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Neutrosophic Overlap Function and Its Derived Neutrosophic Residual Implication

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Abstract: A new concept of neutrosophic overlap function is given, furthermore a neutrosophic residual implication derived from it is also introduced. Firstly, we give new concept of neutrosophic overlap function and some classical examples which are introduced on the lattice. Secondly, the concept of representable neutrosophic overlap function and its pertinent examples are given, meanwhile the general method of constructing representable neutrosophic overlap function by using intuitionistic overlap function is given. Finally, neutrosophic residual implication induced by neutrosophic overlap function and its basic properties are studied.

Keywords: neutrosophic overlap function; lattice; representable neutrosophic overlap function; neutrosophic residual implication

1. Introduction

In 1998, smarandache added an independent membership degree of uncertainty to intuitionistic fuzzy set (IFS)[1], thus putting forward the neutrosophic set(NS) initially. NS is such a robust formal frame extending that concepts of typical set, fuzzy set, IFS and interval-valued IFS from philosophical viewpoint. Because IFS and interval-valued IFS that can solely address incomplete information, but can't address uncertainty and the lack of consistent information which exists in reality. Hence, NS is introduced. Its uncertainty can be explicitly quantified, and its true affiliation, uncertain affiliation and false affiliation are expressed independent. However, its application is hard to solve the actual problems, some scholars have brought forward that notion of single-valued NS[2], as one specific case of NS. And those relevant contents of using single-valued NS to address decision-making issues is as follows[3-8].

Since the triangular norm has a broad range of applications in solving pragmatic issues, it is also important to study the wide range of forms of the triangular norm in applications. The overlap function is a generalization of triangular norm that fulfils continuity[9]. Bustince et al. gave accurate definition of overlap(grouping) function in[10,11]. Over the past period of time, overlap function and grouping function evolved rapidly in theory and practice. See the following literature[12-16] for the rich achievements in the field of theoretical research about overlap function and grouping function. In decision problems, image processing and other fields of wide application see the following literature[17-20]. In an effort to better handle inconclusive information, some scholars extend the overlap function [12,21] into the IFS, while introducing the method at [22].

Fuzzy implication and fuzzy residual implication play an integral part in traditional fuzzy logic. Fuzzy implication [23] generalizes classical implication into fuzzy logic via the consideration of truth values varying in [0, 1] as opposed to {0, 1}. Fuzzy implication is one of the important components of fuzzy logic and acts as a very crucial part in some fields, such as image processing, fuzzy control, data mining see the following literature [24-26], etc. Based on the wide application of fuzzy implication, it is necessary to research it from the theoretical viewpoint [19]. There are a few various models of fuzzy implication for example the R-implication induced by triangular norm[27], (S-N)-implication induced by triangular conorm and fuzzy negation[28], etc. Because overlap function is closely related to triangular norm, in view of the research of neutrosophic triangular norm on neutrosophic fuzzy residual implication, and referring to the research of neutrosophic triangular norm derived residual implication in Hu and Zhang [18], it is natural to consider the neutrosophic residual implication(NRI) induced by neutrosophic overlap function.

The second section mainly introduces the basic knowledge that needs to be used, such as overlap function, grouping function and NS etc. And in the third section, the new concept and related examples of neutrosophic overlap function are given. In addition, the notions and relevant examples of representable neutrosophic overlap function and non-representable neutrosophic overlap function are presented, respectively. Furthermore, the new concept of neutrosophic negation and De Morgan neutrosophic triple which can express the dual relationship between neutrosophic overlap function and neutrosophic grouping function is introduced. The general method of constructing representable neutrosophic overlap functions by intuitionistic overlap functions is given. The fourth section focuses on NRI induced by neutrosophic overlap function, and concludes that every NRI induced by neutrosophic overlap function must be a neutrosophic implication. The final section summarizes the research content.

2. Preliminaries

Definition 2.1 ([29]) O is referred to as an overlap function, if the binary map O: [0, 1] × [0, 1]→[0, 1] fulfils prerequisites below, $\forall s, t \in [0, 1]$:

(a) O fulfils exchangeability;

(b) $O(s, t) = 0$ when and only when $st = 0$;

(c) $O(s, t) = 1$ when and only when $st = 1$;

(d) $O(s, t) \leq_1 O(u, v)$ if $s \leq_1 u$, $t \leq_1 v$;

(e) O fulfils continuity.

Example 2.1 The bivariate functions below are overlap functions, $\forall s, t \in [0, 1]$:

(a) $O_{\mathrm{mM}}(s,t) = \min(s,t)\max(s^2,t^2)$;

(b) $O_p(s,t) = s^p t^p$, for $p > 0$ and $p \neq 1$;

(c) $O_{\mathrm{DB}}(s,t) = \begin{cases} \dfrac{2st}{s+t}, & \text{if } s+t \neq 0, \\ 0, & \text{if } s+t = 0. \end{cases}$;

(d) $O_{\min}(s,t) = \min\{\sqrt{s}, \sqrt{t}\}$.

Definition 2.2 ([30]) The bivariate function G: [0, 1] × [0, 1]→[0, 1] is referred to as the grouping function, when it fulfils prerequisites below, $\forall s, t \in [0, 1]$:

(a) G fulfils exchangeability;

(b) $G(s, t) = 0$ when and only when $s= 0$ and $t= 0$;

(c) $G(s, t) = 1$ when and only when $s=1$ or $t = 1$;

(d) $G(s, t) \leq_1 G(u, v)$ if $s \leq_1 u$, $t \leq_1 v$;

(e) G fulfils continuity.

Example 2.2 The bivariate functions below are grouping functions, $\forall s, t\in [0, 1]$:

(a) $G_{\mathrm{mM}}(s,t)=1-\min(1-s,1-t)\max((1-s)^2,(1-t)^2)$;

(b) $G_p(s,t)=1-(1-s)^p(1-t)^p$, for $p>0$ and $p\neq 1$;

(c) $G_{\mathrm{DB}}(s,t)=\begin{cases}\dfrac{s+t-2st}{2-s-t}, & \text{if } s+t\neq 2,\\ 1, & \text{if } s+t=2.\end{cases}$;

(d) $G_{\min}(s,t)=1-\min\{\sqrt{1-s},\sqrt{1-t}\}$.

Definition 2.3 ([31]) An affiliation function $\mu_E(s)$ and a non-affiliation function $\nu_E(s)$ portray an IFS E in S. S is a set that is not empty. And the IFS E be denoted as

$$E=\{(s,\mu_E(s),\nu_E(s))\mid s\in S\}.$$

In which $\mu_E(s), \nu_E(s)\in [0, 1]$ and satisfies the term of $0\leq \mu_E(s)+\nu_E(s)\leq 1$.

Definition 2.4 ([2]) Truth-affiliation function $T_E(s)$, uncertainty-affiliation function $U_E(s)$ and falsity -affiliation function $F_E(s)$ portray the single-valued NS E in S. S is a set that is not empty. And the single-valued NS E is defined as

$$E=\{\langle s,T_E(s),U_E(s),F_E(s)\rangle\mid s\in S\}.$$

In which $T_E(s), U_E(s), F_E(s)\in [0, 1]$ and satisfies the term of $0 \leq T_E(s)+U_E(s)+F_E(s) \leq 3$.

Definition 2.5 ([32]) The overlap function is a map O: $L^2\rightarrow L$ on $(L; \leq_L)$ which fulfils monotonicity, commutative and continuity, while it fulfils $O(s, 0_L)= 0_L$, $\forall s\in L$; the grouping function is a map G: $L^2\rightarrow L$ on $(L; \leq_L)$ which fulfils monotonicity, commutative and continuity, while it fulfils $G(s, 1_L)= 1_L$, $\forall s\in L$.

Definition 2.6 ([18]) Define the set D^* in the following way,

$$D^*=\{s=(s_1,s_2,s_3)\mid s_1,s_2,s_3\in[0,1]\}.$$

If $s\in D^*$, as above, then s has three components s_1, s_2 and s_3.

$\forall s, t\in D^*$, where $s= (s_1, s_2, s_3)$, t is analogous to s. $\leq_1$ on D^* is defined as the order relation below,

$$s\leq_1 t \text{ iff } s_1\leq t_1,\ s_2\geq t_2,\ s_3\geq t_3.$$

Proposition 2.1 ([18]) $(D^*; \leq_1)$ is a complete lattice.

Definition 2.7 ([18]) The supplement of z is written as below, $\forall s\in D^*$,

$$s^c= (s_3, 1-s_2, s_1).$$

In particular, $1_{D^*}= (1, 0, 0)$ and $0_{D^*}= (0, 1, 1)$ represent the maximum and minimum in $(D^*; \leq_1)$, respectively.

