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Interior and closure in anti-minimal and anti-biminimal spaces in the frame of anti-topology

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Abstract. The main goal of this project is to analyze the structure of anti-biminimal spaces through the lens of the notions of interior and closure. Anti-biminimal spaces can be considered as generalizations of anti-bitopological spaces that have been introduced and studied earlier. On the other hand, they can be viewed as counterparts of biminimal spaces. Finally, they are spaces equipped with two anti-minimal structures. Thus, we show some basic results on the latter. In particular, we refer to the concepts of density, nowhere density and rarity in anti-minimal spaces.

In general, anti-topology is a structure κ in which at least one classical axiom is totally false. In this paper, we consider the first axiom. Hence, $\emptyset$ and X do not belong to κ.

Keywords: anti-biminimal space, biminimal space, anti-topology

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1. Introduction

We know that topology on some non-empty universe X is defined as a family $\tau \subseteq P(X)$ that is closed under finite intersections and arbitrary unions. The elements of this family of subsets are called *open* sets. Moreover, we always assume that $\emptyset$ and X are open too. This definition can be considered as a generalization of the idea of open intervals on real line or open balls on real plane. Using this concept mathematicians defined many notions that are used in other branches of mathematics. In particular, continuity is very important in analysis.

Many modern topologists try to reconstruct typical topological notions (like continuity, density, nowhere density, compactness, connectedness, separation etc.) in weaker or just different frameworks than the one mentioned above. For example, we can assume that closure of our family under finite intersections is not necessary and thus we obtain supra topological spaces (see [13]). Instead of this, we can eliminate necessity of closure under unions (to get infra topological spaces, see [3] and [14]). One can reject both these conditions: this leads to minimal structures, like in [15]. We can even assume that X may not be open. These are weak structures (see [6]), where the only requirement is that $\emptyset$ is open. If weak structure is closed under arbitrary unions then it can be considered as a generalized topological space in the sense of Császár (see [5]). Finally, we can assume that τ is arbitrary (we can use the notion of generalized weak structure here), see [1] and [7].

We have different approaches too. The whole direction of our present research is based on Smarandache's suggestion that we should investigate anti-algebras. They are based on the idea that some classical requirements (like closure under finite intersections in case of topologies) are *forbidden*. Initial papers on this concept are e.g. [17], [18], [19] and [20].

In general, Smarandache invented six new types of topologies recently (that is, in the years 2019 - 2022). These are: refined neutrosophic topology, refined neutrosophic crisp topology, neutro-topology, anti-topology, super-hyper topology and neutrosophic super-hyper-topology. The last two are based on the idea of the n-th power set of a given non-empty set (be it classical or neutrosophic). As for the neutro-topological structures, they are based on the assumption that at least one of the classical topological axioms is partially true, partially indeterminate, and partially false. They have been studied e.g. by Şahin et al. in [16]. There was also another paper by Khaklary and Chandra Ray, see [10].

Refined structures are characterized by the fact that truth, falsity and indeterminacy can be split into arbitrarily many subcomponents (depending on applications and needs). This leads to the idea of refined fuzzy, refined intuitionistic fuzzy and then refined neutrosophic set.

Finally, we have the idea of anti-topology. Anti-topology means a topology where at least one of its classical axioms is totally false. In particular, it is possible that any non-trivial intersection or union of the elements of τ is *beyond* τ . Moreover, we may assume $\emptyset$ and X are never open. Such anti-topologies are anti-chains of sets. These spaces have been already studied e.g. in [21], [22]. Moreover, anti-bitopological spaces equipped with two anti-topologies have been investigated in [9].

As for the present paper, it is devoted to the initial study of anti-minimal and anti-weak structures. Anti-biminimal structures are presented too.

2. **Basic notions**

Let us introduce several basic notions that will be used extensively throughout the paper. The first two are somewhat classical, the next two refer to the general program of anti-algebra.

By "non-trivial" we mean intersection or union that engages at least two different sets.

Then we say that such a family is "anti-closed" under these operations.

