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An Application of Pentagonal Neutrosophic Linear Programming for Stock Portfolio Optimization

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Abstract: The Linear programming problems (LPP) have been widely applied to many real-world problems. In this study, a formulation of stock portfolio problem is proposed. The problem is formulated by involving neutrosophic pentagonal fuzzy numbers (NPFN) in the rate of risked return, expected return rate and portfolio risk amount. Based on score function, the problem is transformed to its corresponding crisp form. A solution algorithm is investigated to provide the decision of the portfolio investment joined with investors in savings and securities. The main features of this study are: the investor can choose freely the risk coefficients to maximize the expected returns; also, the investors may determine their strategies under consideration of their own conditions. The optimal return rate is obtained by using TORA software. An example is introduced to indicate the efficiency and reliability of the technique.

Keywords: Portfolio; Investment: Stock Portfolio Investment; Pentagonal Fuzzy Numbers; Score Function, TORA Software; Neutrosophic Pentagonal Fuzzy Return Rate.

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1. Introduction

Portfolio optimization is one of the essential problems in asset management of financial, its main goal is to minimize the risk of an investment by dividing it into many assets expected to fluctuate independently (Elton et al., 2009). A portfolio is a set of financial assets like cash equivalents, stocks, commodities, currencies and bonds. Portfolio can also include non-publicly tradable securities as, arts, private investment and real estate. Portfolio are directly held by investors and/ or managed by money managers and financial professionals [1]. Skrinjaric and Sego [2] applied Grey Relational Analysis (GRA) method to study the performance for a sample of stocks under various factors.

Fuzzy set theory initiated by Zadeh [3] has gained a great attention of researchers to solve real-life issues, like the supervision of economic threat. It permits us to illustrate and control vagueness in decision-support system. The imprecise facts of assets reports and the vagueness associated with the behavior of monetary markets can also be considered by means of fuzzy quantities or constraints. Fuzzy numerical data may be described using the phenomena of fuzzy subsets of R, are fuzzy numbers. Dubois and Prade [4] used a fuzzification principle to extended algebraic operations on real numbers to fuzzy numbers (FN).

Portfolio selection (PS) is the problem where investor selects the optimal portfolio from a set of possible portfolios. Also, it focuses on the optimal investment of one's wealth for maximizing profitable return and minimizing risk control [5]. According to lack of clarity of the real-world applications, the exact return of each security cannot be predetermined. The theory of optimal portfolios has been developed by Markowitz [6],where he has firstly introduced the mean-variance models. The PS problem is typically a LPP when all return of securities is constants. Numerous studies for PS have been done in the last few decades such as [7 –15]. Many researchers studied stock price assessment, in [16] Lindberg introduced new parameterization of the drift rates to modify the n stock Black-choles model, and solved Markowitz' continuous time PS in this framework.

Neutrosophic set (NS) theory was introduced by Smarandache [17] it is a generalization of fuzzy set; each element of NS has a truth, indeterminacy and falsity membership function. So, NS can describe inaccurate and maladjusted information effectively. Neutrosophic linear programming (NLP) problem is a LP problem that contains at least one neutrosophic coefficient or parameter. The NLP problem is more efficient than regular LP problems due to imperfect data. Many researchers studied NLP problems; Hussein et al. [18] transformed the NLP problem into its corresponding crisp model. Abdel-Basset et al. [19] proposed a novel method for solving a fully NLP problem. Ahmed [20], developed a new method for solving LR- type NLP problems. Ahmad et al. [21] developed a method for solving bipolar single-valued NLP problem. In [22], Bera studies the applications of NLP in real life. Das and Dash [23], introduced a modified Solution for NLP Problems with Mixed Constraints. Thamaraiselvi and Santhi [24] presented a new method for optimizing a real-life transportation problem in neutrosophic environment.

