Neutrosophic Sets and Systems, Vol. 32, 2020
University of New Mexico

Arithmetic and Geometric Operators of Pentagonal Neutrosophic Number and its Application in Mobile Communication Service Based MCGDM Problem

Avishek Chakraborty [1,2*],Baisakhi Banik [3],Sankar Prasad Mondal[4] and Shariful Alam[2]

[1]Department of Basic Science, Narula Institute of Technology, Agarpara, Kolkata-700109, India.
Email- avishek.chakraborty@nit.ac.in
[2]Department of Mathematics, Indian Institute of Engineering Science and Technology, Shibpur, Howrah-711103, India.
Email- salam50in@yahoo.co.in
[3]Department of Basic Science, Techno Engineering College (Banipur), Habra, W.Bengal-743233, India.
Email-baisakhibanik14@gmail.com.
[4]Department of Applied Science, MaulanaAbulKalam Azad University of Technology, West Bengal, Haringhata-721249, Nadia,WestBengal, India. Email-sankar.res07@gmail.com
*Correspondence: avishek.chakraborty@nit.ac.in.

Abstract: In this paper, the theory of pentagonal neutrosophic number has been studied in a disjunctive frame of reference. Moreover, the dependency and independency of the membership functions for the pentagonal neutrosophic number are also classified here. Additionally, the development of a new score function and its computation have been formulated in distinct rational perspectives. Further, weighted arithmetic averaging operator and weighted geometric averaging operator in the pentagonal neutrosophic environment are introduced here using an influx of different logical & innovative thought. Also, a multi-criteria group decision-making problem (MCGDM) in a mobile communication system is formulated in this paper as an application in the pentagonal neutrosophic arena. Lastly, the sensitivity analysis portion reflects the variation of this noble work.

Keywords: Pentagonal neutrosophic number, Weighted arithmetic and geometric averaging operator, Score functions, MCGDM.

1. Introduction

1.1 Neutrosophic Sets

Handling the notion of vagueness and uncertainty concepts, fuzzy set theory is a dominant field, was first presented by Zadeh [1] in his paper (1965).Vagueness theory has a salient feature for solving engineering and statistical problem very lucidly. It has a great impact on social-science, networking, decision making and numerous kinds of realistic problems. On the basis of ideas of Zadeh's research paper, Atanassov [2] invented the prodigious concept of intuitionistic fuzzy set where he precisely interpreted the idea of membership as well as non membership function very aptly. Further, researchers developed the formulation of triangular [3], trapezoidal [4], pentagonal [5] fuzzy numbers in uncertainty arena. Also, Liu & Yuan [6] established the concept of the triangular intuitionistic fuzzy set;Ye [7] put forth the basic idea of trapezoidal intuitionistic fuzzy set in the research field. Naturally, the question arises, how can we evolve the idea of uncertainty concepts in mathematical modelling? Researchers have invented disjunctive kinds of methodologies to define elaborately the concepts and have suggested some new kinds of ambivalent parameters. To deal with those kinds of problems, the decision-makers' choice varies in different areas. F. Smarandache [8] in 1998 generated the concept of a neutrosophic set having three different integrants namely, (i) truthiness, (ii) indeterminacies, and (iii) falseness. Each and every characteristic of the neutrosophic set are very pertinent factors in our real-life models. Later, Wang et al. [9] proceeded with the idea of a single typed neutrosophic set, which is very productive to sort out the solution of any complicated kind of problem. Recently, Chakraborty et al. [10, 11] conceptualized the dynamic idea of triangular and trapezoidal neutrosophic numbers in the research domain and applied it in different real-life problem. Also, Maity et al. [12] built the perception of ranking and defuzzification in a completely different type of attributes. To handle human decision making procedure on the basis of positive and negative sides, Bosc and Pivert [13] cultivated the notion of bipolarity. With that continuation, Lee [14] elucidated the perception of bipolar fuzzy set in their research article. Further, Kang and Kang [15] broadened this concept into semi-groups and group structures field. As research proceeded, Deli et al. [16] germinated the idea of a bipolar neutrosophic set and used it as an implication to a decision-making related problem. Broumi et al. [17] produced the idea of bipolar neutrosophic graph theory and, subsequently, Ali and Smarandache [18] put forth the concept of the uncertain complex neutrosophic set. Chakraborty [19] introduced the triangular bipolar number in different aspects. In succession; Wang et al. [20] also introduced the idea of operators in a bipolar neutrosophic set and applied it in a decision-making problem. The multi-criteria decision making (MCDM) problem is a supreme interest to the researchers who deal with the decision scientific analysis. Presently, it is more acceptable in such issues where a group of criteria is utilized. Such cases of problems relating to multi-criteria group decision making (MCGDM) have shown its fervent influence. Also MCDM has broad applications in disjunctive fields under various uncertainty contexts.We can find many applications and development of neutrosophic theory in multi-criteria decision making problem in the literature surveys presented in [21–25], graph theory [26-30], optimization techniques [31-33] etc. In this current era, Basset [34-40] presented some worthy articles related to neutrosophic sphere and applied it in many different well-known fields.Also, K.Mondal [41,42] successfully applied the notion of neutrosophic number

in faculty recruitment MCDM problem in education purpose. Recently, the viewpoint of plithogenic set is being constructed by Abdel [43] and it has an immense influential motivation in impreciseness field in various sphere of research field. Also, Chakraborty [44] developed the conception of cylindrical neutrosophic number is minimal tree problem.

Neutrosophic concept is very fruitful & vibrant in a realistic approach in the recent research field. R. Helen [45] first germinated the idea of the pentagonal fuzzy number then Christi [46] utilized the conception of pentagonal fuzzy number into pentagonal intuitionistic number and skillfully applied it to solve a transportation problem. Additionally, Chakraborty [47, 48] put forward the notion of pentagonal neutrosophicnumber and its different and disjunctive representation in transportation problem and graph-theoretical research arenas. Subsequently, Karaaslan [51-56] put forth some innovative idea on multi-attribute decision making in neutrosophic domain. Also, Karaaslan [57-61] presented the notion of soft set theory with the appropriate justification of neutrosophic fuzzy number. Recently, Broumi et.al [62-66] manifested the conception of the graph-theoretical shortest path problem under neutrosophic environment. Further, Broumi [67] implemented the concept of neutrosophic membership functions using MATLAB programming. A few works [68-71] are also established recently, based on impreciseness domain.

In this article, we mainly focus on the different representation of pentagonal neutrosophic number and its dependency, independency portions. We generate a new logical score function for crispification of pentagonal neutrosophic number. Additionally, we introduce two different logical operators namely i) pentagonal neutrosophic weighted arithmetic averaging operator (PNWAA), ii) pentagonal neutrosophic weighted geometric averaging operator (PNWGA) and established its theoretical developments along with its different properties. Also, we discussed the utility of these operators in real-life problems. Later, we consider a mobile communication based MCGDM problem in neutrosophic domain and solve it using the established two operators & score function.Sensitivity analysis of this problem is also addressed here which will show distinct results in different aspects. Finally, comparison analysis is performed here with the established methods which give an important impact in the research arena. This noble thought will help us to solve a plethora of daily life problems in uncertainty arena.

1.2 Motivation for the study

With the advent of vagueness theory the arena of numerous realistic mathematical modeling, engineering structural issues, multi-criteria problem have immensely achieved a productive and impulsive effect.Naturally it is very intriguing to the researchers that if someone sheds light on the pentagonal neutrosophic number then what will be it in the form of linearity and its classification? Based on this perception we impose three components on a pentagonal neutrosophic number i.e. truthiness, indeterminacy and falsity. Proceeding with the PNNWAA and PNNWGA operators and based on the score function of pentagonal neutrosophic numbers, an MCGDM method is built up and some interesting and worthy conclusions are tried to extract from this research article.

1.3 Novelties of the work

Recently, researchers are utmost persevere to develop theories connecting neutrosophic field and constantly try to generate its distinct application in various sphere of neutrosophic arena. However, justifying all the perspectives related to pentagonal neutrosophic fuzzy set theory; numerous theories and problems are yet to be solved. In this research article our ultimate objective is to shed light some unfocussed points in the pentagonal domain.

(1) Classification of Pentagonal Neutrosophic Number.
(2) Illustrative demonstration of aggregation operations and geometric operations on Pentagonal Neutrosophic Number's.
(3) Proposed new score function and its utility.
(4) Execute the idea of Pentagonal Neutrosophic Number's in MCGDM problem.

2. Preliminaries

Definition 2.1: Fuzzy Set: [1] Let $\tilde{A}$ be a set such that $\tilde{A} = \{(\beta, \alpha_{\tilde{A}}(\beta)) : \beta \epsilon A, \alpha_{\tilde{A}}(\beta) \epsilon [0,1]\}$ which is normally denoted by this ordered pair$(\beta, \alpha_{\tilde{A}}(\beta))$, here β is a member of the set$A$and $0 \leq \alpha_{\tilde{A}}(\beta) \leq 1$, then set $\tilde{A}$ is called a fuzzy set.

