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# Generalizations of Neutrosophic Subalgebras in $BCK/BCI$-Algebras Based on Neutrosophic Points

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**Abstract:** Saeid and Jun introduced the notion of neutrosophic points, and studied neutrosophic subalgebras of several types in $BCK/BCI$-algebras by using the notion of neutrosophic points (see [4] and [6]). More general form of neutrosophic points is considered in this paper, and generalizations of Saeid and Jun's results are discussed. The concepts of $(\in, \in \vee q_{(k_T,k_I,k_F)})$-neutrosophic subalgebra, $(q_{(k_T,k_I,k_F)}, \in \vee q_{(k_T,k_I,k_F)})$-neutrosophic subalgebra and $(\in, q_{(k_T,k_I,k_F)})$-neutrosophic subalgebra are introduced, and several properties are investigated. Characterizations of $(\in, \in \vee q_{(k_T,k_I,k_F)})$-neutrosophic subalgebra are discussed.

**Keywords:** $(\in, \in \vee q_{(k_T,k_I,k_F)})$-neutrosophic subalgebra; $(q_{(k_T,k_I,k_F)}, \in \vee q_{(k_T,k_I,k_F)})$-neutrosophic subalgebra; $(\in, q_{(k_T,k_I,k_F)})$-neutrosophic subalgebra.

## 1 Introduction

As a generalization of fuzzy sets, Atanassov [1] introduced the degree of nonmembership/falsehood (f) in 1986 and defined the intuitionistic fuzzy set. As a more general platform which extends the notions of the classic set and fuzzy set, intuitionistic fuzzy set and interval valued (intuitionistic) fuzzy set, Smarandache introduced the notion of neutrosophic sets (see [7, 8]), which is useful mathematical tool for dealing with incomplete, inconsistent and indeterminate information. For further particulars on neutrosophic set theory, we refer the readers to the site

http://fs.gallup.unm.edu/FlorentinSmarandache.htm

Jun [4] introduced the notion of $(\Phi, \Psi)$-neutrosophic subalgebra of a $BCK/BCI$-algebra $X$ for $\Phi, \Psi \in \{\in, q, \in \vee q\}$, and investigated related properties. He provided characterizations of an $(\in, \in)$-neutrosophic subalgebra and an $(\in, \in \vee q)$-neutrosophic subalgebra, and considered conditions for a neutrosophic set to be a $(q, \in \vee q)$-neutrosophic subalgebra. Saeid and Jun [6] gave relations between an $(\in, \in \vee q)$-neutrosophic subalgebra and a $(q, \in \vee q)$-neutrosophic subalgebra, and investigated properties on neutrosophic $q$-subsets and neutrosophic $\in \vee q$-subsets.

The purpose of this article is to give an algebraic tool of neutrosophic set theory which can be used in applied sciences, for example, decision making problems, medical sciences etc. We consider a general form of neutrosophic points, and then we discuss generalizations of the papers [4] and [6]. As a generalization of $(\in, \in \vee q)$-neutrosophic subalgebras, we introduce the notions of $(\in, \in \vee q_{(k_T,k_I,k_F)})$-neutrosophic subalgebra, and $(\in, q_{(k_T,k_I,k_F)})$-neutrosophic subalgebra in $BCK/BCI$-algebras, and investigate several properties. We discuss characterizations of $(\in, \in \vee q_{(k_T,k_I,k_F)})$-neutrosophic subalgebra. We consider relations between $(\in, \in)$-neutrosophic subalgebra, $(\in, q_{(k_T,k_I,k_F)})$-neutrosophic subalgebra and $(\in, \in \vee q_{(k_T,k_I,k_F)})$-neutrosophic subalgebra.

## 2 Preliminaries

By a *$BCI$-algebra*, we mean a set $X$ with a binary operation $*$ and the special element $0$ satisfying the conditions (see [3, 5]):

(a1) $(\forall x, y, z \in X)(((x * y) * (x * z)) * (z * y) = 0)$,

(a2) $(\forall x, y \in X)((x * (x * y)) * y = 0)$,

(a3) $(\forall x \in X)(x * x = 0)$,

(a4) $(\forall x, y \in X)(x * y = y * x = 0 \Rightarrow x = y)$.

If a $BCI$-algebra $X$ satisfies the axiom

(a5) $0 * x = 0$ for all $x \in X$,

then we say that $X$ is a *$BCK$-algebra* (see [3, 5]). A nonempty subset $S$ of a $BCK/BCI$-algebra $X$ is called a *subalgebra* of $X$ (see [3, 5]) if $x * y \in S$ for all $x, y \in S$.

The collection of all $BCK$-algebras and all $BCI$-algebras are denoted by $\mathcal{B}_K(X)$ and $\mathcal{B}_I(X)$, respectively. Also $\mathcal{B}(X) := \mathcal{B}_K(X) \cup \mathcal{B}_I(X)$.

We refer the reader to the books [3] and [5] for further information regarding $BCK/BCI$-algebras.

Let $X$ be a non-empty set. A neutrosophic set (NS) in $X$ (see [7]) is a structure of the form:

$$A := \{\langle x; A_T(x), A_I(x), A_F(x)\rangle \mid x \in X\} \tag{2.1}$$

where $A_T$, $A_I$ and $A_F$ are a truth membership function, an indeterminate membership function and a false membership function, respectively, from $X$ into the unit interval $[0,1]$. The neutrosophic set (2.1) will be denoted by $A = (A_T, A_I, A_F)$.

Given a neutrosophic set $A = (A_T, A_I, A_F)$ in a set $X$, $\alpha, \beta \in (0,1]$ and $\gamma \in [0,1)$, we consider the following sets (see [4]):

$$\begin{aligned}
&T_\in(A;\alpha) := \{x \in X \mid A_T(x) \geq \alpha\},\\
&I_\in(A;\beta) := \{x \in X \mid A_I(x) \geq \beta\},\\
&F_\in(A;\gamma) := \{x \in X \mid A_F(x) \leq \gamma\},\\
&T_q(A;\alpha) := \{x \in X \mid A_T(x) + \alpha > 1\},\\
&I_q(A;\beta) := \{x \in X \mid A_I(x) + \beta > 1\},\\
&F_q(A;\gamma) := \{x \in X \mid A_F(x) + \gamma < 1\},\\
&T_{\in\vee\, q}(A;\alpha) := \{x \in X \mid A_T(x) \geq \alpha \text{ or } A_T(x) + \alpha > 1\},\\
&I_{\in\vee\, q}(A;\beta) := \{x \in X \mid A_I(x) \geq \beta \text{ or } A_I(x) + \beta > 1\},\\
&F_{\in\vee\, q}(A;\gamma) := \{x \in X \mid A_F(x) \leq \gamma \text{ or } A_F(x) + \gamma < 1\}.
\end{aligned}$$

We say $T_\in(A;\alpha)$, $I_\in(A;\beta)$ and $F_\in(A;\gamma)$ are *neutrosophic* $\in$*-subsets*; $T_q(A;\alpha)$, $I_q(A;\beta)$ and $F_q(A;\gamma)$ are *neutrosophic* $q$*-subsets*; and $T_{\in\vee\, q}(A;\alpha)$, $I_{\in\vee\, q}(A;\beta)$ and $F_{\in\vee\, q}(A;\gamma)$ are *neutrosophic* $\in \vee\, q$*-subsets*. It is clear that

$$T_{\in\vee\, q}(A;\alpha) = T_\in(A;\alpha) \cup T_q(A;\alpha), \tag{2.2}$$
$$I_{\in\vee\, q}(A;\beta) = I_\in(A;\beta) \cup I_q(A;\beta), \tag{2.3}$$
$$F_{\in\vee\, q}(A;\gamma) = F_\in(A;\gamma) \cup F_q(A;\gamma). \tag{2.4}$$

Given $\Phi, \Psi \in \{\in, q, \in\vee\, q\}$, a neutrosophic set $A = (A_T, A_I, A_F)$ in $X \in \mathcal{B}(X)$ is called a $(\Phi, \Psi)$*-neutrosophic subalgebra* of $X$ (see [4]) if the following assertions are valid.

$$\begin{aligned}
&x \in T_\Phi(A;\alpha_x),\ y \in T_\Phi(A;\alpha_y)\\
&\qquad \Rightarrow\ x * y \in T_\Psi(A;\alpha_x \wedge \alpha_y),\\
&x \in I_\Phi(A;\beta_x),\ y \in I_\Phi(A;\beta_y)\\
&\qquad \Rightarrow\ x * y \in I_\Psi(A;\beta_x \wedge \beta_y),\\
&x \in F_\Phi(A;\gamma_x),\ y \in F_\Phi(A;\gamma_y)\\
&\qquad \Rightarrow\ x * y \in F_\Psi(A;\gamma_x \vee \gamma_y)
\end{aligned} \tag{2.5}$$

for all $x, y \in X$, $\alpha_x, \alpha_y, \beta_x, \beta_y \in (0,1]$ and $\gamma_x, \gamma_y \in [0,1)$.

