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The Property (P) and New Fixed Point Results on Ordered Metric Spaces in Neutrosophic Theory

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Abstract. In this manuscript, We introduce some fixed point results for some contractive type mappings on complete ordered triangular neutrosophic metric spaces, and review many existing results in the literature. Furthermore, we use our results to obtain the property (P).

Keywords: the property (P), Contractive mapping, neutrosophic fixed Point,fixed point, neutrosophic topology, neutrosophic theory, neutrosophic cone metric space, complete ordered triangular neutrosophic metric space,.

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1. Introduction and Preliminaries

Zadeh [1] defined the concept of fuzzy set in 1965. After that, many authors have introduced and discussed several notions of generalizations of this fundamental concept. In the year 1968 Chang [3] initiated and study fuzzy topological spaces. In particular, the concept of intuitionistic fuzzy sets (IFSs for short) was first investigated by Atanassov [4]. This concept was extended and modified to intuitionistic L-fuzzy setting by Stoeva and Atanassov [5], which currently known by "intuitionistic L-topological spaces". Using the cocept of intuitionistic fuzzy sets, the concept of intuitionistic fuzzy topological space was introduced by Coker [6, 15, 16]. In diverse latest papers, F. Smarandache modified the concepts of intuitionistic fuzzy sets and different styles of sets to obtained neutrosophic sets (NSs for short) [7]. F. Smarandache and A. Al Shumrani obtained the concept of neutrosophic topology on the non-general and standard interval [8, 9]. Several authors was extended this principle with many applications (see [10, 19–23]). Recently, Alomari and Smarandache [11, 12] introduce and discussed the concepts of continuity in neutrosophic topology, neutrosophic closed and open sets in neutrosophic topological space, they also defined the notion of neutrosophic connectedness and neutrosophic mapping.

W. Al-Omeri et al, [13] introduce the concept of neutrosophic metric space. That is a generalization of intuitionistic fuzzy metric space due to Veeramani and George [17]. Zhang and Huang [18] focused on this new notion of cone metric space and they discussed some fixed point theorems for contractive type mappings. In 2019 wadei Al-Omeri et al. [13] introduced a new concept known by "neutrosophic cone metric space" which is generalized the corresponding concept of intuitionistic fuzzy metric space.

In intuitionistic fuzzy metric space, Bag et al [2] extended the concept of $(\emptyset, \Psi)$-weak contraction, then by using the altering distance function he proved some fixed point theorems. Metric fixed point and cone metric space results are played a remarkable role in the study of (Φ, Ψ)-weak contraction to neutrosophic cone metric space

The purpose of this paper is to introduce a new results about the property (P). In addition, some fixed point consequences with the aid of combine all the principles of these papers for a few contractive type mappings for such mappings in entire metric spaces on complete ordered triangular neutrosophic metric spaces.

2. Historical Background

In this part, we have studied some basic notions such as continuous t-norm, induced topology and neutrosophic cone metric space (NCMS, shortly) the which is defined as $\tau(\Sigma, \Xi)$. A sequence $\{u_m\}$ in an neutrosophic cone metric space $(\Sigma, \Xi, \Theta, \bigotimes, \diamond)$ is said to be Cauchy when every $z > 0$ and $\epsilon > 0$, there exists a natural number m_0 such that $M(u_m, u_n, z) > 1 - \epsilon$ and $N(u_m, u_n, z) < \epsilon$ for all $m, n \geq m_0$. Also, $(\Sigma, \Xi, \Theta, \bigotimes, \diamond)$ is said to be complete NCMS when each Cauchy sequence in NCMS is convergent with respect $\tau(\Sigma, \Xi)$.

Definition 2.1. [13] For any neutrosophic metric space $(\Sigma, \Xi, \Theta, \bigotimes, \diamond)$, the sequence $\{x_n\}$ is said to be neutrosophic cone contractive sequence if there exists $q \in (0, 1)$ such that

$$\frac{1}{\Xi(\epsilon_{1n+1}, \epsilon_{1n+2}, m)} - 1 \leq q(\frac{1}{\Xi(\epsilon_1, \epsilon_{1n+1}, m)} - 1)$$

,

$$\Theta(\epsilon_{1n+1}, \epsilon_{1n+2}, m) \leq q\Theta(\epsilon_1, \epsilon_{1n+1}, m) \text{ for every } n \in \Theta.$$

.

Definition 2.2. [13] Let $(\Sigma, \Xi, \Theta, \bigotimes, \diamond)$ be a neutrosophic CMS and an identity mapping $k : \Sigma \to \Sigma$. Then k is said to be neutrosophic cone contractive if there exists $0 < q < 1$ such that

$$\frac{1}{\Xi(k(\epsilon_1),k(\epsilon_2),m)}-1\leq q(\frac{1}{\Xi(\epsilon_1,\epsilon_2,m)}-1)$$

$$\Theta(k(\epsilon_1),k(\epsilon_2),m)\leq q\Theta(\epsilon_1,\epsilon_2,m)$$

for each $\epsilon_1, \epsilon_2 \in \Sigma$ and $m \gg 0_\Theta$. The constant q is said to be contractive constant of k.

Definition 2.3. [14] For any neutrosophic CMS $(\Sigma, \Xi, \Theta, \bigotimes, \diamond)$ and the mappings $\mathcal{H}, \mathcal{T} : \Sigma \to \Sigma$. Then the mapping $\mathcal{H}$ is called neutrosophic (Φ, Ψ)-weak contraction with respect to $\mathcal{T}$ if there exists an alternating distance function Φ and a function $\Psi : [0, \infty) \longrightarrow [0, \infty)$ with $\Psi(s) > 0$ for $\Psi(s) = 0$ and $s > 0$ such that

$$\Phi(\frac{1}{\Xi(\mathcal{H}(\epsilon_1),\mathcal{H}(\epsilon_2),\mathcal{H}(\epsilon_3),m)}-1_\Theta)\leq\Psi(\frac{1}{\Xi(\mathcal{T}(\epsilon_1),\mathcal{T}(\epsilon_2),\mathcal{T}(\epsilon_3),m)}-1_\Theta). \tag{2.1}$$

hold for all $\epsilon_1, \epsilon_2, \epsilon_3 \in \Xi$ and every $m \gg \Theta$. If $\mathcal{T}$ is the identity map, then $\mathcal{H}$ is called neutrosophic (Φ, Ψ)-weak contraction mapping.

Example 2.4. Let $\Sigma = [0, \infty)$ and $d(r, s) = |r - s|$. Define the self-map Γ on Σ and $\beta : \Sigma \times \Sigma \longrightarrow [0, \infty)$, respectively by the formulas $\Gamma_r = \sqrt{r}$, and $\beta(r, s) = exp(r - s)$, whenever $r \geq s$ and $\beta(r, s) = 0$ whenever $r < s$ for all $r, s \in \Sigma$. Then Γ is β-admissible

Definition 2.5. [13] Let $(\Sigma, \Xi, \Theta, \bigotimes, \diamond)$ be a neutrosophic cone metric space. The cone metric (Σ, Ξ) is said to be triangular when

$$\frac{1}{\Xi(u,v,n)}-1\leq\frac{1}{\Xi(u,w,n)}-1+\frac{1}{\Xi(w,v,n)}-1$$

,

$$\Theta(u,v,n)\leq\Theta(u,w,n)+\Theta(w,v,n) \text{ for all } u,v,n\in\Sigma \text{ and } n>0$$

A self-map $\mathcal{H} : \Sigma \to \Sigma$ is said to be orbitally continuous at ϵ_1 when for every sequence $\{x(i)\}_{i\geq1}$ with $\mathcal{H}^{x(i)}\epsilon_1 \to b$ for few $b \in \Sigma$, we have $\mathcal{H}^{x(i)+1} \to \mathcal{H}_b$. By [14], here $\mathcal{H}^{m+1} = \mathcal{H}(\mathcal{H}^m)$. Finally, we define the orbit of $\mathcal{H}$ at ϵ_1 by $O(\epsilon_1, \infty) := \{\epsilon_1, \mathcal{H}\epsilon_1, \mathcal{H}^2\epsilon_1, ..., \mathcal{H}^n\epsilon_1, ..\}$.

We say that $\mathcal{H}$ has the strongly similar property whilst $(\mathcal{H}^{n-1}y, \mathcal{H}^n y) \in \Sigma_{\ll}$ for each $n \geq 1$ and $m \geq 2$, where $y \in F(\mathcal{H}^m)$.

3. Existence result

In this part, we have studied some special mappings of discontinuity.