Proposition 2.2 ([18]) $s\wedge_1 t$ is defined as maximum lower bound of *s*, *t*, and expressed as inf(*s*, *t*); $s\vee_1 t$ is defined as minimum upper bound of *s*, *t*, and expressed as sup(*s*, *t*), $\forall s, t\in D^*$.

3. Neutrosophic overlap function

This section proposes new concept of neutrosophic overlap function and provides relevant examples, giving the concept and examples of representable and non-representable neutrosophic overlap function. Finally, a new method for constructing representable neutrosophic overlap functions through intuitionistic fuzzy overlap function(IFO) is proposed.

Definition 3.1 A neutrosophic overlap function is a map $\boldsymbol{O}$: $D^*\times D^*\to D^*$ which fulfils prerequisites below, $\forall s, t, u, v\in D^*$:

(NO1) $\boldsymbol{O}$ fulfils exchangeability;
(NO2) $\boldsymbol{O}(s, t) \leq_1 \boldsymbol{O}(u, v)$ if $s \leq_1 u$, $t \leq_1 v$;
(NO3) $\boldsymbol{O}(0_{D^*}, t) = 0_{D^*}$ or $\boldsymbol{O}(s, 0_{D^*}) = 0_{D^*}$;
(NO4) $\boldsymbol{O}(1_{D^*}, 1_{D^*}) = 1_{D^*}$;
(NO5) $\boldsymbol{O}$ fulfils continuity.

Definition 3.2 A neutrosophic grouping function is a map $\boldsymbol{G}$: $D^*\times D^*\to D^*$ which fulfils prerequisites below, $\forall s, t, u, v \in D^*$:

(NG1) $\boldsymbol{G}$ fulfils exchangeability;
(NG2) $\boldsymbol{G}(s, t) \leq_1 \boldsymbol{G}(u, v)$ if $s \leq_1 u$, $t \leq_1 v$;
(NG3) $\boldsymbol{G}(1_{D^*}, t) = 1_{D^*}$ or $\boldsymbol{G}(s, 1_{D^*}) = 1_{D^*}$;
(NG4) $\boldsymbol{G}(0_{D^*}, 0_{D^*}) = 0_{D^*}$;
(NG5) $\boldsymbol{G}$ fulfils continuity.

Example 3.1 The following binary functions are neutrosophic overlap functions, $\forall s, t\in D^*$

(1) $\boldsymbol{O}_{\mathrm{mM}}(s,t) = (O_{\mathrm{mM}}(s_1,t_1), G_{\mathrm{mM}}(s_2,t_2), G_{\mathrm{mM}}(s_3,t_3))$

$$=(\min(s_1,t_1)\max(s_1^2,t_1^2), 1-\min(1-s_2,1-t_2)\max((1-s_2)^2,(1-t_2)^2), 1-\min(1-s_3,1-t_3)\max((1-s_3)^2,(1-t_3)^2));$$

(2) $\boldsymbol{O}_p(s,t) = (O_p(s_1,t_1), G_p(s_2,t_2), G_p(s_3,t_3)) = (s_1^p t_1^p, 1-(1-s_2)^p(1-t_2)^p, 1-(1-s_3)^p(1-t_3)^p)$, for $p>0$ and $p\neq 1$;

(3) $\boldsymbol{O}_{\mathrm{DB}}(s,t) = (O_{\mathrm{DB}}(s_1,t_1), G_{\mathrm{DB}}(s_2,t_2), G_{\mathrm{DB}}(s_3,t_3)) = (\frac{2s_1t_1}{s_1+t_1}, \frac{s_2+t_2-2s_2t_2}{2-s_2-t_2}, \frac{s_3+t_3-2s_3t_3}{2-s_3-t_3})$, for $s_1\neq t_1\neq 0$, $s_2\neq t_2\neq 1$ and $s_3\neq t_3\neq 1$;

(4) $\boldsymbol{O}_{\min}(s,t) = (O_{\min}(s_1,t_1), G_{\min}(s_2,t_2), G_{\min}(s_3,t_3))$

$$=(\min\{\sqrt{s_1},\sqrt{t_1}\}, 1-\min\{\sqrt{1-s_2},\sqrt{1-t_2}\}, 1-\min\{\sqrt{1-s_3},\sqrt{1-t_3}\}).$$

Example 3.2 The following binary functions are neutrosophic grouping functions, $\forall s, t\in D^*$

(1) $\boldsymbol{G}_{\mathrm{mM}}(s,t) = (G_{\mathrm{mM}}(s_1,t_1), O_{\mathrm{mM}}(s_2,t_2), O_{\mathrm{mM}}(s_3,t_3))$

$$=(1-\min(1-s_1,1-t_1)\max((1-s_1)^2,(1-t_1)^2, \min(s_2,t_2)\max(s_2^2,t_2^2), \min(s_3,t_3)\max(s_3^2,t_3^2));$$

(2) $\boldsymbol{G}_p(s,t) = (G_p(s_1,t_1), O_p(s_2,t_2), O_p(s_3,t_3)) = (1-(1-s_1)^p(1-t_1)^p, s_2^p t_2^p, s_3^p t_3^p)$, for $p>0$ and $p\neq 1$;

(3) $\boldsymbol{G}_{\mathrm{DB}}(s,t)=(G_{\mathrm{DB}}(s_1,t_1),O_{\mathrm{DB}}(s_2,t_2),O_{\mathrm{DB}}(s_3,t_3))=(\frac{s_1+t_1-2s_1t_1}{2-s_1-t_1},\frac{2s_2t_2}{s_2+t_2},\frac{2s_3t_3}{s_3+t_3})$, for $s_1\neq t_1\neq 1$, $s_2\neq t_2\neq 0$ and $s_3\neq t_3\neq 0$;

(4) $\boldsymbol{G}_{\min}(s,t)=(G_{\min}(s_1,t_1),O_{\min}(s_2,t_2),O_{\min}(s_3,t_3))$

$=(1-\min\{\sqrt{1-s_1},\sqrt{1-t_1}\},\min\{\sqrt{s_2},\sqrt{t_2}\},\min\{\sqrt{s_3},\sqrt{t_3}\})$.

Theorem 3.1 Let $\boldsymbol{O}$ is a bivariate operation on D^*, $\forall s, t\in D^*$

$$\boldsymbol{O}(s,t)=(O(s_1,t_1),G_1(s_2,t_2),G_2(s_3,t_3)).$$

Thus $\boldsymbol{O}$ is a neutrosophic overlap function, where O is the overlap function, G_1 and G_2 are grouping functions on [0, 1].

Proof. $\forall s, u, c, v\in D^*$ in which $s=(s_1, s_2, s_3)$, t, u and v are analogous to s.

(NO1) Since G_1 and G_2 are grouping functions, O is the overlap function, then $O(s_1, t_1) = O(t_1, s_1)$, $G_1(t_2, s_2) = G_1(s_2, t_2)$ and $G_2(t_3, s_3) = G_2(s_3, t_3)$, thus $\boldsymbol{O}$ fulfils exchangeability.

(NO2) Let $s\leq_1 u$ and $t\leq_1 v$, then $O(s_1, t_1) \leq O(u_1, v_1)$, $G_1(s_2, t_2) \geq G_1(u_2, v_2)$, $G_2(s_3, t_3) \geq G_2(u_3, v_3)$. Therefore, $\boldsymbol{O}(s, t) \leq_1 \boldsymbol{O}(u, v)$.

(NO3) $\boldsymbol{O}(0_{D^*}, t)= (O(0, t_1), G_1(1, t_2), G_2(1, t_3))=(0, 1, 1)= 0_{D^*}$, and $\boldsymbol{O}(s, 0_{D^*})= (O(s_1, 0), G_1(s_2, 1), G_2(s_3, 1))=(0, 1, 1)= 0_{D^*}$.

(NO4) $\boldsymbol{O}(1_{D^*}, 1_{D^*})= (O(1, 1), G_1(0, 0), G_2(0, 0))= (1, 0, 0)= 1_{D^*}$.

(NO5) Firstly, we prove left continuous. That is, prove that this equation $\boldsymbol{O}(s, \vee_{i\in I}t_i)= \vee_{i\in I}\boldsymbol{O}(s, t_i)$ holds. Because the overlap function and the grouping function are continuous, so $O(s, \vee_{i\in I}t_i)= \vee_{i\in I}O(s, t_i)$, $G_1(s, \vee_{i\in I}t_i)= \vee_{i\in I}G_1(s, t_i)$ and $G_2(s, \vee_{i\in I}t_i)= \vee_{i\in I}G_2(s, t_i)$ is valid.

