The neutrosophic vector spaces- another approach

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Abstract. In this paper, we introduce and define a new version of a neutrosophic vector space. Indeed, this approach is a generalization of the notion of fuzzy vector space. Also, we study the neutrosophic subspace and linear independence. Furthermore, this study builds the basis and dimension of a neutrosophic vector space. Finally, we investigate the properties of the introduced notations.

Keywords: Neutrosophic vector space, basis, dimension.

1. Introduction

After Zadeh [16] introduced fuzzy sets, this fundamental concept has been generalized for a variety of purposes. Atanassov [4,5] first proposed the concept of intuitionistic fuzzy sets (IFSs) in 1986. The notion of a neutrosophic set (NS) was introduced by Smarandache [14,15]. As a generalization of the fuzzy and intuitionistic sets, the theory of neutrosophic sets is expected to play an important role in modern mathematics in general. Since 2005, the concept of the neutrosophic set has gotten a lot of attention [7, 9, 11–13], and it has a lot of applications [1, 2, 6, 10].

Additionally, a number of works have been published by researchers to extend the classical and fuzzy mathematical notions to the context of neutrosophic fuzzy mathematics. The difficulty in such generalizations lies in how to choose the most rational generalization among many available approaches. The concept of a neutrosophic vector space was introduced in [3]. In this study, we provide a new definition of the neutrosophic vector space, which represents the rational generalization of the fuzzy vector space. Also, we study the purely algebraic properties of neutrosophic vector spaces. Furthermore, we establish the idea of a neutrosophic basis and show that it exists in a large class of neutrosophic vector spaces.

The work is conceived as follows. In Section 2, some basic concepts in our study are recorded. Section 3 is devoted to introducing the new definition for a neutrosophic vector space as an extension of fuzzy vector space. Also, we introduced and studied some of the concepts. The conclusion remarks are reached in section 4.

In order to clarify the picture, we present here the following standard notation. V will denote a vector space over any field $\mathbb{K}$.

2. Basic concepts

In this section, we will go over certain definitions and outcomes that will be used in the next section.

Definition 2.1. [8] Let V be a vector space over $\mathbb{K}$. A fuzzy vector space is the pair $\overline{V} = (V, \mu)$ with the property that $\forall a, b \in R$ and $u, v \in V$, we have

$$\mu(au + bv) > \mu(u) \wedge \mu(v),$$

where $\mu : V \to [0, 1]$.

Definition 2.2. [15] Let N be the universe set. A neutrosophic set $\mathcal{N}$ on N (NS $\mathcal{N}$) is defined as:

$$\mathcal{N} = \{< a, \mu(a), \gamma(a), \zeta(a) > | a \in N\}.$$

where $\mu, \gamma, \zeta : N \to [0, 1]$.

Definition 2.3. [8] A set A is said to be upper well ordered if for all non-empty subsets $B \subset A$, $\sup B \in B$

Definition 2.4. [8] A subset $A \subset [0, 1]$ is said to have an increasing monotonic limit $x \in [0, 1]$ if and only if x is a limit of a monotonically increasing sequence in A.

Proposition 2.5. *[8] A set $A \subset [0, 1]$ is without any increasing monotonic limits iff it is upper well ordered.*

Proposition 2.6. *[8] All upper well ordered subsets of $[0, 1]$ are countable.*

3. Main result

In this section, we present a new definition of a neutrosophic vector space and give examples. Also, we derive some properties concerning this definition. In addition, we define neutrosophic linear independence and investigate certain properties. Lastly, the neutrosophic basis and dimension are defined and studied.

Definition 3.1. Neutrosohic vector space is a quaternary $\overline{V} = (V, \mu, \gamma, \zeta)$ where V is a vector space over arbitrary field $\mathbb{K}$ with

$$\mu : V \to [0,1],$$
$$\gamma : V \to [0,1],$$
$$\zeta : V \to [0,1],$$

with the following properties

$$\mu(au+bv) \geq \mu(u) \wedge \mu(v),$$
$$\gamma(au+bv) \leq \gamma(u) \vee \gamma(v),$$
$$\zeta(au+bv) \leq \zeta(u) \vee \zeta(v),$$

where $u, v \in V$ and $a, b \in \mathbb{K}$.