The rest of the paper is outlined as follows:

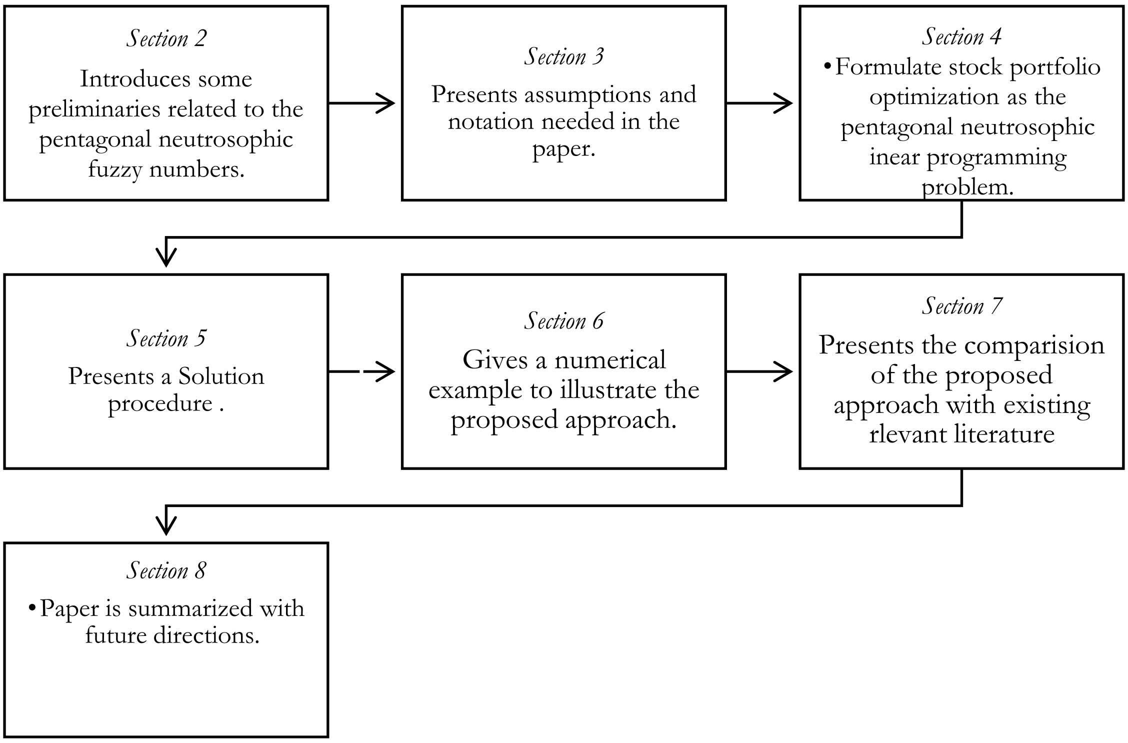

Fig.1. Rest of the paper

2. Preliminaries

In this section, some essential definitions and terminologies are recalled from fuzzy-like literature for proper understanding of the proposed work.

Definition 1. [3] A fuzzy set $\tilde{\wp}$ defined on the set of real numbers $\mathcal{R}$ is said to be fuzzy numbers when its membership function $\mu_{\tilde{\wp}}(x)$: $\mathcal{R} \to [0,1]$, have the following properties:

1. $\mu_{\tilde{\wp}}(x)$ is an upper semi- continuous membership function;
2. $\tilde{\wp}$ is convex fuzzy set, i.e., $\mu_{\tilde{\wp}}(\mathcal{F}x + (1-\mathcal{F})y) \geq \min\left\{\mu_{\tilde{\wp}}(x), \mu_{\tilde{\wp}}(y)\right\}$ for all $x, y \in \mathcal{R}; 0 \leq \mathcal{F} \leq 1$;
3. $\tilde{\wp}$ is normal, i.e., $\exists x_0 \in \mathcal{R}$ such that $\mu_{\tilde{\wp}}(x_0) = 1$;
4. Supp $(\tilde{\wp}) = \left\{x \in \mathcal{R}: \mu_{\tilde{\wp}}(x) > 0\right\}$ is the support of $\tilde{\wp}$, and the closure Cl (Supp$(\tilde{\wp})$) is a compact set.

