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Operations on Neutrosophic Vague Soft Graphs

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Abstract: This article concerns with the neutrosophic vague soft graphs for treating neutrosophic vague soft information by employing the theory of neutrosophic vague soft sets with graphs. Operations like Cartesian product, cross product, lexicographic product, strong product and composition of neutrosophic vague soft graphs are established. The proposed concepts are explained with examples.

Keywords: Cartesian product; Neutrosophic vague soft graph; Operations on neutrosophic vague soft graph.
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1 Introduction

Nowadays, the great success of neutrosophic sets in modelling natural phenomena is that its efficiency to hold incomplete data, and handling of indeterminate information. It is the base of neutrosophic logic, a multiple value logic that generalizes the fuzzy logic that carries with paradoxes, contradictions, antitheses, antinomies, invented by the author Smarandache [8, 21, 22]. For example, suppose there are hundred patients to check a pandemic during testing. In that time, there are thirty patients having positive, fifty will have negative, and twenty are undecided or yet to come. By employing the neutrosophic concepts it can be expressed as $x(0.3,0.2,0.5)$. Hence the neutrosophic field arises to hold the indeterminacy data more accurately. It generalizes many concepts from the philosophical viewpoint. The single-valued neutrosophic set is the generalization of intuitionistic fuzzy sets and is used expediently to deal with real-world problems, especially in decision support [23]. The computation of belief in that element (truth), they disbelieve in that element (falsehood) and the indeterminacy part of that element with the sum of these three components are strictly less than 1. Neutrosophic set and related notions have shown applications in many different fields. In the definition of single valued neutrosophic set, the indeterminacy value is quantified explicitly and truth-membership, indeterminacy membership, and false-membership are defined completely independent with the sum of these values lies between 0 and 3 (see [21]-[23]). The indeterminacy function is considered as an individual term and each element x is characterized by a truth-membership function $\mathcal{T}_A(x)$, an indeterminacy membership function $\mathcal{I}_A(x)$ and a falsity-membership function $\mathcal{F}_A(x)$, each of that from the non-standard unit interval $]0^-,1^+[$. Despite the neutrosophic indeterminacy is independent of the truth and falsity-membership values, but it is more general than the hesitation margin of intuitionistic fuzzy sets. It's not sure whether the indeterminacy values relevant to a particular element correspond to hesitant values about its belonging or not belonging to it. In another way, if a person identifies an indeterminacy membership $\mathcal{I}_A(x)$ with a specific event x, it becomes difficult to understand whether the person's degree of uncertainty regarding the event's occurrence is $\mathcal{I}_A(x)$ or whether the person's degree of uncertainty regarding the event's non-occurrence is $\mathcal{I}_A(x)$. As a result, some authors prefer to model the indeterminacy's behaviour in the same way they similar to truth-membership, others may prefer to model it, in the same way, they similar to falsity-membership. Wang et al. [23] initiated the concept of a single-valued neutrosophic set and provide its various properties.

Molodtsov [20] successfully proposed a completely new theory namely soft set theory by using classical sets in 1999 and after that, there has been a rapid development of interest in soft sets and their various applications [7, 9]. This theory provides a parametrized point of view for uncertainty modelling and soft computing. Vague sets are considered as a particular case of context-dependent fuzzy sets. Vague sets are studied by Gau and Buehrer [24] as an extension of fuzzy set theory. Neutrosophic soft rough graphs with applications are established in [5]. Neutrosophic soft relations and neutrosophic refined relations with their properties are studied in [8, 18]. Recently, the generalization of neutrosophic graphs are developed in [25]. Also, neutrosophic soft graphs, neutrosophic graphs, co-neutrosophic graphs,single valued neutrosophic graphs are established in [2, 10, 12]. Neutrosophic vague set is first invented by the author in [6]. In [3], the authors studied the notion of neutrosophic vague soft expert set as a combination of soft expert set and neutrosophic vague set to the substantial improvement in decision making. Also, the neutrosophic vague soft set is studied by him [4] with application in decision making problems. In [2], the certain notions, including neutrosophic soft graphs, strong neutrosophic soft graphs, complete neutrosophic soft graphs are discussed. Neutrosophic vague graphs are introduced in [15]. Intuitionistic bipolar neutrosophic set and its application to graphs are established in [16]. Motivated by papers [2, 4, 6, 15, 17], we introduce the concept of operations on neutrosophic vague soft graphs. The major contributions of this work are as follows:

- In this paper, we present a novel frame work for handling neutrosophic vague soft information by combining the theory of neutrosophic vague soft sets with graphs.
- The operations on neutrosophic vague soft graphs are established and this manuscript makes the first attempt in this domain. Some basic definitions regarding to neutrosophic vague graphs are explained with example.
- Results on the Cartesian product, cross product, lexicographic product, strong product and composition of neutrosophic vague graph are illustrated with examples.
- The validity of the developed method is verifying in multi-attribute decision-making method based on neutrosophic vague soft graphs.
- Finally, a conclusion is elaborated with future direction.

The paper is organised as follows: Some elementary definitions and results are provided in Section 2. Operations on neutrosophic vague soft graphs with example are established in Section 3. In Section 4, the multi-attribute decision making method is solved for neutrosophic vague soft graphs, in that, the solving procedure is based on the score function S_{ij} [25]. Finally, the advantages and limitations of the proposed concepts are given.

2 Preliminaries

In this section, basic definitions and example are given.

Definition 2.1 *[20] Let $\mathcal{U}$ be the universe of discourse and $\mathcal{P}$ be the universe of all possible parameters related to the objects in $\mathcal{U}$. Each parameters are considered to be attributes, characteristics of objects in $\mathcal{U}$. The pair $(\mathcal{U},\mathcal{P})$ is also known as a soft universe. The power set of $\mathcal{U}$ is dentoted as $\rho(\mathcal{U})$*

Definition 2.2 *[20] A pair (F, A) is called soft set over $\mathcal{U}$, where $A \subseteq P$, F is a set- valued function $F: A \to \rho(\mathcal{U})$. In other words, a soft set over $\mathcal{U}$ is a parametrized family of subsets of $\mathcal{U}$.*

By means of parametrization, a soft set produces a series of approximate descriptions of a complicated object being perceived from various points of view. It is apparent that a soft set $F_A = (F, A)$ over a universe $\mathcal{U}$. For any parameter $\epsilon \in A$, the subset $F(\epsilon) \subseteq \mathcal{U}$ may be interpreted as the set of ϵ-approximate elements.

Definition 2.3 *[19] Let $\mathcal{U}$ be an initial universe and P be a set of parameters. Consider $A \subseteq P$. Let $p(\mathcal{U})$ denotes the set of all neutrosophic sets of $\mathcal{U}$. The collection of (F, A) is termed to be neutrosophic soft set over $\mathcal{U}$, where F is a mapping given by $F: A \to P(\mathcal{U})$.*

Definition 2.4 *[24] A vague set A on a non empty set X is a pair (T_A, F_A), where $T_A: X \to [0,1]$ and $F_A: X \to [0,1]$ are true membership and false membership functions, respectively, such that*

$$0 \leq T_A(x) + F_A(y) \leq 1 \text{ for any } x \in X.$$

Let X and Y be two non-empty sets. A vague relation R of X to Y is a vague set R on $X \times Y$ that is $R = (T_R, F_R)$, where $T_R: X \times Y \to [0,1], F_R: X \times Y \to [0,1]$ and satisfy the condition:

$$0 \leq T_R(x,y) + F_R(x,y) \leq 1 \text{ for any } x, y \in X.$$

Definition 2.5 *[7] Let $G^* = (V, E)$ be a graph. A pair $G = (J, K)$ is called a vague graph on G^*, where $J = (T_J, F_J)$ is a vague set on V and $K = (T_K, F_K)$ is a vague set on $E \subseteq V \times V$ such that for each $xy \in E$,*

$$T_K(xy) \leq \min(T_J(x), T_J(y)) \text{ and } F_K(xy) \geq \max(T_J(x), F_J(y)).$$

Definition 2.6 *[21] Let X be a space of points (objects), with generic elements in X denoted by x. A single valued neutrosophic set A in X is characterised by truth-membership function $T_A(x)$, indeterminacy-membership function $I_A(x)$ and falsity-membership-function $F_A(x)$,*

For each point x in X, $T_A(x), F_A(x), I_A(x) \in [0,1]$. Also

$$A = \{x, T_A(x), F_A(x), I_A(x)\} \text{ and } 0 \leq T_A(x) + I_A(x) + F_A(x) \leq 3.$$