Definition 2.2: Neutrosophic Set:[8] A set $\tilde{A}_{Neu}$ in the domain of discourse A, most commonly stated as $\in$ is called a neutrosophic set if $\tilde{A}_{Neu} = \{\langle \in; [\varphi_{\widetilde{A_{Neu}}}(\in), \gamma_{\widetilde{A_{Neu}}}(\in), \delta_{\widetilde{A_{Neu}}}(\in)] \rangle : \in \epsilon A\}$,where $\varphi_{\widetilde{A_{Neu}}}(\in): A \to]-0,1+[$ symbolizes the index of confidence, $\gamma_{\widetilde{A_{Neu}}}(\in): A \to]-0,1+[$symbolizes the index of uncertainty and $\delta_{\widetilde{A_{Neu}}}(\in): A \to]-0,1+[$symbolizes the degree of falseness in the decision making procedure. Where,$[\varphi_{\widetilde{A_{Neu}}}(\in), \gamma_{\widetilde{A_{Neu}}}(\in), \delta_{\widetilde{A_{Neu}}}(\in)]$ satisfies the in the equation

$$-0 \leq \varphi_{\widetilde{A_{Neu}}}(\in) + \gamma_{\widetilde{A_{Neu}}}(\in) + \delta_{\widetilde{A_{Neu}}}(\in) \leq 3+.$$

Definition 2.3: Single Typed Neutrosophic Number: [8]Single Typed Neutrosophic Number $(\tilde{n})$ is denoted as $\tilde{n} = \langle [(u^1, v^1, w^1, x^1); \alpha], [(u^2, v^2, w^2, x^2); \beta], [(u^3, v^3, w^3, x^3); \gamma] \rangle$where$\alpha, \beta, \gamma \in [0,1]$, where$(\varphi_{\tilde{n}}): \mathbb{R} \to [0, \alpha]$, $(\gamma_{\tilde{n}}): \mathbb{R} \to [\beta, 1]$ and $(\delta_{\tilde{n}}): \mathbb{R} \to [\gamma, 1]$ is given as:

$$\varphi_{\tilde{n}}(\in) = \begin{cases} €_{\widetilde{nl}}(\in) \; when \; u^1 \leq \in \leq v^1 \\ \alpha \quad when \; v^1 \leq \in \leq w^1 \\ €_{\widetilde{nu}}(\in) \; when \; w^1 \leq \in \leq x^{1\prime} \\ 0 \qquad otherwise \end{cases}$$

$$£_{\tilde{n}}(\in) = \begin{cases} \gamma_{\widetilde{nl}}(\in) \; when \; u^2 \leq \in \leq v^2 \\ \beta \quad when \; v^2 \leq \in \leq w^2 \\ \gamma_{\widetilde{nu}}(\in) \; when \; w^2 \leq \in \leq x^2 \\ 1 \qquad otherwise \end{cases}$$

and

$$\delta_{\tilde{n}}(\in) = \begin{cases} \mu_{\widetilde{nl}}(\in) \; when \; u^3 \leq \in \leq v^3 \\ \gamma \quad when \; v^3 \leq \in \leq w^3 \\ \mu_{\widetilde{nu}}(\in) \; when \; w^3 \leq \in \leq x^3 \\ 1 \qquad otherwise \end{cases}$$

Definition 2.4: Single-Valued Neutrosophic Set:[9] A Neutrosophic set in the definition 2.2 is $\tilde{A}_{Neu}$said to be a Single-Valued Neutrosophic Set $(\tilde{A}_{Neu})$ if $\in$ is a single-valued independent variable. $\tilde{A}_{Neu} = \{\langle \in; [\alpha_{\tilde{A}_{Neu}}(\in), \beta_{\tilde{A}_{Neu}}(\in), \gamma_{\tilde{A}_{Neu}}(\in)] \rangle : \in \epsilon A\}$, where $\alpha_{\tilde{A}_{Neu}}(\in), \beta_{\tilde{A}_{Neu}}(\in) \& \gamma_{\tilde{A}_{Neu}}(\in)$ denote the idea of accuracy, ambiguity and falsity membership functions respectively.$\widetilde{SnS}$is named as neut-convex, which implies that $\widetilde{SnS}$ is a subset of R by satisfying the following criterion:

i. $\alpha_{\tilde{A}_{Neu}}\langle \delta a_1 + (1-\delta) a_2 \rangle \geq min\langle \alpha_{\tilde{A}_{Neu}}(a_1), \alpha_{\tilde{A}_{Neu}}(a_2) \rangle$

ii. $\beta_{\tilde{A}_{Neu}}\langle \delta a_1 + (1-\delta) a_2 \rangle \leq max\langle \beta_{\tilde{A}_{Neu}}(a_1), \beta_{\tilde{A}_{Neu}}(a_2) \rangle$

iii. $\gamma_{\tilde{A}_{Neu}}\langle \delta a_1 + (1-\delta) a_2 \rangle \leq max\langle \gamma_{\tilde{A}_{Neu}}(a_1), \gamma_{\tilde{A}_{Neu}}(a_2) \rangle$

where$a_1 \& a_2 \epsilon \mathbb{R} and \delta \epsilon [0,1]$

3. Single Type Linear Pentagonal Neutrosophic Number:

In this section we introduce different type single type linear pentagonal neutrosophic number. For the help of the researchers we pictorially draw the following block diagram as follows:

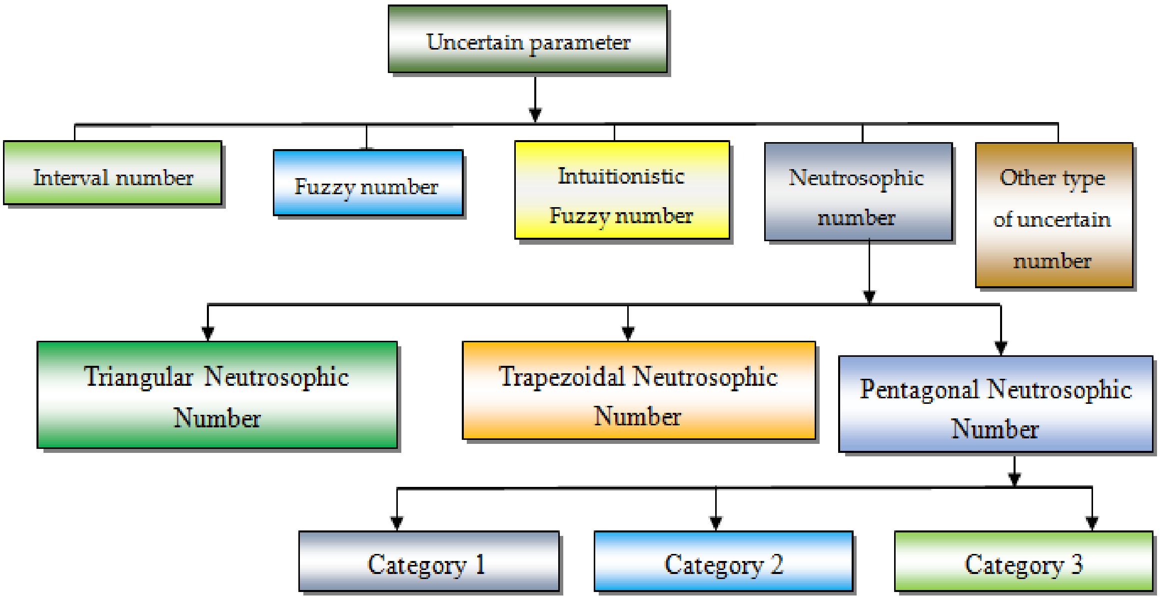

Figure 3.1: Block diagram for a different type of uncertain numbers and their categories

Definition 3.1: Single-Valued Pentagonal Neutrosophic Number: [47]A Single-Valued Pentagonal Neutrosophic Number $(\widetilde{N_{Pen}})$ is defined as$\widetilde{N_{Pen}} =$ $\langle[(h_1, h_2, h_3, h_4, h_5); \pi], [(h_1, h_2, h_3, h_4, h_5); \mu], [(h_1, h_2, h_3, h_4, h_5); \sigma]\rangle$, where$\pi, \mu, \sigma \in [0,1]$. The accuracy membership function$(\tau_{\tilde{S}}): \mathbb{R} \to [0, \pi]$, the ambiguity membership function $(\vartheta_{\tilde{S}}): \mathbb{R} \to [\rho, 1]$ and the falsity membership function $(\varepsilon_{\tilde{S}}): \mathbb{R} \to [\sigma, 1]$ are defined by:

$$\tau_{\tilde{S}}(x) = \begin{cases} \frac{\pi(x-h_1)}{(h_2-h_1)} & when\ h_1 \le x \le h_2 \\ \frac{\pi(x-h_2)}{(h_3-h_2)} & when\ h_2 \le x < h_3 \\ \pi & when\ x = h_3 \\ \frac{\pi(h_4-x)}{(h_4-h_3)} & when\ h_3 < x \le h_4 \\ \frac{\pi(h_4-x)}{(h_5-h_4)} & when\ h_4 \le x \le h_5 \\ 0 & otherwise \end{cases},$$

$$\vartheta_{\tilde{S}}(x) = \begin{cases} \frac{h_2 - x + \mu(x - h_1)}{(h_2 - h_1)} & when\ h_1 \le x \le h_2 \\ \frac{h_3 - x + \mu(x - h_2)}{(h_3 - h_2)} & when\ h_2 \le x < h_3 \\ \mu & when\ x = h_3 \\ \frac{x - h_3 + \mu(h_4 - x)}{(h_4 - h_3)} & when\ h_3 < x \le h_4 \\ \frac{x - h_4 + \mu(h_5 - x)}{(h_5 - h_4)} & when\ h_4 \le x \le h_5 \\ 1 & otherwise \end{cases}$$

and

$$\varepsilon_{\tilde{S}}(x) = \begin{cases} \dfrac{h_2 - x + \sigma(x - h_1)}{(h_2 - h_1)} & when\ h_1 \le x \le h_2 \\ \dfrac{h_3 - x + \sigma(x - h_2)}{(h_3 - h_2)} & when\ h_2 \le x < h_3 \\ \sigma & when\ x = h_3 \\ \dfrac{x - h_3 + \sigma(h_4 - x)}{(h_4 - h_3)} & when\ h_3 < x \le h_4 \\ \dfrac{x - h_4 + \sigma(h_5 - x)}{(h_5 - h_4)} & when\ h_4 \le x \le h_5 \\ 1 & otherwise \end{cases}$$

4. Proposed Score Function:

Score function of a pentagonal neutrosophic number entirely depends on the value of truth membership indicator degree, falsity membership indicator degree and uncertainty membership indicator degree. The necessity of score function is to draw a comparison or transfer a pentagonal neutrosophic fuzzy number into a crisp number. In this section, we will generate a score function as follows.For any Pentagonal Single typed Neutrosophic Number (PSNN)

$$\tilde{A}_{Pt} = (s_1, s_2, s_3, s_4, s_5; \pi, \mu, \sigma)$$

We define the score function as

$$S_{Pt} = \frac{1}{15}(s_1 + s_2 + s_3 + s_4 + s_5) \times (2 + \pi - \sigma - \mu)$$

Here, S_{Pt} belongs to [0,1].