Definition 2. [25]A fuzzy number $\widetilde{A}_{\wp} = (r, s, t, u, v), r \leq s \leq t \leq u \leq v$, on $\mathcal{R}$ is said to be a pentagonal fuzzy number if its membership function is:

$$\mu_{\tilde{A}_P} = \begin{cases} 0, & x < r, \\ w_1\left(\frac{x-r}{s-r}\right), & r \le x \le s, \\ 1-(1-w_1)\left(\frac{x-s}{t-s}\right), & s \le x \le t \\ 1, & x = t, \\ 1-(1-w_2)\left(\frac{u-x}{u-t}\right), & t \le x \le u, \\ w_2\left(\frac{v-x}{v-u}\right), & u \le x \le v, \\ 0, & x > v. \end{cases} \quad (1)$$

The graphical representation of the pentagonal fuzzy number is illustrated in the following figure

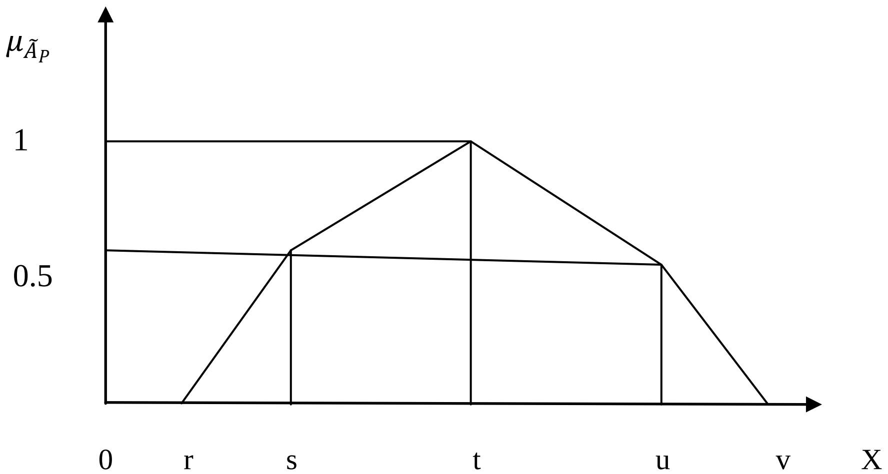

Fig.2. Graphical Representation of Pentagonal Fuzzy number [25]

Definition 3. [17] A neutrosophic set $\tilde{B}^N$ of non-empty set $\mathcal{X}$ is defined as

$\tilde{B}^N = \{\langle x;\ I_{\tilde{B}^N}(x), J_{\tilde{B}^N}(x), V_{\tilde{B}^N}(x)\rangle : x \in \mathcal{X}, I_{\tilde{B}^N}(x), J_{\tilde{B}^N}(x), V_{\tilde{B}^N}(x) \in]0_-, 1^+[\}$, where $I_{\tilde{B}^N}(x), J_{\tilde{B}^N}(x)$, and $V_{\tilde{B}^N}(x)$ are truth membership function, an indeterminacy- membership function, and a falsity-membership function and there is no restriction on the sum of $I_{\tilde{B}^N}(x), J_{\tilde{B}^N}(x)$, and $V_{\tilde{B}^N}(x)$, so $0^- \le \text{Sup}\{I_{\tilde{B}^N}(x)\} + \text{Sup}\{J_{\tilde{B}^N}(x)\} + \text{Sup}\{V_{\tilde{B}^N}(x)\} \le 3^+$, and $]0^-, 1^+[$ is a nonstandard unit interval.

Definition 4. [17] A single- valued neutrosophic set $\tilde{B}^{SVN}$ of a non-empty set $\mathcal{X}$ is defined as

$\bar{B}^{SVN} = \{\langle x, I_{\tilde{B}^N}(x), J_{\tilde{B}^N}(x), V_{\tilde{B}^N}(x)\rangle : x \in X\}$, where $I_{\tilde{B}^N}(x), J_{\tilde{B}^N}(x)$, and $V_{\tilde{B}^N}(x) \in [0,1]$ for each $x \in \mathcal{X}$ and $0 \le I_{\tilde{B}^N}(x) + J_{\tilde{B}^N}(x) + V_{\tilde{B}^N}(x) \le 3$.