Definition 2.7 *[1] A single valued neutrosophic graph is defined as a pair $G = (J, K)$ where*

(i) $V = \{v_1, v_2, .., v_n\}$ such that $T_J: V \to [0,1]$, $I_J: V \to [0,1]$ and $F_J: V \to [0,1]$ denote the degree of truth-membership function, indeterminacy function and falsity-membership function, respectively and

$$0 \leq T_J(v) + I_J(v) + F_J(v) \leq 3,$$

(ii) $E \subseteq V \times V$ where $T_K: E \to [0,1]$, $I_K: E \to [0,1]$ and $F_K: E \to [0,1]$ are such that

$$T_K(uv) \leq \min\{T_J(u), T_J(v)\},$$
$$I_K(uv) \leq \min\{I_J(u), I_J(v)\},$$
$$F_K(uv) \leq \max\{F_J(u), F_J(v)\},$$
$$\text{and } 0 \leq T_K(uv) + I_K(uv) + F_K(uv) \leq 3, \ \forall uv \in E.$$

Definition 2.8 *[21] A Neutrosophic set A is contained in another neutrosophic set B, (i.e) $A \subseteq B$ if $\forall x \in X, T_A(x) \leq T_B(x), I_A(x) \geq I_B(x)$ and $F_A(x) \geq F_B(x)$.*

Definition 2.9 *[6] A Neutrosophic Vague Set A_{NV} (NVS in short) on the universe of discourse X written as*

$$A_{NV} = \{\langle x, \hat{T}_{A_{NV}}(x), \hat{I}_{A_{NV}}(x), \hat{F}_{A_{NV}}(x)\rangle, x \in X\},$$

whose truth-membership, indeterminacy membership and falsity-membership functions are defined as

$$\hat{T}_{A_{NV}}(x) = [T^-(x), T^+(x)], \hat{I}_{A_{NV}}(x) = [I^-(x), I^+(x)] \text{and } \hat{F}_{A_{NV}}(x) = [F^-(x), F^+(x)],$$

where $T^+(x) = 1 - F^-(x), F^+(x) = 1 - T^-(x)$, and $0 \leq T^-(x) + I^-(x) + F^-(x) \leq 2$.

Definition 2.10 *[6] The complement of NVS A_{NV} is denoted by A^c_{NV} and it is defined by*

$$\hat{T}^c_{A_{NV}}(x) = [1 - T^+(x), 1 - T^-(x)],$$
$$\hat{I}^c_{A_{NV}}(x) = [1 - I^+(x), 1 - I^-(x)],$$
$$\hat{F}^c_{A_{NV}}(x) = [1 - F^+(x), 1 - F^-(x)].$$

Definition 2.11 *[6] Let A_{NV} and B_{NV} be two NVSs of the universe U. If for all $u_i \in U$,*

$$\hat{T}_{A_{NV}}(u_i) \leq \hat{T}_{B_{NV}}(u_i), \hat{I}_{A_{NV}}(u_i) \geq \hat{I}_{B_{NV}}(u_i), \hat{F}_{A_{NV}}(u_i) \geq \hat{F}_{B_{NV}}(u_i).$$

Then, the NVSs, A_{NV} are included in B_{NV}, denoted by $A_{NV} \subseteq B_{NV}$ where $1 \leq i \leq n$.

Definition 2.12 *[6] The union of two NVSs A_{NV} and B_{NV} is an NVSs, C_{NV}, written as $C_{NV} = A_{NV} \cup B_{NV}$, whose truth membership function, indeterminacy-membership function and false-membership function are related to those of A_{NV} and B_{NV} by*

$$\hat{T}_{C_{NV}}(x) = [\max(T^-_{A_{NV}}(x), T^-_{B_{NV}}(x)), \max(T^+_{A_{NV}}(x), T^+_{B_{NV}}(x))]$$
$$\hat{I}_{C_{NV}}(x) = [\min(I^-_{A_{NV}}(x), I^-_{B_{NV}}(x)), \min(I^+_{A_{NV}}(x), I^+_{B_{NV}}(x))]$$
$$\hat{F}_{C_{NV}}(x) = [\min(F^-_{A_{NV}}(x), F^-_{B_{NV}}(x)), \min(F^+_{A_{NV}}(x), F^+_{B_{NV}}(x))].$$

Definition 2.13 *[6] The intersection of two NVSs, A_{NV} and B_{NV} is an NVSs C_{NV}, written as $C_{NV} = A_{NV} \cap B_{NV}$, whose truth-membership function, indeterminacy-membership function and false-membership function are related to those of A_{NV} and B_{NV} by*

$$\hat{T}_{C_{NV}}(x) = [\min(T^-_{A_{NV}}(x), T^-_{B_{NV}}(x)), \min(T^+_{A_{NV}}(x), T^+_{B_{NV}}(x))]$$
$$\hat{I}_{C_{NV}}(x) = [\max(I^-_{A_{NV}}(x), I^-_{B_{NV}}(x)), \max(I^+_{A_{NV}}(x), I^+_{B_{NV}}(x))]$$
$$\hat{F}_{C_{NV}}(x) = [\max(F^-_{A_{NV}}(x), F^-_{B_{NV}}(x)), \max(F^+_{A_{NV}}(x), F^+_{B_{NV}}(x))].$$

Definition 2.14 *[19] Let $\mathcal{U}$ be a universe, E a set of parameters and $A \subseteq E$. A collection of pairs (F, A) is called a neutrosophic vague soft set (NVSS) over $\mathcal{U}$ where F is a mapping given by $F: A \to NV(\mathcal{U})$ and $NV(\mathcal{U})$ denotes the set of all neutrosophic vague subsets of $\mathcal{U}$.*

Definition 2.15 *[2] A neutrosophic soft graph $G = (G^*, J, K, R)$ is an ordered four tuple if it satisfies the following conditions:*

- $G^* = (V, E)$ is a simple graph,
- R is a non-empty set of parameters,
- (J, R) is a neutrosophic soft set over V,
- (K, R) is a neutrosophic soft set over E,
- $(J(e), K(e))$ is a neutrosophic graph of G^*, that is,

$$T_{K(e)}(ab) \leq \min\{T^-_{J(e)}(a), T^-_{J(e)}(b)\},$$
$$I_{K(e)}(ab) \leq \min\{I^-_{J(e)}(a), I^-_{J(e)}(b)\},$$
$$F_{K(e)}(ab) \leq \max\{F^-_{J(e)}(a), F^-_{J(e)}(b)\}$$

such that,

$$0 \leq T_{K(e)}(ab) + I_{K(e)}(ab) + T_{K(e)}(ab) \leq 3 \ \ \forall e \in R, a, b \in V.$$

For convenience, the neutrosophic graph $(J(e), K(e))$ is denoted by $H(e)$. A neutrosophic vague soft graph is a parametrized family of neutrosophic graphs.

Definition 2.16 *[15] Let $G^* = (R, S)$ be a graph. A pair $G = (A, B)$ is called a neutrosophic vague graph (NVG) on G^* or a neutrosophic vague graph where $A = (\hat{T}_A, \hat{I}_A, \hat{F}_A)$ is a neutrosophic vague set on R and $B = (\hat{T}_B, \hat{I}_B, \hat{F}_B)$ is a neutrosophic vague set $S \subseteq R \times R$ where*

(1)$R = \{v_1, v_2, \ldots, v_n\}$ such that $T^-_A: R \to [0,1], I^-_A: R \to [0,1], F^-_A: R \to [0,1]$ satisfies the condition $F^-_A = [1 - T^+_A]$, and $T^+_A: R \to [0,1], I^+_A: R \to [0,1], F^+_A: R \to [0,1]$ which satisfies the condition $F^+_A = [1 - T^-_A]$, denote the degrees of truth membership, indeterminacy membership and falsity membership of the element $v_i \in R$, and

$$0 \leq T^-_A(v_i) + I^-_A(v_i) + F^-_A(v_i) \leq 2$$
$$0 \leq T^+_A(v_i) + I^+_A(v_i) + F^+_A(v_i) \leq 2.$$