So we can get

$$\begin{aligned}\boldsymbol{O}(s,\vee_{i\in I}t_i)&=(O(s_1,\vee_{i\in I}s_{i1}),G_1(s_2,\vee_{i\in I}t_{i2}),G_2(s_3,\vee_{i\in I}t_{i3}))\\&=(\vee_{i\in I}O(s_1,t_{i1}),\vee_{i\in I}G_1(s_2,t_{i2}),\vee_{i\in I}G_2(s_3,t_{i3}))\\&=\vee_{i\in I}\boldsymbol{O}(s,t_i)\end{aligned}$$

In this way, show that $\boldsymbol{O}$ is left continuous.

Likewise, $\boldsymbol{O}(s, \wedge_{i\in I}t_i)= \wedge_{i\in I}\boldsymbol{O}(s, t_i)$ is simple to prove, state clearly that $\boldsymbol{O}$ is right continuous. To sum up, $\boldsymbol{O}$ is shown to be a neutrosophic overlap function.

Theorem 3.2 $\boldsymbol{G}$ is a bivariate operation on D^*, $\forall s, t\in D^*$

$$\boldsymbol{G}(s,t)=(G(s_1,t_1),O_1(s_2,t_2),O_2(s_3,t_3)).$$

Then $\boldsymbol{G}$ can be called the neutrosophic grouping function, where G is the grouping function, O_1 and O_2 are overlap functions on [0, 1].

Proof. The procedure for proving analogy **Theorem 3.1.**

Above **Theorem 3.1** supplies the measure for constructing neutrosophic overlap functions using overlap function O and grouping functions G_1, G_2 which are defined on [0, 1]. But it requires a condition that $\boldsymbol{O}= (O, G_1, G_2)$ holds. According to this condition, we bring in the concept of representable neutrosophic overlap function.

Definition 3.3 A neutrosophic overlap function $\boldsymbol{O}$ is referred to as representable, when and only when, there exists O which is an overlap function on [0, 1] and G_1, G_2 which are grouping functions on [0, 1] satisfying, $\forall s, t\in D^*$

$$\boldsymbol{O}(s,t)=(O(s_1,t_1),G_1(s_2,t_2),G_2(s_3,t_3)).$$

Example 3.3 The representable neutrosophic overlap function is shown below, $\forall s, t \in D^*$

$$\boldsymbol{O}(s,t) = (O_{\mathrm{DB}}(s_1,t_1), G_p(s_2,t_2), G_{\mathrm{mM}}(s_3,t_3)).$$

Proof. The first step verifies that $\boldsymbol{O}$ is a neutrosophic overlap function holds. $\forall s, u, t, v \in D^*$ in which $s=(s_1, s_2, s_3)$, t, u and v are analogous to s.

(NO1) Let $G_1= G_p$, $G_2= G_{\mathrm{mM}}$ (p=2) are grouping functions, $O= O_{\mathrm{DB}}$ is an overlap function on [0, 1]. Since $O_{\mathrm{DB}}(s_1, t_1)= O_{\mathrm{DB}}(t_1, s_1)$, $G_p(s_2, t_2)= G_p(t_2, s_2)$, $G_{\mathrm{mM}}(s_3, t_3)= G_{\mathrm{mM}}(s_3, t_3)$, thus $\boldsymbol{O}$ fulfils exchangeability.

(NO2) Let $s \leq_1 u$ and $t \leq_1 v$, then $O_{\mathrm{DB}}(s_1, t_1) \leq O_{\mathrm{DB}}(u_1, v_1)$, $G_p(s_2, t_2) \geq G_p(u_2, v_2)$, $G_{\mathrm{mM}}(s_3, t_3) \geq G_{\mathrm{mM}}(u_3, v_3)$. Therefore, $\boldsymbol{O}(s, t) \leq_1 \boldsymbol{O}(u, v)$.

(NO3) $\boldsymbol{O}(0_{D^*}, t)= (O_{\mathrm{DB}}(0, t_1), G_p(1, t_2), G_{\mathrm{mM}}(1, t_3))= (0, 1, 1)= 0_{D^*}$, and $\boldsymbol{O}(s, 0_{D^*})= (O_{\mathrm{DB}}(s_1, 0), G_p(s_2, 1), G_{\mathrm{mM}}(s_3, 1))= (0, 1, 1)= 0_{D^*}$.

(NO4) $\boldsymbol{O}(1_{D^*}, 1_{D^*})= (O_{\mathrm{DB}}(1, 1), G_p(0, 0), G_{\mathrm{mM}}(0, 0))= (1, 0, 0)= 1_{D^*}$.

(NO5) Firstly, we prove left continuous. That is, prove that this equation $\boldsymbol{O}(s, \vee_{i\in I} t_i)= \vee_{i\in I}\boldsymbol{O}(s, t_i)$ holds. Because of an overlap function O_{DB} and grouping functions G_p and G_{mM} are continuous, so $O_{\mathrm{DB}}(s, \vee_{i\in I} t_i)= \vee_{i\in I} O_{\mathrm{DB}}(s, t_i)$, $G_p(s, \vee_{i\in I} t_i)= \vee_{i\in I} G_p(s, t_i)$ and $G_{\mathrm{mM}}(s, \vee_{i\in I} t_i)= \vee_{i\in I} G_{\mathrm{mM}}(s, t_i)$ is valid.

So we can get

$$\begin{aligned}\boldsymbol{O}(s,\vee_{i\in I} t_i) &= (O_{DB}(s_1,\vee_{i\in I} t_{i1}), G_p(s_2,\vee_{i\in I} t_{i2}), G_{\mathrm{mM}}(s_3,\vee_{i\in I} t_{i3}))\\ &= (\vee_{i\in I} O_{DB}(s_1,t_{i1}), \vee_{i\in I} G_p(s_2,t_{i2}), \vee_{i\in I} G_{\mathrm{mM}}(s_3,t_{i3}))\\ &= \vee_{i\in I}\boldsymbol{O}(s,t_i)\end{aligned}$$

In this way, show that $\boldsymbol{O}$ is left continuous.

Likewise, $\boldsymbol{O}(s, \wedge_{i\in I} t_i)= \wedge_{i\in I}\boldsymbol{O}(s, t_i)$ is simple to prove, state clearly that $\boldsymbol{O}$ is right continuous. To sum up, $\boldsymbol{O}$ is shown to be the neutrosophic overlap function.

Finally, it is simple to show that fulfils $\boldsymbol{O}(s, t)= (O_{\mathrm{DB}}(s_1, t_1), G_p(s_2, t_2), G_{\mathrm{mM}}(s_3, t_3))$, so it must be the representable neutrosophic overlap function.

Definition 3.4 The neutrosophic overlap function $\boldsymbol{O}$ is known as standard representable, when and only when, there exists G which is a grouping function on [0, 1] and O which is an overlap function on [0, 1] satisfying, $\forall s, t \in D^*$

$$\boldsymbol{O}(s,t) = (O(s_1,t_1), G(s_2,t_2), G(s_3,t_3)).$$

Example 3.4 The standard representable neutrosophic overlap function is as follows, $\forall s, t \in D^*$

$$\boldsymbol{O}(s,t) = (O_{\mathrm{DB}}(s_1,t_1), G_p(s_2,t_2), G_p(s_3,t_3)).$$

Proof. This procedure for proving analogy **Example 3.3.**

Definition 3.5 The N-dual representable neutrosophic overlap function $\boldsymbol{O}$ by the following being defined by, $\forall s, t \in D^*$

$$\boldsymbol{O}(s,t) = (O(s_1,t_1), G(s_2,t_2), G(s_3,t_3)).$$

O and G has dual relation as follows,

$$O(s,t) = 1 - G(1-s, 1-t).$$

Example 3.5 The N-dual representable neutrosophic overlap function is as follows, $\forall s, t \in D^*$

$$\boldsymbol{O}(s,t) = (O_p(s_1,t_1), G_p(s_2,t_2), G_p(s_3,t_3)).$$

Proof. This procedure for proving analogy **Example 3.3**.

Definition 3.6 $\boldsymbol{G}$ is referred to as representable neutrosophic grouping function, when and only when, there obtains the grouping function G and the overlap functions O_1, O_2 on [0, 1] satisfying, $\forall s, t \in D^*$

$$\boldsymbol{G}(s,t) = (G(s_1,t_1), O_1(s_2,t_2), O_2(s_3,t_3)).$$

Other concepts can be derived from the analogy of the neutrosophic overlap function.

