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Neutrosophic General Machine: A Group-Based Study

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Abstract. Interval-valued neutrosophic sets have been shown to provide a limited platform for computational complexity, but neutrosophic sets are suitable for it. The neutrosophic setsare a suitable mechanism for interpreting the philosophical problems of real-life, but not for scientific problems because it is difficult to consolidate. This study aims to develop the notion of neutrosophic single-valued general machine over a finite group, which is knows as "group neutrosophic general machine", for simplicity, GNGM. After that, we present the notions of max-min GNGM, single-valued neutrosophic subgroup (SVNSG) and single-valued neutrosophic normal subgroup (SVNNSG). We show that if there exists a homomorphism between two GNGMs $\mathcal{M}_1$ and $\mathcal{M}_2$ and M is a single-valued neutrosophic fundamental (SVNF) of $\mathcal{M}_2$, then $f^{-1}(M)$ is a SVNF of $\mathcal{M}_1$. Also, if M is a singe-valued neutrosophic kernel (SVNK) of $\mathcal{M}_2$, then $f^{-1}(M)$ is a SVNK of $\mathcal{M}_1$. Moreover, we prove that if M is a SVNK and N is a SVNF of $\mathcal{M}$, then $M * N$ is a SVNF of $\mathcal{M}$. In addition, we show that if M and N are SVNK of $\mathcal{M}$, then $M * N$ is a SVNK of $\mathcal{M}$.

Keywords: Neutrosophic set, Automata, Intuitionistic set, Submachine, General fuzzy automata

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1. **Introduction**

The concept of 'fuzzy' together with some other notions in mathematics and other areas were fuzzified by Zadeh [16] in 1965. Within this real, among the first investigations was the concept of fuzzy automaton suggested by Wee [15] and Santos [8]. Doostfatemeh and Kremer [4] introduced the concept of general fuzzy automata.

An intuitionistic fuzzy set can be viewed as an alternative approach when available information is not sufficient to define the impreciseness by the conventional fuzzy set. Subsequently, as a generalization, the concept of intuitionistic general fuzzy automata has been introduced and studied by Shamsizadeh and Zahedi [10]. For more details see the recent literature as [2, 3, 5, 9, 11, 12].

Neutrosophy deals with origin, nature and scope of neutralities, as well as their interactions with different ideational spectra. Neutrosophy is the basis of neutrosophic sets (derivative of neutrosophy). A neutrosophic set is a general framework which generalizes the concept of fuzzy set, interval valued fuzzy set, and intuitionistic fuzzy set. Wang et al. [14] introduced single valued neutrosophic sets which is a neutrosophic set defined in the range [0, 1]. Wang et.al: [13] introduced the notion of interval-valued neutrosophic sets. Interval-valued neutrosophic sets have been shown that fuzzy sets provides limited platform for computational complexity but neutrosophic sets is suitable for it. The neutrosophic sets is an appropriate mechanism for interpreting real-life philosophical problems but not for scientific problems since it is difficult to consolidate.

The concept of interval neutrosophic finite state machine was introduced by Tahir Mahmood [7]. In 2019 [6] Kavikumar introduced the notion of neutrosophic general fuzzy automata.

The present paper is organized as follows: Section 2 encompasses preliminary information pertaining to the content of the paper. In Section 3 we present the notion of neutrosophic single-valued general automata over a finite group which is knows as "group neutrosophic general machine" (GNGM). Moreover, we give the notions of max-min GNGM, single-valued neutrosophic subgroup (SVNSG) and single-valued neutrosophic normal subgroup (SVNNSG). We prove that if there exists a homomorphism between two GNGMs $\mathcal{M}_1$ and $\mathcal{M}_2$ and M be a single-valued neutrosophic fundamental (SVNF) of $\mathcal{M}_2$, then $f^{-1}(M)$ is a SVNF of $\mathcal{M}_1$. Also, if M is a singe-valued neutrosophic kernel (SVNK) of $\mathcal{M}_2$, then $f^{-1}(M)$ is a SVNK of $\mathcal{M}_1$. Moreover, we prove that if M is a SVNK and N is a SVNF of $\mathcal{M}$, then $M * N$ is a SVNF of $\mathcal{M}$. In addition, we show that if M and N are SVNK of $\mathcal{M}$, then $M * N$ is a SVNK of $\mathcal{M}$.

2. Preliminaries

In this section, some concepts and definitions related to single-valued neutrosophy and automata are introduced.

Definition 2.1. [4] A general fuzzy automaton (GFA) is considered as: $\tilde{F} = (Q, \Sigma, \tilde{R}, Z, \tilde{\delta}, \omega, F_1, F_2)$, where (i) Q is a finite set of states, $Q = \{q_1, q_2, ..., q_n\}$, (ii) Σ is a finite set of input symbols, $\Sigma = \{a_1, a_2, ..., a_m\}$, (iii) $\tilde{R}$ is the set of fuzzy start states, $\tilde{R} \subseteq \tilde{P}(Q)$, (iv) Z is a finite set of output symbols, $Z = \{b_1, b_2, ..., b_k\}$, (v) $\omega : Q \to Z$ is the output function, (vi) $\tilde{\delta} : (Q \times [0, 1]) \times \Sigma \times Q \to [0, 1]$ is the augmented transition function. (vii) Function $F_1 : [0, 1] \times [0, 1] \to [0, 1]$ is called membership assignment function. (viii) $F_2 : [0, 1]^* \to [0, 1]$, is called multi-membership resolution function.

Definition 2.2. Let Σ be a space of points (objects), with a generic element in Σ denoted by x. A neutrosophic set A in Σ is characterized by a truth-membership function T_A, an indeterminacy-membership function I_A and a falsity-membership function F_A. $T_A(x), I_A(x)$ and $F_A(x)$ are real standard or non-standard subsets of $]0^-, 1^+[$. That is $T_A : \Sigma \to]0^-, 1^+[, I_A : \Sigma \to]0^-, 1^+[, F_A : \Sigma \to]0^-, 1^+[$. There is no restriction on the sum of $T_A(x), I_A(x)$ and $F_A(x)$, so $0^- \leq \sup T_A(x) + \sup I_A(x) + \sup F_A(x) \leq 3^+$.

Definition 2.3. Single-valued neutrosophic set is the immediate results of neutrosophic set if it is defined over standard unit interval $[0, 1]$ instead of the non-standard unit interval $]0^-, 1^+[$. A single-valued neutrosophic subset (SVNS) A of Q is defined by $SVNS(A) = \{(x, T_A(x), I_A(x), F_A(x)) \big| x \in \Sigma\}$, where $T_A(x), I_A(x), F_A(x) : \Sigma \to [0, 1]$ such that $0 \leq \sup T_A(x) + \sup I_A(x) + \sup F_A(x) \leq 3$.

Definition 2.4. [1] The support of a single-valued neutrosophic set $A = \{(x, T_A(x), I_A(x), F_A(x)) \big| x \in X\}$ is denoted by $supp(A)$, defined by $supp(A) = \{x \in A \big| T_A(x) \neq 0, I_A(x) \neq 0, F_A(x) \neq 0\}$. The support of a single-valued neutrosophic set is a crisp set.

3. Single-Valued Neutrosophic General Machine Over a Finite Group

Definition 3.1. A group single-valued neutrosophic general machine (GNGM) is a seven-tuple machine $\mathcal{M} = (Q, *, \Sigma, \tilde{R}, \tilde{\delta}, E_1, E_2)$, such that $(Q, *)$ is a finite group and

1. Q is called the set of states,
2. Σ is a finite set of input symbols,
3. $\tilde{R} \subseteq \tilde{P}(Q)$ is the set of single-valued neutrosophic initial states,
4. $\tilde{\delta} : (Q \times [0, 1] \times [0, 1] \times [0, 1]) \times \Sigma \times Q \to [0, 1] \times [0, 1] \times [0, 1]$ is the single-valued neutrosophic augmented transition function,
5. $E_1 = (E_1^T, E_1^I, E_1^F)$, where $E_1^T : [0, 1] \times [0, 1] \to [0, 1]$ is a t-norm and it is called the truth-membership assignment function. $E_1^T(T, T_\delta)$ is motivated by two parameters T and T_δ, where T is the truth-membership value of a predecessor and T_δ is the truth-membership value of the transition. Also, $E_1^I : [0, 1] \times [0, 1] \to [0, 1]$ is a t-norm and it is called the indeterminacy-membership function. $E_1^I(I, I_\delta)$ is motivated by two parameters I and I_δ, where I is the indeterminacy-membership value of a predecessor and I_δ is the indeterminacy-membership value of the transition. Moreover, $E_1^F : [0, 1] \times [0, 1] \to [0, 1]$ is a t-conorm and it is called the falsity-membership function. $E_1^F(F, F_\delta)$ is motivated by two parameters F and F_δ, where F is the falsity-membership value of a predecessor and F_δ is the falsity-membership value of the transition.