4.1Relationship between any two pentagonal neutrosophic fuzzy numbers:

Let us consider any two pentagonal neutrosophic fuzzy number defined as follows

$\tilde{A}_{Pt1} = (s_{Pt11}, s_{Pt12}, s_{Pt13}, s_{Pt14}, s_{Pt15}; \pi_{Pt1}, \mu_{Pt1}, \sigma_{Pt1})and\tilde{A}_{Pt2} = (s_{Pt21}, s_{Pt22}, s_{Pt23}, s_{Pt24}, s_{Pt25}; \pi_{Pt2}, \mu_{Pt2}, \sigma_{Pt2})$

The score function for the are

$$S_{Pt1} = \frac{1}{15}(s_{Pt11} + s_{Pt12} + s_{Pt13} + s_{Pt14} + s_{Pt15}) \times (2 + \pi_{Pt1} - \sigma_{Pt1} - \mu_{Pt1})$$

and

$$S_{Pt2} = \frac{1}{15}(s_{Pt21} + s_{Pt22} + s_{Pt23} + s_{Pt24} + s_{Pt25}) \times (2 + \pi_{Pt2} - \sigma_{Pt2} - \mu_{Pt2})$$

Then we can say the following

1) $\tilde{A}_{Pt1} > \tilde{A}_{Pt2}$ if $S_{Pt1} > S_{Pt2}$
2) $\tilde{A}_{Pt1} < \tilde{A}_{Pt2}$if $S_{Pt1} < S_{Pt2}$
3) $\tilde{A}_{Pt1} = A_{Pt2}$ if $S_{Pt1} = S_{Pt2}$

Table 4.1: Numerical Examples

Pentagonal Neutrosophic Number ($\tilde{A}_{Pt}$)	Score Value (S_{Pt})	Ordering
$\tilde{A}_{Pt1} =< (0.2, 0.3, 0.4, 0.5, 0.6; 0.4, 0.5, 0.6) >$	0.17333	$A_{Pt4} > A_{Pt2} > A_{Pt1} > A_{Pt3}$
$\tilde{A}_{Pt2} =< (0.35, 0.4, 0.45, 0.5, 0.55; 0.6, 0.3, 0.4) >$	0.28500	
$\tilde{A}_{Pt3} =< (0.15, 0.2, 0.25, 0.3, 0.35; 0.6, 0.4, 0.5) >$	0.14167	
$\tilde{A}_{Pt4} =< (0.7, 0.75, 0.8, 0.85, 0.9; 0.3, 0.2, 0.6) >$	0.40000	

4.1 Basic Operations for pentagonal neutrosophic fuzzy number:

Let $\widetilde{p_1} = \langle (c_1, c_2, c_3, c_4, c_5); \pi_{\widetilde{p_1}}, \mu_{\widetilde{p_1}}, \sigma_{\widetilde{p_1}} \rangle$ and $\widetilde{p_2} = \langle (d_1, d_2, d_3, d_4, d_5); \pi_{\widetilde{p_2}}, \mu_{\widetilde{p_2}}, \sigma_{\widetilde{p_2}} \rangle$ be two IPFNs and $\alpha \geq 0$. Then the following operational relations hold:

4.1.1 Addition:

$$\widetilde{p_1} + \widetilde{p_2} = \langle (c_1 + d_1, c_2 + d_2, c_3 + d_3, c_4 + d_4, c_5 + d_5); \pi_{\widetilde{p_1}} + \pi_{\widetilde{p_2}} - \pi_{\widetilde{p_1}}\pi_{\widetilde{p_2}}, \mu_{\widetilde{p_1}}\mu_{\widetilde{p_2}}, \sigma_{\widetilde{p_1}}\sigma_{\widetilde{p_2}} \rangle$$

4.1.2Multipliction:

$$\widetilde{p_1}\widetilde{p_2} = \langle (c_1 d_1, c_2 d_2, c_3 d_3, c_4 d_4, c_5 d_5);\ \pi_{\widetilde{p_1}}\pi_{\widetilde{p_2}}, \mu_{\widetilde{p_1}} +, \mu_{\widetilde{p_2}} - \mu_{\widetilde{p_1}}\mu_{\widetilde{p_2}}, \sigma_{\widetilde{p_1}} + \sigma_{\widetilde{p_2}} - \sigma_{\widetilde{p_1}}\sigma_{\widetilde{p_2}} \rangle$$

4.1.3 Multiplication by scalar:

$$\alpha\widetilde{p_1} = \langle (\alpha c_1, \alpha c_2, \alpha c_3, \alpha c_4, \alpha c_5); 1 - (1 - \pi_{\widetilde{p_1}})^{\alpha}, \mu_{\widetilde{p_1}}{}^{\alpha}, \sigma_{\widetilde{p_1}}{}^{\alpha}) \rangle$$

4.1.4 Power:

$$\widetilde{p_1}^{\alpha} = \langle (c_1{}^{\alpha}, c_2{}^{\alpha}, c_3{}^{\alpha}, c_4{}^{\alpha}, c_5{}^{\alpha}); \pi_{\widetilde{p_1}}{}^{\alpha}, (1 - \mu_{\widetilde{p_1}})^{\alpha}, (1 - \sigma_{\widetilde{p_1}})^{\alpha} \rangle$$

5. Arithmetic and Geometric Operators:

5.1 Two weighted aggregation operators of Pentagonal Neutrosophic Numbers

Aggregation operators are such pertinent tool for aggregating information to tactfully handle the decision making procedure, this section generates a brief understanding between two weighted aggregation operators to aggregate PNNs as a generalization of the weighted aggregation operators for PNNs, which are broadly and aptly used in decision making.

5.1.1 Pentagonal neutrosophic weighted arithmetic averaging operator

Let $\tilde{p}_j = \langle (c_{j1}, c_{j2}, c_{j3}, c_{j4}, c_{j5});\ \pi_{\widetilde{p_1}}, \mu_{\widetilde{p_1}}, \sigma_{\widetilde{p_1}} \rangle (j = 1,2,3, \dots, n)$ be a set of PNNs, then a PNWAA operator is defined as follows:

$$PNWAA\ (\tilde{p}_1, \tilde{p}_2, \dots, \tilde{p}_n) = \sum_{j=1}^{n} \omega_j\, \tilde{p}_j \qquad (5.1)$$

whereω_j is the weight of $\tilde{p}_j (j = 1,2,3, \dots, n)$ such that $\omega_j > 0$ and $\sum_{j=1}^{n} \omega_j = 1$.

In accordance with the result ofSection 4.1 and equation (5.1) we can introduce the following theorems:

Theorem 5.1. Let $\tilde{p}_j = \langle (c_{j1}, c_{j2}, c_{j3}, c_{j4}, c_{j5});\ \pi_{\widetilde{p_1}}, \mu_{\widetilde{p_1}}, \sigma_{\widetilde{p_1}} \rangle (j = 1,2,3, \dots, n)$ be a set of PNNs, then according to Section 4.1 and equation (5.1) we can give the following PNWAA operator

$$PNWAA\ (\tilde{p}_1, \tilde{p}_2, \dots, \tilde{p}_n) = \sum_{j=1}^{n} \omega_j\, \tilde{p}_j$$

$$= \langle \left(\sum_{j=1}^{n} \omega_j\, c_{j1}, \sum_{j=1}^{n} \omega_j\, c_{j2}, \sum_{j=1}^{n} \omega_j\, c_{j3}, \sum_{j=1}^{n} \omega_j\, c_{j4}, \sum_{j=1}^{n} \omega_j\, c_{j5} \right); 1 - \prod_{j=1}^{n} (1 - \pi_{\widetilde{p_j}})^{\omega_j}, \prod_{j=1}^{n} \mu_{\widetilde{p_j}}{}^{\omega_j}, \prod_{j=1}^{n} \sigma_{\widetilde{p_j}}{}^{\omega_j} \rangle$$

Where ω_j is the weight of $\tilde{p}_j (j = 1,2,3, \dots, n)$ such that $\omega_j > 0$ and $\sum_{j=1}^{n} \omega_j = 1$.

Theorem 5.1 can be proved with the help of mathematical induction.

Proof: When $n = 2$ then,

$$\omega_1\tilde{p}_1 = \langle (\omega_1 c_{11}, \omega_1 c_{12}, \omega_1 c_{13}, \omega_1 c_{14}, \omega_1 c_{15});\ 1 - \left(1 - \pi_{\widetilde{p_1}}\right)^{\omega_1} \sigma_{\widetilde{p_2}}{}^{\omega_2}, \mu_{\widetilde{p_1}}{}^{\omega_1}, \sigma_{\widetilde{p_1}}{}^{\omega_1} \rangle$$

and $\omega_2\tilde{p}_2 = \langle (\omega_2 c_{21}, \omega_2 c_{22}, \omega_2 c_{23}, \omega_2 c_{24}, \omega_2 c_{25});\ 1 - \left(1 - \pi_{\widetilde{p_2}}\right)^{\omega_2}, \mu_{\widetilde{p_2}}{}^{\omega_2}, \sigma_{\widetilde{p_2}}{}^{\omega_2} \rangle$

Thus, $PNWAA\ (\widetilde{p_1}, \widetilde{p_2}) = \omega_1 \tilde{p}_1 + \omega_2 \tilde{p}_2$

$=\langle(\omega_1 c_{11} + \omega_2 c_{21} + \omega_1 c_{12} + \omega_2 c_{22} + \omega_1 c_{13} + \omega_2 c_{23} + \omega_1 c_{14} + \omega_2 c_{24} + \omega_1 c_{15} + \omega_2 c_{25}); 1 - \left(1-\pi_{\widetilde{p_1}}\right)^{\omega_1} + 1 - \left(1-\pi_{\widetilde{p_2}}\right)^{\omega_2}(1-\left(1-\pi_{\widetilde{p_1}}\right)^{\omega_1})(1-\left(1-\pi_{\widetilde{p_2}}\right)^{\omega_2}), \mu_{\widetilde{p_1}}{}^{\omega_1}\mu_{\widetilde{p_2}}{}^{\omega_2}, \sigma_{\widetilde{p_1}}{}^{\omega_1}\sigma_{\widetilde{p_2}}{}^{\omega_2}\rangle$

When applying $n = k$, by applying equation (5.1) , we get

$PNWAA\ (\tilde{p}_1, \tilde{p}_2, \ldots, \tilde{p}_k) = \sum_{j=1}^{k} \omega_j\, \tilde{p}_j$(5.2)

$= \langle(\sum_{j=1}^{k} \omega_j\, c_{j1}, \sum_{j=1}^{k} \omega_j\, c_{j2}, \sum_{j=1}^{k} \omega_j\, c_{j3}, \sum_{j=1}^{k} \omega_j\, c_{j4}, \sum_{j=1}^{k} \omega_j\, c_{j5}\ );\ 1 - \prod_{j=1}^{k}(1 - \pi_{\widetilde{p_j}})^{\omega_j}\ , \prod_{j=1}^{k} \mu_{\widetilde{p_j}}{}^{\omega_j}, \prod_{j=1}^{k} \sigma_{\widetilde{p_j}}{}^{\omega_j}\rangle$