# 3 Generalizations of $(\in, \in\vee q)$-neutrosophic subalgebras

In what follows, let $k_T$, $k_I$ and $k_F$ denote arbitrary elements of $[0,1)$ unless otherwise specified. If $k_T$, $k_I$ and $k_F$ are the same number in $[0,1)$, then it is denoted by $k$, i.e., $k = k_T = k_I = k_F$.

Given a neutrosophic set $A = (A_T, A_I, A_F)$ in a set $X$, $\alpha, \beta \in (0,1]$ and $\gamma \in [0,1)$, we consider the following sets:

$$\begin{aligned}
&T_{q_{k_T}}(A;\alpha) := \{x \in X \mid A_T(x) + \alpha + k_T > 1\},\\
&I_{q_{k_I}}(A;\beta) := \{x \in X \mid A_I(x) + \beta + k_I > 1\},\\
&F_{q_{k_F}}(A;\gamma) := \{x \in X \mid A_F(x) + \gamma + k_F < 1\},\\
&T_{\in\vee\, q_{k_T}}(A;\alpha) := \{x \in X \mid A_T(x) \geq \alpha \text{ or }\\
&\qquad\qquad A_T(x) + \alpha + k_T > 1\},\\
&I_{\in\vee\, q_{k_I}}(A;\beta) := \{x \in X \mid A_I(x) \geq \beta \text{ or }\\
&\qquad\qquad A_I(x) + \beta + k_I > 1\},\\
&F_{\in\vee\, q_{k_F}}(A;\gamma) := \{x \in X \mid A_F(x) \leq \gamma \text{ or }\\
&\qquad\qquad A_F(x) + \gamma + k_F < 1\}.
\end{aligned}$$

We say $T_{q_{k_T}}(A;\alpha)$, $I_{q_{k_I}}(A;\beta)$ and $F_{q_{k_F}}(A;\gamma)$ are *neutrosophic* $q_k$*-subsets*; and $T_{\in\vee\, q_{k_T}}(A;\alpha)$, $I_{\in\vee\, q_{k_I}}(A;\beta)$ and $F_{\in\vee\, q_{k_F}}(A;\gamma)$ are *neutrosophic* $(\in\vee\, q_k)$*-subsets*. For $\Phi \in \{\in, q, q_k, q_{k_T}, q_{k_I}, q_{k_F}, \in\vee\, q, \in\vee\, q_k, \in\vee\, q_{k_T}, \in\vee\, q_{k_I}, \in\vee\, q_{k_F}\}$, the element of $T_\Phi(A;\alpha)$ (resp., $I_\Phi(A;\beta)$ and $F_\Phi(A;\gamma)$) is called a *neutrosophic* $T_\Phi$*-point* (resp., *neutrosophic* $I_\Phi$*-point* and *neutrosophic* $F_\Phi$*-point*) with value $\alpha$ (resp., $\beta$ and $\gamma$).

It is clear that

$$T_{\in\vee\, q_{k_T}}(A;\alpha) = T_\in(A;\alpha) \cup T_{q_{k_T}}(A;\alpha), \tag{3.1}$$
$$I_{\in\vee\, q_{k_I}}(A;\beta) = I_\in(A;\beta) \cup I_{q_{k_I}}(A;\beta), \tag{3.2}$$
$$F_{\in\vee\, q_{k_F}}(A;\gamma) = F_\in(A;\gamma) \cup F_{q_{k_F}}(A;\gamma). \tag{3.3}$$

Given a neutrosophic set $A = (A_T, A_I, A_F)$ in a set $X$, $\alpha, \beta \in (0,1]$ and $\gamma \in [0,1)$, we consider the following sets:

$$T^*_\in(A;\alpha) := \{x \in X \mid A_T(x) > \alpha\}, \tag{3.4}$$
$$I^*_\in(A;\beta) := \{x \in X \mid A_I(x) > \beta\}, \tag{3.5}$$
$$F^*_\in(A;\gamma) := \{x \in X \mid A_F(x) < \gamma\}. \tag{3.6}$$

**Proposition 3.1.** *For any neutrosophic set $A = (A_T, A_I, A_F)$ in a set $X$, $\alpha, \beta \in (0,1]$ and $\gamma \in [0,1)$, we have*

$$\alpha \leq \tfrac{1-k}{2} \ \Rightarrow\ T_{q_k}(A;\alpha) \subseteq T^*_\in(A;\alpha), \tag{3.7}$$
$$\beta \leq \tfrac{1-k}{2} \ \Rightarrow\ I_{q_k}(A;\beta) \subseteq I^*_\in(A;\beta), \tag{3.8}$$
$$\gamma \geq \tfrac{1-k}{2} \ \Rightarrow\ F_{q_k}(A;\gamma) \subseteq F^*_\in(A;\gamma), \tag{3.9}$$
$$\alpha > \tfrac{1-k}{2} \ \Rightarrow\ T_\in(A;\alpha) \subseteq T_{q_k}(A;\alpha), \tag{3.10}$$
$$\beta > \tfrac{1-k}{2} \ \Rightarrow\ I_\in(A;\beta) \subseteq I_{q_k}(A;\beta), \tag{3.11}$$
$$\gamma < \tfrac{1-k}{2} \ \Rightarrow\ F_\in(A;\gamma) \subseteq F_{q_k}(A;\gamma). \tag{3.12}$$

*Proof.* If $\alpha \leq \frac{1-k}{2}$, then $1 - \alpha \geq \frac{1+k}{2}$ and $\alpha \leq 1 - \alpha$. Assume that $x \in T_{q_k}(A;\alpha)$. Then $A_T(x) + k > 1 - \alpha \geq \frac{1+k}{2}$, and so $A_T(x) > \frac{1+k}{2} - k = \frac{1-k}{2} \geq \alpha$. Hence $x \in T^*_\in(A;\alpha)$. Similarly, we have the result (3.8). Suppose that $\gamma \geq \frac{1-k}{2}$ and let $x \in F_{q_k}(A;\gamma)$. Then $A_F(x) + \gamma + k < 1$, and thus

$$A_F(x) < 1 - \gamma - k \leq 1 - \tfrac{1-k}{2} - k = \tfrac{1-k}{2} \leq \gamma.$$

Hence $x \in F^*_\in(A;\gamma)$. Suppose that $\alpha > \frac{1-k}{2}$. If $x \in T_\in(A;\alpha)$, then

$$A_T(x) + \alpha + k \geq 2\alpha + k > 2 \cdot \tfrac{1-k}{2} + k = 1$$

and so $x \in T_{q_k}(A;\alpha)$. Hence $T_\in(A;\alpha) \subseteq T_{q_k}(A;\alpha)$. Similarly, we can verify that if $\beta > \frac{1-k}{2}$, then $I_\in(A;\beta) \subseteq I_{q_k}(A;\beta)$. Suppose that $\gamma < \frac{1-k}{2}$. If $x \in F_\in(A;\gamma)$, then $A_F(x) \leq \gamma$, and thus

$$A_F(x) + \gamma + k \leq 2\gamma + k < 2 \cdot \tfrac{1-k}{2} + k = 1,$$

that is, $x \in F_{q_k}(A;\gamma)$. Hence $F_\in(A;\gamma) \subseteq F_{q_k}(A;\gamma)$. □

**Definition 3.2.** A neutrosophic set $A = (A_T, A_I, A_F)$ in $X \in \mathcal{B}(X)$ is called an $(\in, \in \vee q_{(k_T,k_I,k_F)})$-*neutrosophic subalgebra* of $X$ if

$$\begin{aligned} & x \in T_\in(A;\alpha_x),\ y \in T_\in(A;\alpha_y) \\ & \qquad \Rightarrow\ x*y \in T_{\in\vee q_{k_T}}(A;\alpha_x \wedge \alpha_y), \\ & x \in I_\in(A;\beta_x),\ y \in I_\in(A;\beta_y) \\ & \qquad \Rightarrow\ x*y \in I_{\in\vee q_{k_I}}(A;\beta_x \wedge \beta_y), \\ & x \in F_\in(A;\gamma_x),\ y \in F_\in(A;\gamma_y) \\ & \qquad \Rightarrow\ x*y \in F_{\in\vee q_{k_F}}(A;\gamma_x \vee \gamma_y) \end{aligned} \tag{3.13}$$

for all $x, y \in X$, $\alpha_x, \alpha_y, \beta_x, \beta_y \in (0,1]$ and $\gamma_x, \gamma_y \in [0,1)$.

An $(\in, \in \vee q_{(k_T,k_I,k_F)})$-neutrosophic subalgebra with $k_T = k_I = k_F = k$ is called an $(\in, \in \vee q_k)$-*neutrosophic subalgebra*.