Example 3.2. Let $\mathbb{R}^2$ be a vector space over a field $\mathbb{R}$, then $\overline{V} = (\mathbb{R}^2, \mu, \gamma, \zeta)$ is a neutrosophic vector space over a field $\mathbb{R}$, where

$$\mu(s,t) = \begin{cases} 1 & \text{if } s = t = 0 \\ \frac{1}{2} & \text{if } s = 0, t \in \mathbb{R} - \{0\} \\ \frac{1}{4} & \text{if } s \in \mathbb{R}, t = 0 \end{cases}$$

$$\gamma(s,t) = \begin{cases} \frac{1}{4} & \text{if } s = t = 0 \\ \frac{1}{2} & \text{if } s = 0, t \in \mathbb{R} - \{0\} \\ \frac{3}{4} & \text{if } s \in \mathbb{R}, t = 0 \end{cases}$$

$$\zeta(s,t) = \begin{cases} \frac{1}{3} & \text{if } s = t = 0 \\ \frac{1}{2} & \text{if } s = 0, t \in \mathbb{R} - \{0\} \\ \frac{5}{6} & \text{if } s \in \mathbb{R}, t = 0 \end{cases}$$

Proposition 3.3. *If $\overline{V} = (V, \mu, \gamma, \zeta)$ is a neutrosophic vector space over a field $\mathbb{K}$, then*

(i) $\mu(au) = \mu(u)$, $\forall a \in \mathbb{K} - \{0\}$.

(ii) $\gamma(au) = \gamma(u)$, $\forall a \in \mathbb{K} - \{0\}$.

(iii) $\zeta(au) = \zeta(u)$, $\forall a \in \mathbb{K} - \{0\}$.

(iv) If $u, v \in V$ and $\mu(u) > \mu(v)$, then $\mu(u+v) = \mu(v)$.

(v) If $u, v \in V$ and $\gamma(u) < \gamma(v)$, then $\gamma(u+v) = \gamma(v)$.

(vi) If $u, v \in V$ and $\zeta(u) < \zeta(v)$, then $\zeta(u+v) = \zeta(v)$.

Proof. We prove only (v) since the remainder are well-known.
Since $\gamma(u) < \gamma(v)$ we have $\gamma(v) \leq \gamma(u+v)$. Also, $\gamma[(u+v)-v] = \gamma(u) \leq \gamma(u+v) \vee \gamma(v)$.
Since $\gamma(u) < \gamma(v)$ we have $\gamma(u+v) \leq \gamma(u)$. Consequently $\gamma(u+v) = \gamma(v)$. □

Proposition 3.4. *Let $\overline{V} = (V, \mu, \gamma, \zeta)$ be a neutrosophic vector space over $\mathbb{K}$ with $\mu(u) \neq \mu(v)$, $\gamma(u) \neq \gamma(v)$ and $\zeta(u) \neq \zeta(v)$, then*

$$\mu(u+v) = \mu(u) \wedge \mu(v),$$

$$\gamma(u+v) = \gamma(u) \vee \gamma(v),$$

$$\zeta(u+v) = \zeta(u) \vee \zeta(v),$$

where $u, v \in V$.

Proof. It is clear from Proposition 3.3. □

Proposition 3.5. *Let* $\overline{V} = (V, \mu, \gamma, \zeta)$ *be a neutrosophic vector space over a field* $\mathbb{K}$*, then*

(i) $\mu(0) = \sup_{u \in V} \mu(u) = \sup[\mu(V)]$.

(ii) $\gamma(0) = \inf_{u \in V} \gamma(u) = \inf[\gamma(V)]$.

(iii) $\zeta(0) = \inf_{u \in V} \zeta(u) = \inf[\zeta(V)]$.