Theorem 3.1. *Let $(\Sigma, \Xi, \Theta, \bigotimes, \diamond)$ be a complete ordered triangular NMS, $\delta \in (0,1)$ and $\mathcal{H}$ a self-map on Σ satisfying*

$$min\left\{\frac{[1-\Xi(\mathcal{H}u,\mathcal{H}v,t)]^2}{\Xi^2(\mathcal{H}u,\mathcal{H}v,t)}, \frac{[1-\Xi(u,v,t)][1-\Xi(\mathcal{H}u,\mathcal{H}v,t)]}{\Xi(u,v,t)\Xi(\mathcal{H}u,\mathcal{H}v,t)}, \frac{[1-\Xi(v,\mathcal{H}v,t)]^2}{\Xi^2(v,\mathcal{H}v,t)}\right\}$$
$$-min\left\{\frac{[1-\Xi(u,\mathcal{H}u,t)]^2}{\Xi^2(u,\mathcal{H}u,t)}, \frac{[1-\Xi(v,\mathcal{H}v,t)][1-\Xi(u,\mathcal{H}v,t)]}{\Xi(v,\mathcal{H}v,t)\Xi(u,\mathcal{H}v,t)}, \frac{[1-\Xi(v,\mathcal{H}u,t)]^2}{\Xi^2(v,\mathcal{H}u,t)}\right\}$$
$$\leq \delta\frac{[1-\Xi(u,\mathcal{H}u,t)][1-\Xi(v,\mathcal{H}v,t)]}{\Xi(u,\mathcal{H}u,t)\Xi(v,\mathcal{H}v,t)}.$$

Thus, for all $u, v \in \Sigma_{\ll}$.if $\mathcal{T}$ has the strongly comparable property, then $\mathcal{T}$ has the property (P). Moreover,If there exists $u_0 \in \Sigma$ such that $(\mathcal{H}^{m-1}u_0, \mathcal{H}^m u_0) \in \Sigma_{\ll}$ for all $m \geq 1$ and $\mathcal{H}$ is orbitally continuous at u_0, then $\mathcal{T}$ has a fixed point.

Proof. To prove that $\mathcal{H}$ has the property (P). Let $n \geq 2$ be given and $u \in T(\mathcal{H}_n)$. Since $\mathcal{H}$ has the strongly comparable property, we can put $x = \mathcal{H}^{m-1}u$ and $u = \mathcal{H}^m u$ in the condition. Then we have

$$min\left\{\frac{[1-\Xi(\mathcal{H}^m u,\mathcal{H}^{m+1}u,t)]^2}{\Xi^2(\mathcal{H}^m u,\mathcal{H}^{m+1}u,t)}, \frac{[1-\Xi(\mathcal{H}^{m-1}u,\mathcal{H}^m u,t)][1-\Xi(\mathcal{H}^m u,\mathcal{H}^{m+1}u,t)]}{\Xi(\mathcal{H}^{m-1}u,\mathcal{H}^m u,t)\Xi(\mathcal{H}^m u,\mathcal{H}^{m+1}u,t)}\right\}$$
$$\leq \delta\frac{[1-\Xi(\mathcal{H}^{m-1}u,\mathcal{H}^m u,t)][1-\Xi(\mathcal{H}^m u,\mathcal{H}^{m+1}u,t)]}{\Xi(\mathcal{H}^{m-1}u,\mathcal{H}^m u,t)\Xi(\mathcal{H}^m u,\mathcal{H}^{m+1}u,t)}.$$

Therefore,

$$min\left\{\frac{[1-\Xi(u,\mathcal{H}u,t)]^2}{\Xi^2(u,\mathcal{H}u,t)}, \frac{[1-\Xi(\mathcal{H}^{m-1}u,u,t)][1-\Xi(u,\mathcal{H}u,t)]}{\Xi(\mathcal{H}^{m-1}u,u,t)\Xi(u,\mathcal{H}u,t)}\right\},$$
$$\leq \delta\frac{[1-\Xi(\mathcal{H}^{m-1}u,u,t)][1-\Xi(u,\mathcal{H}u,t)]}{\Xi(\mathcal{H}^{m-1}u,u,t)\Xi(u,\mathcal{H}u,t)}.$$

and so we get two cases.

Case I.
$$\frac{[1-\Xi(u,\mathcal{H}u,t)]^2}{\Xi^2(u,\mathcal{H}u,t)} \leq \delta\frac{[1-\Xi(\mathcal{H}^{n-1}u,u,t)][1-\Xi(u,\mathcal{H}u,t)]}{\Xi(\mathcal{H}^{n-1}u,u,t)\Xi(u,\mathcal{H}u,t)}.$$

We claim that $\frac{1}{\Xi(u,\mathcal{H}u,t)} - 1 = 0$. If $\frac{1}{\Xi(u,\mathcal{H}u,t)} - 1 > 0$. Then $\frac{1}{\Xi(u,\mathcal{H}u,t)} - 1 = \frac{1}{\Xi(\mathcal{H}^m u,\mathcal{H}^{m+1}u,t)} - 1 \leq \delta\frac{1}{\Xi(\mathcal{H}^{m-1}u,\mathcal{H}^m u,t)} - 1$.

Again, by putting $x = \mathcal{H}^{m-2}u$ and $y = \mathcal{H}^{m-1}u$ in condition, we obtain

$$min\left\{\frac{[1-\Xi(\mathcal{H}^{m-1}u,\mathcal{H}^m u,t)]^2}{\Xi^2(\mathcal{H}^{m-1}u,\mathcal{H}^m u,t)}, \frac{[1-\Xi(\mathcal{H}^{m-2}u,\mathcal{H}^{m-1}u,t)][1-\Xi(\mathcal{H}^{m-1}u,\mathcal{H}^m u,t)]}{\Xi(\mathcal{H}^{m-2}u,\mathcal{H}^{m-1}u,t)\Xi(\mathcal{H}^{m-1}u,\mathcal{H}^m u,t)}\right\}$$
$$\leq \delta\frac{[1-\Xi(\mathcal{H}^{m-2}u,\mathcal{H}^{m-1}u,t)][1-\Xi(\mathcal{H}^{m-1}u,\mathcal{H}^m u,t)]}{\Xi(\mathcal{H}^{m-2}u,\mathcal{H}^{m-1}u,t)\Xi(\mathcal{H}^{m-1}u,\mathcal{H}^m u,t)}.$$

Again, we get two cases. Let

$$min\left\{\frac{[1-\Xi(\mathcal{H}^{m-1}u,\mathcal{H}^m u,t)]^2}{\Xi^2(\mathcal{H}^{m-1}u,\mathcal{H}^m u,t)} \leq \delta\frac{[1-\Xi(\mathcal{H}^{m-2}u,\mathcal{H}^{m-1}u,t)][1-\Xi(\mathcal{H}^{m-1}u,\mathcal{H}^m u,t)]}{\Xi(\mathcal{H}^{m-2}u,\mathcal{H}^{m-1}u,t)\Xi(\mathcal{H}^{m-1}u,\mathcal{H}^m u,t)}\right\}.$$

If $\frac{1}{\Xi(\mathcal{H}^{m-1}u,\mathcal{H}^m u,t)} - 1 = 0$, then $\mathcal{H}^{m-1}u = u$ and so $u = \mathcal{H}^m u = \mathcal{H}u$. If $\frac{1}{\Xi(\mathcal{H}^{m-1}u,\mathcal{H}^m u,t)} - 1 > 0$, then $\frac{1}{\Xi(\mathcal{H}^{m-1}u,\mathcal{H}^{m}u,t)}-1\leq\delta\big[\frac{1}{\Xi(\mathcal{H}^{m-2}u,\mathcal{H}^{m-1}u,t)}-1\big]$. Now, let

$$\begin{aligned}&\frac{[1-\Xi(\mathcal{H}^{m-2}u,\mathcal{H}^{m-1}u,t)][1-\Xi(\mathcal{H}^{m-1}u,\mathcal{H}^{m}u,t)]}{\Xi(\mathcal{H}^{m-2}u,\mathcal{H}^{m-1}u,t)\Xi(\mathcal{H}^{m-1}u,\mathcal{H}^{m}u,t)}\\&\leq\delta\frac{[1-\Xi(\mathcal{H}^{m-2}u,\mathcal{H}^{m-1}u,t)][1-\Xi(\mathcal{H}^{m-1}u,\mathcal{H}^{m}u,t)]}{\Xi(\mathcal{H}^{m-2}u,\mathcal{H}^{m-1}u,t)\Xi(\mathcal{H}^{m-1}u,\mathcal{H}^{m}u,t)}.\end{aligned}$$

