Weighted aggregation operators of single-valued neutrosophic linguistic neutrosophic sets and their decision-making method

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Abstract: Multiple attribute decision-making (MADM) problems often contain quantitative and qualitative information that is inconsistent, uncertain, and incomplete. However, existing evaluation methods can only perform quantitative or qualitative processing on all attribute data, which easily leads to some information loss. In order to deal with MADM problems more effectively, this paper proposes a single-valued neutrosophic linguistic neutrosophic element (SvNLNE), which consists of a single-valued neutrosophic number for quantitative expression and a linguistic neutrosophic number for qualitative description. This paper also provides the fundamental operations of SvNLNEs, the SvNLNE score and accuracy functions for sorting the elements, and the SvNLNE weighted arithmetic averaging (SvNLNEWAA) and geometric averaging (SvNLNEWGA) operators for information aggregation. Finally, some MADM approaches are developed based on the SvNLNEWAA and SvNLNEWGA operators, and their application and rationality are further illustrated by an investment case in the SvNLNE setting.

Keywords: multi-attribute decision making; single-valued neutrosophic linguistic neutrosophic element; single-valued neutrosophic linguistic neutrosophic element weighted arithmetic averaging operator; single-valued neutrosophic linguistic neutrosophic element weighted geometric averaging operator

1. Introduction

In complex decision-making (DM) environments, conflicting quantitative and qualitative attribute data often need to be considered to optimize the selection of alternatives. Among them, quantitative information is usually expressed as numerical variables, while qualitative information is usually depicted as linguistic variables because linguistic items are more suitable to describe human cognition of objective things. In recent decades, many theories and methods on numerical and linguistic DM methods have been proposed for the DM problem. To handle uncertain and incomplete quantitative information, the fuzzy set [1] was firstly defined with a numerical membership degree. Then, the intuitionistic fuzzy set (IFS) [2] was presented by appending a numerical non-membership degree, and the interval-valued IFS [3] was represented by the interval-valued membership and non-membership degrees. Recently, for further comprehensive expression of the incomplete, uncertain and inconsistent data in DM problems, the simplified neutrosophic set (SNS) [4] that implied the definitions of single-valued neutrosophic set (SvNS) [5] and interval neutrosophic set (IvNS) [6] was put forward as a subclass of neutrosophic set (NS) [7] by constraining the membership degrees of truth, indeterminacy and falsity in the standard range of [0,1]. Since then, various aggregation operators and multi-attribute DM (MADM) methods of SvNSs/IvNSs/SNSs have been presented [4,8-10], and some extended neutrosophic sets, such as the neutrosophic cubic set (NCS) [11], the simplified neutrosophic indeterminate set (SNIS) [12], the neutrosophic Z-numbers [13] and the consistency neutrosophic set (CNS) [14], have also been proposed for specific applications. However, the numerical variable-based DM methods described above are more suitable for dealing with quantitative information than qualitative information. Thus, in terms of human thinking and expression habits, the linguistic neutrosophic number (LNN) [15], which is generalized from the concepts of linguistic variable (LV) [16], interval linguistic variable (ILV) [17] and linguistic intuitionistic fuzzy number (LIFN) [18], was proposed as a new branch of NS [7] to represent incomplete, indeterminate and inconsistent qualitative information using linguistic membership degrees of truth, falsity and uncertainty. Then, various aggregation operators and MADM methods of LNNs have been presented for linguistic DM problems [19-21]. Other extended sets, such as the linguistic neutrosophic uncertain number (LNUN) [22], have also been introduced to satisfy special applications. Unfortunately, theories and methods based on linguistic variables are more suitable for solving qualitative DM problems than quantitative DM problems.

In practical MADM applications, there is often quantitative and qualitative attribute information that needs to be evaluated together. However, existing DM methods can only make final decisions on a single type of information, but cannot handle multiple types of information. Especially for MADM problems with incomplete, inconsistent and indeterminate information, the single-valued neutrosophic number (SvNN) is only used for quantitative processing, while the LNN is only used for qualitative processing. Therefore, to overcome the limitations of existing DM approaches and better satisfy the preferences of the evaluators, this paper defines the single-valued neutrosophic linguistic neutrosophic set/element (SvNLNS/SvNLNE) as a combination of SvNN and LNN to uniformly describe quantitative and qualitative information and proposes the basic operational laws of SvNLNE. Then, this paper puts forward a SvNLNE weighted arithmetic averaging (SvNLNEWAA) operator and a SvNLNE weighted geometric averaging (SvNLNEWGA) operator, and further develops MADM approaches based on the presented operators in the SvNLNE setting.

In the construction of the paper, the preliminaries of SvNNs and LNNs are first reviewed in Section 2. The concepts, the fundamental operations, and the score and accuracy functions of SvNLNSs are put forward in Section 3. Then, two aggregation operators of SvNLNEWAA and SvNLNEWGA are presented and proved in Section 4. In Section 5, a new MADM method with SvNLNE information is developed by applying the proposed SvNLNEWAA and SvNLNEWGA operators. Finally, comparative analysis and conclusions are given in Sections 6 and 7, respectively.

2. Preliminaries of SvNNs and LNNs

This section introduces the concepts and operational relations of SvNNs and LNNs.

2.1 SvNNs

Definition 1 [5]. Set E as a fixed universal set. Then, a SvNS H in E can be given by

$$H=\left\{\left(\delta,\left\langle T(\delta),U(\delta),V(\delta)\right\rangle\right)\mid\delta\in E\right\},$$

Where $\langle T(\delta), U(\delta), V(\delta)\rangle$ is the SvNN for $\delta \in E$ satisfying the condition of $T(\delta), U(\delta), V(\delta) \in [0,1]$, and can simply be written as $h_\delta = \langle T_\delta, U_\delta, V_\delta\rangle$.

Definition 2 [5]. Assuming $h_{\delta1} = \langle T_{\delta1}, U_{\delta1}, V_{\delta1}\rangle$ and $h_{\delta2} = \langle T_{\delta2}, U_{\delta2}, V_{\delta2}\rangle$ are two SvNNs and $\sigma > 0$, there are the following relations:

(1) $h_{\delta 1} \oplus h_{\delta 2} = \langle T_{\delta 1} + T_{\delta 2} - T_{\delta 1}T_{\delta 2}, U_{\delta 1}U_{\delta 2}, V_{\delta 1}V_{\delta 2} \rangle$;

(2) $h_{\delta 1} \otimes h_{\delta 2} = \langle T_{\delta 1}T_{\delta 2}, U_{\delta 1} + U_{\delta 2} - U_{\delta 1}U_{\delta 2}, V_{\delta 1} + V_{\delta 2} - V_{\delta 1}V_{\delta 2} \rangle$;

(3) $\sigma h_{\delta 1} = \left\langle 1-(1-T_{\delta 1})^{\sigma}, U_{\delta 1}^{\sigma}, V_{\delta 1}^{\sigma} \right\rangle$;

(4) $h_{\delta 1}^{\sigma} = \left\langle T_{\delta 1}^{\sigma}, 1-(1-U_{\delta 1})^{\sigma}, 1-(1-V_{\delta 1})^{\sigma} \right\rangle$.

