator, NeutroOperator, AntiOperator

Let *U* be a universe of discourse and *H* a non-empty subset of *U*.
Let $n \geq 1$ be an integer, and ω be an operator defined as:
$\omega: H^n \to H$
Let's take a random *n-tuple* $(x_1, x_2, ..., x_n) \in H^n$.
There are three possible cases:

i. $\omega(x_1, x_2, ..., x_n) \in H$ and $\omega(x_1, x_2, ..., x_n)$ is a determinate (clear, certain, unique) output; this is called degree of well-defined (inner-defined), or degree of Truth (*T*).

ii. $\omega(x_1, x_2, ..., x_n)$ is an indeterminate (unclear, uncertain, undefined, not unique) output; this is called degree of Indeterminacy (*I*).

iii. $\omega(x_1, x_2, ..., x_n) \in U - H$; this is called degree of outer-defined (since the output is outside of *H*), or degree of Falsehood (*F*).

Consequently, one has a Neutrosophic Triplet of the form

<Operator, NeutroOperator, AntiOperator>

defined as follows [12, 13, 14]:

10.1. (Classical) Operator

For any *n-tuple* $(x_1, x_2, ..., x_n) \in H^n$, one has $\omega(x_1, x_2, ..., x_n) \in H$ and $\omega(x_1, x_2, ..., x_n)$ is a determinate (clear, certain, unique) output. Therefore *(T, I, F) = (1, 0, 0).*

10.2. NeutroOperator

There are some *n-tuples* $(x_1, x_2, ..., x_n) \in H^n$ such that $\omega(x_1, x_2, ..., x_n) \in H$ and $\omega(x_1, x_2, ..., x_n)$ are determinate (clear, certain, unique) outputs (degree of truth T);

other *n-tuples* $(y_1, y_2, ..., y_n) \in H^n$ such that $\omega(y_1, y_2, ..., y_n) \in H$ and $\omega(y_1, y_2, ..., y_n)$ are indeterminate (unclear, uncertain, not unique) output (degree of indeterminacy I);

and other *n-tuples* $(z_1, z_2, ..., z_n) \in H^n$ such that $\omega(z_1, z_2, ..., z_n) \in U - H$ (degree of falsehood F);

where $(T, I, F) \neq \{(1,0,0),(0,0,1)\}$ that represent the first (Classical Operator), and respectively the third case (AntiOperator).

10.3. AntiOperator

For any *n-tuple* $(x_1, x_2, ..., x_n) \in H^n$, one has $\omega(x_1, x_2, ..., x_n) \in U - H$. Therefore *(T, I, F)* = (0, 0, 1).

11. Particular Cases of Operators

11.1. Determinate Operator

A *Determinate Operator* is an operator whose degree of indeterminacy *I = 0*,
while the degree of truth *T = 1* and degree of falsehood *F = 0*.

Therefore, only the Classical Operator is a Determinate Operator.

11.2. IndetermOperator

As a subclass of the above NeutroOperator, there is the *IndetermOperator* (*Indeterminate Operator*), which is an operator that has some degree of indeterminacy (*I > 0).*

12. Applications of the IndetermOperators to the Soft Sets

Let *H* be a set of finite number of houses (or, in general, objects, items, etc.):

$H = \{h_1, h_2, ..., h_n\} \cup \{\emptyset\}, 1 \leq n < \infty$,
where h_1 = house1, h_2 = house2, etc.
and ∅ is the empty (or null) element (no house).

13. Determinate and Indeterminate Soft Operators

Let us define four soft operators on H.

13.1. joinAND

joinAND, or put together, denoted by ⩕, defined as:

x ⩕ y = x and y, or put together x and y; herein the conjunction "and" has the common sense from the natural language.

$x \Cap y = \{x, y\}$ is a set of two objects.

For example:

h1 ⩕ h2 = house1 ⩕ house2 = house1 and house2

= put together *house1* and *house2 = {house1, house2} =* $\{h_1, h_2\}$.

joinAND is a Determinate Soft Operator since one gets one clear (certain) output.

13.2. disjoinOR

disjoinOR, or separate in parts, denoted by ⩔, defined as:

x disjoinOR $y = x \Cup y = \{x\}$, or $\{y\}$, or both $\{x, y\}$

= x, or y, or both x and y;

herein, similarly, the disjunction "or" (and the conjunction "and" as well) have the common sense from the natural language.

But there is some indeterminacy (uncertainty) to choose among three alternatives.

For example:

$h_1 \Cup h_2$ = *house1* ⩔ *house2* = *house1*, or *house2*, or both houses together *{house1 and house2}.*

disjoinOR is an IndetermSoft Operator, since it does not have a clear unique output, but three possible alternative outputs to choose from.

13.3. exclusiveOR

exclusiveOR, meaning either one, or the other**;** it is an IndetermSoft Operator (to choose among two alternatives).

$h_1 \Cup_E h_2$ = either h_1, or h_2, and no both $\{h_1, h_2\}$.

13.4. NOT

NOT, or no, or sub-negation/sub-complement, denoted by ⫬, where

$\text{NOT}(h) = ⫬\, h = no\ h$, in other words all elements from H, except h, either single elements, or two elements, …, or $n-1$ elements from $H - \{h\}$, or the empty element $\emptyset$.

The "not" negation has the common sense from the natural language; when we say "not John" that means "someone else" or "many others".

13.4.1. Theorem 1

Let the cardinal of the set *H-{h}* be $|H - \{h\}| = m \geq 1$.

Then $NOT(h) = \{x, x \in P(H - \{h\})\}$ and the cardinal $|\text{NOT(h)}| = 2^{n-1}$.

Proof:

Because *NOT(h)* means all elements from H, except h,
either by single elements, or by two elements, …, or by $n-1$ elements from $H - \{h\}$, or the empty element $\emptyset$, then one obtains:

$C_{n-1}^1 + C_{n-1}^2 + \cdots + C_{n-1}^{n-1} + 1 = (2^{n-1} - 1) + 1 = 2^{n-1}$ possibilities (alternatives to h).

The *NOT* operator has as output a multitude of sub-negations (or sub-complements).

NOT is also an IndetermSoft Operator.

