Neutrosophic Sets and Systems, Vol. 51, 2022
University of New Mexico

Quadripartitioned Neutrosophic Graph Structures

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Abstract: The quadripartitioned neutrosophic set is the partition of indeterminacy function of the neutrosophic set into contradiction part and ignorance part. In this work, the concept of quadripartitioned neutrosophic graph structures and its properties are invented. The strong, tree, $\phi -$ permutation and $\phi -$ complement of quadripartitioned neutrosophic graph structure are investigated. The operations like Cartesian Product, cross product, lexicographic product, composition in graph structures and join operations are established.

Keywords: Quadripartitioned neutrosophic graph, quadripartitioned neutrosophic graph structure, ϕ permutation, ϕ complement, Operations

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1. Introduction

The intuitionistic fuzzy sets represent a novel component in the fuzzy sets, namely a non-membership function. However, some limits only allow for the storage of incomplete data when interpreting the degree of true and false membership functions, but the handling of indeterminate data is still possible. Can we look at an example where ten patients are being tested for a pandemic? Three patients will have a positive result, five will have a negative result, and two will be uncertain or have yet to be determined throughout that period. It can be stated as $x(0.3,0.2,0.5)$ using neutrosophic notions. Using the neutrosophic set, one can classify the environment as cold as truth, moderate as indeterminacy, and hot as false for a clear comprehension. As a result, the neutrosophic field emerges to hold the indeterminacy data. From a philosophical standpoint, it generalises the aforementioned sets. The single-valued neutrosophic set is a generalisation of intuitionistic fuzzy sets that can be utilised to solve real-world problems, particularly in decision support. The sum of the three components of belief in that element (truth), disbelief in that element (falsehood), and the indeterminacy part of that element is strictly less than 1. Smarandache [36, 38] and references therein propose neutrosophic sets as the foundation of neutrosophic logic, a multiple value logic that generalises fuzzy logic and deals with paradoxes, contradictions, antitheses, and antinomies.

In the situation of neutrosophic sets, indeterminacy is considered as a distinct concept, and each component is defined by a truth-membership function, an indeterminacy membership function, and a falsity-membership function, all of which are obtained from the non-standard unit interval $]0^-, 1^+[$. Ignoring the fact that neutrosophic indeterminacy is independent of truth and falsity-membership values, it is more general than the hesitation margin of intuitionistic fuzzy sets. It is unclear whether the indeterminacy values relevant to a specific element correspond to hesitant values about its belonging or non-belonging to it. As a result, some authors prefer to model the indeterminacy's behaviour in the same way they similar to truth-membership, others may prefer to model it in the same way they similar to falsity-membership. Wang et al. [43] initiated the concept of a single valued neutrosophic set and provide its various properties. It has been widely applied in various fields, such as information fusion in which data are combined from different sensors [10], control theory [1], image processing [12], medical diagnosis [42], decision making [41], and graph theory [4, 8, 15-18, 25, 35], etc. When the indeterminacy portion of the netrosophic set is divided into two parts, we get four components: 'Contradiction' (both true and false) and 'Unknown' (neither true nor false), that is $\mathbb{T}, \mathbb{C}, \mathbb{U}$ and $\mathbb{F}$ which defines a new set called 'quadripartitioned single valued neutrosophic set', introduced by Chatterjee., et al. [11]. This study is completely based on "Belnap's four valued logic" [9] and Smarandache's "Four Numerical valued neutrosophic logic" [39]. By employing the concept of Quadripartitioned neutrosophic set, this paper presents the quadripartitioned neutrosophic graphs structure. Operations on single-valued neutrosophic graph structures are studied in [2, 6]. Motivated by the above mentioned works, to the best of authors' knowledge, there is no work reported on the concepts of quadripartioned single valued neutrosophic graphs with application. The major contributions in this work are foregrounded as follows:

1. The notions of Quadripartitioned Neutrosophic Graph Structure (QNGS) and its properties are introduced.
2. In addition, the complete, strong and complement of QNGS are defined.
3. Furthermore, the ϕ −permutation and ϕ −complement of QNGS are investigated. The proposed concepts are illustrated with examples.
4. The operations like Cartesian Product, cross product, lexicographic product, composition in graph structures and join operations are established.

2. Preliminaries

Definition 2.1 *A graph structure* $\mathfrak{G} = (\mathcal{P}, \mathfrak{R}_1, \mathfrak{R}_2, .. \mathfrak{R}_n)$ *consists of a non-empty set* $\mathcal{V}$ *together with relation* $\mathfrak{R}_1, \mathfrak{R}_2, \dots, \mathfrak{R}_n$ *on* $\mathcal{P}$ *which are mutually disjoint such that each* $\mathfrak{R}_i$, $1 \leq i \leq n$, *is symmetric and irreflexive.*

Definition 2.2 *A neutrosophic set* $\mathcal{N}$ *on a universal set* $\mathcal{P}$ *is an object of the form*
$\mathcal{N} = \{(p, \mathfrak{T}_{\mathcal{N}}(p), \mathfrak{I}_{\mathcal{N}}(p), \mathfrak{F}_{\mathcal{N}}(p): p \in \mathcal{P})\}$, where $\mathcal{T}_{\mathcal{N}}, \mathcal{I}_{\mathcal{N}}, \mathcal{F}_{\mathcal{N}}: \mathcal{P} \to \ ]0^-, 1^+[$ and $0^- \leq \mathfrak{T}_{\mathcal{N}}(p), \mathfrak{I}_{\mathcal{N}}(p), \mathfrak{F}_{\mathcal{N}}(p) \leq 3^+$.

Definition 2.3 *A single valued neutrosophic set* $\mathcal{N}$ *on a universal set* $\mathcal{P}$ *is an object of the form*
$\mathcal{N} = \{(p, \mathfrak{T}_{\mathcal{N}}(p), \mathfrak{I}_{\mathcal{N}}(p), \mathfrak{F}_{\mathcal{N}}(p): p \in \mathcal{P})\}$, where $\mathcal{T}_{\mathcal{N}}, \mathcal{I}_{\mathcal{N}}, \mathcal{F}_{\mathcal{N}}: \mathcal{P} \to [0,1]$ and $0 \leq \mathfrak{T}_{\mathcal{N}}(p), \mathfrak{I}_{\mathcal{N}}(p), \mathfrak{F}_{\mathcal{N}}(p) \leq 3$.

Definition 2.4 *[3] A neutrosophic graph is defined as a pair* $G^* = (V, E)$ *where*

(i) $V = \{v_1, v_2, .., v_n\}$ such that $\mathcal{T}_A: V \to [0,1]$, $\mathcal{I}_A: V \to [0,1]$ and $\mathcal{F}_A: V \to [0,1]$ denote the degree of truth-membership function, indeterminacy function and falsity-membership function, respectively and

$$0 \leq \mathcal{T}_A(v) + \mathcal{I}_A(v) + \mathcal{F}_A(v) \leq 3, \ \forall \quad v \in V.$$

(ii) $E \subseteq V \times V$ where $\mathcal{T}_B: E \to [0,1]$, $\mathcal{I}_B: E \to [0,1]$ and $\mathcal{F}_B: E \to [0,1]$ are such that

$$\mathcal{T}_B(uv) \leq \min\{T_A(u), T_A(v)\},$$
$$\mathcal{I}_B(uv) \leq \min\{I_A(u), I_A(v)\},$$
$$\mathcal{F}_B(uv) \leq \max\{F_A(u), F_A(v)\},$$
$$\forall u, v \in V.$$

For more details about the following definitions and results, see the article [11].

Definition 2.5 *Let $\mathcal{X}$ be a non-empty set. A quadripartitioned neutrosopohic set (QSVNS) $\mathcal{A}$ over $\mathcal{R}$ characterizes each elements x in $\mathcal{X}$ by a truth membership function $\mathcal{T}_{\mathcal{A}}$, a contradiction membership function $\mathcal{C}_{\mathcal{A}}$, an ignorance membership function $\mathcal{U}_{\mathcal{A}}$ and a false membership function $\mathcal{F}_{\mathcal{A}}$ such that for each $x \in \mathcal{R}, \mathcal{T}_{\mathcal{A}}, \mathcal{C}_{\mathcal{A}}, \mathcal{U}_{\mathcal{A}}, \mathcal{F}_{\mathcal{A}} \in [0,1]$ and $0 \leq \mathcal{T}_{\mathcal{A}}(r) + \mathcal{C}_{\mathcal{A}}(r) + \mathcal{U}_{\mathcal{A}}(r) + \mathcal{F}_{\mathcal{A}}(r) \leq 4$.*

Remark 2.6 *A QSVNS $\mathfrak{A}$, can be decomposed to yields two SVNS say, $\mathfrak{A}_t$ and $\mathfrak{A}_f$ where the respective membership functions of both these sets are defined as*

$$\mathcal{T}_{\mathfrak{A}_t}(r) = \mathcal{T}_{\mathfrak{A}}(r) = \mathcal{T}_{\mathfrak{A}_f}(r)$$
$$\mathcal{I}_{\mathfrak{A}_t}(r) = \mathcal{C}_{\mathfrak{A}}(r), \quad \mathcal{I}_{\mathfrak{A}_f}(r) = \mathcal{U}_{\mathfrak{A}}(r)$$
$$\mathcal{F}_{\mathfrak{A}_t}(r) = \mathcal{F}_{\mathfrak{A}}(r) = \mathcal{F}_{\mathfrak{A}_f}(r), \quad \forall r \in \mathcal{R}.$$

In this respect to needs to be stated that while performing set-theoretic operations over these SVNS, behavior of $\mathcal{I}_{\mathfrak{A}_t}$ is treated similar to that of $\mathcal{T}_{\mathfrak{A}_t}$ while the behavior of $\mathcal{I}_{\mathfrak{A}_f}$ is modeled in a way similar to that of $\mathcal{F}_{\mathfrak{A}_f}$.

Definition 2.7 *A QSVNS is said to be an absolute QSVNS, denoted by $\mathfrak{A}$, if its is membership values are respectively defined as $\mathcal{T}_{\mathfrak{A}}(r) = 1$, $\mathcal{C}_{\mathfrak{A}}(r) = 1$, $\mathcal{U}_{\mathfrak{A}}(r) = 0$ and $\mathcal{F}_{\mathfrak{A}}(r) = 0$.*

Definition 2.8 *Consider two QSVNS $\mathfrak{A}$ and $\mathfrak{B}$, over $\mathcal{R}$. $\mathfrak{A}$ is said to be contained in $\mathfrak{B}$, denoted by $\mathfrak{A} \subseteq \mathfrak{B}$ if, and only, if $\mathcal{T}_{\mathfrak{A}}(r) \leq \mathcal{T}_{\mathfrak{B}}(r)$, $\mathcal{C}_{\mathfrak{A}}(r) \leq \mathcal{C}_{\mathfrak{B}}(r)$, $\mathcal{U}_{\mathfrak{A}}(r) \geq \mathcal{U}_{\mathfrak{B}}(r)$ and $\mathcal{F}_{\mathfrak{A}}(r) \geq \mathcal{F}_{\mathfrak{B}}(r)$.*

Definition 2.9 *The complement of a QSVNS $\mathfrak{A}$, is denoted by $\mathfrak{A}^c$ and is defined as*

$$\mathfrak{A}^c = \sum_{i=1}^{n} \langle \mathcal{F}_{\mathfrak{A}}(r_i), \mathcal{U}_{\mathfrak{A}}(r_i), \mathcal{C}_{\mathfrak{A}}(r_i), \mathcal{T}_{\mathfrak{A}}(r_i) \rangle, \qquad \forall r_i \in \mathcal{R}.$$

i.e. $\mathcal{T}_{\mathfrak{A}^c}(r_i) = \mathcal{F}_{\mathfrak{A}}(r_i)$, $\mathcal{C}_{\mathfrak{A}^c}(r_i) = \mathcal{U}_{\mathfrak{A}}(r_i)$

$\mathcal{U}_{\mathfrak{A}^c}(r_i) = \mathcal{C}_{\mathfrak{A}}(r_i)$, $\mathcal{F}_{\mathfrak{A}^c}(r_i) = \mathcal{T}_{\mathfrak{A}}(r_i)$, $\forall r_i \in \mathcal{R}$.

Definition 2.10 *The union of two QSVNS $\mathfrak{A}$ and $\mathfrak{B}$ is denoted by $\mathfrak{A} \cup \mathfrak{B}$ and is defined as*

$$\mathfrak{A} \cup \mathfrak{B} = \sum_{i=1}^{n} \langle \mathcal{T}_{\mathfrak{A}}(r_i) \vee \mathcal{T}_{\mathfrak{B}}(r_i), \mathcal{C}_{\mathfrak{A}}(r_i) \vee \mathcal{C}_{\mathfrak{B}}(r_i)$$
$$\mathcal{U}_{\mathfrak{A}}(r_i) \wedge \mathcal{U}_{\mathfrak{B}}(r_i), \mathcal{F}_{\mathfrak{A}}(r_i) \wedge \mathcal{F}_{\mathfrak{B}}(r_i) \rangle / \mathcal{R}.$$

Definition 2.11 *The intersection of two QSVNS $\mathfrak{A}$ and $\mathfrak{B}$ is denoted by $\mathfrak{A} \cap \mathfrak{B}$ and is defined as*

$$\mathfrak{A} \cap \mathfrak{B} = \sum_{i=1}^{n} \langle \mathcal{T}_{\mathfrak{A}}(r_i) \wedge \mathcal{T}_{\mathfrak{B}}(r_i), \mathcal{C}_{\mathfrak{A}}(r_i) \wedge \mathcal{C}_{\mathfrak{B}}(r_i)$$
$$\mathcal{U}_{\mathfrak{A}}(r_i) \vee \mathcal{U}_{\mathfrak{B}}(r_i), \mathcal{F}_{\mathfrak{A}}(r_i) \vee \mathcal{F}_{\mathfrak{B}}(r_i) \rangle / \mathcal{R}$$

3. Quadripartitioned Neutrosophic Graph structure

Definition 3.1 *Let $\mathbb{R}$ be a non-empty set and $\mathbb{E}_1, \mathbb{E}_2, \ldots, \mathbb{E}_n$ relation on $\mathbb{R}$. $\mathfrak{G} = (\mathfrak{A}, \mathfrak{B}_1, \mathfrak{B}_2, \ldots, \mathfrak{B}_n)$ is called a quadripartioned neutrosophic graph structure if*

$$\mathfrak{A} = \{n, \mathbb{T}_i(l), \mathbb{C}_i(l), \mathbb{U}_i(l), \mathbb{F}_i(l) : n \in \mathbb{R}\}$$

is a quadripartitioned neutrosophic set on $\mathbb{R}$ and

$$\mathfrak{B}_i = \{(k,l), \mathbb{T}(k,l), \mathbb{I}(k,l), \mathbb{U}(k,l), \mathbb{F}(k,l) : n \in \mathbb{E}_i\}$$

is a quadripartitioned neutrosophic set on $\mathbb{E}_i$ such that

$$\mathbb{T}_i(k,l) \leq \min\{\mathbb{T}(k), \mathbb{T}(l)\},$$
$$\mathbb{C}_i(k,l) \leq \min\{\mathbb{C}(k), \mathbb{C}(l)\},$$
$$\mathbb{U}_i(k,l) \leq \max\{\mathbb{U}(k), \mathbb{U}(l)\},$$
$$\mathbb{F}_i(k,l) \leq \max\{\mathbb{F}(k), \mathbb{F}(l)\},$$
$$\forall m, n \in \mathbb{R}.$$

$$0 \leq \mathbb{T}_i(k,l) + \mathbb{C}_i(k,l) + \mathbb{U}_i(k,l) + \mathbb{F}_i(k,l) \leq 4. \text{ for all } (k,l) \in \mathbb{E}_i$$

where $\mathbb{R}$ and $\mathbb{E}_i$ $(i = 1,2,\ldots,n)$ are underlying vertex and underlying i-edge sets of $\mathbb{G}$, respectively.

Example 3.2 *Let* $\mathfrak{G}^* = (\mathbb{R}, \mathbb{E}_1, \mathbb{E}_2)$ *be a graph structure* $\mathfrak{G} = \{q_1, q_2, q_3, q_4, q_5, q_5, q_6\}$, $\mathbb{E}_1 = \{q_1q_6, q_2q_3, q_3q_4, q_4q_5\}$, $\mathbb{E}_2 = \{q_1q_2, q_5q_6, q_4q_6, q_1q_3\}$. *Now we can define quadripartitioned neutrosophic sets* $\mathfrak{A}, \mathfrak{B}_1, \mathfrak{B}_2$ *on* $\mathbb{R}, \mathbb{E}_1, \mathbb{E}_2$ *respectively,*

Let $\mathfrak{A} = \{(q_1, 0.3,0.7,0.7,0.4), (q_2, 0.4,0.7,0.6,0.6), (q_3, 0.4,0.4,0.3,0.2)$
$(q_4, 0.5,0.6,0.7,0.4), (q_5, 0.3,0.4,0.7,0.8), (q_6, 0.4,0.3,0.4,0.3)\}$

$\mathfrak{B}_1 = \{(q_1q_6, 0.2,0.1,0.4,0.3), (q_2q_3, 0.3,0.4,0.5,0.5), (q_3q_4, 0.3,0.3,0.5,0.1), (q_4q_5, 0.3,0.4,0.7,0.4)\}$

$\mathfrak{B}_2 = \{(q_1q_2, 0.3,0.2,0.6,0.3), (q_5q_6, 0.2,0.3,0.3,0.2), (q_4q_6, 0.3,0. ,0.6,0.2), (q_1q_3, 0.2,0.2,0.2,0.2)\}$

By direct calculation, it is easy to show that $\mathfrak{G} = (\mathfrak{A}, \mathfrak{B}_1, \mathfrak{B}_2)$ is a QNGS of $\mathfrak{G}^*$ is shown in figure 1

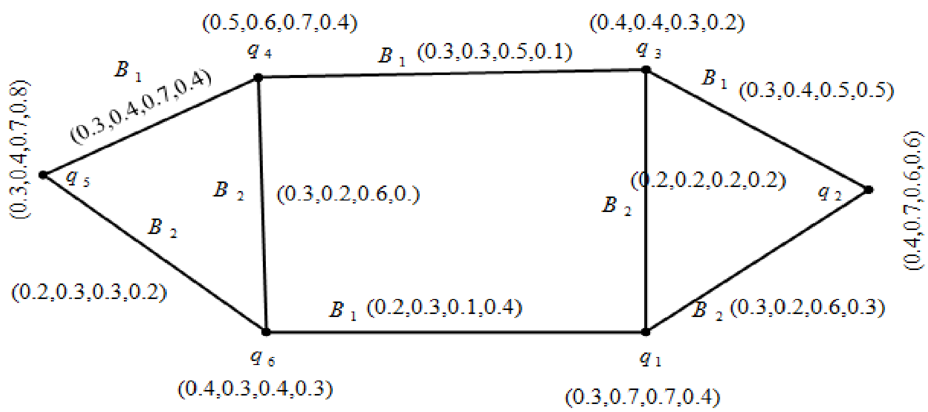

Figure 1: QUADRIPARTITIONED NEUTROSOPHIC GRAPH STRUCTURE

Definition 3.3 *Let* $\mathfrak{G} = (\mathfrak{A}, \mathfrak{B}_1, \mathfrak{B}_2, \ldots, \mathfrak{B}_n)$ *be a QNGS of* $\mathfrak{G}^*$. *If* $\mathcal{H} = (\mathfrak{A}', \mathfrak{B}'_1, \mathfrak{B}'_2, \ldots, \mathfrak{B}'_n)$ *is a QNGS of* $\mathfrak{G}^*$ *such that*

$$\mathbb{T}'(l) \leq \mathbb{T}(l), \mathbb{C}'(l) \leq \mathbb{C}(l), \mathbb{U}'(l) \geq \mathbb{U}(l), \mathbb{F}'(l) \geq \mathbb{F}(l)$$

for all $n \in \mathbb{R}$,

$$\mathbb{T}'_i(k,l) \leq \mathbb{T}_i(k,l), \mathbb{C}'_i(k,l) \leq \mathbb{C}_i(k,l), \mathbb{U}'_i(k,l) \geq \mathbb{U}_i(k,l), \mathbb{F}'_i(k,l) \geq \mathbb{F}_i(k,l)$$

for all $m, n \in \mathbb{E}_i$, where $i = 1,2,\ldots,n$. Then $\mathcal{H}$ is called a quadripartitioned neutrosophic subgraph structure of QNGS $\mathbb{G}$.

Example 3.4 *Consider a graph structure* $\mathfrak{G}^* = (\mathbb{R}, \mathbb{E}_1, \mathbb{E}_2)$ *and let* $(\mathfrak{A}, \mathfrak{B}_1, \mathfrak{B}_2)$ *be quadripartitioned neutrosophic subsets of* $(\mathbb{R}, \mathbb{E}_1, \mathbb{E}_2)$ *respectively, such that*

$\mathfrak{A} = \{(n_1, 0.8,0.6,0.5,0.4), (n_2, 0.7,0.6,0.5,0.4), (n_3, 0.6,0.8,0.4,0.4), (n_4, 0.5,0.5,0.3,0.4)\}$
$\mathfrak{B}_1 = \{(n_1n_2, 0.6,0.5,0.4,0.3), (n_2n_4, 0.3,0.3,0.4,0.3)\}$,

$\mathfrak{B}_2 = \{(n_3 n_4, 0,4,0.3,0.3,0.3), (n_1 n_4, 0.4,0.4,0.5,0.3)\}$

Direct calculations show that $\mathfrak{G} = (\mathfrak{A}, \mathfrak{B}_1, \mathfrak{B}_2)$ is a QNGS of $\mathfrak{G}^*$ as presented in Figure 3.

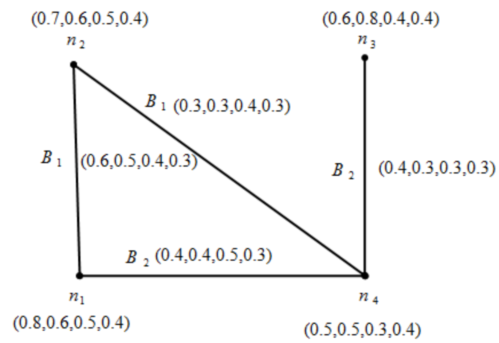

Figure 2: QUADRIPARTITIONED NEUTROSOPHIC GRAPH STRUCTURE

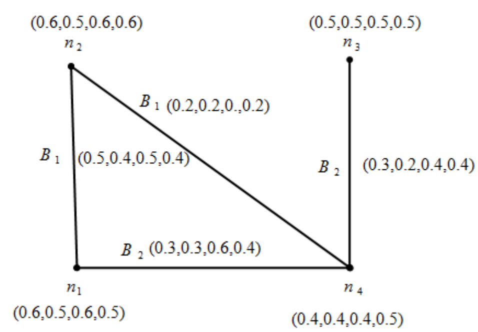

Figure 3: QUADRIPARTITIONED NEUTROSOPHIC SUBGRAPH STRUCTURE

Definition 3.5 *A QNGS* $\mathcal{H} = (\mathfrak{A}', \mathfrak{B}'_1, \mathfrak{B}'_2, \ldots, \mathfrak{B}'_n)$ *is called an induced subgraph structure of* $\mathfrak{G}$ *by a subset* $\mathcal{R}$ *of* $\mathcal{X}$ *if*

$$\mathbb{T}'(l) = \mathbb{T}(l), \mathbb{C}'(l) = \mathbb{C}(l), \mathbb{U}'(l) = \mathbb{U}(l), \mathbb{F}'(l) = \mathbb{F}(l)$$

for all $n \in \mathbb{E}$,

$$\mathbb{T}'_i(k,l) = \mathbb{T}_i(k,l), \mathbb{C}'_i(k,l) = \mathbb{C}_i(k,l), \mathbb{U}'_i(k,l) = \mathbb{U}_i(k,l), \mathbb{F}'_i(k,l) = \mathbb{F}_i(k,l)$$

for all $m, n \in \mathbb{E}_i$, where $i = 1,2,\ldots,n$.

Definition 3.6 *A QNGS* $\mathcal{H} = (\mathfrak{A}', \mathfrak{B}'_1, \mathfrak{B}'_2, \ldots, \mathfrak{B}'_n)$ *is said to be a spanning subgraph structure of* $\mathfrak{G}$ *when* $\mathfrak{A}' = \mathfrak{A}$ *and*

$$\mathbb{T}'_i(k,l) \leq \mathbb{T}_i(k,l), \mathbb{C}'_i(k,l) \leq \mathbb{C}_i(k,l), \mathbb{U}'_i(k,l) \geq \mathbb{U}_i(k,l), \mathbb{F}'_i(k,l) \geq \mathbb{F}_i(k,l)$$

$i = 1,2,\ldots,n$.