2.2 *LNNs*

Definition 3 [15]. Set E as a fixed universal set and $L = \{l_s \mid s = 0, 1, \cdots, r\}$ as a linguistic term set (LTS) whose odd cardinality is $r + 1$. Then, a linguistic neutrosophic set Z in E can be given by

$$Z = \left\{ \left(\delta, \left\langle l_{\tau(\delta)}, l_{u(\delta)}, l_{v(\delta)} \right\rangle \right) \mid \delta \in E \right\},$$

where $\left\langle l_{\tau(\delta)}, l_{u(\delta)}, l_{v(\delta)} \right\rangle$ is a LNN for $\delta \in E$, containing the linguistic variables of truth, indeterminacy and falsity $l_{\tau(\delta)}, l_{u(\delta)}, l_{v(\delta)} \in L$. Then, the LNN $\left\langle l_{\tau(\delta)}, l_{u(\delta)}, l_{v(\delta)} \right\rangle$ can be simply denoted as $z_{\delta} = \left\langle l_{\tau_{\delta}}, l_{u_{\delta}}, l_{v_{\delta}} \right\rangle$.

Definition 4 [15]. Assuming $z_{\delta 1} = \left\langle l_{\tau_{\delta 1}}, l_{u_{\delta 1}}, l_{v_{\delta 1}} \right\rangle$ and $z_{\delta 2} = \left\langle l_{\tau_{\delta 2}}, l_{u_{\delta 2}}, l_{v_{\delta 2}} \right\rangle$ are two LNNs in L and $\sigma > 0$, there exist the following relations:

(1) $z_{\delta 1} \oplus z_{\delta 2} = \left\langle l_{\tau_{\delta 1} + \tau_{\delta 2} - \frac{\tau_{\delta 1}\tau_{\delta 2}}{r}}, l_{\frac{u_{\delta 1}u_{\delta 2}}{r}}, l_{\frac{v_{\delta 1}v_{\delta 2}}{r}} \right\rangle$;

(2) $z_{\delta 1} \otimes z_{\delta 2} = \left\langle l_{\frac{\tau_{\delta 1}\tau_{\delta 2}}{r}}, l_{u_{\delta 1} + u_{\delta 2} - \frac{u_{\delta 1}u_{\delta 2}}{r}}, l_{v_{\delta 1} + v_{\delta 2} - \frac{v_{\delta 1}v_{\delta 2}}{r}} \right\rangle$;

(3) $\sigma z_{\delta 1} = \left\langle l_{r - r\left(1 - \frac{\tau_{\delta 1}}{r}\right)^{\sigma}}, l_{r\left(\frac{u_{\delta 1}}{r}\right)^{\sigma}}, l_{r\left(\frac{v_{\delta 1}}{r}\right)^{\sigma}} \right\rangle$;

(4) $z_{\delta 1}^{\sigma} = \left\langle l_{r\left(\frac{\tau_{\delta 1}}{r}\right)^{\sigma}}, l_{r - r\left(1 - \frac{u_{\delta 1}}{r}\right)^{\sigma}}, l_{r - r\left(1 - \frac{v_{\delta 1}}{r}\right)^{\sigma}} \right\rangle$.

3. SvNLNSs

Definition 5. Set E as a universal set and $L = \{l_s \mid s = 0, 1, \cdots, r\}$ as a LTS with an odd cardinality $r + 1$. Then, a SvNLNS H can be defined as

$$H = \left\{ \left(\delta, \langle T(\delta), U(\delta), V(\delta) \rangle, \left\langle l_{\tau(\delta)}, l_{u(\delta)}, l_{v(\delta)} \right\rangle \right) \middle| \delta \in E \right\},$$

where $\langle T(\delta), U(\delta), V(\delta) \rangle$ for $\delta \in E$ is a SvNN depicted independently by the truth, indeterminacy and falsity numerical variables $T(\delta), U(\delta), V(\delta) \in [0, 1]$, and $\left\langle l_{\tau(\delta)}, l_{u(\delta)}, l_{v(\delta)} \right\rangle$ for $\delta \in E$ is a LNN described independently by the truth, indeterminacy and falsity linguistic variables $l_{\tau(\delta)}, l_{u(\delta)}, l_{v(\delta)} \in L$ with $\tau(\delta), u(\delta), v(\delta) \in [0, r]$.

Then, the element $\left(\delta, \langle T(\delta), U(\delta), V(\delta) \rangle, \left\langle l_{\tau(\delta)}, l_{u(\delta)}, l_{v(\delta)} \right\rangle \right)$ of H can be simply represented by $\xi_{\delta} = \left(\langle T_{\delta}, U_{\delta}, V_{\delta} \rangle, \left\langle l_{\tau_{\delta}}, l_{u_{\delta}}, l_{v_{\delta}} \right\rangle \right)$ for $T_{\delta}, U_{\delta}, V_{\delta} \in [0, 1]$, $l_{\tau_{\delta}}, l_{u_{\delta}}, l_{v_{\delta}} \in L$, and $\tau_{\delta}, u_{\delta}, v_{\delta} \in [0, r]$, called SvNLNE. It is obvious that SvNLNS is composed of SvNNs and LNNs in E.

Definition 6. Set $\xi_{\delta 1} = \left(\langle T_{\delta 1}, U_{\delta 1}, V_{\delta 1} \rangle, \left\langle l_{\tau_{\delta 1}}, l_{u_{\delta 1}}, l_{v_{\delta 1}} \right\rangle \right)$ and $\xi_{\delta 2} = \left(\langle T_{\delta 2}, U_{\delta 2}, V_{\delta 2} \rangle, \left\langle l_{\tau_{\delta 2}}, l_{u_{\delta 2}}, l_{v_{\delta 2}} \right\rangle \right)$ as two SvNLNEs. Then there exist the following relations:

(1) $\xi_{\delta 1} \subseteq \xi_{\delta 2} \Leftrightarrow T_{\delta 1} \le T_{\delta 2}$, $U_{\delta 1} \ge U_{\delta 2}$, $V_{\delta 1} \ge V_{\delta 2}$, $l_{\tau_{\delta 1}} \le l_{\tau_{\delta 2}}$, $l_{u_{\delta 1}} \ge l_{u_{\delta 2}}$, and $l_{v_{\delta 1}} \ge l_{v_{\delta 2}}$;

(2) $\xi_{\delta 1} = \xi_{\delta 2} \Leftrightarrow \xi_{\delta 1} \subseteq \xi_{\delta 2}$ and $\xi_{\delta 1} \supseteq \xi_{\delta 2}$, i.e., $T_{\delta 1} = T_{\delta 2}$, $U_{\delta 1} = U_{\delta 2}$, $V_{\delta 1} = V_{\delta 2}$, $l_{\tau_{\delta 1}} = l_{\tau_{\delta 2}}$, $l_{u_{\delta 1}} = l_{u_{\delta 2}}$, and $l_{v_{\delta 1}} = l_{v_{\delta 2}}$;

(3) $\xi_{\delta 1} \oplus \xi_{\delta 2} = \left(\left\langle T_{\delta 1} + T_{\delta 2} - T_{\delta 1}T_{\delta 2}, U_{\delta 1}U_{\delta 2}, V_{\delta 1}V_{\delta 2} \right\rangle, \left\langle l_{\tau_{\delta 1}+\tau_{\delta 2}-\frac{\tau_{\delta 1}\tau_{\delta 2}}{r}}, l_{\frac{u_{\delta 1}u_{\delta 2}}{r}}, l_{\frac{v_{\delta 1}v_{\delta 2}}{r}} \right\rangle \right)$;

(4) $\xi_{\delta 1} \otimes \xi_{\delta 2} = \left(\left\langle T_{\delta 1}T_{\delta 2}, U_{\delta 1} + U_{\delta 2} - U_{\delta 1}U_{\delta 2}, V_{\delta 1} + V_{\delta 2} - V_{\delta 1}V_{\delta 2} \right\rangle, \left\langle l_{\frac{\tau_{\delta 1}\tau_{\delta 2}}{r}}, l_{u_{\delta 1}+u_{\delta 2}-\frac{u_{\delta 1}u_{\delta 2}}{r}}, l_{v_{\delta 1}+v_{\delta 2}-\frac{v_{\delta 1}v_{\delta 2}}{r}} \right\rangle \right)$;