13.4.2. Example

Let $H = \{x_1, x_2, x_3, x_4\}$

Then,

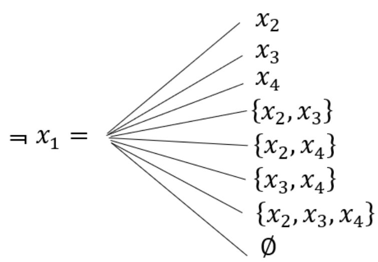
$NOT(x_1) = ⫬\, x_1$ = either x_2, or x_3, or x_4,

or $\{x_2, x_3\}$, or $\{x_2, x_4\}$, or $\{x_3, x_4\}$, or $\{x_2, x_3, x_4\}$,

or $\emptyset$;

therefore $C_3^1 + C_3^2 + C_3^3 + 1 = 3 + 3 + 1 + 1 = 8 = 2^3$ possibilities/alternatives.

Graphic representations:



Or another representation (equivalent to the above) is below:

$$\neg x_1 = \begin{cases} x_2 \\ x_3 \\ x_4 \\ \{x_2, x_3\} \\ \{x_2, x_4\} \\ \{x_3, x_4\} \\ \{x_2, x_3, x_4\} \\ \phi \end{cases}$$

The *NOT* operator is equivalent to $(2^{n-1} - 1)$ *OR* disjunctions (from the natural language).

14. Similarities between IndetermSoft Operators and Classical Operators

(i) joinAND is similar to the classical logic AND operator (∧) in the following way.

Let *A, B, C* be propositions, where $C = A \wedge B$.

Then the proposition *C* is true, if both: *A* = true, and *B* = true.

(ii) disjoinOR is also similar to the classical logic OR operator (∨) in the following way.

Let *A, B, D* be propositions, where $D = A \vee B$.

Then the proposition *D* is true if:

either *A* = true,

or *B* = true,

or both *A* = true and *B* = true

(therefore, one has three possibilities).

(iii) exclusiveOR is also similar to the classical logic exclusive OR operator ($\vee_E$) in the following way.

Let *A, B, D* be propositions, where D = A $\vee_E$ B

Then the proposition *D* is true if:

either *A* = true,

or *B* = true,

and not both *A* and *B* are true simultaneously

(therefore, one has two possibilities).

(iv) NOT resembles the classical set, or complement operator (¬), in the following way.

Let *A, B, C, D* be four sets, whose intersections two by two are empty, from the universe of discourse $\mathcal{U} = A \cup B \cup C \cup D$.

Then $\neg A = NotA = \mathcal{U} \setminus A$ = the complement of A with respect to $\mathcal{U}$.

While $\neg A$ has only one exact output ($\mathcal{U} \setminus A$) in the classical set theory, the NOT operator ⫬ A has 8 possible outcomes: either the empty set ($\emptyset$), or *B*, or *C*, or *D*, or $\{B, C\}$, or $\{B, D\}$, or $\{C, D\}$, or or $\{B, C, D\}$.

15. Properties of Operators

Let $x, y, z \in$ H(⩓,⩔, ⩔E, ⫬).

15.1. Well-Defined Operators

Let consider the set H closed under these four operators: H(⩓,⩔, ⩔E, ⫬).

Therefore, for any $x, y \in H$ one has:

x ⩓ y ∈ H(⩓,⩔, ⩔E, ⫬), because $\{x, y\} \in H$(⩓,⩔, ⩔E, ⫬),

and x ⩔ y ∈ H(⩓,⩔, ⩔E, ⫬), because each of $\{x\}$, $\{y\}$, $\{x, y\} \in H$(⩓,⩔, ⩔E, ⫬),

also x ⩔E y ∈ H(⩓,⩔, ⩔E, ⫬), because each of $\{x\}$, $\{y\} \in H$(⩓,⩔, ⩔E, ⫬),

Then the NOT operator is also well-defined because it is equivalent to a multiple of disjoinOR operators.

Thus:

⩓ : $H^2 \to H$(⩓,⩔, ⩔E, ⫬)

⩔ : $H^2 \to H$(⩓,⩔, ⩔E, ⫬)

⩔E: $H^2 \to H$(⩓,⩔, ⩔E, ⫬)

⫬ : $H \to H$(⩓,⩔, ⩔E, ⫬)

15.2. Commutativity

x ⩓ y = y ⩓ x, and x ⩔ y = y ⩔ x, and x ⩔E y = y ⩔E x

Proof

x ⩓ y = $\{x, y\} = \{y, x\}$ = y ⩓ x

x ⩔ y = ($\{x\}$, or $\{y\}$, or $\{x, y\}$) = ($\{y\}$ or $\{x\}$, or $\{y, x\}$) = y ⩔ x

x ⩔E y = (either $\{x\}$, or $\{y\}$, but not both x and y) =

= (either $\{y\}$, or $\{x\}$, but not both y and x) = y ⩔E x.

15.3. Associativity

x ⩓ (y ⩓ z) = (x ⩓ y) ⩓ z,

and x ⩔ (y ⩔ z) = (x ⩔ y) ⩔ z, and x ⩔E (y ⩔E z = (x ⩔E y) ⩔E z

Proof

x ⩓ (y ⩓ z) = $\{x, y$ ⩓ $z\}$ = $\{x, \{y, z\}\}$

= $\{x, y, z\} = \{\{x, y\}, z\}$

= (x ⩓ y) ⩓ z.

x ⩔ (y ⩔ z) = (x ⩔ y) ⩔ z

x or (y or z) = x or $\begin{cases} y \\ z \\ y \text{ or } z \end{cases}$ = x or $\begin{cases} y \\ z \\ y \text{ or } z \begin{cases} y \\ z \\ \{y, z\} \end{cases} \end{cases}$

$$x \text{ or } y = \begin{cases} x \\ y \\ \{x, y\} \end{cases}$$

$$= \begin{cases} x \text{ or } z = \begin{cases} x \\ z \\ \{x, z\} \end{cases} \\ x \text{ or } y \\ x \text{ or } z \\ x \text{ or } \{y, z\} = \begin{cases} x \\ \{y, z\} \\ \{x, y, z\} \end{cases} \end{cases}$$

$$= x, y, z, \{x, y\}, \{x, z\}, \{y, z\}, \{x, y, z\}.$$

$$(x \text{ or } y) \text{ or } z = \begin{cases} x \\ y \\ \{x, y\} \end{cases} \text{ or } z = \begin{cases} x \text{ or } z \begin{cases} x \\ z \\ \{x, z\} \end{cases} \\ y \text{ or } z \begin{cases} y \\ z \\ \{y, z\} \end{cases} \\ \{x, z\} \text{ or } z \begin{cases} \{x, y\} \\ z \\ \{x, y, z\} \end{cases} \end{cases}$$

$$= x, y, z, \{x, y\}, \{x, z\}, \{y, z\}, \{x, y, z\}.$$

Therefore, *(x or y) or z = x or (y or z) = x, y, z, {x, y}, {y, z}, {z, x}, {x, y, z}* with $2^3 - 1 = 8 - 1 = 7$ possibilities.