Definition 2.1. (compare [15]). Assume that X is a non-empty universe and $\mathfrak{m} \subseteq P(X)$. If $\emptyset, X \in \mathfrak{m}$ then we say that $\mathfrak{m}$ is a *minimal structure* on X. We say that an ordered pair $(X, \mathfrak{m})$ is a *minimal structure space*.

Clearly, every topology on X is a minimal structure too. In particular, anti-discrete topology is the simplest example of minimal structure.

Definition 2.2. (compare [4]). Assume that X is a non-empty universe and $\mathfrak{m}^1$, $\mathfrak{m}^2$ are two minimal structures on X. Then we say that an ordered pair $(X, \mathfrak{m}^1, \mathfrak{m}^2)$ is a *biminimal structure space*.

Remark 2.3. If the structures mentioned in the last definition are weak structures (and not necessarily minimal), then the whole space is called *biweak structure*. Such spaces have been investigated e.g. in [11].

As it was announced earlier, the next two definitions can be considered as negations of the former two.

Definition 2.4. Assume that X is a non-empty universe and $\kappa \subseteq P(X)$. If $\emptyset, X \notin \kappa$, then we say that κ is an *anti-minimal structure* on X. We say that an ordered pair (X, κ) is an *anti-minimal structure space*. The elements of κ are called κ-*open* sets and their complements are κ-*closed*. The set of all κ-closed sets is denoted with κ_{Cl}.

In the light of our earlier considerations, anti-minimal structure is an example of anti-topology (since the first axiom of topology is totally false).

Definition 2.5. Assume that X is a non-empty universe and κ^1, κ^2 are two anti-minimal structures on X. Then we say that an ordered pair (X, κ^1, κ^2) is an *anti-biminimal structure space*.

One can easily give many examples of anti-minimal and anti-biminimal structures. Some of them will be presented throughout the paper. Clearly, anti-minimal structures can be closed under some operations (like non-empty intersections or unions that do not lead to X). They can be anti-closed under these operations too.

We can define closure and interior in terms of anti-minimal structures. Both the definitions below are standard.

Definition 2.6. Assume that (X, κ) is an anti-minimal structure space and $A \subseteq X$. We say that κ-*interior* of A is the following set: $\kappa Int(A) = \bigcup\{B \in \kappa; B \subseteq A\}$.

Definition 2.7. Assume that (X, κ) is an anti-minimal structure space and $A \subseteq X$. We say that κ-*closure* of A is the following set: $\kappa Cl(A) = \bigcap\{B \in \kappa_{Cl}; A \subseteq B\}$.

Remark 2.8. Note that anti-minimal structures have one interesting property. Due to the fact that $\emptyset$ and X are never κ-open (nor κ-closed) we can define four non-trivial sets:

(1) $\kappa Ynt(A) = \bigcap\{B \in \kappa; B \subseteq A\}$ (*subinterior* of A).
(2) $\kappa Kl(A) = \bigcup\{B \in \kappa_{Cl}; A \subseteq B\}$ (*superclosure* of A).
(3) $\kappa Ent(A) = \bigcup\{B \in \kappa; B \subseteq A\}$ (*superinterior* of A).
(4) $\kappa Gl(A) = \bigcap\{B \in \kappa_{Cl}; A \subseteq B\}$ (*subclosure* of A).

This will be analyzed in our further research. Clearly, similar operators can be defined even for topological spaces but with some more or less additional assumptions (like e.g. "the intersection of all *non-empty* open sets contained in A"), while in anti-minimal structures they are more natural. Note (for example) that if our space is *not* closed under unions, then $\kappa Ent(A)$ may be different than $\bigcup \kappa$ (because the union of all open sets need not to be open).

The following properties of κ-interior and κ-closure are true just because they are true for any generalized weak structure (and anti-minimal structure is a generalized weak one, without any doubt). The reader can compare this e.g. with [1].