**Lemma 3.3** ([4]). *A neutrosophic set $A = (A_T, A_I, A_F)$ in $X \in \mathcal{B}(X)$ is an $(\in, \in)$-neutrosophic subalgebra of $X$ if and only if it satisfies:*

$$(\forall x, y \in X) \left( \begin{array}{l} A_T(x*y) \geq A_T(x) \wedge A_T(y) \\ A_I(x*y) \geq A_I(x) \wedge A_I(y) \\ A_F(x*y) \leq A_F(x) \vee A_F(y) \end{array} \right). \tag{3.14}$$

**Theorem 3.4.** *If $A = (A_T, A_I, A_F)$ is an $(\in, \in)$-neutrosophic subalgebra of $X \in \mathcal{B}(X)$, then neutrosophic $q_k$-subsets $T_{q_{k_T}}(A;\alpha)$, $I_{q_{k_I}}(A;\beta)$ and $F_{q_{k_F}}(A;\gamma)$ are subalgebras of $X$ for all $\alpha, \beta \in (0,1]$ and $\gamma \in [0,1)$ whenever they are nonempty.*

*Proof.* Let $x, y \in T_{q_{k_T}}(A;\alpha)$. Then $A_T(x) + \alpha + k_T > 1$ and $A_T(y) + \alpha + k_T > 1$. It follows from Lemma 3.3 that

$$\begin{aligned} & A_T(x*y) + \alpha + k_T \geq (A_T(x) \wedge A_T(y)) + \alpha + k_T \\ & = (A_T(x) + \alpha + k_T) \wedge (A_T(y) + \alpha + k_T) > 1 \end{aligned}$$

and so that $x*y \in T_{q_{k_T}}(A;\alpha)$. Hence $T_{q_{k_T}}(A;\alpha)$ is a subalgebra of $X$. Similarly, we can prove that $I_{q_{k_I}}(A;\beta)$ is a subalgebra of $X$. Now let $x, y \in F_{q_{k_F}}(A;\gamma)$. Then $A_F(x) + \gamma + k_F < 1$ and $A_F(y) + \gamma + k_F < 1$, which imply from Lemma 3.3 that

$$\begin{aligned} & A_F(x*y) + \gamma + k_F \leq (A_F(x) \vee A_F(y)) + \gamma + k_F \\ & = (A_F(x) + \gamma + k_F) \vee (A_F(y) + \gamma + k_F) < 1. \end{aligned}$$

Hence $x*y \in F_{q_{k_F}}(A;\gamma)$ and so $F_{q_{k_F}}(A;\gamma)$ is a subalgebra of $X$. □

**Corollary 3.5.** *If $A = (A_T, A_I, A_F)$ is an $(\in, \in)$-neutrosophic subalgebra of $X \in \mathcal{B}(X)$, then neutrosophic $q_k$-subsets $T_{q_k}(A;\alpha)$, $I_{q_k}(A;\beta)$ and $F_{q_k}(A;\gamma)$ are subalgebras of $X$ for all $\alpha, \beta \in (0,1]$ and $\gamma \in [0,1)$ whenever they are nonempty.*

If we take $k_T = k_I = k_F = 0$ in Theorem 3.4, then we have the following corollary.

**Corollary 3.6** ([4]). *If $A = (A_T, A_I, A_F)$ is an $(\in, \in)$-neutrosophic subalgebra of $X \in \mathcal{B}(X)$, then neutrosophic $q$-subsets $T_q(A;\alpha)$, $I_q(A;\beta)$ and $F_q(A;\gamma)$ are subalgebras of $X$ for all $\alpha, \beta \in (0,1]$ and $\gamma \in [0,1)$ whenever they are nonempty.*

**Definition 3.7.** A neutrosophic set $A = (A_T, A_I, A_F)$ in $X \in \mathcal{B}(X)$ is called a $(q_{(k_T,k_I,k_F)}, \in \vee q_{(k_T,k_I,k_F)})$-*neutrosophic subalgebra* of $X$ if

$$\begin{aligned} & x \in T_{q_{k_T}}(A;\alpha_x),\ y \in T_{q_{k_T}}(A;\alpha_y) \\ & \qquad \Rightarrow\ x*y \in T_{\in\vee q_{k_T}}(A;\alpha_x \wedge \alpha_y), \\ & x \in I_{q_{k_I}}(A;\beta_x),\ y \in I_{q_{k_I}}(A;\beta_y) \\ & \qquad \Rightarrow\ x*y \in I_{\in\vee q_{k_I}}(A;\beta_x \wedge \beta_y), \\ & x \in F_{q_{k_F}}(A;\gamma_x),\ y \in F_{q_{k_F}}(A;\gamma_y) \\ & \qquad \Rightarrow\ x*y \in F_{\in\vee q_{k_F}}(A;\gamma_x \vee \gamma_y) \end{aligned} \tag{3.15}$$

for all $x, y \in X$, $\alpha_x, \alpha_y, \beta_x, \beta_y \in (0,1]$ and $\gamma_x, \gamma_y \in [0,1)$.

A $(q_{(k_T,k_I,k_F)}, \in \vee q_{(k_T,k_I,k_F)})$-neutrosophic subalgebra with $k_T = k_I = k_F = k$ is called a $(q_k, \in \vee q_k)$-*neutrosophic subalgebra*.

**Theorem 3.8.** *If $A = (A_T, A_I, A_F)$ is a $(q_{(k_T,k_I,k_F)}, \in \vee q_{(k_T,k_I,k_F)})$-neutrosophic subalgebra of $X \in \mathcal{B}(X)$, then neutrosophic $q_k$-subsets $T_{q_{k_T}}(A;\alpha)$, $I_{q_{k_I}}(A;\beta)$ and $F_{q_{k_F}}(A;\gamma)$ are subalgebras of $X$ for all $\alpha \in (\frac{1-k_T}{2}, 1]$, $\beta \in (\frac{1-k_I}{2}, 1]$ and $\gamma \in [0, \frac{1-k_F}{2})$ whenever they are nonempty.*

*Proof.* Let $x, y \in T_{q_{k_T}}(A;\alpha)$ for $\alpha \in (\frac{1-k_T}{2}, 1]$. Then $x*y \in T_{\in\vee\, q_{k_T}}(A;\alpha)$, that is, $x*y \in T_\in(A;\alpha)$ or $x*y \in T_{q_{k_T}}(A;\alpha)$. If $x*y \in T_\in(A;\alpha)$, then $x*y \in T_{q_{k_T}}(A;\alpha)$ by (3.10). Therefore $T_{q_{k_T}}(A;\alpha)$ is a subalgebra of $X$. Similarly, we prove that $I_{q_{k_I}}(A;\beta)$ is a subalgebra of $X$. Let $x, y \in F_{q_{k_F}}(A;\gamma)$ for $\gamma \in [0, \frac{1-k_F}{2})$. Then $x*y \in F_{\in\vee\, q_{k_F}}(A;\gamma)$, and so $x*y \in F_\in(A;\gamma)$ or $x*y \in F_{q_{k_F}}(A;\gamma)$. If $x*y \in F_\in(A;\gamma)$, then $x*y \in F_{q_{k_F}}(A;\gamma)$ by (3.12). Hence $F_{q_{k_F}}(A;\gamma)$ is a subalgebra of $X$. □

Taking $k_T = k_I = k_F = 0$ in Theorem 3.8 induces the following corollary.

**Corollary 3.9** ([4]). *If $A = (A_T, A_I, A_F)$ is a $(q, \in \vee q)$-neutrosophic subalgebra of $X \in \mathcal{B}(X)$, then neutrosophic $q$-subsets $T_q(A;\alpha)$, $I_q(A;\beta)$ and $F_q(A;\gamma)$ are subalgebras of $X$ for all $\alpha, \beta \in (0.5, 1]$ and $\gamma \in [0, 0, 5)$ whenever they are nonempty.*

We provide characterizations of an $(\in, \in \vee q_{(k_T,k_I,k_F)})$-neutrosophic subalgebra.