x $⩔_E$ (y $⩔_E$ z) =

either x, or (y ⩔E z), and no both x and (y ⩔E z) = either x, or (y, or z, and no both y and z), and no both x and (either y or z) = either x, or y, or z, and no both {y, z}, and (no x and no (either y or z)) = either x, or y, or z, and no {y, z}, no {x, y}, no {x, z} = (x ⩔E y) ⩔E z

15.4. Distributivity of joinAND over disjoinOR and exclusiveOR

$x ⩓ (y ⩔ z) = (x ⩓ y) ⩔ (x ⩓ z)$

P roof

$x ⩓ (y ⩔ z) = x$ and $(y$ or $z) = x$ and $(y,$ or $z,$ or $\{y, z\})$

= x and y, or x and z, or x and $\{y, z\}$

= $\{x, y\}$, or $\{x, z\}$, or $\{x, y, z\}$

= $\{z, y\}$, $\{x, z\}$, $\{x, y, z\}$.

$(x ⩓ y) ⩔ (x ⩓ z) = \{x, y\}$

or $\{x, z\} = \{x, y\}, \{x, z\}, \{x, y, x, z\} = \{x, y\}, \{x, z\}, \{x, y, z\}$.

x ⩓ (y ⩔E z) = x and (either y, or z, and no both {y, z}) = either x and y, or x and z, and x and no both {y, z} = either {x, y}, or {x, z}, and {x, no { y, z} } =

= either {x, y}, or {x, z}, and no {x, y, z} = (x ⩓ y) ⩔E (x ⩓ z)

15.5. No distributivity of disjoinOR and exclusiveOR over joinAND

$x ⩔ (y ⩓ z) \neq (x ⩔ y) ⩓ (x ⩔ z)$

$x ⩔ (y ⩓ z) = x$ or $(y$ and $z) = x$ *or* $\{y, z\} = x, \{y, z\}, \{x, y, z\}$

But

$(x ⩔ y) ⩓ (x ⩔ z) = (x, y, \{x, y\})$ and $(x, z, \{x, z\})$

= $\{x, x\}$, $\{x, z\}$, $\{x, z\}$, $\{y, z\}$, $\{y, x\}$, $\{x, y, z\}$, $\{x, y\}$, $\{x, y, z\}$, $\{x, y, z\}$

= x, $\{x, y\}$, $\{x, z\}$, $\{y, z\}$, $\{x, y, z\}$.

Whence in general $x, \{y, z\}, \{x, y, z\} \neq x, \{x, y\}, \{x, z\}, \{y, z\}, \{x, y, z\}$.

While in classical Boolean Algebra the distribution of <u>or</u> over <u>and</u> is valid:

$$x \vee (y \wedge z) = (x \vee y) \wedge (x \vee z).$$

x $\mathbb{V}_E$ *(y* $\mathbb{A}$ *z) = either x, or {y, z}, and no {x, y, z}* $\neq$

$\neq$ *(x* $\mathbb{V}_E$ *y)* $\mathbb{A}$ *(x* $\mathbb{V}_E$ *y) = (either x, or y, and no {x, y}) and (either x, or z, and no {x, z})*

15.6. Idempotence

$x \mathbb{A} x = \{x, x\} = x$

$x \mathbb{V} x =$ either x, or x, or $\{x, x\}$

$= x$, or x, or x

$= x$.

$x \mathbb{V}_E x$ = either x, or x, and no $\{x, x\}$ = impossible.

15.6.1. Theorem 2

Let $x_1, x_2, \dots, x_n \in (H, \mathbb{A}, \mathbb{V}, \Rightarrow)$, for $n \geq 2$. Then:

(i) $x_1 \mathbb{A}\ x_2 \mathbb{A} \dots \mathbb{A} x_n = \{x_1, x_2, \dots, x_n\}$,

and

(ii) $x_1 \mathbb{V} x_2 \mathbb{V} \dots \mathbb{V} x_n = x_1, x_2, \dots, x_n$,

$\{x_1, x_2\}, \{x_1, x_3\}, \dots, \{x_{n-1}, x_n\}$,

$\{x_1, x_2, x_3\}, \dots$

$\dots \dots \dots \dots \dots \dots$

$\{x_1, x_2, \dots, x_{n-1}\}, \dots$

$\{x_1, x_2, \dots, x_{n-1}, x_n\}$.

There are: $C_n^1 + C_n^2 + \cdots + C_n^{n-1} + C_n^n = 2^n - 1$ possibilities/alternatives.

The bigger is *n*, the bigger the indeterminacy.

(iii) $x_1 \mathbb{V}_E\ x_2\ \mathbb{V}_E \dots \mathbb{V}_E x_n = x_1, x_2, \dots, x_n =$

= either x_1*, or* x_2*, …, or* x_n*,*

and no two or more variables be true simultaneously.

There are: $C_n^1 = n$ possibilities.

The bigger is *n*, the bigger the indeterminacy due to many alternatives.

Proof

(i) The joinAND equality is obvious.

(ii) The disjoinOR outputs from the fact that for the disjunction of n proposition to be true, it is enough to have at least one which is true. As such, we may have only one proposition true, or only two propositions true, and so on, only $n - 1$ propositions true, up to all n propositions true.

(iii) It is obvious.

15.7. The classical Boolean Absorption Law1

$x \wedge (x \vee y) = x$ does not work in this structure, since $x \mathbb{A} (x \mathbb{V} y) \neq x$.