In this case we should have $\frac{1}{\Xi(\mathcal{H}^{m-2}u,\mathcal{H}^{m-1}u,t)}-1=0$ or $\frac{1}{\Xi(\mathcal{H}^{m-1}u,\mathcal{H}^{m}u,t)}-1=0$ (and so $u=\mathcal{H}u$), because if $\frac{1}{\Xi(\mathcal{H}^{m-2}u,\mathcal{H}^{m-1}u,t)}-1>0$ and $\frac{1}{\Xi(\mathcal{H}^{m-1}u,\mathcal{H}^{m}u,t)}-1>0$, then we get $\delta\geq 1$ which is a contradiction. By continuing this process, we have

$$\begin{aligned}&\frac{1}{\Xi(u,\mathcal{H}u,t)}-1=\frac{1}{\Xi(\mathcal{H}^{m}u,\mathcal{H}^{m+1}u,t)}-1\leq\delta\left[\frac{1}{\Xi(\mathcal{H}^{m-1}u,\mathcal{H}^{m}u,t)}-1\right]\\&\leq\delta^2\left[\frac{1}{\Xi(\mathcal{H}^{m-2}u,\mathcal{H}^{m-1}u,t)}-1\right]\leq...\leq\delta^m\left[\frac{1}{\Xi(u,\mathcal{H}u,t)}-1\right]\end{aligned}$$

which leads us to $\delta\geq 1$ which is a contradiction. Thus, in this case we obtain $\frac{1}{\Xi(u,\mathcal{H}u,t)}-1$ and so $\mathcal{H}u=u$

Case II. $$\frac{[1-\Xi(\mathcal{H}^{m-1}u,u,t)][1-\Xi(u,\mathcal{H}u,t)]}{\Xi(\mathcal{H}^{m-1}u,u,t)\Xi(u,\mathcal{H}u,t)}\leq\delta\frac{[1-\Xi(\mathcal{H}^{m-1}u,u,t)][1-\Xi(u,\mathcal{H}u,t)]}{\Xi(\mathcal{H}^{m-1}u,u,t)\Xi(u,\mathcal{H}u,t)}.$$

In this case, we should have $\frac{1}{\Xi(\mathcal{H}^{m-1}u,\mathcal{H}u,t)}-1=0$ or $\frac{1}{\Xi(u,\mathcal{H}u,t)}-1=0$ (and so $u=\mathcal{H}u$) In fact, if $\frac{1}{\Xi(\mathcal{H}^{m-1}u,\mathcal{H}u,t)}-1>0$ and $\frac{1}{\Xi(u,\mathcal{H}u,t)}-1>0$, then $\delta\geq 1$ which is a contradiction. Therefore, we consequence that $T(\mathcal{H}m)\subseteq T(\mathcal{H})$. Therefore, $\mathcal{H}$ has the property

Now, define $u_{n+1}=\mathcal{H}u_n=\mathcal{H}^{n+1}u_0$ for all $n\geq 0$. If $u_{n0}=u_{n_0-1}$ for some natural number n_0, then $u_n=u_{n_0}$ for all $n\geq n_0$ and u_{n0} is a fixed point of $\mathcal{H}$. Suppose that $u_n\neq u_{n-1}$ for all $n\geq 1$. Now for each $n\geq 1$, by using the hypotheses, we can put $u=u_{n-1}$ and $y=u_n$ in the condition. Therefore we obtain

$$\begin{aligned}&min\left\{\frac{[1-\Xi(u_m,u_{m+1},t)]^2}{\Xi^2(u_m,u_{m+1},t)},\frac{[1-\Xi(u_{m-1},u_m,t)][1-\Xi(u_m,u_{m+1},t)]}{\Xi(u_{m-1},u_m,t)\Xi(u_m,u_{m+1},t)}\right\}\\&\leq\delta\frac{[1-\Xi(u_{m-1},u_m,t)][1-\Xi(u_m,u_{m+1},t)]}{\Xi(u_{m-1},u_m,t)\Xi(u_m,u_{m+1},t)}.\end{aligned}$$

Since $\delta\leq 1$

$$\begin{aligned}&min\left\{\frac{[1-\Xi(u_m,u_{m+1},t)]^2}{\Xi^2(u_m,u_{m+1},t)},\frac{[1-\Xi(u_{m-1},u_m,t)][1-\Xi(u_m,u_{m+1},t)]}{\Xi(u_{m-1},u_m,t)\Xi(u_m,u_{m+1},t)}\right\}\\&=\frac{[1-\Xi(u_m,u_{m+1},t)]^2}{\Xi^2(u_m,u_{m+1},t)}.\end{aligned}$$

Hence

$$\frac{1}{\Xi^2(u_m,u_{m+1},t)}-1\leq\delta\left(\frac{1}{\Xi^2(u_{m-1},u_m,t)}-1\right).$$

By continuing this process we obtain

$$\frac{1}{\Xi^2(u_m,u_{m+1},t)}-1\leq\delta^m\left(\frac{1}{\Xi^2(u_0,u_1,t)}-1\right),$$

for all $m \geq 1$. Thus for each natural number k we have

$$\frac{1}{\Xi^2(u_m, u_{m+k}, t)} - 1 \leq \sum_{i=m}^{m+k-1} \left(\frac{1}{\Xi^2(u_i, u_{i+1}, t)} - 1 \right) \leq \sum_{i=m}^{m+k-1} \delta^i \left(\frac{1}{\Xi^2(u_0, u_1, t)} - 1 \right)$$
$$\leq \frac{\delta^m}{1-\delta} \left(\frac{1}{\Xi^2(u_0, u_1, t)} - 1 \right).$$

Then, $\{u_m\}$ is a Cauchy sequence. If $(\Sigma, \Xi, \Theta, \bigotimes, \diamond)$ is a complete NMS, then there exists $v \in \Sigma$ such that $u_m \to v$. Since $\mathcal{H}$ is orbitally continuous, $u_{m+1} = \mathcal{H}_{u_m} \to \mathcal{H}_v$. This implies that $\mathcal{H}_v = v$. □

Theorem 3.2. *Let $(\Sigma, \Xi, \Theta, \bigotimes, \diamond)$ be a complete ordered triangular neutrosophic metric space, $c \in [0, 1)$, $b \geq 0$, n a nonnegative integer and $\mathcal{H}$ a selfmap on Σ satisfy the condition*

$$\frac{[1 - \Xi(\mathcal{H}^{n+1}u, \mathcal{H}^{n+2}v, t)]^2}{\Xi^2(\mathcal{H}^{n+1}u, \mathcal{H}^{n+2}v, t)} \leq c \frac{[1 - \Xi(\mathcal{H}^n u, \mathcal{H}^{n+1}u, t)][1 - \Xi(\mathcal{H}^{n+1}v, \mathcal{H}^{n+2}v, t)]}{\Xi(\mathcal{H}^n u, \mathcal{H}^{n+1}u, t)\Xi(\mathcal{H}^{n+1}v, \mathcal{H}^{n+2}v, t)}$$
$$+ b \frac{[1 - \Xi(\mathcal{H}^n u, \mathcal{H}^{n+2}v, t)][1 - \Xi(\mathcal{H}^{n+1}v, \mathcal{H}^{n+1}u, t)]}{\Xi(\mathcal{H}^n u, \mathcal{H}^{n+2}v, t)\Xi(\mathcal{H}^{n+1}v, \mathcal{H}^{n+1}u, t)}.$$

for all $u, v \in \Sigma_{\ll}$. Suppose that there exists $u_0 \in \Sigma$ such that $(\mathcal{H}^{m-1}x_0, \mathcal{H}^m u_0) \in \Sigma_{\ll}$ for all $m \geq 1$. If $\mathcal{H}$ is orbitally continuous at u_0 or $n = 0$, then $\mathcal{H}$ has a fixed point. Moreover, $\mathcal{H}$ has a unique fixed point whenever $b < 1$. If $\mathcal{H}$ has the strongly comparable property, then $\mathcal{H}$ has the property (P).