When $n = k + 1$, by applying equations (5.1) and (5.2) we can yield

$PNWAA\ (\tilde{p}_1, \tilde{p}_2, \ldots, \tilde{p}_{k+1}) = \sum_{j=1}^{k+1} \omega_j\, \tilde{p}_j$(5.3)

$= \langle(\sum_{j=1}^{k+1} \omega_j\, c_{j1}, \sum_{j=1}^{k+1} \omega_j\, c_{j2}, \sum_{j=1}^{k+1} \omega_j\, c_{j3}, \sum_{j=1}^{k+1} \omega_j\, c_{j4}, \sum_{j=1}^{k+1} \omega_j\, c_{j5}\ );\ 1 - \prod_{j=1}^{k}\left(1-\pi_{\widetilde{p_j}}\right)^{\omega_j} + 1 -$

$\left(1-\pi_{\widetilde{p_{k+1}}}\right)^{\omega_{k+1}}, \prod_{j=1}^{k+1} \mu_{\widetilde{p_j}}{}^{\omega_j}, \prod_{j=1}^{k+1} \sigma_{\widetilde{p_j}}{}^{\omega_j}\rangle$

$= \langle(\sum_{j=1}^{k+1} \omega_j\, c_{j1}, \sum_{j=1}^{k+1} \omega_j\, c_{j2}, \sum_{j=1}^{k+1} \omega_j\, c_{j3}, \sum_{j=1}^{k+1} \omega_j\, c_{j4}, \sum_{j=1}^{k+1} \omega_j\, c_{j5}\ ); 1 - \prod_{j=1}^{k+1}(1 - \pi_{\widetilde{p_j}})^{\omega_j}\ , \prod_{j=1}^{k+1} \mu_{\widetilde{p_j}}{}^{\omega_j}, \prod_{j=1}^{k+1} \sigma_{\widetilde{p_j}}{}^{\omega_j}\rangle$

This completes the proof.

Obviously, the $PNWAA$ operator satisfies the following properties:

i) Idempotency: Let $\tilde{p}_j (j = 1,2,3, \ldots, n)$ be a set of PNNs. If $\tilde{p}_j (j = 1,2,3, \ldots, n)$ is equal , i.e. $\tilde{p}_j = \tilde{p}$ for j=1,2,3,….,n then $PNWAA\ (\tilde{p}_1, \tilde{p}_2, \ldots, \tilde{p}_n) = \tilde{p}$.

Proof: Since $\tilde{p}_j = \tilde{p}$ for $j = 1,2,3, \ldots, n$ we have,

$$PNWAA\ (\tilde{p}_1, \tilde{p}_2, \ldots, \tilde{p}_n) = \sum_{j=1}^{n} \omega_j\, \tilde{p}_j$$

$=\langle(\sum_{j=1}^{n} \omega_j\, c_{j1}, \sum_{j=1}^{n} \omega_j\, c_{j2}, \sum_{j=1}^{n} \omega_j\, c_{j3}, \sum_{j=1}^{n} \omega_j\, c_{j4}, \sum_{j=1}^{n} \omega_j\, c_{j5}\ );\ 1 - \prod_{j=1}^{n}(1 - \pi_{\widetilde{p_j}})^{\omega_j}\ , \prod_{j=1}^{n} \mu_{\widetilde{p_j}}{}^{\omega_j}, \prod_{j=1}^{n} \sigma_{\widetilde{p_j}}{}^{\omega_j}\rangle$

$= \langle\left(c_1 \sum_{j=1}^{n} \omega_j, c_2 \sum_{j=1}^{n} \omega_j, c_3 \sum_{j=1}^{n} \omega_j, c_4 \sum_{j=1}^{n} \omega_j, c_5 \sum_{j=1}^{n} \omega_j\right); (1 - \left(1 - \pi_{\widetilde{p_j}}\right))^{\sum_{j=1}^{n} \omega_j}, \mu_{\widetilde{p_j}}{}^{\sum_{j=1}^{n} \omega_j}, \sigma_{\widetilde{p_j}}{}^{\sum_{j=1}^{n} \omega_j}\rangle$

$=\langle(\ c_1, c_2, c_3, c_4, c_5); 1 - (1 - \pi_{\widetilde{p_1}}), \mu_{\widetilde{p_1}}, \sigma_{\widetilde{p_1}}\rangle\ = \tilde{p}$

ii) Boundedness: Let $\tilde{p}_j$(j=1,2,3,….,n) be a set of PNNs and let

$\tilde{p}^- = \langle\left(min_j(c_{j1}\ ), min_j(c_{j2}\ ), min_j(c_{j3}\ ), min_j(c_{j4}\ ), min_j(c_{j5}\ )\right); min_j\left(\pi_{\widetilde{p_j}}\right), max_j\left(\mu_{\widetilde{p_j}}\right), max_j\left(\sigma_{\widetilde{p_j}}\right)\rangle$

and $\tilde{p}^+ = \langle\left(max_j(c_{j1}\ ), max_j(c_{j2}\ ), max_j(c_{j3}\ ), max_j(c_{j4}\ ), max_j(c_{j5}\ )\right); max_j\left(\pi_{\widetilde{p_j}}\right), min_j\left(\mu_{\widetilde{p_j}}\right), min_j\left(\sigma_{\widetilde{p_j}}\right)\rangle$

Then $\tilde{p}^- \le PNWAA\ (\tilde{p}_1, \tilde{p}_2, \ldots, \tilde{p}_n) \le \tilde{p}^+$.

Proof: Since the minimum PNN is $\tilde{p}^-$ and the maximum is $\tilde{p}^+$there is $\tilde{p}^- \le \tilde{p}_j \le \tilde{p}^+$. Thus there is $\sum_{j=1}^{n} \omega_j \tilde{p}^- \le \sum_{j=1}^{n} \omega_j \tilde{p}_j \le \sum_{j=1}^{n} \omega_j \tilde{p}^+$.According to the above property (i) there is $\tilde{p}^- \le \sum_{j=1}^{n} \omega_j \tilde{p}_j \le \tilde{p}^+$,
i.e.,$\tilde{p}^- \le$ PNWAA $(\tilde{p}_1, \tilde{p}_2, \dots, \tilde{p}_n) \le \tilde{p}^+$.

iii) Monotonicity: Let $\tilde{p}_j (j = 1,2,3, \dots, n)$ be a set of PNNs. If $\tilde{p}_j \le \tilde{p}_j{}^*$ for j= j=1,2,3,….,n, then

PNWAA $(\tilde{p}_1, \tilde{p}_2, \dots, \tilde{p}_n) \le$ PNWAA$(\tilde{p}_1^*, \tilde{p}_2^*, \tilde{p}_3^*, \tilde{p}_4^*, \tilde{p}_5^*,)$

Proof: Since $\tilde{p}_j \le \tilde{p}_j{}^*$ for $j = j = 1,2,3, \dots, n$ there is $\sum_{j=1}^{n} \omega_j \tilde{p}_j \le \sum_{j=1}^{n} \omega_j \tilde{p}_j{}^*$ i.e. PNWAA $(\tilde{p}_1, \tilde{p}_2, \dots, \tilde{p}_n) \le$ PNWAA$(\tilde{p}_1^*, \tilde{p}_2^*, \tilde{p}_3^*, \tilde{p}_4^*, \tilde{p}_5^*)$.Thus we complete the proofs of all the properties.

5.2 Pentagonal neutrosophic weighted geometric averaging operator

Let $\tilde{p}_j = \langle (c_{j1}, c_{j2}, c_{j3}, c_{j4}, c_{j5}); \pi_{\widetilde{p_1}}, \mu_{\widetilde{p_1}}, \sigma_{\widetilde{p_1}} \rangle (j = 1,2,3, \dots, n)$ be a set of PNNs, then a PNWGAA operator is defined as follows:

$PNWGA\ (\tilde{p}_1, \tilde{p}_2, \dots, \tilde{p}_n) = \prod_{j=1}^{n} \widetilde{p_j{}^{\omega_j}}$ (5.4)

whereω_j is the weight of $\tilde{p}_j (j = 1,2,3, \dots, n)$ such that $\omega_j > 0$ and $\sum_{j=1}^{n} \omega_j = 1$.

Theorem 5.2. Let $\tilde{p}_j =< (c_{j1}, c_{j2}, c_{j3}, c_{j4}, c_{j5}); \pi_{\widetilde{p_1}}, \mu_{\widetilde{p_1}}, \sigma_{\widetilde{p_1}} > (j = 1,2,3, \dots, n)$ be a set of PNNs, then according to Section 4.1 and equation (5.4) we can give the following PNWGA operator

$PNWGA\ (\tilde{p}_1, \tilde{p}_2, \dots, \tilde{p}_n) = \prod_{j=1}^{n} \widetilde{p_j{}^{\omega_j}}$ (5.5)

$= \langle (\prod_{j=1}^{n} c_{j1}{}^{\omega_j}, \prod_{j=1}^{n} c_{j2}{}^{\omega_j}, \prod_{j=1}^{n} c_{j3}{}^{\omega_j}, \prod_{j=1}^{n} c_{j4}{}^{\omega_j}, \prod_{j=1}^{n} c_{j5}{}^{\omega_j}; \prod_{j=1}^{n} \pi_{\widetilde{p_j}}{}^{\omega_j}, 1 - \prod_{j=1}^{n} (1 - \mu_{\widetilde{p_j}})^{\omega_j}, 1 - \prod_{j=1}^{n} (1 - \sigma_{\widetilde{p_j}})^{\omega_j} \rangle$

whereω_j is the weight of $\tilde{p}_j (j = 1,2,3, \dots, n)$ such that $\omega_j > 0$ and $\sum_{j=1}^{n} \omega_j = 1$.

By the similar proof manner of Theorem 5.1 we can prove the Theorem 5.2 which is not repeated here.

Obviously, the PNWGA operator satisfies the following properties:

i) Idempotency: Let $\tilde{p}_j (j = 1,2,3, \dots, n)$ be a set of PNNs.