Proof

$$\begin{aligned} x \mathbb{A} (x \mathbb{V} y) &= x \text{ and } (x \text{ or } y) \\ &= x \text{ and } \begin{cases} x \\ y \\ \{x, y\} \end{cases} \\ &= \{x, x\} \text{ or } \{x, y\} \text{ or } \{x, x, y\} \\ &= x \text{ or } \{x, y\} \text{ or } \{x, y\} \\ &= x \text{ or } \{x, y\} \end{aligned}$$

$$= \begin{cases} x \\ \{x, y\} \\ \{x, x, y\} \end{cases} = \begin{cases} x \\ \{x, y\} \\ \{x, y\} \end{cases} = \begin{cases} x \\ \{x, y\} \end{cases} \neq x.$$

But this one work:

$x \curlywedge$ (x $\curlyvee_E$ y) = *x and (either x, or y, and no {x, y}) =*

= (x and x), or (x and y), and (x and no{x, y}) = x.

15.8. The classical Boolean Absorption Law2

$x \vee (x \wedge y) = x$ does not work in this structure, since $x \curlyvee (x \curlywedge y) \neq x$.

Proof

$x \curlywedge (x \curlyvee y) = x$ and (x or y)

x or (x and y) = x or $\{x, y\}$

$$= \begin{cases} x \\ \{x, y\} \\ \{x, x, y\} \end{cases} = \begin{cases} x \\ \{x, y\} \\ \{x, y\} \end{cases}$$

$$= \begin{cases} x \\ \{x, y\} \end{cases} \neq x.$$

But this one work:

x $\curlyvee_E$ *(x* $\curlywedge$ y *) =* (either x), or {x, y}, and (no {x, y}) = x.

15.9. Annihilators and Identities for IndetermSoft Algebra

While 0 is an annihilator for conjunction $\wedge$ in the classical Boolean Algebra, $x \wedge 0 = 0$, in IndetermSoft Algebra $\emptyset$ is an identity for $\curlywedge$, while for the others it does not work.

Proof

$x \curlywedge \emptyset = x$ and $\emptyset$

= x and nothing

= x put together with nothing

= x.

15.10. $\emptyset$ is neither an identity, nor an annihilator for disjoinOR nor for exclusiveOR

While 0 is an identity for the $\vee$ in the classical Boolean Algebra, $x \vee 0 = x$ in IndetermAlgebra $\emptyset$ is neither an identity, nor an annihilator.

Proof

$x \curlyvee \emptyset = x$, or $\emptyset$ (nothing), or $\{x, \emptyset\}$

= x, or $\emptyset$, or x

= x, or $\emptyset$.

x $\curlyvee_E$ $\emptyset$ = either x, or $\emptyset$, and no {x, $\emptyset$}.

15.11. The negation of $\emptyset$ has multiple solutions

While in the classical Boolean Algebra the negation of 0 is 1 (one solution only), $\neg 0 = 1$, in IndetermAlgebra the negation of $\emptyset$ has multiple solutions.

Proof

$⫬ \emptyset = NOT(\emptyset)$,

= not nothing

= one or more elements from the set that the operator ⫬ is defined upon.

Example

Let $H = \{x_1, x_2, x_3\} \cup \emptyset$.

Then, $⫬ \emptyset = x_1$, or x_2, or x_3, or $\{x_1, x_2\}$, or $\{x_1, x_3\}$, or $\{x_2, x_3\}$, or $\{x_1, x_2, x_3\}$, therefore 7 alternative solutions.

15.12. The Double Negation is invalid on IndetermSoft Algebra

While in the classical Boolean Algebra the Double Negation Law is valid: $\neg(\neg x) = x$, in IndetermAlgebra it is not true:

In general, $⫬ (⫬ x) \neq x$.

Proof

A counter-example:

Let $H = \{x_1, x_2, x_3\} \cup \emptyset$.

$⫬ x_1$ = what is not x_1 or does not contain x_1

= x_2, x_3, $\{x_2, x_3\}$, $\emptyset$.

Thus one has 4 different values of the negation of x_1.

Let us choose $⫬ x_1 = x_2$; then $⫬ (⫬ x_1) = x_2 = (x_1, x_3, \{x_1, x_3\}, \emptyset) \neq x_1$.

Similarly for taking other values of $⫬ x_1$.

Let $H = \{x_1, x_2, \dots, x_n\} \cup \emptyset$, $n \geq 2$. Let $x \in H$.

Minimum and Maximum elements with respect to the relation of inclusion are:

$\emptyset$ = the empty (null) element

and respectively

$$x_1 \Cap x_2 \Cap \dots \Cap x_n \overset{notation}{=} \{x_1, x_2, \dots, x_n\} = H,$$

but in the Boolean Algebra they are 0 and 1 respectively.

15.13. The whole set H is an annihilator for joinAND

While in the classical Boolean Algebra the identity for $\wedge$ is 1, since $x \wedge 1 = x$, in the IndetermSoft Algebra for $\Cap$ there is an annihilator H, since $x \Cap H = H$, since $x \Cap H = \{x_1, x_2, \dots, x_n, x\} = H$, because $x \in H$ so x is one of $x_1, x_2, \dots, x_n$.

16. The maximum *(H)* is neither annihilator nor identity

While in the classical Boolean Algebra the annihilator for $\vee$ is 1, because $x \vee 1 = 1$, in the IndetermSoft Algebra for $\Cup$ the maximum H is neither annihilator nor identity,

$x \Cup H = x$ or $H = x$, H, $\{x, H\} = x, H, H = x, H$.

$x \Cup_E H$ = either x, or H, and (no x and no H).

17. Complementation1

In the classical Boolean Algebra, Complementation1 is: $x \wedge \neg x = 0$.

In the IndetermSoft Algebra, $x \Cap (⫬ x) \neq \emptyset$, and $x \Cap (⫬ x) \neq H$.

Counter-Example

$M = \{x_1, x_2, x_3\} \cup \emptyset$

$⫬ x_1 = x_2, x_3, \{x_2, x_3\}, \emptyset$

$x_1 \Cap (⫬ x_1) = x_1 \Cap (x_2, x_3, \{x_2, x_3\}, \emptyset) =$

= (x_1 and x_2) or (x_1 or x_3)

or (x_1 and $\{x_2, x_3\}$)

or (x_1 and $\emptyset$) =

$= (x_1, \{x_1, x_2\}, \{x_1, x_3\}, \{x_1, x_2, x_3\}) \neq \emptyset \neq M$.