(2) $S \subseteq R \times R$ where

$$T^-_B: R \times R \to [0,1], I^-_B: R \times R \to [0,1], F^-_B: R \times R \to [0,1]$$
$$T^+_B: R \times R \to [0,1], I^+_B: R \times R \to [0,1], F^+_B: R \times R \to [0,1]$$

denote the degrees of truth membership, indeterminacy membership and falsity membership of the element $v_i, v_j \in S$, respectively and such that,

$$0 \leq T_B^-(v_i v_j) + I_B^-(v_i v_j) + F_B^-(v_i v_j) \leq 2$$
$$0 \leq T_B^+(v_i v_j) + I_B^+(v_i v_j) + F_B^+(v_i v_j) \leq 2,$$

such that

$$T_B^-(v_i v_j) \leq \min\{T_A^-(v_i), T_A^-(v_j)\}$$
$$I_B^-(v_i v_j) \leq \min\{I_A^-(v_i), I_A^-(v_j)\}$$
$$F_B^-(v_i v_j) \leq \max\{F_A^-(v_i), F_A^-(v_j)\},$$

and similarly

$$T_B^+(v_i v_j) \leq \min\{T_A^+(v_i), T_A^+(v_j)\}$$
$$I_B^+(v_i v_j) \leq \min\{I_A^+(v_i), I_A^+(v_j)\}$$
$$F_B^+(v_i v_j) \leq \max\{F_A^+(v_i), F_A^+(v_j)\}.$$

Example 2.17 *Consider a neutrosophic vague graph $G = (A, B)$ such that $A = \{a, b, c\}$ and $B = \{ab, bc, ca\}$ are defined by*

$$\hat{a} = T[0.5,0.5], I[0.4,0.3], F[0.5,0.5], \qquad \hat{b} = T[0.4,0.6], I[0.7,0.3], F[0.4,0.6],$$
$$\hat{c} = T[0.4,0.4], I[0.5,0.3], F[0.6,0.6]$$

where $\hat{a}, \hat{b}, \hat{c}$ are the neutrosophic vague sets on A. Now, $\hat{a} = (a^-, a^+), \hat{b} = (b^-, b^+), \hat{c} = (c^-, c^+)$.

$$a^- = (0.5,0.4,0.4), b^- = (0.4,0.7,0.4), c^- = (0.4,0.5,0.6)$$
$$a^+ = (0.5,0.3,0.5), b^+ = (0.6,0.3,0.6), c^+ = (0.4,0.3,0.6).$$

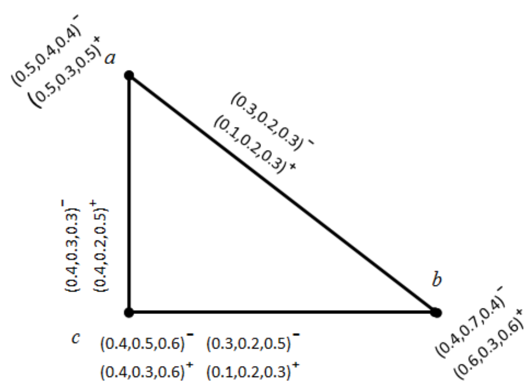


Figure 1 NEUTROSOPHIC VAGUE GRAPH

Definition 2.18 *A partial neutrosophic vague subgraph of netrosophic vague graph $G = (A, B)$ is a neutrosophic vague graph $G^* = (V', E')$ such that*

- $V' \subseteq V$ where $\hat{T}'_A(v_i) \leq \hat{T}_A(v_i)$, $\hat{I}'_A(v_i) \leq \hat{I}_A(v_i)$ and $\hat{F}'_A(v_i) \geq \hat{F}_A(v_i)$ for all $v_i \in V$.
- $E' \subseteq E$ where $\hat{T}'_B(v_i, v_j) \leq \hat{T}_B(v_i, v_j)$, $\hat{I}'_B(v_i, v_j) \leq \hat{I}_B(v_i, v_j)$ and $\hat{F}'_B(v_i, v_j) \geq \hat{F}_B(v_i, v_j)$ for all $(v_i, v_j) \in E$.

3 Operations on Neutrosophic Vague Soft Graphs

In this section, the results on operations of neutrosophic vague soft graphs with example are established.

Let $\mathcal{U}$ be an initial universe and P be the set of all parameters. $P(\mathcal{U})$ denotes the set of all neutrosophic vague soft sets of $\mathcal{U}$. Let A be a subset of P. A pair (F, A) is called a neutrosophic vague soft set over $\mathcal{U}$. Let $P(V)$ denotes the set of all neutrosophic vague sets of V and $P(E)$ denotes the set of all neutrosophic vague sets of E.

Definition 3.1 *A neutrosophic vague soft graph* $G = (G^*, J, K, R)$ *is an ordered four tuple if it satisfies the following conditions:*

- $G^* = (V, E)$ is a simple graph,
- R is a non-empty set of parameters,
- (J, R) is a neutrosophic vague soft set over V,
- (K, R) is a neutrosophic vague soft set over E,
- $(J(e), K(e))$ is a neutrosophic vague graph of G^*, that is,

$$T^-_{K(e)}(ab) \le \min\{T^-_{J(e)}(a), T^-_{J(e)}(b)\},$$
$$I^-_{K(e)}(ab) \le \min\{I^-_{J(e)}(a), I^-_{J(e)}(b)\},$$
$$F^-_{K(e)}(ab) \le \max\{F^-_{J(e)}(a), F^-_{J(e)}(b)\}$$
$$T^+_{K(e)}(ab) \le \min\{T^+_{J(e)}(a), T^+_{J(e)}(b)\},$$
$$I^+_{K(e)}(ab) \le \min\{I^+_{J(e)}(a), I^+_{J(e)}(b)\},$$
$$F^+_{K(e)}(ab) \le \max\{F^+_{J(e)}(a), F^+_{J(e)}(b)\}$$

such that,

$$0 \le T^-_{K(e)}(ab) + I^-_{K(e)}(ab) + F^-_{K(e)}(ab) \le 2,$$
$$0 \le T^+_{K(e)}(ab) + I^+_{K(e)}(ab) + F^+_{K(e)}(ab) \le 2, \quad \forall e \in R,\ a, b \in V.$$

For the convenience, the neutrosophic vague graph $(J(e), K(e))$ is denoted by $H(e)$. A neutrosophic vague soft graph is a parametrized family of neutrosophic vague graphs.

Definition 3.2 *Let* $G_1 = (J_1, K_1, R)$ *and* $G_2 = (J_2, K_2, S)$ *be two neutrosophic vague soft graphs of* G^*. *Then* G_1 *is neutrosophic vague soft subgraph of* G_2 *if*

- $R \subseteq S$.
- $H_1(e)$ partial neutrosophic vague subgraph of $H_2(e)$ for all $e \in R$.

Example 3.3 *Consider a simple graph* $G^* = (V, E)$ *such that* $V = \{v_1, v_2, v_3, v_4\}$ *and*

$$E = \{v_1v_2, v_1v_3, v_1v_4, v_2v_4, v_3v_4\}.$$

Let $R = \{e_1, e_2\}$ be a set of parameters and let (J, R) be a neutrosophic vague soft set over V with neutrosophic approximation function $J: R \to \rho(V)$ defined by

$$J(e_1) = \hat{v}_1 = T[0.4,0.4], I[0.3,0.4], F[0.6,0.6], \quad \hat{v}_2 = T[0.3,0.7], I[0.3,0.5], F[0.3,0.7],$$
$$\hat{v}_3 = T[0.5,0.6], I[0.4,0.2], F[0.4,0.5], \quad \hat{v}_4 = T[0.8,0.3], I[0.5,0.6], F[0.7,0.2]$$
$$J(e_1) = v_1^- = (0.4,0.3,0.6), v_2^- = (0.3,0.3,0.3), v_3^- = (0.5,0.4,0.4), v_4^- = (0.8,0.5,0.7)$$
$$J(e_1) = v_1^+ = (0.4,0.4,0.6), v_2^+ = (0.7,0.5,0.7), v_3^+ = (0.6,0.2,0.5), v_4^+ = (0.2,0.6,0.2).$$
$$J(e_2) = \hat{v}_1 = T[0.5,0.4], I[0.4,0.5], F[0.6,0.5], \quad \hat{v}_2 = T[0.4,0.6], I[0.5,0.6], F[0.4,0.6],$$
$$\hat{v}_3 = T[0.6,0.6], I[0.4,0.4], F[0.4,0.4], \quad \hat{v}_4 = T[0.7,0.3], I[0.6,0.3], F[0.7,0.3]$$
$$J(e_2) = v_1^- = (0.5,0.4,0.6), v_2^- = (0.3,0.3,0.3), v_3^- = (0.5,0.4,0.4), v_4^- = (0.8,0.5,0.7)$$
$$J(e_2) = v_1^+ = (0.4,0.5,0.5), v_2^+ = (0.7,0.5,0.7), v_3^+ = (0.6,0.2,0.5), v_4^+ = (0.3,0.6,0.2).$$