Definition 5. [23] Let $\tau_{\tilde{p}}, \phi_{\tilde{p}}, \omega_{\tilde{p}} \in [0,1]$ and $r, s, t, u, v \in \mathbb{R}$ such that $r \le s \le t \le u \le v$. Then a single-valued pentagonal fuzzy neutrosophic set (SVPFN), $\tilde{p}^{PN} = \langle (r, s, t, u, v);\ \tau_{\tilde{p}}, \phi_{\tilde{p}}, \omega_{\tilde{p}}\rangle$ is a special neutrosophic set on $\mathcal{R}$, whose truth-membership, hesitant- membership, and falsity-membership functions are

$$\tau_{\tilde{p}^{PN}}(x) = \begin{cases} 0, & x < r; \\ \tau_{\tilde{p}^{PN}}\left(\frac{1}{2}\frac{1}{(s-r)^2}(x-r)^2\right), & r \le x \le s; \\ \tau_{\tilde{p}^{PN}}\left(\frac{1}{2}\frac{1}{(t-s)^2}(x-t)^2+1\right), & s \le x \le t; \\ \tau_{\tilde{p}^{PN}}\left(\frac{1}{2}\frac{1}{(u-t)^2}(x-t)^2+1\right), & t \le x \le u; \\ \tau_{\tilde{p}^{PN}}\left(\frac{1}{2}\frac{1}{(v-u)^2}(x-v)^2\right), & u \le x \le v; \\ 0, & x > v. \end{cases} \quad (2)$$

$$\phi_{\tilde{p}^{PN}} = \begin{cases} 0, & x < r; \\ \phi_{\tilde{p}^{PN}}\left(\frac{1}{2}\frac{1}{(s-r)^2}(x-r)^2\right), & r \le x \le s; \\ \phi_{\tilde{p}^{PN}}\left(\frac{1}{2}\frac{1}{(t-s)^2}(x-t)^2+1\right), & s \le x \le t; \\ \phi_{\tilde{p}^{PN}}\left(\frac{1}{2}\frac{1}{(u-t)^2}(x-t)^2+1\right), & t \le x \le u; \\ \phi_{\tilde{p}^{PN}}\left(\frac{1}{2}\frac{1}{(v-u)^2}(x-v)^2\right), & u \le x \le v; \\ 0, & x > v. \end{cases} \quad (3)$$

$$\omega_{\tilde{p}^{PN}} = \begin{cases} 0, & x < r; \\ \omega_{\tilde{p}^{PN}}\left(\frac{1}{2}\frac{1}{(s-r)^2}(x-r)^2\right), & r \le x \le s; \\ \omega_{\tilde{p}^{PN}}\left(\frac{1}{2}\frac{1}{(t-s)^2}(x-t)^2+1\right), & s \le x \le t; \\ \omega_{\tilde{p}^{PN}}\left(\frac{1}{2}\frac{1}{(u-t)^2}(x-t)^2+1\right), & t \le x \le u; \\ \omega_{\tilde{p}^{PN}}\left(\frac{1}{2}\frac{1}{(v-u)^2}(x-v)^2\right), & u \le x \le v; \\ 0, & x > v. \end{cases} \quad (4)$$

Where $\tau_{\tilde{p}^{PN}}, \phi_{\tilde{p}^{PN}}$, and $\omega_{\tilde{p}^{PN}}$ denote the maximum truth, minimum-hesitant, and minimum falsity membership degrees, respectively. SVPFN $\tilde{p}^{PN} = \langle (r, s, t, u, v) : \tau_{\tilde{p}^{PN}}, \phi_{\tilde{p}^{PN}}, \omega_{\tilde{p}^{PN}} \rangle$ may express in ill-defined quantity about p, which is approximately similar to $[s, u]$.

Definition 6. [25]

Let $\tilde{p}^{PN} = \langle (r, s, t, u, v);\ \tau_{\tilde{p}^{PN}}, \phi_{\tilde{p}^{PN}}, \omega_{\tilde{p}^{PN}} \rangle$ and $\tilde{q}^{PN} = \langle (r^*, s^*, t^*, u^*, v^*);\ \tau_{\tilde{q}^{PN}}, \phi_{\tilde{q}^{PN}}, \omega_{\tilde{q}^{PN}} \rangle$ be two single-valued PFNs, the arithmetic operations on $\tilde{p}^{PN}$ and $\tilde{q}^{PN}$ are:

1. $\tilde{p}^{PN} \oplus \tilde{q}^{PN} = \langle (r + r^*, s + s^*, t + t^*, u + u^*, v + v^*);\ \tau_{\tilde{p}^{PN}} \wedge \tau_{\tilde{q}^{PN}}, \phi_{\tilde{p}^{PN}} \vee \phi_{\tilde{q}^{PN}}, \omega_{p^{PN}} \vee$ ωqPN,
2. $\tilde{p}^{PN} \ominus \tilde{q}^{PN} = \langle (r - v^*, s - u^*, t - t^*, u - s^*, v - r^*);\ \tau_{\tilde{p}^{PN}} \wedge \tau_{\tilde{q}^{PN}}, \phi_{\tilde{p}^{PN}} \vee \phi_{\tilde{q}^{PN}}, \omega_{p^{PN}} \vee \omega_{\tilde{q}^{PN}} \rangle$,

3. $\tilde{p}^{PN} \otimes \tilde{q}^{PN} = \frac{1}{5}\gamma_{q} \langle (r, s, t, u, v);\ \tau_{\tilde{p}^{PN}} \wedge \tau_{\tilde{q}^{PN}}, \phi_{\tilde{p}^{PN}} \vee \phi_{\tilde{q}^{PN}}, \omega_{p^{PN}} \vee \omega_{\tilde{q}^{PN}} \rangle, \gamma_{q} = \frac{1}{3}(r^{*} + s^{*} +$

$t* + u* + v*2 + \tau q PN - \phi q PN \neq 0$,

4. $\tilde{p}^{PN} \oslash \tilde{q}^{PN} = \frac{5}{\gamma_{q}} \langle (r, s, t, u, v);\ \tau_{\tilde{p}^{PN}} \wedge \tau_{\tilde{q}^{PN}}, \phi_{\tilde{p}^{PN}} \vee \phi_{\tilde{q}^{PN}}, \omega_{p^{PN}} \vee \omega_{\tilde{q}^{PN}} \rangle, \gamma_{q} \neq 0,$

5. $m\tilde{p}^{PN} = \begin{cases} \langle (mr, ms, mt, mu, mv);\ \tau_{\tilde{p}^{PN}}, \phi_{\tilde{p}^{PN}}, \omega_{\tilde{p}^{PN}} \rangle, m > 0, \\ \langle (mv, mu, mt, ms, mr);\ \tau_{\tilde{p}^{PN}}, \phi_{\tilde{p}^{PN}}, \omega_{\tilde{p}^{PN}} \rangle, m < 0, \end{cases}$

6. $\tilde{p}^{PN^{-1}} = \langle \left(\frac{1}{v}, \frac{1}{u}, \frac{1}{t}, \frac{1}{s}, \frac{1}{r}\right);\ \tau_{\tilde{p}^{PN}}, \phi_{\tilde{p}^{PN}}, \omega_{\tilde{p}^{PN}} \rangle, \tilde{p}^{PN} \neq 0.$

Definition7. [26] Let $\tilde{p}^{PN} = \langle (r, s, t, u, v); \tau_{\tilde{p}^{PN}}, \phi_{\tilde{p}^{PN}}, \omega_{\tilde{p}^{PN}} \rangle$ be a single- valued pentagonal fuzzy neutrosophic numbers, then

1. Accuracy function $AC(\tilde{p}^{PN}) = \left(\frac{1}{15}\right)(r + s + t + u + v) * [2 + \tau_{\tilde{p}^{PN}} - \phi_{\tilde{p}^{PN}}].$

2. Score function $SC(\tilde{p}^{PN}) = \left(\frac{1}{15}\right)(r + s + t + u + v) * \left[2 + \tau_{\tilde{p}^{PN}} - \phi_{\tilde{p}^{PN}} - \omega_{\tilde{p}^{PN}}\right].$

Definition 8. [27] The order relations between $\tilde{p}^{PN}$ and $\tilde{q}^{PN}$ based on $SC(\tilde{p}^{NP})$ and $AC(\tilde{q}^{NP})$ are defined as