(5) $\sigma\xi_{\delta 1} = \left(\left\langle 1-(1-T_{\delta 1})^{\sigma}, U_{\delta 1}^{\sigma}, V_{\delta 1}^{\sigma} \right\rangle, \left\langle l_{r-r\left(1-\frac{\tau_{\delta 1}}{r}\right)^{\sigma}}, l_{r\left(\frac{u_{\delta 1}}{r}\right)^{\sigma}}, l_{r\left(\frac{v_{\delta 1}}{r}\right)^{\sigma}} \right\rangle \right)$ for σ > 0;

(6) $\xi_{\delta 1}^{\sigma} = \left(\left\langle T_{\delta 1}^{\sigma}, 1-(1-U_{\delta 1})^{\sigma}, 1-(1-V_{\delta 1})^{\sigma} \right\rangle, \left\langle l_{r\left(\frac{\tau_{\delta 1}}{r}\right)^{\sigma}}, l_{r-r\left(1-\frac{u_{\delta 1}}{r}\right)^{\sigma}}, l_{r-r\left(1-\frac{v_{\delta 1}}{r}\right)^{\sigma}} \right\rangle \right)$ for $\sigma > 0$.

To compare SvNLNEs, the score and accuracy functions for SvNLNEs and their sorting approaches are given by the definitions below.

Definition 7. Set ξ = (< T, U, V>, <l_τ, l_u, l_v>) as SvNLNE. Then, its score and accuracy functions are

$$F(\xi) = \frac{1}{2}\left(\frac{2+T-U-V}{3} + \frac{2r+\tau-u-v}{3r} \right) \text{ for } F(\xi) \in [0, 1], \quad (1)$$

$$G(\xi) = \frac{1}{2}\left[T - V + \frac{\tau - v}{r} \right] \text{ for } G(\xi) \in [-1, 1]. \quad (2)$$

Definition 8. Let $\xi_{\delta 1} = \left(\left\langle T_{\delta 1}, U_{\delta 1}, V_{\delta 1} \right\rangle, \left\langle l_{\tau_{\delta 1}}, l_{u_{\delta 1}}, l_{v_{\delta 1}} \right\rangle \right)$ and $\xi_{\delta 2} = \left(\left\langle T_{\delta 2}, U_{\delta 2}, V_{\delta 2} \right\rangle, \left\langle l_{\tau_{\delta 2}}, l_{u_{\delta 2}}, l_{v_{\delta 2}} \right\rangle \right)$ be two SvNLNEs, then based on the score and accuracy values of $F(\xi_{\delta\varsigma})$ and $G(\xi_{\delta\varsigma})$ (ς =1, 2), the ranking approaches are given below:

(1) If $F(\xi_{\delta 1}) > F(\xi_{\delta 2})$, then $\xi_{\delta 1} > \xi_{\delta 2}$;

(2) If $F(\xi_{\delta 1}) = F(\xi_{\delta 2})$ and $G(\xi_{\delta 1}) > G(\xi_{\delta 2})$, then $\xi_{\delta 1} > \xi_{\delta 2}$;

(3) If $F(\xi_{\delta 1}) = F(\xi_{\delta 2})$ and $G(\xi_{\delta 1}) = G(\xi_{\delta 2})$, then $\xi_{\delta 1} = \xi_{\delta 2}$.

4. Aggregation Operators of SvNLNEs

4.1 SvNLNEWAA Operator

Theorem 1. Set $\xi_\varsigma = \left(\left\langle T_\varsigma, U_\varsigma, V_\varsigma \right\rangle, \left\langle l_{\tau_\varsigma}, l_{u_\varsigma}, l_{v_\varsigma} \right\rangle \right)$ (ς = 1, 2, ⋯, η) as a collection of SvNLNEs. Then the SvNLNEWAA operator can be represented as

$$SvNLNEWAA(\xi_1, \xi_2, \cdots, \xi_\eta) = \sum_{\varsigma=1}^{\eta} \sigma_\varsigma \xi_\varsigma = \begin{pmatrix} \left\langle 1 - \prod_{\varsigma=1}^{\eta}(1-T_\varsigma)^{\sigma_\varsigma}, \prod_{\varsigma=1}^{\eta} U_\varsigma^{\sigma_\varsigma}, \prod_{\varsigma=1}^{\eta} V_\varsigma^{\sigma_\varsigma}, \right\rangle, \\ \left\langle l_{r-r\prod_{\varsigma=1}^{\eta}(1-\frac{\tau_\varsigma}{r})^{\sigma_\varsigma}}, l_{r\prod_{\varsigma=1}^{\eta}(\frac{u_\varsigma}{r})^{\sigma_\varsigma}}, l_{r\prod_{\varsigma=1}^{\eta}(\frac{v_\varsigma}{r})^{\sigma_\varsigma}} \right\rangle \end{pmatrix}, \quad (3)$$

where $\sigma_\varsigma \in [0, 1]$ is the weight of ξ_ς (ς=1, 2, ⋯, η) with $\sum_{\varsigma=1}^{\eta} \sigma_\varsigma = 1$.

Proof. (1) It is straightforward that the theorem is valid when η = 1;

(2) When η = 2, from the relation (5) of Definition 6, we can obtain

$$\sigma_1\xi_1 = \left(\left\langle 1-(1-T_1)^{\sigma_1}, U_1^{\sigma_1}, V_1^{\sigma_1} \right\rangle, \left\langle l_{r-r\left(1-\frac{\tau_1}{r}\right)^{\sigma_1}}, l_{r\left(\frac{u_1}{r}\right)^{\sigma_1}}, l_{r\left(\frac{v_1}{r}\right)^{\sigma_1}} \right\rangle \right),$$

$$\sigma_2\xi_2=\left(\left\langle 1-(1-T_2)^{\sigma_2},U_2^{\sigma_2},V_2^{\sigma_2}\right\rangle,\left\langle l_{r-r\left(1-\frac{\tau_2}{r}\right)^{\sigma_2}},l_{r\left(\frac{u_2}{r}\right)^{\sigma_2}},l_{r\left(\frac{v_2}{r}\right)^{\sigma_2}}\right\rangle\right).$$

From the relation (3) of Definition 6, the SvNLNEWAA aggregation result is

$$SvNLNEWAA(\xi_1,\xi_2)=\sigma_1\xi_1\oplus\sigma_2\xi_2$$
$$=\left(\begin{array}{l}\left\langle 1-(1-T_1)^{\sigma_1}+1-(1-T_2)^{\sigma_2}-\left[1-(1-T_1)^{\sigma_1}\right]\left[1-(1-T_2)^{\sigma_2}\right],U_1^{\sigma_1}U_2^{\sigma_2},V_1^{\sigma_1}V_2^{\sigma_2}\right\rangle,\\ \left\langle l_{r-r\left(1-\frac{\tau_1}{r}\right)^{\sigma_1}+r-r\left(1-\frac{\tau_2}{r}\right)^{\sigma_2}-\frac{\left[r-r\left(1-\frac{\tau_1}{r}\right)^{\sigma_1}\right]\left[r-r\left(1-\frac{\tau_2}{r}\right)^{\sigma_2}\right]}{r}},l_{\frac{r\left(\frac{u_1}{r}\right)^{\sigma_1}r\left(\frac{u_2}{r}\right)^{\sigma_2}}{r}},l_{\frac{r\left(\frac{v_1}{r}\right)^{\sigma_1}r\left(\frac{v_2}{r}\right)^{\sigma_2}}{r}}\right\rangle\end{array}\right)$$
$$=\left(\left\langle 1-\prod_{\varsigma=1}^{2}(1-T_\varsigma)^{\sigma_\varsigma},\prod_{\varsigma=1}^{2}U_\varsigma^{\sigma_\varsigma},\prod_{\varsigma=1}^{2}V_\varsigma^{\sigma_\varsigma}\right\rangle,\left\langle l_{r-r\prod_{\varsigma=1}^{2}\left(1-\frac{\tau_\varsigma}{r}\right)^{\sigma_\varsigma}},l_{r\prod_{\varsigma=1}^{2}\left(\frac{u_\varsigma}{r}\right)^{\sigma_\varsigma}},l_{r\prod_{\varsigma=1}^{2}\left(\frac{v_\varsigma}{r}\right)^{\sigma_\varsigma}}\right\rangle\right).$$