   In this definition, the process that takes place upon the transition from the state q_i to

q_j on an input a_k is represented by:

$$T^{t+1}(q_j) = \tilde{\delta}_1((q_i, T^t(q_i), I^t(q_i), F^t(q_i)), a_k, q_j) = E_1^T(T^t(q_i), \delta_1(q_i, a_k, q_j)),$$
$$I^{t+1}(q_j) = \tilde{\delta}_2((q_i, T^t(q_i), I^t(q_i), F^t(q_i)), a_k, q_j) = E_1^I(I^t(q_i), \delta_2(q_i, a_k, q_j)),$$
$$F^{t+1}(q_j) = \tilde{\delta}_3((q_i, T^t(q_i), I^t(q_i), F^t(q_i)), a_k, q_j) = E_1^F(T^t(q_i), \delta_3(q_i, a_k, q_j)),$$

where

$$\tilde{\delta}((q_i, T^t(q_i), I^t(q_i), F^t(q_i)), a_k, q_j) = (\tilde{\delta}_1((q_i, T^t(q_i), I^t(q_i), F^t(q_i)), a_k, q_j), \tilde{\delta}_2((q_i, T^t(q_i), I^t(q_i), F^t(q_i)), a_k, q_j), \tilde{\delta}_3((q_i, T^t(q_i), I^t(q_i), F^t(q_i)), a_k, q_j)),$$

and

$$\delta(q_i, a_k, q_j) = (\delta_1(q_i, a_k, q_j), \delta_2(q_i, a_k, q_j), \delta_3(q_i, a_k, q_j)).$$

6. $E_2 = (E_2^T, E_2^I, E_2^F)$, where $E_2^T : [0,1]^* \to [0,1]$ is a T-conorm and it is called multi-truth-membership function, $E_2^I : [0,1]^* \to [0,1]$ is a T-conorm and it is called multi-indetermincy-membership function, $E_2^F : [0,1]^* \to [0,1]$ is a T-norm and it is called multi-falsity-membership function.

Example 3.2. Let the GNGM $\mathcal{M} = (Q, *, \Sigma, \tilde{\delta}, \tilde{R}, E_1, E_2)$ such that $Q = \{q_0, q_1, q_2\}, \Sigma = \{a\}$, $\tilde{R} = \{(q_0, 0.4, 0.7, 0.3)\}$ and δ is defined as follows:

$$\delta(q_0, a, q_0) = (0.6, 0.7, 1), \qquad \delta(q_0, a, q_1) = (0.7, 0.5, 0.5),$$
$$\delta(q_0, a, q_2) = (0.9, 0.7, 0.4), \qquad \delta(q_1, a, q_1) = (0.4, 0.5, 0.2),$$
$$\delta(q_1, a, q_2) = (0.3, 0.7, 0.6), \qquad \delta(q_2, a, q_0) = (0.7, 0.9, 0.6),$$
$$\delta(q_2, a, q_1) = (0.7, 1, 1), \qquad \delta(q_2, a, q_2) = (0.6, 0.9, 0.5).$$

Now, we can consider E_1 as follows:
1. $E_1^T = T \wedge T_\delta,\ E_1^I = I \wedge I_\delta,\ E_1^F = F \vee F_\delta$,

$$T^{t+1}(q_m) = \bigvee_{i=1}^{n} E_1^T(T^t(q_i), \delta_1(q_i, a_k, q_m)),$$
$$I^{t+1}(q_m) = \bigvee_{i=1}^{n} E_1^I(I^t(q_i), \delta_2(q_i, a_k, q_m)),$$
$$F^{t+1}(q_m) = \bigwedge_{i=1}^{n} E_3^F(F^t(q_i), \delta_3(q_i, a_k, q_m)),$$

2. $E_1^T = T.T_\delta,\ E_1^I = I.I_\delta,\ E_1^F = F + F_\delta - F.F_\delta,$

$$T^{t+1}(q_m) = \bigvee_{i=1}^{n} E_1^T(T^t(q_i), \delta_1(q_i, a_k, q_m)),$$
$$I^{t+1}(q_m) = \bigvee_{i=1}^{n} E_1^I(I^t(q_i), \delta_2(q_i, a_k, q_m)),$$
$$F^{t+1}(q_m) = \bigwedge_{i=1}^{n} E_3^F(F^t(q_i), \delta_3(q_i, a_k, q_m)),$$

3. $E_1^T = T \wedge T_\delta,\ E_1^I = I \wedge I_\delta,\ E_1^F = F \vee F_\delta,$

$$T^{t+1}(q_m) = T_p(T_p(T^t(q_i), \delta_1(q_i, a_k, q_m))),$$
$$I^{t+1}(q_m) = T_p(T_p(I^t(q_i), \delta_2(q_i, a_k, q_m)),$$
$$F^{t+1}(q_m)) = S_p(S_p E_3^F(F^t(q_i), \delta_3(q_i, a_k, q_m))),$$

where T_p is the product t-norm and S_p is the product t-conorm.
If we choose the case 1, then we have

$$T^{t_1}(q_0) = E_1^T(T^{t_0}(q_0), \delta_1(q_0, a, q_0)) = 0.4 \wedge 0.6 = 0.4,$$
$$I^{t_1}(q_0) = E_1^I(I^{t_0}(q_0), \delta_2(q_0, a, q_0)) = 0.7 \wedge 0.7 = 0.7,$$
$$F^{t_1}(q_0) = E_1^F(F^{t_0}(q_0), \delta_3(q_0, a, q_0)) = 0.3 \vee 1 = 1,$$
$$T^{t_1}(q_1) = E_1^T(T^{t_0}(q_0), \delta_1(q_0, a, q_1)) = 0.4 \wedge 0.7 = 0.4,$$
$$I^{t_1}(q_1) = E_1^I(I^{t_0}(q_0), \delta_2(q_0, a, q_1)) = 0.7 \wedge 0.5 = 0.5,$$
$$F^{t_1}(q_1) = E_1^F(F^{t_0}(q_0), \delta_3(q_0, a, q_1)) = 0.3 \vee 0.5 = 0.5,$$
$$T^{t_1}(q_2) = E_1^T(T^{t_0}(q_0), \delta_1(q_0, a, q_2)) = 0.4 \wedge 0.9 = 0.4,$$
$$I^{t_1}(q_2) = E_1^I(I^{t_0}(q_0), \delta_2(q_0, a, q_2)) = 0.7 \wedge 0.7 = 0.7,$$
$$F^{t_1}(q_2) = E_1^F(F^{t_0}(q_0), \delta_3(q_0, a, q_2)) = 0.3 \vee 0.4 = 0.4,$$
$$T^{t_2}(q_0) = E_1^T(T^{t_1}(q_0), \delta_1(q_0, a, q_0)) \vee E_1^T(T^{t_1}(q_2), \delta_1(q_2, a, q_0)) = (0.4 \wedge 0.6) \vee (0.4 \wedge 0.7) = 0.4,$$
$$I^{t_2}(q_0) = E_1^I(I^{t_1}(q_0), \delta_2(q_0, a, q_0)) \vee E_1^I(I^{t_1}(q_2), \delta_2(q_2, a, q_0)) = (0.7 \wedge 0.7) \vee (0.7 \wedge 0.9) = 0.7,$$
$$F^{t_2}(q_0) = E_1^F(F^{t_1}(q_0), \delta_3(q_0, a, q_0)) \wedge E_1^F(F^{t_1}(q_2), \delta_3(q_2, a, q_0)) = (1 \vee 1) \wedge (0.4 \vee 0.6) = 0.6,$$
$$T^{t_2}(q_1) = E_1^T(T^{t_1}(q_0), \delta_1(q_0, a, q_1)) \vee E_1^T(T^{t_1}(q_1), \delta_1(q_1, a, q_1)) \vee E_1^T(T^{t_1}(q_2), \delta_1(q_2, a, q_1))$$
$$= (0.4 \wedge 0.7) \vee (0.4 \wedge 0.4) \vee (0.4 \wedge 0.7) = 0.4,$$