Let (K, R) be a neutrosophic vague soft set over E with neutrosophic approximation function $K: R \to \rho(E)$ defined by

$$K(e_1) = \{(v_1v_2)^- = (0.3,0.2,0.5)^-, (v_1v_2)^+ = (0.3,0.3,0.6)^+, (v_1v_3)^- = (0.4,0.3,0.4)^-, (v_1v_3)^+ = (0.3,0.2,0.5)^+, (v_1v_4)^- = (0.3,0.3,0.5)^-, (v_1v_4)^+ = (0.1,0.2,0.3)^+\}$$
$$K(e_2) = \{(v_1v_2)^- = (0.4,0.3,0.5)^-, (v_1v_2)^+ = (0.3,0.1,0.4)^+, (v_1v_3)^- = (0.2,0.2,0.5)^-, (v_1v_3)^+ = (0.3,0.5,0.7)^+, (v_1v_4)^- = (0.4,0.3,0.6)^-, (v_1v_4)^+ = (0.3,0.2,0.5)^+\}.$$

Clearly, $H(e_1) = (J(e_1), K(e_1))$ and $H(e_2) = (J(e_2), K(e_2))$ are neutrosophic vague graphs corresponding to the parameters e_1 and e_2 respectively as shown in Figure 2

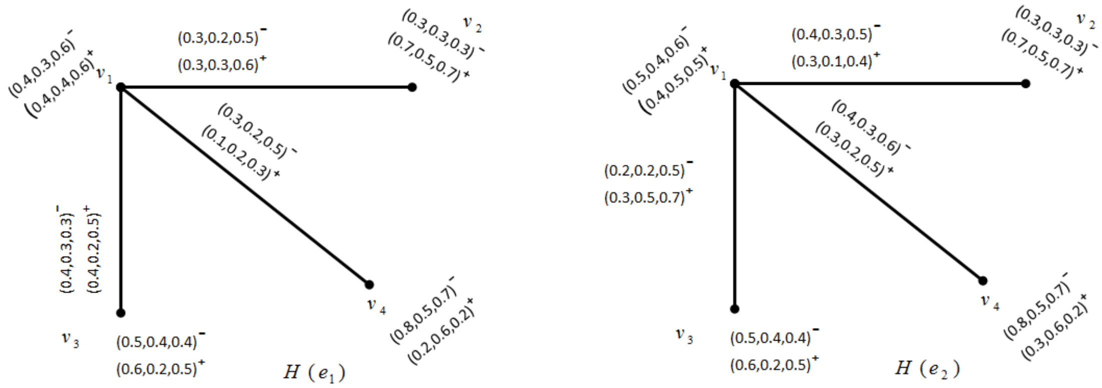


Figure 2: Neutrosophic vague soft graph

Definition 3.4 *The neutrosophic vague soft graph* $G_1 = (G^*, J_1, K_1, A)$ *is called spanning neutrosophic vague soft subgraph of* $G = (G^*, J, K, B)$ *if*

- $A \subset B$.
- $\hat{T}_{J_1(e)}(v) = \hat{T}_{J(e)}(v)$, $\hat{I}_{J_1(e)}(v) = \hat{I}_{J(e)}(v)$ $\hat{F}_{J_1(e)}(v) = \hat{F}_{J(e)}(v)$ for all $e \in A, v \in V$.

Definition 3.5 *Let* $G_1 = (J_1, K_1, R)$ *and* $G_2 = (J_2, K_2, S)$ *be two neutrosophic vague soft graphs of* $G_1^* = (V_1, E_1)$ *and* $G_2^* = (V_2, E_2)$*, respectively. The Cartesian product of* G_1 *and* G_2 *is* $G = G_1 \times G_2 = (J, K, R \times S)$*, where* $(J = J_1 \times J_2, R \times S)$ *is a neutrosophic vague soft set over* $V = V_1 \times V_2$, $(K = K_1 \times K_2, R \times S)$ *is a neutrsophic vague soft set over* $E = \{((u, v_1), (u, v_2)) : u \in V_1, (v_1, v_2) \in E_2\} \cup \{((u_1, v), (u_2, v)) : v \in V_2, (u_1, u_2) \in E_1\}$ *such that,*

$(i)\ \hat{T}_{J(a,b)}(u, v) = \hat{T}_{J_1(a)}(u) \wedge \hat{T}_{J_2(b)}(v)$,
$\hat{I}_{J(a,b)}(u, v) = \hat{I}_{J_1(a)}(u) \wedge \hat{I}_{J_2(b)}(v)$,
$\hat{F}_{J(a,b)}(u, v) = \hat{F}_{J_1(a)}(u) \vee \hat{F}_{J_2(b)}(v)$,
$\forall (u, v) \in V, (a, b) \in R \times S$.
$(ii)\ \hat{T}_{K(a,b)}((u, v_1), (u, v_2)) = \hat{T}_{J_1(a)}(u) \wedge \hat{T}_{K_2(b)}(v_1, v_2)$,
$\hat{I}_{K(a,b)}((u, v_1), (u, v_2)) = \hat{I}_{J_1(a)}(u) \wedge \hat{I}_{K_2(b)}(v_1, v_2)$,
$\hat{F}_{K(a,b)}((u, v_1), (u, v_2)) = \hat{F}_{J_1(a)}(u) \vee \hat{F}_{K_2(b)}(v_1, v_2)$,
$\forall u \in V_1, (v_1, v_2) \in E_2$.
$(iii)\ \hat{T}_{K(a,b)}((u_1, v), (u_2, v)) = \hat{T}_{J_2(a)}(v) \wedge \hat{T}_{K_2(b)}(u_1, u_2)$,
$\hat{I}_{K(a,b)}((u_1, v), (u_2, v)) = \hat{I}_{J_2(a)}(v) \wedge \hat{I}_{K_2(b)}(u_1, u_2)$,
$\hat{F}_{K(a,b)}((u_1, v), (u_2, v)) = \hat{F}_{J_2(a)}(v) \vee \hat{F}_{K_2(b)}(u_1, u_2)$,
$\forall v \in V_2, (u_1, u_2) \in E_1$.

Theorem 3.6 *The Cartesian product of two neutrosophic vague soft graphs is a neutrosophic vague soft graph.*

Proof. Let $G_1 = (J_1, K_1, R)$ and $G_2 = (J_2, K_2, S)$ be two neutrosophic vague soft graphs of $G_1^* = (V_1, E_1)$ and $G_2^* = (V_2, E_2)$, respectively. Let $G = G_1 \times G_2 = (J, K, R \times S)$ be the Cartesian product of G_1 and G_2. We claim that $G = (J, K, R \times S)$ is a neutrosophic vague soft graph and $(H, R \times S) = \{(J_1 \times J_2)(a_i, b_j), (K_1 \times K_2)(a_i, b_j)\}$ $\forall a_i \in R, b_j \in S$ for $i = 1,2,\ldots,m, j = 1,2,\ldots,n$ are neutrosophic vague graphs of G.