1. If $SC(\tilde{p}^{PN}) > SC(\tilde{q}^{NP})$, then $\tilde{p} > \tilde{q}$,
2. If $SC(\tilde{p}^{PN}) < SC(\tilde{q}^{NP})$, then $\tilde{p} < \tilde{q}$,
3. If $SC(\tilde{p}^{PN}) = SC(\tilde{q}^{NP})$, then

   i. If $AC(\tilde{p}^{PN}) < AC(\tilde{q}^{NP})$, then $\tilde{p} < \tilde{q}$,

   ii. If $AC(\tilde{p}^{PN}) > AC(\tilde{q}^{NP})$, then $\tilde{p} > \tilde{q}$,

   iii. If $AC(\tilde{p}^{PN}) = AC(\tilde{q}^{NP})$, then $\tilde{p} = \tilde{q}$.

3. Assumptions and Notations

3.1 Assumptions

In reality, small changes influence in selecting portfolio, since the investment environment is quite sensitive. For facilitating problem formulation, we assumed that:

1) The securities are evaluated based on the expected return rate and the loss-risk rate;
2) Securities are imperfect and can be divided;
3) In the course of transaction, there is no need to pay for transactions;
4) Investors must obey the assumptions of avoiding risk and of non-satisfaction;

5) During the investment period, the interest rate of the bank is fixed;

6) The operation of short selling is not allowed;

7) There are n different risk securities.

3.2 Notations

r_0: Bank interest rate;

r_i: Expected return rates, $i = 1, 2, \dots, n$;

A_{ij}: Risked return rates, $i = 1, 2, \dots, n, j = 1, 2, \dots, m$;

x_0: Proportion of total investments during the investment period

x_i: Proportion of funds invested in the secondary securities, $i = 1, 2, \dots, n$;

R: Total expected return rate;

$\mathfrak{b}$: Risk coefficient of portfolio investment;

V: Maximum value of all securities risks.

4. Formulation of the Problem

Consider the stock problem introduced by Yin [28]. The expected rate of return of a combination of investments, takes the form:

$$R = \sum_{i=0}^{n} r_i x_i$$

Investors aim to maximize investments interest and minimize risk in their risk securities. The risk coefficient of portfolio $\mathfrak{b}$indicates the market risk. In case of $\mathfrak{b} > 1$, risk of stock portfolio is more than the average value of the market risk; in the case of $\mathfrak{b} < 1$, the risk of stock portfolio is less than the average value of market risk; when $\mathfrak{b} = 1$, the average market risk and stock portfolio risk are equal. The maximum value of all securities risks, denoted

$$V = \max(A_1 x_1, A_2 x_2, \dots, A_n x_n)$$

Now we can formulate the following linear programming model:

$$\max R = \sum_{i=0}^{n} r_i x_i$$

$$s.t.\begin{cases} Ax \leq \mathfrak{b} \\ \sum_{i=0}^{n} x_i = 1 \\ x_i \geq 0, \quad i = 1,2,\dots,n \end{cases} \quad (5)$$

The above model is the classical linear programming problem. For more generalization and flexibility, it is more reasonable to describe r_i, $\mathfrak{b}_i$ and A_i as pentagonal fuzzy neutrosophic numbers. So, we set up the following model:

$$\max \tilde{R}^{NP} = r_0 x_0 + \sum_{i=1}^{n} \tilde{r}_i^{NP} x_i$$

$$s.t.\begin{cases} \tilde{A}^{NP} x \leq \tilde{\mathfrak{b}}^{NP} \\ \sum_{j=0}^{n} x_j = 1 \\ x_j \geq 0, \quad j = 1,2,\dots,n;\ i = 1,2,\dots,m. \end{cases} \quad (6)$$

5. Solution Procedure of Pentagonal Fuzzy Neutrosophic LPP

In this section we will illustrate the solution procedure of the pentagonal fuzzy neutrosophic linear programming problem. The model associated with pentagonal fuzzy neutrosophic numbers in expected return rates, risk loss rates and risk coefficients