In recent years, there have been many extensions of overlap functions. However, due to the limitations of existing definitions in addressing practical issues by using intuitionistic fuzzy information, scholars have proposed IFO. In the preceding paragraphs, the representable neutrosophic overlap function is proposed and further the below propositions propose a method to construct new representable neutrosophic overlap function(grouping function) with IFO (intuitionistic fuzzy grouping function).

Proposition 3.1 Where $s=(s_1, s_3)$, $t=(t_1, t_3)$, $\forall s, t\in L$. **O** is an IFO while satisfying $\mathbf{O}(s, t)=(O(s_1, t_1), G_2(s_3, t_3))$, with O being an overlap function on [0, 1], G_2 being a grouping function on [0, 1]. Suppose G_1 is a grouping function on [0, 1] satisfying

$$0 \le O(s_1,t_1)+G_1(s_2,t_2)+G_2(s_3,t_3) \le 3\,.$$

Then $\boldsymbol{O}(s, t)=(O(s_1, t_1), G_1(s_2, t_2), G_2(s_3, t_3))$ is called the representable neutrosophic overlap function, $\forall s, t\in D^*$.

Proof. First, we can get $\mathbf{O}(s, t)=(O(s_1, t_1), G_2(s_3, t_3))$ which is an IFO, and then we add another grouping function G_1, satisfying $0 \le O(s_1, t_1)+G_1(s_2, t_2)+G_2(s_3, t_3) \le 3$.

$\forall s, u, t, v\in D^*$ in which $s=(s_1, s_2, s_3)$, t, u and v are analogous to s.

(NO1) Since $O(s_1, t_1)=O(t_1, s_1)$, $G_2(s_3, t_3)=G_2(t_3, s_3)$, $G_1(s_2, t_2)=G_1(s_2, t_2)$, then $\boldsymbol{O}(s, t)=\boldsymbol{O}(t, s)$, thus it shown that $\boldsymbol{O}$ fulfils exchangeability.

(NO2) Let $s\le_1 u$ and $t\le_1 v$, then $O(s_1, t_1)\le O(u_1, v_1)$, $G_2(s_3, t_3)\ge G_2(u_3, v_3)$, $G_1(s_2, t_2)\ge G_1(u_2, v_2)$. Therefore, $\boldsymbol{O}(s, t)\le_1 \boldsymbol{O}(u, v)$.

(NO3) $\boldsymbol{O}(0_{D^*}, t)=(O(0, t_1), G_1(1, t_2), G_2(1, t_3))=(0, 1, 1)=0_{D^*}$, and $\boldsymbol{O}(s, 0_{D^*})=(O(s_1, 0), G_1(s_2, 1), G_2(s_3, 1))=(0, 1, 1)=0_{D^*}$.

(NO4) $\boldsymbol{O}(1_{D^*}, 1_{D^*})=(O(1, 1), G_1(0, 0), G_2(0, 0))=(1, 0, 0)=1_{D^*}$.

(NO5) Firstly, we prove left continuous. That is, prove that this equation $\boldsymbol{O}(s, \vee_{i\in I}t_i)=\vee_{i\in I}\boldsymbol{O}(s, t_i)$ holds. Because the overlap function O and the grouping functions G_2, G_1 are continuous, $O(s, \vee_{i\in I}t_i)=\vee_{i\in I}O(s, t_i)$, $G_2(s, \vee_{i\in I}t_i)=\vee_{i\in I}G_2(s, t_i)$ and $G_1(s, \vee_{i\in I}t_i)=\vee_{i\in I}G_1(s, t_i)$ is holding.

So we can get

$$\begin{aligned}\mathcal{O}(s,\vee_{i\in I}t_i) &= (O(s_1,\vee_{i\in I}t_{i1}),G_1(s_2,\vee_{i\in I}t_{i2}),G_2(s_3,\vee_{i\in I}t_{i3}))\\ &= (\vee_{i\in I}O(s_1,t_{i1}),\vee_{i\in I}G_1(s_2,t_{i2}),\vee_{i\in I}G_2(s_3,t_{i3}))\\ &= \vee_{i\in I}\mathcal{O}(s,t_i)\end{aligned}$$

In this way, show that $\boldsymbol{O}$ is left continuous.

Likewise, $\boldsymbol{O}(s, \wedge_{i\in I}t_i)=\wedge_{i\in I}\boldsymbol{O}(s, t_i)$ is simple to prove, state clearly that $\boldsymbol{O}$ is right continuous. To sum up, $\boldsymbol{O}$ is shown to be the neutrosophic overlap function.

It is simple to show that satisfies $\boldsymbol{O}(s, t)=(O(s_1, t_1), G_1(s_2, t_2), G_2(s_3, t_3))$, so $\boldsymbol{O}$ is a representable neutrosophic overlap function.

Proposition 3.2 Where $s=(s_1, s_3)$, $t=(t_1, t_3)$, $\forall s, t\in L$. **G** is an intuitionistic fuzzy grouping function, while satisfying the fact that $\mathbf{G}(s, t)=(G(s_1, t_1), O_2(s_3, t_3))$, with G being a grouping function on [0, 1], O_2 being an overlap function on [0, 1]. Suppose O_1 is an overlap function on [0, 1] satisfying,

$$0 \le G(s_1,t_1)+O_1(s_2,t_2)+O_2(s_3,t_3) \le 3\,.$$

Then $\boldsymbol{G}(s, t)=(G(s_1, t_1), O_1(s_2, t_2), O_2(s_3, t_3))$ is a representable neutrosophic grouping function, $\forall s, t\in D^*$.

Proof. This procedure for proving analogy **Proposition 3.1.**

The dual relation between triangular norm and triangular conorm in relation to fuzzy negation can be characterized by De Morgan triple, which is a proper expression for the relationship between triangular norm, triangular conorm and fuzzy negation [18]. There are also corresponding studies on NS. Based on the close connection between triangular norm and overlap function, one can naturally consider De Morgan neutrosophic triple about neutrosophic overlap function, neutrosophic grouping function and neutrosophic fuzzy negation. First, neutrosophic fuzzy negation as an extension of fuzzy negation can be denoted by the method below.

Definition 3.7 ([18]) A neutrosophic fuzzy negaton is a map $\boldsymbol{N}: D^*\rightarrow D^*$ that fulfils prerequisites below:

(a) $\boldsymbol{N}(t)\geq_1 \boldsymbol{N}(s)$, $\forall s, t\in D^*$ such as $t \leq_1 s$;

(b) $\boldsymbol{N}(0_{D^*}) = 1_{D^*}$;

(c) $\boldsymbol{N}(1_{D^*}) = 0_{D^*}$.

$\boldsymbol{N}$ is referred to as the involutive neutrosophic negaton when and only when that fulfils $\boldsymbol{N}(\boldsymbol{N}(s))= s$, $\forall s\in D^*$.

A neutrosophic negaton $\boldsymbol{N}$g: $D^*\rightarrow D^*$ satisfies the following is said to be the involutive neutrosophic negaton, where $s= (s_1, s_2, s_3)$, $\forall s\in D^*$,

$$\boldsymbol{N}g(s_1, s_2, s_3)= (s_3, 1\text{-}s_2, s_1)$$

Further, we define such $\boldsymbol{N}$g as the standard neutrosophic negaton.

Definition 3.8 $\boldsymbol{O}$, $\boldsymbol{G}$ and $\boldsymbol{N}$ are a neutrosophic overlap function, a neutrosophic grouping function, and a neutrosophic negation, respectively.

For this triple ($\boldsymbol{O}$, $\boldsymbol{N}$, $\boldsymbol{G}$) if the conditions below holding true, $\forall s, t\in D^*$,

$$\boldsymbol{N}(\boldsymbol{O}(s, t))= \boldsymbol{G}(\boldsymbol{N}(s), \boldsymbol{N}(t));$$

$$\boldsymbol{N}(\boldsymbol{G}(s, t))= \boldsymbol{O}(\boldsymbol{N}(s), \boldsymbol{N}(t)).$$

Then such the triple is referred to as De Morgan neutrosophic triple. In addition, $\boldsymbol{O}$ and $\boldsymbol{G}$ have a dual relationship in relation to $\boldsymbol{N}$.

Theorem 3.3 Suppose neutrosophic negaton $\boldsymbol{N}$ is involutory, that it fulfils $\boldsymbol{N}(\boldsymbol{N}(s))= s$, $\forall s\in D^*$.