Proof. Define $u_1 = \mathcal{H}^{n+1}u_0$ and $u_{m+1} = \mathcal{H}x_m$ for all $m \geq 1$. Then

$$\frac{[1 - \Xi(u_1, u_2, t)]^2}{\Xi^2(u_1, u_2, t)} = \frac{[1 - \Xi(\mathcal{H}^{n+1}u_0, \mathcal{H}^{n+2}u_0, t)]^2}{\Xi^2(\mathcal{H}^{n+1}u_0, \mathcal{H}^{n+2}u_0, t)}$$
$$\leq c \frac{[1 - \Xi(\mathcal{H}^n u_0, \mathcal{H}^{n+1}u_0, t)][1 - \Xi(\mathcal{H}^{n+1}u_0, \mathcal{H}^{n+2}u_0, t)]}{\Xi(\mathcal{H}^n u_0, \mathcal{H}^{n+1}u_0, t)\Xi(\mathcal{H}^{n+1}u_0, \mathcal{H}^{n+2}u_0, t)}$$
$$+ b \frac{[1 - \Xi(\mathcal{H}^n u_0, \mathcal{H}^{n+2}u_0, t)][1 - \Xi(\mathcal{H}^{n+1}u_0, \mathcal{H}^{n+1}u_0, t)]}{\Xi(\mathcal{H}^n u_0, \mathcal{H}^{n+2}u_0, t)\Xi(\mathcal{H}^{n+1}u_0, \mathcal{H}^{n+1}u_0, t)}.$$
$$= c \frac{[1 - \Xi(\mathcal{H}^n u_0, \mathcal{H}^{n+1}u_0, t)][1 - \Xi(\mathcal{H}^{n+1}u_0, \mathcal{H}^{n+2}u_0, t)]}{\Xi(\mathcal{H}^n u_0, \mathcal{H}^{n+1}u_0, t)\Xi(\mathcal{H}^{n+1}u_0, \mathcal{H}^{n+2}u_0, t)}$$
$$= c \frac{[1 - \Xi(\mathcal{H}^n u_0, \mathcal{H}^{n+1}u_0, t)][1 - \Xi(u_1, u_2, t)]}{\Xi(\mathcal{H}^n u_0, \mathcal{H}^{n+1}u_0, t)\Xi(u_1, u_2, t)}.$$

If $\frac{1}{\Xi(u_1,u_2,t)}-1=0$, then $\mathcal{H}u_1=u_2=u_1$ and so $\mathcal{H}$ has a fixed point. If $\frac{1}{\Xi(u_1,u_2,t)}-1>0$, then $\frac{1}{\Xi(u_1,u_2,t)}-1\leq c\frac{1}{\Xi(\mathcal{H}^n u_0,u_1,t)}-1$. Similarly, we have.

$$\begin{aligned}\frac{[1-\Xi(u_2,u_3,t)]^2}{\Xi^2(u_2,u_3,t)}&=\frac{[1-\Xi(\mathcal{H}^{n+2}u_0,\mathcal{H}^{n+3}u_0,t)]^2}{\Xi^2(\mathcal{H}^{n+2}u_0,\mathcal{H}^{n+3}u_0,t)}\\
&\leq c\frac{[1-\Xi(\mathcal{H}^{n+1}u_0,\mathcal{H}^{n+2}u_0,t)][1-\Xi(\mathcal{H}^{n+2}u_0,\mathcal{H}^{n+3}u_0,t)]}{\Xi(\mathcal{H}^{n+1}u_0,\mathcal{H}^{n+2}u_0,t)\Xi(\mathcal{H}^{n+2}u_0,\mathcal{H}^{n+3}u_0,t)}\\
&+b\frac{[1-\Xi(\mathcal{H}^{n+1}u_0,\mathcal{H}^{n+3}u_0,t)][1-\Xi(\mathcal{H}^{n+2}u_0,\mathcal{H}^{n+2}u_0,t)]}{\Xi(\mathcal{H}^{n+1}u_0,\mathcal{H}^{n+3}u_0,t)\Xi(\mathcal{H}^{n+2}u_0,\mathcal{H}^{n+2}u_0,t)}.\\
&=c\frac{[1-\Xi(\mathcal{H}^{n+1}u_0,\mathcal{H}^{n+2}u_0,t)][1-\Xi(\mathcal{H}^{n+2}u_0,\mathcal{H}^{n+3}u_0,t)]}{\Xi(\mathcal{H}^{n+1}u_0,\mathcal{H}^{n+2}u_0,t)\Xi(\mathcal{H}^{n+2}u_0,\mathcal{H}^{n+3}u_0,t)}\\
&=c\frac{[1-\Xi(u_1,u_2,t)][1-\Xi(u_2,u_3,t)]}{\Xi(u_1,u_2,t)\Xi(u_2,u_3,t)}.\end{aligned}$$

If $\frac{1}{\Xi(u_1,u_2,t)}-1=0$, then $\mathcal{H}u_2=u_3=u_2$ and so $\mathcal{H}$ has a fixed point. If $\frac{1}{\Xi(u_2,u_3,t)}-1>0$, then $\frac{1}{\Xi(u_2,u_3,t)}-1\leq c\big[\frac{1}{\Xi(u_1,u_2,t)}-1\big]$ and so $\frac{1}{\Xi(u_2,u_3,t)}-1\leq c^2\big[\frac{1}{\Xi(\mathcal{H}^n u_0,u_1,t)}-1\big]$. By continuing this process we get that $\frac{1}{\Xi(u_n,u_{n+1},t)}-1\leq c^n\big[\frac{1}{\Xi(\mathcal{H}^n u_0,u_1,t)}-1\big]$ for all $m\geq 1$. This implies that $\{u_m\}$ is a Cauchy sequence. Since $(\Sigma,\Xi,\Theta,\bigotimes,\diamond)$ is a complete neutrosophic metric space, there exists $x\in\Sigma$ such that $u_m\to x$. If $\mathcal{H}$ is orbitally continuous, then $\mathcal{H}u_m\to\mathcal{H}x$. Hence, $\mathcal{H}x=x$.

If $n=0$, then for each $m\geq 2$ we have

$$\begin{aligned}\frac{[1-\Xi(\mathcal{H}x,\mathcal{H}^m u_0,t)]^2}{\Xi^2(\mathcal{H}x,\mathcal{H}^m u_0,t)}&\leq c\frac{[1-\Xi(x,\mathcal{H}x,t)][1-\Xi(\mathcal{H}u_{m-2},\mathcal{H}^2u_{m-2},t)]}{\Xi(x,\mathcal{H}x,t)\Xi(\mathcal{H}u_{m-2},\mathcal{H}^2u_{m-2},t)}\\
+b\frac{[1-\Xi(x,\mathcal{H}^2u_{m-2},t)][1-\Xi(\mathcal{H}u_{m-2},\mathcal{H}x,t)]}{\Xi(x,\mathcal{H}^2u_{m-2},t)\Xi(\mathcal{H}u_{m-2},\mathcal{H}x,t)}.\end{aligned}$$

Since $u_m\to x$, we have

$$\frac{1}{\Xi(\mathcal{H}x,x,t)}-1\leq b\frac{[1-\Xi(x,x,t)][1-\Xi(x,\mathcal{H}x,t)]}{\Xi(x,x,t)\Xi(x,\mathcal{H}x,t)}=0$$

and so $\mathcal{H}x=x$. Now, we show that $\mathcal{H}$ has a unique fixed point whenever $b<1$. Let x and y be fixed points of $\mathcal{H}$. Then, we have

$$\begin{aligned}\left(\frac{1}{\Xi(x,y,t)}-1\right)^2&=\left(\frac{1}{\Xi(\mathcal{H}^{n+1}x,\mathcal{H}^{n+2}y,t)}-1\right)^2\\
&\leq c\frac{[1-\Xi(\mathcal{H}^n x,\mathcal{H}^{n+1}x,t)][1-\Xi(\mathcal{H}^{n+1}y,\mathcal{H}^{n+2}y,t)]}{\Xi(\mathcal{H}^n x,\mathcal{H}^{n+1}x,t)\Xi(\mathcal{H}^{n+1}y,\mathcal{H}^{n+2}y,t)}\\
&+b\frac{[1-\Xi(\mathcal{H}^n x,\mathcal{H}^{n+2}y,t)][1-\Xi(\mathcal{H}^{n+1}y,\mathcal{H}^{n+1}x,t)]}{\Xi(\mathcal{H}^n x,\mathcal{H}^{n+2}y,t)\Xi(\mathcal{H}^{n+1}y,\mathcal{H}^{n+1}x,t)}=b\left(\frac{1}{\Xi(x,y,t)}-1\right)^2.\end{aligned}$$