If $\tilde{p}_j (j = 1,2,3, \dots, n)$ is equal , i.e. $\tilde{p}_j = \tilde{p}$ for $j = 1,2,3, \dots, n$ then PNWGA $(\tilde{p}_1, \tilde{p}_2, \dots, \tilde{p}_n) = \tilde{p}$.

ii) Boundedness: Let $\tilde{p}_j (j = 1,2,3, \dots, n)$ be a set of PNNs and let

$\tilde{p}^- = \langle (min_j(c_{j1}), min_j(c_{j2}), min_j(c_{j3}), min_j(c_{j4}), min_j(c_{j5}); min_j \left(\pi_{\widetilde{p_j}}\right), max_j \left(\mu_{\widetilde{p_j}}\right), max_j \left(\sigma_{\widetilde{p_j}}\right) \rangle$

and

$\tilde{p}^+ = \langle (max_j(c_{j1}), max_j(c_{j2}), max_j(c_{j3}), max_j(c_{j4}), max_j(c_{j5}); max_j \left(\pi_{\widetilde{p_j}}\right), min_j \left(\mu_{\widetilde{p_j}}\right), min_j \left(\sigma_{\widetilde{p_j}}\right) \rangle$

Then $\tilde{p}^- \le$ PNWGA $(\tilde{p}_1, \tilde{p}_2, \dots, \tilde{p}_n) \le \tilde{p}^+$.

iii) Monotonicity: Let $\tilde{p}_j (j = 1,2,3, \dots, n)$ be a set of PNNs. If $\tilde{p}_j \le \tilde{p}_j{}^*$for $j = j = 1,2,3, \dots, n$, then

PNWGA $(\tilde{p}_1, \tilde{p}_2, \dots, \tilde{p}_n) \le$ PNWGA$(\tilde{p}_1^*, \tilde{p}_2^*, \tilde{p}_3^*, \tilde{p}_4^*, \tilde{p}_5^*)$

As the proofs of these properties are similar to the proofs of the above properties, so we don't repeat them.

6. Multi-Criteria Group Decision Making Problem in Pentagonal Neutrosophic Environment

Multi-criteria group decision-making problem is one of the reliable, logistical and mostly used topics in this current era. The main goal of this process is to find out the best alternatives among a finite number of distinct alternatives based on finite different attribute values. Such decision-making program may be raised powerfully by the methods of multi-criteria group decision analysis (MCGDA) which is extremely beneficial to produce decision counselling and offers procedure benefits in terms of upgraded decision attributes,delivers improvised communication techniques and enrichesresolutions of decision-makers.The execution process is not so much easy to evaluate in the pentagonal neutrosophic environment. Using some mathematical operators, score function technique, we developed an algorithm to tackle this MCGDM problem.

In this section, we consider a multi-criteria group decision-making problem based on mobile communication provider services in which we need to select the best service according to different opinions from people. The developed algorithm is described briefly as follows:

6.1 Illustration of the MCGDM problem

We consider the problem as follows:

Suppose $G = \{ G_1, G_2, G_3 \dots\dots\dots.. G_m\}$ is a distinctive alternative set and $H = \{ H_1, H_2, H_3 \dots\dots\dots.. H_n\}$ is the distinctive attribute set respectively. Let $\omega = \{ \omega_1, \omega_2, \omega_3 \dots\dots\dots.. \omega_n\}$ be the corresponding weight set attributes where each $\omega \geq 0$ and also satisfies the relation$\sum_{i=1}^{n} \omega_i = 1$. Thus we consider the set of decision-maker $\lambda = \{ \lambda_1, \lambda_2, \lambda_3 \dots\dots\dots.. \lambda_K\}$ associated with alternatives whose weight vector is stated as $\Omega = \{\Omega_1, \Omega_2, \Omega_3 \dots\dots\dots.. \Omega_k\}$ where each $\Omega_i \geq 0$ and also satisfies the relation$\sum_{i=1}^{k} \Omega_i = 1$, this weight vector will be chosen in accordance with the decision-makers capability of judgment, experience, innovative thinking power etc.

6.2 Normalisation Algorithm of MCGDM Problem:

Step 1: Composition of Decision Matrices

Here, we construct all decision matrices proposed by the decision maker's choice connected with finite alternatives and finite attribute functions. The interesting fact is that the member's s_{ij} for each matrix are of pentagonal neutrosophic numbers. Thus, we finalize the matrix and is given as follows:

$$X^K = \begin{pmatrix} . & H_1 & H_2 & H_3 & . & . & . & H_n \\ G_1 & s_{11}^k & s_{12}^k & s_{13}^k & . & .. & . & s_{1n}^k \\ G_2 & s_{21}^k & s_{22}^k & s_{23}^k & . & . & . & s_{2n}^k \\ G_3 & . & . & . & . & . & . & . \\ . & \vdots & . & . & . & . & . & . \\ G_m & s_{m1}^k & s_{m2}^k & s_{m3}^k & . & . & . & s_{mn}^k \end{pmatrix} \quad (6.1)$$

Step 2: Composition of Single decision matrix

For generating a single group decision matrix X we have promoted the logical pentagonal neutrosophic weighted arithmetic averaging operator (PNWAA) as, $s'_{ij} = \sum_{j=1}^{n} \omega_j\, s_{ij}^{k}$, for individual decision matrix X^k, where$k = 1,2,3 \dots .n$. hence, we finalize the matrix and defined as follows:

$$X = \begin{pmatrix} . & H_1 & H_2 & H_3 & . & . & . & H_n \\ G_1 & s'_{11} & s'_{12} & s'_{13} & . & .. & . & s'_{1n} \\ G_2 & s'_{21} & s'_{22} & s'_{23} & . & . & . & s'_{2n} \\ G_3 & \cdot & \cdot & \cdot & . & . & . & \cdot \\ . & \vdots & . & . & . & . & . & . \\ G_m & s'_{m1} & s'_{m2} & s'_{m3} & . & . & . & s'_{mn} \end{pmatrix} \quad (6.2)$$

Step 3: Composition of leading matrix

To illustrate the single decision matrix we have promoted the logical pentagonal neutrosophic weighted geometric averaging operator (PNWGA) as, $s''_{ij} = \prod_{j=1}^{n} \widetilde{s_{ij}}^{\omega_j}$ for each individual column and finally, we construct the decision matrix as below,

$$X = \begin{pmatrix} . & H_1 \\ G_1 & s''_{11} \\ G_2 & s''_{21} \\ \vdots & \vdots \\ G_m & s''_{m1} \end{pmatrix} \quad (6.3)$$

Step 4: Ranking

Now, considering the score value and transforming the matrix (6.3) into crisp form, we can evaluate the best substitute corresponding to the best attributes. We align the values as increasing order according to their score values and then detect the best fit result. The best result will be the highest magnitude and the worst ones will be the least one.

6.3.1 Flowchart:

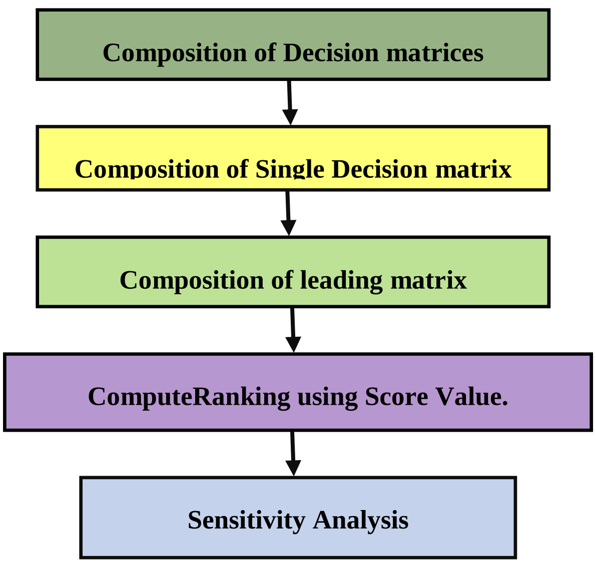

Figure 6.3.1: Flowchart for the problem

6.3 Illustrative Example:

Here, we consider a mobile communication service provider based problem in which there are three different companies are accessible. Among those companies, our problem is to find out the best mobile communication service provider in a logical and meaningful way. Normally, mobile communication service providers mostly depend on attributes such as Service & Reliability, Price & Availability, and Quality & Features of the system. Here, we also consider three different categories of people i) youth age ii) adult age iii) old age people as a decision-maker. According to their opinions we formulate the different decision matrices in the pentagonal neutrosophic environment described below:

$$G_1 = \; Mobilecommunicationserviceprovider\ 1,$$
$$G_2 = Mobilecommunicationserviceprovider\ 2,$$
$$G_3 = Mobilecommunicationserviceprovider\ 3$$

are the alternatives.

Also

$$H_1 = \text{Service \& Reliability},$$
$$H_2 = \text{Price \& Availability},$$
$$H_3 = \text{Quality \& Features}$$

arethe attributes.

Let, $D_1 = Youthagepeople$, $D_2 = Adultagepeople, D_3 = Senioragepeople$ having weight allocation $D = \{\, 0.31, 0.35, 0.34 \,\}$ and the weight allocation in different attribute function is Δ= {0.3,0.4,0.3}.A verbal matrix is built up by the decision maker's to assist the classification of the decision matrix. Attribute vs. Verbal Phrase matrix is given below in Table 6.3.1. The total MCGDM problem is graphically described as below:

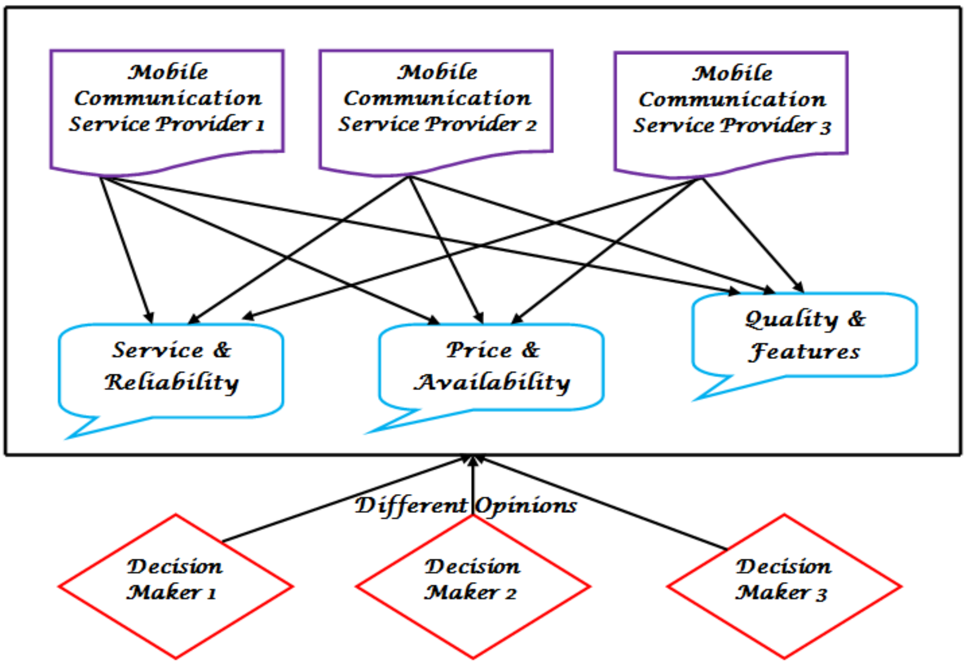

Table 6.3.1: List of Verbal Phrase

Sl no.	Attribute	Verbal phrase
	Quantitative Attributes	
1	Service & Reliability	Very High (VH), High (L), Intermediate (I), Small (S), Very small (VS)
2	Price & Availability	Very high (VH), High (H), Mid (M), Low (L), Very low (VL)
3	Quality & Features	Very high (VH), High (H), Standard (SD), Low (L), Very low (VL)