Lemma 2.9. *Assume that* (X, κ) *is an anti-minimal structure space and* $A, B \subseteq X$. *Then:*

(1) $\kappa Int(A) \subseteq A$.
(2) *If* $A \in \kappa$, *then* $\kappa Int(A) = A$.
(3) *If* $A \subseteq B$, *then* $\kappa Int(A) \subseteq \kappa Int(B)$.
(4) $\kappa Int(\kappa Int(A)) = \kappa Int(A)$.
(5) $A \subseteq \kappa Cl(A)$.
(6) *If* $A \in \kappa_{Cl}$, *then* $\kappa Cl(A) = A$.
(7) *If* $A \subseteq B$, *then* $\kappa Cl(A) \subseteq \kappa Cl(B)$.
(8) $\kappa Cl(\kappa Cl(A)) = \kappa Cl(A)$.
(9) $-\kappa Int(A) = \kappa Cl(-A)$.
(10) $\kappa Int(-A) = -\kappa Cl(A)$.
(11) $x \in \kappa Int(A)$ *if and only if there is* $U \in \kappa$ *such that* $x \in U \subseteq A$.
(12) $x \in \kappa Cl(A)$ *if and only if* $U \cap A \neq \emptyset$ *for any* $U \in \kappa$ *such that* $x \in U$.
(13) $\kappa Int(A \cap B) \subseteq \kappa Int(A) \cap \kappa Int(B)$.

Remark 2.10. Note that the converses of Lemma 2.9 (2), (6) need not to be true. For example, let $X = \{a, b, c, d\}$ and $\kappa = \{\{a, b\}, \{b, c\}, \{c, d\}\}$. Now let $A = \{a, b, c\}$. Then $\kappa Int(A) = \{a, b\} \cup \{b, c\} = A \notin \kappa$. Besides, this particular κ is an example of anti-topological space. This is because it is anti-closed under unions and intersections (any union and intersection of two different sets from κ leads beyond κ). Moreover, in the definition of anti-topological space (see [21]) we always assume that $\emptyset, X \notin \kappa$. Hence, each anti-topological space is an anti-minimal structure too.

Remark 2.11. The converse of Lemma 2.9 (13) may not be true. Consider $X = \{1, 2, 3, 4, 5\}$ with $\kappa = \{\{1, 3\}, \{2\}, \{3, 4\}\}$, where $A = \{1, 2, 3\}$ and $B = \{2, 3, 4\}$. One can easily check that $\kappa Int(A) \cap \kappa Int(B) = A \cap B = \{2, 3\} \nsubseteq \{2\} = \kappa Int(\{2, 3\}) = \kappa Int(A \cap B)$.

3. On density and related notions

We can briefly discuss the concepts of density, nowhere density and rarity (leaving more detailed theorems for the further research).

Definition 3.1. Let (X, κ) be an anti-minimal structure and $A \subseteq X$. Then we say that A is:

(1) κ-*dense* if and only if $\kappa Cl(A) = X$.
(2) κ-*nowhere dense* if and only if $\kappa Int(\kappa Cl(A)) = \emptyset$.
(3) Strongly κ-*nowhere dense* if and only if for any κ-open set B there is some κ-open $U \subseteq B$ such that $A \cap U = \emptyset$.
(4) κ-*rare* if and only if $\kappa Int(A) = \emptyset$.

Remark 3.2. Note that if A is κ-dense in an anti-minimal space, then it is equivalent with saying that the set $Z = \{C \in \kappa_{Cl}; A \subseteq C\}$ is empty. Note that X is never κ-open. Analogously, A is κ-rare in anti-minimal space if and only if the set $J = \{C \in \kappa; C \subseteq A\}$ is empty.

We can prove equivalent characterization of κ-dense sets.

Theorem 3.3. *Let (X, κ) be an anti-minimal structure. Let $A \subseteq X$. Then A is κ-dense if and only if $A \cap B \neq \emptyset$ for any $B \in \kappa$.*

Proof. ($\Rightarrow$). We have $\kappa Cl(A) = X$ and $B \in \kappa$. Assume that $A \cap B = \emptyset$. Now let $x \in B$ (there must be some $x \in B$ because B is non-empty as a member of κ). Hence $x \in X = \kappa Cl(A)$. By Lemma 2.9 (13) we obtain $A \cap B \neq \emptyset$.