Definition 3.7 *Let* $\mathfrak{G} = (\mathfrak{A}, \mathfrak{B}_1, \mathfrak{B}_2, \ldots, \mathfrak{B}_n)$ *be an QNGS of* $\mathfrak{G}^*$. *Then* $kl \in \mathbb{E}_i$ *is called* $\mathfrak{B}_i$ *edge if* $\mathbb{T}_i(k,l) > 0$ *or* $\mathbb{C}_i(k,l) > 0$ *or* $\mathbb{U}_i(k,l) > 0$ *or* $\mathbb{F}_i(k,l) > 0$ *all the four conditions hold. Consequently, support of* $\mathfrak{B}_i$ *is defined as:*

$$supp(\mathfrak{B}_i) = \{kl \in \mathfrak{B}_i : \mathbb{T}_i(k,l) > 0\} \cup \{kl \in \mathfrak{B}_i : \mathbb{C}_i(k,l) > 0\} \cup \{kl \in \mathfrak{B}_i : \mathbb{U}_i(k,l) > 0\} \cup \{kl \in \mathfrak{B}_i : \mathbb{F}_i(k,l) > 0\}, i = 1,2,\ldots,n.$$

Definition 3.9 $\mathfrak{B}_i$ *−path in a QNGS* $\mathfrak{G} = (\mathfrak{A}, \mathfrak{B}_1, \mathfrak{B}_2, \ldots, \mathfrak{B}_n)$ *is a sequence of different nodes* $n_1, n_2, \ldots, n_m$ *(except choice that* $n_m = n_1$*) in* $\mathcal{X}$, *such that* $n_{j-1}n_j$ *is a quadripartitioned neutrosophic* $\mathfrak{B}_i$*-edge, for all* $j = 2, \ldots, m$.

Definition 3.10 *A QNGS* $\mathfrak{G} = (\mathfrak{A}, \mathfrak{B}_1, \mathfrak{B}_2, \ldots, \mathfrak{B}_n)$ *is called* $\mathfrak{B}_i$*- strong for some* $i \in \{1,2,3, \ldots n\}$ *if*

$$\mathbb{T}_i(k,l) = \min\{\mathbb{T}(k), \mathbb{T}(l)\},$$
$$\mathbb{C}_i(k,l) = \min\{\mathbb{C}(k), \mathbb{C}(l)\},$$
$$\mathbb{U}_i(k,l) = \max\{\mathbb{U}(k), \mathbb{U}(l)\}, \text{and}$$
$$\mathbb{F}_i(k,l) = \max\{\mathbb{F}(k), \mathbb{F}(l)\}, \forall mn \in supp(\mathfrak{B}_i).$$

Further, QNGS $\mathfrak{G}$ is said to be strong if it is $\mathfrak{B}_i$- strong for all $i \in \{1,2,\ldots,n.\}$

Definition 3.11 *A QNGS* $\mathfrak{G} = (\mathfrak{A}, \mathfrak{B}_1, \mathfrak{B}_2, .., \mathfrak{B}_n)$ *is said to be complete if* $\mathfrak{G}$ *is a strong QNGS,* $supp(\mathfrak{B}_i) \neq \phi$ *for all* $i = 1,2,\ldots,n$ *and for all pair of nodes* $k, l \in \mathcal{X}$, kl *is a* $\mathfrak{B}_i$ *edge for some* i.

Definition 3.12 *Let* $\mathfrak{G} = (\mathfrak{A}, \mathfrak{B}_1, \mathfrak{B}_2, .., \mathfrak{B}_n)$ *be a QNGS. Now truth strength, contradiction strength, ignorance strength and false strength of a* $\mathfrak{B}_i$*-path* $P_{\mathfrak{B}_i} = n_1, n_2, \ldots, n_m$ *are denoted by* $T.P_{\mathfrak{B}_i}, C.P_{\mathfrak{B}_i}, U.P_{\mathfrak{B}_i}$ *and* $F.P_{\mathfrak{B}_i}$, *respectively, and defined as*

$$T.P_{\mathfrak{B}_i} = \bigwedge_{j=2}^{m} [\mathbb{T}^P_{\mathfrak{B}_i}(n_{j-1}n_j)],$$
$$C.P_{\mathfrak{B}_i} = \bigwedge_{j=2}^{m} [\mathbb{C}^P_{\mathfrak{B}_i}(n_{j-1}n_j)],$$
$$U.P_{\mathfrak{B}_i} = \bigvee_{j=2}^{m} [\mathbb{U}^P_{\mathfrak{B}_i}(n_{j-1}n_j)],$$
$$F.P_{\mathfrak{B}_i} = \bigvee_{j=2}^{m} [\mathbb{F}^P_{\mathfrak{B}_i}(n_{j-1}n_j)].$$

Definition 3.13 *Suppose* $\mathfrak{G} = (\mathfrak{A}, \mathfrak{B}_1, \mathfrak{B}_2, .., \mathfrak{B}_n)$ *is a QNGS. Then*

1. $\mathfrak{B}_i$- truth strength of connectedness between m and n is defined as: $\mathbb{T}^{\infty}_{\mathfrak{B}_i}(kl) = \bigvee_{j\geq 1} \{\mathbb{T}^j_{\mathfrak{B}_i}(kl)\}$ such that $\mathbb{T}^j_{\mathfrak{B}_i}(kl) = (\mathbb{T}^{j-1}_{\mathfrak{B}_i} \circ \mathbb{T}^1_{\mathfrak{B}_i})(kl)$ for $j \geq 2$ and
$$\mathbb{T}^2_{\mathfrak{B}_i}(kl) = (\mathbb{T}^1_{\mathfrak{B}_i} \circ \mathbb{T}^1_{\mathfrak{B}_i})(kl) = \bigvee_{z} (\mathbb{T}^1_{B_i}(mz) \wedge \mathbb{T}^1_{B_i}(zn)).$$
2. $\mathfrak{B}_i$- contradiction strength of connectedness between m and n is defined as: $\mathbb{C}^{\infty}_{\mathfrak{B}_i}(kl) = \bigvee_{j\geq 1} \{\mathbb{C}^j_{\mathfrak{B}_i}(kl)\}$ such that $\mathbb{C}^j_{\mathfrak{B}_i}(kl) = (\mathbb{C}^1_{\mathfrak{B}_i} \circ \mathbb{C}^1_{\mathfrak{B}_i})(kl)$ for $j \geq 2$ and
$$\mathbb{C}^2_{\mathfrak{B}_i}(kl) = (\mathbb{C}^1_{\mathfrak{B}_i} \circ \mathbb{C}^1_{\mathfrak{B}_i})(kl) = \vee_z (\mathbb{C}^1_{B_i}(mz) \wedge \mathbb{C}^1_{B_i}(zn)).$$
3. $\mathfrak{B}_i$- ignorance strength of connectedness between m and n is defined as: $\mathbb{U}^{\infty}_{\mathfrak{B}_i}(kl) = \bigwedge_{j\geq 1} \{\mathbb{U}^j_{\mathfrak{B}_i}(kl)\}$ such that $\mathbb{U}^j_{\mathfrak{B}_i}(kl) = (\mathbb{U}^1_{\mathfrak{B}_i} \circ \mathbb{U}^1_{\mathfrak{B}_i})(kl)$ for $j \geq 2$ and
$$\mathbb{U}^2_{\mathfrak{B}_i}(kl) = (\mathbb{U}^1_{\mathfrak{B}_i} \circ \mathbb{U}^1_{\mathfrak{B}_i})(kl) = \bigwedge_{z} (\mathbb{U}^1_{B_i}(mz) \vee \mathbb{U}^1_{B_i}(zn)).$$
4. $\mathfrak{B}_i$- false strength of connectedness between m and n is defined as: $\mathbb{F}^{\infty}_{\mathfrak{B}_i}(kl) = \bigwedge_{j\geq 1} \{\mathbb{F}^j_{\mathfrak{B}_i}(kl)\}$ such that $\mathbb{F}^j_{\mathfrak{B}_i}(kl) = (\mathbb{F}^1_{\mathfrak{B}_i} \circ \mathbb{F}^1_{\mathfrak{B}_i})(kl)$ for $j \geq 2$ and
$$\mathbb{F}^2_{\mathfrak{B}_i}(kl) = (\mathbb{F}^1_{\mathfrak{B}_i} \circ \mathbb{F}^1_{\mathfrak{B}_i})(kl) = \bigwedge_{z} (\mathbb{F}^1_{B_i}(mz) \vee \mathbb{F}^1_{B_i}(zn)).$$

Definition 3.14 *A QNGS* $\mathfrak{G} = (\mathfrak{A}, \mathfrak{B}_1, \mathfrak{B}_2, \dots, \mathfrak{B}_n)$ *is a* $\mathfrak{B}_i$*-cycle if*

$$(supp(\mathfrak{A}), supp(\mathfrak{B}_1), supp(\mathfrak{B}_2), \dots, supp(\mathfrak{B}_n)) \text{is a} \mathfrak{B}_i - \text{cycle}.$$

Definition 3.15 *A QNGS* $\mathfrak{G} = (\mathfrak{A}, \mathfrak{B}_1, \mathfrak{B}_2, \dots, \mathfrak{B}_n)$ *is a* $\mathfrak{B}_i$*-cycle (for some* i*) if* $\mathfrak{G}$ *is a* $\mathfrak{B}_i$*-cycle, no unique* $\mathfrak{B}_i$*-edge* kl *belongs to 1* $\mathfrak{G}$ *with*

$$\mathbb{T}_{B_i}(kl) = \min\{\mathbb{T}_{B_i}(rs): rs \in \mathbb{E}_i = \text{supp}(\mathfrak{B}_i)\},$$
$$\mathbb{C}_{B_i}(kl) = \min\{\mathbb{C}_{B_i}(rs): rs \in \mathbb{E}_i = \text{supp}(\mathfrak{B}_i)\},$$
$$\mathbb{U}_{B_i}(kl) = \max\{\mathbb{C}_{B_i}(rs): rs \in \mathbb{E}_i = \text{supp}(\mathfrak{B}_i)\},$$
$$\mathbb{F}_{B_i}(kl) = \max\{\mathbb{F}_{B_i}(rs): rs \in \mathbb{E}_i = \text{supp}(\mathfrak{B}_i)\}.$$

Definition 3.16 *Let* $\mathfrak{G} = (\mathfrak{A}, \mathfrak{B}_1, \mathfrak{B}_2, \dots, \mathfrak{B}_n)$ *be a QNGS and* q *be a node in* $\mathfrak{G}$*. Let* $(\mathfrak{A}', \mathfrak{B}'_1, \mathfrak{B}'_2, \dots, \mathfrak{B}'_n)$ *be a QNGS induced by* $\mathcal{X}\backslash\{q\}$ *such that, for all* $m \neq q,\ o \neq q,$

$$\mathbb{T}_{A'}(q) = \mathbb{C}_{A'}(q) = 0 = \mathbb{U}_{A'}(q) = \mathbb{F}_{A'}(q),$$
$$\mathbb{T}_{B'_i}(qm) = \mathbb{C}_{B'_i}(qm) = 0 = \mathbb{U}_{B'_i}(qm) = \mathbb{F}_{B'_i}(qm), \forall \quad \text{edges} \quad qm \in \mathfrak{G}$$
$$\mathbb{T}_{A'}(m) = \mathbb{T}_A(m), \mathbb{C}_{A'}(m) = \mathbb{C}_A(m), \mathbb{U}_{A'}(m) = \mathbb{U}_A(m), \mathbb{F}_{A'}(m) = \mathbb{F}_A(m),$$
$$\mathbb{T}_{B'_i}(mo) = \mathbb{T}_{B_i}(mo), \mathbb{C}_{B'_i}(mo) = \mathbb{C}_{B_i}(mo), \mathbb{U}_{B'_i}(mo) = \mathbb{U}_{B_i}(mo), \mathbb{F}_{B'_i}(mo) = \mathbb{F}_{B_i}(mo).$$

Now q is quadripartitioned neutrosophic $\mathfrak{B}_i$ cut vertex for some i if

$$\mathbb{T}^{\infty}_{B_i}(mo) > \mathbb{T}^{\infty}_{B'_i}(mo), \mathbb{C}^{\infty}_{B_i}(mo) > \mathbb{C}^{\infty}_{B'_i}(mo), \mathbb{U}^{\infty}_{B_i}(mo) > \mathbb{U}^{\infty}_{B'_i}(mo), \mathbb{F}^{\infty}_{B_i}(mo) > \mathbb{F}^{\infty}_{B'_i}(mo)$$

for some $m, o \in \mathcal{X}\backslash\{q\}$. Note that q is a

- $\mathfrak{B}_i$- $\mathbb{T}$ quadripartitioned neutrosophic cut node if $\mathbb{T}^{\infty}_{B_i}(mo) > \mathbb{T}^{\infty}_{B'_i}(mo)$.
- $\mathfrak{B}_i$- $\mathbb{C}$ quadripartitioned neutrosophic cut node if $\mathbb{C}^{\infty}_{B_i}(mo) > \mathbb{C}^{\infty}_{B'_i}(mo)$.
- $\mathfrak{B}_i$- $\mathbb{U}$ quadripartitioned neutrosophic cut node if $\mathbb{U}^{\infty}_{B_i}(mo) > \mathbb{U}^{\infty}_{B'_i}(mo)$.
- $\mathfrak{B}_i$- $\mathbb{F}$ quadripartitioned neutrosophic cut node if $\mathbb{F}^{\infty}_{B_i}(mo) > \mathbb{F}^{\infty}_{B'_i}(mo)$.

Definition 3.17 *Suppose* $\mathfrak{G} = (\mathfrak{A}, \mathfrak{B}_1, \mathfrak{B}_2, \dots, \mathfrak{B}_n)$ *be a QNGS and* kl *be* $\mathfrak{B}_i$*-edge. Let* $(\mathfrak{A}', \mathfrak{B}'_1, \mathfrak{B}'_2, \dots, \mathfrak{B}'_n)$ *be a quadripartitioned neutrosophic graph spanning subgraph structure of* $\mathfrak{G}$ *with for all lines* $kl \neq rs,$

$$\mathbb{T}_{B'_i}(kl) = \mathbb{C}_{B'_i}(kl) = 0 = \mathbb{U}_{B'_i}(kl) = \mathbb{F}_{B'_i}(kl),$$
$$\mathbb{T}_{B'_i}(rs) = \mathbb{T}_{B_i}(rs), \mathbb{C}_{B'_i}(rs) = \mathbb{C}_{B_i}(rs), \mathbb{U}_{B'_i}(rs) = \mathbb{U}_{B_i}(rs), \mathbb{F}_{B'_i}(rs) = \mathbb{F}_{B_i}(rs).$$

Then kl is quadripartitioned neutrosophic $\mathfrak{B}_i$-bridge if

$$\mathbb{T}^{\infty}_{B_i}(mo) > \mathbb{T}^{\infty}_{B'_i}(mo), \mathbb{C}^{\infty}_{B_i}(mo) > \mathbb{C}^{\infty}_{B'_i}(mo), \mathbb{U}^{\infty}_{B_i}(mo) > \mathbb{U}^{\infty}_{B'_i}(vw), \mathbb{F}^{\infty}_{B_i}(mo) > \mathbb{F}^{\infty}_{B'_i}(mo)$$

for some $m, o \in \mathcal{X}$. Note kl is a

- $\mathfrak{B}_i$- $\mathbb{T}$ quadripartitioned neutrosophic bridge if $\mathbb{T}^{\infty}_{B_i}(mo) > \mathbb{T}^{\infty}_{B'_i}(mo)$.
- $\mathfrak{B}_i$- $\mathbb{C}$ quadripartitioned neutrosophic bridge if $\mathbb{C}^{\infty}_{B_i}(mo) > \mathbb{C}^{\infty}_{B'_i}(mo)$.
- $\mathfrak{B}_i$- $\mathbb{U}$ quadripartitioned neutrosophic bridge if $\mathbb{U}^{\infty}_{B_i}(mo) > \mathbb{U}^{\infty}_{B'_i}(mo)$.
- $\mathfrak{B}_i$- $\mathbb{F}$ quadripartitioned neutrosophic bridge if $\mathbb{F}^{\infty}_{B_i}(mo) > \mathbb{F}^{\infty}_{B'_i}(mo)$.

Definition 3.18 *A QNGS* $\mathfrak{G} = (\mathfrak{A}, \mathfrak{B}_1, \mathfrak{B}_2, \ldots, \mathfrak{B}_n)$ *is a* $\mathfrak{B}_i$ *tree if*

$$(supp(\mathfrak{A}), supp(\mathfrak{B}_i), supp(\mathfrak{B}_2, \ldots, supp(\mathfrak{B}_n))$$

is a $\mathfrak{B}_i$-tree. In otherwords, $\mathfrak{G}$ is a $\mathfrak{B}_i$-tree provided a subgraph of $\mathfrak{G}$ induced by $supp(\mathfrak{B}_i)$ produces a tree.

Definition 3.19 *A QNGS* $\mathfrak{G} = (\mathfrak{A}, \mathfrak{B}_1, \mathfrak{B}_2, \ldots, \mathfrak{B}_n)$ *is a* $\mathfrak{B}_i$*- tree if* $\mathfrak{G}$ *has a quadripartitioned neutrosophic spanning subgraph structure* $\mathcal{H} = (\mathfrak{A}', \mathfrak{B}'_1, \mathfrak{B}'_2, \ldots, \mathfrak{B}'_n)$ *with for every* $\mathfrak{B}_i$*-edges* kl *not belongs to* $\mathcal{H}$, $\mathcal{H}$ *is a* $\mathfrak{B}'_i$*-tree,*

$$\mathbb{T}^{\infty}_{B_i}(kl) < \mathbb{T}^{\infty}_{B'_i}(kl), \mathbb{C}^{\infty}_{B_i}(kl) < \mathbb{C}^{\infty}_{B'_i}(kl), \mathbb{U}^{\infty}_{B_i}(kl) > \mathbb{U}^{\infty}_{B'_i}(kl), \mathbb{F}^{\infty}_{B_i}(kl) > \mathbb{F}^{\infty}_{B'_i}(kl)$$

In particular, $\mathfrak{G}$ is a:

- $\mathfrak{B}_i$- $\mathbb{T}$ quadripartitioned neutrosophic tree if $\mathbb{T}^{\infty}_{B_i}(kl) < \mathbb{T}^{\infty}_{B'_i}(kl)$.
- $\mathfrak{B}_i$- $\mathbb{C}$ quadripartitioned neutrosophic tree if $\mathbb{C}^{\infty}_{B_i}(kl) < \mathbb{C}^{\infty}_{B'_i}(kl)$.
- $\mathfrak{B}_i$- $\mathbb{U}$ quadripartitioned neutrosophic tree if $\mathbb{U}^{\infty}_{B_i}(kl) > \mathbb{U}^{\infty}_{B'_i}(kl)$.
- $\mathfrak{B}_i$- $\mathbb{F}$ quadripartitioned neutrosophic bridge if $\mathbb{F}^{\infty}_{B_i}(kl) > \mathbb{F}^{\infty}_{B'_i}(kl)$.

Definition 3.20 *A QNGS* $\mathfrak{G}_1 = (\mathfrak{A}_1, \mathfrak{B}_{11}, \mathfrak{B}_{12}, \ldots, \mathfrak{B}_{1n})$ *of the graph structure* $\mathfrak{G}^*_1 = (\mathbb{R}_1, \mathbb{E}_{11}, \mathbb{E}_{12}, \ldots, \mathbb{E}_{1n})$ *is isomorphic to QNGS* $\mathfrak{G}_2 = (\mathfrak{A}_2, \mathfrak{B}_{21}, \mathfrak{B}_{22}, \ldots, \mathfrak{B}_{2n})$ *of graph structure* $\mathfrak{G}^*_2 = (\mathbb{R}_2, \mathbb{E}_{21}, \mathbb{E}_{22}, \ldots, \mathbb{E}_{2n})$ *if* $f: \mathbb{R}_1 \to \mathbb{R}_2$ *is a bijection and the conditions below are fulfilled:*

$$\mathbb{T}_{A_1}(k) = \mathbb{T}_{A_2}(f(k)), \mathbb{C}_{A_1}(k) = \mathbb{C}_{A_2}(f(k)), \mathbb{U}_{A_1}(k) = \mathbb{U}_{A_2}(f(k)), \mathbb{F}_{A_1}(k) = \mathbb{F}_{A_2}(f(k)),$$

for all $m \in \mathbb{R}_1$ and

$$\mathbb{T}_{B_{1i}}(kl) = \mathbb{T}_{B_{2\phi(i)}}(f(k)f(l)), \mathbb{C}_{B_{1i}}(kl) = \mathbb{C}_{B_{2\phi(i)}}(f(k)f(l)),$$
$$\mathbb{U}_{B_{1i}}(kl) = \mathbb{U}_{B_{2\phi(i)}}(f(k)f(l)), \mathbb{F}_{B_{1i}}(kl) = \mathbb{F}_{B_{2\phi(i)}}(f(k)f(l)),$$

for all $kl \in \mathbb{E}_{1i}$ and $i = 1,2,..,n$.

Definition 3.21 *A QNGS* $\mathfrak{G}_1 = (\mathfrak{A}_1, \mathfrak{B}_{11}, \mathfrak{B}_{12}, \ldots, \mathfrak{B}_{1n})$ *of the graph structure* $\mathfrak{G}^*_1 = (\mathbb{R}_1, \mathbb{E}_{11}, \mathbb{E}_{12}, \ldots, \mathbb{E}_{1n})$ *is identical to QNGS* $\mathfrak{G}_2 = (\mathfrak{A}_2, \mathfrak{B}_{21}, \mathfrak{B}_{22}, \ldots, \mathfrak{B}_{2n})$ *of graph structure* $\mathfrak{G}^*_2 = (\mathbb{R}_2, \mathbb{E}_{21}, \mathbb{E}_{22}, \ldots, \mathbb{E}_{2n})$ *if* $f: \mathbb{R}_1 \to \mathbb{R}_2$ *is a bijection and the conditions below are fulfilled:*

$$\mathbb{T}_{A_1}(k) = \mathbb{T}_{A_2}(f(k)), \mathbb{C}_{A_1}(k) = \mathbb{C}_{A_2}(f(k)), \mathbb{U}_{A_1}(k) = \mathbb{U}_{A_2}(f(k)), \mathbb{F}_{A_1}(k) = \mathbb{F}_{A_2}(f(k)),$$

for all $m \in \mathbb{R}_1$ and

$$\mathbb{T}_{B_{1i}}(kl) = \mathbb{T}_{B_{2i}}(f(k)f(l)), \mathbb{C}_{B_{1i}}(kl) = \mathbb{C}_{B_2}(f(k)f(l)),$$
$$\mathbb{U}_{B_{1i}}(kl) = \mathbb{U}_{B_{2i}}(f(k)f(l)), \mathbb{F}_{B_{1i}}(kl) = \mathbb{F}_{B_{2i}}(f(k)f(l)),$$

for all $kl \in \mathbb{E}_{1i}$ and $i = 1,2,..,n$.