$$
\begin{aligned}
I^{t_2}(q_1) &= E_1^I(I^{t_1}(q_0), \delta_2(q_0, a, q_1)) \vee E_1^I(I^{t_1}(q_1), \delta_2(q_1, a, q_1)) \vee E_1^I(I^{t_1}(q_2), \delta_2(q_2, a, q_1)) \\
&= (0.7 \wedge 0.5) \vee (0.5 \wedge 0.5) \vee (0.7 \wedge 1) = 0.7, \\
F^{t_2}(q_1) &= E_1^F(F^{t_1}(q_0), \delta_3(q_0, a, q_1)) \wedge E_1^F(F^{t_1}(q_1), \delta_3(q_1, a, q_1)) \wedge E_1^F(F^{t_1}(q_2), \delta_3(q_2, a, q_1)) \\
&= (1 \vee 0.5) \wedge (0.5 \vee 0.2) \wedge (0.4 \vee 1) = 0.5, \\
T^{t_2}(q_2) &= E_1^T(T^{t_1}(q_0), \delta_1(q_0, a, q_2)) \vee E_1^T(T^{t_1}(q_1), \delta_1(q_1, a, q_2)) \vee E_1^T(T^{t_1}(q_2), \delta_1(q_2, a, q_2)) \\
&= (0.4 \wedge 0.9) \vee (0.4 \wedge 0.3) \vee (0.4 \wedge 0.6) = 0.4, \\
I^{t_2}(q_2) &= E_1^I(I^{t_1}(q_0), \delta_2(q_0, a, q_2)) \vee E_1^I(I^{t_1}(q_1), \delta_2(q_1, a, q_2)) \vee E_1^I(I^{t_1}(q_2), \delta_2(q_2, a, q_2)) \\
&= (0.7 \wedge 0.7) \vee (0.5 \wedge 0.7) \vee (0.7 \wedge 0.9) = 0.7, \\
F^{t_2}(q_2) &= E_1^F(F^{t_1}(q_0), \delta_3(q_0, a, q_2)) \wedge E_1^F(F^{t_1}(q_1), \delta_3(q_1, a, q_2)) \wedge E_1^F(F^{t_1}(q_2), \delta_3(q_2, a, q_2)) \\
&= (1 \vee 0.4) \wedge (0.5 \vee 0.6) \wedge (0.4 \vee 0.5) = 0.5.
\end{aligned}
$$

Clearly, we can see that there are three simultaneous transition to the action states q_0, q_1 and q_2 at time t_2.

Definition 3.3. Let $\mathcal{M} = (Q, *, \Sigma, \tilde{\delta}, \tilde{R}, E_1, E_2)$ be a GNGM. We define max-min GNGM of the form $\mathcal{M} = (Q, *, \Sigma, \tilde{\delta}^*, \tilde{R}, E_1, E_2)$ such that $\tilde{\delta}^* : Q_{act} \times \Sigma^* \times Q \to [0,1] \times [0,1] \times [0,1]$, where $Q_{act} = \{Q_{act}(t_0), Q_{act}(t_1), ...\}$ and for every $i \geq 0$,

$$\tilde{\delta}_1^*((q, T^t(q), I^t(q), F^t(q)), \Lambda, p) = \begin{cases} 1 & \text{if } p = q \\ 0 & \text{otherwise} \end{cases},$$

$$\tilde{\delta}_2^*((q, T^t(q), I^t(q), F^t(q)), \Lambda, p) = \begin{cases} 1 & \text{if } p = q \\ 0 & \text{otherwise} \end{cases},$$

$$\tilde{\delta}_3^*((q, T^t(q), I^t(q), F^t(q)), \Lambda, p) = \begin{cases} 0 & \text{if p=q} \\ 1 & \text{otherwise} \end{cases},$$

and for every $i \geq 1, \tilde{\delta}^*((q, T^t(q), I^t(q), F^t(q)), a, p) = \tilde{\delta}((q, T^t(q), I^t(q), F^t(q)), a, p)$ and recursively,

$$
\begin{aligned}
&\tilde{\delta}_1^*((q, T^t(q), I^t(q), F^t(q)), a_1a_2...a_n, p) = \vee\{\tilde{\delta}_1((q, T^t(q), I^t(q), F^t(q)), a_1, p_1) \wedge ... \\
&\quad \wedge \tilde{\delta}_1((p_{n-1}, T^t(p_{n-1}), I^t(p_{n-1}), F^t(p_{n-1})), a_n, p) \big| p_1 \in Q_{act}(t_1), ..., p_{n-1} \in Q_{act}(t_{n-1})\},
\end{aligned}
$$

$$
\begin{aligned}
&\tilde{\delta}_2^*((q, T^t(q), I^t(q), F^t(q)), a_1a_2...a_n, p) = \vee\{\tilde{\delta}_2((q, T^t(q), I^t(q), F^t(q)), a_1, p_1) \wedge ... \\
&\quad \wedge \tilde{\delta}_2((p_{n-1}, T^t(p_{n-1}), I^t(p_{n-1}), F^t(p_{n-1})), a_n, p) \big| p_1 \in Q_{act}(t_1), ..., p_{n-1} \in Q_{act}(t_{n-1})\},
\end{aligned}
$$

$$
\begin{aligned}
&\tilde{\delta}_3^*((q, T^t(q), I^t(q), F^t(q)), a_1a_2...a_n, p) = \wedge\{\tilde{\delta}_3((q, T^t(q), I^t(q), F^t(q)), a_1, p_1) \vee ... \\
&\quad \vee \tilde{\delta}_3((p_{n-1}, T^t(p_{n-1}), I^t(p_{n-1}), F^t(p_{n-1})), a_n, p) \big| p_1 \in Q_{act}(t_1), ..., p_{n-1} \in Q_{act}(t_{n-1})\},
\end{aligned}
$$

in which $a_i \in \Sigma$, for all $1 \leq i \leq n$ and assuming that the entered in put at time t_i is a_i, for $1 \leq i \leq n-1$.

In the rest of paper, instead of max-min GNGM we say that GNGM.

Definition 3.4. A SVNS N of Q is called a single-valued neutrosophic subgroup (SVNSG) of Q if the following properties hold:

(1) $T_N(p*q) \geq T_N(p) \wedge T_N(q)$,
(2) $I_N(p*q) \geq I_N(p) \wedge I_N(q)$,
(3) $F_N(p*q) \leq F_N(p) \vee F_N(q)$,
(4) $(T_N(p), I_N(p), F_N(p)) = (T_N(p^{-1}), I_N(p^{-1}), F_N(p^{-1}))$,

for every $p, q \in Q$.

Definition 3.5. A SVNSG N of Q is called a single-valued neutrosophic normal subgroup (SVNNSG) of Q if for every $p, q \in Q$, $T_N(p*q*p^{-1}) \geq T_N(q), I_N(p*q*p^{-1}) \geq I_N(q), F_N(p*q*p^{-1}) \leq F_N(q)$.

Definition 3.6. Let N and M be two SVNS of Q. Then product $N*M$ is defined as follows: $N*M(p) = (T_{N*M}(p), I_{N*M}(p), F_{N*M}(p))$, where

$$T_{N*M}(p) = \vee\{T_N(r) \wedge T_M(s) \big| r, s \in Q, p = r*s\},$$
$$I_{N*M}(p) = \vee\{I_N(r) \wedge I_M(s) \big| r, s \in Q, p = r*s\},$$
$$F_{N*M}(p) = \wedge\{F_N(r) \vee F_M(s) \big| r, s \in Q, p = r*s\},$$

for every $p \in Q$.

Definition 3.7. Let N and M be two SVNSG of Q such that $N \subseteq M$. Then N is called SVNNSG of M if for every $p, q \in Q$, we have

$$T_N(p*q*p^{-1}) \geq T_N(q) \wedge T_M(p),$$
$$I_N(p*q*p^{-1}) \geq I_N(q) \wedge I_M(p),$$
$$F_N(p*q*p^{-1}) \leq F_N(q) \vee F_M(p).$$

Let N and M be two SVNSG of Q such that N is a SVNNSG of M. Then $supp(N)$ is a SVNNSG of $supp(M)$, too.

Definition 3.8. Let N and M be two SVNS of the groups Q and P, respectively. Let $f: Q \to P$ be a group homomorphism. Then the SVNSs $f(N)$ of P and $f^{-1}(M)$ of Q are defined as follows:

$$f(N)(p) = \begin{cases} (T_{f(N)}(p), I_{f(N)}(p), F_{f(N)}(p)) & \text{if } f^{-1}(P) \neq \emptyset \\ (0,0,1) & \text{if } f^{-1}(P) = \emptyset \end{cases}, \tag{1}$$

where $T_{f(N)}(p) = \vee\{T_N(q)\big| q \in Q\text{and } f(q) = p\}$, $I_{f(N)}(p) = \vee\{I_N(q)\big| q \in Q\text{and } f(q) = p\}$ and $F_{f(N)}(p) = \wedge\{F_N(q)\big| q \in Q\text{and } f(q) = p\}$, for every $p \in P$. Also, $f^{-1}(M)(q) = (T_M(f(q)), I_M(f(q)), F_M(f(q)))$, for every $q \in Q$.

Definition 3.9. Let M and N be two SVNS of the groups Q and P, respectively. A function $f : Q \to P$ is called a weak homomorphism from M into N if the following conditions hold:

(1) f is an epimorphism,
(2) $f(M) \subseteq N$.