(a) Assume $\boldsymbol{G}$ is the neutrosophic grouping function, $\boldsymbol{O}$ is expressed in the following form

$$\boldsymbol{O}(s, t)= \boldsymbol{N}(\boldsymbol{G}(\boldsymbol{N}(s), \boldsymbol{N}(t))).$$

Then $\boldsymbol{O}$ is the neutrosophic overlap function. Moreover, ($\boldsymbol{O}$, $\boldsymbol{N}$, $\boldsymbol{G}$) is De Morgan neutrosophic triple.

(b) Assume $\boldsymbol{O}$ is the neutrosophic overlap function, $\boldsymbol{G}$ is expressed in the following form

$$\boldsymbol{G}(s, t)= \boldsymbol{N}(\boldsymbol{O}(\boldsymbol{N}(s), \boldsymbol{N}(t))).$$

Then $\boldsymbol{G}$ is the neutrosophic grouping function. Moreover, ($\boldsymbol{O}$, $\boldsymbol{N}$, $\boldsymbol{G}$) is De Morgan neutrosophic triple.

Proof. (a) Suppose $\boldsymbol{N}$, $\boldsymbol{G}$ are the involutory neutrosophic negaton and the neutrosophic grouping function, respectively. $\forall s, u, t, v\in D^*$ in which $s= (s_1, s_2, s_3)$, t, u and v are analogous to s.

(NO1) It is pretty simple to justify that $\boldsymbol{O}(s, t)= \boldsymbol{N}(\boldsymbol{G}(\boldsymbol{N}(s), \boldsymbol{N}(t))) = \boldsymbol{N}(\boldsymbol{G}(\boldsymbol{N}(t), \boldsymbol{N}(s)))= \boldsymbol{O}(t, s)$, $\boldsymbol{O}$ fulfils exchangeability.

(NO2) Let $s\leq_1 u$ and $t\leq_1 v$, $\boldsymbol{O}(s, t)= \boldsymbol{N}(\boldsymbol{G}(\boldsymbol{N}(s), \boldsymbol{N}(t)))$, $\boldsymbol{O}(u, v)= \boldsymbol{N}(\boldsymbol{G}(\boldsymbol{N}(u), \boldsymbol{N}(v)))$, because $\boldsymbol{N}$ is non-increasing, then $\boldsymbol{N}(s) \geq_1 \boldsymbol{N}(u)$, $\boldsymbol{N}(t) \geq_1 \boldsymbol{N}(v)$. Moreover $\boldsymbol{G}(s, t) \leq_1 \boldsymbol{G}(u, v)$ and when $s \leq_1 u$, $t \leq_1 v$, then $\boldsymbol{G}(\boldsymbol{N}(s), \boldsymbol{N}(t)) \geq_1 \boldsymbol{G}(\boldsymbol{N}(u), \boldsymbol{N}(v))$. Hence $\boldsymbol{N}(\boldsymbol{G}(\boldsymbol{N}(s), \boldsymbol{N}(t))) \leq_1 \boldsymbol{N}(\boldsymbol{G}(\boldsymbol{N}(u), \boldsymbol{N}(v)))$, then $\boldsymbol{O}(s, t) \leq_1 \boldsymbol{O}(u, v)$.

(NO3) $\boldsymbol{O}(0_{D^*}, t)= \boldsymbol{N}(\boldsymbol{G}(\boldsymbol{N}(0_{D^*}), \boldsymbol{N}(t)))= \boldsymbol{N}(\boldsymbol{G}(1_{D^*}, \boldsymbol{N}(t)))= \boldsymbol{N}(1_{D^*})= 0_{D^*}$, similarly $\boldsymbol{O}(s, 0_{D^*})= \boldsymbol{N}(\boldsymbol{G}(\boldsymbol{N}(s), \boldsymbol{N}(0_{D^*})))= 0_{D^*}$.

(NO4) $\boldsymbol{O}(1_{D^*}, 1_{D^*})= \boldsymbol{N}(\boldsymbol{G}(\boldsymbol{N}(1_{D^*}), \boldsymbol{N}(1_{D^*})))= \boldsymbol{N}(\boldsymbol{G}(0_{D^*}, 0_{D^*}))= \boldsymbol{N}(0_{D^*})= 1_{D^*}$.

(NO5) Firstly, we prove left continuous. That is, prove that this equation $\boldsymbol{O}(s, \vee_{i\in I}t_i)= \vee_{i\in I}\boldsymbol{O}(s, t_i)$ holds. As a result of $\boldsymbol{O}(s, \vee_{i\in I}t_i)= \boldsymbol{N}(\boldsymbol{G}(\boldsymbol{N}(s), \boldsymbol{N}(\vee_{i\in I}t_i)))= \boldsymbol{N}(\boldsymbol{G}(\boldsymbol{N}(s), \wedge_{i\in I}\boldsymbol{N}(t_i)))= \boldsymbol{N}(\wedge_{i\in I}\boldsymbol{G}(\boldsymbol{N}(s), \boldsymbol{N}(t_i))) = \vee_{i\in I}\boldsymbol{N}(\boldsymbol{G}(\boldsymbol{N}(s), \boldsymbol{N}(t_i))) = \vee_{i\in I}\boldsymbol{O}(s, t_i)$. Then we could get $\boldsymbol{O}(s, \vee_{i\in I}t_i)= \vee_{i\in I}\boldsymbol{O}(s, t_i)$. In this way, show that $\boldsymbol{O}$ is left continuous.

Likewise, $\boldsymbol{O}(s, \wedge_{i\in I}t_i)= \wedge_{i\in I}\boldsymbol{O}(s, t_i)$ is simple to prove, state clearly that $\boldsymbol{O}$ is right continuous. To sum up, $\boldsymbol{O}(s, t)$ is shown to be the neutrosophic overlap function.

Moreover, ($\boldsymbol{O}$, $\boldsymbol{N}$, $\boldsymbol{G}$) is the De Morgan neutrosophic triple.

(b) Likewise, suppose $\boldsymbol{O}$ is the neutrosophic overlap function and that $\boldsymbol{G}$ can be shown to be the neutrosophic grouping function, ($\boldsymbol{O}$, $\boldsymbol{N}$, $\boldsymbol{G}$) would be the De Morgan neutrosophic triple.

Example 3.6 The following functions are the neutrosophic overlap(grouping) functions, which are dual in relation to $\boldsymbol{N}$g, $\forall s, t \in D^*$,

(1) $\boldsymbol{O}_{\text{mM}}(s,t)=(O_{\text{mM}}(s_1,t_1),G_{\text{mM}}(s_2,t_2),G_{\text{mM}}(s_3,t_3))$ and $\boldsymbol{G}_{\text{mM}}(s,t)=(G_{\text{mM}}(s_1,t_1),O_{\text{mM}}(s_2,t_2),O_{\text{mM}}(s_3,t_3))$;

In fact, $\boldsymbol{O}_{\text{mM}}(\boldsymbol{N}(s), \boldsymbol{N}(t))= \boldsymbol{O}_{\text{mM}}((s_3, 1-s_2, s_1), (t_3, 1-t_2, t_1))= (O_{\text{mM}}(s_3, t_3), G_{\text{mM}}(1-s_2, 1-t_2), G_{\text{mM}}(s_1, t_1))$, then $\boldsymbol{N}(\boldsymbol{O}_{\text{mM}}(\boldsymbol{N}(s), \boldsymbol{N}(t)))= \boldsymbol{N}(O_{\text{mM}}(s_3, t_3), G_{\text{mM}}(1-s_2, 1-t_2), G_{\text{mM}}(s_1, t_1))= (G_{\text{mM}}(s_1, t_1), 1-G_{\text{mM}}(1-s_2, 1-t_2), O_{\text{mM}}(s_3, t_3))= (G_{\text{mM}}(s_1, t_1), O_{\text{mM}}(s_2, t_2), O_{\text{mM}}(s_3, t_3))= \boldsymbol{G}_{\text{mM}}(s, t)$. Thus, $\boldsymbol{O}_{\text{mM}}$ and $\boldsymbol{G}_{\text{mM}}$ are dual with respect to $\boldsymbol{N}$g.