Definition 3.22 *Let* $\mathfrak{G}_1 = (\mathfrak{A}_1, \mathfrak{B}_{11}, \mathfrak{B}_{12}, \ldots, \mathfrak{B}_{1n})$ *be a QNGS and* ϕ *−permutation on* $\mathfrak{B}_1, \mathfrak{B}_2, .., \mathfrak{B}_n$ *and on* $\{1,2,\ldots,n\}$ *defined by* $\phi(\mathfrak{B}_i) = \mathfrak{B}_j$ *if and only if* $\phi(i) = j$ *for every* i*. If* $kl \in \mathfrak{B}_i$ *for some* i *and*

$$\mathbb{T}_{\mathfrak{B}^{\phi}_i}(kl) = \mathbb{T}_{\mathfrak{A}}(k) \wedge \mathbb{T}_{\mathfrak{A}}(l) - \bigvee_{j \neq 1} \mathbb{T}_{\phi(\mathbb{B}_j)}(kl)$$
$$\mathbb{C}_{\mathfrak{B}^{\phi}_i}(kl) = \mathbb{C}_{\mathfrak{A}}(k) \wedge \mathbb{C}_{\mathfrak{A}}(l) - \bigvee_{j \neq 1} \mathbb{C}_{\phi(\mathfrak{B}_j)}(kl)$$
$$\mathbb{U}_{\mathfrak{B}^{\phi}_i}(kl) = \mathbb{U}_{\mathfrak{A}}(k) \vee \mathbb{U}_{\mathfrak{A}}(l) - \bigwedge_{j \neq 1} \mathbb{U}_{\phi(\mathbb{B}_j)}(kl)$$

$$\mathbb{F}_{\mathbb{B}_i^{\phi}}(kl) = \mathbb{F}_{\mathfrak{A}}(k) \vee \mathbb{F}_{\mathfrak{A}}(l) - \wedge_{j \neq 1} \mathbb{F}_{\phi(\mathbb{B}_j)}(kl), i = 1,2,\ldots,n,$$

then $kl \in \mathfrak{B}_k^{\phi}$, where k is selected such that

$$\mathbb{T}_{\mathbb{B}_k^{\phi}}(kl) \geq \mathbb{T}_{\mathfrak{B}_i^{\phi}}(kl),$$
$$\mathbb{C}_{\mathfrak{B}_k^{\phi}}(kl) \geq \mathbb{C}_{\mathfrak{B}_i^{\phi}}(kl),$$
$$\mathbb{U}_{\mathfrak{B}_k^{\phi}}(kl) \geq \mathbb{U}_{\mathfrak{B}_i^{\phi}}(kl),$$
$$\mathbb{F}_{\mathfrak{B}_k^{\phi}}(kl) \geq \mathbb{F}_{\mathfrak{B}_i^{\phi}}(kl).$$

then quadripartitined neutrosophic graph structure $(\mathfrak{A}, \mathfrak{B}_1^{\phi}, \mathfrak{B}_2^{\phi}, \ldots, \mathfrak{B}_n^{\phi})$ is called ϕ- complement of $\mathfrak{G}$ and dentoted by $\mathfrak{G}^{\phi c}$

Proposition 3.23 *ϕ-complement of a QNGS $\mathfrak{G} = (\mathfrak{A}, \mathfrak{B}_1, \mathfrak{B}_2, \ldots, \mathfrak{B}_n)$ is always a strong QNGS. Further, if $\phi(i) = k$, where $i, k \in \{1,2,\ldots,n\}$ then for all $\mathfrak{B}_k$-edges in quadripartitioned neutrosophic graphic structure $(\mathfrak{A}, \mathfrak{B}_1, \mathfrak{B}_2, \ldots, \mathfrak{B}_n)$ become $\mathfrak{B}_i^{\phi}$-edges in $(\mathfrak{A}^{\phi}, \mathfrak{B}_1^{\phi}, \mathfrak{B}_2^{\phi}, \ldots, \mathfrak{B}_n^{\phi})$.*

Proof. We know that,

$$\mathbb{T}_{\mathfrak{B}_i^{\phi}}(kl) = \mathbb{T}_{\mathfrak{A}}(k) \wedge \mathbb{T}_{\mathfrak{A}}(l) - \vee_{j \neq i} \mathbb{T}_{\phi(\mathbb{B}_j)}(kl),$$
$$\mathbb{C}_{\mathfrak{B}_i^{\phi}}(kl) = \mathbb{C}_{\mathfrak{A}}(k) \wedge \mathbb{C}_{\mathfrak{A}}(l) - \vee_{j \neq i} \mathbb{C}_{\phi(\mathbb{B}_j)}(kl),$$
$$\mathbb{U}_{\mathfrak{B}_i^{\phi}}(kl) = \mathbb{U}_{\mathfrak{A}}(k) \vee \mathbb{U}_{\mathfrak{A}}(l) - \wedge_{j \neq i} \mathbb{U}_{\phi(\mathbb{B}_j)}(kl),$$
$$\mathbb{F}_{\mathfrak{B}_i^{\phi}}(kl) = \mathbb{F}_{\mathfrak{A}}(k) \vee \mathbb{F}_{\mathfrak{A}}(l) - \wedge_{j \neq i} \mathbb{F}_{\phi(\mathbb{B}_j)}(kl),$$

for $i \in 1,2,\ldots,n$. Due to the expression of truthness in ϕ -complement, $\mathbb{T}_{\mathfrak{A}}(k) \wedge \mathbb{T}_{\mathfrak{A}}(l) \geq 0$, $\vee_{j \neq i} \mathbb{T}_{\phi(\mathbb{B}_j)}(kl) \geq 0$ and $\mathbb{T}_{\mathfrak{B}_i}(kl) \leq \mathbb{T}_{\mathfrak{A}}(k) \wedge \mathbb{T}_{\mathfrak{A}}(l)$, for all $\mathfrak{B}_i$, now $\vee_{j \neq i} \mathbb{T}_{\phi(\mathbb{B}_j)}(kl) \leq \mathbb{T}_{\mathfrak{A}}(k) \wedge \mathbb{T}_{\mathfrak{A}}(l)$ which implies that

$$\mathbb{T}_{\mathfrak{A}}(k) \wedge \mathbb{T}_{\mathfrak{A}}(l) - \vee_{j \neq i} \mathbb{T}_{\phi(\mathbb{B}_j)}(kl) \geq 0$$

Hence, $\mathbb{T}_{\mathfrak{B}_i^{\phi}}(kl) \geq 0$ for every i. Further, $\mathbb{T}_{\mathfrak{B}_i^{\phi}}(kl)$ attains its maximum provided $\vee_{j \neq i} \mathbb{T}_{\phi(\mathbb{B}_j)}(kl) \geq 0$ is zero. Clearly, when $\phi(\mathfrak{B}_i) = \mathfrak{B}_k$ and kl is a $\mathfrak{B}_k$-edge then $\vee_{j \neq i} \mathbb{T}_{\phi(\mathbb{B}_j)}(kl)$ gets zero value. So

$\mathbb{T}_{\mathfrak{B}_i^{\phi}}(kl) = \mathbb{T}_{\mathfrak{A}}(k) \wedge \mathbb{T}_{\mathfrak{A}}(l)$, for some $(kl) \in \mathfrak{B}_k$, $\phi(\mathfrak{B}_i) = \mathfrak{B}_k$

Similarly, we have

$\mathbb{C}_{\mathfrak{B}_i^{\phi}}(kl) = \mathbb{C}_{\mathfrak{A}}(k) \wedge \mathbb{C}_{\mathfrak{A}}(l)$, for some $(kl) \in \mathfrak{B}_k$, $\phi(\mathfrak{B}_i) = \mathfrak{B}_k$

$\mathbb{U}_{\mathfrak{B}_i^{\phi}}(kl) = \mathbb{C}_{\mathfrak{A}}(k) \vee \mathbb{U}_{\mathfrak{A}}(l)$, for some $(kl) \in \mathfrak{B}_k$, $\phi(\mathfrak{B}_i) = \mathfrak{B}_k$

$\mathbb{F}_{\mathfrak{B}_i^{\phi}}(kl) = \mathbb{F}_{\mathfrak{A}}(k) \vee \mathbb{F}_{\mathfrak{A}}(l)$, for some $(kl) \in \mathfrak{B}_k$, $\phi(\mathfrak{B}_i) = \mathfrak{B}_k$.

Likewise, the expression of falsity in ϕ- complement:

$$\mathbb{F}_{\mathfrak{A}}(k) \vee \mathbb{F}_{\mathfrak{A}}(l) \geq 0, \wedge_{j \neq i} \mathbb{F}_{\phi(\mathfrak{B}_j)}(kl) \geq 0 and \mathbb{F}_{\mathfrak{B}i}(kl) \leq \mathbb{F}_{\mathfrak{A}}(k) \vee \mathbb{F}_{\mathfrak{A}}(l) \forall \mathfrak{B}_i$$

Then

$$\wedge_{j \neq 1} \mathbb{F}_{\phi(\mathfrak{B}_j)}(kl) \leq \mathbb{F}_{\mathfrak{A}}(k) \vee \mathbb{F}_{\mathfrak{A}}(l)$$

yields,

$$\mathbb{F}_{\mathfrak{A}}(k) \vee \mathbb{F}_{\mathfrak{A}}(l) - \wedge_{j \neq i} \mathbb{F}_{\phi(\mathfrak{B}_j)}(kl) \geq 0$$

Therefore, $\mathbb{T}_{\mathfrak{B}_i^{\phi}}(kl)$ is non-negative for all i . Morevoer, $\mathbb{T}_{\mathfrak{B}_i^{\phi}}(kl)$ reaches its maximum when $\wedge_{j \neq i} \mathbb{F}_{\phi(\mathfrak{B}_j)}(kl)$ becomes zero. It is clear that when $\phi(\mathfrak{B}_i) = \mathfrak{B}_k$ and kl is a $\mathfrak{B}_k$ edge then $\wedge_{j \neq i} \mathbb{F}_{\phi(\mathfrak{B}_j)}(kl)$ gets zero value. So

$$\mathbb{F}_{\mathfrak{B}_i^{\phi}}(kl) = \mathbb{F}_{\mathfrak{A}}(k) \vee \mathbb{F}_{\mathfrak{A}}(l) \quad for \quad (kl) \in \mathfrak{B}_k, \phi(\mathfrak{B}i) = \mathfrak{B}_k$$

Definition 3.24 *Let* $\mathfrak{G} = (\mathfrak{A}, \mathfrak{B}_1, \mathfrak{B}_2, \ldots, \mathfrak{B}_n)$ *be a QNGS and* ϕ *be a permutation on* $\{1,2,\ldots,n\}$ *then*

- If $\mathfrak{G}$ is isomorphic to $\mathfrak{G}^{\phi c}$, then $\mathfrak{G}$ is called self-complementary.
- If $\mathfrak{G}$ is identical to $\mathfrak{G}^{\phi c}$, then $\mathfrak{G}$ is called strong-self-complementary.

Definition 3.25 *Suppose* $\mathfrak{G} = (\mathfrak{A}, \mathfrak{B}_1, \mathfrak{B}_2, \ldots, \mathfrak{B}_n)$ *be a QNGS. Then*

- If $\mathfrak{G}$ is isomorphic to $\mathfrak{G}^{\phi c}$, for all permutation ϕ on $\{1,2,\ldots,n\}$, then $\mathfrak{G}$ is totally self complementary.
- If $\mathfrak{G}$ is identical to $\mathfrak{G}^{\phi c}$, for all permutation ϕ on $\{1,2,\ldots,n\}$, then $\mathfrak{G}$ is totally strong self complementary.

Remark 3.26 *All strong QNGSs are self complementary or totally self-complementary QNGSs.*

Theorem 3.27*A QNGSs is totally self-complementary if and only if it is strong QNGS.*

Proof. Consider a strong QNGS $\mathfrak{G}$ and permutation ϕ on $\{1,2,\ldots,n\}$. In the view of Proposition 3.22, ϕ-complement of a QNGS $\mathfrak{G} = (\mathfrak{A}, \mathfrak{B}_1, \mathfrak{B}_2, \ldots, \mathfrak{B}_n)$ is always a strong QNGS. Moreover, if $\phi(i) = k$, here $i, k \in \{1,2,\ldots,n\}$, then every $\mathfrak{B}_k$ lines in QNGSs $(\mathfrak{A}, \mathfrak{B}_1, \mathfrak{B}_2, \ldots, \mathfrak{B}_n)$ becomes $\mathfrak{B}_i^{\phi}$ -edges in $(\mathfrak{A}^{\phi}, \mathfrak{B}_1^{\phi}, \mathfrak{B}_2^{\phi}, \ldots, \mathfrak{B}_n^{\phi})$. It yields

$$\mathbb{T}_{\mathfrak{B}_k}(kl) = \mathbb{T}_{\mathfrak{A}}(k) \wedge \mathbb{T}_{\mathfrak{A}}(l) = \mathbb{T}_{\mathfrak{B}_i^{\phi}}(kl)$$

$$\mathbb{C}_{\mathfrak{B}_k}(kl) = \mathbb{C}_{\mathfrak{A}}(k) \wedge \mathbb{C}_{\mathfrak{A}}(l) = \mathbb{C}_{\mathfrak{B}_i^{\phi}}(kl)$$

$$\mathbb{U}_{\mathfrak{B}_k}(kl) = \mathbb{U}_{\mathfrak{A}}(k) \vee \mathbb{U}_{\mathfrak{A}}(l) = \mathbb{U}_{\mathfrak{B}_i^{\phi}}(kl)$$

$$\mathbb{F}_{\mathfrak{B}_k}(kl) = \mathbb{F}_{\mathfrak{A}}(k) \vee \mathbb{F}_{\mathfrak{A}}(l) = \mathbb{F}_{\mathfrak{B}_i^{\phi}}(kl)$$

Thus, in identity mapping $f: \mathcal{X} \to \mathcal{X}$, $\mathfrak{G}$ and $\mathfrak{G}^{\phi}$ are isomorphic with

$$\mathbb{T}_{\mathfrak{A}}(k) = \mathbb{T}_{\mathfrak{A}}(f(k)), \mathbb{C}_{\mathfrak{A}}(k) = \mathbb{C}_{\mathfrak{A}}(f(k)),$$

$$\mathbb{U}_{\mathfrak{A}}(k) = \mathbb{U}_{\mathfrak{A}}(f(k)), \mathbb{F}_{\mathfrak{A}}(k) = \mathbb{F}_{\mathfrak{A}}(f(k)),$$

$$\mathbb{T}_{\mathfrak{B}_k}(kl) = \mathbb{T}_{\mathfrak{B}_k^{\phi}}(f(k)f(l)) = \mathbb{T}_{\mathfrak{B}_k^{\phi}}(kl),\ \mathbb{C}_{\mathfrak{B}_k}(kl) = \mathbb{C}_{\mathfrak{B}_k^{\phi}}(f(k)f(l)) = \mathbb{C}_{\mathfrak{B}_k^{\phi}}(kl),$$

$$\mathbb{U}_{\mathfrak{B}_k}(kl) = \mathbb{U}_{\mathfrak{B}_k^{\phi}}(f(k)f(l)) = \mathbb{U}_{\mathfrak{B}_k^{\phi}}(kl),\ \mathbb{F}_{\mathfrak{B}_k}(kl) = \mathbb{F}_{\mathfrak{B}_k^{\phi}}(f(k)f(l)) = \mathbb{F}_{\mathfrak{B}_k^{\phi}}(kl),$$

for all $kl \in \mathcal{E}_k$, $\phi^{-1}(k) = i$ and $k = 1,,\ldots,n$. It holds for all permutation ϕ on $\{1,2,\ldots,n\}$. Thus, $\mathfrak{G}$ is totally self-complementary QNGS. Conversely, suppose for all permutation ϕ on $\{1,2,..,n\}$ $\mathfrak{G}$ is isomorphic to $\mathfrak{G}^{\phi}$. Then according to the definition of isomorphism of QNGSs and ϕ -complement of QNGS,

$$\mathbb{T}_{\mathfrak{B}_k}(kl) = \mathbb{T}_{\mathfrak{B}_k^{\phi}}(f(k)f(l)) = \mathbb{T}_{\mathfrak{A}}(f(k)) \wedge \mathbb{T}_{\mathfrak{A}}(f(l)) = \mathbb{T}_{\mathfrak{A}}(k) \wedge \mathbb{T}_{\mathfrak{A}}(l)$$
$$\mathbb{C}_{\mathfrak{B}_k}(kl) = \mathbb{C}_{\mathfrak{B}_k^{\phi}}(f(k)f(l)) = \mathbb{C}_{\mathfrak{A}}(f(k)) \wedge \mathbb{C}_{\mathfrak{A}}(f(l)) = \mathbb{C}_{\mathfrak{A}}(k) \wedge \mathbb{C}_{\mathfrak{A}}(l)$$
$$\mathbb{U}_{\mathfrak{B}_k}(kl) = \mathbb{U}_{\mathfrak{B}_k^{\phi}}(f(k)f(l)) = \mathbb{U}_{\mathfrak{A}}(f(k)) \vee \mathbb{U}_{\mathfrak{A}}(f(l)) = \mathbb{U}_{\mathfrak{A}}(k) \vee \mathbb{U}_{\mathfrak{A}}(l)$$
$$\mathbb{F}_{\mathfrak{B}_k}(kl) = \mathbb{F}_{\mathfrak{B}_k^{\phi}}(f(k)f(l)) = \mathbb{F}_{\mathfrak{A}}(f(k)) \vee \mathbb{F}_{\mathfrak{A}}(f(l)) = \mathbb{F}_{\mathfrak{A}}(k) \vee \mathbb{F}_{\mathfrak{A}}(l).$$

for all $kl \in \mathcal{E}_k$ and $k = 1,2,..,n$. Hence, $\mathfrak{G}$ is strong QNGS.

Remark 3.28 *All self-complementary QNGS is totally self-complementary.*

Theorem3.39 *If $\mathfrak{G}^* = (\mathcal{X}, \mathcal{E}_1, \mathcal{E}_2, \ldots, \mathcal{E}_n)$ is totally strong self-complementary QNGS and $A = \langle \mathbb{T}_{\mathfrak{A}}, \mathbb{C}_{\mathfrak{A}}, \mathbb{U}_{\mathfrak{A}}, \mathbb{F}_{\mathfrak{A}} \rangle$ is a quadripartitioned neutrosophic subset of $\mathcal{X}$ here $\mathbb{T}_{\mathfrak{A}}, \mathbb{C}_{\mathfrak{A}}, \mathbb{U}_{\mathfrak{A}}, \mathbb{F}_{\mathfrak{A}}$ are constant value functions, then a strong QNGS of $\mathfrak{G}^*$ with quadripartitioned neutrosophic node set $\mathfrak{A}$ is always a totally strong self-complementary QNGS.*

Proof. Let the four constants be $p, q, r, s \in [0,1]$, such that $\mathbb{T}_{\mathfrak{A}}(k) = p$ $\mathbb{C}_{\mathfrak{A}}(k) = q$, $\mathbb{U}_{\mathfrak{A}}(k) = r$, $\mathbb{F}_{\mathfrak{A}}(k) = s$ for all $m \in \mathcal{X}$. Because $\mathfrak{G}^*$ is totally self-complementary strong QNGS, hence there exists a bijection $f: \mathcal{X} \to \mathcal{X}$ for permutation ϕ^{-1} on $\{1,2,\ldots,n\}$, with for any $\mathbb{E}_k$- edge $(kl), (f(k)f(l))$ [an $\mathbb{E}_i$-line in $\mathfrak{G}^*$] is an $\mathbb{E}_k$ line in $\mathfrak{G}^{*\phi^{-1}c}$. Thus, for all $\mathfrak{B}_k$- edge $(kl), (f(k)f(l))$ [an $\mathfrak{B}_i$-edge in $\mathfrak{G}$] is a $\mathfrak{B}_k^{\phi}$- edge in $\mathfrak{G}^{*\phi^{-1}c}$. Further, $\mathfrak{G}$ is strong QNGS. Hence

$$\mathbb{T}_{\mathfrak{A}}(k) = p = \mathbb{T}_{\mathfrak{A}}(f(k)), \mathbb{C}_{\mathfrak{A}}(k) = q = \mathbb{C}_{\mathfrak{A}}(f(k)),$$
$$\mathbb{U}_{\mathfrak{A}}(k) = r = \mathbb{U}_{\mathfrak{A}}(f(k)), \mathbb{F}_{\mathfrak{A}}(k) = s = \mathbb{F}_{\mathfrak{A}}(f(k)), \forall m \in \mathcal{X},$$

$$\mathbb{T}_{\mathfrak{B}_k}(kl) = \mathbb{T}_{\mathfrak{A}}(k) \wedge \mathbb{T}_{\mathfrak{A}}(l) = \mathbb{T}_{\mathfrak{A}}(f(k)) \wedge \mathbb{T}_{\mathfrak{A}}(f(l)) = \mathbb{T}_{\mathfrak{B}_i^{\phi}}(f(k)f(l))$$
$$\mathbb{C}_{\mathfrak{B}_k}(kl) = \mathbb{C}_{\mathfrak{A}}(k) \wedge \mathbb{C}_{\mathfrak{A}}(l) = \mathbb{T}_{\mathfrak{A}}(f(k)) \wedge \mathbb{C}_{\mathfrak{A}}(f(l)) = \mathbb{C}_{\mathfrak{B}_i^{\phi}}(f(k)f(l))$$
$$\mathbb{U}_{\mathfrak{B}_k}(kl) = \mathbb{U}_{\mathfrak{A}}(k) \vee \mathbb{U}_{\mathfrak{A}}(l) = \mathbb{U}_{\mathfrak{A}}(f(k)) \vee \mathbb{U}_{\mathfrak{A}}(f(l)) = \mathbb{U}_{\mathfrak{B}_i^{\phi}}(f(k)f(l))$$
$$\mathbb{F}_{\mathfrak{B}_k}(kl) = \mathbb{F}_{\mathfrak{A}}(k) \vee \mathbb{F}_{\mathfrak{A}}(l) = \mathbb{F}_{\mathfrak{A}}(f(k)) \vee \mathbb{F}_{\mathfrak{A}}(f(l)) = \mathbb{F}_{\mathfrak{B}_i^{\phi}}(f(k)f(l)).$$

for every $kl \in \mathbb{E}_i$ and $i = 1,2,\ldots n$. This leads to $\mathfrak{G}$ is self complementary strong QNGS. All permutation ϕ and ϕ^{-1} on $\{1,2,\ldots,n\}$ fulfils the above arguments, hence $\mathfrak{G}$ is totally strong self-complementary QNGS. Converse of the theorem may not be true.

Definition 3.40 *Let $\mathfrak{G}_{n1} = (\mathfrak{Q}_1, \mathfrak{Q}_{11}, \mathfrak{Q}_{12}, \ldots . \mathfrak{Q}_{1n})$ and $\mathfrak{G}_{n2} = (\mathfrak{Q}_2, \mathfrak{Q}_{21}, \mathfrak{Q}_{22}, \ldots . \mathfrak{Q}_{2n})$ be QNGS. The Cartesian product of $\mathfrak{G}_1$ and $\mathfrak{G}_2$ denoted by*

$$\mathfrak{G}_{n1} \times \mathfrak{G}_{n2} = (\mathfrak{Q}_1 \times \mathfrak{Q}_2, \mathfrak{Q}_{11} \times \mathfrak{Q}_{21}, \mathfrak{Q}_{12} \times \mathfrak{Q}_{22}, \ldots, \mathfrak{Q}_{1n} \times \mathfrak{Q}_{2n}),$$

is defined by the following:

(i) $\mathbb{T}_{\mathfrak{Q}1\times\mathfrak{Q}2}(rs) = (\mathbb{T}_{\mathfrak{Q}1} \times \mathbb{T}_{\mathfrak{Q}2})(rs) = \mathbb{T}_{\mathfrak{Q}1}(r) \wedge \mathbb{T}_{\mathfrak{Q}2}(s).$
$\mathbb{C}_{\mathfrak{Q}1\times\mathfrak{Q}2}(rs) = (\mathbb{C}_{\mathfrak{Q}1} \times \mathbb{C}_{\mathfrak{Q}2})(rs) = \mathbb{C}_{\mathfrak{Q}1}(r) \wedge \mathbb{C}_{\mathfrak{Q}2}(s).$
$\mathbb{U}_{\mathfrak{Q}1\times\mathfrak{Q}2}(rs) = (\mathbb{U}_{\mathfrak{Q}1} \times \mathbb{U}_{\mathfrak{Q}2})(rs) = \mathbb{U}_{\mathfrak{Q}1}(r) \vee \mathbb{U}_{\mathfrak{Q}2}(s).$
$\mathbb{F}_{\mathfrak{Q}1\times\mathfrak{Q}2}(rs) = (\mathbb{F}_{\mathfrak{Q}1} \times \mathbb{F}_{\mathfrak{Q}2})(rs) = \mathbb{F}_{\mathfrak{Q}1}(r) \vee \mathbb{F}_{\mathfrak{Q}2}(s).$
$\forall rs \in S_1 \times S_2.$

(ii) $\mathbb{T}_{(\mathfrak{Q}_{1i}\times\mathfrak{Q}_{2i})}(rs_1)(rs_2) = (\mathbb{T}_{\mathfrak{Q}_{1i}} \times \mathbb{T}_{\mathfrak{Q}_{2i}})(rs_1)(rs_2) = \mathbb{T}_{Q1}(r) \wedge \mathbb{T}_{Q2i}(s_1 s_2)$
$\mathbb{C}_{(\mathfrak{Q}_{1i}\times\mathfrak{Q}_{2i})}(rs_1)(rs_2) = (\mathbb{C}_{\mathfrak{Q}_{1i}} \times \mathbb{C}_{\mathfrak{Q}_{2i}})(rs_1)(rs_2) = \mathbb{C}_{Q1}(r) \wedge \mathbb{C}_{Q2i}(s_1 s_2)$
$\mathbb{U}_{(\mathfrak{Q}_{1i}\times\mathfrak{Q}_{2i})}(rs_1)(rs_2) = (\mathbb{U}_{\mathfrak{Q}_{1i}} \times \mathbb{U}_{\mathfrak{Q}_{2i}})(rs_1)(rs_2) = \mathbb{U}_{Q1}(r) \vee \mathbb{U}_{Q2i}(s_1 s_2)$
$\mathbb{F}_{(\mathfrak{Q}_{1i}\times\mathfrak{Q}_{2i})}(rs_1)(rs_2) = (\mathbb{F}_{\mathfrak{Q}_{1i}} \times \mathbb{F}_{\mathfrak{Q}_{2i}})(rs_1)(rs_2) = \mathbb{F}_{Q1}(r) \vee \mathbb{F}_{Q2i}(s_1 s_2)$
$\forall r \in S_1, s_1 s_2 \in S_{2i},$