Proof. We prove only (ii) since (i) and (iii) are well-known.
Let $u \in V$, then $\gamma(0) = \gamma(0u) \leq \gamma(u)$. □

Definition 3.6. Let W be a subspace of a vector space V. Then $(W, \mu_W, \gamma_W, \zeta_W)$ is called neutrosophic subspace of a neutrosophic vector (V, μ, γ, ζ) if the following conditions are satisfied:

(i) $\mu_W(x-y) \geq \mu_W(x) \wedge \mu_W(y)$.

(ii) $\mu_W(cx) = \mu_W(x)$.

(iii) $\gamma_W(x-y) \leq \gamma_W(x) \vee \gamma_W(y)$.

(iv) $\gamma_W(cx) = \gamma_W(x)$.

(v) $\zeta_W(x-y) \leq \zeta_W(x) \vee \zeta_W(y)$.

(vi) $\zeta_W(cx) = \zeta_W(x)$.

Definition 3.7. Let $\overline{V}_1 = (V, \mu_1, \gamma_1, \zeta_1)$ and $\overline{V}_2 = (V, \mu_2, \gamma_2, \zeta_2)$ be two neutrosophic vector spaces over $\mathbb{K}$, then

(i) The intersection of $\overline{V}_1$ and $\overline{V}_2$ define as follows: $\overline{V}_1 \cap \overline{V}_2 = (V, \mu_1 \wedge \mu_2, \gamma_1 \vee \gamma_2, \zeta_1 \vee \zeta_2)$

(ii) The sum of $\overline{V}_1$ and $\overline{V}_2$ define as follows: $\overline{V}_1 + \overline{V}_2 = (V, \mu_1 + \mu_2, \gamma_1 + \gamma_2, \zeta_1 + \zeta_2)$,
where $(\mu_1 + \mu_2)(a) = sup\{\mu_1(a) \wedge \mu_2(a-v)\}$, $(\gamma_1 + \gamma_2)(a) = inf\{\gamma_1(a) \vee \gamma_2(a-v)\}$, $(\zeta_1 + \zeta_2)(a) = inf\{\zeta_1(a) \vee \zeta_2(a-v)\}$ and $a = u + v$.

Proposition 3.8. *Let* $\overline{W}_i = (V, \mu_i, \gamma_i, \zeta_i)$ *be a set of family neutrosophic subspaces over a field* $\mathbb{K}$ *with* $i \in I = 1, 2, ..., n$*, then* $\cap_{i \in I} \overline{W}_i$ *is a neutrosophic vector space over a field* $\mathbb{K}$.

Proposition 3.9. *Let* $\overline{W}_i = (V, \mu_i, \gamma_i, \zeta_i)$ *be a set of family neutrosophic subspaces over a field* $\mathbb{K}$ *with* $i \in I = 1, 2, ..., n$*, then* $\sum_{i=1}^{n} \overline{W}_i$ *is a neutrosophic vector space over a field* $\mathbb{K}$.

Proof. We use contradiction to prove this result. Firstly, assume that

$$(\mu_1 + \mu_2 + ... + \mu_n)(x+y) < (\mu_1 + \mu_2 + ... + \mu_n)(x) \wedge (\mu_1 + \mu_2 + ... + \mu_n)(y).$$

Thus, there exists $u_1+u_2+...+u_{n+1}$ and $v_1+v_2+...+v_{n+1}$ such that for all $z_1+z_2+...+z_{n+1}$ we have

$$\mu_1(z_1)\wedge\mu_2(z_2)\wedge...\wedge\mu_n(x+y-z_1-z_2-...-z_n)<[\mu_1(u_1)\wedge\mu_2(u_2)\wedge...\wedge\mu_n(x-u_1-u_2-...-u_n)]$$
$$\wedge[\mu_1(v_1)\wedge\mu_2(v_2)\wedge...\wedge\mu_n(y-v_1-v_2-...-v_n)]\rightarrow(\star)$$