**Theorem 3.10.** *Given a neutrosophic set $A = (A_T, A_I, A_F)$ in $X \in \mathcal{B}(X)$, the following are equivalent.*

(1) *$A = (A_T, A_I, A_F)$ is an $(\in, \in \vee q_{(k_T,k_I,k_F)})$-neutrosophic subalgebra of $X$.*

(2) *$A = (A_T, A_I, A_F)$ satisfies the following assertion.*

$$\begin{aligned} A_T(x*y) &\geq \bigwedge\{A_T(x), A_T(y), \tfrac{1-k_T}{2}\} \\ A_I(x*y) &\geq \bigwedge\{A_I(x), A_I(y), \tfrac{1-k_I}{2}\} \\ A_F(x*y) &\leq \bigvee\{A_F(x), A_F(y), \tfrac{1-k_F}{2}\} \end{aligned} \tag{3.16}$$

*for all $x, y \in X$.*

*Proof.* Let $A = (A_T, A_I, A_F)$ be an $(\in, \in \vee q_{(k_T,k_I,k_F)})$-neutrosophic subalgebra of $X$. Assume that there exist $a, b \in X$ such that

$$A_T(a*b) < \bigwedge\{A_T(a), A_T(b), \tfrac{1-k_T}{2}\}.$$

If $A_T(a) \wedge A_T(b) < \frac{1-k_T}{2}$, then $A_T(a*b) < A_T(a) \wedge A_T(b)$. Hence

$$A_T(a*b) < \alpha_t \leq A_T(a) \wedge A_T(b)$$

for some $\alpha_t \in (0,1]$. It follows that $a \in T_\in(A;\alpha_t)$ and $b \in T_\in(A;\alpha_t)$ but $a*b \notin T_\in(A;\alpha_t)$. Moreover,

$$A_T(a*b) + \alpha_t < 2\alpha_t < 1 - k_T,$$

and so $a*b \notin T_{q_{k_T}}(A;\alpha_t)$. Thus $a*b \notin T_{\in\vee\, q_{k_T}}(A;\alpha_t)$, a contradiction. If $A_T(a) \wedge A_T(b) \geq \frac{1-k_T}{2}$, then $a \in T_\in(A; \frac{1-k_T}{2})$, $b \in T_\in(A; \frac{1-k_T}{2})$ and $a*b \notin T_\in(A; \frac{1-k_T}{2})$. Also,

$$A_T(a*b) + \tfrac{1-k_T}{2} < \tfrac{1-k_T}{2} + \tfrac{1-k_T}{2} = 1 - k_T,$$

i.e., $a*b \notin T_{q_{k_T}}(A; \frac{1-k_T}{2})$. Hence $a*b \notin T_{\in\vee\, q_{k_T}}(A; \frac{1-k_T}{2})$, a contradiction. Consequently,

$$A_T(x*y) \geq \bigwedge\{A_T(x), A_T(y), \tfrac{1-k_T}{2}\}$$

for all $x, y \in X$. Similarly, we know that

$$A_I(x*y) \geq \bigwedge\{A_I(x), A_I(y), \tfrac{1-k_I}{2}\}$$

for all $x, y \in X$. Suppose that there exist $a, b \in X$ such that

$$A_F(a*b) > \bigvee\{A_F(a), A_F(b), \tfrac{1-k_F}{2}\}.$$

Then $A_F(a*b) > \gamma_F \geq \bigvee\{A_F(a), A_F(b), \frac{1-k_F}{2}\}$ for some $\gamma_F \in [0,1)$. If $A_F(a) \vee A_F(b) \geq \frac{1-k_F}{2}$, then

$$A_F(a*b) > \gamma_F \geq A_F(a) \vee A_F(b)$$

which implies that $a, b \in F_\in(A;\gamma_F)$ and $a*b \notin F_\in(A;\gamma_F)$. Also,

$$A_F(a*b) + \gamma_F > 2\gamma_F \geq 1 - k_F,$$

that is, $a*b \notin F_{q_{k_F}}(A;\gamma_F)$. Thus $a*b \notin F_{\in\vee\, q_{k_F}}(A;\gamma_F)$, which is a contradiction. If $A_F(a) \vee A_F(b) < \frac{1-k_F}{2}$, then $a, b \in F_\in(A; \frac{1-k_F}{2})$ and $a*b \notin F_\in(A; \frac{1-k_F}{2})$. Also,

$$A_F(a*b) + \tfrac{1-k_F}{2} > \tfrac{1-k_F}{2} + \tfrac{1-k_F}{2} = 1 - k_F$$

and so $a*b \notin F_{q_{k_F}}(A; \frac{1-k_F}{2})$. Hence $a*b \notin F_{\in\vee\, q_{k_F}}(A; \frac{1-k_F}{2})$, a contradiction. Therefore

$$A_F(x*y) \leq \bigvee\{A_F(x), A_F(y), \tfrac{1-k_F}{2}\}$$

for all $x, y \in X$.

Conversely, let $A = (A_T, A_I, A_F)$ be a neutrosophic set in $X$ which satisfies the condition (3.16). Let $x, y \in X$ and $\beta_x, \beta_y \in (0,1]$ be such that $x \in I_\in(A;\beta_x)$ and $y \in I_\in(A;\beta_y)$. Then

$$A_I(x*y) \geq \bigwedge\{A_I(x), A_I(y), \tfrac{1-k_I}{2}\} \geq \bigwedge\{\beta_x, \beta_y, \tfrac{1-k_I}{2}\}.$$

Suppose that $\beta_x \leq \frac{1-k_I}{2}$ or $\beta_y \leq \frac{1-k_I}{2}$. Then $A_I(x*y) \geq \beta_x \wedge \beta_y$, and so $x*y \in I_\in(A;\beta_x \wedge \beta_y)$. Now, assume that $\beta_x > \frac{1-k_I}{2}$ and $\beta_y > \frac{1-k_I}{2}$. Then $A_I(x*y) \geq \frac{1-k_I}{2}$, and so

$$A_I(x*y) + \beta_x \wedge \beta_y > \tfrac{1-k_I}{2} + \tfrac{1-k_I}{2} = 1 - k_I,$$

that is, $x*y \in I_{q_{k_I}}(A;\beta_x \wedge \beta_y)$. Hence

$$x*y \in I_{\in\vee\, q_{k_I}}(A;\beta_x \wedge \beta_y).$$

Similarly, we can verify that if $x \in T_\in(A;\alpha_x)$ and $y \in T_\in(A;\alpha_y)$, then $x*y \in T_{\in\vee\, q_{k_T}}(A;\alpha_x \wedge \alpha_y)$. Finally, let $x, y \in X$ and $\gamma_x, \gamma_y \in [0,1)$ be such that $x \in F_\in(A;\gamma_x)$ and $y \in F_\in(A;\gamma_y)$. Then

$$A_F(x*y) \leq \bigvee\{A_F(x), A_F(y), \tfrac{1-k_F}{2}\} \leq \bigvee\{\gamma_x, \gamma_y, \tfrac{1-k_F}{2}\}.$$

If $\gamma_x \geq \frac{1-k_F}{2}$ or $\gamma_y \geq \frac{1-k_F}{2}$, then $A_F(x*y) \leq \gamma_x \vee \gamma_y$ and thus $x*y \in F_\in(A;\gamma_x \vee \gamma_y)$. If $\gamma_x < \frac{1-k_F}{2}$ and $\gamma_y < \frac{1-k_F}{2}$, then $A_F(x*y) \leq \frac{1-k_F}{2}$. Hence

$$A_F(x*y) + \gamma_x \vee \gamma_y < \tfrac{1-k_F}{2} + \tfrac{1-k_F}{2} = 1 - k_F,$$

that is, $x*y \in F_{q_{k_F}}(A;\gamma_x \vee \gamma_y)$. Thus

$$x*y \in F_{\in\vee\, q_{k_F}}(A;\gamma_x \vee \gamma_y).$$

Therefore $A = (A_T, A_I, A_F)$ is an $(\in, \in \vee\, q_{k_F})$-neutrosophic subalgebra of $X$. □

**Corollary 3.11** ([4]). *A neutrosophic set $A = (A_T, A_I, A_F)$ in $X \in \mathcal{B}(X)$ is an $(\in, \in \vee\, q)$-neutrosophic subalgebra of $X$ if and only if it satisfies:*

$$(\forall x, y \in X) \left( \begin{array}{l} A_T(x*y) \geq \bigwedge\{A_T(x), A_T(y), 0.5\} \\ A_I(x*y) \geq \bigwedge\{A_I(x), A_I(y).0.5\} \\ A_F(x*y) \leq \bigvee\{A_F(x), A_F(y), 0.5\} \end{array} \right).$$

*Proof.* It follows from taking $k_T = k_I = k_F = 0$ in Theorem 3.10. □

**Theorem 3.12.** *Let $A = (A_T, A_I, A_F)$ be a neutrosophic set in $X \in \mathcal{B}(X)$. Then $A = (A_T, A_I, A_F)$ is an $(\in, \in \vee q_{(k_T,k_I,k_F)})$-neutrosophic subalgebra of $X$ if and only if neutrosophic $\in$-subsets $T_\in(A;\alpha)$, $I_\in(A;\beta)$ and $F_\in(A;\gamma)$ are subalgebras of $X$ for all $\alpha \in (0, \frac{1-k_T}{2}]$, $\beta \in (0, \frac{1-k_I}{2}]$ and $\gamma \in [\frac{1-k_F}{2}, 1)$ whenever they are nonempty.*

*Proof.* Assume that $A = (A_T, A_I, A_F)$ is an $(\in, \in \vee q_{(k_T,k_I,k_F)})$-neutrosophic subalgebra of $X$. Let $\beta \in (0, \frac{1-k_I}{2}]$ and $x, y \in I_\in(A;\beta)$. Then $A_I(x) \geq \beta$ and $A_I(y) \geq \beta$. It follows from Theorem 3.10 that

$$A_I(x * y) \geq \bigwedge\{A_I(x), A_I(y), \tfrac{1-k_I}{2}\} \geq \beta \wedge \tfrac{1-k_I}{2} = \beta$$

and so that $x * y \in I_\in(A;\beta)$. Hence $I_\in(A;\beta)$ is a subalgebra of $X$ for all $\beta \in (0, \frac{1-k_I}{2}]$. Similarly, we know that $T_\in(A;\alpha)$ is a subalgebra of $X$ for all $\alpha \in (0, \frac{1-k_T}{2}]$. Let $\gamma \in [\frac{1-k_F}{2}, 1)$ and $x, y \in F_\in(A;\gamma)$. Then $A_F(x) \leq \gamma$ and $A_F(y) \leq \gamma$. Using Theorem 3.10 implies that

$$A_F(x * y) \leq \bigvee\{A_F(x), A_F(y), \tfrac{1-k_F}{2}\} \leq \gamma \vee \tfrac{1-k_F}{2} = \gamma.$$

Hence $x * y \in F_\in(A;\gamma)$, and therefore $F_\in(A;\gamma)$ is a subalgebra of $X$ for all $\gamma \in [\frac{1-k_F}{2}, 1)$.