(iii) $\mathbb{T}_{(\mathfrak{Q}_{1i}\times\mathfrak{Q}_{2i})}(r_1 s)(r_2 s) = (\mathbb{T}_{\mathfrak{Q}_{1i}} \times \mathbb{T}_{\mathfrak{Q}_{2i}})(r_1 s)(r_2 s) = \mathbb{T}_{Q2}(s) \wedge \mathbb{T}_{Q1i}(r_1 r_2)$
$\mathbb{C}_{(\mathfrak{Q}_{1i}\times\mathfrak{Q}_{2i})}(r_1 s)(r_2 s) = (\mathbb{C}_{\mathfrak{Q}_{1i}} \times \mathbb{C}_{\mathfrak{Q}_{2i}})(r_1 s)(r_2 s) = \mathbb{C}_{Q2}(s) \wedge \mathbb{C}_{Q1i}(r_1 r_2)$
$\mathbb{U}_{(\mathfrak{Q}_{1i}\times\mathfrak{Q}_{2i})}(r_1 s)(r_2 s) = (\mathbb{U}_{\mathfrak{Q}_{1i}} \times \mathbb{U}_{\mathfrak{Q}_{2i}})(r_1 s)(r_2 s) = \mathbb{U}_{Q2}(s) \vee \mathbb{U}_{Q1i}(r_1 r_2)$
$\mathbb{F}_{(\mathfrak{Q}_{1i}\times\mathfrak{Q}_{2i})}(r_1 s)(r_2 s) = (\mathbb{F}_{\mathfrak{Q}_{1i}} \times \mathbb{F}_{\mathfrak{Q}_{2i}})(r_1 s)(r_2 s) = \mathbb{F}_{Q2}(s) \vee \mathbb{F}_{Q1i}(r_1 r_2)$
$\forall s \in S_2, r_1 r_2 \in S_{1i}.$

Theorem 3.41 *The Cartesian product $\mathfrak{G}_{n1} \times \mathfrak{G}_{n2} = (\mathfrak{Q}_1 \times \mathfrak{Q}_2, \mathfrak{Q}_{11} \times \mathfrak{Q}_{21}, \mathfrak{Q}_{12} \times \mathfrak{Q}_{22}, \ldots, \mathfrak{Q}_{1n} \times \mathfrak{Q}_{2n})$ of two QNGS of the GS $\mathfrak{G}_1$ and $\mathfrak{G}_2$ is a QNGs of $\mathfrak{G}_1 \times \mathfrak{G}_2$.*

Proof. According to the definition of Cartesian product there are two cases:
Case1: when $r \in S_1, r_1 r_2 \in S_{2i}$

$$\begin{aligned}
&\mathbb{T}_{(\mathfrak{Q}_{1i}\times\mathfrak{Q}_{2i})}(rs_1)(rs_2) = \mathbb{T}_{Q1}(r) \wedge \mathbb{T}_{Q2i}(s_1 s_2)\\
&\leq \mathbb{T}_{Q1}(r) \wedge [\mathbb{T}_{Q2}(s_1) \wedge \mathbb{T}_{Q2}(s_2)]\\
&= [\mathbb{T}_{Q1}(r) \wedge \mathbb{T}_{Q2}(s_1)] \wedge [\mathbb{T}_{Q1}(r) \wedge \mathbb{T}_{Q2}(s_2)]\\
&= \mathbb{T}_{(Q_1\times Q_2)}(rs_1) \wedge \mathbb{T}_{(Q_1\times Q_2)}(rs_2).
\end{aligned}$$

$$\begin{aligned}
&\mathbb{C}_{(\mathfrak{Q}_{1i}\times\mathfrak{Q}_{2i})}(rs_1)(rs_2) = \mathbb{C}_{Q1}(r) \wedge \mathbb{C}_{Q2i}(s_1 s_2)\\
&\leq \mathbb{C}_{Q1}(r) \wedge [\mathbb{C}_{Q2}(s_1) \wedge \mathbb{C}_{Q2}(s_2)]\\
&= [\mathbb{C}_{Q1}(r) \wedge \mathbb{C}_{Q2}(s_1)] \wedge [\mathbb{C}_{Q1}(r) \wedge \mathbb{C}_{Q2}(s_2)]\\
&= \mathbb{C}_{(Q_1\times Q_2)}(rs_1) \wedge \mathbb{C}_{(Q_1\times Q_2)}(rs_2).
\end{aligned}$$

$$\begin{aligned}
&\mathbb{U}_{(\mathfrak{Q}_{1i}\times\mathfrak{Q}_{2i})}(rs_1)(rs_2) = \mathbb{U}_{Q1}(r) \vee \mathbb{U}_{Q2i}(s_1 s_2)\\
&\leq \mathbb{U}_{Q1}(r) \vee [\mathbb{F}_{Q2}(s_1) \vee \mathbb{U}_{Q2}(s_2)]\\
&= [\mathbb{U}_{Q1}(r) \vee \mathbb{U}_{Q2}(s_1)] \vee [\mathbb{U}_{Q1}(r) \vee \mathbb{U}_{Q2}(s_2)]\\
&= \mathbb{U}_{(Q_1\times Q_2)}(rs_1) \vee \mathbb{U}_{(Q_1\times Q_2)}(rs_2).
\end{aligned}$$

$$\begin{aligned}
&\mathbb{F}_{(\mathfrak{Q}_{1i}\times\mathfrak{Q}_{2i})}(rs_1)(rs_2) = \mathbb{F}_{Q1}(r) \vee \mathbb{F}_{Q2i}(s_1 s_2)\\
&\leq \mathbb{F}_{Q1}(r) \vee [\mathbb{F}_{Q2}(s_1) \vee \mathbb{F}_{Q2}(s_2)]\\
&= [\mathbb{F}_{Q1}(r) \vee \mathbb{F}_{Q2}(s_1)] \vee [\mathbb{F}_{Q1}(r) \vee \mathbb{F}_{Q2}(s_2)]\\
&= \mathbb{F}_{(Q_1\times Q_2)}(rs_1) \vee \mathbb{F}_{(Q_1\times Q_2)}(rs_2).
\end{aligned}$$

for $rs_1, rs_2 \in S_1 \times S_2$.

Case 2: when $r \in S_2, s_1 s_2 \in S_{1i}$

$$\begin{aligned}
&\mathbb{T}_{(\mathfrak{Q}_{1i}\times\mathfrak{Q}_{2i})}(s_1 r)(s_2 r) = \mathbb{T}_{Q2}(r) \wedge \mathbb{T}_{Q1i}(r_1 r_2)\\
&\leq \mathbb{T}_{Q2}(r) \wedge [\mathbb{T}_{Q1}(s_1) \wedge \mathbb{T}_{Q2}(s_2)]\\
&= [\mathbb{T}_{Q2}(r) \wedge \mathbb{T}_{Q1}(s_1)] \wedge [\mathbb{T}_{Q2}(r) \wedge \mathbb{T}_{Q1}(s_2)]\\
&= \mathbb{T}_{(Q_1\times Q_2)}(s_1 r) \wedge \mathbb{T}_{(Q_1\times Q_2)}(s_2 r).
\end{aligned}$$

$$\begin{aligned}
&\mathbb{C}_{(\mathfrak{Q}_{1i}\times\mathfrak{Q}_{2i})}(s_1 r)(s_2 r) = \mathbb{C}_{Q2}(r) \wedge \mathbb{C}_{Q1i}(r_1 r_2)\\
&\leq \mathbb{C}_{Q2}(r) \wedge [\mathbb{C}_{Q1}(s_1) \wedge \mathbb{C}_{Q2}(s_2)]\\
&= [\mathbb{C}_{Q2}(r) \wedge \mathbb{C}_{Q1}(s_1)] \wedge [\mathbb{C}_{Q2}(r) \wedge \mathbb{C}_{Q1}(s_2)]\\
&= \mathbb{C}_{(Q_1\times Q_2)}(s_1 r) \wedge \mathbb{C}_{(Q_1\times Q_2)}(s_2 r).
\end{aligned}$$

$$\begin{aligned}
&\mathbb{U}_{(\mathfrak{Q}_{1i}\times\mathfrak{Q}_{2i})}(s_1 r)(s_2 r) = \mathbb{U}_{Q2}(r) \vee \mathbb{U}_{Q1i}(r_1 r_2)\\
&\leq \mathbb{U}_{Q2}(r) \vee [\mathbb{U}_{Q1}(s_1) \vee \mathbb{U}_{Q2}(s_2)]\\
&= [\mathbb{U}_{Q2}(r) \wedge \mathbb{U}_{Q1}(s_1)] \vee [\mathbb{U}_{Q2}(r) \wedge \mathbb{U}_{Q1}(s_2)]\\
&= \mathbb{U}_{(Q_1\times Q_2)}(s_1 r) \vee \mathbb{U}_{(Q_1\times Q_2)}(s_2 r).
\end{aligned}$$

$$\begin{aligned}
&\mathbb{F}_{(\mathfrak{Q}_{1i}\times\mathfrak{Q}_{2i})}(s_1 r)(s_2 r) = \mathbb{F}_{Q2}(r) \vee \mathbb{F}_{Q1i}(r_1 r_2)\\
&\leq \mathbb{F}_{Q2}(r) \vee [\mathbb{F}_{Q1}(s_1) \vee \mathbb{F}_{Q2}(s_2)]\\
&= [\mathbb{F}_{Q2}(r) \wedge \mathbb{F}_{Q1}(s_1)] \vee [\mathbb{F}_{Q2}(r) \wedge \mathbb{F}_{Q1}(s_2)]\\
&= \mathbb{F}_{(Q_1\times Q_2)}(s_1 r) \vee \mathbb{F}_{(Q_1\times Q_2)}(s_2 r).
\end{aligned}$$

for $s_1 r, s_2 r \in S_1 S_2$.

Hence Proved.

Definition 3.42 *Let* $\mathfrak{G}_{n1} = (\mathfrak{Q}_1, \mathfrak{Q}_{11}, \mathfrak{Q}_{12}, \ldots . \mathfrak{Q}_{1n)}$ *and* $\mathfrak{G}_{n2} = (\mathfrak{Q}_2, \mathfrak{Q}_{21}, \mathfrak{Q}_{22}, \ldots . \mathfrak{Q}_{2n)}$ *be QNGS. The cross product of* $\mathfrak{G}_1$ *and* $\mathfrak{G}_2$ *denoted by*

$\mathfrak{G}_{n1} \star \mathfrak{G}_{n2} = (\mathfrak{Q}_1 \star \mathfrak{Q}_2, \mathfrak{Q}_{11} \star \mathfrak{Q}_{21}, \mathfrak{Q}_{12} \star \mathfrak{Q}_{22}, \ldots, \mathfrak{Q}_{1n} \star \mathfrak{Q}_{2n})$,

is defined by the following:

(i) $\mathbb{T}_{\mathfrak{Q}1\star\mathfrak{Q}2}(rs) = (\mathbb{T}_{\mathfrak{Q}1} \star \mathbb{T}_{\mathfrak{Q}2})(rs) = \mathbb{T}_{\mathfrak{Q}1}(r) \wedge \mathbb{T}_{\mathfrak{Q}2}(s)$.

$\mathbb{C}_{\mathfrak{Q}1\star\mathfrak{Q}2}(rs) = (\mathbb{C}_{\mathfrak{Q}1} \star \mathbb{C}_{\mathfrak{Q}2})(rs) = \mathbb{C}_{\mathfrak{Q}1}(r) \wedge \mathbb{C}_{\mathfrak{Q}2}(s)$.

$\mathbb{U}_{\mathfrak{Q}1\star\mathfrak{Q}2}(rs) = (\mathbb{U}_{\mathfrak{Q}1} \star \mathbb{U}_{\mathfrak{Q}2})(rs) = \mathbb{U}_{\mathfrak{Q}1}(r) \vee \mathbb{U}_{\mathfrak{Q}2}(s)$.

$\mathbb{F}_{\mathfrak{Q}1\star\mathfrak{Q}2}(rs) = (\mathbb{F}_{\mathfrak{Q}1} \star \mathbb{F}_{\mathfrak{Q}2})(rs) = \mathbb{F}_{\mathfrak{Q}1}(r) \vee \mathbb{F}_{\mathfrak{Q}2}(s)$.

$\forall rs \in S_1 \star S_2$.

(ii) $\mathbb{T}_{(\mathfrak{Q}_{1i}\star\mathfrak{Q}_{2i})}(r_1s_1)(r_2s_2) = (\mathbb{T}_{\mathfrak{Q}_{1i}} \star \mathbb{T}_{\mathfrak{Q}_{2i}})(r_1s_1)(r_2s_2) = \mathbb{T}_{Q1i}(r_1r_2) \wedge \mathbb{T}_{Q2i}(s_1s_2)$

$\mathbb{C}_{(\mathfrak{Q}_{1i}\star\mathfrak{Q}_{2i})}(r_1s_1)(r_2s_2) = (\mathbb{C}_{\mathfrak{Q}_{1i}} \star \mathbb{C}_{\mathfrak{Q}_{2i}})(r_1s_1)(r_2s_2) = \mathbb{C}_{Q1i}(r_1r_2) \wedge \mathbb{C}_{Q2i}(s_1s_2)$

$\mathbb{U}_{(\mathfrak{Q}_{1i}\star\mathfrak{Q}_{2i})}(r_1s_1)(r_2s_2) = (\mathbb{U}_{\mathfrak{Q}_{1i}} \star \mathbb{U}_{\mathfrak{Q}_{2i}})(r_1s_1)(r_2s_2) = \mathbb{U}_{Q1i}(r_1r_2) \vee \mathbb{U}_{Q2i}(s_1s_2)$

$\mathbb{F}_{(\mathfrak{Q}_{1i}\star\mathfrak{Q}_{2i})}(r_1s_1)(r_2s_2) = (\mathbb{F}_{\mathfrak{Q}_{1i}} \star \mathbb{F}_{\mathfrak{Q}_{2i}})(r_1s_1)(r_2s_2) = \mathbb{F}_{Q1i}(r_1r_2) \vee \mathbb{F}_{Q2i}(s_1s_2)$

$\forall r_1r_2 \in S_{1i}, s_1s_2 \in S_{2i}$,

Theorem 3.43 *The cross product* $\mathfrak{G}_{n1} \star \mathfrak{G}_{n2} = (\mathfrak{Q}_1 \star \mathfrak{Q}_2, \mathfrak{Q}_{11} \star \mathfrak{Q}_{21}, \mathfrak{Q}_{12} \star \mathfrak{Q}_{22}, \ldots, \mathfrak{Q}_{1n} \star \mathfrak{Q}_{2n})$ *of two QNGS of the GS* $\mathfrak{G}_1$ *and* $\mathfrak{G}_2$ *is a QNGs of* $\mathfrak{G}_1 \star \mathfrak{G}_2$.

Proof. For all $r_1s_1, r_2s_2 \in S_1 \star S_2$

$\mathbb{T}_{(\mathfrak{Q}_{1i}\star\mathfrak{Q}_{2i})}(r_1s_1)(r_2s_2) = (\mathbb{T}_{\mathfrak{Q}_{1i}} \star \mathbb{T}_{\mathfrak{Q}_{2i}})(r_1s_1)(r_2s_2) = \mathbb{T}_{Q1i}(r_1r_2) \wedge \mathbb{T}_{Q2i}(s_1s_2)$
$\leq [\mathbb{T}_{Q1}(r_1) \wedge \mathbb{T}_{Q1}(r_2)] \wedge [\mathbb{T}_{Q2}(s_1) \wedge \mathbb{T}_{Q2}(s_2)]$
$= [\mathbb{T}_{Q1}(r_1) \wedge \mathbb{T}_{Q2}(s_1)] \wedge [\mathbb{T}_{Q1}(r_2) \wedge \mathbb{T}_{Q2}(s_2)]$
$= \mathbb{T}_{(\mathfrak{Q}_1\star\mathfrak{Q}_2)}(r_1s_1) \wedge \mathbb{T}_{(\mathfrak{Q}_1\star\mathfrak{Q}_2)}(r_2s_2)$,

$\mathbb{C}_{(\mathfrak{Q}_{1i}\star\mathfrak{Q}_{2i})}(r_1s_1)(r_2s_2) = (\mathbb{C}_{\mathfrak{Q}_{1i}} \star \mathbb{C}_{\mathfrak{Q}_{2i}})(r_1s_1)(r_2s_2) = \mathbb{C}_{Q1i}(r_1r_2) \wedge \mathbb{C}_{Q2i}(s_1s_2)$
$\leq [\mathbb{C}_{Q1}(r_1) \wedge \mathbb{C}_{Q1}(r_2)] \wedge [\mathbb{C}_{Q2}(s_1) \wedge \mathbb{C}_{Q2}(s_2)]$
$= [\mathbb{C}_{Q1}(r_1) \wedge \mathbb{C}_{Q2}(s_1)] \wedge [\mathbb{C}_{Q1}(r_2) \wedge \mathbb{C}_{Q2}(s_2)]$
$= \mathbb{C}_{(\mathfrak{Q}_1\star\mathfrak{Q}_2)}(r_1s_1) \wedge \mathbb{C}_{(\mathfrak{Q}_1\star\mathfrak{Q}_2)}(r_2s_2)$,

$\mathbb{U}_{(\mathfrak{Q}_{1i}\star\mathfrak{Q}_{2i})}(r_1s_1)(r_2s_2) = (\mathbb{U}_{\mathfrak{Q}_{1i}} \star \mathbb{U}_{\mathfrak{Q}_{2i}})(r_1s_1)(r_2s_2) = \mathbb{U}_{Q1i}(r_1r_2) \vee \mathbb{U}_{Q2i}(s_1s_2)$
$\leq [\mathbb{U}_{Q1}(r_1) \vee \mathbb{U}_{Q1}(r_2)] \vee [\mathbb{U}_{Q2}(s_1) \vee \mathbb{U}_{Q2}(s_2)]$
$= [\mathbb{U}_{Q1}(r_1) \vee \mathbb{U}_{Q2}(s_1)] \vee [\mathbb{U}_{Q1}(r_2) \vee \mathbb{U}_{Q2}(s_2)]$
$= \mathbb{U}_{(\mathfrak{Q}_1\star\mathfrak{Q}_2)}(r_1s_1) \vee \mathbb{U}_{(\mathfrak{Q}_1\star\mathfrak{Q}_2)}(r_2s_2)$,

$\mathbb{F}_{(\mathfrak{Q}_{1i}\star\mathfrak{Q}_{2i})}(r_1s_1)(r_2s_2) = (\mathbb{F}_{\mathfrak{Q}_{1i}} \star \mathbb{F}_{\mathfrak{Q}_{2i}})(r_1s_1)(r_2s_2) = \mathbb{F}_{Q1i}(r_1r_2) \vee \mathbb{F}_{Q2i}(s_1s_2)$
$\leq [\mathbb{F}_{Q1}(r_1) \vee \mathbb{F}_{Q1}(r_2)] \vee [\mathbb{F}_{Q2}(s_1) \vee \mathbb{F}_{Q2}(s_2)]$
$= [\mathbb{F}_{Q1}(r_1) \vee \mathbb{F}_{Q2}(s_1)] \vee [\mathbb{F}_{Q1}(r_2) \vee \mathbb{F}_{Q2}(s_2)]$
$= \mathbb{F}_{(\mathfrak{Q}_1\star\mathfrak{Q}_2)}(r_1s_1) \vee \mathbb{F}_{(\mathfrak{Q}_1\star\mathfrak{Q}_2)}(r_2s_2)$,

for $i \in 1,2,\ldots,n$. This gives required result.

Definition 3.44 *Let* $\mathfrak{G}_{n1} = (\mathfrak{Q}_1, \mathfrak{Q}_{11}, \mathfrak{Q}_{12}, \ldots . \mathfrak{Q}_{1n)}$ *and* $\mathfrak{G}_{n2} = (\mathfrak{Q}_2, \mathfrak{Q}_{21}, \mathfrak{Q}_{22}, \ldots . \mathfrak{Q}_{2n)}$ *be QNGS. The lexicographic product of* $\mathfrak{G}_1$ *and* $\mathfrak{G}_2$ *denoted by*

$\mathfrak{G}_{n1} \bullet \mathfrak{G}_{n2} = (\mathfrak{Q}_1 \bullet \mathfrak{Q}_2, \mathfrak{Q}_{11} \bullet \mathfrak{Q}_{21}, \mathfrak{Q}_{12} \bullet \mathfrak{Q}_{22}, \ldots, \mathfrak{Q}_{1n} \bullet \mathfrak{Q}_{2n})$,
is defined by the following:

(i) $\mathbb{T}_{\mathfrak{Q}1\bullet\mathfrak{Q}2}(rs) = (\mathbb{T}_{\mathfrak{Q}1} \bullet \mathbb{T}_{\mathfrak{Q}2})(rs) = \mathbb{T}_{\mathfrak{Q}1}(r) \wedge \mathbb{T}_{\mathfrak{Q}2}(s)$.
$\mathbb{C}_{\mathfrak{Q}1\bullet\mathfrak{Q}2}(rs) = (\mathbb{C}_{\mathfrak{Q}1} \bullet \mathbb{C}_{\mathfrak{Q}2})(rs) = \mathbb{C}_{\mathfrak{Q}1}(r) \wedge \mathbb{C}_{\mathfrak{Q}2}(s)$.
$\mathbb{U}_{\mathfrak{Q}1\bullet\mathfrak{Q}2}(rs) = (\mathbb{U}_{\mathfrak{Q}1} \bullet \mathbb{U}_{\mathfrak{Q}2})(rs) = \mathbb{U}_{\mathfrak{Q}1}(r) \vee \mathbb{U}_{\mathfrak{Q}2}(s)$.
$\mathbb{F}_{\mathfrak{Q}1\bullet\mathfrak{Q}2}(rs) = (\mathbb{F}_{\mathfrak{Q}1} \bullet \mathbb{F}_{\mathfrak{Q}2})(rs) = \mathbb{F}_{\mathfrak{Q}1}(r) \vee \mathbb{F}_{\mathfrak{Q}2}(s)$.
$\forall rs \in S_1 \bullet S_2$.

(ii) $\mathbb{T}_{(\mathfrak{Q}_{1i}\bullet\mathfrak{Q}_{2i})}(rs_1)(rs_2) = (\mathbb{T}_{\mathfrak{Q}_{1i}} \bullet \mathbb{T}_{\mathfrak{Q}_{2i}})(rs_1)(rs_2) = \mathbb{T}_{Q1}(r) \wedge \mathbb{T}_{Q2i}(s_1s_2)$
$\mathbb{C}_{(\mathfrak{Q}_{1i}\bullet\mathfrak{Q}_{2i})}(rs_1)(rs_2) = (\mathbb{C}_{\mathfrak{Q}_{1i}} \bullet \mathbb{C}_{\mathfrak{Q}_{2i}})(rs_1)(rs_2) = \mathbb{C}_{Q1}(r) \wedge \mathbb{C}_{Q2i}(s_1s_2)$
$\mathbb{U}_{(\mathfrak{Q}_{1i}\bullet\mathfrak{Q}_{2i})}(rs_1)(rs_2) = (\mathbb{U}_{\mathfrak{Q}_{1i}} \bullet \mathbb{U}_{\mathfrak{Q}_{2i}})(rs_1)(rs_2) = \mathbb{U}_{Q1}(r) \vee \mathbb{U}_{Q2i}(s_1s_2)$
$\mathbb{F}_{(\mathfrak{Q}_{1i}\bullet\mathfrak{Q}_{2i})}(rs_1)(rs_2) = (\mathbb{F}_{\mathfrak{Q}_{1i}} \bullet \mathbb{F}_{\mathfrak{Q}_{2i}})(rs_1)(rs_2) = \mathbb{F}_{Q1}(r) \vee \mathbb{F}_{Q2i}(s_1s_2)$
$\forall r \in S_1, s_1s_2 \in S_{2i}$,

(iii) $\mathbb{T}_{(\mathfrak{Q}_{1i}\bullet\mathfrak{Q}_{2i})}(r_1s_1)(r_2s_2) = (\mathbb{T}_{\mathfrak{Q}_{1i}} \bullet \mathbb{T}_{\mathfrak{Q}_{2i}})(r_1s_1)(r_2s_2) = \mathbb{T}_{Q1i}(r_1r_2) \wedge \mathbb{T}_{Q2i}(s_1s_2)$
$\mathbb{C}_{(\mathfrak{Q}_{1i}\bullet\mathfrak{Q}_{2i})}(r_1s_1)(r_2s_2) = (\mathbb{C}_{\mathfrak{Q}_{1i}} \bullet \mathbb{C}_{\mathfrak{Q}_{2i}})(r_1s_1)(r_2s_2) = \mathbb{C}_{Q1i}(r_1r_2) \wedge \mathbb{C}_{Q2i}(s_1s_2)$
$\mathbb{U}_{(\mathfrak{Q}_{1i}\bullet\mathfrak{Q}_{2i})}(r_1s_1)(r_2s_2) = (\mathbb{U}_{\mathfrak{Q}_{1i}} \bullet \mathbb{U}_{\mathfrak{Q}_{2i}})(r_1s_1)(r_2s_2) = \mathbb{U}_{Q1i}(r_1r_2) \vee \mathbb{U}_{Q2i}(s_1s_2)$
$\mathbb{F}_{(\mathfrak{Q}_{1i}\bullet\mathfrak{Q}_{2i})}(r_1s_1)(r_2s_2) = (\mathbb{F}_{\mathfrak{Q}_{1i}} \bullet \mathbb{F}_{\mathfrak{Q}_{2i}})(r_1s_1)(r_2s_2) = \mathbb{F}_{Q1i}(r_1r_2) \vee \mathbb{F}_{Q2i}(s_1s_2)$
$\forall r_1r_2 \in S_{1i}, s_1s_2 \in S_{2i}$,

Theorem 3.45 *The lexicographic product* $\mathfrak{G}_{n1} \bullet \mathfrak{G}_{n2} = (\mathfrak{Q}_1 \bullet \mathfrak{Q}_2, \mathfrak{Q}_{11} \bullet \mathfrak{Q}_{21}, \mathfrak{Q}_{12} \star \mathfrak{Q}_{22}, \ldots, \mathfrak{Q}_{1n} \bullet \mathfrak{Q}_{2n})$ *of two QNGS of the GS* $\mathfrak{G}_1$ *and* $\mathfrak{G}_2$ *is a QNGs of* $\mathfrak{G}_1 \bullet \mathfrak{G}_2$.