but

$$[\mu_1(u_1)\wedge\mu_2(u_2)\wedge...\wedge\mu_n(x-u_1-u_2-...-u_n)]\wedge[\mu_1(v_1)\wedge\mu_2(v_2)\wedge...\wedge\mu_n(y-v_1-v_2-...-v_n)]=$$
$$\mu_1(u_1)\wedge\mu_1(v_1)\wedge\mu_2(u_2)\wedge\mu_2(v_2)\wedge...\wedge\mu_n(x-u_1-u_2-...-u_n)]\wedge\mu_n(y-v_1-v_2-...-v_n)\leq$$
$$\mu_1(u_1\wedge v_1)\wedge\mu_1(u_2\wedge v_2)\wedge....\wedge\mu_n(x+y-u_1-v_1-u_2-v_2...-u_n-v_n).$$

Therefore, there exists $z_i=u_i+v_i$ for which $(\star)$ is false, this we have a contradiction and therefore $(\mu_1+\mu_2+...+\mu_n)(x+y)\geq(\mu_1+\mu_2+...+\mu_n)(x)\wedge(\mu_1+\mu_2+...+\mu_n)(y)$.

Secondly, suppose that

$$(\gamma_1+\gamma_2+...+\gamma_n)(x+y)>(\gamma_1+\gamma_2+...+\gamma_n)(x)\vee(\gamma_1+\gamma_2+...+\gamma_n)(y).$$

Thus, there exists $u_1+u_2+...+u_{n+1}$ and $v_1+v_2+...+v_{n+1}$ such that for all $z_1+z_2+...+z_{n+1}$ we have

$$\gamma_1(z_1)\vee\ \gamma_2(z_2)\vee...\vee\gamma_n(x+y-z_1-z_2-...-z_n)>[\gamma_1(u_1)\vee\gamma_2(u_2)\vee...\vee\gamma_n(x-u_1-u_2-...-u_n)]$$
$$\vee[\gamma_1(v_1)\vee\gamma_2(v_2)\vee...\vee\gamma_n(y-v_1-v_2-...-v_n)]\rightarrow(\star\star)$$

but

$$[\gamma_1(u_1)\vee\gamma_2(u_2)\vee...\vee\gamma_n(x-u_1-u_2-...-u_n)]\vee[\gamma_1(v_1)\vee\gamma_2(v_2)\vee...\vee\gamma_n(y-v_1-v_2-...-v_n)]=$$
$$\gamma_1(u_1)\vee\gamma_1(v_1)\vee\gamma_2(u_2)\vee\gamma_2(v_2)\vee...\vee\gamma_n(x-u_1-u_2-...-u_n)]\vee\gamma_n(y-v_1-v_2-...-v_n)\geq$$
$$\gamma_1(u_1\vee v_1)\vee\gamma_1(u_2\vee v_2)\vee....\vee\gamma_n(x+y-u_1-v_1-u_2-v_2...-u_n-v_n).$$

Therefore, there exists $z_i=u_i+v_i$ for which $(\star\star)$ is false, this we have a contradiction and therefore $(\gamma_1+\gamma_2+...+\gamma_n)(x+y)\leq(\gamma_1+\gamma_2+...+\gamma_n)(x)\vee(\gamma_1+\gamma_2+...+\gamma_n)(y)$. Similarly, we find $(\gamma_1+\gamma_2+...+\gamma_n)(x+y)\leq(\gamma_1+\gamma_2+...+\gamma_n)(x)\vee(\gamma_1+\gamma_2+...+\gamma_n)(y)$. □

Now, we proceed to characterize the neutrosophic linear independence.