(3) Let $\eta=\mu$, the aggregation result of SvNLNS is

$$SvNLNEWAA(\xi_1,\xi_2,\cdots,\xi_\mu)=\sum_{\varsigma=1}^{\mu}\sigma_\varsigma\xi_\varsigma$$
$$=\left(\left\langle 1-\prod_{\varsigma=1}^{\mu}(1-T_\varsigma)^{\sigma_\varsigma},\prod_{\varsigma=1}^{\mu}U_\varsigma^{\sigma_\varsigma},\prod_{\varsigma=1}^{\mu}V_\varsigma^{\sigma_\varsigma}\right\rangle,\left\langle l_{r-r\prod_{\varsigma=1}^{\mu}\left(1-\frac{\tau_\varsigma}{r}\right)^{\sigma_\varsigma}},l_{r\prod_{\varsigma=1}^{\mu}\left(\frac{u_\varsigma}{r}\right)^{\sigma_\varsigma}},l_{r\prod_{\varsigma=1}^{\mu}\left(\frac{v_\varsigma}{r}\right)^{\sigma_\varsigma}}\right\rangle\right).$$

(4) Let $\eta=\mu+1$, the aggregation result of SvNLNS is

$$SvNLNEWAA(\xi_1,\xi_2,\cdots,\xi_{\mu+1})=\sum_{\varsigma=1}^{\mu}\sigma_\varsigma\xi_\varsigma\oplus\sigma_{\mu+1}\xi_{\mu+1}$$
$$=\left(\begin{array}{l}\left\langle\begin{array}{l}1-\prod_{\varsigma=1}^{\mu}(1-T_\varsigma)^{\sigma_\varsigma}+1-(1-T_{\mu+1})^{\sigma_{\mu+1}}-\left[1-\prod_{\varsigma=1}^{\mu}(1-T_\varsigma)^{\sigma_\varsigma}\right]\left[1-(1-T_{\mu+1})^{\sigma_{\mu+1}}\right],\\ \left(\prod_{\varsigma=1}^{\mu}U_\varsigma^{\sigma_\varsigma}\right)U_{\mu+1}^{\sigma_{\mu+1}},\left(\prod_{\varsigma=1}^{\mu}V_\varsigma^{\sigma_\varsigma}\right)V_{\mu+1}^{\sigma_{\mu+1}}\end{array}\right\rangle,\\ \left\langle\begin{array}{l}l_{r-r\prod_{\varsigma=1}^{\mu}\left(1-\frac{\tau_\varsigma}{r}\right)^{\sigma_\varsigma}+r-r\left(1-\frac{\tau_{\mu+1}}{r}\right)^{\sigma_{\mu+1}}-\frac{\left[r-r\prod_{\varsigma=1}^{\mu}\left(1-\frac{\tau_\varsigma}{r}\right)^{\sigma_\varsigma}\right]\left[r-r\left(1-\frac{\tau_{\mu+1}}{r}\right)^{\sigma_{\mu+1}}\right]}{r}},\\ l_{\frac{\left[r\prod_{\varsigma=1}^{\mu}\left(\frac{u_\varsigma}{r}\right)^{\sigma_\varsigma}\right]r\left(\frac{u_{\mu+1}}{r}\right)^{\sigma_{\mu+1}}}{r}},l_{\frac{\left[r\prod_{\varsigma=1}^{\mu}\left(\frac{v_\varsigma}{r}\right)^{\sigma_\varsigma}\right]r\left(\frac{v_{\mu+1}}{r}\right)^{\sigma_{\mu+1}}}{r}}\end{array}\right\rangle\end{array}\right)$$
$$=\left(\left\langle 1-\prod_{\varsigma=1}^{\mu+1}(1-T_\varsigma)^{\sigma_\varsigma},\prod_{\varsigma=1}^{\mu+1}U_\varsigma^{\sigma_\varsigma},\prod_{\varsigma=1}^{\mu+1}V_\varsigma^{\sigma_\varsigma}\right\rangle,\left\langle l_{r-r\prod_{\varsigma=1}^{\mu+1}\left(1-\frac{\tau_\varsigma}{r}\right)^{\sigma_\varsigma}},l_{r\prod_{\varsigma=1}^{\mu+1}\left(\frac{u_\varsigma}{r}\right)^{\sigma_\varsigma}},l_{r\prod_{\varsigma=1}^{\mu+1}\left(\frac{v_\varsigma}{r}\right)^{\sigma_\varsigma}}\right\rangle\right).$$

Thus, Theorem 1 is proved to be valid for any η.

Additionally, for the SvNLNE collection given by $\xi_\varsigma=\left(\left\langle T_\varsigma,U_\varsigma,V_\varsigma\right\rangle,\left\langle l_{\tau_\varsigma},l_{u_\varsigma},l_{v_\varsigma}\right\rangle\right)$ (ς = 1, 2, ⋯, η), the SvNLNEWAA operator implies some properties:

(1) Idempotency: There is $SvNLNEWAA(\xi_1, \xi_2, \cdots, \xi_\eta) = \xi$ when $\xi_\varsigma = \xi$ is satisfied for ς = 1, 2, ⋯, η.

(2) Boundedness: Assume $\xi^- = \left(\left\langle \min_\varsigma(T_\varsigma), \max_\varsigma(U_\varsigma), \max_\varsigma(V_\varsigma)\right\rangle, \left\langle \min_\varsigma(l_{\tau_\varsigma}), \max_\varsigma(l_{u_\varsigma}), \max_\varsigma(l_{v_\varsigma})\right\rangle\right)$ and $\xi^+ = \left(\left\langle \max_\varsigma(T_\varsigma), \min_\varsigma(U_\varsigma), \min_\varsigma(V_\varsigma)\right\rangle, \left\langle \max_\varsigma(l_{\tau_\varsigma}), \min_\varsigma(l_{u_\varsigma}), \min_\varsigma(l_{v_\varsigma})\right\rangle\right)$ represent the minimum and maximum SvNLNEs for ς= 1, 2, …, η, then $\xi^- \le SvNLNEWAA(\xi_1, \xi_2, \cdots, \xi_\eta) \le \xi^+$.

(3) Monotonicity: There is $SvNLNEWAA(\xi_1, \xi_2, \cdots, \xi_\eta) \le SvNLNEWAA(\xi_1^*, \xi_2^*, \cdots, \xi_\eta^*)$ when the condition of $\xi_\varsigma \le \xi_\varsigma^*$ is satisfied for ς= 1, 2, ⋯, η.