Proof. According to the definition of lexicographic product there are two cases:
Case 1: when $r \in S_1, s_1s_2 \in S_{2i}$

$$\begin{aligned}
&\mathbb{T}_{(\mathfrak{Q}_{1i}\bullet\mathfrak{Q}_{2i})}(rs_1)(rs_2) = \mathbb{T}_{Q1}(r) \wedge \mathbb{T}_{Q2i}(s_1s_2)\\
&\leq \mathbb{T}_{Q1}(r) \wedge [\mathbb{T}_{Q2}(s_1) \wedge \mathbb{T}_{Q2}(s_2)]\\
&= [\mathbb{T}_{Q1}(r) \wedge \mathbb{T}_{Q2}(s_1)] \wedge [\mathbb{T}_{Q1}(r) \wedge \mathbb{T}_{Q2}(s_2)]\\
&= \mathbb{T}_{(Q_1\bullet Q_2)}(rs_1) \wedge \mathbb{T}_{(Q_1\bullet Q_2)}(rs_2).
\end{aligned}$$

$$\begin{aligned}
&\mathbb{C}_{(\mathfrak{Q}_{1i}\bullet\mathfrak{Q}_{2i})}(rs_1)(rs_2) = \mathbb{C}_{Q1}(r) \wedge \mathbb{C}_{Q2i}(s_1s_2)\\
&\leq \mathbb{C}_{Q1}(r) \wedge [\mathbb{C}_{Q2}(s_1) \wedge \mathbb{C}_{Q2}(s_2)]\\
&= [\mathbb{C}_{Q1}(r) \wedge \mathbb{C}_{Q2}(s_1)] \wedge [\mathbb{C}_{Q1}(r) \wedge \mathbb{C}_{Q2}(s_2)]\\
&= \mathbb{C}_{(Q_1\bullet Q_2)}(rs_1) \wedge \mathbb{C}_{(Q_1\bullet Q_2)}(rs_2).
\end{aligned}$$

$$\begin{aligned}
&\mathbb{U}_{(\mathfrak{Q}_{1i}\bullet\mathfrak{Q}_{2i})}(rs_1)(rs_2) = \mathbb{U}_{Q1}(r) \vee \mathbb{U}_{Q2i}(s_1s_2)\\
&\leq \mathbb{U}_{Q1}(r) \vee [\mathbb{U}_{Q2}(s_1) \vee \mathbb{U}_{Q2}(s_2)]\\
&= [\mathbb{U}_{Q1}(r) \vee \mathbb{U}_{Q2}(s_1)] \vee [\mathbb{U}_{Q1}(r) \vee \mathbb{U}_{Q2}(s_2)]\\
&= \mathbb{U}_{(Q_1\bullet Q_2)}(rs_1) \vee \mathbb{U}_{(Q_1\bullet Q_2)}(rs_2).
\end{aligned}$$

$$\begin{aligned}
&\mathbb{F}_{(\mathfrak{Q}_{1i}\bullet\mathfrak{Q}_{2i})}(rs_1)(rs_2) = \mathbb{F}_{Q1}(r) \vee \mathbb{F}_{Q2i}(s_1s_2)\\
&\leq \mathbb{F}_{Q1}(r) \vee [\mathbb{F}_{Q2}(s_1) \vee \mathbb{F}_{Q2}(s_2)]\\
&= [\mathbb{F}_{Q1}(r) \vee \mathbb{F}_{Q2}(s_1)] \vee [\mathbb{F}_{Q1}(r) \vee \mathbb{F}_{Q2}(s_2)]\\
&= \mathbb{F}_{(Q_1\bullet Q_2)}(rs_1) \vee \mathbb{F}_{(Q_1\bullet Q_2)}(rs_2).
\end{aligned}$$

for $rs_1, rs_2 \in S_1 \bullet S_2$.
Case 2: For all $r_1s_1, r_2s_2 \in S_1 \bullet S_2$

$$\begin{aligned}
&\mathbb{T}_{(\mathfrak{Q}_{1i}\bullet\mathfrak{Q}_{2i})}(r_1s_1)(r_2s_2) = (\mathbb{T}_{\mathfrak{Q}_{1i}} \bullet \mathbb{T}_{\mathfrak{Q}_{2i}})(r_1s_1)(r_2s_2) = \mathbb{T}_{Q1i}(r_1r_2) \wedge \mathbb{T}_{Q2i}(s_1s_2)\\
&\leq [\mathbb{T}_{Q1}(r_1) \wedge \mathbb{T}_{Q1}(r_2)] \wedge [\mathbb{T}_{Q2}(s_1) \wedge \mathbb{T}_{Q2}(s_2)]
\end{aligned}$$

$= [\mathbb{T}_{Q1}(r_1) \wedge \mathbb{T}_{Q2}(s_1)] \wedge [\mathbb{T}_{Q1}(r_2) \wedge \mathbb{T}_{Q2}(s_2)]$
$= \mathbb{T}_{(\mathfrak{Q}_1 \bullet \mathfrak{Q}_2)}(r_1 s_1) \wedge \mathbb{T}_{(\mathfrak{Q}_1 \bullet \mathfrak{Q}_2)}(r_2 s_2),$

$\mathbb{C}_{(\mathfrak{Q}_{1i} \bullet \mathfrak{Q}_{2i})}(r_1 s_1)(r_2 s_2) = (\mathbb{C}_{\mathfrak{Q}_{1i}} \bullet \mathbb{C}_{\mathfrak{Q}_{2i}})(r_1 s_1)(r_2 s_2) = \mathbb{C}_{Q1i}(r_1 r_2) \wedge \mathbb{C}_{Q2i}(s_1 s_2)$
$\leq [\mathbb{C}_{Q1}(r_1) \wedge \mathbb{C}_{Q1}(r_2)] \wedge [\mathbb{C}_{Q2}(s_1) \wedge \mathbb{C}_{Q2}(s_2)]$
$= [\mathbb{C}_{Q1}(r_1) \wedge \mathbb{C}_{Q2}(s_1)] \wedge [\mathbb{C}_{Q1}(r_2) \wedge \mathbb{C}_{Q2}(s_2)]$
$= \mathbb{C}_{(\mathfrak{Q}_1 \bullet \mathfrak{Q}_2)}(r_1 s_1) \wedge \mathbb{C}_{(\mathfrak{Q}_1 \bullet \mathfrak{Q}_2)}(r_2 s_2),$

$\mathbb{U}_{(\mathfrak{Q}_{1i} \bullet \mathfrak{Q}_{2i})}(r_1 s_1)(r_2 s_2) = (\mathbb{U}_{\mathfrak{Q}_{1i}} \bullet \mathbb{U}_{\mathfrak{Q}_{2i}})(r_1 s_1)(r_2 s_2) = \mathbb{U}_{Q1i}(r_1 r_2) \vee \mathbb{U}_{Q2i}(s_1 s_2)$
$\leq [\mathbb{U}_{Q1}(r_1) \vee \mathbb{U}_{Q1}(r_2)] \vee [\mathbb{U}_{Q2}(s_1) \vee \mathbb{U}_{Q2}(s_2)]$
$= [\mathbb{U}_{Q1}(r_1) \vee \mathbb{U}_{Q2}(s_1)] \vee [\mathbb{U}_{Q1}(r_2) \vee \mathbb{U}_{Q2}(s_2)]$
$= \mathbb{U}_{(\mathfrak{Q}_1 \bullet \mathfrak{Q}_2)}(r_1 s_1) \vee \mathbb{U}_{(\mathfrak{Q}_1 \bullet \mathfrak{Q}_2)}(r_2 s_2),$

$\mathbb{F}_{(\mathfrak{Q}_{1i} \bullet \mathfrak{Q}_{2i})}(r_1 s_1)(r_2 s_2) = (\mathbb{F}_{\mathfrak{Q}_{1i}} \bullet \mathbb{F}_{\mathfrak{Q}_{2i}})(r_1 s_1)(r_2 s_2) = \mathbb{F}_{Q1i}(r_1 r_2) \vee \mathbb{F}_{Q2i}(s_1 s_2)$
$\leq [\mathbb{F}_{Q1}(r_1) \vee \mathbb{F}_{Q1}(r_2)] \vee [\mathbb{F}_{Q2}(s_1) \vee \mathbb{F}_{Q2}(s_2)]$
$= [\mathbb{F}_{Q1}(r_1) \vee \mathbb{F}_{Q2}(s_1)] \vee [\mathbb{F}_{Q1}(r_2) \vee \mathbb{F}_{Q2}(s_2)]$
$= \mathbb{F}_{(\mathfrak{Q}_1 \bullet \mathfrak{Q}_2)}(r_1 s_1) \vee \mathbb{F}_{(\mathfrak{Q}_1 \bullet \mathfrak{Q}_2)}(r_2 s_2),$

for $i \in 1,2,\ldots,n$. This gives required result.

Definition 3.46 *Let* $\mathfrak{G}_{n1} = (\mathfrak{Q}_1, \mathfrak{Q}_{11}, \mathfrak{Q}_{12}, \ldots. \mathfrak{Q}_{1n})$ *and* $\mathfrak{G}_{n2} = (\mathfrak{Q}_2, \mathfrak{Q}_{21}, \mathfrak{Q}_{22}, \ldots. \mathfrak{Q}_{2n})$ *be QNGS. The strong product of* $\mathfrak{G}_1$ *and* $\mathfrak{G}_2$ *denoted by*

$\mathfrak{G}_{n1} \boxtimes \mathfrak{G}_{n2} = (\mathfrak{Q}_1 \boxtimes \mathfrak{Q}_2, \mathfrak{Q}_{11} \boxtimes \mathfrak{Q}_{21}, \mathfrak{Q}_{12} \boxtimes \mathfrak{Q}_{22}, \ldots, \mathfrak{Q}_{1n} \boxtimes \mathfrak{Q}_{2n}),$

is defined by the following:

(i) $\mathbb{T}_{\mathfrak{Q}1 \boxtimes \mathfrak{Q}2}(rs) = (\mathbb{T}_{\mathfrak{Q}1} \boxtimes \mathbb{T}_{\mathfrak{Q}2})(rs) = \mathbb{T}_{\mathfrak{Q}1}(r) \wedge \mathbb{T}_{\mathfrak{Q}2}(s).$
$\mathbb{C}_{\mathfrak{Q}1 \boxtimes \mathfrak{Q}2}(rs) = (\mathbb{C}_{\mathfrak{Q}1} \boxtimes \mathbb{C}_{\mathfrak{Q}2})(rs) = \mathbb{C}_{\mathfrak{Q}1}(r) \wedge \mathbb{C}_{\mathfrak{Q}2}(s).$
$\mathbb{U}_{\mathfrak{Q}1 \boxtimes \mathfrak{Q}2}(rs) = (\mathbb{U}_{\mathfrak{Q}1} \boxtimes \mathbb{U}_{\mathfrak{Q}2})(rs) = \mathbb{U}_{\mathfrak{Q}1}(r) \vee \mathbb{U}_{\mathfrak{Q}2}(s).$
$\mathbb{F}_{\mathfrak{Q}1 \boxtimes \mathfrak{Q}2}(rs) = (\mathbb{F}_{\mathfrak{Q}1} \boxtimes \mathbb{F}_{\mathfrak{Q}2})(rs) = \mathbb{F}_{\mathfrak{Q}1}(r) \vee \mathbb{F}_{\mathfrak{Q}2}(s).$
$\forall rs \in S_1 \boxtimes S_2.$

(ii) $\mathbb{T}_{(\mathfrak{Q}_{1i} \boxtimes \mathfrak{Q}_{2i})}(rs_1)(rs_2) = (\mathbb{T}_{\mathfrak{Q}_{1i}} \boxtimes \mathbb{T}_{\mathfrak{Q}_{2i}})(rs_1)(rs_2) = \mathbb{T}_{Q1}(r) \wedge \mathbb{T}_{Q2i}(s_1 s_2)$
$\mathbb{C}_{(\mathfrak{Q}_{1i} \boxtimes \mathfrak{Q}_{2i})}(rs_1)(rs_2) = (\mathbb{C}_{\mathfrak{Q}_{1i}} \boxtimes \mathbb{C}_{\mathfrak{Q}_{2i}})(rs_1)(rs_2) = \mathbb{C}_{Q1}(r) \wedge \mathbb{C}_{Q2i}(s_1 s_2)$
$\mathbb{U}_{(\mathfrak{Q}_{1i} \boxtimes \mathfrak{Q}_{2i})}(rs_1)(rs_2) = (\mathbb{U}_{\mathfrak{Q}_{1i}} \boxtimes \mathbb{U}_{\mathfrak{Q}_{2i}})(rs_1)(rs_2) = \mathbb{U}_{Q1}(r) \vee \mathbb{U}_{Q2i}(s_1 s_2)$
$\mathbb{F}_{(\mathfrak{Q}_{1i} \boxtimes \mathfrak{Q}_{2i})}(rs_1)(rs_2) = (\mathbb{F}_{\mathfrak{Q}_{1i}} \boxtimes \mathbb{F}_{\mathfrak{Q}_{2i}})(rs_1)(rs_2) = \mathbb{F}_{Q1}(r) \vee \mathbb{F}_{Q2i}(s_1 s_2)$
$\forall r \in S_1, s_1 s_2 \in S_{2i},$

(iii) $\mathbb{T}_{(\mathfrak{Q}_{1i} \boxtimes \mathfrak{Q}_{2i})}(r_1 s)(r_2 s) = (\mathbb{T}_{\mathfrak{Q}_{1i}} \boxtimes \mathbb{T}_{\mathfrak{Q}_{2i}})(r_1 s)(r_2 s) = \mathbb{T}_{Q2}(s) \wedge \mathbb{T}_{Q1i}(r_1 r_2)$
$\mathbb{C}_{(\mathfrak{Q}_{1i} \boxtimes \mathfrak{Q}_{2i})}(r_1 s)(r_2 s) = (\mathbb{C}_{\mathfrak{Q}_{1i}} \boxtimes \mathbb{C}_{\mathfrak{Q}_{2i}})(r_1 s)(r_2 s) = \mathbb{C}_{Q2}(s) \wedge \mathbb{C}_{Q1i}(r_1 r_2)$
$\mathbb{U}_{(\mathfrak{Q}_{1i} \boxtimes \mathfrak{Q}_{2i})}(r_1 s)(r_2 s) = (\mathbb{U}_{\mathfrak{Q}_{1i}} \boxtimes \mathbb{U}_{\mathfrak{Q}_{2i}})(r_1 s)(r_2 s) = \mathbb{U}_{Q2}(s) \vee \mathbb{U}_{Q1i}(r_1 r_2)$
$\mathbb{F}_{(\mathfrak{Q}_{1i} \boxtimes \mathfrak{Q}_{2i})}(r_1 s)(r_2 s) = (\mathbb{F}_{\mathfrak{Q}_{1i}} \boxtimes \mathbb{F}_{\mathfrak{Q}_{2i}})(r_1 s)(r_2 s) = \mathbb{F}_{Q2}(s) \vee \mathbb{F}_{Q1i}(r_1 r_2)$
$\forall s \in S_2, r_1 r_2 \in S_{1i}.$

(iv) $\mathbb{T}_{(\mathfrak{Q}_{1i} \boxtimes \mathfrak{Q}_{2i})}(r_1 s_1)(r_2 s_2) = (\mathbb{T}_{\mathfrak{Q}_{1i}} \boxtimes \mathbb{T}_{\mathfrak{Q}_{2i}})(r_1 s_1)(r_2 s_2) = \mathbb{T}_{Q1i}(r_1 r_2) \wedge \mathbb{T}_{Q2i}(s_1 s_2)$
$\mathbb{C}_{(\mathfrak{Q}_{1i} \boxtimes \mathfrak{Q}_{2i})}(r_1 s_1)(r_2 s_2) = (\mathbb{C}_{\mathfrak{Q}_{1i}} \boxtimes \mathbb{C}_{\mathfrak{Q}_{2i}})(r_1 s_1)(r_2 s_2) = \mathbb{C}_{Q1i}(r_1 r_2) \wedge \mathbb{C}_{Q2i}(s_1 s_2)$
$\mathbb{U}_{(\mathfrak{Q}_{1i} \boxtimes \mathfrak{Q}_{2i})}(r_1 s_1)(r_2 s_2) = (\mathbb{U}_{\mathfrak{Q}_{1i}} \boxtimes \mathbb{U}_{\mathfrak{Q}_{2i}})(r_1 s_1)(r_2 s_2) = \mathbb{U}_{Q1i}(r_1 r_2) \vee \mathbb{U}_{Q2i}(s_1 s_2)$
$\mathbb{F}_{(\mathfrak{Q}_{1i} \boxtimes \mathfrak{Q}_{2i})}(r_1 s_1)(r_2 s_2) = (\mathbb{F}_{\mathfrak{Q}_{1i}} \boxtimes \mathbb{F}_{\mathfrak{Q}_{2i}})(r_1 s_1)(r_2 s_2) = \mathbb{F}_{Q1i}(r_1 r_2) \vee \mathbb{F}_{Q2i}(s_1 s_2)$
$\forall r_1 r_2 \in S_{1i}, s_1 s_2 \in S_{2i},$

Theorem 3.47 *The strong product* $\mathfrak{G}_{n1} \boxtimes \mathfrak{G}_{n2} = (\mathfrak{Q}_1 \boxtimes \mathfrak{Q}_2, \mathfrak{Q}_{11} \boxtimes \mathfrak{Q}_{21}, \mathfrak{Q}_{12} \boxtimes \mathfrak{Q}_{22}, \ldots, \mathfrak{Q}_{1n} \boxtimes \mathfrak{Q}_{2n})$ *of two QNGS of the GS* $\mathfrak{G}_1$ *and* $\mathfrak{G}_2$ *is a QNGs of* $\mathfrak{G}_1 \boxtimes \mathfrak{G}_2$.

Proof. According to the definition of strong product there are three cases:
Case1: when $r \in S_1, s_1s_2 \in S_{2i}$

$$\begin{aligned}
&\mathbb{T}_{(\mathfrak{Q}_{1i}\boxtimes\mathfrak{Q}_{2i})}(rs_1)(rs_2) = \mathbb{T}_{Q1}(r) \wedge \mathbb{T}_{Q2i}(s_1s_2)\\
&\leq \mathbb{T}_{Q1}(r) \wedge [\mathbb{T}_{Q2}(s_1) \wedge \mathbb{T}_{Q2}(s_2)]\\
&= [\mathbb{T}_{Q1}(r) \wedge \mathbb{T}_{Q2}(s_1)] \wedge [\mathbb{T}_{Q1}(r) \wedge \mathbb{T}_{Q2}(s_2)]\\
&= \mathbb{T}_{(Q_1\boxtimes Q_2)}(rs_1) \wedge \mathbb{T}_{(Q_1\boxtimes Q_2)}(rs_2).
\end{aligned}$$

$$\begin{aligned}
&\mathbb{C}_{(\mathfrak{Q}_{1i}\boxtimes\mathfrak{Q}_{2i})}(rs_1)(rs_2) = \mathbb{C}_{Q1}(r) \wedge \mathbb{C}_{Q2i}(s_1s_2)\\
&\leq \mathbb{C}_{Q1}(r) \wedge [\mathbb{C}_{Q2}(s_1) \wedge \mathbb{C}_{Q2}(s_2)]\\
&= [\mathbb{C}_{Q1}(r) \wedge \mathbb{C}_{Q2}(s_1)] \wedge [\mathbb{C}_{Q1}(r) \wedge \mathbb{C}_{Q2}(s_2)]\\
&= \mathbb{C}_{(Q_1\boxtimes Q_2)}(rs_1) \wedge \mathbb{C}_{(Q_1\boxtimes Q_2)}(rs_2).
\end{aligned}$$

$$\begin{aligned}
&\mathbb{U}_{(\mathfrak{Q}_{1i}\boxtimes\mathfrak{Q}_{2i})}(rs_1)(rs_2) = \mathbb{U}_{Q1}(r) \vee \mathbb{U}_{Q2i}(s_1s_2)\\
&\leq \mathbb{U}_{Q1}(r) \vee [\mathbb{F}_{Q2}(s_1) \vee \mathbb{U}_{Q2}(s_2)]\\
&= [\mathbb{U}_{Q1}(r) \vee \mathbb{U}_{Q2}(s_1)] \vee [\mathbb{U}_{Q1}(r) \vee \mathbb{U}_{Q2}(s_2)]\\
&= \mathbb{U}_{(Q_1\boxtimes Q_2)}(rs_1) \vee \mathbb{U}_{(Q_1\boxtimes Q_2)}(rs_2).
\end{aligned}$$

$$\begin{aligned}
&\mathbb{F}_{(\mathfrak{Q}_{1i}\boxtimes\mathfrak{Q}_{2i})}(rs_1)(rs_2) = \mathbb{F}_{Q1}(r) \vee \mathbb{F}_{Q2i}(s_1s_2)\\
&\leq \mathbb{F}_{Q1}(r) \vee [\mathbb{F}_{Q2}(s_1) \vee \mathbb{F}_{Q2}(s_2)]\\
&= [\mathbb{F}_{Q1}(r) \vee \mathbb{F}_{Q2}(s_1)] \vee [\mathbb{F}_{Q1}(r) \vee \mathbb{F}_{Q2}(s_2)]\\
&= \mathbb{F}_{(Q_1\boxtimes Q_2)}(rs_1) \vee \mathbb{F}_{(Q_1\boxtimes Q_2)}(rs_2).
\end{aligned}$$

for $rs_1, rs_2 \in S_1 \boxtimes S_2$.
Case 2: when $r \in S_2, s_1s_2 \in S_{1i}$

$$\begin{aligned}
&\mathbb{T}_{(\mathfrak{Q}_{1i}\boxtimes\mathfrak{Q}_{2i})}(s_1r)(s_2r) = \mathbb{T}_{Q2}(r) \wedge \mathbb{T}_{Q1i}(s_1s_2)\\
&\leq \mathbb{T}_{Q2}(r) \wedge [\mathbb{T}_{Q1}(s_1) \wedge \mathbb{T}_{Q2}(s_2)]\\
&= [\mathbb{T}_{Q2}(r) \wedge \mathbb{T}_{Q1}(s_1)] \wedge [\mathbb{T}_{Q2}(r) \wedge \mathbb{T}_{Q1}(s_2)]\\
&= \mathbb{T}_{(Q_1\boxtimes Q_2)}(s_1r) \wedge \mathbb{T}_{(Q_1\boxtimes Q_2)}(s_2r).
\end{aligned}$$

$$\begin{aligned}
&\mathbb{C}_{(\mathfrak{Q}_{1i}\boxtimes\mathfrak{Q}_{2i})}(s_1r)(s_2r) = \mathbb{C}_{Q2}(r) \wedge \mathbb{C}_{Q1i}(s_1s_2)\\
&\leq \mathbb{C}_{Q2}(r) \wedge [\mathbb{C}_{Q1}(s_1) \wedge \mathbb{C}_{Q2}(s_2)]\\
&= [\mathbb{C}_{Q2}(r) \wedge \mathbb{C}_{Q1}(s_1)] \wedge [\mathbb{C}_{Q2}(r) \wedge \mathbb{C}_{Q1}(s_2)]\\
&= \mathbb{C}_{(Q_1\boxtimes Q_2)}(s_1r) \wedge \mathbb{C}_{(Q_1\boxtimes Q_2)}(s_2r).
\end{aligned}$$

$$\begin{aligned}
&\mathbb{U}_{(\mathfrak{Q}_{1i}\boxtimes\mathfrak{Q}_{2i})}(s_1r)(s_2r) = \mathbb{U}_{Q2}(r) \vee \mathbb{U}_{Q1i}(s_1s_2)\\
&\leq \mathbb{U}_{Q2}(r) \vee [\mathbb{U}_{Q1}(s_1) \vee \mathbb{U}_{Q2}(s_2)]\\
&= [\mathbb{U}_{Q2}(r) \wedge \mathbb{U}_{Q1}(s_1)] \vee [\mathbb{U}_{Q2}(r) \wedge \mathbb{U}_{Q1}(s_2)]\\
&= \mathbb{U}_{(Q_1\boxtimes Q_2)}(s_1r) \vee \mathbb{U}_{(Q_1\boxtimes Q_2)}(s_2r).
\end{aligned}$$

$$\begin{aligned}
&\mathbb{F}_{(\mathfrak{Q}_{1i}\boxtimes\mathfrak{Q}_{2i})}(s_1r)(s_2r) = \mathbb{F}_{Q2}(r) \vee \mathbb{F}_{Q1i}(s_1s_2)\\
&\leq \mathbb{F}_{Q2}(r) \vee [\mathbb{F}_{Q1}(s_1) \vee \mathbb{F}_{Q2}(s_2)]\\
&= [\mathbb{F}_{Q2}(r) \wedge \mathbb{F}_{Q1}(s_1)] \vee [\mathbb{F}_{Q2}(r) \wedge \mathbb{F}_{Q1}(s_2)]
\end{aligned}$$

$= \mathbb{F}_{(Q_1 \boxtimes Q_2)}(s_1 r) \vee \mathbb{F}_{(Q_1 \boxtimes Q_2)}(s_2 r).$

for $s_1 r, s_2 r \in S_1 \boxtimes S_2$.