Consider,

$\hat{T}_{K(a_i,b_j)}((u, v_1), (u, v_2)) = \min\{\hat{T}_{J_1(a_i)}(u), \hat{T}_{K_2(b_j)}(v_1, v_2)\}$
for $i = 1,2,..,m, j = 1,2,\ldots n$
$\leq \min\{\hat{T}_{J_1(a_i)}(u), \min\{\hat{T}_{J_2(b_j)}(v_1), \hat{T}_{J_2(b_j)}(v_2)\}\}$

$= \min\{\min\{\hat{T}_{J_1(a_i)}(u), \hat{T}_{J_2(b_j)}(v_1)\}, \min\{\hat{T}_{J_1(a_i)}(u), \hat{T}_{J_2(b_j)}(v_2)\}\}$
$\hat{T}_{K(a_i,b_j)}((u,v_1),(u,v_2)) \le \min\{(\hat{T}_{J_1(a_i)} \times \hat{T}_{J_2(b_j)})(u,v_1), (\hat{T}_{J_1(a_i)} \times \hat{T}_{J_2(b_j)})(u,v_2)\}$
for $i = 1,2,\ldots m, j = 1,2,\ldots,n,$

$\hat{I}_{K(a_i,b_j)}((u,v_1),(u,v_2)) = \min\{\hat{I}_{J_1(a_i)}(u), \hat{I}_{K_2(b_j)}(v_1,v_2)\}$
for$i = 1,2,..,m, j = 1,2,\ldots n$
$\le \min\{\hat{I}_{J_1(a_i)}(u), \min\{\hat{I}_{J_2(b_j)}(v_1), \hat{I}_{J_2(b_j)}(v_2)\}\}$
$= \min\{\min\{\hat{I}_{J_1(a_i)}(u), \hat{I}_{J_2(b_j)}(v_1)\}, \min\{\hat{I}_{J_1(a_i)}(u), \hat{I}_{J_2(b_j)}(v_2)\}\}$
$\hat{I}_{K(a_i,b_j)}((u,v_1),(u,v_2)) \le \min\{(\hat{I}_{J_1(a_i)} \times \hat{I}_{J_2(b_j)})(u,v_1), (\hat{I}_{J_1(a_i)} \times \hat{I}_{J_2(b_j)})(u,v_2)\}$
for$i = 1,2,\ldots m, j = 1,2,\ldots,n,$

$\hat{F}_{K(a_i,b_j)}((u,v_1),(u,v_2)) = \max\{\hat{F}_{J_1(a_i)}(u), \hat{I}_{K_2(b_j)}(v_1,v_2)\}$
for$i = 1,2,..,m, j = 1,2,\ldots n$
$\le \max\{\hat{F}_{J_1(a_i)}(u), \max\{\hat{F}_{J_2(b_j)}(v_1), \hat{F}_{J_2(b_j)}(v_2)\}\}$
$= \max\{\max\{\hat{F}_{J_1(a_i)}(u), \hat{F}_{J_2(b_j)}(v_1)\}, \max\{\hat{F}_{J_1(a_i)}(u), \hat{F}_{J_2(b_j)}(v_2)\}\}$
$\hat{F}_{K(a_i,b_j)}((u,v_1),(u,v_2)) \le \max\{(\hat{F}_{J_1(a_i)} \times \hat{F}_{J_2(b_j)})(u,v_1), (\hat{F}_{J_1(a_i)} \times \hat{F}_{J_2(b_j)})(u,v_2)\}$
for$i = 1,2,\ldots m, j = 1,2,\ldots,n,$

Similarly,

$\hat{T}_{K(a_i,b_j)}((u_1,v),(u_2,v)) \le \min\{(\hat{T}_{J_1(a_i)} \times \hat{T}_{J_2(b_j)})(u_1,v), (\hat{T}_{J_1(a_i)} \times \hat{T}_{J_2(b_j)})(u_2,v)\}$
for$i = 1,2,\ldots m, j = 1,2,\ldots,n,$

$\hat{I}_{K(a_i,b_j)}((u_1,v),(u_2,v)) \le \min\{(\hat{I}_{J_1(a_i)} \times \hat{I}_{J_2(b_j)})(u_1,v), (\hat{I}_{J_1(a_i)} \times \hat{I}_{J_2(b_j)})(u_2,v)\}$
for$i = 1,2,\ldots m, j = 1,2,\ldots,n,$

$\hat{F}_{K(a_i,b_j)}((u_1,v),(u_2,v)) \le \max\{(\hat{F}_{J_1(a_i)} \times \hat{F}_{J_2(b_j)})(u_1,v), (\hat{F}_{J_1(a_i)} \times \hat{F}_{J_2(b_j)})(u_2,v)\}$
for$i = 1,2,\ldots m, j = 1,2,\ldots,n,$

Hence $G = (J, K, R \times S)$ is a neutrosophic vague soft graph.

Definition 3.7 *The cross product of* G_1 *and* G_2 *is defined as a neutrosophic vague soft graphs of* $G = G_1 \odot G_2 = (J, K, R \times S)$*, where* $(J, R \times S)$ *is a neutrosophic vague soft set over* $V = V_1 \times V_2$, $(K, R \times S)$ *is a neutrsophic vague soft set over* $E = \{((u_1,v_1),(u_2,v_2)) : (u_1,u_2) \in E_1, (v_1,v_2) \in E_2\}$ *such that*

$(i)\hat{T}_{J(a,b)}(u,v) = \hat{T}_{J_1(a)}(u) \wedge \hat{T}_{J_2(b)}(v),$
$\hat{I}_{J(a,b)}(u,v) = \hat{I}_{J_1(a)}(u) \wedge \hat{I}_{J_2(b)}(v),$
$\hat{F}_{J(a,b)}(u,v) = \hat{F}_{J_1(a)}(u) \vee \hat{F}_{J_2(b)}(v),$
$\forall (u,v) \in V, (a,b) \in R \times S.$

$(ii)\hat{T}_{K(a,b)}((u_1,v_1),(u_2,v_2)) = \hat{T}_{K_1(a)}(u_1,u_2) \wedge \hat{T}_{K_2(b)}(v_1,v_2),$
$\hat{I}_{K(a,b)}((u_1,v_1),(u_2,v_2)) = \hat{I}_{K_1(a)}(u_1,u_2) \wedge \hat{I}_{K_2(b)}(v_1,v_2),$
$\hat{F}_{K(a,b)}((u_1,v_1),(u_2,v_2)) = \hat{F}_{K_1(a)}(u_1,u_2) \vee \hat{F}_{K_2(b)}(v_1,v_2),$
$\forall (u_1,u_2) \in E_1, (v_1,v_2) \in E_2.$

$H(a,b) = H_1(a) \odot H_2(b)$ for all $(a,b) \in R \times S$ are neutrosophic vague graphs of G.

Theorem 3.8 *The cross product of two neutrosophic vague soft graphs is a neutrosophic vague soft graph.*

Proof. Let $G_1 = (J_1, K_1, R)$ and $G_2 = (J_2, K_2, S)$ be two neutrosophic vague soft graphs of $G_1^* = (V_1, E_1)$ and $G_2^* = (V_2, E_2)$, respectively. Let $G = G_1 \odot G_2 = (J, K, R \times S)$ be the cross product of G_1 and G_2. We claim that $G=(J,K,R\times S)$ is a neutrosophic vague soft graph and $(H,R\times S)=\{J_1 \circledcirc J_2(a_i,b_j), K_1 \circledcirc K_2(a_i,b_j)\}$ $\forall a_i\in R, b_j\in S$ for $i=1,2,\dots,m, j=1,2,\dots,n$ are neutrosophic vague graphs of G.

Consider,

$$\hat{T}_{K_{(a_i,b_j)}}((u_1,v_1),(u_2,v_2))=\min\{\hat{T}_{K_1(a_i)}(u_1,u_2),\hat{T}_{K_2(b_j)}(v_1,v_2)\}$$
for $i=1,2,..,m, j=1,2,\dots n$
$$\le \min\{\{\hat{T}_{J_1(a_i)}(u_1),\hat{T}_{J_1(a_i)}(u_2)\},\min\{\hat{T}_{J_2(b_j)}(v_1),\hat{T}_{J_2(b_j)}(v_2)\}\}$$
$$=\min\{\min\{\hat{T}_{J_1(a_i)}(u_1),\hat{T}_{J_2(b_j)}(v_1)\},\min\{\hat{T}_{J_1(a_i)}(u_2),\hat{T}_{J_2(b_j)}(v_2)\}\}$$
$$\hat{T}_{K_{(a_i,b_j)}}((u_1,v_1),(u_2,v_2))\le \min\{(\hat{T}_{J_1(a_i)}\circledcirc \hat{T}_{J_2(b_j)})(u_1,v_1),(\hat{T}_{J_1(a_i)}\circledcirc \hat{T}_{J_2(b_j)})(u_2,v_2)\}$$
for $i=1,2,\dots m, j=1,2,\dots,n$,

$$\hat{I}_{K_{(a_i,b_j)}}((u_1,v_1),(u_2,v_2))=\min\{\hat{I}_{K_1(a_i)}(u_1,u_2),\hat{I}_{K_2(b_j)}(v_1,v_2)\}$$
for $i=1,2,..,m, j=1,2,\dots n$
$$\le \min\{\{\hat{I}_{J_1(a_i)}(u_1),\hat{I}_{J_1(a_i)}(u_2)\},\min\{\hat{I}_{J_2(b_j)}(v_1),\hat{I}_{J_2(b_j)}(v_2)\}\}$$
$$=\min\{\min\{\hat{I}_{J_1(a_i)}(u_1),\hat{I}_{J_2(b_j)}(v_1)\},\min\{\hat{I}_{J_1(a_i)}(u_2),\hat{I}_{J_2(b_j)}(v_2)\}\}$$
$$\hat{I}_{K_{(a_i,b_j)}}((u_1,v_1),(u_2,v_2))\le \min\{(\hat{I}_{J_1(a_i)}\circledcirc \hat{I}_{J_2(b_j)})(u_1,v_1),(\hat{I}_{J_1(a_i)}\circledcirc \hat{I}_{J_2(b_j)})(u_2,v_2)\}$$
for $i=1,2,\dots m, j=1,2,\dots,n$,