Hence, $\frac{1}{\Xi(x,y,t)}-1=0$ because $b<1$. Thus, $x=y$ and so $\mathcal{H}$ has a unique fixed point. Finally, we prove that $\mathcal{H}$ has the property (P) whenever $\mathcal{H}$ has the strongly comparable property. Let $m2$ be given and $y\in T(\mathcal{H}m)$. We consider the following cases. **Case I**. $n=0$. In this case, we have

$$\begin{aligned}\left(\frac{1}{\Xi(y,\mathcal{H}y,t)}-1\right)^2 =&\left(\frac{1}{\Xi(\mathcal{H}(\mathcal{H}^{m-1}y),\mathcal{H}^2(\mathcal{H}^{m-1}y),t)}-1\right)^2\\ \leq& c\frac{[1-\Xi(\mathcal{H}^{m-1}y,\mathcal{H}^m y,t)][1-\Xi(\mathcal{H}^m y,\mathcal{H}^{m+1}y,t)]}{\Xi(\mathcal{H}^{m-1}y,\mathcal{H}^m y,t)\Xi(\mathcal{H}^m y,\mathcal{H}^{m+1}y,t)}\\ &+b\frac{[1-\Xi(\mathcal{H}^{m-1}y,\mathcal{H}^{m+1}y,t)][1-\Xi(\mathcal{H}^m y,\mathcal{H}^m y,t)]}{\Xi(\mathcal{H}^{m-1}y,\mathcal{H}^{m+1}y,t)\Xi(\mathcal{H}^m y,\mathcal{H}^m y,t)}\\ =&c\frac{[1-\Xi(\mathcal{H}^{m-1}y,y,t)][1-\Xi(y,\mathcal{H}y,t)]}{\Xi(\mathcal{H}^{m-1}y,y,t)\Xi(y,\mathcal{H}y,t)}.\end{aligned}$$

If $\frac{1}{\Xi(y,\mathcal{H}y,t)}-1=0$ then $\mathcal{H}y=y$. If $\frac{1}{\Xi(x,y,t)}-1>0$, then $\frac{1}{\Xi(\mathcal{H}^m y,\mathcal{H}^{m+1}y,t)}-1\leq c\left(\frac{1}{\Xi(\mathcal{H}^{m+1}y,\mathcal{H}^m y,t)}-1\right)$. By using a similar argument as in Theorem 3.1 and continuing the process, we obtain

$$\begin{aligned}&\frac{1}{\Xi(y,\mathcal{H}y,t)}-1=\frac{1}{\Xi(\mathcal{H}^m y,\mathcal{H}^{m+1}y,t)}-1\leq c\left[\frac{1}{\Xi(\mathcal{H}^{m-1}y,\mathcal{H}^m y,t)}-1\right]\\ &\leq c^2\left[\frac{1}{\Xi(\mathcal{H}^{m-2}y,\mathcal{H}^{m-1}y,t)}-1\right]\leq \dots\leq c^m\left[\frac{1}{\Xi(y,\mathcal{H}y,t)}-1\right].\end{aligned}$$

Since $c<1$, $\mathcal{H}y=y$.

Case II. $n\geq 1$ and $m\leq n$. In this case, choose a natural number μ and an integer number $0\leq \nu<m$ such that $n+1=\mu m+\nu$. Then, we have $\mathcal{H}^m(\mathcal{H}^{m-\nu}y)=\mathcal{H}^{n+1}(\mathcal{H}^{n-\nu}y)=y$, and so

$$\begin{aligned}\left(\frac{1}{\Xi(y,\mathcal{H}y,t)}-1\right)^2 =&\left(\frac{1}{\Xi(\mathcal{H}^{n+1}(\mathcal{H}^{m-\nu}y),\mathcal{H}^{n+2}(\mathcal{H}^{m-\nu}y),t)}-1\right)^2\\ \leq& c\frac{[1-\Xi(\mathcal{H}^{n}(\mathcal{H}^{m-\nu}y),\mathcal{H}^{n+1}(\mathcal{H}^{m-\nu}y),t)][1-\Xi(\mathcal{H}^{n+1}(\mathcal{H}^{m-\nu}y),\mathcal{H}^{n+2}(\mathcal{H}^{m-\nu}y),t)]}{\Xi(\mathcal{H}^{n}(\mathcal{H}^{m-\nu}y),\mathcal{H}^{n+1}(\mathcal{H}^{m-\nu}y),t)\Xi(\mathcal{H}^{n+1}(\mathcal{H}^{m-\nu}y),\mathcal{H}^{n+2}(\mathcal{H}^{m-\nu}y),t)}\\ &+b\frac{[1-\Xi(\mathcal{H}^{n}(\mathcal{H}^{m-\nu}y),\mathcal{H}^{n+2}(\mathcal{H}^{m-\nu}y),t)][1-\Xi(\mathcal{H}^{n+1}(\mathcal{H}^{m-\nu}y),\mathcal{H}^{n+1}(\mathcal{H}^{m-\nu}y),t)]}{\Xi(\mathcal{H}^{n}(\mathcal{H}^{m-\nu}y),\mathcal{H}^{n+2}(\mathcal{H}^{m-\nu}y),t)\Xi(\mathcal{H}^{n+1}(\mathcal{H}^{m-\nu}y),\mathcal{H}^{n+1}(\mathcal{H}^{m-\nu}y),t)}\\ =&c\frac{[1-\Xi(\mathcal{H}^{m-1}y,y,t)][1-\Xi(y,\mathcal{H}y,t)]}{\Xi(\mathcal{H}^{m-1}y,y,t)\Xi(y,\mathcal{H}y,t)}.\end{aligned}$$

If $\frac{1}{\Xi(y,\mathcal{H}y,t)}-1=0$ then $\mathcal{H}y=y$. If $\frac{1}{\Xi(x,\mathcal{H}y,t)}-1>0$, then $\frac{1}{\Xi(\mathcal{H}^m y,\mathcal{H}^{m+1}y,t)}-1\leq c\left(\frac{1}{\Xi(\mathcal{H}^{m-1}y,\mathcal{H}^m y,t)}-1\right)$. By using a similar argument as in Theorem 3.1, we obtain

$$\begin{aligned}&\frac{1}{\Xi(y,\mathcal{H}y,t)}-1=\frac{1}{\Xi(\mathcal{H}^m y,\mathcal{H}^{m+1}y,t)}-1\leq c\left[\frac{1}{\Xi(\mathcal{H}^{m-1}y,\mathcal{H}^m y,t)}-1\right]\\ &\leq c^2\left[\frac{1}{\Xi(\mathcal{H}^{m-2}y,\mathcal{H}^{m-1}y,t)}-1\right]\leq \dots\leq c^m\left[\frac{1}{\Xi(y,\mathcal{H}y,t)}-1\right].\end{aligned}$$

Since $c<1$, $\mathcal{H}y=y$. Thus, $T(\mathcal{H}^m)\subseteq T(\mathcal{H})$. Therefore, $\mathcal{H}$ has the property (P). □

Definition 3.3. Let $(\Sigma, \Xi, \Theta, \bigotimes, \diamond)$ be a neutrosophic metric space and $\mathcal{H}$ a selfmap on Σ. Then, $\mathcal{H}$ is said to be a convex contraction of order 2 if there exist $r, s \in (01)$ and $t > 0$ with $r + s < 1$ such that

$$\frac{1}{\Xi(\mathcal{H}^2 u, \mathcal{H}^2 v, t)} - 1 \leq r\left[\frac{1}{\Xi(\mathcal{H}u, \mathcal{H}v, t)} - 1\right] + s\left[\frac{1}{\Xi(u, v, t)} - 1\right]$$

for all $u, v \in \Sigma$. Also, $\mathcal{H}$ is said to be a convex contraction of order 2 if there exist $r_1, r_2, s_1, s_2 \in (0,1)$ with $r_1 + r_2 + s_1 + s_2 < 1$ such that

$$\frac{1}{\Xi(\mathcal{H}^2 u, \mathcal{H}^2 v, t)} - 1 \leq r_1\left[\frac{1}{\Xi(u, \mathcal{H}u, t)} - 1\right] + r_2\left[\frac{1}{\Xi(\mathcal{H}u, \mathcal{H}^2 u, t)} - 1\right]$$
$$+ s_1\left[\frac{1}{\Xi(v, \mathcal{H}v, t)} - 1\right] + s_2\left[\frac{1}{\Xi(\mathcal{H}v, \mathcal{H}^2 v, t)} - 1\right]$$

$\forall u, v \in \Sigma$

Theorem 3.4. *Let $(\Sigma, \Xi, \Theta, \bigotimes, \diamond)$ be a complete order triangular NMS, $r, s \in (0,1)$ with $r + s < 1$ and $\mathcal{H}$ an orbitally continuous selfmap on Σ satisfy the condition*

$$\frac{1}{\Xi(\mathcal{H}^2 u, \mathcal{H}^2 v, t)} - 1 \leq r\left[\frac{1}{\Xi(\mathcal{H}u, \mathcal{H}v, t)} - 1\right] + s\left[\frac{1}{\Xi(u, v, t)} - 1\right]$$

for all $u, v \in \Sigma_{\ll}$, then $\mathcal{H}$ has a unique fixed point. Also, $T(\mathcal{H}) = T(\mathcal{H}^2)$.