Case 3: For all $r_1 r_2 \in S_{1i}, s_1 s_2 \in S_{2i}$

$$\mathbb{T}_{(\mathfrak{Q}_{1i} \boxtimes \mathfrak{Q}_{2i})}(r_1 s_1)(r_2 s_2) = (\mathbb{T}_{\mathfrak{Q}_{1i}} \boxtimes \mathbb{T}_{\mathfrak{Q}_{2i}})(r_1 s_1)(r_2 s_2) = \mathbb{T}_{Q1i}(r_1 r_2) \wedge \mathbb{T}_{Q2i}(s_1 s_2)$$
$$\leq [\mathbb{T}_{Q1}(r_1) \wedge \mathbb{T}_{Q1}(r_2)] \wedge [\mathbb{T}_{Q2}(s_1) \wedge \mathbb{T}_{Q2}(s_2)]$$
$$= [\mathbb{T}_{Q1}(r_1) \wedge \mathbb{T}_{Q2}(s_1)] \wedge [\mathbb{T}_{Q1}(r_2) \wedge \mathbb{T}_{Q2}(s_2)]$$
$$= \mathbb{T}_{(\mathfrak{Q}_1 \boxtimes \mathfrak{Q}_2)}(r_1 s_1) \wedge \mathbb{T}_{(\mathfrak{Q}_{11} \boxtimes \mathfrak{Q}_2)}(r_2 s_2),$$

$$\mathbb{C}_{(\mathfrak{Q}_{1i} \boxtimes \mathfrak{Q}_{2i})}(r_1 s_1)(r_2 s_2) = (\mathbb{C}_{\mathfrak{Q}_{1i}} \boxtimes \mathbb{C}_{\mathfrak{Q}_{2i}})(r_1 s_1)(r_2 s_2) = \mathbb{C}_{Q1i}(r_1 r_2) \wedge \mathbb{C}_{Q2i}(s_1 s_2)$$
$$\leq [\mathbb{C}_{Q1}(r_1) \wedge \mathbb{C}_{Q1}(r_2)] \wedge [\mathbb{C}_{Q2}(s_1) \wedge \mathbb{C}_{Q2}(s_2)]$$
$$= [\mathbb{C}_{Q1}(r_1) \wedge \mathbb{C}_{Q2}(s_1)] \wedge [\mathbb{C}_{Q1}(r_2) \wedge \mathbb{C}_{Q2}(s_2)]$$
$$= \mathbb{C}_{(\mathfrak{Q}_1 \boxtimes \mathfrak{Q}_2)}(r_1 s_1) \wedge \mathbb{C}_{(\mathfrak{Q}_{11} \boxtimes \mathfrak{Q}_2)}(r_2 s_2),$$

$$\mathbb{U}_{(\mathfrak{Q}_{1i} \boxtimes \mathfrak{Q}_{2i})}(r_1 s_1)(r_2 s_2) = (\mathbb{U}_{\mathfrak{Q}_{1i}} \boxtimes \mathbb{U}_{\mathfrak{Q}_{2i}})(r_1 s_1)(r_2 s_2) = \mathbb{U}_{Q1i}(r_1 r_2) \vee \mathbb{U}_{Q2i}(s_1 s_2)$$
$$\leq [\mathbb{U}_{Q1}(r_1) \vee \mathbb{U}_{Q1}(r_2)] \vee [\mathbb{U}_{Q2}(s_1) \vee \mathbb{U}_{Q2}(s_2)]$$
$$= [\mathbb{U}_{Q1}(r_1) \vee \mathbb{U}_{Q2}(s_1)] \vee [\mathbb{U}_{Q1}(r_2) \vee \mathbb{U}_{Q2}(s_2)]$$
$$= \mathbb{U}_{(\mathfrak{Q}_1 \boxtimes \mathfrak{Q}_2)}(r_1 s_1) \vee \mathbb{U}_{(\mathfrak{Q}_{11} \boxtimes \mathfrak{Q}_2)}(r_2 s_2),$$

$$\mathbb{F}_{(\mathfrak{Q}_{1i} \boxtimes \mathfrak{Q}_{2i})}(r_1 s_1)(r_2 s_2) = (\mathbb{F}_{\mathfrak{Q}_{1i}} \boxtimes \mathbb{F}_{\mathfrak{Q}_{2i}})(r_1 s_1)(r_2 s_2) = \mathbb{F}_{Q1i}(r_1 r_2) \vee \mathbb{F}_{Q2i}(s_1 s_2)$$
$$\leq [\mathbb{F}_{Q1}(r_1) \vee \mathbb{F}_{Q1}(r_2)] \vee [\mathbb{F}_{Q2}(s_1) \vee \mathbb{F}_{Q2}(s_2)]$$
$$= [\mathbb{F}_{Q1}(r_1) \vee \mathbb{F}_{Q2}(s_1)] \vee [\mathbb{F}_{Q1}(r_2) \vee \mathbb{F}_{Q2}(s_2)]$$
$$= \mathbb{F}_{(\mathfrak{Q}_1 \boxtimes \mathfrak{Q}_2)}(r_1 s_1) \vee \mathbb{F}_{(\mathfrak{Q}_{11} \boxtimes \mathfrak{Q}_2)}(r_2 s_2),$$

for $i \in 1,2,\ldots,n$. This gives required result.

Definition 3.48 *Let* $\mathfrak{G}_{n1} = (\mathfrak{Q}_1, \mathfrak{Q}_{11}, \mathfrak{Q}_{12}, \ldots . \mathfrak{Q}_{1n})$ *and* $\mathfrak{G}_{n2} = (\mathfrak{Q}_2, \mathfrak{Q}_{21}, \mathfrak{Q}_{22}, \ldots . \mathfrak{Q}_{2n})$ *be QNGS. The composition product of* $\mathfrak{G}_1$ *and* $\mathfrak{G}_2$ *denoted by*

$\mathfrak{G}_{n1} \circ \mathfrak{G}_{n2} = (\mathfrak{Q}_1 \circ \mathfrak{Q}_2, \mathfrak{Q}_{11} \circ \mathfrak{Q}_{21}, \mathfrak{Q}_{12} \circ \mathfrak{Q}_{22}, \ldots, \mathfrak{Q}_{1n} \circ \mathfrak{Q}_{2n})$,

is defined by the following:

(i) $\mathbb{T}_{\mathfrak{Q}1 \circ \mathfrak{Q}2}(rs) = (\mathbb{T}_{\mathfrak{Q}1} \circ \mathbb{T}_{\mathfrak{Q}2})(rs) = \mathbb{T}_{\mathfrak{Q}1}(r) \wedge \mathbb{T}_{\mathfrak{Q}2}(s)$.
$\mathbb{C}_{\mathfrak{Q}1 \circ \mathfrak{Q}2}(rs) = (\mathbb{C}_{\mathfrak{Q}1} \circ \mathbb{C}_{\mathfrak{Q}2})(rs) = \mathbb{C}_{\mathfrak{Q}1}(r) \wedge \mathbb{C}_{\mathfrak{Q}2}(s)$.
$\mathbb{U}_{\mathfrak{Q}1 \circ \mathfrak{Q}2}(rs) = (\mathbb{U}_{\mathfrak{Q}1} \circ \mathbb{U}_{\mathfrak{Q}2})(rs) = \mathbb{U}_{\mathfrak{Q}1}(r) \vee \mathbb{U}_{\mathfrak{Q}2}(s)$.
$\mathbb{F}_{\mathfrak{Q}1 \circ \mathfrak{Q}2}(rs) = (\mathbb{F}_{\mathfrak{Q}1} \circ \mathbb{F}_{\mathfrak{Q}2})(rs) = \mathbb{F}_{\mathfrak{Q}1}(r) \vee \mathbb{F}_{\mathfrak{Q}2}(s)$.
$\forall rs \in S_1 \circ S_2$.

(ii) $\mathbb{T}_{(\mathfrak{Q}_{1i} \circ \mathfrak{Q}_{2i})}(rs_1)(rs_2) = (\mathbb{T}_{\mathfrak{Q}_{1i}} \circ \mathbb{T}_{\mathfrak{Q}_{2i}})(rs_1)(rs_2) = \mathbb{T}_{Q1}(r) \wedge \mathbb{T}_{Q2i}(s_1 s_2)$
$\mathbb{C}_{(\mathfrak{Q}_{1i} \circ \mathfrak{Q}_{2i})}(rs_1)(rs_2) = (\mathbb{C}_{\mathfrak{Q}_{1i}} \circ \mathbb{C}_{\mathfrak{Q}_{2i}})(rs_1)(rs_2) = \mathbb{C}_{Q1}(r) \wedge \mathbb{C}_{Q2i}(s_1 s_2)$
$\mathbb{U}_{(\mathfrak{Q}_{1i} \circ \mathfrak{Q}_{2i})}(rs_1)(rs_2) = (\mathbb{U}_{\mathfrak{Q}_{1i}} \circ \mathbb{U}_{\mathfrak{Q}_{2i}})(rs_1)(rs_2) = \mathbb{U}_{Q1}(r) \vee \mathbb{U}_{Q2i}(s_1 s_2)$
$\mathbb{F}_{(\mathfrak{Q}_{1i} \circ \mathfrak{Q}_{2i})}(rs_1)(rs_2) = (\mathbb{F}_{\mathfrak{Q}_{1i}} \circ \mathbb{F}_{\mathfrak{Q}_{2i}})(rs_1)(rs_2) = \mathbb{F}_{Q1}(r) \vee \mathbb{F}_{Q2i}(s_1 s_2)$
$\forall r \in S_1, s_1 s_2 \in S_{2i}$,

(iii) $\mathbb{T}_{(\mathfrak{Q}_{1i} \circ \mathfrak{Q}_{2i})}(r_1 s)(r_2 s) = (\mathbb{T}_{\mathfrak{Q}_{1i}} \circ \mathbb{T}_{\mathfrak{Q}_{2i}})(r_1 s)(r_2 s) = \mathbb{T}_{Q2}(s) \wedge \mathbb{T}_{Q1i}(r_1 r_2)$
$\mathbb{C}_{(\mathfrak{Q}_{1i} \circ \mathfrak{Q}_{2i})}(r_1 s)(r_2 s) = (\mathbb{C}_{\mathfrak{Q}_{1i}} \circ \mathbb{C}_{\mathfrak{Q}_{2i}})(r_1 s)(r_2 s) = \mathbb{C}_{Q2}(s) \wedge \mathbb{C}_{Q1i}(r_1 r_2)$
$\mathbb{U}_{(\mathfrak{Q}_{1i} \circ \mathfrak{Q}_{2i})}(r_1 s)(r_2 s) = (\mathbb{U}_{\mathfrak{Q}_{1i}} \circ \mathbb{U}_{\mathfrak{Q}_{2i}})(r_1 s)(r_2 s) = \mathbb{U}_{Q2}(s) \vee \mathbb{U}_{Q1i}(r_1 r_2)$
$\mathbb{F}_{(\mathfrak{Q}_{1i} \circ \mathfrak{Q}_{2i})}(r_1 s)(r_2 s) = (\mathbb{F}_{\mathfrak{Q}_{1i}} \circ \mathbb{F}_{\mathfrak{Q}_{2i}})(r_1 s)(r_2 s) = \mathbb{F}_{Q2}(s) \vee \mathbb{F}_{Q1i}(r_1 r_2)$
$\forall s \in S_2, r_1 r_2 \in S_{1i}$.

$(iv)\quad \mathbb{T}_{(\mathfrak{Q}_{1i}\circ\mathfrak{Q}_{2i})}(r_1s_1)(r_2s_2) = (\mathbb{T}_{\mathfrak{Q}_{1i}} \circ \mathbb{T}_{\mathfrak{Q}_{2i}})(r_1s_1)(r_2s_2) = \mathbb{T}_{Q1i}(r_1r_2) \wedge \mathbb{T}_{Q2}(s_1) \wedge \mathbb{T}_{Q2}(s_2)$
$\mathbb{C}_{(\mathfrak{Q}_{1i}\circ\mathfrak{Q}_{2i})}(r_1s_1)(r_2s_2) = (\mathbb{C}_{\mathfrak{Q}_{1i}} \circ \mathbb{C}_{\mathfrak{Q}_{2i}})(r_1s_1)(r_2s_2) = \mathbb{C}_{Q1i}(r_1r_2) \wedge \mathbb{C}_{Q2}(s_1) \wedge \mathbb{C}_{Q2}(s_2)$
$\mathbb{U}_{(\mathfrak{Q}_{1i}\circ\mathfrak{Q}_{2i})}(r_1s_1)(r_2s_2) = (\mathbb{U}_{\mathfrak{Q}_{1i}} \circ \mathbb{U}_{\mathfrak{Q}_{2i}})(r_1s_1)(r_2s_2) = \mathbb{U}_{Q1i}(r_1r_2) \vee \mathbb{U}_{Q2}(s_1) \vee \mathbb{U}_{Q2}(s_2)$
$\mathbb{F}_{(\mathfrak{Q}_{1i}\circ\mathfrak{Q}_{2i})}(r_1s_1)(r_2s_2) = (\mathbb{F}_{\mathfrak{Q}_{1i}} \circ \mathbb{F}_{\mathfrak{Q}_{2i}})(r_1s_1)(r_2s_2) = \mathbb{F}_{Q1i}(r_1r_2) \vee \mathbb{F}_{Q2}(s_1) \vee \mathbb{F}_{Q2}(s_2)$
$\forall r_1r_2 \in S_{1i}, s_1s_2 \in S_{2i}$ suchthat $s_1 \neq s_2$.

Theorem 3.49 *The Composition product* $\mathfrak{G}_{n1} \circ \mathfrak{G}_{n2} = (\mathfrak{Q}_1 \circ \mathfrak{Q}_2, \mathfrak{Q}_{11} \circ \mathfrak{Q}_{21}, \mathfrak{Q}_{12} \circ \mathfrak{Q}_{22}, \ldots, \mathfrak{Q}_{1n} \circ \mathfrak{Q}_{2n})$ *of two QNGS of the GS* $\mathfrak{G}_1$ *and* $\mathfrak{G}_2$ *is a QNGs of* $\mathfrak{G}_1 \circ \mathfrak{G}_2$.

Proof. According to the definition of composition product there are three cases:
Case1: when $r \in S_1, s_1s_2 \in S_{2i}$

$\mathbb{T}_{(\mathfrak{Q}_{1i}\circ\mathfrak{Q}_{2i})}(rs_1)(rs_2) = \mathbb{T}_{Q1}(r) \wedge \mathbb{T}_{Q2i}(s_1s_2)$
$\leq \mathbb{T}_{Q1}(r) \wedge [\mathbb{T}_{Q2}(s_1) \wedge \mathbb{T}_{Q2}(s_2)]$
$= [\mathbb{T}_{Q1}(r) \wedge \mathbb{T}_{Q2}(s_1)] \wedge [\mathbb{T}_{Q1}(r) \wedge \mathbb{T}_{Q2}(s_2)]$
$= \mathbb{T}_{(Q_1\circ Q_2)}(rs_1) \wedge \mathbb{T}_{(Q_1\circ Q_2)}(rs_2)$.

$\mathbb{C}_{(\mathfrak{Q}_{1i}\circ\mathfrak{Q}_{2i})}(rs_1)(rs_2) = \mathbb{C}_{Q1}(r) \wedge \mathbb{C}_{Q2i}(s_1s_2)$
$\leq \mathbb{C}_{Q1}(r) \wedge [\mathbb{C}_{Q2}(s_1) \wedge \mathbb{C}_{Q2}(s_2)]$
$= [\mathbb{C}_{Q1}(r) \wedge \mathbb{C}_{Q2}(s_1)] \wedge [\mathbb{C}_{Q1}(r) \wedge \mathbb{C}_{Q2}(s_2)]$
$= \mathbb{C}_{(Q_1\circ Q_2)}(rs_1) \wedge \mathbb{C}_{(Q_1\circ Q_2)}(rs_2)$.

$\mathbb{U}_{(\mathfrak{Q}_{1i}\circ\mathfrak{Q}_{2i})}(rs_1)(rs_2) = \mathbb{U}_{Q1}(r) \vee \mathbb{U}_{Q2i}(s_1s_2)$
$\leq \mathbb{U}_{Q1}(r) \vee [\mathbb{F}_{Q2}(s_1) \vee \mathbb{U}_{Q2}(s_2)]$
$= [\mathbb{U}_{Q1}(r) \vee \mathbb{U}_{Q2}(s_1)] \vee [\mathbb{U}_{Q1}(r) \vee \mathbb{U}_{Q2}(s_2)]$
$= \mathbb{U}_{(Q_1\circ Q_2)}(rs_1) \vee \mathbb{U}_{(Q_1\circ Q_2)}(rs_2)$.

$\mathbb{F}_{(\mathfrak{Q}_{1i}\circ\mathfrak{Q}_{2i})}(rs_1)(rs_2) = \mathbb{F}_{Q1}(r) \vee \mathbb{F}_{Q2i}(s_1s_2)$
$\leq \mathbb{F}_{Q1}(r) \vee [\mathbb{F}_{Q2}(s_1) \vee \mathbb{F}_{Q2}(s_2)]$
$= [\mathbb{F}_{Q1}(r) \vee \mathbb{F}_{Q2}(s_1)] \vee [\mathbb{F}_{Q1}(r) \vee \mathbb{F}_{Q2}(s_2)]$
$= \mathbb{F}_{(Q_1\circ Q_2)}(rs_1) \vee \mathbb{F}_{(Q_1\circ Q_2)}(rs_2)$.

for $rs_1, rs_2 \in S_1 \circ S_2$.
Case 2: when $r \in S_2, s_1s_2 \in S_{1i}$

$\mathbb{T}_{(\mathfrak{Q}_{1i}\circ\mathfrak{Q}_{2i})}(s_1r)(s_2r) = \mathbb{T}_{Q2}(s) \wedge \mathbb{T}_{Q1i}(s_1s_2)$
$\leq \mathbb{T}_{Q2}(r) \wedge [\mathbb{T}_{Q1}(s_1) \wedge \mathbb{T}_{Q2}(s_2)]$
$= [\mathbb{T}_{Q2}(r) \wedge \mathbb{T}_{Q1}(s_1)] \wedge [\mathbb{T}_{Q2}(r) \wedge \mathbb{T}_{Q1}(s_2)]$
$= \mathbb{T}_{(Q_1\circ Q_2)}(s_1r) \wedge \mathbb{T}_{(Q_1\circ Q_2)}(s_2r)$.

$\mathbb{C}_{(\mathfrak{Q}_{1i}\circ\mathfrak{Q}_{2i})}(s_1r)(s_2r) = \mathbb{C}_{Q2}(s) \wedge \mathbb{C}_{Q1i}(s_1s_2)$
$\leq \mathbb{C}_{Q2}(r) \wedge [\mathbb{C}_{Q1}(s_1) \wedge \mathbb{C}_{Q2}(s_2)]$
$= [\mathbb{C}_{Q2}(r) \wedge \mathbb{C}_{Q1}(s_1)] \wedge [\mathbb{C}_{Q2}(r) \wedge \mathbb{C}_{Q1}(s_2)]$
$= \mathbb{C}_{(Q_1\circ Q_2)}(s_1r) \wedge \mathbb{C}_{(Q_1\circ Q_2)}(s_2r)$.

$\mathbb{U}_{(\mathfrak{Q}_{1i}\circ\mathfrak{Q}_{2i})}(s_1r)(s_2r) = \mathbb{U}_{Q2}(s) \vee \mathbb{U}_{Q1i}(s_1s_2)$
$\leq \mathbb{U}_{Q2}(r) \vee [\mathbb{U}_{Q1}(s_1) \vee \mathbb{U}_{Q2}(s_2)]$
$= [\mathbb{U}_{Q2}(r) \wedge \mathbb{U}_{Q1}(s_1)] \vee [\mathbb{U}_{Q2}(r) \wedge \mathbb{U}_{Q1}(s_2)]$

$= \mathbb{U}_{(Q_1 \circ Q_2)}(s_1 r) \vee \mathbb{U}_{(Q_1 \circ Q_2)}(s_2 r).$

$\mathbb{F}_{(\mathfrak{Q}_{1i} \circ \mathfrak{Q}_{2i})}(s_1 r)(s_2 r) = \mathbb{F}_{Q2}(s) \vee \mathbb{F}_{Q1i}(s_1 s_2)$
$\leq \mathbb{F}_{Q2}(r) \vee [\mathbb{F}_{Q1}(s_1) \vee \mathbb{F}_{Q2}(s_2)]$
$= [\mathbb{F}_{Q2}(r) \wedge \mathbb{F}_{Q1}(s_1)] \vee [\mathbb{F}_{Q2}(r) \wedge \mathbb{F}_{Q1}(s_2)]$
$= \mathbb{F}_{(Q_1 \circ Q_2)}(s_1 r) \vee \mathbb{F}_{(Q_1 \circ Q_2)}(s_2 r).$

for $s_1 r, s_2 r \in S_1 S_2$.

Case 3: For all $r_1 r_2 \in S_{1i}, s_1 s_2 \in S_2$ such that $s_1 \neq s_2$
$\mathbb{T}_{(\mathfrak{Q}_{1i} \circ \mathfrak{Q}_{2i})}(r_1 s_1)(r_2 s_2) = (\mathbb{T}_{\mathfrak{Q}_{1i}} \circ \mathbb{T}_{\mathfrak{Q}_{2i}})(r_1 s_1)(r_2 s_2) = \mathbb{T}_{Q1i}(r_1 r_2) \wedge \mathbb{T}_{Q2}(s_1) \wedge \mathbb{T}_{Q2}(s_2)$
$\leq [\mathbb{T}_{Q1}(r_1) \wedge \mathbb{T}_{Q1}(r_2)] \wedge \mathbb{T}_{Q2}(s_1) \wedge \mathbb{T}_{Q2}(s_2)$
$= [\mathbb{T}_{Q1}(r_1) \wedge \mathbb{T}_{Q2}(s_1)] \wedge [\mathbb{T}_{Q1}(r_2) \wedge \mathbb{T}_{Q2}(s_2)]$
$= \mathbb{T}_{(\mathfrak{Q}_1 \circ \mathfrak{Q}_2)}(r_1 s_1) \wedge \mathbb{T}_{(\mathfrak{Q}_1 \circ \mathfrak{Q}_2)}(r_2 s_2)$

$\mathbb{C}_{(\mathfrak{Q}_{1i} \circ \mathfrak{Q}_{2i})}(r_1 s_1)(r_2 s_2) = (\mathbb{C}_{\mathfrak{Q}_{1i}} \circ \mathbb{C}_{\mathfrak{Q}_{2i}})(r_1 s_1)(r_2 s_2) = \mathbb{C}_{Q1i}(r_1 r_2) \wedge \mathbb{C}_{Q2}(s_1) \wedge \mathbb{C}_{Q2}(s_2)$
$\leq [\mathbb{C}_{Q1}(r_1) \wedge \mathbb{C}_{Q1}(r_2)] \wedge \mathbb{C}_{Q2}(s_1) \wedge \mathbb{C}_{Q2}(s_2)$
$= [\mathbb{C}_{Q1}(r_1) \wedge \mathbb{C}_{Q2}(s_1)] \wedge [\mathbb{C}_{Q1}(r_2) \wedge \mathbb{C}_{Q2}(s_2)]$
$= \mathbb{C}_{(\mathfrak{Q}_1 \circ \mathfrak{Q}_2)}(r_1 s_1) \wedge \mathbb{C}_{(\mathfrak{Q}_1 \circ \mathfrak{Q}_2)}(r_2 s_2)$

$\mathbb{U}_{(\mathfrak{Q}_{1i} \circ \mathfrak{Q}_{2i})}(r_1 s_1)(r_2 s_2) = (\mathbb{U}_{\mathfrak{Q}_{1i}} \circ \mathbb{U}_{\mathfrak{Q}_{2i}})(r_1 s_1)(r_2 s_2) = \mathbb{U}_{Q1i}(r_1 r_2) \vee \mathbb{U}_{Q2}(s_1) \vee \mathbb{U}_{Q2}(s_2)$
$\leq [\mathbb{U}_{Q1}(r_1) \vee \mathbb{U}_{Q1}(r_2)] \vee \mathbb{U}_{Q2}(s_1) \vee \mathbb{U}_{Q2}(s_2)$
$= [\mathbb{U}_{Q1}(r_1) \vee \mathbb{U}_{Q2}(s_1)] \vee [\mathbb{U}_{Q1}(r_2) \vee \mathbb{U}_{Q2}(s_2)]$
$= \mathbb{U}_{(\mathfrak{Q}_1 \circ \mathfrak{Q}_2)}(r_1 s_1) \vee \mathbb{U}_{(\mathfrak{Q}_1 \circ \mathfrak{Q}_2)}(r_2 s_2)$

$\mathbb{F}_{(\mathfrak{Q}_{1i} \circ \mathfrak{Q}_{2i})}(r_1 s_1)(r_2 s_2) = (\mathbb{F}_{\mathfrak{Q}_{1i}} \circ \mathbb{F}_{\mathfrak{Q}_{2i}})(r_1 s_1)(r_2 s_2) = \mathbb{F}_{Q1i}(r_1 r_2) \vee \mathbb{F}_{Q2}(s_1) \vee \mathbb{F}_{Q2}(s_2)$
$\leq [\mathbb{F}_{Q1}(r_1) \vee \mathbb{F}_{Q1}(r_2)] \vee \mathbb{F}_{Q2}(s_1) \vee \mathbb{F}_{Q2}(s_2)$
$= [\mathbb{F}_{Q1}(r_1) \vee \mathbb{F}_{Q2}(s_1)] \vee [\mathbb{F}_{Q1}(r_2) \vee \mathbb{F}_{Q2}(s_2)]$
$= \mathbb{F}_{(\mathfrak{Q}_1 \circ \mathfrak{Q}_2)}(r_1 s_1) \vee \mathbb{F}_{(\mathfrak{Q}_1 \circ \mathfrak{Q}_2)}(r_2 s_2)$

for $r_1 s_1, r_2 s_2 \in S_1 \circ S_2$. Hence proved.