If f is an isomorphism from Q onto P, then we say that f is a weak isomorphism from M to N.

Definition 3.10. Let $\mathcal{M}_i = (Q_i, *, \Sigma_i, \tilde{R}_i, \tilde{\delta}^i, E_1, E_2), i = 1, 2$, be two GNGM. A pair of functions (f, g), where $f : Q_1 \to Q_2$ and $g : \Sigma_1 \to \Sigma_2$ is called a homomorphism from $\mathcal{M}_1$ into $\mathcal{M}_2$ and written $(f, g) : \mathcal{M}_1 \to \mathcal{M}_2$ if the following conditions hold:

(1) f is a group homomorphism,
(2) $\tilde{\delta}_1^1((q, T^t(q), I^t(q), F^t(q)), a, p) \leq \tilde{\delta}_1^2((f(q), T^t(f(q)), I^t(f(q)), F^t(f(q))), g(a), f(q))$,
(3) $\tilde{\delta}_2^1((q, T^t(q), I^t(q), F^t(q)), a, p) \leq \tilde{\delta}_2^2((f(q), T^t(f(q)), I^t(f(q)), F^t(f(q))), g(a), f(q))$,
(4) $\tilde{\delta}_3^1((q, T^t(q), I^t(q), F^t(q)), a, p) \geq \tilde{\delta}_3^2((f(q), T^t(f(q)), I^t(f(q)), F^t(f(q))), g(a), f(q))$,

for every $p, q \in Q$ and $a \in \Sigma_i$.

The pair (f, g) is called a strong homomorphism from $\mathcal{M}_1$ into $\mathcal{M}_2$ if it satisfies 1 of Definition 3.10, and the following conditions holds:

(1) $\tilde{\delta}_1^2((f(q), T^t(f(q)), I^t(f(q)), F^t(f(q))), g(a), f(p)) = \vee\{\tilde{\delta}_1^1((q, T^t(q), I^t(q), F^t(q)), a, r)\big| f(r) = f(p), r \in Q_1\}$,
(2) $\tilde{\delta}_2^2((f(q), T^t(f(q)), I^t(f(q)), F^t(f(q))), g(a), f(p)) = \vee\{\tilde{\delta}_2^1((q, T^t(q), I^t(q), F^t(q)), a, r)\big| f(r) = f(p), r \in Q_1\}$,
(3) $\tilde{\delta}_3^2((f(q), T^t(f(q)), I^t(f(q)), F^t(f(q))), g(a), f(p)) = \wedge\{\tilde{\delta}_3^1((q, T^t(q), I^t(q), F^t(q)), a, r)\big| f(r) = f(p), r \in Q_1\}$,

for every $p, q \in Q$, $a \in \Sigma_i$ and $z \in Z$.

In Definition 3.10, if $\Sigma_1 = \Sigma_2$ and g is the identity map of Σ_1, then we say that f is a homomorphism (strong homomorphism) from $\mathcal{M}_1$ into $\mathcal{M}_2$ and we write $f : \mathcal{M}_1 \to \mathcal{M}_2$.

Definition 3.11. Let $\mathcal{M} = (Q, *, \Sigma, \tilde{R}, \tilde{\delta}, E_1, E_2)$ be a GNGM. A SVNS M of Q is called a single-valued neutrosophic kernel (SVNK) of δ if the following conditions hold:

(1) M is a SVNS of Q,
(2) $T_M(p) \geq \delta_1(q, a, p) \wedge T_M(q)$,
(3) $I_M(p) \geq \delta_2(q, a, p) \wedge I_M(q)$,
(4) $F_M(p) \leq \delta_3(q, a, p) \vee F_M(q)$,

for every $p, q \in Q$ and $a \in \Sigma$.

Theorem 3.12. *Let $\mathcal{M}$ be a GNGM. A SVNNSG M of Q is a SVNK of $\mathcal{M}$ if and only if the following conditions hold:*

(1) $T_M(p * r^{-1}) \geq$
$\tilde{\delta}_1^*((q*k, T^t(q*k), I^t(q*k), F^t(q*k)), x, p) \wedge \tilde{\delta}_1^*((q, T^{t'}(q), I^{t'}(q), F^{t'}(q)), x, r) \wedge T_M(k)$,

(2) $I_M(p * r^{-1}) \geq$
$\tilde{\delta}_2^*((q*k, T^t(q*k), I^t(q*k), F^t(q*k)), x, p) \wedge \tilde{\delta}_2^*((q, T^{t'}(q), I^{t'}(q), F^{t'}(q)), x, r) \wedge I_M(k)$,

(3) $F_M(p * r^{-1}) \leq$
$\tilde{\delta}_3^*((q*k, T^t(q*k), I^t(q*k), F^t(q*k)), x, p) \vee \tilde{\delta}_3^*((q, T^{t'}(q), I^{t'}(q), F^{t'}(q)), x, r) \vee F_M(k)$,

for every $p, q, k, r \in Q$ and $x \in \Sigma^*$.

Proof. Let M be a SVNK of $\mathcal{M}$. We prove the claim by induction on $|x| = n$. Let $n = 0$. Then $x = \Lambda$. If $p = q * k$ and $r = q$. So,

$$\begin{aligned}
&\tilde{\delta}_1^*((q*k, T^t(q*k), I^t(q*k), F^t(q*k)), \Lambda, p) \wedge \tilde{\delta}_1^*((q, T^{t'}(q), I^{t'}(q), F^{t'}(q)), \Lambda, r) \wedge T_M(k) \\
&\qquad \leq T_M(k) \leq T_M(q*k*q^{-1}), \\
&\tilde{\delta}_2^*((q*k, T^t(q*k), I^t(q*k), F^t(q*k)), \Lambda, p) \wedge \tilde{\delta}_2^*((q, T^{t'}(q), I^{t'}(q), F^{t'}(q)), \Lambda, r) \wedge I_M(k) \\
&\qquad \leq I_M(k) \leq I_M(q*k*q^{-1}), \\
&\tilde{\delta}_3^*((q*k, T^t(q*k), I^t(q*k), F^t(q*k)), \Lambda, p) \vee \tilde{\delta}_3^*((q, T^{t'}(q), I^{t'}(q), F^{t'}(q)), \Lambda, r) \vee F_M(k) \\
&\qquad \geq F_M(k) \geq F_M(q*k*q^{-1}).
\end{aligned}$$

If $p \neq q * k$ or $r \neq q$. Then

$$\begin{aligned}
&\tilde{\delta}_1^*((q*k, T^t(q*k), I^t(q*k), F^t(q*k)), \Lambda, p) \wedge \tilde{\delta}_1^*((q, T^{t'}(q), I^{t'}(q), F^{t'}(q)), \Lambda, r) \wedge T_M(k) \\
&\qquad = 0 \leq T_M(p*r^{-1}), \\
&\tilde{\delta}_2^*((q*k, T^t(q*k), I^t(q*k), F^t(q*k)), \Lambda, p) \wedge \tilde{\delta}_2^*((q, T^{t'}(q), I^{t'}(q), F^{t'}(q)), \Lambda, r) \wedge I_M(k) \\
&\qquad = 0 \leq I_M(p*r^{-1}), \\
&\tilde{\delta}_3^*((q*k, T^t(q*k), I^t(q*k), F^t(q*k)), \Lambda, p) \vee \tilde{\delta}_3^*((q, T^{t'}(q), I^{t'}(q), F^{t'}(q)), \Lambda, r) \vee F_M(k) \\
&\qquad = 1 \geq F_M(p*r^{-1}).
\end{aligned}$$