Conversely, suppose that the nonempty neutrosophic $\in$-subsets $T_\in(A;\alpha)$, $I_\in(A;\beta)$ and $F_\in(A;\gamma)$ are subalgebras of $X$ for all $\alpha \in (0, \frac{1-k_T}{2}]$, $\beta \in (0, \frac{1-k_I}{2}]$ and $\gamma \in [\frac{1-k_F}{2}, 1)$. If there exist $a, b \in X$ such that

$$A_T(a * b) < \bigwedge\{A_T(a), A_T(b), \tfrac{1-k_T}{2}\},$$

then $a, b \in T_\in(A;\alpha_T)$ by taking

$$\alpha_T := \bigwedge\{A_T(a), A_T(b), \tfrac{1-k_T}{2}\}.$$

Since $T_\in(A;\alpha_T)$ is a subalgebra of $X$, it follows that $a * b \in T_\in(A;\alpha_T)$, that is, $A_T(a * b) \geq \alpha_T$. This is a contradiction, and hence

$$A_T(x * y) \geq \bigwedge\{A_T(x), A_T(y), \tfrac{1-k_T}{2}\}$$

for all $x, y \in X$. Similarly, we can verify that

$$A_I(x * y) \geq \bigwedge\{A_I(x), A_I(y), \tfrac{1-k_I}{2}\}$$

for all $x, y \in X$. Now, assume that there exist $a, b \in X$ such that

$$A_F(a * b) > \bigvee\{A_F(a), A_F(b), \tfrac{1-k_F}{2}\}.$$

Then $A_F(a * b) > \gamma_F \geq \bigvee\{A_F(a), A_F(b), \frac{1-k_F}{2}\}$ for some $\gamma_F \in [\frac{1-k_F}{2}, 1)$. Hence $a, b \in F_\in(A;\gamma_F)$, and so $a * b \in F_\in(A;\gamma_F)$ since $F_\in(A;\gamma_F)$ is a subalgebra of $X$. It follows that $A_F(a * b) \leq \gamma_F$ which is a contradiction. Thus

$$A_F(x * y) \leq \bigvee\{A_F(x), A_F(y), \tfrac{1-k_F}{2}\}$$

for all $x, y \in X$. Therefore $A = (A_T, A_I, A_F)$ is an $(\in, \in \vee q_{(k_T,k_I,k_F)})$-neutrosophic subalgebra of $X$ by Theorem 3.10. □

**Corollary 3.13.** *Let $A = (A_T, A_I, A_F)$ be a neutrosophic set in $X \in \mathcal{B}(X)$. Then $A = (A_T, A_I, A_F)$ is an $(\in, \in \vee q)$-neutrosophic subalgebra of $X$ if and only if neutrosophic $\in$-subsets $T_\in(A;\alpha)$, $I_\in(A;\beta)$ and $F_\in(A;\gamma)$ are subalgebras of $X$ for all $\alpha, \beta \in (0, 0.5]$ and $\gamma \in [0.5, 1)$ whenever they are nonempty.*

*Proof.* It follows from taking $k_T = k_I = k_F = 0$ in Theorem 3.12. □

**Theorem 3.14.** *Every $(\in, \in)$-neutrosophic subalgebra is an $(\in, \in \vee q_{(k_T,k_I,k_F)})$-neutrosophic subalgebra.*

*Proof.* Straightforward. □

The converse of Theorem 3.14 is not true as seen in the following example.

**Example 3.15.** Consider a $BCI$-algebra $X = \{0, a, b, c\}$ with the binary operation $*$ which is given in Table 1 (see [5]).

Table 1: Cayley table for the binary operation "$*$"

| $*$ | $0$ | $a$ | $b$ | $c$ |
|---|---|---|---|---|
| $0$ | $0$ | $a$ | $b$ | $c$ |
| $a$ | $a$ | $0$ | $c$ | $b$ |
| $b$ | $b$ | $c$ | $0$ | $a$ |
| $c$ | $c$ | $b$ | $a$ | $0$ |

Let $A = (A_T, A_I, A_F)$ be a neutrosophic set in $X \in \mathcal{B}_I(X)$ defined by Table 2

Table 2: Tabular representation of "$A = (A_T, A_I, A_F)$"

| $X$ | $A_T(x)$ | $A_I(x)$ | $A_F(x)$ |
|---|---|---|---|
| $0$ | 0.6 | 0.5 | 0.2 |
| $a$ | 0.7 | 0.3 | 0.6 |
| $b$ | 0.3 | 0.6 | 0.6 |
| $c$ | 0.3 | 0.3 | 0.4 |

If $k_T = 0.36$, then

$$T_\in(A;\alpha) = \begin{cases} X & \text{if } \alpha \in (0, 0.3], \\ \{0, a\} & \text{if } \alpha \in (0.3, 0.32]. \end{cases}$$

If $k_I = 0.32$, then

$$I_\in(A;\beta) = \begin{cases} X & \text{if } \beta \in (0, 0.3], \\ \{0, b\} & \text{if } \beta \in (0.3, 0.34]. \end{cases}$$

If $k_F = 0.36$, then

$$F_\in(A;\gamma) = \begin{cases} \{0\} & \text{if } \gamma \in [0.32, 0.4), \\ \{0, c\} & \text{if } \gamma \in [0.4, 0.6), \\ X & \text{if } \gamma \in [0.6, 1]. \end{cases}$$

We know that $T_\in(A;\alpha)$, $I_\in(A;\beta)$ and $F_\in(A;\gamma)$ are subalgebras of $X$ for all $\alpha \in (0, 0.32]$, $\beta \in (0, 0.34]$ and $\gamma \in [0.32, 1)$. It follows from Theorem 3.12 that $A = (A_T, A_I, A_F)$ is an $(\in, \in \vee q_{(k_T,k_I,k_F)})$-neutrosophic subalgebra of $X$ for $k_T = 0.36$, $k_I = 0.32$ and $k_F = 0.36$. Since

$$A_T(0) = 0.6 < 0.7 = A_T(a) \wedge A_T(a)$$

and/or

$$A_I(0) = 0.5 > 0.3 = A_I(c) \vee A_I(c),$$

we know that $A = (A_T, A_I, A_F)$ is not an $(\in, \in)$-neutrosophic subalgebra of $X$ by Lemma 3.3.

**Definition 3.16.** A neutrosophic set $A = (A_T, A_I, A_F)$ in $X \in \mathcal{B}(X)$ is called an $(\in, q_{(k_T,k_I,k_F)})$*-neutrosophic subalgebra* of $X$ if

$$\begin{aligned} & x \in T_\in(A;\alpha_x),\ y \in T_\in(A;\alpha_y) \\ & \qquad \Rightarrow\ x * y \in T_{q_{k_T}}(A;\alpha_x \wedge \alpha_y), \\ & x \in I_\in(A;\beta_x),\ y \in I_\in(A;\beta_y) \\ & \qquad \Rightarrow\ x * y \in I_{q_{k_I}}(A;\beta_x \wedge \beta_y), \\ & x \in F_\in(A;\gamma_x),\ y \in F_\in(A;\gamma_y) \\ & \qquad \Rightarrow\ x * y \in F_{q_{k_F}}(A;\gamma_x \vee \gamma_y) \end{aligned} \tag{3.17}$$

for all $x, y \in X$, $\alpha_x, \alpha_y, \beta_x, \beta_y \in (0, 1]$ and $\gamma_x, \gamma_y \in [0, 1)$.

An $(\in, q_{(k_T,k_I,k_F)})$-neutrosophic subalgebra with $k_T = k_I = k_F = k$ is called an $(\in, q_k)$*-neutrosophic subalgebra*.