3.1. *Neutrosophic linear independence*

Definition 3.10. Let $\overline{V}=(V,\mu,\gamma,\zeta)$ be a neutrosophic vector space over a field $\mathbb{K}$. We say that a finite set of vectors $\{u_i\}_{i=1}^n$ is a neutrosophic linear independence in $\overline{V}$ iff $\{u_i\}_{i=1}^n$ is linear independence in V and $\forall\ \{a_i\}_{i=1}^n\subset\mathbb{K}$,

$$\mu(\sum_{i=1}^n a_iu_i)=\bigwedge_{i=1}^n\mu(a_iu_i),$$
$$\gamma(\sum_{i=1}^n a_iu_i)=\bigvee_{i=1}^n\gamma(a_iu_i),$$
$$\zeta(\sum_{i=1}^n a_iu_i)=\bigvee_{i=1}^n\zeta(a_iu_i).$$

A set of vectors is neutrosophic linearly independent in $\overline{V}$ if all finite subsets of it are neutrosophic linearly independent in $\overline{V}$.

Example 3.11. Let $\overline{V}$ be a neutrosophic vector space which define in Example 3.20. The set of vectors $\{x = (2, 0), y = (-2, 1)\}$ is linearly independent. Also, it easy to checked

$$\mu(x) = \mu(y) \text{ and } \mu(x + y) > \mu(x),$$

$$\gamma(x) = \gamma(y) \text{ and } \gamma(x + y) < \gamma(x),$$

$$\zeta(x) = \zeta(y) \text{ and } \zeta(x + y) < \zeta(x).$$

This set is not neutrosophic linearly independent in $\overline{V}$.

Proposition 3.12. *If $\overline{V} = (V, \mu, \gamma, \zeta)$ is a neutrosophic vector space over a field $\mathbb{K}$, then any set of vectors $\{x_i\}_{i=1}^n \subset V - \{0\}$ which has distinct μ, γ, ζ-values is linearly and neutrosophic linearly independent.*

Proof. We use mathematical induction to prove this proposition. By [8], μ-values are linearly and neutrosophic linearly independent. We now show that γ-values are both linearly and neutrosophic linearly independent. In the case $n = 1$ we find the statement is true. Also, suppose that the statement is true for n. Assume that $\{x_i\}_{i=1}^{n+1}$ is a set of vectors in $V \backslash \{0\}$ with distinct γ-values. According to the inductive hypothesis we have $\{x_i\}_{i=1}^n$ is neutrosophic linearly independent. Suppose that $\{x_i\}_{i=1}^{n+1}$ is not linearly independent and thus $x_{n+1} = \sum_{i \in S} a_i x_i$ where $S \subset \{1, \ldots, n\}, S \neq \emptyset$ and for all $i \in S, a_i \neq 0$. Then

$$\gamma(x_{n+1}) = \bigvee_{\epsilon} \gamma(a_i x_i) = \bigvee_{\epsilon} \gamma(x_i)$$

and hence $\gamma(x_{n+1}) \in \{\gamma(x_i)\}_{i=1}^n$. This contradicts the fact that $\{x_i\}_{i=1}^{n+1}$ has distinct γ-values. Therefore $\{x_i\}_{i=1}^{n+1}$ is linearly independent. Finally Propositions 3.3 (ii), 3.4 and 3.5 (ii) clearly show that $\{x_i\}_{i=1}^{n+1}$ is neutrosophic linearly independent. □

We conclude this section by providing definitions of the neutrosophic base and dimension and looking at some properties.

3.2. *Neutrosophic basis and dimension*

Definition 3.13. The linearly independent basis for $\overline{V}$ is the neutrosophic basis of a neutrosophic vector space $\overline{V} = (V, \mu, \gamma, \zeta)$ over a field $\mathbb{K}$.

The following theorem illustrates how a neutrosophic basis may be used to create a large class of neutrosophic vector spaces.

Theorem 3.14. *Let V be a vector space with basis $B = \{v_\aleph\}_{\aleph \in I}$, constants $\mu_0, \gamma_0, \zeta_0 \in (0,1]$ and any sets of constants $\{\mu_\aleph\}_{\aleph\in I}, \{\gamma_\aleph\}_{\aleph\in I}, \{\zeta_\aleph\}_{\aleph\in I} \subset (0,1]$ such that $\mu_0 \geq \mu_\aleph, \gamma_0 \geq \gamma_\aleph, \zeta_0 \geq \zeta_\aleph \;\forall\; \aleph \in I$. Define*