($\Leftarrow$). Let $A \cap B \neq \emptyset$ for any $B \in \kappa$. Suppose that A is not κ-dense. Then $\kappa Cl(A) \neq X$. Hence, for some $D \in \kappa_{Cl}$, $A \subseteq D$. But $D \neq X$. Then $-D = X \setminus D \in \kappa$ and $A \cap (-D) = \emptyset$. This is contradiction. □

The theorem above can be proved for generalized weak structures too. However, we should assume that B is non-empty (while in anti-minimal structures it is clear by the very definition of κ).

Remark 3.4. Consider $X = \{1, 2, 3, 4\}$ and $\kappa = \{\{1, 2\}, \{2, 3\}, \{2, 3, 4\}, \{1, 3, 4\}\}$. Then $\kappa_{Cl} = \{\{3, 4\}, \{1, 4\}, \{1\}, \{2\}\}$. Hence, $A = \{1, 2\}$ and $B = \{2, 3\}$ are both κ-open and κ-dense. However, their intersection, that is $A \cap B = \{2\}$ is not κ-dense. This is because $\{2\}$ is κ-closed, so its κ-closure is just $\{2\}$ itself.

Note that the remark above shows us that the situation is different than e.g. in topological spaces. Recall that in topological spaces we can prove that if A, B are dense and at least B is open, then their intersection is dense too. However, we can prove the following lemma and theorem.

Lemma 3.5. *Let (X, κ) be an anti-minimal structure that is closed under non-empty finite intersections. If A is κ-dense, then $\kappa Cl(U) = \kappa Cl(A \cap U)$ for any $U \in \kappa$.*

Proof. Clearly, $U \cap A \subseteq U$. Now we use Lemma 2.9 (8) to say that $\kappa Cl(U \cap A) \subseteq \kappa Cl(U)$. Let $x \in \kappa Cl(U)$. Now it must be that $W \cap U \neq \emptyset$ for any $W \in \kappa$ such that $x \in W$ (that is, for any κ-open neighborhood of x). Because of the closure of our κ under non-empty finite intersections, $W \cap U \in \kappa$. Hence, by Lemma 2.9 (13), $(W \cap U) \cap A \neq \emptyset$. Thus, $x \in \kappa Cl(U \cap A)$. □

Theorem 3.6. *Let (X, κ) be an anti-minimal space that is closed under non-empty finite intersections. Suppose that $A, B \subseteq X$ are both κ-dense and B is κ-open. Then $A \cap B$ is κ-dense.*

Proof. We already know that if A is κ-dense, then $A \cap U \neq \emptyset$ for any $U \in \kappa$. Now let $V \in \kappa$. There is some $x \in V$. But $x \in X = \kappa Cl(B)$. Hence, $B \cap V \neq \emptyset$. Moreover, $B \cap V \in \kappa$ (because of the closure under non-empty finite intersections). Hence, $A \cap (B \cap V) \neq \emptyset$. But this means that $A \cap B$ is κ-dense. Note that we can write $(A \cap B) \cap V \neq \emptyset$ to emphasize this fact. □

The fact that we distinguish between nowhere density and strong nowhere density is important. In topological spaces these two notions are equivalent but not here.

On the one hand we can prove the following theorem.

Theorem 3.7. *Every κ-strongly nowhere dense set in anti-minimal structure is κ-nowhere dense too.*

Proof. Suppose that A is κ-strongly nowhere dense but $\kappa Int(\kappa Cl(A)) \neq \emptyset$. Then there is some $x \in \kappa Int(\kappa Cl(A))$. In particular, it means that $x \in \kappa Cl(A)$. Then for any $V \in \kappa$ such that $x \in V$, $V \cap A \neq \emptyset$. However, from the property of κ-strongly nowhere density of A we infer that for any $B \in \kappa$ we can find $U \in \kappa$, $U \subseteq B$ such that $A \cap U = \emptyset$. Thus we obtain contradiction. □

However, the converse is not necessarily true.

Example 3.8. Let $X = \{a, b, c, d\}$, $\kappa = \{\{a, b\}, \{b, c\}, \{c, d\}\}$. Then $\kappa_{Cl} = \{\{c, d\}, \{a, d\}, \{a, b\}\}$. Take $A = \{a, d\}$. Its κ-closure is just $\{a, d\} = A$ but then we see that $\kappa Int(A) = \emptyset$ (there are no κ-open sets contained in A). However, A it not strongly nowhere dense. Take $B = \{a, b\}$. We see that $A \cap B = \{a\} \neq \emptyset$.