$$\hat{F}_{K_{(a_i,b_j)}}((u_1,v_1),(u_2,v_2))=\min\{\hat{F}_{K_1(a_i)}(u_1,u_2),\hat{F}_{K_2(b_j)}(v_1,v_2)\}$$
for $i=1,2,..,m, j=1,2,\dots n$
$$\le \min\{\{\hat{F}_{J_1(a_i)}(u_1),\hat{F}_{J_1(a_i)}(u_2)\},\min\{\hat{F}_{J_2(b_j)}(v_1),\hat{F}_{J_2(b_j)}(v_2)\}\}$$
$$=\min\{\min\{\hat{F}_{J_1(a_i)}(u_1),\hat{F}_{J_2(b_j)}(v_1)\},\min\{\hat{F}_{J_1(a_i)}(u_2),\hat{F}_{J_2(b_j)}(v_2)\}\}$$
$$\hat{F}_{K_{(a_i,b_j)}}((u_1,v_1),(u_2,v_2))\le \min\{(\hat{F}_{J_1(a_i)}\circledcirc \hat{F}_{J_2(b_j)})(u_1,v_1),(\hat{F}_{J_1(a_i)}\circledcirc \hat{F}_{J_2(b_j)})(u_2,v_2)\}$$
for $i=1,2,\dots m, j=1,2,\dots,n$,

Hence $G=(J,K,R\times S)$ is a neutrosophic vague soft graph.

Definition 3.9 *The lexicographic product of G_1 and G_2 is defined as a neutrosophic vague soft graphs of $G=G_1\odot G_2=(J,K,R\times S)$, where $(J,R\times S)$ is a neutrosophic vague soft set over $V=V_1\times V_2$, $(K,R\times S)$ is a neutrosophic vague soft set over $E=\{((u,v_1),(u,v_2)): u\in V_1,(v_1,v_2)\in E_2\}\cup\{((u_1,v_1),(u_2,v_2)):(u_1,u_2)\in E_1,(v_1,v_2)\in E_2\}$ such that*

$(i)\hat{T}_{J(a,b)}(u,v)=\hat{T}_{J_1(a)}(u)\wedge \hat{T}_{J_2(b)}(v)$,
$\hat{I}_{J(a,b)}(u,v)=\hat{I}_{J_1(a)}(u)\wedge \hat{I}_{J_2(b)}(v)$,
$\hat{F}_{J(a,b)}(u,v)=\hat{F}_{J_1(a)}(u)\vee \hat{F}_{J_2(b)}(v), \forall (u,v)\in V,(a,b)\in R\times S$.
$(ii)\hat{T}_{K(a,b)}((u,v_1),(u,v_2))=\hat{T}_{J_1(a)}(u)\wedge \hat{T}_{K_2(b)}(v_1,v_2)$,
$\hat{I}_{K(a,b)}((u,v_1),(u,v_2))=\hat{I}_{J_1(a)}(u)\wedge \hat{I}_{K_2(b)}(v_1,v_2)$,
$\hat{F}_{K(a,b)}((u,v_1),(u,v_2))=\hat{F}_{J_1(a)}(u)\vee \hat{F}_{K_2(b)}(v_1,v_2), \forall u\in V_1,(v_1,v_2)\in E_2$.
$(iii)\hat{T}_{K(a,b)}((u_1,v_1),(u_2,v_2))=\hat{T}_{K_1(a)}(u_1,u_2)\wedge \hat{T}_{K_2(b)}(v_1,v_2)$,
$\hat{I}_{K(a,b)}((u_1,v_1),(u_2,v_2))=\hat{I}_{K_1(a)}(u_1,u_2)\wedge \hat{I}_{K_2(b)}(v_1,v_2)$,
$\hat{F}_{K(a,b)}((u_1,v_1),(u_2,v_2))=\hat{F}_{K_1(a)}(u_1,u_2)\vee \hat{F}_{K_2(b)}(v_1,v_2), \forall (u_1,u_2)\in E_1,(v_1,v_2)\in E_2$.

$H(a,b)=H_1(a)\odot H_2(b)$ for all $(a,b)\in R\times S$ are neutrosophic vague graphs of G.

Theorem 3.10 *The lexicographic product of two neutrosophic vague soft graphs is a neutrosophic vague soft graph.*

Proof. Similar to the proof of Theorem 3.8.

Definition 3.11 *The strong product of G_1 and G_2 is defined as a neutrosophic vague soft graphs of $G=G_1\otimes G_2=(J,K,R\times S)$, where $(J,R\times S)$ is a neutrosophic vague soft set over $V=V_1\times V_2$, $(K,R\times S)$ is a neutrsophic* *vague soft set over* $E = \{((u, v_1), (u, v_2)): (u) \in V_1, (v_1, v_2) \in E_2\} \cup \{((u_1, v), (u_2, v)): (v) \in V_2, (u_1, u_2) \in E_1\}, \cup \{((u_1, v_1), (u_2, v_2)): (u_1, u_2) \in E_1, (v_1, v_2 \in E_2)\}$ *such that*

$(i)\quad \hat{T}_{J(a,b)}(u, v) = \hat{T}_{J_1(a)}(u) \wedge \hat{T}_{J_2(b)}(v),$
$\hat{I}_{J(a,b)}(u, v) = \hat{I}_{J_1(a)}(u) \wedge \hat{T}_{J_2(b)}(v),$
$\hat{F}_{J(a,b)}(u, v) = \hat{F}_{J_1(a)}(u) \vee \hat{T}_{J_2(b)}(v), \forall (u, v) \in V, (a, b) \in R \times S.$
$(ii)\quad \hat{T}_{K(a,b)}((u, v_1), (u, v_2)) = \hat{T}_{J_1(a)}(u) \wedge \hat{T}_{K_2(b)}(v_1, v_2),$
$\hat{I}_{K(a,b)}((u, v_1), (u, v_2)) = \hat{I}_{J_1(a)}(u) \wedge \hat{I}_{K_2(b)}(v_1, v_2),$
$\hat{F}_{K(a,b)}((u, v_1), (u, v_2)) = \hat{F}_{J_1(a)}(u) \vee \hat{F}_{K_2(b)}(v_1, v_2), \forall u \in V_1, (v_1, v_2) \in E_2.$
$(iii)\quad \hat{T}_{K(a,b)}((u_1, v), (u_2, v)) = \hat{T}_{J_2(b)}(v) \wedge \hat{T}_{K_2(a)}(u_1, u_2),$
$\hat{I}_{K(a,b)}((u_1, v), (u_2, v)) = \hat{I}_{J_2(b)}(v) \wedge \hat{I}_{K_2(a)}(u_1, u_2),$
$\hat{F}_{K(a,b)}((u_1, v), (u_2, v)) = \hat{F}_{J_2(b)}(v) \vee \hat{F}_{K_2(a)}(u_1, u_2), \forall v \in V_2, (u_1, u_2) \in E_1.$
$(iv)\quad \hat{T}_{K(a,b)}((u_1, v_1), (u_2, v_2)) = \hat{T}_{K_1(a)}(u_1, u_2) \wedge \hat{T}_{K_2(b)}(v_1, v_2),$
$\hat{I}_{K(a,b)}((u_1, v_1), (u_2, v_2)) = \hat{I}_{K_1(a)}(u_1, u_2) \wedge \hat{I}_{K_2(b)}(v_1, v_2),$
$\hat{F}_{K(a,b)}((u_1, v_1), (u_2, v_2)) = \hat{F}_{K_1(a)}(u_1, u_2) \vee \hat{F}_{K_2(b)}(v_1, v_2), \forall (u_1, u_2) \in E_1, (v_1, v_2) \in E_2.$

$H(a, b) = H_1(a) \otimes H_2(b)$ for all $(a, b) \in R \times S$ are neutrosophic vague graphs of G.