Table 6.3.2: Relationship between Verbal Phrase and PNN

Verbal Phase	Linguistic Pentagonal Neutrosophic Number (PNN)
Very Low (VL)	$<(0.1,0.1,0.1,0.1,0.1;0.4,0.4,0.4)>$
Low (L)	$<(0.2,0.3,0.4,0.5,0.6;0.5,0.3,0.3)>$
Moderate (M)	$<(0.4,0.5,0.6,0.7,0.8;0.7,0.2,0.2)>$
Little High (LH)	$<(0.5,0.6,0.7,0.8,0.9;0.75,0.18,0.18)>$
High (H)	$<(0.6,0.7,0.8,0.9,1.0;0.8,0.15,0.15)>$
Very High (VH)	$<(1.0,1.0,1.0,1.0,1.0;0.95,0.05,0.05)>$

Step 1

In accordance with finite alternatives and finite attribute functions the decision matrices are constructed by the proposal of decision maker's choice. The noteworthy fact is that the entity s_{ij} for each matrix are of pentagonal neutrosophic numbers. Finally, the matrices are presented as follows:

$$D^1 = \begin{pmatrix} . & H_1 & H_2 & H_3 \\ G_1 & <0.2,0.3,0.4,0.5,0.6;0.4,0.6,0.5> & <0.1,0.2,0.3,0.4,0.5;0.5,0.6,0.7> & <0.3,0.4,0.5,0.6,0.7;0.6,0.3,0.3> \\ G_2 & <0.15,0.25,0.35,0.45,0.5;0.5,0.6,0.5> & <0.3,0.4,0.5,0.6,0.7;0.7,0.3,0.5> & <0.4,0.5,0.55,0.6,0.7;0.8,0.7,0.3> \\ G_3 & <0.4,0.5,0.6,0.7,0.8;0.6,0.4,0.3> & <0.25,0.3,0.35,0.4,0.45;0.4,0.6,0.5> & <0.35,0.4,0.45,0.5,0.55;0.6,0.3,0.4> \end{pmatrix}$$

$$Youth's\ opinion$$

$$D^2 = \begin{pmatrix} . & H_1 & H_2 & H_3 \\ G_1 & <0.15,0.2,0.25,0.3,0.35;0.6,0.4,0.5> & <0.1,0.15,0.3,0.35,0.4;0.7,0.5,0.3> & <0.7,0.75,0.8,0.85,0.9;0.3,0.2,0.6> \\ G_2 & <0.2,0.25,0.3,0.35,0.4;0.7,0.5,0.4> & <0.2,0.25,0.3,0.4,0.45;0.6,0.3,0.3> & <0.4,0.5,0.55,0.6,0.7;0.8,0.7,0.4> \\ G_3 & <0.3,0.35,0.4,0.45,0.5;0.7,0.5,0.3> & <0.5,0.55,0.6,0.7,0.8;0.5,0.6,0.7> & <0.6,0.7,0.75,0.8,0.9;0.6,0.5,0.6> \end{pmatrix}$$

$$Adult's\ Opinion$$

$$D^3 = \begin{pmatrix} . & H_1 & H_2 & H_3 \\ G_1 & <0.2,0.25,0.3,0.4,0.45;0.6,0.3,0.3> & <0.2,0.3,0.4,0.5,0.6;0.4,0.6,0.5> & <0.7,0.75,0.8,0.85,0.9;0.3,0.2,0.6> \\ G_2 & <0.3,0.4,0.5,0.6,0.7;0.7,0.3,0.5> & <0.6,0.7,0.75,0.8,0.9;0.6,0.5,0.6> & <0.7,0.75,0.8,0.85,0.9;0.3,0.2,0.6> \\ G_3 & <0.3,0.35,0.4,0.45,0.5;0.7,0.5,0.3> & <0.4,0.5,0.55,0.6,0.7;0.8,0.7,0.3> & <0.15,0.2,0.25,0.3,0.35;0.6,0.4,0.5> \end{pmatrix}$$

$$Senior\ 's\ Opinion$$

Step 2: Composition of Single decision matrix

In this step we generate a single group decision matrix M and have incorporated the idea of logical pentagonal neutrosophic weighted arithmetic averaging operator (PNWAA) as, $s'_{ij} = \sum_{j=1}^{n} \omega_j \, s_{ij}^{k}$, for individual decision matrix D^k, where$k = 1,2,3 \ldots .n$. Thus we finalize the matrix which is presented as follows:

$$M = \begin{pmatrix} . & H_1 & H_2 & H_3 \\ G_1 & <0.18,0.25,0.31,0.4,0.46;1.00,0.41,0.42> & <0.13,0.22,0.33,0.42,0.50;0.99,0.56,0.46> & <0.58,0.64,0.70,0.77,0.84;0.98,0.23,0.43> \\ G_2 & <0.22,0.30,0.38,0.47,0.53;1.00,0.44,0.46> & <0.38,0.45,0.52,0.60,0.68;1.00,0.36,0.44> & <0.42,0.48,0.53,0.58,0.65;1.00,0.40,0.39> \\ G_3 & <0.33,0.40,0.46,0.53,0.59;1.00,0.47,0.30> & <0.39,0.46,0.51,0.57,0.66;1.00,0.41,0.47> & <0.37,0.48,0.49,0.58,0.60;1.00,0.40,0.50> \end{pmatrix}$$

Step 3: Composition of leading matrix

To define the single decision matrix we have employed the concept of the logical pentagonal neutrosophic weighted geometric averaging operator (PNWGA) as, $s''_{ij} = \prod_{j=1}^{n} \widetilde{s_{ij}}^{\omega_j}$ for each individual column and finally, we present the decision matrix as below

$$M = \begin{pmatrix} \langle 0.26, 0.35, 0.44, 0.56, 0.60; 0.99,0.98,0.99 \rangle \\ \langle 0.33, 0.41, 0.48, 0.55, 0.62; 1.00,0.98,0.99 \rangle \\ \langle 0.36,0.43,0.48,0.54,0.62; 1.00,0.99,0.99 \rangle \end{pmatrix}$$

Step 4: Ranking

Now, we examine the proposed score value for crispification of the PNN into a real number, thus we get the ultimate decision matrix as

$$M = \begin{pmatrix} <0.1503> \\ <0.1641> \\ <0.1652> \end{pmatrix}$$

Here, ordering is $0.1503 < 0.1641 < 0.1652$. Hence, the ranking of the mobile communication service provider is $G_3 > G_2 > G_1$.

6.4 Results and Sensitivity Analysis

To understand how the attribute weights of each criterion affect the relative matrix and their ranking a sensitivity analysis is done. The basic idea of sensitivity analysis is to exchange weights of the attribute values keeping the rest of the terms are fixed. The below table is the evaluation table which shows the sensitivity results.

Attribute Weight	Final Decision Matrix	Ordering
<$(\mathbf{0.3, 0.3, 0.4})$>	$\begin{pmatrix} <0.1367> \\ <0.1617> \\ <0.1650> \end{pmatrix}$	$G_3 > G_2 > G_1$
<$(\mathbf{0.33, 0.35, 0.32})$>	$\begin{pmatrix} <0.1387> \\ <0.1641> \\ <0.1666> \end{pmatrix}$	$G_3 > G_2 > G_1$
<$(\mathbf{0.3, 0.37, 0.33})$>	$\begin{pmatrix} <0.1394> \\ <0.1621> \\ <0.1692> \end{pmatrix}$	$G_3 > G_2 > G_1$
<$(\mathbf{0.45, 0.25, 0.3})$>	$\begin{pmatrix} <0.1415> \\ <0.1641> \\ <0.1699> \end{pmatrix}$	$G_3 > G_2 > G_1$
<$(\mathbf{0.25, 0.45, 0.3})$>	$\begin{pmatrix} <0.1799> \\ <0.1559> \\ <0.1623> \end{pmatrix}$	$G_1 > G_3 > G_2$

$<(\mathbf{0.25}, \mathbf{0.3}, \mathbf{0.45})>$	$\begin{pmatrix} <0.1367> \\ <0.1669> \\ <0.1680> \end{pmatrix}$	$G_3 > G_2 > G_1$
$<(\mathbf{0.4}, \mathbf{0.3}, \mathbf{0.3})>$	$\begin{pmatrix} <0.1544> \\ <0.1675> \\ <0.1666> \end{pmatrix}$	$G_2 > G_3 > G_1$
$<(\mathbf{0.3}, \mathbf{0.4}, \mathbf{0.3})>$	$\begin{pmatrix} <0.1503> \\ <0.1641> \\ <0.1652> \end{pmatrix}$	$G_3 > G_2 > G_1$