So, the result holds for $n = 0$. Now, let the claim holds for every $y \in \Sigma^*$ such that $|y| = n - 1$ and $n > 0$. Let $x \in \Sigma^*, x = ya, y \in \Sigma^*, a \in \Sigma, |y| = n - 1$ and $n > 0$. Then

$$\begin{aligned}
\tilde{\delta}_1^*((q*k,&T^t(q*k), I^t(q*k), F^t(q*k)), x, p) \wedge \tilde{\delta}_1^*((q, T^{t'}(q), I^{t'}(q), F^{t'}(q)), x, r) \wedge T_M(k) \\
&= (\vee\{\tilde{\delta}_1^*((q*k, T^t(q*k), I^t(q*k), F^t(q*k)), y, u) \\
&\wedge \tilde{\delta}_1^*((u, T^{t+n-1}(u), I^{t+n-1}(u), F^{t+n-1}(u)), a, p)\big|u \in Q\}) \\
&\wedge (\vee\{\tilde{\delta}_1^*((q, T^{t'}(q), I^{t'}(q), F^{t'}(q)), y, v) \\
&\wedge \tilde{\delta}_1^*((v, T^{t'+n-1}(v), I^{t'+n-1}(v), F^{t'+n-1}(v)), a, r)\big|v \in Q\}) \wedge T_M(k) \\
&\leq \vee\{\vee\{\tilde{\delta}_1^*((q*k, T^t(q*k), I^t(q*k), F^t(q*k)), y, u) \\
&\wedge \tilde{\delta}_1^*((u, T^{t+n-1}(u), I^{t+n-1}(u), F^{t+n-1}(u)), a, p) \\
&\wedge \tilde{\delta}_1^*((q, T^{t'}(q), I^{t'}(q), F^{t'}(q)), y, v) \\
&\wedge \tilde{\delta}_1^*((v, T^{t'+n-1}(v), I^{t'+n-1}(v), F^{t'+n-1}(v)), a, r) \wedge T_M(k)\big|u \in Q\}\big|v \in Q\} \\
&\leq \vee\{\vee\{T_M(u*v^{-1}) \wedge \tilde{\delta}_1^*((u, T^{t+n-1}(u), I^{t+n-1}(u), F^{t+n-1}(u)), a, p) \\
&\wedge \tilde{\delta}_1^*((v, T^{t'+n-1}(v), I^{t'+n-1}(v), F^{t'+n-1}(v)), a, r)\big|u \in Q\}\big|v \in Q\} \\
&\leq \vee\{\vee\{T_M(v^{-1}*u) \\
&\wedge \tilde{\delta}_1^*((v*v^{-1}*u, T^{t+n-1}(v*v^{-1}*u), I^{t+n-1}(v*v^{-1}*u), F^{t+n-1}(v*v^{-1}*u)), a, p) \\
&\wedge \tilde{\delta}_1^*((v, T^{t'+n-1}(v), I^{t'+n-1}(v), F^{t'+n-1}(v)), a, r)\big|u \in Q\}\big|v \in Q\} \\
&\leq T_M(p*r^{-1}),
\end{aligned}$$

also,

$$\begin{aligned}
\tilde{\delta}_2^*((q*k,&T^t(q*k), I^t(q*k), F^t(q*k)), x, p) \wedge \tilde{\delta}_2^*((q, T^{t'}(q), I^{t'}(q), F^{t'}(q)), x, r) \wedge I_M(k) \\
&= (\vee\{\tilde{\delta}_2^*((q*k, T^t(q*k), I^t(q*k), F^t(q*k)), y, u) \\
&\wedge \tilde{\delta}_2^*((u, T^{t+n-1}(u), I^{t+n-1}(u), F^{t+n-1}(u)), a, p)\big|u \in Q\}) \\
&\wedge (\vee\{\tilde{\delta}_2^*((q, T^{t'}(q), I^{t'}(q), F^{t'}(q)), y, v) \\
&\wedge \tilde{\delta}_2^*((v, T^{t'+n-1}(v), I^{t'+n-1}(v), F^{t'+n-1}(v)), a, r)\big|v \in Q\}) \wedge I_M(k) \\
&\leq \vee\{\vee\{\tilde{\delta}_2^*((q*k, T^t(q*k), I^t(q*k), F^t(q*k)), y, u) \\
&\wedge \tilde{\delta}_2^*((u, T^{t+n-1}(u), I^{t+n-1}(u), F^{t+n-1}(u)), a, p)
\end{aligned}$$

$$\wedge\, \tilde{\delta}_2^*((q, T^{t'}(q), I^{t'}(q), F^{t'}(q)), y, v)$$
$$\wedge\, \tilde{\delta}_2^*((v, T^{t'+n-1}(v), I^{t'+n-1}(v), F^{t'+n-1}(v)), a, r)$$
$$\wedge\, I_M(k) \big| u \in Q\} \big| v \in Q\}$$
$$\leq \vee\{\vee\{I_M(u * v^{-1})$$
$$\wedge\, \tilde{\delta}_2^*((u, T^{t+n-1}(u), I^{t+n-1}(u), F^{t+n-1}(u)), a, p)$$
$$\wedge\, \tilde{\delta}_2^*((v, T^{t'+n-1}(v), I^{t'+n-1}(v), F^{t'+n-1}(v)), a, r) \big| u \in Q\} \big| v \in Q\}$$
$$\leq \vee\{\vee\{I_M(v^{-1} * u)$$
$$\wedge\, \tilde{\delta}_2^*((v * v^{-1} * u, T^{t+n-1}(v * v^{-1} * u), I^{t+n-1}(v * v^{-1} * u), F^{t+n-1}(v * v^{-1} * u)), a, p)$$
$$\wedge\, \tilde{\delta}_2^*((v, T^{t'+n-1}(v), I^{t'+n-1}(v), F^{t'+n-1}(v)), a, r) \big| u \in Q\} \big| v \in Q\}$$
$$\leq I_M(p * r^{-1}),$$

and

$$\tilde{\delta}_3^*((q * k, T^t(q * k), I^t(q * k), F^t(q * k)), x, p) \vee \tilde{\delta}_3^*((q, T^{t'}(q), I^{t'}(q), F^{t'}(q)), x, r) \vee F_M(k)$$
$$= (\wedge\{\tilde{\delta}_3^*((q * k, T^t(q * k), I^t(q * k), F^t(q * k)), y, u)$$
$$\vee\, \tilde{\delta}_3^*((u, T^{t+n-1}(u), I^{t+n-1}(u), F^{t+n-1}(u)), a, p) \big| u \in Q\})$$
$$\vee\, (\wedge\{\tilde{\delta}_3^*((q, T^{t'}(q), I^{t'}(q), F^{t'}(q)), y, v)$$
$$\vee\, \tilde{\delta}_3^*((v, T^{t'+n-1}(v), I^{t'+n-1}(v), F^{t'+n-1}(v)), a, r) \big| v \in Q\}) \vee F_M(k)$$
$$\geq \wedge\{\wedge\{\tilde{\delta}_3^*((q * k, T^t(q * k), I^t(q * k), F^t(q * k)), y, u)$$
$$\vee\, \tilde{\delta}_3^*((u, T^{t+n-1}(u), I^{t+n-1}(u), F^{t+n-1}(u)), a, p)$$
$$\vee\, \tilde{\delta}_3^*((q, T^{t'}(q), I^{t'}(q), F^{t'}(q)), y, v)$$
$$\vee\, \tilde{\delta}_3^*((v, T^{t'+n-1}(v), I^{t'+n-1}(v), F^{t'+n-1}(v)), a, r)$$
$$\vee\, F_M(k) \big| u \in Q\} \big| v \in Q\}$$
$$\geq \wedge\{\wedge\{F_M(u * v^{-1})$$
$$\vee\, \tilde{\delta}_3^*((u, T^{t+n-1}(u), I^{t+n-1}(u), F^{t+n-1}(u)), a, p)$$
$$\vee\, \tilde{\delta}_3^*((v, T^{t'+n-1}(v), I^{t'+n-1}(v), F^{t'+n-1}(v)), a, r) \big| u \in Q\} \big| v \in Q\}$$
$$\geq \wedge\{\wedge\{F_M(v^{-1} * u)$$
$$\vee\, \tilde{\delta}_3^*((v * v^{-1} * u, T^{t+n-1}(v * v^{-1} * u), I^{t+n-1}(v * v^{-1} * u), F^{t+n-1}(v * v^{-1} * u)), a, p)$$
$$\vee\, \tilde{\delta}_3^*((v, T^{t'+n-1}(v), I^{t'+n-1}(v), F^{t'+n-1}(v)), a, r) \big| u \in Q\} \big| v \in Q\}$$
$$\geq F_M(p * r^{-1}),$$

□

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Definition 3.13. Let $\mathcal{M}$ be a GNGM and M be a SVNSM of Q. M is called single-valued neutrosophic fundamental (SVNF) of $\mathcal{M}$ if the following hold:

(1) M is a SVNSG of Q,

(2)

$$T_M(p) \geq \tilde{\delta}_1((q, T^t(q), I^t(q), F^t(q)), x, p) \wedge T_M(q)$$
$$I_M(p) \geq \tilde{\delta}_2((q, T^t(q), I^t(q), F^t(q)), x, p) \wedge I_M(q)$$
$$F_M(p) \leq \tilde{\delta}_3((q, T^t(q), I^t(q), F^t(q)), x, p) \vee F_M(q),$$

for every $p, q \in Q$ and $x \in \Sigma$.

Theorem 3.14. *Let $\mathcal{M}$ be a GNGM and M be a SVNSM of Q. Then M is a SVNF of $\mathcal{M}$ if and only if the following hold:*

$$T_M(p) \geq \tilde{\delta}_1^*((q, T^t(q), I^t(q), F^t(q)), x, p) \wedge T_M(q)$$
$$I_M(p) \geq \tilde{\delta}_2^*((q, T^t(q), I^t(q), F^t(q)), x, p) \wedge I_M(q)$$
$$F_M(p) \leq \tilde{\delta}_3^*((q, T^t(q), I^t(q), F^t(q)), x, p) \vee F_M(q),$$

for every $p, q \in Q$ and $x \in \Sigma^*$.