Proof. Define $u_m = \mathcal{H}^m u_0$ for all $m \geq 1$, $y = \frac{1}{\Xi(\mathcal{H}u_0, \mathcal{H}^2 u_0, t)} - 1 + \frac{1}{\Xi(u_0, \mathcal{H}u_0, t)} - 1$, and $\delta = r + s$. Thus $\frac{1}{\Xi(\mathcal{H}^2 u_0, \mathcal{H}u_0, t)} - 1 \leq y$. Now, by using the assumption, we can put $u = \mathcal{H}u_0$ and $v = u_0$ in the condition. Thus, we obtain

$$\frac{1}{\Xi(\mathcal{H}^3 u_0, \mathcal{H}^2 u_0, t)} - 1 \leq r\left[\frac{1}{\Xi(\mathcal{H}^2 u_0, \mathcal{H}u_0, t)} - 1\right] + s\left[\frac{1}{\Xi(u_0, \mathcal{H}u_0, t)} - 1\right] \leq \delta_y$$

Now, by putting $u = \mathcal{H}^2 u_0$ and $v = \mathcal{H}u_0$ in the condition, we get

$$\frac{1}{\Xi(\mathcal{H}^4 u_0, \mathcal{H}^3 u_0, t)} - 1 \leq r\left[\frac{1}{\Xi(\mathcal{H}^3 u_0, \mathcal{H}^2 u_0, t)} - 1\right] + s\left[\frac{1}{\Xi(\mathcal{H}^2 u_0, u_0, t)} - 1\right]$$
$$\leq r^2\left[\frac{1}{\Xi(\mathcal{H}^2 u_0, \mathcal{H}u_0, t)} - 1\right] + rs\left[\frac{1}{\Xi(u_0, \mathcal{H}u_0, t)} - 1\right] + s\left[\frac{1}{\Xi(\mathcal{H}^2 u_0, u_0, t)} - 1\right] \leq \delta^2 y.$$

Again, by putting $u = \mathcal{H}^3 u_0$ and $v = \mathcal{H}^2 u_0$ in the condition, we obtain

$$\frac{1}{\Xi(\mathcal{H}^5 u_0, \mathcal{H}^4 u_0, t)} - 1 \leq r\left[\frac{1}{\Xi(\mathcal{H}^4 u_0, \mathcal{H}^3 u_0, t)} - 1\right] + s\left[\frac{1}{\Xi(\mathcal{H}^3 u_0, \mathcal{H}^2 u_0, t)} - 1\right]$$
$$\leq (r^3 + rs)\left[\frac{1}{\Xi(\mathcal{H}^2 u_0, \mathcal{H}u_0, t)} - 1\right] + r^2 s\left[\frac{1}{\Xi(u_0, \mathcal{H}u_0, t)} - 1\right]$$
$$+ rs\left[\frac{1}{\Xi(\mathcal{H}^2 u_0, \mathcal{H}u_0, t)} - 1\right] + s\left[\frac{1}{\Xi(u_0, \mathcal{H}u_0, t)} - 1\right]$$
$$= (r^3 + 2rs)\left[\frac{1}{\Xi(\mathcal{H}^2 u_0, \mathcal{H}u_0, t)} - 1\right] + (r^2 s + s^2)\left[\frac{1}{\Xi(u_0, \mathcal{H}u_0, t)} - 1\right] \leq \delta^3 y.$$

By continuing this process, we get $\frac{1}{\Xi(\mathcal{H}^{m+1}u_0,\mathcal{H}^m u_0,t)} - 1 \leq \delta^{m-1}y\ \forall m \geq 3$. This implies that

$$\frac{1}{\Xi^2(\mathcal{H}^n u_0, \mathcal{H}^m u_0, t)} - 1 \leq \sum_{i=n}^{m-1}\left(\frac{1}{\Xi^2(\mathcal{H}^i u_0, \mathcal{H}^{i+1}u_0, t)} - 1\right) \leq \sum_{i=n}^{m-1}\delta^{i-2}y \leq \frac{\delta^{i-2}}{1-\delta}y.$$

for all $m > n \geq 3$. Hence, $\{u_m\}$ is a Cauchy sequence. If there exists $x \in \Sigma$ such that $u_m \to x$. Then $(\Sigma, \Xi, \Theta, \bigotimes, \diamond)$ is a complete neutrosophic metric space. Since $\mathcal{H}$ is orbitally continuous, $\mathcal{H}u_m \to \mathcal{H}x$ and so $\mathcal{H}x = x$. Now, we show that $\mathcal{H}$ mapping has a unique fixed point. Let v and w be fixed points of $\mathcal{H}$. Then

$$\begin{aligned}\frac{1}{\Xi(v,w,t)} - 1 =& \frac{1}{\Xi(\mathcal{H}^2 v, \mathcal{H}^2 w, t)} - 1 \leq r\left[\frac{1}{\Xi(\mathcal{H}v,\mathcal{H}w,t)} - 1\right] + s\left[\frac{1}{\Xi(v,w,t)} - 1\right]\\ =&(r+s)\frac{1}{\Xi(v,w,t)} - 1\end{aligned}$$

Since $r + s < 1$, we get $\mathcal{H}v = v$. □

Theorem 3.5. *Let* $(\Sigma, \Xi, \Theta, \bigotimes, \diamond)$ *be a complete order triangular neutrosophic metric space,* $r_1, r_2, s_1, s_2 \in (0,1)$ *with* $r_1 + r_2 + s_1 + s_2 < 1$ *and* $\mathcal{H}$ *an orbitally continuous selfmap on* Σ *satisfy the condition*

$$\begin{aligned}\frac{1}{\Xi(\mathcal{H}^2 u, \mathcal{H}^2 v, t)} - 1 \leq\ & r_1\left[\frac{1}{\Xi(u,\mathcal{H}u,t)} - 1\right] + r_2\left[\frac{1}{\Xi(\mathcal{H}u,\mathcal{H}^2 u,t)} - 1\right]\\ &+ s_1\left[\frac{1}{\Xi(v,\mathcal{H}v,t)} - 1\right] + s_2\left[\frac{1}{\Xi(\mathcal{H}v,\mathcal{H}^2 v,t)} - 1\right]\end{aligned}$$

$\forall u, v \in \Sigma_{\ll}$. *If there exists* $u_0 \in \Sigma$ *such that* $\mathcal{H}^{m-1}u_0, \mathcal{H}^m u_0) \in \Sigma_{\ll}\ \forall m \geq 1$, *then* $\mathcal{H}$ *has a unique fixed point. Also* $T(\mathcal{H}) = T(\mathcal{H}^2)$.

Proof. Define $u_m = \mathcal{H}^m u_0$, for all $\forall m \geq 1$, and set $y = \frac{1}{\Xi(\mathcal{H}u_0,\mathcal{H}^2 u_0,t)} - 1 + \frac{1}{\Xi(u_0,\mathcal{H}u_0,t)} - 1$ Also, put $\delta = r_1 + r_2 + s_1$ and $\lambda = 1s_2$. We prove that

$$\frac{1}{\Xi(\mathcal{H}^{m+1}u_0, \mathcal{H}^m u_0, t)} - 1 \leq \left(\frac{\delta}{\lambda}\right)^{m-2} y$$

for all $m \geq 3$. Note that

$$\begin{aligned}\frac{1}{\Xi(\mathcal{H}^3 u_0, \mathcal{H}^2 u_0, t)} - 1 \leq\ & r_1\left[\frac{1}{\Xi(u_0,\mathcal{H}u_0,t)} - 1\right] + r_2\left[\frac{1}{\Xi(\mathcal{H}u_0,\mathcal{H}^2 u_0,t)} - 1\right]\\ &+ s_1\left[\frac{1}{\Xi(u_0,\mathcal{H}u_0,t)} - 1\right] + s_2\left[\frac{1}{\Xi(\mathcal{H}^3 u_0,\mathcal{H}^2 u_0,t)} - 1\right]\\ \leq\ & r_1 y + (r_1 + s_1)y + s_2\left[\frac{1}{\Xi(\mathcal{H}^3 u_0,\mathcal{H}^2 u_0,t)} - 1\right].\end{aligned}$$