Definition 3.50 *Let* $\mathfrak{G}_{n1} = (\mathfrak{Q}_1, \mathfrak{Q}_{11}, \mathfrak{Q}_{12}, \ldots. \mathfrak{Q}_{1n})$ *and* $\mathfrak{G}_{n2} = (\mathfrak{Q}_2, \mathfrak{Q}_{21}, \mathfrak{Q}_{22}, \ldots. \mathfrak{Q}_{2n})$ *be QNGS. The union of* $\mathfrak{G}_1$ *and* $\mathfrak{G}_2$ *denoted by*

$$\mathfrak{G}_{n1} \cup \mathfrak{G}_{n2} = (\mathfrak{Q}_1 \cup \mathfrak{Q}_2, \mathfrak{Q}_{11} \cup \mathfrak{Q}_{21}, \mathfrak{Q}_{12} \cup \mathfrak{Q}_{22}, \ldots, \mathfrak{Q}_{1n} \cup \mathfrak{Q}_{2n}),$$

is defined by the following:

(i) $\mathbb{T}_{(\mathfrak{Q}_1 \cup \mathfrak{Q}_2)}(r) = (\mathbb{T}_{\mathfrak{Q}_1} \cup \mathbb{T}_{\mathfrak{Q}_2})(r) = \mathbb{T}_{Q1}(r) \vee \mathbb{T}_{Q2}(r)$
$\mathbb{U}_{(\mathfrak{Q}_1 \cup \mathfrak{Q}_2)}(r) = (\mathbb{U}_{\mathfrak{Q}_1} \cup \mathbb{U}_{\mathfrak{Q}_2})(r) = \mathbb{U}_{Q1}(r) \vee \mathbb{U}_{Q2}(r)$
$\mathbb{C}_{(\mathfrak{Q}_1 \cup \mathfrak{Q}_2)}(r) = (\mathbb{C}_{\mathfrak{Q}_1} \cup \mathbb{C}_{\mathfrak{Q}_2})(r) = \mathbb{C}_{Q1}(r) \wedge \mathbb{C}_{Q2}(r)$
$\mathbb{F}_{(\mathfrak{Q}_1 \cup \mathfrak{Q}_2)}(r) = (\mathbb{F}_{\mathfrak{Q}_1} \cup \mathbb{F}_{\mathfrak{Q}_2})(r) = \mathbb{F}_{Q1}(r) \wedge \mathbb{F}_{Q2}(r),$
$\forall r \in S_1 \cup S_2,$

(ii) $\mathbb{T}_{(\mathfrak{Q}_{1i} \cup \mathfrak{Q}_{2i})}(rs) = (\mathbb{T}_{\mathfrak{Q}_{1i}} \cup \mathbb{T}_{\mathfrak{Q}_{2i}})(rs) = \mathbb{T}_{Q1i}(rs) \vee \mathbb{T}_{Q2i}(rs)$
$\mathbb{C}_{(\mathfrak{Q}_{1i} \cup \mathfrak{Q}_{2i})}(rs) = (\mathbb{C}_{\mathfrak{Q}_{1i}} \cup \mathbb{C}_{\mathfrak{Q}_{2i}})(rs) = \mathbb{C}_{Q1i}(rs) \vee \mathbb{C}_{Q2i}(rs)$
$\mathbb{U}_{(\mathfrak{Q}_{1i} \cup \mathfrak{Q}_{2i})}(rs) = (\mathbb{U}_{\mathfrak{Q}_{1i}} \cup \mathbb{U}_{\mathfrak{Q}_{2i}})(rs) = \mathbb{U}_{Q1i}(rs) \wedge \mathbb{U}_{Q2i}(rs)$
$\mathbb{F}_{(\mathfrak{Q}_{1i} \cup \mathfrak{Q}_{2i})}(rs) = (\mathbb{F}_{\mathfrak{Q}_{1i}} \cup \mathbb{F}_{\mathfrak{Q}_{2i}})(rs) = \mathbb{F}_{Q1i}(rs) \wedge \mathbb{F}_{Q2i}(rs),$

for all $(rs) \in S_{1i} \cup S_{2i}$.

Theorem 3.51 *The union* $\mathfrak{G}_{n1} \cup \mathfrak{G}_{n2} = (\mathfrak{Q}_1 \cup \mathfrak{Q}_2, \mathfrak{Q}_{11} \cup \mathfrak{Q}_{21}, \mathfrak{Q}_{12} \cup \mathfrak{Q}_{22}, \ldots, \mathfrak{Q}_{1n} \cup \mathfrak{Q}_{2n})$ *of two QNGS of the*

GS $\mathfrak{G}_1$ *and* $\mathfrak{G}_2$ *is a QNGs of* $\mathfrak{G}_1 \cup \mathfrak{G}_2$.

Proof. Let $r_1r_2 \in S_{1i} \cup S_{2i}$. Here we consider two cases:

Case 1: when $r_1r_2 \in S_1$, then according to Definition 3.39, $\mathbb{T}_{\mathfrak{Q}_2}(r_1) = \mathbb{T}_{\mathfrak{Q}_2}(r_2) = \mathbb{T}_{\mathfrak{Q}_{2i}}(r_1r_2) = 0$

$$\mathbb{C}_{\mathfrak{Q}_2}(r_1) = \mathbb{C}_{\mathfrak{Q}_2}(r_2) = \mathbb{C}_{\mathfrak{Q}_{2i}}(r_1r_2) = 0$$

$$\mathbb{U}_{\mathfrak{Q}_2}(r_1) = \mathbb{U}_{\mathfrak{Q}_2}(r_2) = \mathbb{U}_{\mathfrak{Q}_{2i}}(r_1r_2) = 1$$

$\mathbb{F}_{\mathfrak{Q}_2}(r_1) = \mathbb{F}_{\mathfrak{Q}_2}(r_2) = \mathbb{F}_{\mathfrak{Q}_{2i}}(r_1r_2) = 1$, so

$$\begin{aligned}
&\mathbb{T}_{(\mathfrak{Q}_{1i} \cup \mathfrak{Q}_{2i})}(r_1r_2) = \mathbb{T}_{Q1i}(r_1r_2) \vee \mathbb{T}_{Q2i}(r_1r_2)\\
&= \mathbb{T}_{Q1i}(r_1r_2) \vee 0\\
&\le [\mathbb{T}_{Q1}(r_1) \wedge \mathbb{T}_{Q1}(r_2)] \vee 0\\
&= [\mathbb{T}_{Q1}(r_1) \vee 0] \wedge [\mathbb{T}_{Q1}(r_2) \vee 0]\\
&= [\mathbb{T}_{Q1}(r_1) \vee \mathbb{T}_{Q2}(r_1)] \wedge [\mathbb{T}_{Q1}(r_2) \vee \mathbb{T}_{Q2}(r_2)]\\
&= \mathbb{T}_{(\mathfrak{Q}_1 \cup \mathfrak{Q}_2)}(r_1) \wedge \mathbb{T}_{(\mathfrak{Q}_1 \cup \mathfrak{Q}_2)}(r_2)
\end{aligned}$$

$$\begin{aligned}
&\mathbb{C}_{(\mathfrak{Q}_{1i} \cup \mathfrak{Q}_{2i})}(r_1r_2) = \mathbb{C}_{Q1i}(r_1r_2) \vee \mathbb{C}_{Q2i}(r_1r_2)\\
&= \mathbb{C}_{Q1i}(r_1r_2) \vee 0\\
&\le [\mathbb{C}_{Q1}(r_1) \wedge \mathbb{C}_{Q1}(r_2)] \vee 0\\
&= [\mathbb{C}_{Q1}(r_1) \vee 0] \wedge [\mathbb{C}_{Q1}(r_2) \vee 0]\\
&= [\mathbb{C}_{Q1}(r_1) \vee \mathbb{C}_{Q2}(r_1)] \wedge [\mathbb{C}_{Q1}(r_2) \vee \mathbb{C}_{Q2}(r_2)]\\
&= \mathbb{C}_{(\mathfrak{Q}_1 \cup \mathfrak{Q}_2)}(r_1) \wedge \mathbb{C}_{(\mathfrak{Q}_1 \cup \mathfrak{Q}_2)}(r_2)
\end{aligned}$$

$$\begin{aligned}
&\mathbb{U}_{(\mathfrak{Q}_{1i} \cup \mathfrak{Q}_{2i})}(r_1r_2) = \mathbb{U}_{Q1i}(r_1r_2) \wedge \mathbb{U}_{Q2i}(r_1r_2)\\
&= \mathbb{U}_{Q1i}(r_1r_2) \vee 1\\
&\le [\mathbb{U}_{Q1}(r_1) \vee \mathbb{U}_{Q1}(r_2)] \wedge 1\\
&= [\mathbb{U}_{Q1}(r_1) \wedge 1] \vee [\mathbb{U}_{Q1}(r_2) \wedge 1]\\
&= [\mathbb{U}_{Q1}(r_1) \wedge \mathbb{U}_{Q2}(r_1)] \vee [\mathbb{U}_{Q1}(r_2) \wedge \mathbb{U}_{Q2}(r_2)]\\
&= \mathbb{U}_{(\mathfrak{Q}_1 \cup \mathfrak{Q}_2)}(r_1) \vee \mathbb{U}_{(\mathfrak{Q}_1 \cup \mathfrak{Q}_2)}(r_2)
\end{aligned}$$

$$\begin{aligned}
&\mathbb{F}_{(\mathfrak{Q}_{1i} \cup \mathfrak{Q}_{2i})}(r_1r_2) = \mathbb{F}_{Q1i}(r_1r_2) \wedge \mathbb{F}_{Q2i}(r_1r_2)\\
&= \mathbb{F}_{Q1i}(r_1r_2) \vee 1\\
&\le [\mathbb{F}_{Q1}(r_1) \vee \mathbb{F}_{Q1}(r_2)] \wedge 1\\
&= [\mathbb{F}_{Q1}(r_1) \wedge 1] \vee [\mathbb{F}_{Q1}(r_2) \wedge 1]\\
&= [\mathbb{F}_{Q1}(r_1) \wedge \mathbb{F}_{Q2}(r_1)] \vee [\mathbb{F}_{Q1}(r_2) \wedge \mathbb{F}_{Q2}(r_2)]\\
&= \mathbb{F}_{(\mathfrak{Q}_1 \cup \mathfrak{Q}_2)}(r_1) \vee \mathbb{F}_{(\mathfrak{Q}_1 \cup \mathfrak{Q}_2)}(r_2)
\end{aligned}$$

For $r_1r_2 \in S_1 \cup S_2$.

Case 2: when $r_1r_2 \in S_2$, then according to Definition 3.39, $\mathbb{T}_{\mathfrak{Q}_1}(r_1) = \mathbb{T}_{\mathfrak{Q}_1}(r_2) = \mathbb{T}_{\mathfrak{Q}_{1i}}(r_1r_2) = 0$

$$\mathbb{C}_{\mathfrak{Q}_1}(r_1) = \mathbb{C}_{\mathfrak{Q}_1}(r_2) = \mathbb{C}_{\mathfrak{Q}_{1i}}(r_1r_2) = 0$$

$$\mathbb{U}_{\mathfrak{Q}_1}(r_1) = \mathbb{U}_{\mathfrak{Q}_1}(r_2) = \mathbb{U}_{\mathfrak{Q}_{1i}}(r_1r_2) = 1$$

$\mathbb{F}_{\mathfrak{Q}_1}(r_1) = \mathbb{F}_{\mathfrak{Q}_1}(r_2) = \mathbb{F}_{\mathfrak{Q}_{1i}}(r_1r_2) = 1$, so

$$\begin{aligned}
&\mathbb{T}_{(\mathfrak{Q}_{1i} \cup \mathfrak{Q}_{2i})}(r_1r_2) = \mathbb{T}_{Q1i}(r_1r_2) \vee \mathbb{T}_{Q2i}(r_1r_2)\\
&= \mathbb{T}_{Q2i}(r_1r_2) \vee 0\\
&\le [\mathbb{T}_{Q2}(r_1) \wedge \mathbb{T}_{Q2}(r_2)] \vee 0\\
&= [\mathbb{T}_{Q2}(r_1) \vee 0] \wedge [\mathbb{T}_{Q2}(r_2) \vee 0]\\
&= [\mathbb{T}_{Q1}(r_1) \vee \mathbb{T}_{Q2}(r_1)] \wedge [\mathbb{T}_{Q1}(r_2) \vee \mathbb{T}_{Q2}(r_2)]\\
&= \mathbb{T}_{(\mathfrak{Q}_1 \cup \mathfrak{Q}_2)}(r_1) \wedge \mathbb{T}_{(\mathfrak{Q}_1 \cup \mathfrak{Q}_2)}(r_2)
\end{aligned}$$

$\mathbb{C}_{(\mathfrak{Q}_{1i}\cup\mathfrak{Q}_{2i})}(r_1r_2) = \mathbb{C}_{Q1i}(r_1r_2) \vee \mathbb{C}_{Q2i}(r_1r_2)$
$= \mathbb{C}_{Q2i}(r_1r_2) \vee 0$
$\leq [\mathbb{C}_{Q2}(r_1) \wedge \mathbb{C}_{Q2}(r_2)] \vee 0$
$= [\mathbb{C}_{Q2}(r_1) \vee 0] \wedge [\mathbb{C}_{Q2}(r_2) \vee 0]$
$= [\mathbb{C}_{Q1}(r_1) \vee \mathbb{C}_{Q2}(r_1)] \wedge [\mathbb{C}_{Q1}(r_2) \vee \mathbb{C}_{Q2}(r_2)]$
$= \mathbb{C}_{(\mathfrak{Q}_1\cup\mathfrak{Q}_2)}(r_1) \wedge \mathbb{C}_{(\mathfrak{Q}_1\cup\mathfrak{Q}_2)}(r_2)$

$\mathbb{U}_{(\mathfrak{Q}_{1i}\cup\mathfrak{Q}_{2i})}(r_1r_2) = \mathbb{U}_{Q1i}(r_1r_2) \wedge \mathbb{U}_{Q2i}(r_1r_2)$
$= \mathbb{U}_{Q2i}(r_1r_2) \vee 1$
$\leq [\mathbb{U}_{Q2}(r_1) \vee \mathbb{U}_{Q2}(r_2)] \wedge 1$
$= [\mathbb{U}_{Q2}(r_1) \wedge 1] \vee [\mathbb{U}_{Q2}(r_2) \wedge 1]$
$= [\mathbb{U}_{Q1}(r_1) \wedge \mathbb{U}_{Q2}(r_1)] \vee [\mathbb{U}_{Q1}(r_2) \wedge \mathbb{U}_{Q2}(r_2)]$
$= \mathbb{U}_{(\mathfrak{Q}_1\cup\mathfrak{Q}_2)}(r_1) \vee \mathbb{U}_{(\mathfrak{Q}_1\cup\mathfrak{Q}_2)}(r_2)$

$\mathbb{F}_{(\mathfrak{Q}_{1i}\cup\mathfrak{Q}_{2i})}(r_1r_2) = \mathbb{F}_{Q1i}(r_1r_2) \wedge \mathbb{F}_{Q2i}(r_1r_2)$
$= \mathbb{F}_{Q2i}(r_1r_2) \vee 1$
$\leq [\mathbb{F}_{Q2}(r_1) \vee \mathbb{F}_{Q2}(r_2)] \wedge 1$
$= [\mathbb{F}_{Q2}(r_1) \wedge 1] \vee [\mathbb{F}_{Q2}(r_2) \wedge 1]$
$= [\mathbb{F}_{Q1}(r_1) \wedge \mathbb{F}_{Q2}(r_1)] \vee [\mathbb{F}_{Q1}(r_2) \wedge \mathbb{F}_{Q2}(r_2)]$
$= \mathbb{F}_{(\mathfrak{Q}_1\cup\mathfrak{Q}_2)}(r_1) \vee \mathbb{F}_{(\mathfrak{Q}_1\cup\mathfrak{Q}_2)}(r_2)$

For $r_1r_2 \in S_1 \cup S_2$. Hence Proved.

Theorem 3.52 *Let* $\mathfrak{G} = (S_1 \cup S_2, S_{11} \cup S_{21}, S_{12} \cup S_{22}, \ldots . S_{1n} \cup S_{2n})$ *to be union of two GSs* $\mathfrak{G}_1 = (S_1, S_{11}, S_{12}, \ldots, S_{1n})$ *and* $\mathfrak{G}_2 = (S_2, S_{21}, S_{22}, \ldots, S_{2n})$*. Then every QNGS* $\mathfrak{G} = (\mathfrak{Q}, \mathfrak{Q}_1, \mathfrak{Q}_2, \ldots . \mathfrak{Q}_n)$ *of* $\mathfrak{G}$ *is union of two QNGs* $\mathfrak{G}_{n1}$ *and* $\mathfrak{G}_{n2}$ *of GS* $\mathfrak{G}_1$ *and* $\mathfrak{G}_2$*, respectively,*

Proof. First we define $\mathfrak{Q}_1, \mathfrak{Q}_2, \mathfrak{Q}_{1i}$ and $\mathfrak{Q}_{2i}$ for $i \in 1,2,\ldots,n$ as

$$\mathbb{T}_{Q1}(r) = \mathbb{T}_Q(r), \mathbb{C}_{Q1}(r) = \mathbb{C}_Q(r), \mathbb{U}_{Q1}(r) = \mathbb{U}_Q(r), \mathbb{F}_{Q1}(r) = \mathbb{F}_Q(r), \text{if} r \in S_1$$
$$\mathbb{T}_{Q2}(r) = \mathbb{T}_Q(r), \mathbb{C}_{Q1}(r) = \mathbb{C}_Q(r), \mathbb{U}_{Q1}(r) = \mathbb{U}_Q(r), \mathbb{F}_{Q1}(r) = \mathbb{F}_Q(r), \text{if} r \in S_2.$$

$$\mathbb{T}_{Q1i}(r_1r_2) = \mathbb{T}_{Qi}(r_1r_2), \mathbb{C}_{Q1i}(r_1r_2) = \mathbb{C}_{Qi}(r_1r_2), \mathbb{U}_{Q1i}(r_1r_2) = \mathbb{U}_{Qi}(r_1r_2), \mathbb{F}_{Q1i}(r_1r_2) = \mathbb{F}_{Qi}(r_1r_2),$$
if $r_1r_2 \in S_{1i}$,

$$\mathbb{T}_{Q2i}(r_1r_2) = \mathbb{T}_{Qi}(r_1r_2), \mathbb{C}_{Q2i}(r_1r_2) = \mathbb{C}_{Qi}(r_1r_2), \mathbb{U}_{Q2i}(r_1r_2) = \mathbb{U}_{Qi}(r_1r_2), \mathbb{F}_{Q2i}(r_1r_2) = \mathbb{F}_{Qi}(r_1r_2),$$
if $r_1r_2 \in S_{2i}$,

Then $\mathfrak{Q} = \mathfrak{Q}_1 \cup \mathfrak{Q}_2$ and $\mathfrak{Q}_i = \mathfrak{Q}_{1i} \cup \mathfrak{Q}_{2i}$, $i \in 1,2,\ldots,n$.

Now for $r_1r_2 \in S_{ki}, k = 1,2, i = 1,2,\ldots,n$

$\mathbb{T}_{Qki}(r_1r_2) = \mathbb{T}_{Q1i}(r_1r_2) \leq \mathbb{T}_{Q1}(r_1) \wedge \mathbb{T}_{Q1}(r_2) = \mathbb{T}_{Qk}(r_1) \wedge \mathbb{T}_{Qk}(r_2)$
$\mathbb{C}_{Qki}(r_1r_2) = \mathbb{C}_{Q1i}(r_1r_2) \leq \mathbb{C}_{Q1}(r_1) \wedge \mathbb{C}_{Q1}(r_2) = \mathbb{C}_{Qk}(r_1) \wedge \mathbb{C}_{Qk}(r_2)$
$\mathbb{U}_{Qki}(r_1r_2) = \mathbb{U}_{Q1i}(r_1r_2) \leq \mathbb{U}_{Q1}(r_1) \vee \mathbb{U}_{Q1}(r_2) = \mathbb{U}_{Qk}(r_1) \vee \mathbb{U}_{Qk}(r_2)$
$\mathbb{F}_{Qki}(r_1r_2) = \mathbb{F}_{Q1i}(r_1r_2) \leq \mathbb{F}_{Q1}(r_1) \vee \mathbb{F}_{Q1}(r_2) = \mathbb{F}_{Qk}(r_1) \vee \mathbb{F}_{Qk}(r_2)$.

i. e

$\mathfrak{G}_{nk} = (\mathfrak{Q}_k, \mathfrak{Q}_{k1}, \ldots . \mathfrak{Q}_{kn})$ isaQNGS of $\mathfrak{G}_k, k = 1,2$. Thus $\mathfrak{G}_{nk} = \mathfrak{Q}, \mathfrak{Q}_1, \mathfrak{Q}_3, \ldots ., \mathfrak{Q}_n$, a QNG of $\mathfrak{G} = \mathfrak{G}_1 \cup \mathfrak{G}_2$ is union of two QNGSs $\mathfrak{G}_{n1}$ and $\mathfrak{G}_{n2}$.