Proof. The proof is similar to proof of Theorem 3.12. □

Theorem 3.15. *Let $\mathcal{M} = (Q, *, \Sigma, \tilde{R}, \tilde{\delta}, E_1, E_2)$ and $\mathcal{M}' = (Q', *, \Sigma, \tilde{R}', \tilde{\delta}', E_1, E_2)$ be two GNGM. Let f be a homomorphism from Q into Q'. If M is a SVNF of $\mathcal{M}'$, then $f^{-1}(M)$ is a SVNF of $\mathcal{M}$.*

Proof. The proof is straightforward. □

Theorem 3.16. *Let $\mathcal{M} = (Q, *, \Sigma, \tilde{R}, \tilde{\delta}, E_1, E_2)$ and $\mathcal{M}' = (Q', *, \Sigma, \tilde{R}', \tilde{\delta}', E_1, E_2)$ be two GNGM. Let f be a homomorphism from Q into Q'. If M is a SVNK of $\mathcal{M}'$, then $f^{-1}(M)$ is a SVNK of $\mathcal{M}$.*

Theorem 3.17. *Let $\mathcal{M} = (Q, *, \Sigma, \tilde{R}, \tilde{\delta}, E_1, E_2)$ and $\mathcal{M}' = (Q', *, \Sigma, \tilde{R}', \tilde{\delta}', E_1, E_2)$ be two GNGM. Let f be a strong homomorphism from $\mathcal{M}$ into $\mathcal{M}'$. If M is a SVNK of $\mathcal{M}$, then $f(M)$ is a SVNK of $\mathcal{M}'$.*

Proof. Let M be a SVNK of $\mathcal{M}$. Then M is a SVNNSG of Q. So, $f(M)$ is a SVNNSG of Q', since f is an epimorphism from Q onto Q'. Let $p', q', r', k' \in Q'$ and $x \in \Sigma$. Then

$$\begin{aligned}
&\tilde{\delta}_1'^*((q_1 * k_1, T^t(q_1 * k_1), I^t(q_1 * k_1), F^t(q_1 * k_1)), x, p_1) \\
&\quad \wedge \tilde{\delta}_1'^*((q_1, T^{t'}(q_1), I^{t'}(q_1), F^{t'}(q_1)), x, r_1) \wedge T_{f(M)}(k_1) \\
&\quad = \tilde{\delta}_1'^*((q_1 * k_1, T^t(q_1 * k_1), I^t(q_1 * k_1), F^t(q_1 * k_1)), x, p) \wedge \tilde{\delta}_1'^*((q_1, T^{t'}(q_1), I^{t'}(q_1), F^{t'}(q_1)), x, r_1) \\
&\quad \wedge (\vee\{T_M(k) \big| k \in Q, f(k) = k_1\}) \\
&\quad = \vee\{\tilde{\delta}_1'^*((q_1 * k_1, T^t(q_1 * k_1), I^t(q_1 * k_1), F^t(q_1 * k_1)), x, p) \\
&\quad \wedge \tilde{\delta}_1'^*((q_1, T^{t'}(q_1), I^{t'}(q_1), F^{t'}(q_1)), x, r_1) \wedge T_M(k) \big| k \in Q, f(k) = k_1\},
\end{aligned}$$

and

$$\begin{aligned}
&\tilde{\delta}_2'^*((q_1 * k_1, T^t(q_1 * k_1), I^t(q_1 * k_1), F^t(q_1 * k_1)), x, p_1) \\
&\quad \wedge \tilde{\delta}_2'^*((q_1, T^{t'}(q_1), I^{t'}(q_1), F^{t'}(q_1)), x, r_1) \wedge I_{f(M)}(k_1) \\
&\quad = \tilde{\delta}_2'^*((q_1 * k_1, T^t(q_1 * k_1), I^t(q_1 * k_1), F^t(q_1 * k_1)), x, p) \wedge \tilde{\delta}_2'^*((q_1, T^{t'}(q_1), I^{t'}(q_1), F^{t'}(q_1)), x, r_1) \\
&\quad \wedge (\vee\{I_M(k) \big| k \in Q, f(k) = k_1\}) \\
&\quad = \vee\{\tilde{\delta}_2'^*((q_1 * k_1, T^t(q_1 * k_1), I^t(q_1 * k_1), F^t(q_1 * k_1)), x, p) \\
&\quad \wedge \tilde{\delta}_2'^*((q_1, T^{t'}(q_1), I^{t'}(q_1), F^{t'}(q_1)), x, r_1) \wedge I_M(k) \big| k \in Q, f(k) = k_1\},
\end{aligned}$$

also,

$$\begin{aligned}
&\tilde{\delta}_3'^*((q_1 * k_1, T^t(q_1 * k_1), I^t(q_1 * k_1), F^t(q_1 * k_1)), x, p_1) \\
&\quad \vee \tilde{\delta}_3'^*((q_1, T^{t'}(q_1), I^{t'}(q_1), F^{t'}(q_1)), x, r_1) \vee F_{f(M)}(k_1) \\
&\quad = \tilde{\delta}_3'^*((q_1 * k_1, T^t(q_1 * k_1), I^t(q_1 * k_1), F^t(q_1 * k_1)), x, p) \vee \tilde{\delta}_3'^*((q_1, T^{t'}(q_1), I^{t'}(q_1), F^{t'}(q_1)), x, r_1) \\
&\quad \vee (\wedge\{F_M(k) \big| k \in Q, f(k) = k_1\}) \wedge \{\tilde{\delta}_3'^*((q_1 * k_1, T^t(q_1 * k_1), I^t(q_1 * k_1), F^t(q_1 * k_1)), x, p) \\
&\quad \vee \tilde{\delta}_3'^*((q_1, T^{t'}(q_1), I^{t'}(q_1), F^{t'}(q_1)), x, r_1) \vee F_M(k) \big| k \in Q, f(k) = k_1\}.
\end{aligned}$$

Now, suppose that $p, q, r, k \in Q$ be such that $f(p) = p_1,\ f(q) = q_1,\ f(r) = r_1$ and $f(k) = k_1$. Then

$$
\begin{aligned}
&\tilde{\delta}_1'^{*}((q_1 * k_1, T^t(q_1 * k_1), I^t(q_1 * k_1), F^t(q_1 * k_1)), x, p_1) \\
&\wedge \tilde{\delta}_1'^{*}((q_1, T^{t'}(q_1), I^{t'}(q_1), F^{t'}(q_1)), x, r_1) \wedge T_{f(M)}(k_1) \\
&= \tilde{\delta}_1'^{*}((f(q * k), T^t(f(q * k)), I^t(f(q * k)), F^t(f(q * k))), x, f(p)) \\
&\wedge \tilde{\delta}_1'^{*}((f(q), T^{t'}(f(q)), I^{t'}(f(q)), F^{t'}(f(q))), x, f(r)) \wedge T_M(k) \\
&= \vee\{\tilde{\delta}_1^{*}((q * k, T^t(q * k), I^t(q * k), F^t(q * k)), x, h) \big| f(h) = f(p), h \in Q\} \\
&\wedge \vee\{\tilde{\delta}_1^{*}((q, T^{t'}(q), I^{t'}(q), F^{t'}(q)), x, l) \big| f(l) = f(r), l \in Q\} \wedge T_M(k) \\
&= \vee\{\vee\{\tilde{\delta}_1^{*}((q * k, T^t(q * k), I^t(q * k), F^t(q * k)), x, h) \wedge \tilde{\delta}_1^{*}((q, T^{t'}(q), I^{t'}(q), F^{t'}(q)), x, l) \\
&\wedge T_M(k) \big| f(h) = f(p), f(l) = f(r), l, h \in Q\}\} \\
&\leq \vee\{\vee\{T_M(h * l^{-1}) \big| h \in Q, f(h) = f(p) \big| l \in Q, f(l) = f(r)\}\} \\
&\leq T(f(M))(p_1 * r_1^{-1}).
\end{aligned}
$$

Similarly,

$$
\begin{aligned}
I(f(M))(p_1 * r_1^{-1}) \geq\ & \tilde{\delta}_2'^{*}((q_1 * k_1, T^t(q_1 * k_1), I^t(q_1 * k_1), F^t(q_1 * k_1)), x, p_1) \\
& \wedge \tilde{\delta}_2'^{*}((q_1, T^{t'}(q_1), I^{t'}(q_1), F^{t'}(q_1)), x, r_1) \wedge I_{f(M)}(k),
\end{aligned}
$$

and

$$
\begin{aligned}
F(f(M))(p_1 * r_1^{-1}) \leq\ & \tilde{\delta}_3'^{*}((q_1 * k_1, T^t(q_1 * k_1), I^t(q_1 * k_1), F^t(q_1 * k_1)), x, p_1) \\
& \vee \tilde{\delta}_3'^{*}((q_1, T^{t'}(q_1), I^{t'}(q_1), F^{t'}(q_1)), x, r_1) \vee F_{f(M)}(k).
\end{aligned}
$$