(2) $\boldsymbol{O}_p(s,t)=(O_p(s_1,t_1),G_p(s_2,t_2),G_p(s_3,t_3))$ and $\boldsymbol{G}_p(s,t)=(G_p(s_1,t_1),O_p(s_2,t_2),O_p(s_3,t_3))$;

(3) $\boldsymbol{O}_{\text{DB}}(s,t)=(O_{\text{DB}}(s_1,t_1),G_{\text{DB}}(s_2,t_2),G_{\text{DB}}(s_3,t_3))$ and $\boldsymbol{G}_{\text{DB}}(s,t)=(G_{\text{DB}}(s_1,t_1),O_{\text{DB}}(s_2,t_2),O_{\text{DB}}(s_3,t_3))$;

(4) $\boldsymbol{O}_{\min}(s,t)=(O_{\min}(s_1,t_1),G_{\min}(s_2,t_2),G_{\min}(s_3,t_3))$ and $\boldsymbol{G}_{\min}(s,t)=(G_{\min}(s_1,t_1),O_{\min}(s_2,t_2),O_{\min}(s_3,t_3))$;

(5) $\boldsymbol{O}(s,t)=(O_{\text{DB}}(s_1,t_1),G_p(s_2,t_2),G_p(s_3,t_3))$ and $\boldsymbol{G}(s,t)=(G_p(s_1,t_1),O_p(s_2,t_2),O_{\text{DB}}(s_3,t_3))$.

We give the following theorem for non-representable neutrosophic overlap function.

Theorem 3.4 Let $\boldsymbol{O}$ be a map on D^* below, $\forall s, t \in D^*$,

$$\boldsymbol{O}(s,t)=\begin{cases}0_{D^*}, & \text{if } s=0_{D^*} \text{ or } t=0_{D^*},\\ 1_{D^*}, & \text{if } s=t=1_{D^*},\\ (s_1t_1, s_3t_3, s_3t_3), & \text{otherwise.}\end{cases}$$

Then $\boldsymbol{O}$ is a non-representable neutrosophic overlap function.

Proof. The first step is to verify that $\boldsymbol{O}$ is the neutrosophic overlap function. $\forall s, u, t, v \in D^*$ in which $s= (s_1, s_2, s_3)$, t, u and v are analogous to s.

(NO1) The proof that $\boldsymbol{O}$ fulfils exchangeability is very straightforward.

(NO2) Let $s \leq_1 u$, $t \leq_1 v$. The obvious one is $\boldsymbol{O}(s, t) \leq_1 \boldsymbol{O}(u, v)$.

(NO3) $\boldsymbol{O}(0_{D^*}, t)= \boldsymbol{O}(s, 0_{D^*})= (0, 1, 1)= 0_{D^*}$.

(NO4) $\boldsymbol{O}(1_{D^*}, 1_{D^*})= (1, 0, 0)= 1_{D^*}$.

(NO5) Firstly, we prove left continuous. That is, prove that this equation $\boldsymbol{O}(s, \vee_{i\in I}t_i)= \vee_{i\in I}\boldsymbol{O}(s, t_i)$ holds. As a result of $\boldsymbol{O}(s, \vee_{i\in I}t_i)= (s_1*\max(t_i), s_3*\max(t_i), s_3*\max(t_i))$; $\vee_{i\in I}\mathcal{O}(s, t_i)= (s_1*t_1, s_3*t_1, s_3*t_1)\vee (s_1*t_2, s_3*t_3, s_3*t_3)\vee (s_1*t_3, s_3*t_3, s_3*t_3)= (s_1*\max(t_i), s_3*\max(t_i), s_3*\max(t_i))$. We can get $\boldsymbol{O}(s, \vee_{i\in I}t_i)= \vee_{i\in I}\boldsymbol{O}(s, t_i)$. Therefore, it is show that $\boldsymbol{O}$ fulfils left continuity.

Likewise, $\boldsymbol{O}(s, \wedge_{i\in I}t_i)= \wedge_{i\in I}\boldsymbol{O}(s, t_i)$ is simple to prove, state clearly that $\boldsymbol{O}$ is right continuous. To sum up, $\boldsymbol{O}(s, t)$ is shown to be the neutrosophic overlap function.

And then, verify that for the representable neutrosophic overlap function $\boldsymbol{O}$ whether there has the overlap function O and grouping functions G_1, G_2 on [0, 1] fulfilling the form $\boldsymbol{O}= (O, G_1, G_2)$.

Have $s= (0.3, 0.5, 0.6)$, $u= (0.3, 0.5, 0.2)$ and $t= (0.4, 0.5, 0.8)$ respectively. From $\boldsymbol{O}(s, t)= (0.12, 0.48, 0.48)$ and $\boldsymbol{O}(u, t)= (0.12, 0.16, 0.16)$. We get $G_1(s_2, t_2)= 0.48$ and $G_1(u_2, t_2)= 0.16$, so $G_1(u_2, t_2)\neq G_1(s_2, t_2)$. Thus, $G_1(s, t)$ is not independent from s_3, which suggests that $\boldsymbol{O}$ is non-representable.

In addition, the neutrosophic grouping function $\boldsymbol{G}$ is the dual of $\boldsymbol{O}$ in relation to the standard neutrosophic negaton $\boldsymbol{N}$g, defined as below, $\forall s, t \in D^*$,

$$\boldsymbol{G}(s,t)=\begin{cases}0_{D^*}, & \text{if } s=t=0_{D^*},\\ 1_{D^*}, & \text{if } s=1_{D^*} \text{ or } t=1_{D^*},\\ (1-(1-s_1)(1-t_1), 1-(1-s_3)(1-t_3), 1-(1-s_3)(1-t_3)), & \text{otherwise.}\end{cases}$$

Then $\boldsymbol{G}$ is a non-representable neutrosophic grouping function.

4. NRI derived from neutrosophic overlap function

This section would bring in the concept of NRI on D^* and research fundamental properties of NRI. First, the notion of neutrosophic implication is introduced on D^*.

Definition 4.1 ([18]) The map $\boldsymbol{I}$: $(D^*)^2 \to D^*$ is known as the neutrosophic implication when it fulfils the prerequisites below, $\forall s, u, t, v \in D^*$:

(a) $\boldsymbol{I}$ is non-increasing for the first variable component (in relation to the order relation $\leq_1$), which means that when $s \leq_1 u$, there is $\boldsymbol{I}(s, t) \geq_1 \boldsymbol{I}(u, t)$;

(b) $\boldsymbol{I}$ is non-decreasing for the second variable component (in relation to the order relation $\leq_1$), which means that when $t \leq_1 v$, there is $\boldsymbol{I}(s, t) \leq_1 \boldsymbol{I}(s, v)$;

(c) $\boldsymbol{I}(0_{D^*}, 0_{D^*}) = 1_{D^*}$;

(d) $\boldsymbol{I}(1_{D^*}, 1_{D^*}) = 1_{D^*}$;

(e) $\boldsymbol{I}(1_{D^*}, 0_{D^*}) = 0_{D^*}$.

Definition 4.2 Suppose $\boldsymbol{I}$: $(D^*)^2 \to D^*$ is a binary map. A neutrosophic overlap function $\boldsymbol{O}$ exists which enables the following condition to hold,

$$\boldsymbol{I}(s,t) = \sup\{h \mid h \in D^*, \boldsymbol{O}(s,h) \leq_1 t\}.$$

Thus such $\boldsymbol{I}$: $(D^*)^2 \to D^*$ is referred to as the NRI.

When $\boldsymbol{I}$ is a NRI derived from a neutrosophic overlap function $\boldsymbol{O}$, it is written as $\boldsymbol{I}_{\boldsymbol{O}}$.

Additionally, a neutrosophic overlap function $\boldsymbol{O}$ fulfils the residual principle, $\forall s, t, h \in D^*$:

$$h \leq_1 \boldsymbol{I}_{\boldsymbol{O}}(s, t) \text{ iff } \boldsymbol{O}(s, h) \leq_1 t.$$