Hence, $\frac{1}{\Xi(\mathcal{H}^3u_0,\mathcal{H}^2u_0,t)} - 1 \leq \left(\frac{\delta}{\lambda}\right)y$. Now, by using the assumption, we can put $u = \mathcal{H}u_0$ and $v = \mathcal{H}^2u_0$ in the condition. Thus, we obtain

$$\begin{aligned}\frac{1}{\Xi(\mathcal{H}^3u_0,\mathcal{H}^4u_0,t)} - 1 &\leq r_1\left[\frac{1}{\Xi(\mathcal{H}u_0,\mathcal{H}^2u_0,t)} - 1\right] + r_2\left[\frac{1}{\Xi(\mathcal{H}^2u_0,\mathcal{H}^3u_0,t)} - 1\right]\\ &+ s_1\left[\frac{1}{\Xi(\mathcal{H}^2u_0,\mathcal{H}^3u_0,t)} - 1\right] + s_2\left[\frac{1}{\Xi(\mathcal{H}^3u_0,\mathcal{H}^4u_0,t)} - 1\right]\\ &\leq r_1y + (r_1+s_1)\frac{r_1+r_2+s_1}{1-s_2}y + s_2\left[\frac{1}{\Xi(\mathcal{H}^3u_0,\mathcal{H}^4u_0,t)} - 1\right].\end{aligned}$$

Hence, $\frac{1}{\Xi(\mathcal{H}^3u_0,\mathcal{H}^4u_0,t)} - 1 \leq \left(\frac{\delta}{\lambda}\right)y$. Similarly, we have

$$\begin{aligned}\frac{1}{\Xi(\mathcal{H}^5u_0,\mathcal{H}^4u_0,t)} - 1 &\leq r_1\left[\frac{1}{\Xi(\mathcal{H}^3u_0,\mathcal{H}^2u_0,t)} - 1\right] + r_2\left[\frac{1}{\Xi(\mathcal{H}^4u_0,\mathcal{H}^3u_0,t)} - 1\right]\\ &+ s_1\left[\frac{1}{\Xi(\mathcal{H}^4u_0,\mathcal{H}^3u_0,t)} - 1\right] + s_2\left[\frac{1}{\Xi(\mathcal{H}^5u_0,\mathcal{H}^4u_0,t)} - 1\right]\\ &\leq r_1\frac{r_1+r_2+s_1}{1-s_2}y + (r_2+s_1)\frac{r_1+r_2+s_1}{1-s_2}y + s_2\left[\frac{1}{\Xi(\mathcal{H}^5u_0,\mathcal{H}^4u_0,t)} - 1\right].\end{aligned}$$

Hence, $\frac{1}{\Xi(\mathcal{H}^5u_0,\mathcal{H}^4u_0,t)} - 1 \leq \left(\frac{\delta}{\lambda}\right)^2 y$. Also, by using the assumption and putting $u = \mathcal{H}^3u_0$ and $v = \mathcal{H}^4u_0$ in the condition, we obtain

$$\begin{aligned}\frac{1}{\Xi(\mathcal{H}^5u_0,\mathcal{H}^6u_0,t)} - 1 &\leq r_1\left[\frac{1}{\Xi(\mathcal{H}^3u_0,\mathcal{H}^4u_0,t)} - 1\right] + r_2\left[\frac{1}{\Xi(\mathcal{H}^4u_0,\mathcal{H}^5u_0,t)} - 1\right]\\ &+ s_1\left[\frac{1}{\Xi(\mathcal{H}^4u_0,\mathcal{H}^5u_0,t)} - 1\right] + s_2\left[\frac{1}{\Xi(\mathcal{H}^5u_0,\mathcal{H}^6u_0,t)} - 1\right]\\ &\leq r_1(\frac{\delta}{\lambda})y + (r_2+s_1)big(\frac{\delta}{\lambda})^2y + s_2\left[\frac{1}{\Xi(\mathcal{H}^5u_0,\mathcal{H}^6u_0,t)} - 1\right]\\ &= (\frac{\delta}{\lambda})^2\left[r_1(\frac{\lambda}{\delta})y + (r_2+s_1)y + (\frac{\lambda}{\delta})^2s_2\left[\frac{1}{\Xi(\mathcal{H}^5u_0,\mathcal{H}^6u_0,t)} - 1\right]\right]\\ &\leq (\frac{\delta}{\lambda})^2\left[r_1(\frac{\lambda}{\delta})y + (r_2+s_1)(\frac{\lambda}{\delta})y + (\frac{\lambda}{\delta})^2s_2\left[\frac{1}{\Xi(\mathcal{H}^5u_0,\mathcal{H}^6u_0,t)} - 1\right]\right]\\ &= (\frac{\delta}{\lambda})^2\left[(\frac{\lambda}{\delta})(r+1+r_2+s_1)v + (\frac{\lambda}{\delta})^2s_2\left[\frac{1}{\Xi(\mathcal{H}^5u_0,\mathcal{H}^6u_0,t)} - 1\right]\right]\\ &\leq (\frac{1}{\lambda})^2\left[(\frac{\lambda}{\delta})(r+1+r_2+s_1)^3v + (\lambda)^2s_2\left[\frac{1}{\Xi(\mathcal{H}^5u_0,\mathcal{H}^6u_0,t)} - 1\right]\right]\end{aligned}$$

which implies

$$(\frac{\lambda}{\delta})(\lambda)^3\left[\frac{1}{\Xi(\mathcal{H}^5u_0,\mathcal{H}^6u_0,t)} - 1\right] \leq 1-(s_2)^3\left[\frac{1}{\Xi(\mathcal{H}^5u_0,\mathcal{H}^6u_0,t)} - 1\right] \leq (\delta)^3y$$

Hence

$$\left[\frac{1}{\Xi(\mathcal{H}^5u_0,\mathcal{H}^6u_0,t)} - 1\right] \leq (\frac{\delta}{\lambda})^3y.$$

By continuing the process, we get $\left[\frac{1}{\Xi(\mathcal{H}^5u_0,\mathcal{H}^6u_0,t)}-1\right]\leq\left(\frac{\delta}{\lambda}\right)^{m-2}$ for all $m\geq 3$. This implies

$$\frac{1}{\Xi^2(\mathcal{H}^n u_0,\mathcal{H}^m u_0,t)}-1\leq\sum_{i=n}^{m-1}\left(\frac{1}{\Xi^2(\mathcal{H}^i u_0,\mathcal{H}^{i+1}u_0,t)}-1\right)\leq\sum_{i=n}^{m-1}\left(\frac{\delta}{\lambda}\right)^{i-2}y\leq\frac{\left(\frac{\delta}{\lambda}\right)^{m-2}}{1-\left(\frac{\delta}{\lambda}\right)}y,$$

for all $mn > m \geq 3$. Hence $\{u_m\}$ is a Cauchy sequence. Since $(\Sigma,\Xi,\Theta,\bigotimes,\diamond)$ is a complete neutrosophic metric space, there exists $x\in\Sigma$ such that $u_m\to x$. Since $\mathcal{H}$ is orbitally continuous, $\mathcal{H}u_m\to\mathcal{H}x$ and so $\mathcal{H}x=x$. Now, we show that $\mathcal{H}$ has a unique fixed point. Let v and w be fixed points of $\mathcal{H}$. Then,

$$\begin{aligned}\frac{1}{\Xi(v,w,t)}-1=&\frac{1}{\Xi(\mathcal{H}^2v,\mathcal{H}^2w,t)}-1\leq r_1\left[\frac{1}{\Xi(\mathcal{H}v,v,t)}-1\right]+r_2\left[\frac{1}{\Xi(\mathcal{H}v,\mathcal{H}^2v,t)}-1\right]\\&+s_1\left[\frac{1}{\Xi(w,\mathcal{H}w,t)}-1\right]+s_2\left[\frac{1}{\Xi(\mathcal{H}w,\mathcal{H}^2w,t)}-1\right]\end{aligned}$$

and so $v=w$. Now, we prove that $T(\mathcal{H})=T(\mathcal{H}^2)$. Let $v\in T(\mathcal{H}^2)$. Then, we have

$$\begin{aligned}\frac{1}{\Xi(v,\mathcal{H}v,t)}-1=&\frac{1}{\Xi(\mathcal{H}^2v,\mathcal{H}^2v,t)}-1\leq r_1\left[\frac{1}{\Xi(\mathcal{H}v,\mathcal{H}^2v,t)}-1\right]+r_2\left[\frac{1}{\Xi(\mathcal{H}^2v,\mathcal{H}^3v,t)}-1\right]\\&+s_1\left[\frac{1}{\Xi(v,\mathcal{H}v,t)}-1\right]+s_2\left[\frac{1}{\Xi(\mathcal{H}v,\mathcal{H}^2v,t)}-1\right]\\&=(r_1+r_2+s_1+s_2)\left[\frac{1}{\Xi(v,\mathcal{H}v,t)}-1\right].\end{aligned}$$

Since $r+s<1$, we get $\mathcal{H}v=v$ □

4. application

Example 4.1. Let $\Sigma=[0,\infty)$, be endowed with $d(u,v)=|u-v|$, $\Xi(u,v,m)=\frac{m}{m+d(u,v)}$ and $\Theta(u,v,m)=\frac{d(u,v)}{m+d(u,v)}$ for all $u,v\in\Sigma$ and $m\geq 0$. Define the selfmap $\mathcal{H}$ on Σ by $\mathcal{H}u=0$ whenever $0\leq u\leq 10$, $\mathcal{H}u=u10$ whenever $10\leq u\leq 11$ and $\mathcal{H}u=1.1$ whenever $u\geq 11$. Then by putting $\delta=\frac{1}{2}$. Therefore, the condition of Theorem 3.1 is satisfied for $\mathcal{H}$.