Proof. (1) Suppose ξ=(<T, U, V>, <l_τ, l_u, l_v>). Since ξ_ς is equal to ξ for ς = 1, 2, ⋯, η, we can obtain

$$SvNLNEWAA(\xi_1, \xi_2, \cdots, \xi_\eta) = \sum_{\varsigma=1}^{\eta} \sigma_\varsigma \xi_\varsigma$$

$$= \left(\left\langle 1-(1-T)^{\sum_{\varsigma=1}^{\eta}\sigma_\varsigma}, U^{\sum_{\varsigma=1}^{\eta}\sigma_\varsigma}, V^{\sum_{\varsigma=1}^{\eta}\sigma_\varsigma} \right\rangle, \left\langle l_{r-r\left(1-\frac{\tau}{r}\right)^{\sum_{\varsigma=1}^{\eta}\sigma_\varsigma}}, l_{r\left(\frac{u}{r}\right)^{\sum_{\varsigma=1}^{\eta}\sigma_\varsigma}}, l_{r\left(\frac{v}{r}\right)^{\sum_{\varsigma=1}^{\eta}\sigma_\varsigma}} \right\rangle\right) = \left(\langle T, U, V\rangle, \langle l_\tau, l_u, l_v\rangle\right) = \xi .$$

(2) Because ξ^- is the minimum SvNLNE and ξ^+ is the maximum SvNLNE, $\xi^- \leqslant \xi_\varsigma \leqslant \xi^+$ can be obtained. Hence, $\sum_{\varsigma=1}^{\eta} \sigma_\varsigma \xi^- \le \sum_{\varsigma=1}^{\eta} \sigma_\varsigma \xi_\varsigma \le \sum_{\varsigma=1}^{\eta} \sigma_\varsigma \xi^+$. According to the property (1), $\sum_{\varsigma=1}^{\eta} \sigma_\varsigma \xi^- = \xi^-$ and

$\sum_{\varsigma=1}^{\eta} \sigma_\varsigma \xi^+ = \xi^+$. Thus, $\xi^- \le SvNLNEWAA(\xi_1, \xi_2, \cdots, \xi_\eta) \le \xi^+$.

(3) Since $\xi_\varsigma \le \xi_\varsigma^*$ for ς = 1, 2, …, η, there exists $\sum_{\varsigma=1}^{\eta} \sigma_\varsigma \xi_\varsigma \le \sum_{\varsigma=1}^{\eta} \sigma_\varsigma \xi_\varsigma^*$. Therefore,

$SvNLNEWAA(\xi_1, \xi_2, \cdots, \xi_\eta) \le SvNLNEWAA(\xi_1^*, \xi_2^*, \cdots, \xi_\eta^*)$.

The properties of the SvNLNEWAA operator are proved above. □

4.2 SvNLNEWGA Operator

Theorem 2. Set $\xi_\varsigma = \left(\langle T_\varsigma, U_\varsigma, V_\varsigma\rangle, \langle l_{\tau_\varsigma}, l_{u_\varsigma}, l_{v_\varsigma}\rangle\right)$ (ς = 1, 2, ⋯, η) as a cluster of SvNLNEs, then the SvNLNEWGA operator is

$$SvNLNEWGA(\xi_1, \xi_2, \cdots, \xi_\eta) = \prod_{\varsigma=1}^{\eta} \xi_\varsigma^{\sigma_\varsigma} = \left(\begin{array}{c} \left\langle \prod_{\varsigma=1}^{\eta} T_\varsigma^{\sigma_\varsigma}, 1-\prod_{\varsigma=1}^{\eta}(1-U_\varsigma)^{\sigma_\varsigma}, 1-\prod_{\varsigma=1}^{\eta}(1-V_\varsigma)^{\sigma_\varsigma} \right\rangle, \\ \left\langle l_{r\prod_{\varsigma=1}^{\eta}\left(\frac{T_\varsigma}{r}\right)^{\sigma_\varsigma}}, l_{r-r\prod_{\varsigma=1}^{\eta}\left(1-\frac{u_\varsigma}{r}\right)^{\sigma_\varsigma}}, l_{r-r\prod_{\varsigma=1}^{\eta}\left(1-\frac{v_\varsigma}{r}\right)^{\sigma_\varsigma}} \right\rangle \end{array} \right), \quad (4)$$

where $\sigma_\varsigma \in$ [0, 1] indicates the weight of ξ_ς with $\sum_{\varsigma=1}^{\eta} \sigma_\varsigma = 1$.

Proof. (1) When η = 1, the theorem 2 is obviously correct;
(2) When η = 2, from the relation (6) of Definition 6, we can obtain

$$\xi_1^{\sigma_1} = \left(\left\langle T_1^{\sigma_1}, 1-(1-U_1)^{\sigma_1}, 1-(1-V_1)^{\sigma_1} \right\rangle, \left\langle l_{r\left(\frac{\tau_1}{r}\right)^{\sigma_1}}, l_{r-r\left(1-\frac{u_1}{r}\right)^{\sigma_1}}, l_{r-r\left(1-\frac{v_1}{r}\right)^{\sigma_1}} \right\rangle\right),$$

$$\xi_2^{\sigma_2}=\left(\left\langle T_2^{\sigma_2},1-(1-U_2)^{\sigma_2},1-(1-V_2)^{\sigma_2}\right\rangle,\left\langle l_{r\left(\frac{\tau_2}{r}\right)^{\sigma_2}},l_{r-r\left(1-\frac{u_2}{r}\right)^{\sigma_2}},l_{r-r\left(1-\frac{v_2}{r}\right)^{\sigma_2}}\right\rangle\right).$$

From the relation (4) of Definition 6, the aggregation result is

$$SvNLNEWGA(\xi_1,\xi_2)=\xi_1^{\sigma_1}\otimes\xi_2^{\sigma_2}$$

$$=\left(\begin{array}{l}\left\langle\begin{array}{l}T_1^{\sigma_1}T_2^{\sigma_2},1-(1-U_1)^{\sigma_1}+1-(1-U_2)^{\sigma_2}-\left[1-(1-U_1)^{\sigma_1}\right]\left[1-(1-U_2)^{\sigma_2}\right],\\ 1-(1-V_1)^{\sigma_1}+1-(1-V_2)^{\sigma_2}-\left[1-(1-V_1)^{\sigma_1}\right]\left[1-(1-V_2)^{\sigma_2}\right]\end{array}\right\rangle,\\ \left\langle\begin{array}{l}l_{\frac{r\left(\frac{\tau_1}{r}\right)^{\sigma_1}r\left(\frac{\tau_2}{r}\right)^{\sigma_2}}{r}},l_{r-r\left(1-\frac{u_1}{r}\right)^{\sigma_1}+r-r\left(1-\frac{u_2}{r}\right)^{\sigma_2}-\frac{\left[r-r\left(1-\frac{u_1}{r}\right)^{\sigma_1}\right]\left[r-r\left(1-\frac{u_2}{r}\right)^{\sigma_2}\right]}{r}},\\ l_{r-r\left(1-\frac{v_1}{r}\right)^{\sigma_1}+r-r\left(1-\frac{v_2}{r}\right)^{\sigma_2}-\frac{\left[r-r\left(1-\frac{v_1}{r}\right)^{\sigma_1}\right]\left[r-r\left(1-\frac{v_2}{r}\right)^{\sigma_2}\right]}{r}}\end{array}\right\rangle\end{array}\right)$$

$$=\left(\begin{array}{l}\left\langle\prod_{\varsigma=1}^{2}T_\varsigma^{\sigma_\varsigma},1-\prod_{\varsigma=1}^{2}(1-U_\varsigma)^{\sigma_\varsigma},1-\prod_{\varsigma=1}^{2}(1-V_\varsigma)^{\sigma_\varsigma}\right\rangle,\\ \left\langle l_{r\prod_{\varsigma=1}^{2}\left(\frac{\tau_\varsigma}{r}\right)^{\sigma_\varsigma}},l_{r-r\prod_{\varsigma=1}^{2}\left(1-\frac{u_\varsigma}{r}\right)^{\sigma_\varsigma}},l_{r-r\prod_{\varsigma=1}^{2}\left(1-\frac{v_\varsigma}{r}\right)^{\sigma_\varsigma}}\right\rangle\end{array}\right).$$