18. Complementation2

In the classical Boolean Algebra, Complementation2 is: $x \vee \neg x = 1$.

In the IndetermSoft Algebra, $x \Vvee \eqcirc x \neq H$, and $x \Vvee \eqcirc x \neq \emptyset$.

Counter-Example

The above $H = \{x_1, x_2, x_3\} \cup \emptyset$

and $\eqcirc x_1 = x_2, x_3, \{x_2, x_3\}, \emptyset$, then

$$x_1 \Vvee \eqcirc x_1 = x_1 \Vvee (x_2, x_3, \{x_2, x_3\}, \emptyset) = \begin{cases} x_1 \\ x_2, x_3, \{x_2, x_3\}, \emptyset \\ x_1, x_2, x_3, \{x_2, x_3\}, \emptyset \end{cases}$$

$$= x_1, \text{or } (x_2, x_3, \{x_2, x_3\}, \emptyset), \text{or } (x_1, x_2, x_3, \{x_2, x_3\}, \emptyset)$$

which is different from H and from $\emptyset$.

And:

$$x_1 \Vvee_E \eqcirc x_1 = x_1 \Vvee_E (x_2, x_3, \{x_2, x_3\}, \emptyset) = \begin{Bmatrix} x_1 \\ x_2, x_3, \{x_2, x_3\}, \phi \end{Bmatrix} \text{ and no } (x_1, x_2, x_3, \{x_2, x_3\}, \phi),$$

which is different from H and from $\emptyset$.

19. De Morgan Law1 in the IndetermSoft Algebra

De Morgan Law1 from Classical Boolean Algebra is:

$\neg(x \vee y) = (\neg x) \wedge (\neg y)$

is also true in the IndetermSoft Algebra:

$\eqcirc (x \Vvee y) = (\eqcirc x) \Wedge (\eqcirc y)$

Proof

$\eqcirc (x \Vvee y) = \eqcirc (x, \text{or } y, \text{or } \{x \text{ and } y\})$

$= \eqcirc x, \text{and } \eqcirc y, \text{and } \eqcirc \{x_1 \text{ and y}\}$

$= \eqcirc x_1, \text{and } \eqcirc y, \text{and } (\eqcirc x, \text{or } \eqcirc \text{y})$

$= \eqcirc x, \text{and } \eqcirc y$

$= (\eqcirc x) \Wedge (\eqcirc y)$.

Example

$M = \{x_1, x_2, x_3\} \cup \emptyset$

$\eqcirc (x_1 \Vvee x_2) \quad = \eqcirc (x_1, \text{or } x_2, \text{or } \{x_1 \text{ and } x_2\})$

$= \eqcirc x_1, \text{and } \eqcirc x_2, \text{and } (\eqcirc x_1 \text{ or } \eqcirc x_2)$

$= \eqcirc x_1, \text{and } \eqcirc x_2$

$= (\eqcirc x_1) \Wedge (\eqcirc x_2)$.

$\eqcirc x_1 = (x_2, x_3, \{x_2, x_3\}, \emptyset)$

$\eqcirc x_2 = (x_1, x_3, \{x_1, x_3\}, \emptyset)$

$(\eqcirc x_1) \Wedge (\eqcirc x_2) = (x_2, x_3, \{x_2, x_3\}, \emptyset) \Wedge (x_1, x_3, \{x_1, x_3\}, \emptyset)$

$= x_1, x_2, x_3, \{x_1, x_3\}, \{x_2, x_3\}, \emptyset$.

20. De Morgan Law2 in the IndetermSoft Algebra

De Morgan Law2 in the classical Boolean Algebra is

$\neg(x \wedge y) = (\neg x) \vee (\neg y)$

is also true in the new structure called IndetermSoft Algebra:

$\eqcirc (x \Wedge y) = (\eqcirc x) \Vvee (\eqcirc y)$

Proof

$\eqcirc (x \Wedge y) = \eqcirc (\{x \text{ and } y\}) = \eqcirc x, \text{or } \eqcirc y, or \{\eqcirc x, and \eqcirc y\} = (\eqcirc x) \Vvee (\eqcirc y)$

Example

$\eqcirc (x_1 \Wedge x_2) \quad = \eqcirc (\{x_1, x_2\})$

$= (\eqcirc x_1, or \eqcirc x_2, \text{or } (\eqcirc x_1 \text{ and } \eqcirc x_2))$

$= (x_2, x_3, \{x_2, x_3\}, \emptyset) =$

or $(x_2, x_3, \{x_2, x_3\}, \emptyset)$
or $(x_1, x_3, \{x_1, x_3\}, \emptyset)$
or $(x_1, x_2, x_3, \{x_1, x_3\}, \{x_2, x_3\}, \emptyset) =$
$= (x_1, x_2, x_3, \{x_1, x_3\}, \{x_2, x_3\}, \emptyset)$

$(\neg x_1) \vee (\neg x_2) = \neg x_1$, or $\neg x_2$, or $(\neg x_1 \wedge \neg x_2) =$
$(x_2, x_3, \{x_2, x_3\}, \emptyset)$
or $(x_1, x_3, \{x_1, x_3\}, \emptyset)$
or $(x_1, x_2, x_3, \{x_1, x_3\}, \{x_2, x_3\}, \emptyset)$
$= (x_1, x_2, x_3, \{x_1, x_3\}, \{x_2, x_3\}, \emptyset)$
$= \neg (x_1 \wedge x_2)$

*

This IndetermSoft Algebra is not a Boolean Algebra because many of Boolean Laws do not work, such as:

- Identity for ⩓
- Identity for ⩔
- Identity for ⩔E
- Annihilator for ⩓
- Annihilator for ⩔
- Annihilator for ⩔E
- Absorption1 [x ⩓ (x ⩔ y) = x]
- Absorption2 [x ⩔ (x ⩓ y) = x]
- Double Negation
- Complementation1 [x ⩓ ⫬ $x = \emptyset$]
- Complementation2 { [x ⩔ ⫬ $x = H$] and [x ⩔E ⫬ $x = H$] }

21. Practical Applications of Soft Set and IndetermSoft Set

Let $H = \{h_1, h_2, h_3, h_4\}$ a set of four houses, and the attribute *a = color*, whose values are *A = {white, green, blue, red}*.