Example 4.1 The functions below are NRIs derived from neutrosophic overlap functions in **Example 3.1**, $\forall s, t \in D^*$,

$$\boldsymbol{I}_{O_{\text{mM}}}(s,t) = \begin{cases} (1,0,0), & \text{if } s_1 \leq t_1, s_2 \geq t_2 \text{ and } s_3 \geq t_3, \\ (1, \max\{1-\sqrt{\frac{1-t_2}{1-s_2}}, 1-\frac{1-t_2}{(1-s_2)^2}\}, \max\{1-\sqrt{\frac{1-t_3}{1-s_3}}, 1-\frac{1-t_3}{(1-s_3)^2}\}), & \text{if } s_1 \leq t_1, s_2 < t_2 \text{ and } s_3 < t_3, \\ (1, 0, \max\{1-\sqrt{\frac{1-t_3}{1-s_3}}, 1-\frac{1-t_3}{(1-s_3)^2}\}), & \text{if } s_1 \leq t_1, s_2 \geq t_2 \text{ and } s_3 < t_3 \\ (1, \max\{1-\sqrt{\frac{1-t_2}{1-s_2}}, 1-\frac{1-t_2}{(1-s_2)^2}\}, 0\}), & \text{if } s_1 \leq t_1, s_2 < t_2 \text{ and } s_3 \geq t_3, \\ (\min\{\sqrt{\frac{t_1}{s_1}}, \frac{t_1}{s_1^2}\}, \max\{1-\sqrt{\frac{1-t_2}{1-s_2}}, 1-\frac{1-t_2}{(1-s_2)^2}\}, \max\{1-\sqrt{\frac{1-t_3}{1-s_3}}, 1-\frac{1-t_3}{(1-s_3)^2}\}), & \text{if } s_1 > t_1, s_2 < t_2 \text{ and } s_3 < t_3, \\ (\min\{\sqrt{\frac{t_1}{s_1}}, \frac{t_1}{s_1^2}\}, 0, 0), & \text{if } s_1 > t_1, s_2 \geq t_2 \text{and } s_3 \geq t_3, \\ (\min\{\sqrt{\frac{t_1}{s_1}}, \frac{t_1}{s_1^2}\}, \max\{1-\sqrt{\frac{1-t_2}{1-s_2}}, 1-\frac{1-t_2}{(1-s_2)^2}\}, 0), & \text{if } s_1 > t_1, s_2 < t_2 \text{ and } s_3 \geq t_3, \\ (\min\{\sqrt{\frac{t_1}{s_1}}, \frac{t_1}{s_1^2}\}, 0, \max\{1-\sqrt{\frac{1-t_3}{1-s_3}}, 1-\frac{1-t_3}{(1-s_3)^2}\}), & \text{if } s_1 > t_1, s_2 \geq t_2 \text{ and } s_3 < t_3. \end{cases}$$

$$\boldsymbol{I}_{O_{p=2}}(s,t)=\begin{cases}(\frac{\sqrt{t_1}}{s_1},1-\frac{\sqrt{1-t_2}}{1-s_2},1-\frac{\sqrt{1-t_3}}{1-s_3}), & \text{if } s_1>\sqrt{t_1}, s_2<1-\sqrt{1-t_2} \text{ and } s_3<1-\sqrt{1-t_3},\\ (\frac{\sqrt{t_1}}{s_1},0,0), & \text{if } s_1>\sqrt{t_1}, s_2\geq 1-\sqrt{1-t_2} \text{ and } s_3\geq 1-\sqrt{1-t_3},\\ (\frac{\sqrt{t_1}}{s_1},1-\frac{\sqrt{1-t_2}}{1-s_2},0), & \text{if } s_1>\sqrt{t_1}, s_2<1-\sqrt{1-t_2} \text{ and } s_3\geq 1-\sqrt{1-t_3},\\ (\frac{\sqrt{t_1}}{s_1},0,1-\frac{\sqrt{1-t_3}}{1-s_3}), & \text{if } s_1>\sqrt{t_1}, s_2\geq 1-\sqrt{1-t_2} \text{ and } s_3<1-\sqrt{1-t_3},\\ (1,1-\frac{\sqrt{1-t_2}}{1-s_2},1-\frac{\sqrt{1-t_3}}{1-s_3}), & \text{if } s_1\leq\sqrt{t_1}, s_2<1-\sqrt{1-t_2} \text{ and } s_3<1-\sqrt{1-t_3},\\ (1,0,0), & \text{if } s_1\leq\sqrt{t_1}, s_2\geq 1-\sqrt{1-t_2} \text{ and } s_3\geq 1-\sqrt{1-t_3},\\ (1,0,1-\frac{\sqrt{1-t_3}}{1-s_3}), & \text{if } s_1\leq\sqrt{t_1}, s_2\geq 1-\sqrt{1-t_2} \text{ and } s_3<1-\sqrt{1-t_3},\\ (1,1-\frac{\sqrt{1-t_2}}{1-s_2},0), & \text{if } s_1\leq\sqrt{t_1}, s_2<1-\sqrt{1-t_2} \text{ and } s_3\geq 1-\sqrt{1-t_3}.\end{cases}$$

$$\boldsymbol{I}_{O_{\min}}(s,t)=\begin{cases}(1,0,0), & \text{if } s_1\leq {t_1}^2, s_2\geq 1-(1-t_2)^2 \text{ and } s_3\geq 1-(1-t_3)^2,\\ (1,1-(1-t_2)^2,0), & \text{if } x_1\leq {y_1}^2, s_2<1-(1-t_2)^2 \text{ and } s_3\geq 1-(1-t_3)^2,\\ (1,0,1-(1-t_3)^2), & \text{if } x_1\leq {y_1}^2, s_2\geq 1-(1-t_2)^2 \text{ and } s_3<1-(1-t_3)^2,\\ (1,1-(1-t_2)^2,1-(1-t_3)^2), & \text{if } x_1\leq {y_1}^2, s_2<1-(1-t_2)^2 \text{ and } s_3<1-(1-t_3)^2,\\ ({t_1}^2,0,0), & \text{if } s_1>{t_1}^2, s_2\geq 1-(1-t_2)^2 \text{ and } s_3\geq 1-(1-t_3)^2,\\ ({t_1}^2,1-(1-t_2)^2,0), & \text{if } x_1>{y_1}^2, s_2<1-(1-t_2)^2 \text{ and } s_3\geq 1-(1-t_3)^2,\\ ({t_1}^2,0,1-(1-t_3)^2), & \text{if } x_1>{y_1}^2, s_2\geq 1-(1-t_2)^2 \text{ and } s_3<1-(1-t_3)^2,\\ ({t_1}^2,1-(1-t_2)^2,1-(1-t_3)^2), & \text{if } x_1>{y_1}^2, s_2<1-(1-t_2)^2 \text{ and } s_3<1-(1-t_3)^2.\end{cases}$$

$$\boldsymbol{I}_{O_{DB}}(s,t)=\begin{cases}(\min\{1,\frac{s_1t_1}{2s_1-t_1}\},\frac{2t_2-s_2t_2-s_2}{1-2s_2+t_2},\frac{2t_3-s_3t_3-s_3}{1-2s_3+t_3}), & \text{if } s_1>\frac{1}{2}t_1, s_2<\frac{1}{2}+\frac{1}{2}t_2 \text{ and } s_3<\frac{1}{2}+\frac{1}{2}t_3,\\ (\min\{1,\frac{s_1t_1}{2s_1-t_1}\},0,\frac{2t_3-s_3t_3-s_3}{1-2s_3+t_3}), & \text{if } s_1>\frac{1}{2}t_1, 1>s_2\geq\frac{1}{2}+\frac{1}{2}t_2 \text{ and } s_3<\frac{1}{2}+\frac{1}{2}t_3,\\ (\min\{1,\frac{s_1t_1}{2s_1-t_1}\},\frac{2t_2-s_2t_2-s_2}{1-2s_2+t_2},0), & \text{if } s_1>\frac{1}{2}t_1, s_2<\frac{1}{2}+\frac{1}{2}t_2 \text{ and } 1>s_3\geq\frac{1}{2}+\frac{1}{2}t_3,\\ (\min\{1,\frac{s_1t_1}{2s_1-t_1}\},0,0), & \text{if } s_1>\frac{1}{2}t_1, 1>s_2\geq\frac{1}{2}+\frac{1}{2}t_2 \text{ and } 1>s_3\geq\frac{1}{2}+\frac{1}{2}t_3,\\ (1,\frac{2t_2-s_2t_2-s_2}{1-2s_2+t_2},\frac{2t_3-s_3t_3-s_3}{1-2s_3+t_3}), & \text{if } 0<s_1\leq\frac{1}{2}t_1, s_2<\frac{1}{2}+\frac{1}{2}t_2 \text{ and } s_3<\frac{1}{2}+\frac{1}{2}t_3,\\ (1,0,\frac{2t_3-s_3t_3-s_3}{1-2s_3+t_3}), & \text{if } 0<s_1\leq\frac{1}{2}t_1, 1>s_2\geq\frac{1}{2}+\frac{1}{2}t_2 \text{ and } s_3<\frac{1}{2}+\frac{1}{2}t_3,\\ (1,\frac{2t_2-s_2t_2-s_2}{1-2s_2+t_2},0), & \text{if } 0<s_1\leq\frac{1}{2}t_1, s_2<\frac{1}{2}+\frac{1}{2}t_2 \text{ and } 1>s_3\geq\frac{1}{2}+\frac{1}{2}t_3,\\ (1,0,0), & \text{if } 0<s_1\leq\frac{1}{2}t_1, 1>s_2\geq\frac{1}{2}+\frac{1}{2}t_2 \text{ and } 1>s_3\geq\frac{1}{2}+\frac{1}{2}t_3,\\ (0,1,1) & \text{if } s_1=t_1=0, s_2=t_2=1 \text{ and } s_3=t_3=1.\end{cases}$$

Theorem 4.1 Suppose $\boldsymbol{O}$ is the neutrosophic overlap function on D^*, $\forall s, t\in D^*$,

$$\boldsymbol{I}_O(s,t)=\sup\{h \mid h\in D^*, \boldsymbol{O}(s,h)\leq_1 t\}.$$

Thus $\boldsymbol{I}_O$ is a neutrosophic implication.