(3) Let $\eta=\mu$, the aggregation result of SvNLNS is

$$SvNLNEWGA(\xi_1,\xi_2,\cdots,\xi_\mu)=\prod_{\varsigma=1}^{\mu}\xi_\varsigma^{\sigma_\varsigma}=\left(\begin{array}{l}\left\langle\prod_{\varsigma=1}^{\mu}T_\varsigma^{\sigma_\varsigma},1-\prod_{\varsigma=1}^{\mu}\left(1-U_\varsigma\right)^{\sigma_\varsigma},1-\prod_{\varsigma=1}^{\mu}\left(1-V_\varsigma\right)^{\sigma_\varsigma}\right\rangle,\\ \left\langle l_{r\prod_{\varsigma=1}^{\mu}\left(\frac{\tau_\varsigma}{r}\right)^{\sigma_\varsigma}},l_{r-r\prod_{\varsigma=1}^{\mu}\left(1-\frac{u_\varsigma}{r}\right)^{\sigma_\varsigma}},l_{r-r\prod_{\varsigma=1}^{\mu}\left(1-\frac{v_\varsigma}{r}\right)^{\sigma_\varsigma}}\right\rangle\end{array}\right).$$

(4) Let $\eta=\mu+1$, the aggregation result of SvNLNS is

$$SvNLNEWGA(\xi_1,\xi_2,\cdots,\xi_{\mu+1})=\prod_{\varsigma=1}^{\mu}\xi_\varsigma^{\sigma_\varsigma}\otimes\xi_{\mu+1}^{\sigma_{\mu+1}}$$

$$=\left(\begin{array}{l}\left\langle\begin{array}{l}\left(\prod_{\varsigma=1}^{\mu}T_\varsigma^{\sigma_\varsigma}\right)T_{\mu+1}^{\sigma_{\mu+1}},1-\prod_{\varsigma=1}^{\mu}\left(1-U_\varsigma\right)^{\sigma_\varsigma}+1-\left(1-U_{\mu+1}\right)^{\sigma_{\mu+1}}-\left[1-\prod_{\varsigma=1}^{\mu}\left(1-U_\varsigma\right)^{\sigma_\varsigma}\right]\left[1-\left(1-U_{\mu+1}\right)^{\sigma_{\mu+1}}\right],\\ 1-\prod_{\varsigma=1}^{\mu}\left(1-V_\varsigma\right)^{\sigma_\varsigma}+1-\left(1-V_{\mu+1}\right)^{\sigma_{\mu+1}}-\left[1-\prod_{\varsigma=1}^{\mu}\left(1-V_\varsigma\right)^{\sigma_\varsigma}\right]\left[1-\left(1-V_{\mu+1}\right)^{\sigma_{\mu+1}}\right]\end{array}\right\rangle,\\ \left\langle\begin{array}{l}l_{\frac{r\prod_{\varsigma=1}^{\mu}\left(\frac{\tau_\varsigma}{r}\right)^{\sigma_\varsigma}r\left(\frac{\tau_{\mu+1}}{r}\right)^{\sigma_{\mu+1}}}{r}},l_{r-r\prod_{\varsigma=1}^{\mu}\left(1-\frac{u_\varsigma}{r}\right)^{\sigma_\varsigma}+r-r\left(1-\frac{u_{\mu+1}}{r}\right)^{\sigma_{\mu+1}}-\frac{\left[r-r\prod_{\varsigma=1}^{\mu}\left(1-\frac{u_\varsigma}{r}\right)^{\sigma_\varsigma}\right]\left[r-r\left(1-\frac{u_{\mu+1}}{r}\right)^{\sigma_{\mu+1}}\right]}{r}},\\ l_{r-r\prod_{\varsigma=1}^{\mu}\left(1-\frac{v_\varsigma}{r}\right)^{\sigma_\varsigma}+r-r\left(1-\frac{v_{\mu+1}}{r}\right)^{\sigma_{\mu+1}}-\frac{\left[r-r\prod_{\varsigma=1}^{\mu}\left(1-\frac{v_\varsigma}{r}\right)^{\sigma_\varsigma}\right]\left[r-r\left(1-\frac{v_{\mu+1}}{r}\right)^{\sigma_{\mu+1}}\right]}{r}}\end{array}\right\rangle\end{array}\right)$$

$$=\left(\begin{array}{l}\left\langle\prod_{\varsigma=1}^{\mu+1}T_\varsigma^{\sigma_\varsigma},1-\prod_{\varsigma=1}^{\mu+1}\left(1-U_\varsigma\right)^{\sigma_\varsigma},1-\prod_{\varsigma=1}^{\mu+1}\left(1-V_\varsigma\right)^{\sigma_\varsigma}\right\rangle,\\ \left\langle l_{r\prod_{\varsigma=1}^{\mu+1}\left(\frac{\tau_\varsigma}{r}\right)^{\sigma_\varsigma}},l_{r-r\prod_{\varsigma=1}^{\mu+1}\left(1-\frac{u_\varsigma}{r}\right)^{\sigma_\varsigma}},l_{r-r\prod_{\varsigma=1}^{\mu+1}\left(1-\frac{v_\varsigma}{r}\right)^{\sigma_\varsigma}}\right\rangle\end{array}\right).$$

Thus, Eq.(4) is proved to be valid for any η.

Additionally, for the group of SvNLNEs given by $\xi_\varsigma=\left(\left\langle T_\varsigma,U_\varsigma,V_\varsigma\right\rangle,\left\langle l_{\tau_\varsigma},l_{u_\varsigma},l_{v_\varsigma}\right\rangle\right)$ (ς = 1, 2, …, η), there are some properties of the SvNLNEWGA operator:

(1) Idempotency: There is $SvNLNEWGA(\xi_1, \xi_2, \cdots, \xi_\eta) = \xi$ when $\xi_\varsigma = \xi$ is satisfied for $\varsigma = 1, 2, \cdots, \eta$.

(2) Boundedness: Assume $\xi^- = \left(\left\langle \min_\varsigma(T_\varsigma), \max_\varsigma(U_\varsigma), \max_\varsigma(V_\varsigma)\right\rangle, \left\langle \min_\varsigma\left(l_{\tau_\varsigma}\right), \max_\varsigma\left(l_{u_\varsigma}\right), \max_\varsigma\left(l_{v_\varsigma}\right)\right\rangle\right)$ and $\xi^+ = \left(\left\langle \max_\varsigma(T_\varsigma), \min_\varsigma(U_\varsigma), \min_\varsigma(V_\varsigma)\right\rangle, \left\langle \max_\varsigma\left(l_{\tau_\varsigma}\right), \min_\varsigma\left(l_{u_\varsigma}\right), \min_\varsigma\left(l_{v_\varsigma}\right)\right\rangle\right)$ represents the minimum and maximum SvNLNEs for $\varsigma = 1, 2, \ldots, \eta$, then $\xi^- \le SvNLNEWGA(\xi_1, \xi_2, \cdots, \xi_\eta) \le \xi^+$.

(3) Monotonicity: There is $SvNLNEWGA\left(\xi_1, \xi_2, \cdots, \xi_\eta\right) \le SvNLNEWGA\left(\xi_1^*, \xi_1^*, \cdots, \xi_\eta^*\right)$ when the condition of $\xi_\varsigma \le \xi_\varsigma^*$ ($\varsigma = 1, 2, \cdots, \eta$) is satisfied.

Since the property proof of the SvNLNEWGA operator is similar to that of the SvNLNEWAA operator, it is omitted here. □

5. MADM Method in the SvNLNE Setting

In this section, by applying the SvNLNEWAA and SvNLNEWGA operators, a novel MADM method is developed to solve DM problems with quantitative and qualitative information.