$$\mu(u) = \bigwedge_{i=1}^{N} \mu(v_{\aleph_i}) = \bigwedge_{i=1}^{N} \mu_{\aleph_i} \textit{ and } \mu(0) > \mu_0,$$

$$\gamma(u) = \bigvee_{i=1}^{N} \gamma(v_{\aleph_i}) = \bigvee_{i=1}^{N} \gamma_{\aleph_i} \textit{ and } \gamma(0) > \gamma_0,$$

$$\zeta(u) = \bigvee_{i=1}^{N} \zeta(v_{\aleph_i}) = \bigvee_{i=1}^{N} \zeta_{\aleph_i} \textit{ and } \zeta(0) > \zeta_0,$$

where $u \in V$ with $u = \sum_{i=1}^{N} c_i v_{\aleph_i}$ and μ, γ, ζ is well-defined. We assert that $\overline{V} = (V, \mu, \gamma, \zeta)$ neutrosophic vector space with basis B.

Proof. Let $u, w \in V - \{0\}$, then we can write $u,\ w$ in a unique formula

$$u = \sum_{i \in C \cup D_u} c_i v_{\aleph_i},$$

$$w = \sum_{i \in C \cup D_w} d_i v_{\aleph_i},$$

where $C \cap D_u = \phi$, $C \cap D_w = \phi$ and $c_i,\ d_i \in \mathbb{R} - \{0\}$. Suppose that $a,\ b \in \mathbb{R} - \{0\}$ and $au + bw \neq 0$. Let $Z = \{i \in C : ac_i + bd_i = 0\}$ and $N = C - \{Z\}$. Now, the proof boils down to showing that

$$\mu(au + bw) \geq \mu(u) \wedge \mu(w), \tag{3.21}$$

$$\gamma(au + bw) \leq \gamma(u) \vee \gamma(w), \tag{3.22}$$

$$\zeta(au + bw) \leq \zeta(u) \vee \zeta(w), \tag{3.23}$$

we prove only 3.23 and 3.23, since 3.21 see [8]. Now,

$$\begin{aligned}\gamma(au + bw) &= \gamma(\sum_{i \in C}(ac_i + bd_i)v_i + \sum_{i \in D_u} ac_i v_i + \sum_{i \in D_w} bd_i v_i) \\ &= \gamma(\sum_{i}^{N}(ac_i + bd_i)v_i + \sum_{i \in D_u} ac_i v_i + \sum_{i \in D_w} bd_i v_i)\end{aligned}$$

All of the coefficients according to the above linear combination are non-zero, and thus by definition of γ we have

$$\begin{aligned}\gamma(au+bw) &= \left(\bigvee_{i\in N}\gamma\left(v_{\aleph_i}\right)\right)\vee\left(\bigvee_{i\in D_u}\gamma\left(v_{\aleph_i}\right)\right)\vee\left(\bigvee_{i\in D_w}\gamma\left(v_{\aleph_i}\right)\right)\\ &= \left(\bigvee_{i\in N}\gamma_{\aleph_i}\right)\vee\left(\bigvee_{i\in D_u}\gamma_{\aleph_i}\right)\vee\left(\bigvee_{i\in D_w}\gamma_{\aleph_i}\right)\\ &= \bigvee_{i\in N\cup D_u\cup D_w}\gamma_{\aleph_i}\\ &\leq \bigvee_{i\in C\cup D_u\cup D_w}\gamma_{\aleph_i} = \left(\bigvee_{i\in C\cup D_u}\gamma_{\aleph_i}\right)\vee\left(\bigvee_{i\in C\cup D_w}\gamma_{\aleph_i}\right)\end{aligned}$$

Therefore if $a, b \neq 0$ and $au+bw \neq 0$ then $\gamma(au+bw) \leq \gamma(u)\vee\gamma(w)$. In case you do $au+bw = 0$, since $\gamma(0) = \mu_0 \leq \inf\gamma(B)$ we must have $\gamma(au+bw) = \gamma(0) \leq \gamma(u)\vee\gamma(w)$.