4. More on anti-biminimal structures

Now let us concentrate on anti-biminimal structures. We would like to determine some specific class of subsets that will be the object of our investigation. Let us assume that i and j will be always interpreted as elements of $\{1, 2\}$. However, first let us express some basic facts.

Remark 4.1. Note that if A and B are anti-minimal structures on some universe X, then $A \cap B$ and $A \cup B$ are anti-minimal structures too.

If A is anti-minimal structure and B is minimal structure, then $A \cup B$ is a minimal structure, while $A \cap B$ is an anti-minimal structure.

Definition 4.2. Let (X, κ^1, κ^2) be an anti-biminimal structure space. Assume that $A \subseteq X$. We say that A is $\kappa^i\kappa^j$-*closed* set if and only if $A = \kappa^i Cl(\kappa^j Cl(A))$. The complement of $\kappa^i\kappa^j$-closed set is called $\kappa^i\kappa^j$-*open*.

Lemma 4.3. *Let (X, κ^1, κ^2) be an anti-biminimal structure space. Then A is an $\kappa^i\kappa^j$-open subset of X if and only if $A = \kappa^i Int(\kappa^j Int(A))$.*

Proof. Assume that A is $\kappa^i\kappa^j$-open. It means that $-A$ is $\kappa^i\kappa^j$-closed. Hence, $\kappa^i Cl(\kappa^j Cl(-A)) = -A$. However, by virtue of the general properties of κ-interior, $\kappa^i Cl(\kappa^j Cl(-A)) = -\kappa^i Int(-\kappa^j Cl(-A)) = -\kappa^i Int(\kappa^j Int(-(-A))) = -\kappa^i Int(\kappa^j Int(A)) = -A$. But then $A = \kappa^i Int(\kappa^j Int(A))$. Now we can repeat the whole reasoning in the opposite direction to obtain the expected result: namely, that $-A$ is $\kappa^i\kappa^j$-closed and thus A is $\kappa^i\kappa^j$-open. □

As we could see, the lemma above is based on the general properties of interior and closure in arbitrary generalized weak structures rather, than on the specific properties of anti-biminimal structure. The same can be said about the next lemma:

Lemma 4.4. *Let (X, κ^1, κ^2) be an anti-biminimal structure space. Assume that $A, B \subseteq X$ are $\kappa^1\kappa^2$-closed subsets of X. Then $A \cap B$ is $\kappa^1\kappa^2$-closed too.*

Proof. By the assumption, $\kappa^1 Cl(\kappa^2 Cl(A)) = A$ and $\kappa^1 Cl(\kappa^2 Cl(B)) = B$. Clearly, $A \cap B \subseteq A$ and $A \cap B \subseteq B$. Thus $\kappa^1 Cl(\kappa^2 Cl(A \cap B)) \subseteq \kappa^1 Cl(\kappa^2 Cl(A))$ and $\kappa^1 Cl(\kappa^2 Cl(A \cap B)) \subseteq$

$\kappa^1 Cl(\kappa^2 Cl(B))$. Then $\kappa^1 Cl(\kappa^2 Cl(A \cap B)) \subseteq \kappa^1 Cl(\kappa^2 Cl(A)) \cap \kappa^1 Cl(\kappa^2 Cl(B)) = A \cap B$. However, on the other hand, $A \cap B \subseteq \kappa^1 Cl(\kappa^2 Cl(A \cap B))$ (by the very definition of closure). Thus, $A \cap B \subseteq \kappa^1 Cl(\kappa^2 Cl(A \cap B)) = A \cap B$ and due to this reason $A \cap B$ is $\kappa^1\kappa^2$-closed. □