Theorem 3.12 *The strong product of two neutrosophic vague soft graphs is a neutrosophic vague soft graph.*

Proof. Similar to the proof of Theorem 3.8.

Definition 3.13 *The composition of* G_1 *and* G_2 *is defined as a a neutrosophic vague soft graphs of* $G = G_1[G_2] = (J, K, R \times S)$, *where* $(J, R \times S)$ *is a neutrosophic vague soft set over* $V = V_1 \times V_2$, $(K, R \times S)$ *is a neutrosophic vague soft set over* $E = \{((u, v_1), (u, v_2)): u \in V_1, (v_1, v_2) \in E_2\} \cup \{((u_1, v), (u_2, v)): v \in V_2, (u_1, u_2) \in E_1\} \cup \{((u_1, v_1), (u_2, v_2)): (u_1, u_2) \in E_1, (v_1 \neq v_2)\}$ *such that*

$(i)\hat{T}_{J(a,b)}(u, v) = \hat{T}_{J_1(a)}(u) \wedge \hat{T}_{J_2(b)}(v),$
$\hat{I}_{J(a,b)}(u, v) = \hat{I}_{J_1(a)}(u) \wedge \hat{I}_{J_2(b)}(v),$
$\hat{F}_{J(a,b)}(u, v) = \hat{F}_{J_1(a)}(u) \vee \hat{F}_{J_2(b)}(v),$
$\forall (u, v) \in V, (a, b) \in R \times S.$
$(ii)\hat{T}_{K(a,b)}((u, v_1), (u, v_2)) = \hat{T}_{J_1(a)}(u) \wedge \hat{T}_{K_2(b)}(v_1, v_2),$
$\hat{I}_{K(a,b)}((u, v_1), (u, v_2)) = \hat{I}_{J_1(a)}(u) \wedge \hat{I}_{K_2(b)}(v_1, v_2),$
$\hat{F}_{K(a,b)}((u, v_1), (u, v_2)) = \hat{F}_{J_1(a)}(u) \vee \hat{F}_{K_2(b)}(v_1, v_2),$
$\forall u \in V_1, (v_1, v_2) \in E_2.$
$(iii)\hat{T}_{K(a,b)}((u_1, v), (u_2, v)) = \hat{T}_{J_2(b)}(v) \wedge \hat{T}_{K_2(a)}(u_1, u_2),$
$\hat{I}_{K(a,b)}((u_1, v), (u_2, v)) = \hat{I}_{J_2(b)}(v) \wedge \hat{I}_{K_2(a)}(u_1, u_2),$
$\hat{F}_{K(a,b)}((u_1, v), (u_2, v)) = \hat{F}_{J_2(b)}(v) \vee \hat{F}_{K_2(a)}(u_1, u_2),$
$\forall v \in V_2, (u_1, u_2) \in E_1.$
$(iv)\hat{T}_{K(a,b)}((u_1, v_1), (u_2, v_2)) = \hat{T}_{K_1(a)}(u_1, u_2) \wedge \hat{T}_{J_2(a)}(v_1) \wedge \hat{T}_{J_2(b)}(v_2),$
$\hat{I}_{K(a,b)}((u_1, v_1), (u_2, v_2)) = \hat{I}_{K_1(a)}(u_1, u_2) \wedge \hat{I}_{J_2(a)}(v_1) \wedge \hat{I}_{J_2(b)}(v_2),$
$\hat{F}_{K(a,b)}((u_1, v_1), (u_2, v_2)) = \hat{F}_{K_1(a)}(u_1, u_2) \vee \hat{F}_{J_2(a)}(v_1) \vee \hat{F}_{J_2(b)}(v_2),$
$\forall (u_1, u_2) \in E_1, where v_1 \neq v_2.$

$H(a, b) = H_1(a)[H_2(b)]$ for all $(a, b) \in R \times S$ are neutrosophic vague graphs of G.

Theorem 3.14 *The composition product of two neutrosophic vague soft graphs is a neutrosophic vague soft graph.*

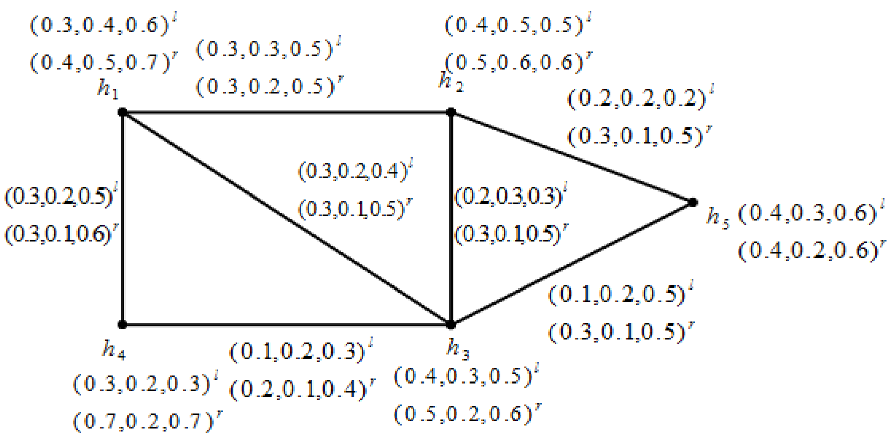
Proof. Similar to the proof of Theorem 3.8.

Application to Decision-making problem:

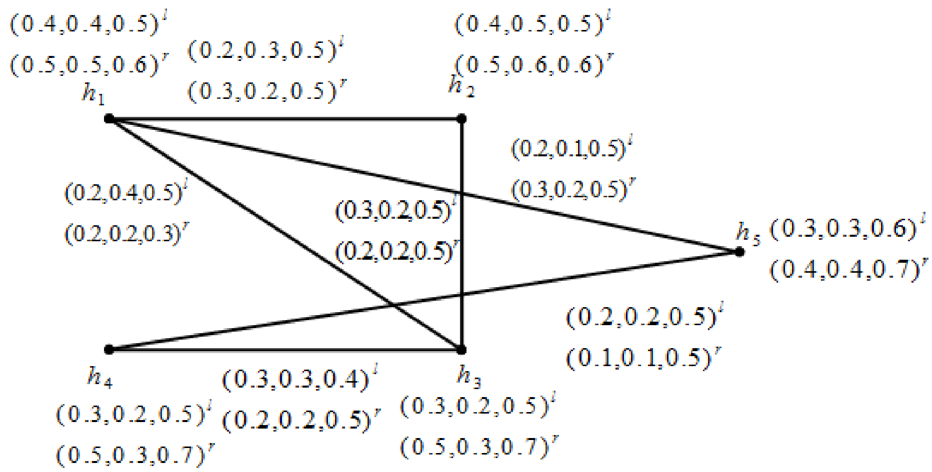
Neutrosophic vague soft set has several applications in decision making problems and used to deal with uncertainties from our different real-life problems. In this section we apply the concept of neutrosophic vague soft sets in a decision-making problem to its graphs and then construct an algorithm for the selection of optimal object based upon given set of information. Suppose that $V = \{h_1, h_2, h_3, h_4, h_5\}$ be the set of five institutions under consideration on which Mr. Z is going to join for his studies on the basis of wishing parameters with 0.5-degree risk value on his risk preferece, with the attributes set $A = \{e_1 = NIRF\ ranking, e_2 = IoE\ Institution, e_3 = University\}$.
(F, A) is the neutrosophic vague soft set on V which describe the value of the students based upon the given parameters $e_1 = NIRF\ ranking, e_2 = IoE\ Institution, e_3 = University,$ respectively.

$$F(e_1) = \{(h_1, (0.3,0.4,0.6)^l\ (0.4,0.5,0.7)^r), (h_2, (0.4,0.5,0.5)^l (0.5,0.6,0.6)^r), (h_3, (0.4,0.3,0.5)^l (0.5,0.4,0.6)^r)$$
$$(h_4, (0.3,0.2,0.3)^l (0.7,0.3,0.7)^r), (h_5, (0.4,0.3,0.5)^l (0.5,0.4,0.6)^r)\}$$
$$F(e_2) = \{(h_1, (0.4,0.4,0.5)^l\ (0.5,0.5,0.6)^r), (h_2, (0.4,0.5,0.5)^l (0.5,0.6,0.6)^r), (h_3, (0.3,0.2,0.5)^l (0.5,0.3,0.7)^r)$$
$$(h_4, (0.3,0.2,0.5)^l (0.5,0.3,0.7)^r), (h_5, (0.3,0.3,0.6)^l (0.4,0.4,0.7)^r)\}$$
$$F(e_3) = \{(h_1, (0.2,0.4,0.7)^l\ (0.3,0.5,0.8)^r), (h_2, (0.3,0.3,0.6)^l (0.4,0.4,0.7)^r), (h_3, (0.2,0.4,0.6)^l (0.4,0.4,0.8)^r)$$
$$(h_4, (0.2,0.3,0.6)^l (0.4,0.3,0.8)^r), (h_5, (0.3,0.4,0.6)^l (0.4,0.4,0.7)^r)\}$$