Without giving up generality, in the case where a or b is zero we may say $a = 0$, then $\gamma(0u+bw) = \gamma(bw) \leq \gamma(u)\vee\gamma(bw) \leq \gamma(u)\vee\gamma(w)$. □

Lemma 3.15. *Let $\overline{V} = (V, \mu, \gamma, \zeta)$ be a neutrosophic vector space over a field $\mathbb{K}$ with $\mu(V), \gamma(V), \zeta(V)$ are upper well ordered and let U be a proper subspace of V, then $\exists\, u \in V/U$ such that*

$$\mu(u+v) = \mu(u)\wedge\mu(v)$$
$$\gamma(u+v) = \gamma(u)\vee\gamma(v)$$
$$\zeta(u+v) = \zeta(u)\vee\zeta(v)$$

where $v \in V$.

Proof. We only prove $\gamma(u+v) = \gamma(u)\vee\gamma(v)$. Since $\gamma(V)$ is upper well ordered we can find $u \in V/U$ such that $\gamma(u) = inf[\gamma(V/U)]$. Now, if $\gamma(u) \neq \gamma(v)$ then $\gamma(u)\vee\gamma(v) = \gamma(u+v)$ by Proposition 3.4. If $\gamma(u) = \gamma(v)$ then $\gamma(u+v) \leq \gamma(u)\vee\gamma(v)$. Also, since $u+v \in V/U$ and $\gamma(u) = inf[\gamma(V/U)]$ we have $\gamma(u+v) \geq \gamma(u)\vee\gamma(v)$ and thus $\gamma(u)\vee\gamma(v) = \gamma(u+v)$. □

Lemma 3.16. *Let $\overline{V} = (V, \mu, \gamma, \zeta)$ be a neutrosophic vector space over a field $\mathbb{K}$ with $\mu(V), \gamma(V), \zeta(V)$ are upper well ordered and let U be a proper subspace of V. Assume that B is a neutrosophic basis for U, then there exists $w \in V\backslash U$ such that $B^* = B\cup\{w\}$ is a neutrosophic basis for $\overline{W} = (W = \langle B^*\rangle, \mu_W, \gamma_W, \zeta_W)$, where $\langle B^*\rangle$ is the vector space spanned by B^*.*

Proof. Suppose that $w \in V\backslash U$ such that $\mu(w) = \sup[\mu(V\backslash U)]$, $\gamma(w) = \inf[\gamma(V\backslash U)]$, and $\zeta(w) = \inf[\zeta(V\backslash U)]$, then by Lemma 3.15, we have w is neutrosophic linearly independent from B. Assume that $B^* = B \cup \{w\}$. Obvious B^* is a neutrosophic basis for $\overline{W} = (W = \langle B^* \rangle, \mu_W, \gamma_W, \zeta_W)$. □

Theorem 3.17. *Let $\overline{V} = (V, \mu, \gamma, \zeta)$ be a neutrosophic vector space over $\mathbb{K}$ which $\mu(V), \gamma(V), \zeta(V)$ are upper well ordered, then $\overline{V}$ has a neutrosophic basis.*

Proof. Suppose that $\overline{V} = (V, \mu, \gamma, \zeta)$ is a neutrosophic vector space over $\mathbb{K}$ which $\mu(V), \gamma(V), \zeta(V)$ are upper well ordered. Let $\vartheta = \{B \subset V \mid B$ is neutrosophic linearly independent $\}$. We find ϑ is partial order by set inclusion. Assume that χ is a totally ordered subset of ϑ and let $A = \bigcup_{B\in\chi} B$. Obviously, A is an upper bound for C. Assume $a_1, \ldots, a_n \in A$. Then there exist $B_{\alpha(1)}, \ldots, B_{\alpha(n)} \in \chi$ such that $a_i \in B_{\alpha(i)}$. Since χ is totally ordered, one of the sets, say $B_{\alpha(k)}$, is a super set of the others. Hence $a_1, \ldots, a_n \in B_{\alpha(k)}$. Since $B_{\alpha(k)}$ is neutrosophic linearly independent $a_1, \ldots, a_n$ are neutrosophic linearly independent. Thus A is upper bound of χ in ϑ. By Zorn's Lemma, there exists a maximal element B^* in ϑ. Suppose that $\langle B^* \rangle = U$ is a proper subspace of V then by Lemma 3.15, there exists $w \in V\backslash U$ such that $B^* \cup \{w\} = B^+$ is a neutrosophic basis for $\overline{W} = (W, \mu_W, \gamma_W, \zeta_W)$. This contradicts the fact that B^* is a maximal element in ϑ. Thus we must have $\langle B^* \rangle = V$ and B^* is a neutrosophic basis for V. □