**Theorem 3.17.** *Every* $(\in, q_{(k_T,k_I,k_F)})$*-neutrosophic subalgebra is an* $(\in, \in \vee q_{(k_T,k_I,k_F)})$*-neutrosophic subalgebra.*

*Proof.* Straightforward. □

The converse of Theorem 3.17 is not true as seen in the following example.

**Example 3.18.** Consider the $BCI$-algebra $X = \{0, a, b, c\}$ and the neutrosophic set $A = (A_T, A_I, A_F)$ which are given in Example 3.15. Taking $k_T = 0.2$, $k_I = 0.3$ and $k_F = 0.24$ imply that

$$T_\in(A;\alpha) = \begin{cases} X & \text{if } \alpha \in (0, 0.3], \\ \{0, a\} & \text{if } \alpha \in (0.3, 0.4], \end{cases}$$

$$I_\in(A;\beta) = \begin{cases} X & \text{if } \beta \in (0, 0.3], \\ \{0, b\} & \text{if } \beta \in (0.3, 0.35], \end{cases}$$

and

$$F_\in(A;\gamma) = \begin{cases} \{0\} & \text{if } \beta \in [0.38, 0.4), \\ \{0, c\} & \text{if } \beta \in [0.4, 0.6), \\ X & \text{if } \beta \in [0.6, 1). \end{cases}$$

Since $X$, $\{0\}$, $\{0, a\}$, $\{0, b\}$ and $\{0, c\}$ are subalgebras of $X$, we know from Theorem 3.12 that $A = (A_T, A_I, A_F)$ is an $(\in, \in \vee q_{(k_T,k_I,k_F)})$-neutrosophic subalgebra of $X$ for $k_T = 0.2$, $k_I = 0.3$ and $k_F = 0.24$. Note that

$$a * b \notin T_{q_{0.2}}(A; 0.25 \wedge 0.4)$$

for $a \in T_\in(A; 0.4)$ and $b \in T_\in(A; 0.25)$,

$$b * c \notin I_{q_{0.3}}(A; 0.5 \wedge 0.27)$$

for $b \in I_\in(A; 0.5)$ and $c \in I_\in(A; 0.27)$, and/or

$$a * c \notin F_{q_{0.24}}(A; 0.6 \vee 0.44)$$

for $a \in F_\in(A; 0.6)$ and $c \in F_\in(A; 0.44)$. Hence $A = (A_T, A_I, A_F)$ is not an $(\in, q_{(0.2,0.3,0.24)})$-neutrosophic subalgebra of $X$.

**Theorem 3.19.** *If* $0 \leq k_T < j_T < 1$, $0 \leq k_I < j_I < 1$ *and* $0 \leq j_F < k_F < 1$, *then every* $(\in, \in \vee q_{(k_T,k_I,k_F)})$*-neutrosophic subalgebra is an* $(\in, \in \vee q_{(j_T,j_I,j_F)})$*-neutrosophic subalgebra.*

*Proof.* Straightforward. □

The following example shows that if $0 \leq k_T < j_T < 1$, $0 \leq k_I < j_I < 1$ and $0 \leq j_F < k_F < 1$, then an $(\in, \in \vee q_{(j_T,j_I,j_F)})$-neutrosophic subalgebra may not be an $(\in, \in \vee q_{(k_T,k_I,k_F)})$-neutrosophic subalgebra.

**Example 3.20.** Let $X$ be the $BCI$-algebra given in Example 3.15 and let $A = (A_T, A_I, A_F)$ be a neutrosophic set in $X$ defined by Table 3

Table 3: Tabular representation of "$A = (A_T, A_I, A_F)$"

| $X$ | $A_T(x)$ | $A_I(x)$ | $A_F(x)$ |
|---|---|---|---|
| 0 | 0.42 | 0.40 | 0.44 |
| $a$ | 0.40 | 0.44 | 0.66 |
| $b$ | 0.48 | 0.36 | 0.66 |
| $c$ | 0.40 | 0.36 | 0.33 |

If $k_T = 0.04$, then

$$T_\in(A;\alpha) = \begin{cases} X & \text{if } \alpha \in (0, 0.40], \\ \{0, b\} & \text{if } \alpha \in (0.40, 0.42], \\ \{b\} & \text{if } \alpha \in (0.42, 0.48]. \end{cases}$$

Note that $T_\in(A;\alpha)$ is not a subalgebra of $X$ for $\alpha \in (0.42, 0.48]$.

If $k_I = 0.08$, then

$$I_\in(A;\beta) = \begin{cases} X & \text{if } \beta \in (0, 0.36], \\ \{0, a\} & \text{if } \beta \in (0.36, 0.40], \\ \{a\} & \text{if } \beta \in (0.40, 0.44], \\ \emptyset & \text{if } \beta \in (0.44, 0.46]. \end{cases}$$

Note that $I_\in(A;\beta)$ is not a subalgebra of $X$ for $\beta \in (0.40, 0.44]$.

If $k_F = 0.42$, then

$$F_\in(A;\gamma) = \begin{cases} \emptyset & \text{if } \gamma \in [0.29, 0.33), \\ \{c\} & \text{if } \gamma \in [0.33, 0.44), \\ \{0, c\} & \text{if } \gamma \in [0.44, 0.66), \\ X & \text{if } \gamma \in [0.66, 1). \end{cases}$$

Note that $F_\in(A;\gamma)$ is not a subalgebra of $X$ for $\gamma \in [0.33, 0.44)$. Therefore $A = (A_T, A_I, A_F)$ is not an $(\in, \in \vee q_{(k_T,k_I,k_F)})$-neutrosophic subalgebra of $X$ for $k_T = 0.04$, $k_I = 0.08$ and $k_F = 0.42$.

If $j_T = 0.16$, then

$$T_\in(A;\alpha) = \begin{cases} X & \text{if } \alpha \in (0, 0.40], \\ \{0, b\} & \text{if } \alpha \in (0.40, 0.42]. \end{cases}$$

If $j_I = 0.20$, then

$$I_\in(A;\beta) = \begin{cases} X & \text{if } \beta \in (0, 0.36], \\ \{0, a\} & \text{if } \beta \in (0.36, 0.40]. \end{cases}$$

If $j_F = 0.12$, then

$$F_\in(A;\gamma) = \begin{cases} \{0, c\} & \text{if } \gamma \in [0.44, 0.66), \\ X & \text{if } \gamma \in [0.66, 1). \end{cases}$$

Therefore $A = (A_T, A_I, A_F)$ is an $(\in, \in \vee q_{(j_T,j_I,j_F)})$-neutrosophic subalgebra of $X$ for $j_T = 0.16$, $j_I = 0.20$ and $j_F = 0.12$.

Given a subset $S$ of $X$, consider a neutrosophic set $A_S = (A_{ST}, A_{SI}, A_{SF})$ in $X$ defined by

$$A_S(x) := \begin{cases} (1, 1, 0) & \text{if } x \in S, \\ (0, 0, 1) & \text{otherwise,} \end{cases}$$

that is,

$$A_{ST}(x) := \begin{cases} 1 & \text{if } x \in S, \\ 0 & \text{otherwise,} \end{cases}$$

$$A_{SI}(x) := \begin{cases} 1 & \text{if } x \in S, \\ 0 & \text{otherwise,} \end{cases}$$

and

$$A_{SF}(x) := \begin{cases} 0 & \text{if } x \in S, \\ 1 & \text{otherwise.} \end{cases}$$

**Theorem 3.21.** *A nonempty subset $S$ of $X \in \mathcal{B}(X)$ is a subalgebra of $X$ if and only if the neutrosophic set $A_S = (A_{ST}, A_{SI}, A_{SF})$ is an $(\in, \in \vee q_{(k_T,k_I,k_F)})$-neutrosophic subalgebra of $X$.*

*Proof.* Let $S$ be a subalgebra of $X$. Then neutrosophic $\in$-subsets $T_\in(A_{ST};\alpha)$, $I_\in(A_{ST};\beta)$ and $F_\in(A_{ST};\gamma)$ are obviously subalgebras of $X$ for all $\alpha \in (0, \frac{1-k_T}{2}]$, $\beta \in (0, \frac{1-k_I}{2}]$ and $\gamma \in [\frac{1-k_F}{2}, 1)$. Hence $A_S = (A_{ST}, A_{SI}, A_{SF})$ is an $(\in, \in \vee q_{(k_T,k_I,k_F)})$-neutrosophic subalgebra of $X$ by Theorem 3.12.