21.1. Soft Set

The function
$F: A \to \mathcal{P}(H)$
where $\mathcal{P}(H)$ is the powerset of H,
is called a classical Soft Set.

For example,
F (white) = h_3, i.e. the house h_3 is painted white;
F (green) = $\{h_1, h_2\}$, i.e. both houses h_1 and h_2 are painted green;
F (blue) = h_4, i.e. the house h_4 is painted blue;
F (red) = $\emptyset$, i.e. no house is painted red.
Therefore, the information about the houses' colors is well-known, certain.

21.2. IndetermSoft Set

But there are many cases in our real life when the information about the attributes' values of the objects (or items – in general) is unclear, uncertain.

That is why we need to extend the classical (Determinate) Soft Set to an Indeterminate Soft Set.

The determinate (exact) soft function
$F: A \to \mathcal{P}(H)$

is extended to an indeterminate soft function

$F: A \rightarrow H(⩕, ⩖, ⩖_E, ⫬)$,

where (⩕, ⩖, ⩖E, ⫬) is a set closed under ⩕, ⩖, ⩖E, and ⫬, and *f(x)* is not always determinate.

For example,

F (white) = h_3 ⩖ h_4,

means the houses h_3 or h_4 are white, but we are not sure which one,
whence one has three possibilities/outcomes/alternatives:

either h_3 is white (and h_4 is not),
or h_4 is white (and h_3 is not),
or both h_3 and h_4 are white.

This is an indeterminate information.

We may also simply write:

$$F\ (white) = \begin{cases} h_3 \\ h_4 \\ \{h_3, h_4\} \end{cases}$$

or *F(white)* = h_3, h_4, $\{h_3, h_4\}$,
where $\{h_3, h_4\}$ means $\{h_3$ and $h_4\}$,
that we read as: either h_3, or h_4, or $\{h_3$ and $h_4\}$.

Another example:

F(blue) = ⫬ h_2, or the house h_2 is not blue,
therefore other houses amongst $\{h_1, h_3, h_4\}$ may be blue,
or no house (∅) may be blue.

This is another indeterminate information.

The negation of h_2 (denoted as NOT(h_2) = ⫬ h_2) is not equal to the classical complement of $C(h_2)$ of the element h_2 with respect to the set H, since

$C(h_2) = H \setminus \{h_2\} = \{h_1, h_3, h_4\}$,

but ⫬ h_2 may be any subset of $H \setminus \{h_2\}$, or any sub-complement of $C(h_2)$,

again many (in this example 8) possible outcomes to choose from:

⫬ h_2 = h_1, h_3, h_4, $\{h_1, h_3\}$, $\{h_1, h_4\}$, $\{h_3, h_4\}$, $\{h_1, h_3, h_4\}$, ∅ =
= either h_1, or h_3, or h_4,
or $\{h_1$ and $h_3\}$, $\{h_1$ and $h_4\}$, $\{h_3$ and $h_4\}$,
or $\{h_1$ and h_3 and $h_4\}$,
or ∅ (null element, i.e. no other house is blue).

The negation (⫬ h_2) produces a higher degree of indeterminacy than the previous unions: (h_3 ⩖ h_4) and respectively (h_3 ⩖E h_4).

The intersection (⩕) is a determinate (certain) operator.

For example,

F (green) = h_1 ⩕ h_2, which is equal to $\{h_1, h_2\}$, i.e. h_1 and h_2 put together, $\{h_1$ and $h_2\}$.

A combination of these operators may occur, so the indeterminate (uncertain) soft function becomes more complex.

Example again.

F (green) = h_1 ⩕ (⫬ h_4), where of course ⫬ $h_4 \neq h_1$, which means that:
the house h_1 is green,
and other houses amongst $\{h_2, h_3\}$ may be blue,
or ∅ (no other house is blue).

h_1 ⩕ (⫬ h_4) = h_1 and (NOTh_4)
= h_1 and ($h_1, h_2, h_3, \{h_1, h_2\}, \{h_1, h_3\}, \{h_2, h_3\}, \{h_1, h_2, h_3\}, \emptyset$)
[one cuts h_1 since ⫬ h_4 suppose to be different from h_1]

$= h_1$ and $(h_2, h_3, \{h_2, h_3\}, \emptyset)$
$= (h_1$ and $h_2)$ or $(h_1$ and $h_3)$
or $(h_1$ and $\{h_2, h_3\})$
or $\emptyset$
$= (h_1$ and $h_2)$ or $(h_1$ and $h_3)$ or $(h_1$ and h_2 and $h_3)$ or $\emptyset$
$\overset{notation}{=} \{h_1, h_2\}, \{h_1, h_3\}, \{h_1, h_2, h_3\}, \emptyset.$

Thus, 4 possibilities.

22. Definitions of <Algebra, NeutroAlgebra, AntiAlgebra>

Let $\mathcal{U}$ be a universe of discourse, and H a non-empty set included in $\mathcal{U}$. Also, H is endowed with some operations and axioms.

22.1. Algebra

An algebraic structure whose all operations are well-defined, and all axioms are totally true, is called a classical Algebraic Structure (or **Algebra**). Whence *(T, I, F) = (1, 0, 0).*

22.2. NeutroAlgebra

If at least one operation or one axiom has some degree of truth (*T*), some degree of indeterminacy (*I*), and some degree of falsehood (*F*), where *(T, I, F)* $\notin \{(1,0,0),(0,0,1)\}$, and no other operation or axiom is totally false (*F = 1*), then this is called a NeutroAlgebra.

22.3. AntiAlgebra

An algebraic structure that has at least one operation that is totally outer-defined (F = 1) or at least one axiom that is totally false (F = 0), is called AntiAlgebra.

23. Definition of IndetermAlgebra

We introduce now for the first time the concept of IntermAlgebra (= Indeterminate Algebra), as a subclass of NeutroAlgebra.

IndetermAlgebra results from real applications, as it will be seen further.

Let $\mathcal{U}$ be a universe of discourse, and H a non-empty set included in $\mathcal{U}$.