Corollary 3.18. *Let $\overline{V} = (V, \mu, \gamma, \zeta)$ be a neutrosophic vector space over a field $\mathbb{K}$ with V finite dimensional, then $\overline{V}$ has a neutrosophic basis.*

Proof. Suppose that $\mu(V)$, $\gamma(V)$ and $\zeta(V)$ are finite and therefore upper well ordered since V is finite dimensional. Thus, $\overline{V}$ has a neutrosophic basis, according to Theorem 3.17. □

In what follows, we define the dimension of neutrosophic vector spaces.

Definition 3.19. Let $\overline{V} = (V, \mu, \gamma, \zeta)$ be a neutrosophic vector space over a field $\mathbb{K}$, then we define the dimension of a neutrosophic space to be

$$dim(\overline{V}) = (dim\ V, \sup_{x \text{ a base for V}} (\sum_{v\in x} \mu(v)), \inf_{x \text{ a base for V}} (\sum_{v\in x} \gamma(v)), \inf_{x \text{ a base for V}} (\sum_{v\in x} \zeta(v))).$$

There is no doubt that the dimension is a function from the class of all neutrosophic vector spaces to $[0, \infty[$. Only when $dim(\overline{V}) = e < \infty$ does a neutrosophic vector space have a finite dimension.

Example 3.20. Let $\mathbb{R}^2$ be a vector space over a field $\mathbb{R}$. It is easily checked that $\overline{V} = (\mathbb{R}^2, \mu, \gamma, \zeta)$ is a neutrosophic vector space over a field $\mathbb{R}$, where

$$\mu(s,t) = \begin{cases} 1 & \text{if } s = t = 0 \\ \frac{1}{3} & \text{if } s = t,\ s \in \mathbb{R} - \{0\} \\ \frac{1}{5} & \text{otherwise} \end{cases}$$

$$\gamma(s,t) = \begin{cases} \frac{1}{15} & \text{if } s = t = 0 \\ \frac{1}{6} & \text{if } s = t,\ s \in \mathbb{R} - \{0\} \\ \frac{1}{3} & \text{other wise} \end{cases}$$

$$\zeta(s,t) = \begin{cases} \frac{1}{9} & \text{if } s = t = 0 \\ \frac{1}{2} & \text{if } s = t,\ s \in \mathbb{R} - \{0\} \\ \frac{1}{3} & \text{otherwise} \end{cases}$$

It is also easy to check that $dim(\mathbb{R}^2, \mu, \gamma, \zeta) = (2, \frac{8}{15}, \frac{3}{6}, \frac{5}{6})$.

4. Conclusions

Recently, it is important and applicable to study neutrosophic sets in the mathematical branch. In this paper, the author has made redefined the concept of neutrosophic vector space as an extension of the definition of fuzzy vector space. Furthermore, this definition was studied in order to define and study linear independence, basis, and dimension. The dimension of a class of neutrosophic vector space will be taken up by the author for future research. As a fuzzy vector space, we couldnt find an example of a neutrosophic vector space without a neutrosophic basis or prove that all neutrosophic vector spaces have one. It is, in my opinion, a difficult problem, and this is an open problem for the next research. However, there is a simple condition that a neutrosophic vector space must satisfy in order to have a neutrosophic basis.

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Competing interests

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