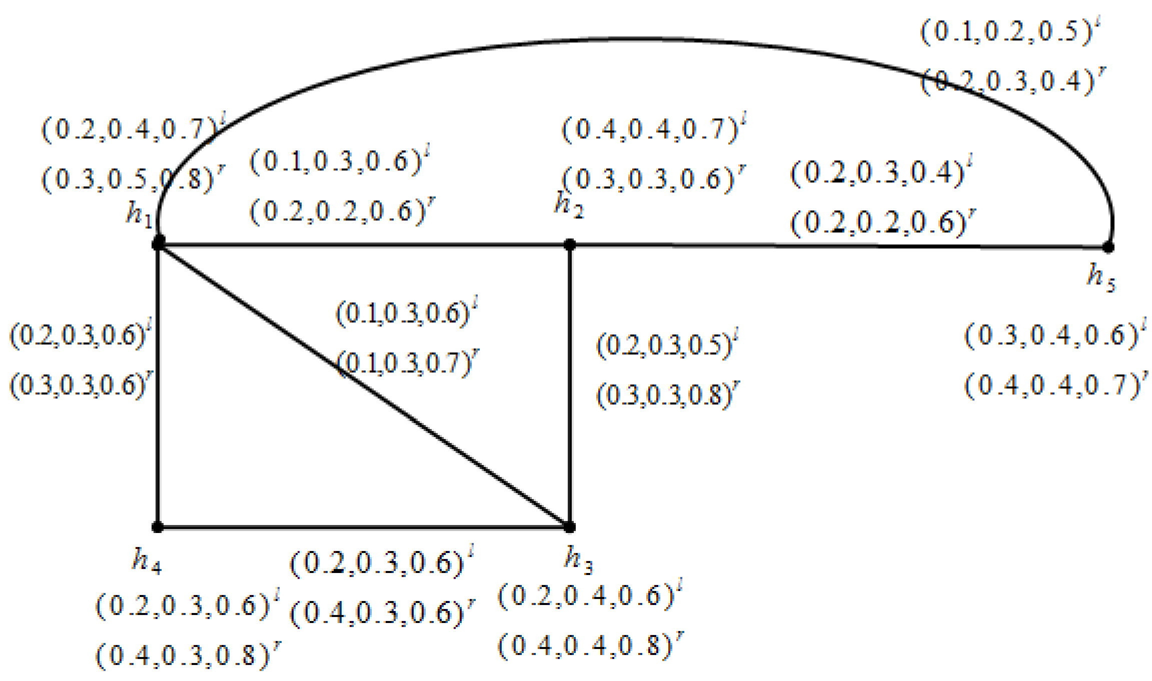
The neutrosophic vague soft graphs $G = (F, K, A)$ corresponding to the parameters e_i for $i = 1,2,3$ are shown in Figure 4.1



$H\ (e_1)$



$H\ (e_2)$



$H\ (e_3)$

Figure 4.1 Neutrosophic vague soft graphs

Table 1 Tabular representation of the Neutrosophic vague soft graph in example 4.1

	e_1	e_2	e_3
h_1	$(0.3,0.4,0.6)^l\ (0.4,0.5,0.7)^r$	$(0.4,0.4,0.5)^l\ (0.5,0.5,0.6)^r$	$(0.2,0.4,0.7)^l\ (0.3,0.5,0.8)^r$
h_2	$(0.4,0.5,0.5)^l\ (0.5,0.6,0.6)^r$	$(0.4,0.5,0.5)^l\ (0.5,0.6,0.6)^r$	$(0.3,0.3,0.6)^l\ (0.4,0.4,0.7)^r$
h_3	$(0.4,0.3,0.5)^l\ (0.5,0.4,0.6)^r$	$(0.3,0.4,0.4)^l\ (0.6,0.6,0.7)^r$	$(0.2,0.4,0.6)^l\ (0.4,0.5,0.8)^r$
h_4	$(0.3,0.2,0.3)^l\ (0.7,0.3,0.7)^r$	$(0.3,0.2,0.5)^l\ (0.5,0.3,0.7)^r$	$(0.2,0.3,0.6)^l\ (0.4,0.3,0.8)^r$
h_5	$(0.4,0.3,0.5)^l\ (0.5,0.4,0.6)^r$	$(0.3,0.3,0.6)^l\ (0.4,0.4,0.7)^r$	$(0.3,0.4,0.6)^l\ (0.4,0.4,0.7)^r$

Table 2 The grade based on Neutrosophic vague soft graph in example 4.1

	e_1	e_2	e_3	G_{min}
h_1	2	2	1	1
h_2	2	2	2	2
h_3	2	2	1	1
h_4	3	2	2	2
h_5	2	2	2	2

Table 3. The resultant neutrosophic vague soft graphs in example 4.1

The score function S_{ij} based on neutrosophic vague soft graph

	e_1	e_2	e_3
h_2	$< 2,0.45,0.55,0.55,-0.65 >$	$< 2,0.45,0.55,0.55,0.55,-65 >$	$< 2,0.35,0.35,0.65,-0.65 >$
h_4	$< 3,0.35,0.25,0.55,-0.45 >$	$< 2,0.55,0.25,0.6,-0.50 >$	$< 2,0.3,0.3,0.7,-0.7 >$
h_5	$< 2,0.45,0.35,0.65,-0.55 >$	$< 2,0.35,0.35,0.65,-0.65 >$	$< 2,0.35,0.4,0.65,-0.75 >$

Table 4. Comparison table for Grade function and Score function, based on e_1

	h_2	h_4	h_5	h_k (min)
h_2	$< 0,0 >$	$< -1,-20 >$	$< 0,-10 >$	$< -1,-20 >$
h_4	$< 1,20 >$	$< 0,0 >$	$< 0,10 >$	$< 0,10 >$
h_5	$< 0,10 >$	$< -1,-10 >$	$< 0,0 >$	$< -1,-10 >$

Table 5. Comparison table for grade function and score function based on e_1 without h_2

	h_4	h_5	h_k(min)
h_4	$< 0,0 >$	$< 1,10 >$	$< 1,10 >$
h_5	$< -1,-10 >$	$< 0,0 >$	$< -1,-10 >$

We get h_2, h_4, h_5 attributes. Similarly, we can get h_4, h_2, h_5 under the attributes e_2 and h_2, h_5, h_4 under the attributes e_3.

Finally, compute the ranking of the research objects under all attributes. Suppose the decision maker assigns weights to each attribute, $a_1 = 0.2, a_2 = 0.3, a_3 = 0.1$. And we can get $A_3 > A_2 > A_1$ from table 6. We consider h_2 is the first superior object, h_4 is the second superior object and h_5 is the third superior object under the E. Therefore, Mr. Z will selected particular institution h_2.

Table 6. The ranking of the objects under all attributes.

	e_1 . 0.2	e_2 . 0.3	e_3 . 0.1	A_i
h_2	1	2	1	0.9
h_4	2	1	3	1
h_5	3	3	2	1.7

Advantages and Limitations:

1. The proposed application is more significant, since it has the method of solving based on the idea of probability in grade function.

2. The developed method is utilised for solving practical decision making problems containing vagueness.

3. The addressed graphs can be extended to the bipolar environment.

4.The challenging one is to handle the vagueness in the application viewpoint of big data. If the indeterminate membership function has the huge data, then it is difficult to handle. This leads to have a massive calculation in the decision-making problems.

Conclusion

Vague sets and neutrosophic soft sets provide a powerful tool to represent the data with uncertain information and have fruitful applications. In this work, neutrosophic vague soft graphs have been developed. This helps the decision-makers more sufficient for taking their input best suit to their domain of reference. Hence, the proposed graphs and their operations have enough capabilities to address the related dependability on the imprecise information. Further, the authors will aim to develop this research to the isomorphic properties of the proposed concepts in future.

Compliance with ethical standards

Conflict of interest: The authors declare that they have no conflict of interest.

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