5.1 Formulation of pentagonal fuzzy neutrosophic LPP

Assume that $\tilde{A}^{NP} = \left(\tilde{A}_{ij}^{NP}\right)_{m\times n}$, $\tilde{b}^{NP} = \left(\tilde{b}_1^{NP}, \tilde{b}_2^{NP}, \dots, \tilde{b}_m^{NP}\right)^{\boldsymbol{T}}$, $\tilde{r}^{NP} = (\tilde{r}_1^{NP}, \tilde{r}_2^{NP}, \dots, \tilde{r}_n^{NP})$ and $X = (x_1, x_2, \dots, x_n)^T$. The following pentagonal neutrosophic linear programming model has been set up:

$$\max \tilde{R}^{NP} = r_0 x_0 + \sum_{j=1}^{n} \tilde{r}_j^{NP} x_j$$

$$s.t.\begin{cases} \sum_{j=1}^{n} \tilde{A}_{ij}^{NP} x_j \leq \tilde{b}_i^{NP} \\ \sum_{j=0}^{n} x_j = 1 \\ x_j \geq 0, \quad j = 1,2,\dots,n;\ i = 1,2,\dots,m. \end{cases} \quad (7)$$

Based on the score function defined in section 2, the pentagonal neutrosophic linear programming model transformed to regular linear programming model which is quite easy and solvable.

$$\max R = r_0 x_0 + \sum_{i=1}^{n} \mathrm{SC}(\tilde{r}_i^{NP}) x_i$$

$$s.t.\begin{cases}\sum_{j=1}^{n} \mathrm{SC}\,(\tilde{A}_{ij}^{NP})x_j \leq \mathrm{SC}(\tilde{b}_i^{NP})\\ \sum_{j=0}^{n} x_j = 1\\ x_j \geq 0, \quad j = 1,2,\dots,n;\ i = 1,2,\dots,m.\end{cases} \quad (8)$$

6. Numerical Example

In this section, a numerical example is studied to demonstrate the proposed approach. Consider the choice of an investor in five available stocks; the first one is a portfolio of bank savings with annual rate of interest $r_0 = 0.07$. The data of the other four stocks are given in the following table 1, table 2 and table 3.

Table 1.Expected return rate %

Stocks	$\tilde{r}^{PN}$
S_1 CNPC (601857)	⟨11.5, 12.0, 12.2, 12.5, 12.9; 1.0, 0.0,0.0 ⟩
S_2 CNPC (600028)	⟨15.8, 16.0, 16.2, 16.5, 16.8; 1.0, 0.0,0.0 ⟩
S_3 Wanke (000002)	⟨13.7, 14.0, 14.3, 14.5, 14.9; 1.0, 0.0,0.0 ⟩
S_4 Poly (600048)	⟨13.0, 13.5, 14.0, 14.5, 15.0; 1.0, 0.0,0.0 ⟩

Table 2. Risk Loss Rate %

$\tilde{A}_{ij}^{NP}$	Risk loss rate
$\tilde{A}_{11}^{NP}$	⟨3.8, 4.0, 5.2, 5.6, 5.9; 1.0, 0.0,0.0 ⟩
$\tilde{A}_{12}^{NP}$	⟨9.0, 10.0, 12.5, 14.0, 16.9; 1.0, 0.0,0.0 ⟩
$\tilde{A}_{13}^{NP}$	⟨3.2, 4.0, 4.8, 5.5, 6.0; 1.0, 0.0,0.0⟩
$\tilde{A}_{14}^{NP}$	⟨8.7, 9.0, 11.9, 14.0, 16.3; 1.0, 0.0,0.0 ⟩
$\tilde{A}_{21}^{NP}$	⟨0.9, 1.0, 1.1, 1.2, 1.3; 1.0, 0.0,0.0 ⟩
$\tilde{A}_{22}^{NP}$	⟨1.39, 1.7, 2.15, 3.0, 3.32; 1.0, 0.0,0.0 ⟩
$\tilde{A}_{23}^{NP}$	⟨1.2, 3.0, 3.2, 4.0, 4.8; 1.0, 0.0,0.0 ⟩
$\tilde{A}_{24}^{NP}$	⟨1.59, 1.8, 2.27, 3.0, 3.3; 1.0, 0.0,0.0 ⟩

Table 3. Risk coefficient%

$\tilde{b}_i^{NP}$	Risk coefficient rate
$\tilde{b}_1^{NP}$	⟨1.2, 1.5, 2.0, 2.2, 2.4; 1.0, 0.0,0.0 ⟩
b_2^{NP}	⟨0.6, 0.9, 2.0, 2.6, 3.0; 1.0, 0.0,0.0 ⟩