Proof. $\forall s, u, t, h, v \in D^*$ with $s=(s_1, s_2, s_3)$, t, h, u and v are analogous to s.

(a) Let $s \leq_1 u$ and since $\boldsymbol{O}$ is non-decreasing, $\{h \mid h \in D^*, \boldsymbol{O}(s, h) \leq_1 t\} \supseteq_1 \{h \mid h \in D^*, \boldsymbol{O}(u, h) \leq_1 t\}$ then $\sup\{h \mid h \in D^*, h \leq_1 \boldsymbol{I}_O(s, t)\} \geq_1 \sup\{h \mid h \in D^*, h \leq_1 \boldsymbol{I}_O(u, t)\}$. Thus $\boldsymbol{I}_O(s, t) \geq_1 \boldsymbol{I}_O(u, t)$. In other words, the first variable of $\boldsymbol{I}_O$ regarding $\leq_1$ is non-increasing.

(b) Let $t \leq_1 v$ and since $\boldsymbol{O}$ is non-decreasing, $\{h \mid h \in D^*, \boldsymbol{O}(s, h) \leq_1 t\} \subseteq_1 \{h \mid h \in D^*, \boldsymbol{O}(s, h) \leq_1 v\}$ then $\sup\{h \mid h \in D^*, h \leq_1 \boldsymbol{I}_O(s, t)\} \leq_1 \sup\{h \mid h \in D^*, h \leq_1 \boldsymbol{I}_O(s, v)\}$. Thus $\boldsymbol{I}_O(s, t) \leq_1 \boldsymbol{I}_O(s, v)$. In other words, the second variable of $\boldsymbol{I}_O$ regarding $\leq_1$ is non-decreasing.

(c) $\boldsymbol{I}_O(0_{D^*}, 0_{D^*}) = \sup\{h \mid h \in D^*, \boldsymbol{O}(0_{D^*}, h) \leq_1 0_{D^*}\} = 1_{D^*}$;

(d) $\boldsymbol{I}_O(1_{D^*}, 1_{D^*}) = \sup\{h \mid h \in D^*, \boldsymbol{O}(1_{D^*}, h) \leq_1 1_{D^*}\} = \sup\{h \mid h \in D^*, h \leq_1 1_{D^*}\} = 1_{D^*}$;

(e) $\boldsymbol{I}_O(1_{D^*}, 0_{D^*}) = \sup\{h \mid h \in D^*, \boldsymbol{O}(1_{D^*}, h) \leq_1 0_{D^*}\} = \sup\{h \mid h \in D^*, h \leq_1 0_{D^*}\} = 0_{D^*}$.

The NRI has the following important properties.

Theorem 4.2 Assume that $\boldsymbol{I}_O$ is NRI, $\boldsymbol{O}$ is neutrosophic overlap function on D^*. $\forall s, t, h \in D^*$, the follows properties are valid,

(1) $\boldsymbol{I}_O(0_{D^*}, t) = 1_{D^*}$;
(2) $\boldsymbol{I}_O(s, 1_{D^*}) = 1_{D^*}$;
(3) $\boldsymbol{I}_O(s, s) = 1_{D^*}$;
(4) $\boldsymbol{I}_O(1_{D^*}, t) = t$;
(5) $\boldsymbol{I}_O(s, t) \geq_1 t$;
(6) $\boldsymbol{I}_O(s, t) = 1_{D^*}$ iff $s \leq_1 t$;
(7) $s \leq_1 \boldsymbol{I}_O(t, h)$ iff $t \leq_1 \boldsymbol{I}_O(s, h)$;
(8) $s \leq_1 \boldsymbol{I}_O(t, \boldsymbol{I}_O(s, t))$.

Proof. $\forall s, t, h \in D^*$ in which $s=(s_1, s_2, s_3)$, t and h are analogous to s.

(1) $\boldsymbol{I}_O(0_{D^*}, t) = 1_{D^*}$ is the same thing as $\boldsymbol{I}_O(0_{D^*}, t) = \sup\{h \mid h \in D^*, \boldsymbol{O}(0_{D^*}, h) \leq_1 t\} = 1_{D^*}$, then $h = 1_{D^*}$ for $\boldsymbol{O}(0_{D^*}, h) \leq_1 t$. Then this formula is proved.

The proofs of **(2)–(4)** is similar to the proof of **(1)**.

(5) $\boldsymbol{I}$ is non-increasing for the first variable component (in relation to the order relation $\leq_1$), then $\boldsymbol{I}_O(s, t) \geq_1 \boldsymbol{I}_O(1_{D^*}, t) = t$.

(6) $\boldsymbol{I}_O(s, t) = 1_{D^*}$ iff $s \leq_1 t$. Let $s \leq_1 t$, $\boldsymbol{O}(1_{D^*}, s) \leq_1 t$, then $\boldsymbol{I}_O(s, t) = 1_{D^*}$. In contrast, let $\boldsymbol{I}_O(s, t) = 1_{D^*}$,thus $\boldsymbol{O}(1_{D^*}, s) \leq_1 t$, hence $s \leq_1 t$.

(7) $s \leq_1 \boldsymbol{I}_O(t, h)$ iff $t \leq_1 \boldsymbol{I}_O(s, h)$. Since $s \leq_1 \boldsymbol{I}_O(t, h)$, $\boldsymbol{O}(t, s) \leq_1 h$. Thus, $t \leq_1 \boldsymbol{I}_O(s, h)$. Likewise, $s \leq_1 \boldsymbol{I}_O(t, h)$ can be proved from $t \leq_1 \boldsymbol{I}_O(s, h)$.

(8) $s \leq_1 \boldsymbol{I}_O(t, \boldsymbol{I}_O(s, t))$. Since $\boldsymbol{O}(t, s) \leq_1 \boldsymbol{O}(s, t)$, then $s \leq_1 \boldsymbol{I}_O(t, \boldsymbol{I}_O(s, t))$.

Example 4.2 These concrete cases about NRI deduced from neutrosophic overlap function as shown in **Example 4.1** are given. And it is readily proved that NRI deduced from neutrosophic overlap functions satisfy the properties characterised by **Theorem 4.2**.

Furthermore, for the non-representable neutrosophic overlap function, using the neutrosophic overlap function got from **Theorem 3.4** as an example, $\forall s, t \in D^*$,

$$\boldsymbol{I}_O(s,t) = \begin{cases} 1_{D^*}, & \text{if } s = 0_{D^*} \text{ or } t = 0_{D^*}; \text{or } s = t = 1_{D^*}, \\ (1, [0,1], \max\{\frac{t_3}{s_2}, \frac{t_3}{s_3}\}), & \text{if } s_1 \leq t_1, \\ (\frac{t_1}{s_1}, [0,1], \max\{\frac{t_3}{s_2}, \frac{t_3}{s_3}\}), & \text{if } s_1 > t_1. \end{cases}$$

Then it follows that $\boldsymbol{I}_O(s, t)$ is a neutrosophic implication, while the properties from **Theorem 4.2** is satisfied.

5. Conclusions

As an important part of NS theory, neutrosophic logic plays an significant part in it. Neutrosophic overlap function, neutrosophic grouping function and neutrosophic implication which are crucial neutrosophic logic operators. For the first kind of inclusion relationship, the definitions of neutrosophic overlap function (neutrosophic grouping function) on (D^*; $\leq_1$) are defined and related examples are given. At the same time, new definitions of representable and non-representable neutrosophic overlap function are proposed. In the next place, based on the close relationship between overlap function and triangular norm, a new description of neutrosophic negation is offered through analogy research, then the dual relationship between neutrosophic overlap function and neutrosophic grouping function on neutrosophic negation is described. Moreover, we show that definition of neutrosophic implication is given based on (D^*; $\leq_1$) and the basic properties of NRI are studied. Finally, the result that NRI induced by neutrosophic overlap function must be neutrosophic implication is proved. Based on these results and some new results [33-42], we consider applying them to generalized neutrosophic overlap function and neutrosophic inference systems of the future.

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