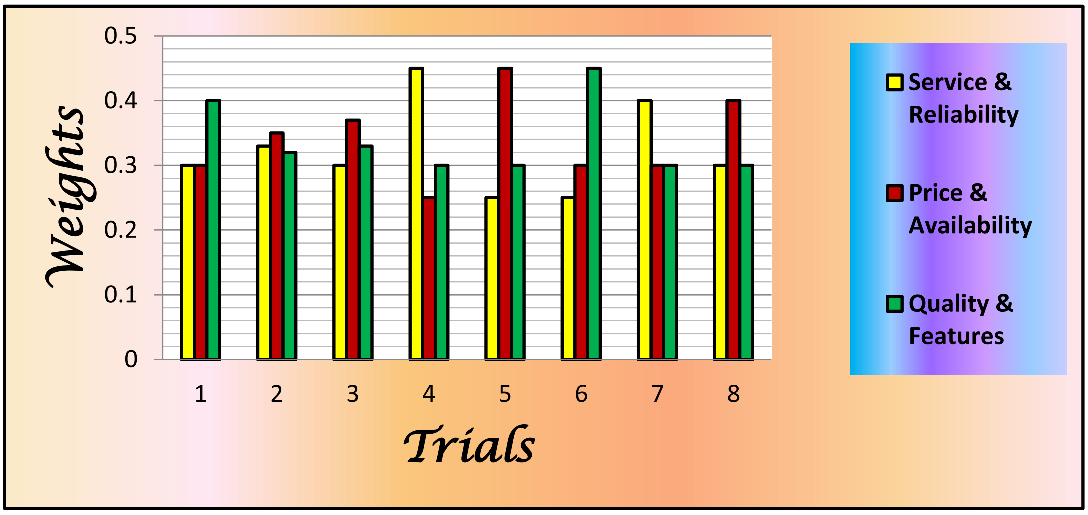

Figure 6.4.1: Sensitivity analysis on attribute functions

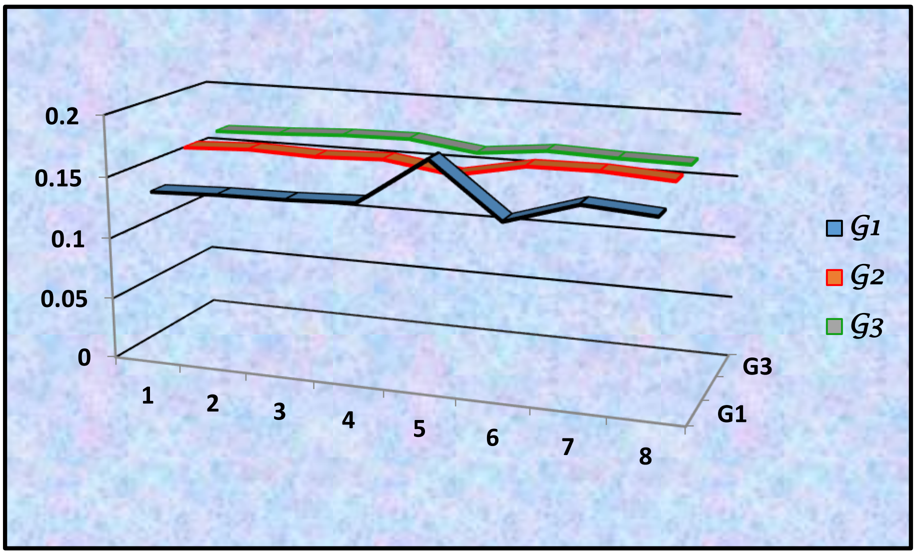

Figure 6.4.2: Best Alternative Mobile Communication Service

6.5 Comparison Table

This section actually contains a comparative study among the established work and proposed work.Comparing with [49,50] , we find that the best service provider among those three and it is noticed that in each case G_3becomes the best mobile communication service provider. The comparison table is given as follows:

Approach	Ranking
Deli[49]	$G_3 > G_2 > G_1$
Garg[50]	$G_3 > G_1 > G_2$
Proposed method	$G_3 > G_2 > G_1$

7. Conclusion and future research scope

The idea of pentagonal neutrosophic number is intriguing, competent and has ample scope of utilization in various research domains. In this research article, we vigorously erect the perception of pentagonal neutrosophic number from different aspects. We also resort to the perception of truthiness, falsity and ambiguity functions in case of pentagonal neutrosophic number when the membership functions are interconnected to each other and a new score function is formulated here. Also, two logical operators have been developed here theoretically as well as applied it in MCGDM problem. Finally we perform a sensitivity analysis and also demonstrate a comparative study with the other results derived from other research articles to enumerate our proposed work and conclude that our result is pretty satisfactory as we consider the pentagonal neutrosophic value in the problem of multi-criteria decision making.

Further, researchers can immensely apply this idea of neutrosophic number in numerous flourishing research fields like an engineering problem, mobile computing problems, diagnoses problem, realistic mathematical modelling, cloud computing issues, pattern recognition problems, an architecture based structural modelling, image processing, linear programming, big data analysis, neural network etc. Apart from these there is an immense scope of application basis works in various fields which can be constructed by taking the help of pentagonal neutrosophic numbers.

Reference

1. Zadeh LA (1965) Fuzzy sets. Information and Control, 8(5): 338- 353.
2. Atanassov K (1986) Intuitionistic fuzzy sets. Fuzzy Sets and Systems 20: 87-96.
3. Yen, K. K.; Ghoshray, S.; Roig, G.; A linear regression model using triangular fuzzy number coefficients, fuzzy sets and system, doi: 10.1016/S0165-0114(97)00269-8.
4. Abbasbandy, S. and Hajjari, T.; A new approach for ranking of trapezoidal fuzzy numbers; Computers and Mathematics with Applications, 57(3)(2009), 413-419.
5. A.Chakraborty, S.P Mondal, A.Ahmadian, N.Senu, D.Dey, S.Alam, S.Salahshour, “The Pentagonal Fuzzy Number: Its Different Representations, Properties, Ranking, Defuzzification and Application in Game Problem”, Symmetry,Vol-11(2), 248; doi:10.3390/sym11020248.
6. Liu F, Yuan XH (2007) Fuzzy number intuitionistic fuzzy set. Fuzzy Systems and Mathematics, 21(1): 88-91.
7. Ye J (2014) prioritized aggregation operators of trapezoidal intuitionistic fuzzy sets and their application to multi criteria decision making, Neural Computing and Applications, 25(6): 1447-1454.
8. Smarandache, F. A unifying field in logics neutrosophy: neutrosophic probability, set and logic.American Research Press, Rehoboth. 1998.
9. H. Wang, F. Smarandache, Q. Zhang and R. Sunderraman, Single valued neutrosophic sets, Multispace and Multistructure**4** (2010), 410–413.
10. A.Chakraborty, S.P Mondal, A.Ahmadian, N.Senu,S.Alam and S.Salahshour, Different Forms of Triangular Neutrosophic Numbers, De-Neutrosophication Techniques, and their Applications, Symmetry, Vol-10, 327; doi:10.3390/sym10080327.

11. A. Chakraborty, S P Mondal, SAlam, A Mahata; Different Linear and Non-linear Form of Trapezoidal Neutrosophic Numbers, De-Neutrosophication Techniques and its Application in Time-Cost Optimization Technique, Sequencing Problem;Rairo Operations Research, doi: 10.1051/ro/2019090.
12. S. Maity, A.Chakraborty, S.K De, S.P.Mondal, S.Alam, A comprehensive study of a backlogging EOQ model with nonlinear heptagonal dense fuzzy environment,Rairo Operations Research;DOI: 10.1051/ro/2018114.
13. P. Bosc and O. Pivert, On a fuzzy bipolar relational algebra,*Information Sciences* **219** (2013), 1–16.
14. K.M. Lee, Bipolar-valued fuzzy sets and their operations,ProcIntConf on Intelligent Technologies, Bangkok, Thailand,2000, pp. 307–312.
15. M.K. Kang and J.G. Kang, Bipolar fuzzy set theory applied to sub-semigroups with operators in semigroups, *J KoreanSoc Math EducSer B Pure Appl Math* **19**(1) (2012),23–35.
16. I. Deli, M. Ali and F. Smarandache, Bipolar Neutrosophic Sets and Their Application Based on Multi-CriteriaDecision Making Problems, *Proceedings of the 2015 InternationalConference on Advanced Mechatronic Systems*,Beijing, China, 2015.
17. S. Broumi, A. Bakali, M. Talea, F. Smarandache and M. Ali, Shortest path problem under bipolar neutrosphic setting, Applied Mechanics and Materials 859 (2016), 59–66.
18. M. Ali and F. Smarandache, Complex neutrosophic set, Neural Computing and Applications **25** (2016),1–18.
19. A. Chakraborty, S. P Mondal, S. Alam ,A. Ahmadian, N. Senu, D. De and S. Salahshour,Disjunctive Representation of Triangular Bipolar Neutrosophic Numbers, De-Bipolarization Technique and Application in Multi-Criteria Decision-Making Problems, Symmetry, 2019, Vol-11(7), 932; doi.org/10.3390/sym11070932.
20. Le Wang,Hong-yuZhang,Jian-qiang Wang, Frank Choquet Bonferroni Mean Operators of BipolarNeutrosophic Sets and Their Application to Multi-criteria Decision-Making Problems Harish Garg, A novel accuracy function under interval-valued Pythagorean fuzzy environment for solving multicriteria decision making problem, Journal of Intelligent & Fuzzy Systems 31 (2016) 529–540, Int. J. Fuzzy Syst. Doi-10.1007/s40815-017-0373-3.
21. Harish Garg, A novel accuracy function under interval-valued Pythagorean fuzzy environment for solving multi criteria decision making problem, Journal of Intelligent & Fuzzy Systems 31 (2016) 529–540.
22. VakkasUlucay,Irfan Deli, Mehmet Sahin, Similarity measures of bipolar neutrosophicsetsand their application to multiple criteria decision making, Neural Comput&Applic DOI 10.1007/s00521-016-2479-1.
23. M. Aslam, S. Abdullah and K. Ullah, *Bipolar Fuzzy Soft Sets And Its Applications in Decision Making Problem*,arXiv:1303.6932v1 [cs. AI] 23, 2013.
24. Le Wang,Hong-yuZhang,Jian-qiang Wang, Frank Choquet Bonferroni Mean Operators of BipolarNeutrosophic Sets and Their Application to Multi-criteria Decision-Making Problems, Int. J. Fuzzy Syst.DOI 10.1007/s40815-017-0373-3.
25. Mumtaz Ali, Le Hoang Son, Irfan Deli and Nguyen Dang Tien, Bipolar neutrosophic soft sets and applications in decision making, Journal of Intelligent & Fuzzy Systems 33 (2017) 4077–4087.
26. S. Broumi, A. Bakali, M. Talea, Prem Kumar Singh, F. Smarandache, Energy and Spectrum Analysis of Interval-valued Neutrosophic graph Using MATLAB, Neutrosophic Set and Systems, vol. 24, Mar 2019, pp. 46-60.
27. P. K. Singh, Interval-valued neutrosophic graph representation of concept lattice and its (α, β, γ)-decomposition, Arabian Journal for Science and Engineering, Year 2018, Vol. 43, Issue 2, pp. 723-74
28. S. Broumi, F. Smarandache, M. Talea and A. Bakali.An Introduction to Bipolar Single Valued Neutrosophic Graph Theory. Applied Mechanics and Materials, vol.841,2016, 184 -191.
29. S. Broumi, M. Talea, A. Bakali, F. Smarandache. Single Valued Neutrosophic, Journal of New Theory. N 10. 2016, pp. 86-101.
30. S. Broumi, M.TaleaA.Bakali, F.Smarandache. On Bipolar Single Valued Neutrosophic Graphs. Journal of Net Theory. N11, 2016, pp. 84-102.
31. A.Kaur and A.Kumar, "A new approach for solving fuzzy transportation problems using generalized trapezoidal fuzzy numbers," Applied soft computing, vol.12, no.3, pp.1201-1213, 2012.
32. M. Mullai and S. Broumi, Neutrosophic Inventory Model without Shortages, Asian Journal of Mathematics and Computer Research, 23(4): 214-219,2018.
33. Yang, P., and Wee, H.,Economic ordering policy of deteriorated item for vendor and buyer: an integrated approach. Production Planning and Control, 11, 2000,474 -480.