If at least one operation or one axiom has some degree of indeterminacy ($I > 0$), the degree of falsehood $F = 0$, and all other operations and axioms are totally true, then H is an IndetermAlgebra.

24. Definition of IndetermSoft Algebra

The set H(⩓, ⩔, ⩔$_E$, ⫬) closed under the following operators:

joinAND (denoted by ⩓), which is a determinate operator;

disjoinOR (denoted by ⩔), which is an indeterminate operator;

exclusiveOR (denoted by ⩔$_E$), which is an indeterminate operator,

and sub-negation/sub-complement *NOT* (denoted by ⫬), which is an indeterminate operator;

is called an IndetermSoft Algebra.

The IndetermSoft Algebra extends the classical Soft Set Algebra.

The IndetermSoft Algebra is a particular case of the IndetermAlgebra, and of the NeutroAlgebra.

The operator *joinAND*

⩓: $H^2 \longrightarrow H$(⩓, ⩔, ⩔$_E$, ⫬)

is <u>determinate</u> (in the classical sense):

$\forall x, y \in H, x \neq y, x \Cap y = x$ joinAND $y = \{x, y\} \in H(\Cap, \Cup, \Cup_E, \lnot)$

therefore, the aggregation of x and y by using the operator ⩕ gives a clear and unique output, i.e. the classical set of two elements: $\{x, y\}$.

But the operator *disjoinOR*

$\Cup: H^2 \longrightarrow H(\Cap, \Cup, \Cup_E, \lnot)$

is <u>indeterminate</u> because:

$$\forall x, y \in H, x \neq y, x \Cup y = x \text{ disjoinOR } y = \begin{cases} \text{either } x \\ \text{or } y \\ \text{or both } \{x \text{ and } y\} \end{cases} = \begin{cases} x \\ y \\ \{x, y\} \end{cases}$$

Thus, the aggregation of x and y by using the operator ⩔ gives an unclear output, with three possible alternative solutions (either x, or y, or $\{x$ and $y\}$).

The exclusiveOR operator is also <u>indeterminate</u>:

$\forall x, y \in H, x \neq y$, x $\Cup_E$ y $= x$ exclusiveOR $y =$ either x, or y, and no {x, y},

therefore two possible solutions:

$\Cup_E: H^2 \longrightarrow H(\Cap, \Cup, \Cup_E, \lnot)$.

Similarly, the operator sub-negation / sub-complement NOT

$\lnot: H \longrightarrow H(\Cap, \Cup, \Cup_E, \lnot)$

is <u>indeterminate</u> because of many elements $x \in H$,

NOT(x) = $\lnot x$ = a part of the complement of x with respect to *H*

= a subset of $H \setminus \{x\}$.

But there are many subsets of $H \setminus \{x\}$, therefore there is an unclear (uncertain, ambiguous) output, with multiple possible alternative solutions.

25. Second Definition of IndetermSoft Set

Let U be a universe of discourse, H a non-empty subset of U, and $H(\Cap, \Cup, \Cup_E, \lnot)$ the IndetermSoft Algebra generated by closing the set H under the operators $\Cap$, $\Cup$, $\Cup_E$, and $\lnot$.

Let *a* be an attribute, with its set of attribute values denoted by A. Then the pair

(F, A), where $F: A \to H(\Cap, \Cup, \Cup_E, \lnot)$, is called an IndetermSoft Set over H.

26. Fuzzy / Intuitionistic Fuzzy / Neutrosophic / and other fuzzy-extension / IndetermSoft Set

One may associate fuzzy / intuitionistic fuzzy / neutrosophic etc. degrees and extend the IndetermSoft Set to some **Fuzzy / Intuitionistic Fuzzy / Neutrosophic / and other fuzzy-extension / IndetermSoft Set**.

26.1. Applications of (Fuzzy/ Intuitionistic Fuzzy / Neutrosophic / and other fuzzy-extension) IndetermSoft Set

Let $H = \{h_1, h_2, h_3, h_4\}$ be a set of four houses, and the IndetermSoft Algebra generated by closing the set H under the previous soft operators, $H(\Cap, \Cup, \Cup_E, \lnot)$.

Let the attribute *c* = *color*, and its attribute values be the set *C* = *{white, green, blue}*.

The IndetermSoft Function $F : A \longrightarrow H(\Cap, \Cup, \Cup_E, \lnot)$ forms an IndetermSoft Set.

Let an element $h \in H$, and one denotes by:

$d^{\circ}(h)$ = any type of degree (either fuzzy, or intuitionistic fuzzy, or neutrosophic, or any other fuzzy-extension) of the element *h*.

We extend the soft operators $\Cap$, $\Cup$, $\Cup_E$, $\lnot$ by assigning some degree $d^0(.) \in [0,1]^p$, where:

$p = 1$ for classical and fuzzy degree, $p = 2$ for intuitionistic fuzzy degree, $p = 3$ for neutrosophic degree, and so on $p = n$ for n-valued refined neutrosophic degree, to the elements involved in the operators, where $\wedge, \vee, \neg$ represent the conjunction, disjunction, and negation respectively of these degrees in their corresponding fuzzy-extension sets or logics.

For examples:

i. From *F(white)* = $h_1 \Cap h_2$ as in IndetermSoft Set, one extends to:

F(white) = $h_1(d_1^\circ) \Cap h_2(d_2^\circ)$, which means the degree (chance) that h_1 be white is d_1° and the degree (chance) that h_2 be white is d_2°, whence:

F(white) = $h_1(d_1^\circ) \Cap h_2(d_2^\circ) = \{h_1, h_2\}(d_1^\circ \wedge d_2^\circ)$

As such, the degree of both houses $\{h_1, h_2\} = \{h_1 \text{ and } h_2\}$ be white is $d_1^\circ \wedge d_2^\circ$.

ii. Similarly, *F(white)* = $h_1(d_1^\circ) \Cup h_2(d_2^\circ) = \{h_1 \text{ or } h_2\}(d_1^\circ \vee d_2^\circ)$,

or the degree of at least one house $\{h_1 \text{ or } h_2\}$ be white is $(d_1^\circ \vee d_2^\circ)$.

iii. *F(white)* = $h_1(d_1^\circ) \Cup_E h_2(d_2^\circ)$ =

= { h_1 *and (no* h_2*)}, or { (no* h_1*) and* h_2 *}, and { (no* h_1*) and (no* h_2*) }*

= (*either* h_1 *is white, or* h_2 *is white, and [no both* $\{h_1, h_2\}$*] are white simultaneously*) has the degree of $(d_1^\circ \vee d_2^\circ) - (d_1^\circ \wedge d_2^\circ)$.

iv. *F(white)* = $(\mathrel{\text{⫤}}h_1)(d_1^\circ)$, which means that the degree (chance) for h_1 not to be white is d_1°.