Definition 3.53 *Let* $\mathfrak{G}_{n1} = (\mathfrak{Q}_1, \mathfrak{Q}_{11}, \mathfrak{Q}_{12}, \ldots . \mathfrak{Q}_{1n)}$ *and* $\mathfrak{G}_{n2} = (\mathfrak{Q}_2, \mathfrak{Q}_{21}, \mathfrak{Q}_{22}, \ldots . \mathfrak{Q}_{2n)}$ *be QNGS and let* $S_1 \cap S_2 = \emptyset$*. The join of* $\mathfrak{G}_{n1}$ *and* $\mathfrak{G}_{n2}$*, denoted by*

$\mathfrak{G}_{n1} + \mathfrak{G}_{n2} = (\mathfrak{Q}_1 + \mathfrak{Q}_2, \mathfrak{Q}_{11} + \mathfrak{Q}_{21}, \mathfrak{Q}_{12} + \mathfrak{Q}_{22}, \ldots, \mathfrak{Q}_{1n} + \mathfrak{Q}_{2n})$,
is defined by following:

$(i)\quad \mathbb{T}_{(\mathfrak{Q}_1+\mathfrak{Q}_2)}(r) = \mathbb{T}_{(\mathfrak{Q}_1\cup\mathfrak{Q}_2)}(r)$
$\mathbb{C}_{(\mathfrak{Q}_1+\mathfrak{Q}_2)}(r) = \mathbb{C}_{(\mathfrak{Q}_1\cup\mathfrak{Q}_2)}(r)$
$\mathbb{U}_{(\mathfrak{Q}_1+\mathfrak{Q}_2)}(r) = \mathbb{U}_{(\mathfrak{Q}_1\cup\mathfrak{Q}_2)}(r)$
$\mathbb{F}_{(\mathfrak{Q}_1+\mathfrak{Q}_2)}(r) = \mathbb{F}_{(\mathfrak{Q}_1\cup\mathfrak{Q}_2)}(r)$
$\forall \quad r \in S_1 \cup S_2,$

$(ii)\quad \mathbb{T}_{(\mathfrak{Q}_{11}+\mathfrak{Q}_{2i})}(rs) = \mathbb{T}_{(\mathfrak{Q}_{1i}\cup\mathfrak{Q}_{2i})}(rs)$
$\mathbb{C}_{(\mathfrak{Q}_{11}+\mathfrak{Q}_{2i})}(rs) = \mathbb{C}_{(\mathfrak{Q}_{1i}\cup\mathfrak{Q}_{2i})}(rs)$
$\mathbb{U}_{(\mathfrak{Q}_{11}+\mathfrak{Q}_{2i})}(rs) = \mathbb{U}_{(\mathfrak{Q}_{1i}\cup\mathfrak{Q}_{2i})}(rs)$
$\mathbb{F}_{(\mathfrak{Q}_{11}+\mathfrak{Q}_{2i})}(rs) = \mathbb{F}_{(\mathfrak{Q}_{1i}\cup\mathfrak{Q}_{2i})}(rs)$
$\forall \quad rs \in S_{1i} \cup S_{2i},$

$(iii)\quad \mathbb{T}_{(\mathfrak{Q}_{1i}+\mathfrak{Q}_{2i})}(rs) = (\mathbb{T}_{\mathfrak{Q}_{1i}} + \mathbb{T}_{\mathfrak{Q}_{2i}})(rs) = \mathbb{T}_{Q1}(r) \wedge \mathbb{T}_{Q2}(s)$
$\mathbb{C}_{(\mathfrak{Q}_{1i}+\mathfrak{Q}_{2i})}(rs) = (\mathbb{C}_{\mathfrak{Q}_{1i}} + \mathbb{C}_{\mathfrak{Q}_{2i}})(rs) = \mathbb{C}_{Q1}(r) \wedge \mathbb{C}_{Q2}(s)$
$\mathbb{U}_{(\mathfrak{Q}_{1i}+\mathfrak{Q}_{2i})}(rs) = (\mathbb{U}_{\mathfrak{Q}_{1i}} + \mathbb{U}_{\mathfrak{Q}_{2i}})(rs) = \mathbb{U}_{Q1}(r) \vee \mathbb{U}_{Q2}(s)$
$\mathbb{F}_{(\mathfrak{Q}_{1i}+\mathfrak{Q}_{2i})}(rs) = (\mathbb{F}_{\mathfrak{Q}_{1i}} + \mathbb{F}_{\mathfrak{Q}_{2i}})(rs) = \mathbb{F}_{Q1}(r) \vee \mathbb{F}_{Q2}(s)$
$\forall \quad r \in S_1, s \in S_2.$

Theorem 3.54 *The join* $\mathfrak{G}_{n1} + \mathfrak{Q}_{n2} = (\mathfrak{Q}_1 + \mathfrak{Q}_2, \mathfrak{Q}_{11} + \mathfrak{Q}_{21}, \mathfrak{Q}_{12} + \mathfrak{Q}_{22}, \ldots, \mathfrak{Q}_{1n} + \mathfrak{Q}_{2n})$ *of two QNG of the GSS* $\mathfrak{G}$ *and* $\mathfrak{G}_2$ *is a QNG of* $\mathfrak{G}_1 + \mathfrak{G}_2$.

Proof. Let $r_1r_2 \in S_{1i} + S_{2i}$. Here we consider three cases:
Case 1: when $r_1r_2 \in S_1$, then according to definition 3.40

$$\mathbb{T}_{\mathfrak{Q}_2}(r_1) = \mathbb{T}_{\mathfrak{Q}_2}(r_2) = \mathbb{T}_{\mathfrak{Q}_{2i}}(r_1r_2) = 0$$
$$\mathbb{C}_{\mathfrak{Q}_2}(r_1) = \mathbb{C}_{\mathfrak{Q}_2}(r_2) = \mathbb{C}_{\mathfrak{Q}_{2i}}(r_1r_2) = 0$$
$$\mathbb{U}_{\mathfrak{Q}_2}(r_1) = \mathbb{U}_{\mathfrak{Q}_2}(r_2) = \mathbb{U}_{\mathfrak{Q}_{2i}}(r_1r_2) = 1$$

$\mathbb{F}_{\mathfrak{Q}_2}(r_1) = \mathbb{F}_{\mathfrak{Q}_2}(r_2) = \mathbb{F}_{\mathfrak{Q}_{2i}}(r_1r_2) = 1$, so

$\mathbb{T}_{(\mathfrak{Q}_{1i}+\mathfrak{Q}_{2i})}(r_1r_2) = \mathbb{T}_{Q1i}(r_1r_2) \vee \mathbb{T}_{Q2i}(r_1r_2)$
$= \mathbb{T}_{Q1i}(r_1r_2) \vee 0$
$\leq [\mathbb{T}_{Q1}(r_1) \wedge \mathbb{T}_{Q1}(r_2)] \vee 0$
$= [\mathbb{T}_{Q1}(r_1) \vee 0] \wedge [\mathbb{T}_{Q1}(r_2) \vee 0]$
$= [\mathbb{T}_{Q1}(r_1) \vee \mathbb{T}_{Q2}(r_1)] \wedge [\mathbb{T}_{Q1}(r_2) \vee \mathbb{T}_{Q2}(r_2)]$
$= \mathbb{T}_{(\mathfrak{Q}_1+\mathfrak{Q}_2)}(r_1) \wedge \mathbb{T}_{(\mathfrak{Q}_1+\mathfrak{Q}_2)}(r_2)$

$\mathbb{C}_{(\mathfrak{Q}_{1i}+\mathfrak{Q}_{2i})}(r_1r_2) = \mathbb{C}_{Q1i}(r_1r_2) \vee \mathbb{C}_{Q2i}(r_1r_2)$
$= \mathbb{C}_{Q1i}(r_1r_2) \vee 0$
$\leq [\mathbb{C}_{Q1}(r_1) \wedge \mathbb{C}_{Q1}(r_2)] \vee 0$
$= [\mathbb{C}_{Q1}(r_1) \vee 0] \wedge [\mathbb{C}_{Q1}(r_2) \vee 0]$
$= [\mathbb{C}_{Q1}(r_1) \vee \mathbb{C}_{Q2}(r_1)] \wedge [\mathbb{C}_{Q1}(r_2) \vee \mathbb{C}_{Q2}(r_2)]$
$= \mathbb{C}_{(\mathfrak{Q}_1+\mathfrak{Q}_2)}(r_1) \wedge \mathbb{C}_{(\mathfrak{Q}_1+\mathfrak{Q}_2)}(r_2)$

$\mathbb{U}_{(\mathfrak{Q}_{1i}+\mathfrak{Q}_{2i})}(r_1r_2) = \mathbb{U}_{Q1i}(r_1r_2) \wedge \mathbb{U}_{Q2i}(r_1r_2)$
$= \mathbb{U}_{Q1i}(r_1r_2) \vee 1$
$\leq [\mathbb{U}_{Q1}(r_1) \vee \mathbb{U}_{Q1}(r_2)] \wedge 1$
$= [\mathbb{U}_{Q1}(r_1) \wedge 1] \vee [\mathbb{U}_{Q1}(r_2) \wedge 1]$

$= [\mathbb{U}_{Q1}(r_1) \wedge \mathbb{U}_{Q2}(r_1)] \vee [\mathbb{U}_{Q1}(r_2) \wedge \mathbb{U}_{Q2}(r_2)]$
$= \mathbb{U}_{(\mathfrak{Q}_1+\mathfrak{Q}_2)}(r_1) \vee \mathbb{U}_{(\mathfrak{Q}_1+\mathfrak{Q}_2)}(r_2)$

$\mathbb{F}_{(\mathfrak{Q}_{1i}+\mathfrak{Q}_{2i})}(r_1r_2) = \mathbb{F}_{Q1i}(r_1r_2) \wedge \mathbb{F}_{Q2i}(r_1r_2)$
$= \mathbb{F}_{Q1i}(r_1r_2) \vee 1$
$\leq [\mathbb{F}_{Q1}(r_1) \vee \mathbb{F}_{Q1}(r_2)] \wedge 1$
$= [\mathbb{F}_{Q1}(r_1) \wedge 1] \vee [\mathbb{F}_{Q1}(r_2) \wedge 1]$
$= [\mathbb{F}_{Q1}(r_1) \wedge \mathbb{F}_{Q2}(r_1)] \vee [\mathbb{F}_{Q1}(r_2) \wedge \mathbb{F}_{Q2}(r_2)]$
$= \mathbb{F}_{(\mathfrak{Q}_1+\mathfrak{Q}_2)}(r_1) \vee \mathbb{F}_{(\mathfrak{Q}_1+\mathfrak{Q}_2)}(r_2)$

For $r_1r_2 \in S_1 + S_2$.

Case 2: when $r_1r_2 \in S_2$, then according to Definition 3.40, $\mathbb{T}_{\mathfrak{Q}_1}(r_1) = \mathbb{T}_{\mathfrak{Q}_1}(r_2) = \mathbb{T}_{\mathfrak{Q}_{1i}}(r_1r_2) = 0$

$\mathbb{C}_{\mathfrak{Q}_1}(r_1) = \mathbb{C}_{\mathfrak{Q}_1}(r_2) = \mathbb{C}_{\mathfrak{Q}_{1i}}(r_1r_2) = 0$

$\mathbb{U}_{\mathfrak{Q}_1}(r_1) = \mathbb{U}_{\mathfrak{Q}_1}(r_2) = \mathbb{U}_{\mathfrak{Q}_{1i}}(r_1r_2) = 1$

$\mathbb{F}_{\mathfrak{Q}_1}(r_1) = \mathbb{F}_{\mathfrak{Q}_1}(r_2) = \mathbb{F}_{\mathfrak{Q}_{1i}}(r_1r_2) = 1$, so

$\mathbb{T}_{(\mathfrak{Q}_{1i}+\mathfrak{Q}_{2i})}(r_1r_2) = \mathbb{T}_{Q1i}(r_1r_2) \vee \mathbb{T}_{Q2i}(r_1r_2)$
$= \mathbb{T}_{Q2i}(r_1r_2) \vee 0$
$\leq [\mathbb{T}_{Q2}(r_1) \wedge \mathbb{T}_{Q2}(r_2)] \vee 0$
$= [\mathbb{T}_{Q2}(r_1) \vee 0] \wedge [\mathbb{T}_{Q2}(r_2) \vee 0]$
$= [\mathbb{T}_{Q1}(r_1) \vee \mathbb{T}_{Q2}(r_1)] \wedge [\mathbb{T}_{Q1}(r_2) \vee \mathbb{T}_{Q2}(r_2)]$
$= \mathbb{T}_{(\mathfrak{Q}_1+\mathfrak{Q}_2)}(r_1) \wedge \mathbb{T}_{(\mathfrak{Q}_1+\mathfrak{Q}_2)}(r_2)$

$\mathbb{C}_{(\mathfrak{Q}_{1i}+\mathfrak{Q}_{2i})}(r_1r_2) = \mathbb{C}_{Q1i}(r_1r_2) \vee \mathbb{C}_{Q2i}(r_1r_2)$
$= \mathbb{C}_{Q2i}(r_1r_2) \vee 0$
$\leq [\mathbb{C}_{Q2}(r_1) \wedge \mathbb{C}_{Q2}(r_2)] \vee 0$
$= [\mathbb{C}_{Q2}(r_1) \vee 0] \wedge [\mathbb{C}_{Q2}(r_2) \vee 0]$
$= [\mathbb{C}_{Q1}(r_1) \vee \mathbb{C}_{Q2}(r_1)] \wedge [\mathbb{C}_{Q1}(r_2) \vee \mathbb{C}_{Q2}(r_2)]$
$= \mathbb{C}_{(\mathfrak{Q}_1+\mathfrak{Q}_2)}(r_1) \wedge \mathbb{C}_{(\mathfrak{Q}_1+\mathfrak{Q}_2)}(r_2)$

$\mathbb{U}_{(\mathfrak{Q}_{1i}+\mathfrak{Q}_{2i})}(r_1r_2) = \mathbb{U}_{Q1i}(r_1r_2) \wedge \mathbb{U}_{Q2i}(r_1r_2)$
$= \mathbb{U}_{Q2i}(r_1r_2) \vee 1$
$\leq [\mathbb{U}_{Q2}(r_1) \vee \mathbb{U}_{Q2}(r_2)] \wedge 1$
$= [\mathbb{U}_{Q2}(r_1) \wedge 1] \vee [\mathbb{U}_{Q2}(r_2) \wedge 1]$
$= [\mathbb{U}_{Q1}(r_1) \wedge \mathbb{U}_{Q2}(r_1)] \vee [\mathbb{U}_{Q1}(r_2) \wedge \mathbb{U}_{Q2}(r_2)]$
$= \mathbb{U}_{(\mathfrak{Q}_1+\mathfrak{Q}_2)}(r_1) \vee \mathbb{U}_{(\mathfrak{Q}_1+\mathfrak{Q}_2)}(r_2)$

$\mathbb{F}_{(\mathfrak{Q}_{1i}+\mathfrak{Q}_{2i})}(r_1r_2) = \mathbb{F}_{Q1i}(r_1r_2) \wedge \mathbb{F}_{Q2i}(r_1r_2)$
$= \mathbb{F}_{Q2i}(r_1r_2) \vee 1$
$\leq [\mathbb{F}_{Q2}(r_1) \vee \mathbb{F}_{Q2}(r_2)] \wedge 1$
$= [\mathbb{F}_{Q2}(r_1) \wedge 1] \vee [\mathbb{F}_{Q2}(r_2) \wedge 1]$
$= [\mathbb{F}_{Q1}(r_1) \wedge \mathbb{F}_{Q2}(r_1)] \vee [\mathbb{F}_{Q1}(r_2) \wedge \mathbb{F}_{Q2}(r_2)]$
$= \mathbb{F}_{(\mathfrak{Q}_1+\mathfrak{Q}_2)}(r_1) \vee \mathbb{F}_{(\mathfrak{Q}_1+\mathfrak{Q}_2)}(r_2)$

For $r_1r_2 \in S_1 + S_2$.

Csse3: $r_1 \in S_1, r_2 \in S_2$, then according to definition 3.42,

$\mathbb{T}_{(\mathfrak{Q}_{1i}+\mathfrak{Q}_{2i})}(r_1r_2) = \mathbb{T}_{Q1}(r_1) \wedge \mathbb{T}_{Q2}(r_2)$
$= [\mathbb{T}_{Q1}(r_1) \vee 0] \wedge [\mathbb{T}_{Q2}(r_2) \vee 0]$
$= [\mathbb{T}_{Q1}(r_1) \vee \mathbb{T}_{Q2}(r_1)] \wedge [\mathbb{T}_{Q2}(r_2) \vee \mathbb{T}_{Q1}(r_2)]$

$= \mathbb{T}_{(\mathfrak{Q}_1+\mathfrak{Q}_2)}(r_1) \wedge \mathbb{T}_{(\mathfrak{Q}_1+\mathfrak{Q}_2)}(r_2)$

$\mathbb{C}_{(\mathfrak{Q}_{1i}+\mathfrak{Q}_{2i})}(r_1r_2) = \mathbb{C}_{Q1}(r_1) \wedge \mathbb{C}_{Q2}(r_2)$
$= [\mathbb{C}_{Q1}(r_1) \vee 0] \wedge [\mathbb{C}_{Q2}(r_2) \vee 0]$
$= [\mathbb{C}_{Q1}(r_1) \vee \mathbb{C}_{Q2}(r_1)] \wedge [\mathbb{C}_{Q2}(r_2) \vee \mathbb{C}_{Q1}(r_2)]$
$= \mathbb{C}_{(\mathfrak{Q}_1+\mathfrak{Q}_2)}(r_1) \wedge \mathbb{C}_{(\mathfrak{Q}_1+\mathfrak{Q}_2)}(r_2)$

$\mathbb{U}_{(\mathfrak{Q}_{1i}+\mathfrak{Q}_{2i})}(r_1r_2) = \mathbb{U}_{Q1}(r_1) \vee \mathbb{U}_{Q2}(r_2)$
$= [\mathbb{U}_{Q1}(r_1) \wedge 0] \vee [\mathbb{U}_{Q2}(r_2) \wedge 0]$
$= [\mathbb{U}_{Q1}(r_1) \wedge \mathbb{U}_{Q2}(r_1)] \vee [\mathbb{U}_{Q2}(r_2) \wedge \mathbb{U}_{Q1}(r_2)]$
$= \mathbb{U}_{(\mathfrak{Q}_1+\mathfrak{Q}_2)}(r_1) \vee \mathbb{U}_{(\mathfrak{Q}_1+\mathfrak{Q}_2)}(r_2)$

$\mathbb{F}_{(\mathfrak{Q}_{1i}+\mathfrak{Q}_{2i})}(r_1r_2) = \mathbb{F}_{Q1}(r_1) \vee \mathbb{F}_{Q2}(r_2)$
$= [\mathbb{F}_{Q1}(r_1) \wedge 0] \vee [\mathbb{F}_{Q2}(r_2) \wedge 0]$
$= [\mathbb{F}_{Q1}(r_1) \wedge \mathbb{F}_{Q2}(r_1)] \vee [\mathbb{F}_{Q2}(r_2) \wedge \mathbb{F}_{Q1}(r_2)]$
$= \mathbb{F}_{(\mathfrak{Q}_1+\mathfrak{Q}_2)}(r_1) \vee \mathbb{F}_{(\mathfrak{Q}_1+\mathfrak{Q}_2)}(r_2)$

For $r_1r_2 \in S_1 + S_2$. Hence proved.

Theorem 3.55 *Let $\mathfrak{G} = (S_1 + S_2, S_{11} + S_{21}, S_{12} + S_{22}, \dots. S_{1n} + S_{2n})$ to be join of two GSs $\mathfrak{G}_1 = (S_1, S_{11}, S_{12}, \dots, S_{1n})$ and $\mathfrak{G}_2 = (S_2, S_{21}, S_{22}, \dots, S_{2n})$. Then every strong QNGS $\mathfrak{G} = (\mathfrak{Q}, \mathfrak{Q}_1, \mathfrak{Q}_2, \dots. \mathfrak{Q}_n)$ of $\mathfrak{G}$ is join of two strong QNGs $\mathfrak{G}_{n1}$ and $\mathfrak{G}_{n2}$ of GS $\mathfrak{G}_1$ and $\mathfrak{G}_2$, respectively,*

Proof. First we define $\mathfrak{Q}_k$ and $\mathfrak{Q}_{ki}$ for $k = 1,2$ and $i = 1,2,\dots,n$ as:

$\mathbb{T}_{\mathfrak{Q}_k}(r) = \mathbb{T}_Q(r), \mathbb{C}_{\mathfrak{Q}_k}(r) = \mathbb{C}_Q(r), \mathbb{U}_{\mathfrak{Q}_k}(r) = \mathbb{U}_Q(r), \mathbb{F}_{\mathfrak{Q}_k}(r) = \mathbb{F}_Q(r), \quad \text{if} \quad r \in S_k$

$\mathbb{T}_{\mathfrak{Q}_{ki}}(r_1r_2) = \mathbb{T}_{Q_i}(r_1r_2), \mathbb{C}_{\mathfrak{Q}_{ki}}(r_1r_2) = \mathbb{C}_{Q_i}(r_1r_2), \mathbb{U}_{\mathfrak{Q}_{ki}}(r_1r_2) = \mathbb{U}_{Q_i}(r_1r_2), \mathbb{F}_{\mathfrak{Q}_{ki}}(r_1r_2) = \mathbb{F}_{Q_i}(r_1r_2), \quad \text{if} \quad r_1r_2 \in S_{ki}$

Now for $r_1r_2 \in S_{ki}, k = 1,2, i = 1,2,\dots,n$

$\mathbb{T}_{\mathfrak{Q}_{ki}}(r_1r_2) = \mathbb{T}_{Q_i}(r_1r_2) = \mathbb{T}_Q(r_1) \wedge \mathbb{T}_Q(r_2) = \mathbb{T}_{\mathfrak{Q}_k}(r_1) \wedge \mathbb{T}_{\mathfrak{Q}_k}(r_2)$
$\mathbb{C}_{\mathfrak{Q}_{ki}}(r_1r_2) = \mathbb{C}_{Q_i}(r_1r_2) = \mathbb{C}_Q(r_1) \wedge \mathbb{C}_Q(r_2) = \mathbb{C}_{\mathfrak{Q}_k}(r_1) \wedge \mathbb{C}_{\mathfrak{Q}_k}(r_2)$
$\mathbb{U}_{\mathfrak{Q}_{ki}}(r_1r_2) = \mathbb{U}_{Q_i}(r_1r_2) = \mathbb{U}_Q(r_1) \vee \mathbb{U}_Q(r_2) = \mathbb{U}_{\mathfrak{Q}_k}(r_1) \vee \mathbb{U}_{\mathfrak{Q}_k}(r_2)$
$\mathbb{F}_{\mathfrak{Q}_{ki}}(r_1r_2) = \mathbb{F}_{Q_i}(r_1r_2) = \mathbb{F}_Q(r_1) \vee \mathbb{F}_Q(r_2) = \mathbb{F}_{\mathfrak{Q}_k}(r_1) \vee \mathbb{F}_{\mathfrak{Q}_k}(r_2)$.

(i.e) $\mathfrak{G}_{nk} = (\mathfrak{Q}_k, \mathfrak{Q}_{k1}, \mathfrak{Q}_{k2}, \dots, \mathfrak{Q}_{kn})$ is a strong QNGS of $\mathfrak{G}_k, k = 1,2$.

Moreover, $\mathfrak{G}_n$ is join of $\mathfrak{G}_{n1}$ and $\mathfrak{G}_{n2}$ as shown: Using Definition 3.39 and 3.42, $Q = Q_1 \cup Q_2 = Q_1 + Q_2$ and $\mathfrak{Q}_i = \mathfrak{Q}_{1i} \cup \mathfrak{Q}_{2i} = \mathfrak{Q}_{1i} + \mathfrak{Q}_{2i}$, $\forall r_1r_2 \in S_{1i} \cap S_{2i}$.

when $r_1r_2 \in S_{1i} + S_{2i} \quad (S_{1i} \cup S_{2i})$, (i.e) $r_1 \in S_1$ and $r_2 \in S_2$

$\mathbb{T}_{\mathfrak{Q}_i}(r_1r_2) = \mathbb{T}_Q(r_1) \wedge \mathbb{T}_Q(r_2) = \mathbb{T}_{Qk}(r_1) \wedge \mathbb{T}_{Qk}(r_2) = \mathbb{T}_{\mathfrak{Q}_{1i}+\mathfrak{Q}_{2i}}(r_1r_2)$
$\mathbb{C}_{\mathfrak{Q}_i}(r_1r_2) = \mathbb{C}_Q(r_1) \wedge \mathbb{C}_Q(r_2) = \mathbb{C}_{Qk}(r_1) \wedge \mathbb{C}_{Qk}(r_2) = \mathbb{C}_{\mathfrak{Q}_{1i}+\mathfrak{Q}_{2i}}(r_1r_2)$
$\mathbb{U}_{\mathfrak{Q}_i}(r_1r_2) = \mathbb{U}_Q(r_1) \vee \mathbb{U}_Q(r_2) = \mathbb{U}_{Qk}(r_1) \vee \mathbb{U}_{Qk}(r_2) = \mathbb{U}_{\mathfrak{Q}_{1i}+\mathfrak{Q}_{2i}}(r_1r_2)$
$\mathbb{F}_{\mathfrak{Q}_i}(r_1r_2) = \mathbb{F}_Q(r_1) \vee \mathbb{F}_Q(r_2) = \mathbb{F}_{Qk}(r_1) \vee \mathbb{F}_{Qk}(r_2) = \mathbb{F}_{\mathfrak{Q}_{1i}+\mathfrak{Q}_{2i}}(r_1r_2)$

Calculation are similar when $r_1 \in S_2$, $r_2 \in S_1$. It is true when $i = 1,2,\dots,n$. Complete the proof.

4. Conclusions

In this work, the concept of quadripartitioned neutrosophic graph structure and its properties have been discussed. The strong, tree, $\phi -$permutation and $\phi -$complement of quadripartitioned neutrosophic graph structure have been studied. The operations like Cartesian Product, cross product, lexicographic product, composition in graph structures and join operations are established. In future, the authors will extend this proposed concept to some applications in decision making and bipolar environment. Wiener index of QNGSs will be studied based on [21, 22]. The proposed concepts are also extended to bipolar QNGSs, interval QNGSs, single valued neutrosophic quadripartitioned hypergraphs and in soft QNGSs.

Funding: This research received no external funding

Conflicts of Interest: The authors declare no conflict of interest.

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Received: July 5, 2022. Accepted: September 20, 2022.