Example 4.2. Let $\Sigma=[0,\infty)$, be endowed with $d(u,v)=|u-v|$, $\Xi(u,v,m)=\frac{m}{m+d(u,v)}$ and $\Theta(u,v,m)=\frac{d(u,v)}{m+d(u,v)}$ for all $u,v\in\Sigma$ and $m\geq 0$. Define the selfmap $\mathcal{H}$ on Σ by $\mathcal{H}u=0$ whenever $0\leq u\leq 100$, $\mathcal{H}u=u100$ whenever $100\leq u\leq 100.1$ and $\mathcal{H}u=0.15$ whenever $u\geq 100.1$. Then by putting $\delta=\frac{1}{2}$ $n=0$. Therefore, the condition of Theorem 3.2 is satisfied for $\mathcal{H}$.

Example 4.3. Let $\Sigma=\{1,3,5\}$, be endowed with $d(u,v)=|u-v|$, $\Xi(u,v,m)=\frac{m}{m+d(u,v)}$ and $\Theta(u,v,m)=\frac{d(u,v)}{m+d(u,v)}$ for all $u,v\in\Sigma$ and $m\geq 0$. Define $\lll=\{(1,1),(3,3),(5,5)\}$ the selfmap $\mathcal{H}$ on Σ by $\mathcal{H}1=3,\mathcal{H}3=1,\mathcal{H}5=5$ Then, by putting $u_0=5$, $r=\frac{1}{2}$and $s=\frac{1}{4}$, we conclude that the condition of Theorem 3.4 is satisfied.

Example 4.4. Let $\Sigma = \{1,3,5\}$, be endowed with $d(u,v) = |u-v|$, $\Xi(u,v,m) = \frac{m}{m+d(u,v)}$ and $\Theta(u,v,m) = \frac{d(u,v)}{m+d(u,v)}$ for all $u,v \in \Sigma$ and $m \geq 0$. Define $\ll = \{(1,1),(3,3),(5,5)\}$ the selfmap $\mathcal{H}$ on Σ by $\mathcal{H}1 = 3, \mathcal{H}3 = 1, \mathcal{H}5 = 5$ Then, by putting $u_0 = 5$, $r_1 = r_2 = s_1 = s_2 = \frac{1}{4}$, it is easy to verify that $\mathcal{H}$ satisfies the conditions of the last theorem 3.5, and so $\mathcal{H}$ has a unique solution.

Here the Cauchy sequence in neutrosophic metric space, complete neutrosophic metric space and complete ordered triangular neutrosophic metric spaces examples are introduced.

Example 4.5. Let $\Sigma = \frac{1}{n} : n \in N$ with the standard metric $d(\mu,\nu) = |\mu-\nu|$. For all $\mu,\nu \in \Sigma$ and $\alpha \in [0,\infty)$, be defined by

$$\Xi(\mu,\nu,\alpha) = \begin{cases} \dfrac{\alpha}{\alpha + d(\mu,\nu)}, \ if \quad \alpha > 0, \\ \quad 0, \qquad\quad if \ \alpha = 0 \end{cases}$$

$$\phi(\mu,\nu,\alpha) = \begin{cases} \dfrac{d(\mu,\nu)}{k\alpha + d(\mu,\nu)}, \ if \quad \alpha > 0, k > 0 \\ \qquad 1, \qquad\quad if \ \alpha = 0 \end{cases}$$

$$\Upsilon(\mu,\nu,\alpha) = \frac{d(\mu,\nu)}{\alpha} \ if \quad \alpha > 0.$$

for all $\mu,\nu \in \Sigma$ and $\alpha > 0$. Then $(\Sigma, \Xi, \phi, \Upsilon, *, \diamondsuit)$ is called complete neutrosophic metric space on Σ, Here $*$ is defined by $\mu * \nu = \mu, \nu$ and $\diamondsuit$ is defined as $\mu \diamondsuit \nu = min\{1, \mu+\nu\}$. Define $\sigma(\mu) = \mu$, $\rho(\nu) = \nu$. Clearly $\sigma(\Sigma) \subseteq \rho(\Sigma)$, Also for $k = \frac{1}{3}$, we get

$$\Xi(\sigma(\mu), \rho(\nu), \frac{\alpha}{3}) = \frac{\frac{\alpha}{3}}{\frac{\alpha}{3} + d(\sigma(\mu), \rho(\nu))} \geq \frac{\alpha}{\alpha + \frac{d(\mu,\nu)}{3}} = \Xi(\sigma(\mu), \rho(\nu)).$$

Example 4.6. For $r > 0$, let $\Xi(y,r) = \frac{r}{r+\|y\|}$, $\phi(y,r) = \frac{\|y\|}{r+\|y\|}$, $\Upsilon(y,r) = \frac{\|y\|}{r}$. Then $(N, V, *, \diamondsuit)$ is an Neutrosophic norm space (NNS). Now,

$$\lim_{\mu,\nu\to\infty} \frac{r}{r+ \parallel y_\mu - y_\nu \parallel} = 1, \lim_{\mu,\nu\to\infty} \frac{\parallel y_\mu - y_\nu \parallel}{r+ \parallel y_\mu - y_\nu \parallel} = 0, \lim_{\mu,\nu\to\infty} \frac{\parallel y_\mu - y_\nu \parallel}{r} = 0.$$

$$\lim_{\mu,\nu\to\infty} \Xi(y_\mu - y_\nu, r) = 1, \lim_{\mu,\nu\to\infty} \phi(y_\mu - y_\nu, r) = 0, \lim_{\mu,\nu\to\infty} \Upsilon(y_\mu - y_\nu, r) = 0, \, as \, r \to \infty.$$

This shows that $\{y_\mu\}$ is a Cauchy sequence in the NNS $(N, V, *, \diamondsuit)$.

Example 4.7. Choose H as natural numbers set. Give the operations $*$ and $\diamondsuit$ as Triangular norms (TN) $\mu * \nu = max\{0, \mu+\nu-1\}$ and Triangular conorms (TC) $\mu \diamondsuit \nu = \mu + \nu - \mu\nu$. for all $\mu,\nu \in H, \alpha > 0$

$$\Xi(\mu,\nu,\alpha) = \begin{cases} \dfrac{\mu}{\nu}, \ if \quad \mu < \nu, \\ \dfrac{\nu}{\mu}, \ if \quad \nu < \mu, \end{cases}$$

$$\Xi(\mu,\nu,\alpha)=\begin{cases}\dfrac{\nu-\mu}{y}, \ if \quad \mu x<\nu,\\ \dfrac{\mu-\nu}{x}, \ if \quad \nu<\mu,\end{cases}$$

$$\Xi(\mu,\nu,\alpha)=\begin{cases}\nu-\mu, \ if \quad \mu<\nu,\\ \mu-\nu, \ if \quad \nu<\mu,\end{cases}$$

Then, $(H,\mathcal{N},*,\diamondsuit)$ is Neutrosophic metric space NMS such that $\mathcal{N}: H\times H\times R^{+}\longrightarrow[0,1]$.

5. Conclusion

In this article, I gave some results about the property (P). Moreover, I study and provide fixed point theorem for such mappings on complete ordered triangular neutrosophic metric spaces (NMS). Also stated and proved some results which extensions from the reference section of this paper of several results as in relevant items, as well as in the literature in general.

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7. Conflict of Interests

The authors declare that there is no conflict of interests regarding this manuscript.

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