Hence, $f(M)$ is a SVNK of $\mathcal{M}'$. □

Theorem 3.18. *Let $\mathcal{M}_i = (Q_i, *, \Sigma, \tilde{R}^i, \tilde{\delta}^i, E_1, E_2), i = 1, 2$ be two GNGM and f be a strong homomorphism from $\mathcal{M}_1$ onto $\mathcal{M}_2$. If M is a SVNF of $\mathcal{M}_1$, then $f(M)$ is a SVNF of $\mathcal{M}_2$.*

Proof. The proof is similar to the proof of Theorem 3.17. □

In the rest of paper for SVNGA $\mathcal{M} = (Q, *, \Sigma, \tilde{R}, \tilde{\delta}, E_1, E_2)$ the following conditions are satisfied:

$$\begin{aligned}
\tilde{\delta}_1^*((p*q, T^t(p*q), I^t(p*q), F^t(p*q)), x, r) &\leq \tilde{\delta}_1^*((p, T^{t'}(p), I^{t'}(p), F^{t'}(p)), x, k) \\
&\wedge \tilde{\delta}_1^*((q, T^{t''}(q), I^{t''}(q), F^{t''}(q)), x, k), \\
\tilde{\delta}_2^*((p*q, T^t(p*q), I^t(p*q), F^t(p*q)), x, r) &\leq \tilde{\delta}_2^*((p, T^{t'}(p), I^{t'}(p), F^{t'}(p)), x, k) \\
&\wedge \tilde{\delta}_2^*((q, T^{t''}(q), I^{t''}(q), F^{t''}(q)), x, k), \\
\tilde{\delta}_3^*((p*q, T^t(p*q), I^t(p*q), F^t(p*q)), x, r) &\geq \tilde{\delta}_3^*((p, T^{t'}(p), I^{t'}(p), F^{t'}(p)), x, k) \\
&\vee \tilde{\delta}_3^*((q, T^{t''}(q), I^{t''}(q), F^{t''}(q)), x, k),
\end{aligned}$$

for every $p, q, r, k \in Q$ and $x \in \Sigma^*$. Also, e will denote the identity element of the group $(Q, *)$.

Theorem 3.19. *Let $\mathcal{M} = (Q, *, \Sigma, \tilde{R}, \tilde{\delta}, E_1, E_2)$ be a GNGM and M be a SVNK of $\mathcal{M}$. Then M is a SVNF of $\mathcal{M}$ if and only if*

1. $T_M(p) \geq \tilde{\delta}_1^*((e, T^t(e), I^t(e), F^t(e)), x, p) \wedge T_M(e)$,
2. $I_M(p) \geq \tilde{\delta}_2^*((e, T^t(e), I^t(e), F^t(e)), x, p) \wedge I_M(e)$,
3. $F_M(p) \leq \tilde{\delta}_3^*((e, T^t(e), I^t(e), F^t(e)), x, p) \vee F_M(e)$,

for every $p \in Q$ and $x \in \Sigma^*$

Proof. Let conditions 1, 2 and 3 are satisfied. Then

$$\begin{aligned}
T_M(p) &= T_M(p * r^{-1} * r) \\
&\geq T_M(p * r^{-1}) \wedge T_M(r) \\
&\geq \tilde{\delta}_1^*((q, T^t(q), I^t(q), F^t(q)), x, p) \wedge \tilde{\delta}_1^*((e, T^{t'}(e), I^{t'}(e), F^{t'}(e)), x, r) \wedge T_M(q) \wedge T_M(r) \\
&\geq \tilde{\delta}_1^*((q, T^t(q), I^t(q), F^t(q)), x, p) \wedge \tilde{\delta}_1^*((e, T^{t'}(e), I^{t'}(e), F^{t'}(e)), x, r) \wedge T_M(e) \wedge T_M(q) \\
&= \tilde{\delta}_1^*((q, T^t(q), I^t(q), F^t(q)), x, p) \wedge T_M(q).
\end{aligned}$$

Since, $T_M(q * q^{-1}) \geq T_M(q) \wedge T_M(q^{-1}) = T_M(q)$, then $T_M(e) \geq T_M(q)$, also

$$\begin{aligned}
\tilde{\delta}_1^*((e, T^{t'}(e), I^{t'}(e), F^{t'}(e)), x, r) &\wedge \tilde{\delta}_1^*((q, T^t(q), I^t(q), F^t(q)), x, r) \\
&\geq \tilde{\delta}_1^*((e*q, T^{t''}(e*q), I^{t''}(e*q), F^{t''}(e*q)), x, r),
\end{aligned}$$

so, $\tilde{\delta}_1^*((e, T^{t'}(e), I^{t'}(e), F^{t'}(e)), x, r) \geq \tilde{\delta}_1^*((e * q, T^{t''}(e * q), I^{t''}(e * q), F^{t''}(e * q)), x, r)$. On the other hand,

$$\begin{aligned} I_M(p) &= I_M(p * r^{-1} * r) \\ &\geq I_M(p * r^{-1}) \wedge I_M(r) \\ &\geq \tilde{\delta}_2^*((q, T^t(q), I^t(q), F^t(q)), x, p) \wedge \tilde{\delta}_2^*((e, T^{t'}(e), I^{t'}(e), F^{t'}(e)), x, r) \wedge I_M(q) \wedge I_M(r) \\ &\geq \tilde{\delta}_2^*((q, T^t(q), I^t(q), F^t(q)), x, p) \wedge \tilde{\delta}_2^*((e, T^{t'}(e), I^{t'}(e), F^{t'}(e)), x, r) \wedge I_M(e) \wedge I_M(q) \\ &\geq \tilde{\delta}_2^*((q, T^t(q), I^t(q), F^t(q)), x, p) \wedge I_M(q). \end{aligned}$$

Also,

$$\begin{aligned} F_M(p) &= F_M(p * r^{-1} * r) \\ &\leq F_M(p * r^{-1}) \vee F_M(r) \\ &\leq \tilde{\delta}_3^*((q, T^t(q), I^t(q), F^t(q)), x, p) \vee \tilde{\delta}_3^*((e, T^{t'}(e), I^{t'}(e), F^{t'}(e)), x, r) \vee F_M(q) \vee F_M(e) \\ &\leq \tilde{\delta}_3^*((q, T^t(q), I^t(q), F^t(q)), x, p) \vee F_M(q). \end{aligned}$$

Hence, M is a SVNF of $\mathcal{M}$. The converse is clear. □

Corollary 3.20. *If M is a SVNK and N is a SVNK of $\mathcal{M}$ such that $M \subseteq N$ and $(T_M(e), I_M(e), F_M(e)) = (T_N(e), I_N(e), F_N(e))$, then M is a SVNF of $\mathcal{M}$.*

Theorem 3.21. *Let M be a SVNK and N be a SVNF of $\mathcal{M}$. Then $M * N$ is a SVNF of $\mathcal{M}$.*

Proof. Since M is a SVNNSG and N is a SVNSG of Q, then $M * N$ is a SVNSG of Q and $M * N = N * M$. Now, we have

$$\begin{aligned} T_{M*N}(p) \geq T_M(p * r^{-1}) \wedge T_N(r) &\geq \tilde{\delta}_1^*((a * b, T^t(a * b), I^t(a * b), F^t(a * b)), x, p) \\ &\wedge \tilde{\delta}_1^*((a, T^{t'}(a), I^{t'}(a), F^{t'}(a)), x, r) \wedge T_M(b) \\ &\wedge \tilde{\delta}_1^*((a, T^{t'}(a), I^{t'}(a), F^{t'}(a)), x, r) \wedge T_N(a) \\ &= \tilde{\delta}_1^*((a * b, T^{t'}(a * b), I^{t'}(a * b), F^{t'}(a * b)), x, p) \\ &\wedge T_M(b) \wedge T_N(a), \end{aligned}$$