The given problem can be formulated in the following model:

$$\max\tilde{R}^{NP} = r_0 x_0 + \sum_{j=1}^{n} \mathrm{SC}(\tilde{r}_i^{NP}) x_j$$

$$s.t. \begin{cases} \mathrm{SC}(\tilde{A}_{11}^{NP})x_1 + \mathrm{SC}(\tilde{A}_{12}^{NP})x_2 + \mathrm{SC}(\tilde{A}_{13}^{NP})x_3 + \mathrm{SC}(\tilde{A}_{14}^{NP})x_4 \le \mathrm{SC}(\tilde{b}_1^{NP}) \\ \mathrm{SC}(\tilde{A}_{21}^{NP})x_1 + \mathrm{SC}(\tilde{A}_{22}^{NP})x_2 + \mathrm{SC}(\tilde{A}_{23}^{NP})x_3 + \mathrm{SC}(\tilde{A}_{24}^{NP})x_4 \le \mathrm{SC}(\tilde{b}_2^{NP}) \\ x_0 + x_1 + x_2 + x_3 + x_4 = 1 \\ x_j \ge 0, \quad 0 \le j \le 4 \end{cases} \quad (9)$$

According to properties and arithmetic operations on pentagonal fuzzy neutrosophic numbers, we obtain the following mathematical model:

$$\max R = 0.07\ x_0 + 0.09165\ x_1 + 0.12195 x_2 + 0.1071 x_3 + 0.105 x_4$$

$$s.t. \begin{cases} 3.675\ x_1 + 9.36\ x_2 + 3.525\ x_3 + 8.985\ x_4 \le 1.395, \\ 0.825\ x_1 + 1.734\ x_2 + 2.43\ x_3 + 1.794\ x_4 \le 1.365, \\ x_0 + x_1 + x_2 + x_3 + x_4 = 1, \\ x_j \ge 0,\ 0 \le j \le 4. \end{cases} \quad (10)$$

The optimal solution is:

$x_0 = 0.6042553,\ x_1 = 0.0, x_2 = 0.0, x_3 = 0.3957447, x_4 = 0.0,$

The optimal value $R = 0.084682$

The obtained results indicate that the optimal investment under the offered information occurred when 60.0426% of all capital is saved to the bank with interest rate 7%and 39.57% of the total capital is invested into security of S_3. This strategy leads to the maximum expected return 8.4682% on the premise of risk coefficients $\tilde{b}_1^{NP}$ and $\tilde{b}_2^{NP}$.

7. Comparative Study

This section, introduces a comparative study between the topics covered by our proposed approach and those studied by some other researchers in related work in solving PS problems.

Table 4. Comparisons with some researcher's contributions

Reference no.	Efficient solution	Environment	Type of number
[28]	NO	Fuzzy	Triangle interval valued
[29]	NO	Neutrosophic	Neutrosophic
[30]	NO	Fuzzy	Fuzzy-valued function
[31]	NO	realistic	Real
[32]	NO	stochastic	random variables
[33]	NO	Fuzzy	Triangle
Our investigation	YES	Neutrosophic	Neutrosophic

8. Conclusion Remarks and Future Work

A formulation of stock portfolio problem involving neutrosophic pentagonal fuzzy numbers in the rate of risked return, expected return rate and portfolio risk amount is proposed. Using score function, the problem is converted to its corresponding crisp form. A solution approach is investigated to provide the decision of the portfolio investment joined with investors in savings and securities. The main advantages of this study are: the freedom in choosing the risk coefficients to maximize the expected returns; also, the investors may select their strategies under consideration of their own conditions. The optimal return rate is obtained using TORA software. A numerical example indicates that the approach is reliable and efficient for studying pentagonal neutrosophic stock portfolio. Future work may include the further extension of this study to other fuzzy- like structure (i. e., interval- valued fuzzy set, Neutrosophic set, Pythagorean fuzzy set, Spherical fuzzy set etc. with more discussion and suggestive comments.

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Conflict of Interest
The authors do not have conflict of interest.

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