For a complex DM problem, m alternatives (given by $R = \{R_1, R_2, R_3, \cdots, R_m\}$) need to be evaluated on η attributes (given by $S = \{s_1, s_2, \cdots, s_\eta\}$) in the SvNLNE setting, where the attribute types may be different. Assume each alternative is evaluated as a SvNLNE $\xi_{\iota\varsigma} = \left(\left\langle T_{\iota\varsigma}, U_{\iota\varsigma}, V_{\iota\varsigma}\right\rangle, \left\langle l_{\tau_{\iota\varsigma}}, l_{u_{\iota\varsigma}}, l_{v_{\iota\varsigma}}\right\rangle\right)$ with $\iota = 1, 2, \cdots, m$ and $\varsigma = 1, 2, \cdots, \eta$. Then, all evaluated values can be further constructed as the SvNLNE decision matrix $E = (\xi_{\iota\varsigma})_{m\times\eta}$.

Then, MADM problems with SvNLNE information can be solved by the SvNLNEWAA and SvNLNEWGA operators along with the SvNLNE score and accuracy functions. Details about the new MADM method are given as below.

Step 1. Standardize the initial evaluation data in the SvNLNE format. For instance, a quantitative attribute data denoted by the SvNN $\xi = (<T, U, V>)$ can be converted into the SvNLNE $\xi' = (<T, U, V>, <l_{T\times r}, l_{U\times r}, l_{V\times r}>)$, and a qualitative attribute data given by the LNN $\xi = (<l_\tau, l_u, l_v>)$ can be transformed into the SvNLNE $\xi' = (<\tau/r, u/r, v/r>, <l_\tau, l_u, l_v>)$. As a result, the initial decision matrix $E = (\xi_{\iota\varsigma})_{m\times\eta}$ can be standardized as $E' = \left(\xi'_{\iota\varsigma}\right)_{m\times\eta}$.

Step 2. Assume $P = \{\sigma_1, \sigma_2, \cdots, \sigma_\eta\}$ is a weight vector that represents the importance of attributes $S = \{s_1, s_2, \cdots, s_\eta\}$, where σ_ς is the weight of $\xi'_{\iota\varsigma}$ ($\varsigma = 1, 2, \cdots, \eta$) with $\sigma_\varsigma \in [0, 1]$ and $\sum_{\varsigma=1}^{\eta}\sigma_\varsigma = 1$. Then, by applying the SvNLNEWAA operator, the aggregation result of ξ'_ι ($\iota = 1, 2, \cdots, m$) is

$$\xi'_\iota = SvNLNEWAA\left(\xi'_{\iota 1}, \xi'_{\iota 2}, \cdots, \xi'_{\iota\eta}\right) = \sum_{\varsigma=1}^{\eta}\sigma_\varsigma \xi'_{\iota\varsigma} = \begin{pmatrix} \left\langle 1 - \prod_{\varsigma=1}^{\eta}\left(1 - T_{\iota\varsigma}\right)^{\sigma_\varsigma}, \prod_{\varsigma=1}^{\eta} U_{\iota\varsigma}^{\sigma_\varsigma}, \prod_{\varsigma=1}^{\eta} V_{\iota\varsigma}^{\sigma_\varsigma} \right\rangle, \\ \left\langle l_{r - r\prod_{\varsigma=1}^{\eta}\left(1 - \frac{\tau_{\iota\varsigma}}{r}\right)^{\sigma_\varsigma}}, l_{r\prod_{\varsigma=1}^{\eta}\left(\frac{u_{\iota\varsigma}}{r}\right)^{\sigma_\varsigma}}, l_{r\prod_{\varsigma=1}^{\eta}\left(\frac{v_{\iota\varsigma}}{r}\right)^{\sigma_\varsigma}} \right\rangle \end{pmatrix}.$$

Similarly, by applying the SvNLNEWGA operator, the aggregation result of ξ'_ι ($\iota = 1, 2, \cdots, m$) is

$$\xi_\iota^{'} = SvNLNEWGA\left(\xi_{\iota 1}^{'}, \xi_{\iota 2}^{'}, \cdots, \xi_{\iota \eta}^{'}\right) = \prod_{\varsigma=1}^{\eta}\left(\xi_{\iota\varsigma}^{'}\right)^{\sigma_\varsigma} = \begin{pmatrix} \left\langle \prod_{\varsigma=1}^{\eta} T_{\iota\varsigma}^{\sigma_\varsigma}, 1-\prod_{\varsigma=1}^{\eta}\left(1-U_{\iota\varsigma}\right)^{\sigma_\varsigma}, 1-\prod_{\varsigma=1}^{\eta}\left(1-V_{\iota\varsigma}\right)^{\sigma_\varsigma} \right\rangle, \\ \left\langle l_{r\prod_{\varsigma=1}^{\eta}\left(\frac{\tau_{\iota\varsigma}}{r}\right)^{\sigma_\varsigma}}, l_{r-r\prod_{\varsigma=1}^{\eta}\left(1-\frac{u_{\iota\varsigma}}{r}\right)^{\sigma_\varsigma}}, l_{r-r\prod_{\varsigma=1}^{\eta}\left(1-\frac{v_{\iota\varsigma}}{r}\right)^{\sigma_\varsigma}} \right\rangle \end{pmatrix}.$$

Step 3: Get the score values of $F\left(\xi_\iota^{'}\right)$ (ι = 1, 2, ⋯, m) by Eq. (1) and the accuracy values of $G\left(\xi_\iota^{'}\right)$ (ι = 1, 2, ⋯, m) by Eq. (2) if necessary.

Step 4: Sort all the alternatives in descending order of the score and accuracy values, then the first one is optimal.

Step 5: End.

6. Example

To illustrate the application of the raised MADM method in the SvNLNE environment, an example of investment decision is given in this section. This example is adapted from references [4,15] and contains both quantitative and qualitative attributes.

A company needs to choose the best of the four alternatives $\vartheta = \{\vartheta_1, \vartheta_2, \vartheta_3, \vartheta_4\}$ that are engaged in electronic devices, weapons, clothing, and construction, respectively. Then, some experts are asked to comprehensively assess the options by considering the attributes $E = \{\delta_1, \delta_2, \delta_3, \delta_4\}$, where δ_1 is the environmental impact, δ_2 is the growth, δ_3 is the risk, and δ_4 is the possible return rate of the investment. The evaluation data can be given in any form of LNN, SvNN, or SvNLNE according to the attribute characteristics and the preferences of the evaluators. Among them, the qualitative information will be evaluated from the LTS L = {l_0 = very low, l_1 = low, l_2 = slight low, l_3 = medium, l_4 = slight high, l_5 = high, l_6 = very high} with the odd cardinality r + 1 = 7. Suppose the original evaluation matrix $M = (\xi_{\iota\varsigma})_{m\times\eta}$ is established as

$$M = \begin{bmatrix} \left(\langle 0.7, 0.2, 0.2\rangle, \langle l_4, l_1, l_1\rangle\right) & \left(\langle l_5, l_1, l_1\rangle\right) & \left(\langle l_5, l_3, l_1\rangle\right) & \left(\langle 0.78, 0.1, 0.2\rangle\right) \\ \left(\langle 0.8, 0.1, 0.1\rangle, \langle l_5, l_2, l_2\rangle\right) & \left(\langle l_5, l_1, l_2\rangle\right) & \left(\langle l_6, l_2, l_1\rangle\right) & \left(\langle 0.8, 0.2, 0.3\rangle\right) \\ \left(\langle 0.75, 0.2, 0.1\rangle, \langle l_4, l_2, l_1\rangle\right) & \left(\langle l_4, l_2, l_1\rangle\right) & \left(\langle l_6, l_1, l_1\rangle\right) & \left(\langle 0.75, 0.1, 0.1\rangle\right) \\ \left(\langle 0.9, 0.1, 0.1\rangle, \langle l_5, l_1, l_1\rangle\right) & \left(\langle l_5, l_2, l_2\rangle\right) & \left(\langle l_4, l_2, l_3\rangle\right) & \left(\langle 0.81, 0.2, 0.1\rangle\right) \end{bmatrix}.$$