Conversely, assume that $A_S = (A_{ST}, A_{SI}, A_{SF})$ is an $(\in, \in \vee q_{(k_T,k_I,k_F)})$-neutrosophic subalgebra of $X$. Let $x, y \in S$. Then

$$\begin{aligned} A_{ST}(x * y) &\geq \bigwedge\{A_{ST}(x), A_{ST}(y), \tfrac{1-k_T}{2}\} \\ &= 1 \wedge \tfrac{1-k_T}{2} = \tfrac{1-k_T}{2}, \end{aligned}$$

$$\begin{aligned} A_{SI}(x * y) &\geq \bigwedge\{A_{SI}(x), A_{SI}(y), \tfrac{1-k_I}{2}\} \\ &= 1 \wedge \tfrac{1-k_I}{2} = \tfrac{1-k_I}{2} \end{aligned}$$

and

$$\begin{aligned} A_{SF}(x * y) &\leq \bigvee\{A_{SF}(x), A_{SF}(y), \tfrac{1-k_F}{2}\} \\ &= 0 \vee \tfrac{1-k_F}{2} = \tfrac{1-k_F}{2}, \end{aligned}$$

which imply that

$$A_{ST}(x * y) = 1, A_{SI}(x * y) = 1 \text{ and } A_{SF}(x * y) = 0.$$

Hence $x * y \in S$, and so $S$ is a subalgebra of $X$. □

**Theorem 3.22.** *Let $S$ be a subalgebra of $X \in \mathcal{B}(X)$. For every $\alpha \in (0, \frac{1-k_T}{2}]$, $\beta \in (0, \frac{1-k_I}{2}]$ and $\gamma \in [\frac{1-k_F}{2}, 1)$, there exists an $(\in, \in \vee q_{(k_T,k_I,k_F)})$-neutrosophic subalgebra $A = (A_T, A_I, A_F)$ of $X$ such that $T_\in(A;\alpha) = S$, $I_\in(A;\beta) = S$ and $F_\in(A;\gamma) = S$.*

*Proof.* Let $A = (A_T, A_I, A_F)$ be a neutrosophic set in $X$ defined by

$$A(x) := \begin{cases} (\alpha, \beta, \gamma) & \text{if } x \in S, \\ (0, 0, 1) & \text{otherwise,} \end{cases}$$

that is,

$$A_T(x) := \begin{cases} \alpha & \text{if } x \in S, \\ 0 & \text{otherwise,} \end{cases}$$

$$A_I(x) := \begin{cases} \beta & \text{if } x \in S, \\ 0 & \text{otherwise,} \end{cases}$$

and

$$A_F(x) := \begin{cases} \gamma & \text{if } x \in S, \\ 1 & \text{otherwise.} \end{cases}$$

Obviously, $T_{\in}(A;\alpha) = S$, $I_{\in}(A;\beta) = S$ and $F_{\in}(A;\gamma) = S$. Suppose that

$$A_T(a*b) < \bigwedge\{A_T(a), A_T(b), \tfrac{1-k_T}{2}\}$$

for some $a,b \in X$. Since $\#\mathrm{Im}(A_T) = 2$, it follows that $\bigwedge\{A_T(a), A_T(b), \frac{1-k_T}{2}\} = \alpha$ and $A_T(a*b) = 0$. Hence $A_T(a) = \alpha = A_T(b)$, and so $a,b \in S$. Since $S$ is a subalgebra of $X$, we have $a*b \in S$. Thus $A_T(a*b) = \alpha$, a contradiction. Therefore

$$A_T(x*y) \geq \bigwedge\{A_T(x), A_T(y), \tfrac{1-k_T}{2}\}$$

for all $x,y \in X$. Similarly, we can verify that

$$A_I(x*y) \geq \bigwedge\{A_I(x), A_I(y), \tfrac{1-k_I}{2}\}$$

for all $x,y \in X$. Assume that there exist $a,b \in X$ such that

$$A_F(a*b) > \bigvee\{A_F(a), A_F(b), \tfrac{1-k_F}{2}\}.$$

Then $A_F(a*b) = 1$ and $\bigvee\{A_F(a), A_F(b), \frac{1-k_F}{2}\} = \gamma$ since $\#\mathrm{Im}(A_F) = 2$. It follows that $A_F(a) = \gamma = A_F(b)$ and so that $a,b \in S$. Hence $a*b \in S$, and so $A_F(a*b) = \gamma$, which is a contradiction. Thus

$$A_F(x*y) \leq \bigvee\{A_F(x), A_F(y), \tfrac{1-k_F}{2}\}$$

for all $x,y \in X$. Therefore $A = (A_T, A_I, A_F)$ is an $(\in, \in \vee q_{(k_T,k_I,k_F)})$-neutrosophic subalgebra of $X$ by Theorem 3.10. □

**Corollary 3.23.** *Let $S$ be a subalgebra of $X \in \mathcal{B}(X)$. For every $\alpha \in (0, 0.5]$, $\beta \in (0, 0.5]$ and $\gamma \in [0.5, 1)$, there exists an $(\in, \in \vee q)$-neutrosophic subalgebra $A = (A_T, A_I, A_F)$ of $X$ such that $T_{\in}(A;\alpha) = S$, $I_{\in}(A;\beta) = S$ and $F_{\in}(A;\gamma) = S$.*

*Proof.* It follows from taking $k_T = k_I = k_F = 0$ in Theorem 3.22. □

**Theorem 3.24.** *Given a neutrosophic set $A = (A_T, A_I, A_F)$ in $X \in \mathcal{B}(X)$, the following are equivalent.*

(1) *$A = (A_T, A_I, A_F)$ is an $(\in, \in \vee q_{(k_T,k_I,k_F)})$-neutrosophic subalgebra of $X$.*

(2) *The neutrosophic $(\in \vee q_k)$-subsets $T_{\in\vee q_{k_T}}(A;\alpha)$, $I_{\in\vee q_{k_I}}(A;\beta)$ and $F_{\in\vee q_{k_F}}(A;\gamma)$ are subalgebras of $X$ for all $\alpha,\beta \in (0,1]$ and $\gamma \in [0,1)$.*

*Proof.* Assume that $A = (A_T, A_I, A_F)$ is an $(\in, \in \vee q_{(k_T,k_I,k_F)})$-neutrosophic subalgebra of $X$. Let $x,y \in I_{\in\vee q_{k_I}}(A;\beta)$ for $\beta \in (0,1]$. Then $A_I(x) \geq \beta$ or $A_I(x) + \beta + k_I > 1$, and $A_I(y) \geq \beta$ or $A_I(y) + \beta + k_I > 1$. Using Theorem 3.10, we have

$$A_I(x*y) \geq \bigwedge\{A_I(x), A_I(y), \tfrac{1-k_I}{2}\}.$$

Case 1. $A_I(x) \geq \beta$ and $A_I(y) \geq \beta$. If $\beta > \frac{1-k_I}{2}$, then

$$A_I(x*y) \geq \bigwedge\{A_I(x), A_I(y), \tfrac{1-k_I}{2}\} = \tfrac{1-k_I}{2},$$

and so $A_I(x*y) + \beta > \frac{1-k_I}{2} + \frac{1-k_I}{2} = 1 - k_I$. Hence $x*y \in I_{q_{k_I}}(A;\beta)$. If $\beta \leq \frac{1-k_I}{2}$, then

$$A_I(x*y) \geq \bigwedge\{A_I(x), A_I(y), \tfrac{1-k_I}{2}\} \geq \beta,$$

and thus $x*y \in I_{\in}(A;\beta)$. Hence

$$x*y \in I_{\in}(A;\beta) \cup I_{q_{k_I}}(A;\beta) = I_{\in\vee q_{k_I}}(A;\beta).$$

Case 2. $A_I(x) \geq \beta$ and $A_I(y) + \beta + k_I > 1$. If $\beta > \frac{1-k_I}{2}$, then

$$\begin{aligned} A_I(x*y) &\geq \bigwedge\{A_I(x), A_I(y), \tfrac{1-k_I}{2}\} \\ &= A_I(y) \wedge \tfrac{1-k_I}{2} > (1-\beta-k_I) \wedge \tfrac{1-k_I}{2} \\ &= 1-\beta-k_I, \end{aligned}$$

and so $x*y \in I_{q_{k_I}}(A;\beta)$. If $\beta \leq \frac{1-k_I}{2}$, then

$$\begin{aligned} A_I(x*y) &\geq \bigwedge\{A_I(x), A_I(y), \tfrac{1-k_I}{2}\} \\ &\geq \bigwedge\{\beta, 1-\beta-k_I, \tfrac{1-k_I}{2}\} = \beta, \end{aligned}$$

and thus $x*y \in I_{\in}(A;\beta)$. Therefore $x*y \in I_{\in\vee q_{k_I}}(A;\beta)$.

Case 3. $A_I(x) + \beta + k_I > 1$ and $A_I(y) \geq \beta$. We have $x*y \in I_{\in\vee q_{k_I}}(A;\beta)$ by the similar way to the Case 2.