34. Abdel-Basset, M., Manogaran, G., Gamal, A., &Smarandache, F. (2019). A group decision making framework based on neutrosophic TOPSIS approach for smart medical device selection. Journal of medical systems.
35. Abdel-Basset, M., Mohamed, M., Elhoseny, M., Chiclana, F., & Zaied, A. E. N. H. (2019). Cosine similarity measures of bipolar neutrosophic set for diagnosis of bipolar disorder diseases. Artificial Intelligence in Medicine, 101, 101735.
36. Gafar, M. G., Elhoseny, M., & Gunasekaran, M. (2018). Modeling neutrosophic variables based on particle swarm optimization and information theory measures for forest fires. The Journal of Supercomputing, 1-18.
37. Abdel-Basset, M., El-hoseny, M., Gamal, A., & Smarandache, F. (2019). A novel model for evaluation Hospital medical care systems based on plithogenic sets. Artificial intelligence in medicine, 100, 101710.
38. Abdel-Baset, M., Chang, V., &Gamal, A. (2019). Evaluation of the green supply chain management practices: A novel neutrosophic approach. Computers in Industry, 108, 210-220.
39. Abdel-Basset, M., Saleh, M., Gamal, A., &Smarandache, F. (2019). An approach of TOPSIS technique for developing supplier selection with group decision making under type-2 neutrosophic number.Applied Soft Computing, 77, 438-452.
40. Abdel-Baset, M., Chang, V., Gamal, A., &Smarandache, F. (2019). An integrated neutrosophic ANP and VIKOR method for achieving sustainable supplier selection: A case study in importing field. Computers in Industry, 106, 94-110.
41. K. Mondal, and S. Pramanik. (2014), Multi-criteria group decision making approach for teacher recruitment in higher education under simplified Neutrosophic environment.Neutrosophic Sets and Systems, 6.
42. A. Gamal, M. Ismail, and F. Smarandache, A Novel Methodology Developing an Integrated ANP: A Neutrosophic Model for Supplier Selection: Infinite Study.
43. Abdel-Basset, M., Mohamed, R., Zaied, A. E. N. H., &Smarandache, F. (2019). A Hybrid PlithogenicDecisionMaking Approach with Quality Function Deployment for Selecting Supply Chain Sustainability Metrics.Symmetry, 11(7), 903.
44. A. Gamal, M. Ismail, and F. Smarandache, A Scientific Decision Framework for Supplier Selection under Neutrosophic Moora Environment: Infinite Study, 2018.
45. R.Helen and G.Uma, A new operation and ranking on pentagon fuzzy numbers, IntJr.of Mathematical Sciences & Applications, Vol. 5, No. 2, 2015, pp 341-346.
46. M.S. Annie Christi, B. Kasthuri; Transportation Problem with Pentagonal Intuitionistic Fuzzy Numbers Solved Using Ranking Technique and Russell's Method; Int. Journal of Engineering Research and Applications; ISSN: 2248-9622, Vol. 6, Issue 2, 2016, pp.82-86.
47. A. Chakraborty, S. Broumi, P.K Singh; Some properties of Pentagonal Neutrosophic Numbers and its Applications in Transportation Problem Environment, Neutrosophic Sets and Systems, vol.28,2019,pp.200-215.
48. A. Chakraborty, S. Mondal, S. Broumi,De-neutrosophication technique of pentagonal neutrosophic number and application in minimal spanning tree; Neutrosophic Sets and Systems; vol. 29, 2019, pp. 1-18, doi : 10.5281/zenodo.3514383.
49. Deli, I.; Ali, M.; Smarandache, F. Bipolar neutrosophic sets and their application based on multi-criteria decision making problems. In Proceedings of the 2015 International Conference on Advanced Mechatronic Systems, Beijing, China, 22–24 August 2015.
50. Garg, H. A novel accuracy function under interval-valued Pythagorean fuzzy environment for solving multi criteria decision making problem. J. Intell. Fuzzy Syst. 2016, 31, 529–540.
51. M. Gulistan, M. Mohammad, F. Karaaslan, S. Kadry, S. Khan, H.A. Wahab, Neutrosophic cubic Heronian mean operators with applications in multiple attribute group decision-making using cosine similarity functions, International Journal of Distributed Sensor Networks, vol. 15, 9, First Published September 30, 2019.
52. F. Karaaslan, K. Hayat,Some new operations on single-valued neutrosophic matrices and their applications in multi-criteria group decision making, Applied Intelligence, 48(2), 2018, 4594–4614.
53. C. Jana, M. Pal, F. Karaaslan, J. Wang, Trapezoidal neutrosophic aggregation operators and its application in multiple attribute decision making process, ScienticaIranica, DOI: 10.24200/SCI.2018.51136.2024.
54. K. Hayat, M. Ali, B. Cao, F. Karaaslan, X. Yang, Another view of group-based generalized intuitionistic fuzzy soft sets: Aggregation operators and multi attribute decision making, Symmetry, 10(12), 2018, 253.

55. F. Karaaslan, Gaussian Single-valued neutrosophic number and its application in multi-attribure decision making, Neutrosophic Sets and Systems, 22, 2018, 101-117.
56. F. Karaaslan, Multi-criteria decision making method based on similarity measures under single-valuedneutrosophic refined and interval neutrosophic refined environments, International Journal of Intelligent Systems, 33(5) 2018, 928-952.
57. F. Karaaslan, Possibility neutrosophic soft sets and PNS-decision making method, Applied Soft Computing, 54, (2017), 403-414.
58. F. Karaaslan; Correlation coefficients of single-valued neutrosophic refined soft sets and their applications in clustering analysis; Neural Computing and Applications,28(9) 2017, 2781–2793.
59. F. Karaaslan, Similarity measure between possibility neutrosophic soft sets and its Applications, U.P.B. Sci. Bull., Series A, Vol. 78, Iss. 3, 2016.
60. F. Karaaslan, Correlation Coefficient between Possibility Neutrosophic Soft Sets, Mathematical Sciences Letters, 5(1), 71-74, 2016.
61. N. Kamal, L. Abdullah, I. Abdullah, S. Alkhazaleh , F. Karaaslan, Multi-Valued Interval Neutrosophic Soft Set: Formulation and Theory, Neutrosophic Sets and Systems 30, 149-170 2019.
62. S. Broumi, A. Dey, M. Talea, A. Bakali, F. Smarandache, D. Nagarajan, M. Lathamaheswari and R. Kumar(2019), "Shortest Path Problem using Bellman Algorithm under Neutrosophic Environment," Complex & Intelligent Systems ,pp-1-8, https://doi.org/10.1007/s40747-019-0101-8.
63. S. Broumi, M.Talea, A. Bakali, F. Smarandache, D.Nagarajan, M. Lathamaheswari and M.Parimala, Shortest path problem in fuzzy, intuitionistic fuzzy and neutrosophic environment: an overview, Complex & Intelligent Systems ,2019,pp 1-8, https://doi.org/10.1007/s40747-019-0098-z.
64. S.Broumi,D. Nagarajan, A. Bakali, M. Talea, F. Smarandache, M. Lathamaheswari, The shortest path problem in interval valued trapezoidal and triangular neutrosophic environment, Complex &Intelligent Systems,2019, pp:1-12, https://doi.org/10.1007/s40747-019-0092-5.
65. S. Broumi, M. Talea, A. Bakali, P. Singh, F. Smarandache: Energy and Spectrum Analysis of Interval Valued Neutrosophic Graph using MATLAB, Neutrosophic Sets and Systems, vol. 24, 2019, pp. 46-60.
66. S. Broumi, A. Bakali, M. Talea, F. Smarandache, K. Kishore, R.Şahin, Shortest Path Proble under Interval Valued Neutrosophic Setting, International Journal of Advanced Trends in Computer Science and Engineering,Volume 8, No.1.1, 2019,pp.216-222.
67. S. Broumi, D. Nagarajan, A. Bakali, M. Talea, F. Smarandache, M.Lathamaheswari, J. Kavikumar: Implementation of NeutrosophicFunctionMembershipsUsing MATLAB Program, Neutrosophic Sets and Systems, vol. 27, 2019, pp. 44-52. DOI: 10.5281/zenodo.3275355.
68. A. Chakraborty, S. Maity, S. P Mondal and S. Alam; Hexagonal Fuzzy Number and its Distinctive Representation, Ranking, Defuzzification Technique and Application in Production Inventory Management Problem; Granular Computing, Springer, doi.org/10.1007/s41066-020-00212-8, (2020).
69. A. Chakraborty; A New Score Function of Pentagonal Neutrosophic Number and its Application in Networking Problem; International Journal of Neutrosophic Science (IJNS), Vol-1(1); pp-35-46; doi-10.5281/zenodo.3679508, 2020.
70. T.S Haque, A. Chakraborty, S. P Mondal and S. Alam; A New Approach to Solve Multi-Criteria Group Decision Making Problems by Exponential Operational Law in Generalised Spherical Fuzzy Environment;
71. A. Chakraborty, S. P Mondal, A. Mahata and S. Alam; Cylindrical Neutrosophic Single- Valued Number and its Application in Networking problem, Multi Criterion Decision Making Problem and Graph Theory; CAAI Transactions on Intelligence Technology , IET Digital, DOI: 10.1049/trit.2019.0083, 2020.

Received: Oct 13, 2019. Accepted: Mar 10, 2020.