($\mathrel{\text{⫤}}h_1$ = *NOT*(h_1) = *either* h_2*, or* h_3*, or* h_4,
or $\{h_2, h_3\}$, $\{h_2, h_4\}$, $\{h_3, h_4\}$,
or $\{h_2, h_3, h_4\}$,
or ϕ *(no house).*

There are 8 alternatives, thus NOT(h_1*) is one of them.*

Let's assume that NOT(h_1*) =* $\{h_3, h_4\}$*. Then the degree of both houses* $\{h_3, h_4\}$ *be white is* $\neg d_1^\circ$.

27. Definition of IndetermHyperSoft Set

Let U be a universe of discourse, H a non-empty subset of U, and H(⩕, ⩖, ⩖E, ⫤) the IndetermSoft Algebra generated by closing the set H under the operators ⩕, ⩖, ⩖E, and ⫤.

Let $a_1, a_2, \ldots a_n$, where $n \geq 1$, be n distinct attributes, whose corresponding attribute values are respectively the sets $A_1, A_2, \ldots, A_n$, with $A_i \cap A_j = \emptyset$ for $i \neq j$, and $i, j \in \{1, 2, \ldots, n\}$. Then the pair $(F, A_1 \times A_2 \times \ldots \times A_n)$, where $A_1 \times A_2 \times \ldots \times A_n$ represents a Cartesian product, with

$F: A_1 \times A_2 \times \ldots \times A_n \to H$(⩕, ⩖, ⩖E, ⫤), is called an IndetermHyperSoft Set.

Similarly, one may associate fuzzy / intuitionistic fuzzy / neutrosophic etc. degrees and extend the IndetermHyperSoft Set to some *Fuzzy / Intuitionistic Fuzzy / Neutrosophic etc. IndetermHyperSoft Set.*

28. Applications of the IndetermHyperSoft Set

Let's again $H = \{h_1, h_2, h_3, h_4\}$ be a set of four houses, and the attribute *c = color*, whose values are *C = {white, green, blue, red}*, and another attribute *p = price*, whose values are *P = {cheap, expensive}*.

The function

$F: C \times P \to \mathcal{P}(H)$

where $\mathcal{P}(H)$ is the powerset of H, is a HyperSoft Set.

$F: C \times P \to H$(⩕, ⩖, ⩖E, ⫤), is called an IndetermHyperSoft Set.

Examples:

F(white, cheap) = h_2 ⩖ h_4

F(green, expensive) = h_1 ⩖E h_2

F(red, expensive) = ⫤ h_3

For a Neutrosophic IndetermHyperSoft Set one has neutrosophic degrees, for example:

F(white, cheap) = h_2*(0.4, 0.2, 0.3)* ⩔ h_4 *(0.5, 0.1, 0.4)*

In the same way as above (Section 26.1), one extends the HyperSoft operators ⩕, ⩔, ⩔$_E$, ⥱ by assigning some degree $d^0(.) \in [0,1]^p$, where: $p = 1$ for classical and fuzzy degree, $p = 2$ for intuitionistic fuzzy degree, $p = 3$ for neutrosophic degree, and so on $p = n$ for n-valued refined neutrosophic degree, to the elements involved in the operators, where $\wedge, \vee, \neg$ represent the conjunction, disjunction, and negation respectively of these degrees in their corresponding fuzzy-extension sets or logics.

29. Definition of Neutrosophic Triplet Commutative Group

Let $\mathcal{U}$ be a universe of discourse, and $(H,*)$ a non-empty set included in $\mathcal{U}$, where * is a binary operation (law) on H.

(i) The operation * on H is well-defined, associative, and commutative.

(ii) For each element $x \in H$ there exist an element $y \in H$, called the neutral of x, such that y is different from the unit element (if any), with $x * y = y * x = x$, and there exist an element

$z \in H$, called the inverse of x, such that $x * z = z * x = y$, then $\langle x, y, z \rangle$ is called a neutrosophic triplet.

Then $(H, *)$ is Neutrosophic Triplet Commutative Group.

In general, a Neutrosophic Triplet Algebra is different from a Classical Algebra.

29.1. Theorem 3

The joinAND Algebra (H, ⩓) and the disjoinOR Algebra (H, ⩔) are Neutrosophic Triplet Commutative Groups.

Proof

We have previously proved that the operators ⩓ and ⩔ are each of them: well-defined, associative, and commutative.

We also proved that the two operators are idempotent:

$\forall x \in H$, x ⩓ $x = x$ and x ⩔ $x = x$.

Therefore, for (H,⩓) and respectively (H,⩔) one has neutrosophic triplets of the form: $\langle x, x, x \rangle$.

30. Enriching the IndetermSoft Set and IndetermHyperSoft Set

The readers are invited to extend this research, since more determinate and indeterminate soft operators may be added to the IndetermSoft Algebra or IndetermHyperSoft Algebra, resulted from, or needed to, various real applications - as such one gets stronger soft and hypersoft structures.

A few suggestions:

F(white) = at least k houses;

or *F(white) = at most k houses;*

or *F(green, small) = between* k_1 *and* k_2 *houses;*

where k, k_1 and k_2 are positive integers, with $k_1 \le k_2$.

Etc.

31. Conclusions

The indeterminate soft operators, presented in this paper, have resulted from our real-world applications. An algebra closed under such operators was called an indeterminate soft algebra.

IndetermSoft Set and IndetermHyperSoft Set, and their corresponding Fuzzy / Intuitionistic Fuzzy / Neutrosophic forms, constructed on this indeterminate algebra, are introduced for the first time as extensions of the classical Soft Set and HyperSoft Set.

Many applications and examples are showed up.

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