Case 4. $A_I(x) + \beta + k_I > 1$ and $A_I(y) + \beta + k_I > 1$. If $\beta > \frac{1-k_I}{2}$, then $1-\beta-k_I < \frac{1-k_I}{2}$, and so

$$A_I(x*y) \geq \bigwedge\{A_I(x), A_I(y), \tfrac{1-k_I}{2}\} > 1-\beta-k_I,$$

i.e., $x*y \in I_{q_{k_I}}(A;\beta)$. If $\beta \leq \frac{1-k_I}{2}$, then

$$\begin{aligned} A_I(x*y) &\geq \bigwedge\{A_I(x), A_I(y), \tfrac{1-k_I}{2}\} \\ &\geq (1-\beta-k_I) \wedge \tfrac{1-k_I}{2} \\ &= \tfrac{1-k_I}{2} \geq \beta, \end{aligned}$$

i.e., $x*y \in I_{\in}(A;\beta)$. Hence $x*y \in I_{\in\vee q_{k_I}}(A;\beta)$. Consequently, $I_{\in\vee q_{k_I}}(A;\beta)$ is a subalgebra of $X$. Similarly, we can prove that if $x,y \in T_{\in\vee q_{k_T}}(A;\alpha)$ for $\alpha \in (0,1]$, then $x*y \in T_{\in\vee q_{k_T}}(A;\alpha)$, that is, $T_{\in\vee q_{k_T}}(A;\alpha)$ is a subalgebra of $X$. Let $x,y \in F_{\in\vee q_{k_F}}(A;\gamma)$ for $\gamma \in [0,1)$. Then $A_F(x) \leq \gamma$ or $A_F(x)+\gamma+k_F < 1$, and $A_F(y) \leq \gamma$ or $A_F(y)+\gamma+k_F < 1$. Using Theorem 3.10, we have

$$A_F(x*y) \leq \bigvee\{A_F(x), A_F(y), \tfrac{1-k_F}{2}\}.$$

Case 1. $A_F(x) \leq \gamma$ and $A_F(y) \leq \gamma$. If $\gamma < \frac{1-k_F}{2}$, then

$$A_F(x*y) \leq \bigvee\{A_F(x), A_F(y), \tfrac{1-k_F}{2}\} = \tfrac{1-k_F}{2},$$

and so $A_F(x*y) + \gamma < \frac{1-k_F}{2} + \frac{1-k_F}{2} = 1 - k_F$. Hence $x*y \in F_{q_{k_F}}(A;\gamma)$. If $\gamma \geq \frac{1-k_F}{2}$, then

$$A_F(x*y) \leq \bigvee\{A_F(x), A_F(y), \tfrac{1-k_F}{2}\} \leq \gamma,$$

and thus $x*y \in F_{\in}(A;\gamma)$. Hence

$$x*y \in F_{\in}(A;\gamma) \cup F_{q_{k_F}}(A;\gamma) = F_{\in\vee\, q_{k_F}}(A;\gamma).$$

Case 2. $A_F(x) \leq \gamma$ and $A_F(y) + \gamma + k_F < 1$. If $\gamma < \frac{1-k_F}{2}$, then

$$\begin{aligned} A_F(x*y) &\leq \bigvee\{A_F(x), A_F(y), \tfrac{1-k_F}{2}\} \\ &= A_F(y) \vee \tfrac{1-k_F}{2} < (1-\gamma-k_F) \vee \tfrac{1-k_F}{2} \\ &= 1-\gamma-k_F, \end{aligned}$$

and so $x*y \in F_{q_{k_F}}(A;\gamma)$. If $\gamma \geq \frac{1-k_F}{2}$, then

$$\begin{aligned} A_F(x*y) &\leq \bigvee\{A_F(x), A_F(y), \tfrac{1-k_F}{2}\} \\ &\leq \bigvee\{\gamma, 1-\gamma-k_F, \tfrac{1-k_F}{2}\} = \gamma, \end{aligned}$$

and thus $x*y \in F_{\in}(A;\gamma)$. Therefore $x*y \in F_{\in\vee\, q_{k_F}}(A;\gamma)$.

Similarly, if $A_I(x) + \beta + k_I < 1$ and $A_I(y) \leq \beta$, then $x*y \in F_{\in\vee\, q_{k_F}}(A;\gamma)$.

Finally, assume that $A_F(x) + \gamma + k_F < 1$ and $A_F(y) + \gamma + k_F < 1$. If $\gamma < \frac{1-k_F}{2}$, then $1-\gamma-k_F > \frac{1-k_F}{2}$, and so

$$A_F(x*y) \leq \bigvee\{A_F(x), A_F(y), \tfrac{1-k_F}{2}\} < 1-\gamma-k_F,$$

i.e., $x*y \in F_{q_{k_F}}(A;\gamma)$. If $\gamma \geq \frac{1-k_F}{2}$, then

$$\begin{aligned} A_F(x*y) &\leq \bigvee\{A_F(x), A_F(y), \tfrac{1-k_F}{2}\} \\ &\leq (1-\gamma-k_F) \vee \tfrac{1-k_F}{2} \\ &= \tfrac{1-k_F}{2} \leq \gamma, \end{aligned}$$

i.e., $x*y \in F_{\in}(A;\gamma)$. Hence $x*y \in F_{\in\vee\, q_{k_F}}(A;\gamma)$. Therefore $F_{\in\vee\, q_{k_F}}(A;\gamma)$ is a subalgebra of $X$.

Conversely, suppose that (2) is valid. If it is possible, let

$$A_T(x*y) < \alpha \leq \bigwedge\{A_T(x), A_T(y), \tfrac{1-k_T}{2}\}$$

for some $\alpha \in (0, \frac{1-k_T}{2})$. Then

$$x, y \in T_{\in}(A;\alpha) \subseteq T_{\in\vee\, q_{k_T}}(A;\alpha),$$

which implies that $x*y \in T_{\in\vee\, q_{k_T}}(A;\alpha)$. Thus $A_T(x*y) \geq \alpha$ or $A_T(x*y) + \alpha + k_T > 1$, a contradiction. Hence

$$A_T(x*y) \geq \bigwedge\{A_T(x), A_T(y), \tfrac{1-k_T}{2}\}$$

for all $x, y \in X$. Similarly, we can verify that

$$A_I(x*y) \geq \bigwedge\{A_I(x), A_I(y), \tfrac{1-k_I}{2}\}$$

for all $x, y \in X$. Now assume that there exist $a, b \in X$ and $\gamma \in (\frac{1-k_F}{2}, 1)$ such that

$$A_F(a*b) > \gamma \geq \bigvee\{A_F(a), A_F(b), \tfrac{1-k_F}{2}\}.$$

Then $a, b \in F_{\in}(A;\gamma) \subseteq F_{\in\vee\, q_{k_F}}(A;\gamma)$, which implies that

$$a*b \in F_{\in\vee\, q_{k_F}}(A;\gamma).$$

Thus $A_F(a*b) \leq \gamma$ or $A_F(a*b) + \gamma + k_F < 1$, which is a contradiction. Hence

$$A_F(x*y) \geq \bigvee\{A_F(x), A_F(y), \tfrac{1-k_F}{2}\}$$

for all $x, y \in X$. Using Theorem 3.10, we conclude that $A = (A_T, A_I, A_F)$ is an $(\in, \in\vee q_{(k_T,k_I,k_F)})$-neutrosophic subalgebra of $X$. □

# 4 Conclusions

Neutrosophic set theory is a nice mathematical tool which can be applied to several fields. The aim of this paper is to consider a general form of neutrosophic points, and to discuss generalizations of the papers [4] and [6]. We have introduce the notions of $(\in, \in\vee q_{(k_T,k_I,k_F)})$-neutrosophic subalgebra, and $(\in, q_{(k_T,k_I,k_F)})$-neutrosophic subalgebra in $BCK/BCI$-algebras, and have investigated several properties. We have discussed characterizations of $(\in, \in\vee q_{(k_T,k_I,k_F)})$-neutrosophic subalgebra. We have considered relations between $(\in, \in)$-neutrosophic subalgebra, $(\in, q_{(k_T,k_I,k_F)})$-neutrosophic subalgebra and $(\in, \in\vee q_{(k_T,k_I,k_F)})$-neutrosophic subalgebra. We hope the idea and result in this paper can be a mathematical tool for dealing with several informations containing uncertainty such as medical diagnosis, decision making, graph theory, etc. So, based on the results in this article, our future research will be focused to solve real-life problems under the opinions of experts in a neutrosophic set environment such as medical diagnosis, decision making, graph theory etc. In particular, Bucolo et al. [2] suggested a generalization of the synchronization principles for the class of array of fuzzy logic chaotic based dynamical systems and evaluated as alternative approach to build locally connected fuzzy complex systems by manipulating both the rules driving the cells and the architecture of the system. We will also try to study complex dynamics through neutrosophic environment. The future works also may use the study neutrosophic set environment on several related algebraic structures, for example, $MV$-algebras, $BL$-algebras, $R_0$-algebras, $EQ$-algebras, equality algebras, $MTL$-algebras etc.

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