In addition, $I_{M*N}(p) \geq \tilde{\delta}_2^*((a * b, T^t(a * b), I^t(a * b), F^t(a * b)), x, p) \wedge I_M(b) \wedge I_N(a)$, since $\tilde{\delta}_2^*((a * b, T^t(a * b), I^t(a * b), F^t(a * b)), x, p) \leq \tilde{\delta}_2^*((a, T^{t'}(a), I^{t'}(a), F^{t'}(a)), x, r)$. Also,

$$\begin{aligned} F_{M*N}(p) &\leq F_M(p * r^{-1}) \vee F_N(r) \\ &\leq \tilde{\delta}_3^*((a * b, T^t(a * b), I^t(a * b), F^t(a * b)), x, p) \vee \tilde{\delta}_3^*((a, T^{t'}(a), I^{t'}(a), F^{t'}(a)), x, r) \\ &\vee F_M(b) \ \vee \tilde{\delta}_3^*((a, T^{t'}(a), I^{t'}(a), F^{t'}(a)), x, r) \vee F_N(a) \\ &\tilde{\delta}_3^*((a * b, T^t(a * b), I^t(a * b), F^t(a * b)), x, p) \vee F_M(b) \vee F_N(a), \end{aligned}$$

for every $a, b, p \in Q$ and $x \in \Sigma^*$. So, for every $p, q \in Q$ and $x \in \Sigma^*$

$$\begin{aligned}T_{M*N}(p) &\geq \vee\{\tilde{\delta}_1^*((a*b, T^t(a*b), I^t(a*b), F^t(a*b)), x, p) \wedge T_M(b) \wedge T_N(a) \big| a, b \in Q, a*b = q\} \\ &= \tilde{\delta}_1^*((q, T^{t'}(q), I^{t'}(q), F^{t'}(q)), x, p) \wedge (\vee\{T_M(b) \wedge T_N(a) \big| a, b \in Q, a*b = q\}) \\ &= \tilde{\delta}_1^*((q, T^{t'}(q), I^{t'}(q), F^{t'}(q)), x, p) \wedge T_{M*N}(q),\end{aligned}$$

clearly, $I_{M*N}(p) \geq \tilde{\delta}_2^*((q, T^{t'}(q), I^{t'}(q), F^{t'}(q)), x, p) \wedge I_{M*N}(q)$ and

$$F_{M*N}(p) \leq \tilde{\delta}_3^*((q, T^{t'}(q), I^{t'}(q), F^{t'}(q)), x, p) \vee F_{M*N}(q).$$

Therefore, $M * N$ is a SVNF of $\tilde{\mathcal{A}}$. □

Theorem 3.22. *If M and N are two SVNKs of $\tilde{\mathcal{M}}$, then $M * N$ is a SVNK of $\tilde{\mathcal{M}}$.*

Proof. Let M and N be two SVNNSGs of Q. Then $M * N$ is a SVNNSG of Q and $M * N = N * M$. Now, we have

$$\begin{aligned}T_{M*N}(p*r^{-1}) &\geq T_M(p*q^{-1}) \wedge T_N(q*r^{-1}) \\ &\geq \tilde{\delta}_1^*((a*b*c, T^t(a*b*c), I^t(a*b*c), F^t(a*b*c)), x, p) \\ &\quad \wedge \tilde{\delta}_1^*((a*b, T^{t'}(a*b), I^{t'}(a*b), F^{t'}(a*b)), x, p) \wedge T_M(c) \\ &\quad \wedge \tilde{\delta}_1^*((a*b, T^{t'}(a*b), I^{t'}(a*b), F^{t'}(a*b)), x, r) \\ &\quad \wedge \tilde{\delta}_1^*((a, T^{t''}(a), I^{t''}(a), F^{t''}(a)), x, r) \wedge I_M(c) \wedge I_N(c) \\ &= \tilde{\delta}_1^*((a*b*c, T^t(a*b*c), I^t(a*b*c), F^t(a*b*c)), x, p) \\ &\quad \wedge \tilde{\delta}_1^*((a, T^{t''}(a), I^{t''}(a), F^{t''}(a)), x, r) \wedge T_M(c) \wedge T_N(b),\end{aligned}$$

and

$$\begin{aligned}I_{M*N}(p*r^{-1}) &\geq I_M(p*q^{-1}) \wedge I_N(q*r^{-1}) \\ &\geq \tilde{\delta}_2^*((a*b*c, T^t(a*b*c), I^t(a*b*c), F^t(a*b*c)), x, p) \\ &\quad \wedge \tilde{\delta}_2^*((a*b, T^{t'}(a*b), I^{t'}(a*b), F^{t'}(a*b)), x, p) \wedge I_M(c) \\ &\quad \wedge \tilde{\delta}_2^*((a*b, T^{t'}(a*b), I^{t'}(a*b), F^{t'}(a*b)), x, r) \\ &\quad \wedge \tilde{\delta}_2^*((a, T^{t''}(a), I^{t''}(a), F^{t''}(a)), x, r) \wedge T_M(c) \wedge T_N(c) \\ &= \tilde{\delta}_2^*((a*b*c, T^t(a*b*c), I^t(a*b*c), F^t(a*b*c)), x, p) \\ &\quad \wedge \tilde{\delta}_2^*((a, T^{t''}(a), I^{t''}(a), F^{t''}(a)), x, r) \wedge I_M(c) \wedge I_N(b),\end{aligned}$$

also,

$$\begin{aligned}F_{M*N}(p*r^{-1}) &\leq \tilde{\delta}_3^*((a*b*c, T^t(a*b*c), I^t(a*b*c), F^t(a*b*c)), x, p) \\ &\quad \vee \tilde{\delta}_3^*((a, T^{t''}(a), I^{t''}(a), F^{t''}(a)), x, r) \vee F_M(c) \vee F_N(b),\end{aligned}$$

for every $a, b, c, p, r, q \in Q$ and $x \in \Sigma^*$.

Now, we have

$$\begin{aligned} T_{M*N}(p*r^{-1}) \geq & \vee\{\tilde{\delta}_1^*((a*b*c, T^t(a*b*c), I^t(a*b*c), F^t(a*b*c)), x, p) \\ & \wedge \tilde{\delta}_1^*((q, T^{t'}(q), I^{t'}(q), F^{t'}(q)), x, r) \wedge T_M(c) \wedge T_N(b) \big| b, c \in Q, b*c = k\} \\ & = (\tilde{\delta}_1^*((q*k, T^{t''}(q*k), I^{t''}(q*k), F^{t''}(q*k)), x, p) \\ & \wedge \tilde{\delta}_1^*((q, T^{t'}(q), I^{t'}(q), F^{t'}(q)), x, r) \\ & \wedge \vee\{T_M(c) \wedge T_N(b) \big| b, c \in Q, b*c = k\}) \\ & = \tilde{\delta}_1^*((q*k, T^{t''}(q*k), I^{t''}(q*k), F^{t''}(q*k)), x, p) \\ & \wedge \tilde{\delta}_1^*((q, T^{t'}(q), I^{t'}(q), F^{t'}(q)), x, r) \wedge T_{M*N}(k), \end{aligned}$$

and

$$\begin{aligned} I_{M*N}(p*r^{-1}) \geq & \tilde{\delta}_2^*((q*k, T^{t''}(q*k), I^{t''}(q*k), F^{t''}(q*k)), x, p) \\ & \wedge \tilde{\delta}_2^*((q, T^{t'}(q), I^{t'}(q), F^{t'}(q)), x, r) \wedge I_{M*N}(k), \end{aligned}$$

also,

$$\begin{aligned} F_{M*N}(p*r^{-1}) \leq & \tilde{\delta}_3^*((q*k, T^{t''}(q*k), I^{t''}(q*k), F^{t''}(q*k)), x, p) \\ & \vee \tilde{\delta}_3^*((q, T^{t'}(q), I^{t'}(q), F^{t'}(q)), x, r) \vee F_{M*N}(k), \end{aligned}$$

Hence, $M * N$ is a SVNK of $\tilde{\mathcal{M}}$. □

Let M be a SVNK of $\tilde{\mathcal{M}}$. Let

$L(M) = \{N \big| N$ is a SVNK of $\tilde{\mathcal{M}}$ such that $(T_M(e), I_M(e), F_M(e)) = (T_N(e), I_N(e), F_N(e))\}$.

Theorem 3.23. *$(L(M), \subseteq)$ is a lattice.*

Proof. Let $B, N \in L(M)$. Then $B \cap N \in L(M)$ and $B \cap N$ is infimum of B and N, for relation $\subseteq$. By Theorem 3.22, $B * N \in L(M)$ and $B * N$ is the supremum of B and N for the relation $\subseteq$. Therefore, $(T(M), \subseteq)$ is a lattice. □

4. Conclusion

In this note, we presented the notion of neutrosophic single-valued general machine over a finite group, which is known as a "group neutrosophic general machine", for simplicity, GNGM. Also, we presented the notions of max-min GNGM, single-valued neutrosophic subgroup (SVNSG) and single-valued neutrosophic normal subgroup (SVNNSG). Moreover, we proved that if M is a SVNK and N is a SVNF of $\mathcal{M}$, then $M * N$ is a SVNF of $\mathcal{M}$.

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