According to the information standardization rules of SvNLNE, the matrix M can be standardized as

$$M^{'} = \begin{bmatrix} \left(\langle 0.7, 0.2, 0.2\rangle, \langle l_4, l_1, l_1\rangle\right) & \left(\langle 0.83, 0.17, 0.17\rangle, \langle l_5, l_1, l_1\rangle\right) \\ \left(\langle 0.8, 0.1, 0.1\rangle, \langle l_5, l_2, l_2\rangle\right) & \left(\langle 0.83, 0.17, 0.33\rangle, \langle l_5, l_1, l_2\rangle\right) \\ \left(\langle 0.75, 0.2, 0.1\rangle, \langle l_4, l_2, l_1\rangle\right) & \left(\langle 0.67, 0.33, 0.17\rangle, \langle l_4, l_2, l_1\rangle\right) \\ \left(\langle 0.9, 0.1, 0.1\rangle, \langle l_5, l_1, l_1\rangle\right) & \left(\langle 0.83, 0.33, 0.33\rangle, \langle l_5, l_2, l_2\rangle\right) \end{bmatrix}$$

$$\begin{bmatrix} \left(\langle 0.83, 0.5, 0.17\rangle, \langle l_5, l_3, l_1\rangle\right) & \left(\langle 0.78, 0.1, 0.2\rangle, \langle l_{4.68}, l_{0.6}, l_{1.2}\rangle\right) \\ \left(\langle 1, 0.33, 0.17\rangle, \langle l_6, l_2, l_1\rangle\right) & \left(\langle 0.8, 0.2, 0.3\rangle, \langle l_{4.8}, l_{1.2}, l_{1.8}\rangle\right) \\ \left(\langle 1, 0.17, 0.17\rangle, \langle l_6, l_1, l_1\rangle\right) & \left(\langle 0.75, 0.1, 0.1\rangle, \langle l_{4.5}, l_{0.6}, l_{0.6}\rangle\right) \\ \left(\langle 0.67, 0.33, 0.5\rangle, \langle l_4, l_2, l_3\rangle\right) & \left(\langle 0.81, 0.2, 0.1\rangle, \langle l_{4.86}, l_{1.2}, l_{0.6}\rangle\right) \end{bmatrix}.$$

Assuming the weight vector P = (0.25, 0.2, 0.25, 0.3) represents the attribute importance of E, the decision process using the SvNLNEWAA operator can be performed as below.

Step 1. By Eq. (3), the aggregated values of SvNLNEWAA for each alternative ϑ_ι (ι = 1, 2, 3, 4) can be obtained as

ξ_1= (<0.7902, 0.197, 0.1842>, <4.7075,1.1291,1.0562>),

ξ_2 = (<1,0.1842,0.201>, <6,1.4937,1.6295>),

ξ_3 = (<1,0.1719,0.1258>, <6,1.1719,0.8579>),

ξ_4 = (<0.8186,0.2116,0.1902>, <4.7631,1.4428,1.297>).

Step 2. By Eq. (1), the score values of $F(\xi_\iota)$ (ι = 1, 2, 3, 4) can be further obtained as

$F(\xi_1)$ =0.8049, $F(\xi_2)$ = 0.849, $F(\xi_3)$ = 0.894, $F(\xi_4)$ = 0.7923.

Step 3. Since $F(\xi_3) > F(\xi_2) > F(\xi_1) > F(\xi_4)$, the ranking of the four alternatives is $\vartheta_3 > \vartheta_2 > \vartheta_1 > \vartheta_4$. Therefore, ϑ_3 is the best choice.

Similarly, the decision steps using the SvNLNEWGA operator can be carried out as below.

Step 1'. By Eq. (4), the aggregated values of SvNLNEWGA for each alternative ϑ_ι (ι = 1, 2, 3, 4) can be obtained as

ξ_1= (<0.7821,0.2571,0.1852>, <4.6358,1.4967,1.0609>),

ξ_2 = (<0.8528,0.2064,0.229>, <5.1695,1.5823,1.7082>),

ξ_3 = (<0.7872,0.1927,0.1306>, <4.5859,1.3721,0.8832>),

ξ_4 = (<0.7966,0.241, 0.2683>, <4.6886,1.5327,1.6932>).

Step 2'. By Eq. (1), the score values of $F(\xi_\iota)$ (ι = 1, 2, 3, 4) are

$F(\xi_1)$ = 0.781, $F(\xi_2)$ = 0.7884, $F(\xi_3)$ = 0.8087, $F(\xi_4)$ = 0.7552.

Step 3'. Since $F(\xi_3) > F(\xi_2) > F(\xi_1) > F(\xi_4)$, the four alternatives are ranked as $\vartheta_3 > \vartheta_2 > \vartheta_1 > \vartheta_4$. Thus, ϑ_3 is also the best choice.

Obviously, the sorting results obtained by the above two operators are the same, and the best options are also the same. Thus, one can choose one of the two operators according to the actual needs.

Different from the existing MADM approaches, the MADM method proposed in this paper handles the incomplete, inconsistent and uncertain data in the form of SvNLNE instead of SvNN or LNN, and uses two novel aggregation operators of SvNLNEWAA and SvNLNEWGA. The SvNLNE composed of SvNN and LNN uses numerical and linguistic variables to represent the truth, uncertainty, and falsity membership degrees of fuzzy information. Hence, it can express mixed information of quantitative and qualitative attributes better than SvNN or LNN that can only depict quantitative or qualitative attribute information. Moreover, the proposed SvNLNEWAA and SvNLNEWGA operators can aggregate SvNNs and LNNs in addition to SvNLNEs, because SvNN and LNN are two special cases of SvNLNE when all attributes are quantitative or qualitative. And the proposed MADM method can handle DM problems in the SvNN and/or LNN setting, while the existing DM methods of SvNN and LNN cannot deal with DM problems under the SvNLNE environment.

All in all, SvNLNE is the further generalization of SvNN and LNN, and the MADM method based on the SvNLNEWAA and SvNLNEWGA operators offers a unified way for complex DM problems with both quantitative and qualitative attributes.

7. Conclusions

This paper originally defined the concept, fundamental operations, and score and accuracy functions of SvNLNE, and then developed the MADM method of the SvNLNE using the proposed SvNLNEWAA and SvNLNEWGA operators. Finally, an investment case proved that the proposed MADM method can effectively solve MADM problems with the SvNLNEs that contain mixed-type or single-type attribute information, overcoming the shortcomings of traditional methods that can only handle single-type attribute data. The research results of this paper enrich the neutrosophic theory and MADM methods.

The paper mainly contributes: (1) The presented SvNLNE can effectively express mixed quantitative and qualitative information for the first time; (2) The proposed SvNLNEWAA and SvNLNEWGA operators can aggregate the hybrid information of SvNN and LNN; (3) The proposed MADM approach of SvNLNS can effectively solve complex DM problems containing qualitative and quantitative attributes, which cannot be satisfactorily processed by existing methods.

Further research will concentrate on the similarity measures of SvNLNEs, the development of novel aggregation operators, and their applications such as pattern recognition and medical diagnosis in the SvNLNE environment.

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