Remark 4.5. Assume that (X, κ^1, κ^2) is an anti-biminimal structure on X and $A, B \subseteq X$ are two $\kappa^1\kappa^2$-closed sets. Then $A \cup B$ does not need to be $\kappa^1\kappa^2$-closed. For example, take $X = \{a, b, c, d, e\}$, $\kappa^1 = \{\{c, d, e\}, \{a, b, e\}, \{a, b, c, d\}\}$ and $\kappa^2 = \{\{c, d, e\}, \{a, b, e\}, \{a, b, d, e\}\}$. Then $\kappa^1_{Cl} = \{\{a, b\}, \{c, d\}, \{e\}\}$ and $\kappa^2_{Cl} = \{\{a, b\}, \{c, d\}, \{c\}\}$. Now take $A = \{a, b\}$ and $B = \{c, d\}$. They are both $\kappa^1\kappa^2$-closed. Let us check their union, that is $\{a, b, c, d\}$. We see that $\kappa^1 Cl(\kappa^2 Cl(\{a, b, c, d\})) = \kappa^1 Cl(\bigcap\{A \in \kappa^2_{Cl}; \{a, b, c, d\} \subseteq A\}) = \kappa^1 Cl(\bigcap \emptyset) = \kappa^1 Cl(X) = \bigcap\{A \in \kappa^1_{Cl}; X \subseteq A\} = \bigcap \emptyset = X \neq \{a, b, c, d\}$.

Besides, let us calculate directly $\kappa^1 Cl(\kappa^2 Cl(A \cap B)) = \kappa^1 Cl(\kappa^2 Cl(\emptyset)) = \kappa^1 Cl(\bigcap\{A \in \kappa^2_{Cl}; \emptyset \subseteq A\}) = \kappa^1 Cl(\{a, b\} \cap \{c, d\} \cap \{c\}) = \kappa^1 Cl(\emptyset) = \bigcap\{A \in \kappa^1_{Cl}; \emptyset \subseteq A\} = \{a, b\} \cap \{c, d\} \cap \{e\} = \emptyset$.

Now, both from the general lemma and from this direct calculation, we know that $\emptyset$ is $\kappa^1\kappa^2$-closed in this particular anti-biminimal structure. However, it is clear (by the very definition of anti-biminimal and anti-minimal structure as such) that $\emptyset \notin \kappa^1_{Cl} \cap \kappa^2_{Cl}$. Hence, we see that $\kappa^1 \cap \kappa^2$ is not identical with the set of all $\kappa^1\kappa^2$-closed sets. This leads us to the next lemma.

Lemma 4.6. *Let (X, κ^1, κ^2) be an anti-biminimal space. Let $A \in \kappa^1_{Cl} \cap \kappa^2_{Cl}$. Then A is $\kappa^1\kappa^2$-closed.*

Proof. $A \in \kappa^2_{Cl}$, hence $\kappa^2 Cl(A) = A$. But $A \in \kappa^1_{Cl}$ too, hence $\kappa^1 Cl(\kappa^2 Cl(A)) = \kappa^1 Cl(A) = A$. □

The converse need not to be true, as we could already seen. However, we should not think that empty set is the only possible counter-example.

Example 4.7. Let (X, κ^1, κ^2) be an anti-biminimal space where $X = \{a, b, c, d, e\}$, $\kappa^1 = \{\{a, d\}, \{e\}\}$ and $\kappa^2 = \{\{a, e\}, \{d\}\}$. Now $\kappa^1_{Cl} = \{\{b, c, e\}, \{a, b, c, d\}\}$ and $\kappa^2_{Cl} = \{\{b, c, d\}, \{a, b, c, e\}\}$. Now take $A = \{b, c\}$. We see that $\kappa^2 Cl(A) = \{b, c, d\} \cap \{a, b, c, e\} = \{b, c\}$. Then $\kappa^1 Cl(\kappa^2 Cl(A)) = \kappa^1 Cl(A) = \{b, c, e\} \cap \{a, b, c, d\} = \{b, c\} = A$. Hence, A is $\kappa^1\kappa^2$-closed but $A \notin \kappa^1_{Cl} \cap \kappa^2_{Cl}$. In fact, A is not even in $\kappa^1_{Cl} \cup \kappa^2_{Cl}$.

Remark 4.8. Let us think about the example above again. Take $A = \{a, b, d\}$. On the one hand, $\kappa^1 Int(A) = \{a, d\}$ and then $\kappa^2 Int(\{a, d\}) = \{d\}$. On the other hand, $\kappa^2 Int(A) = \{d\}$ and $\kappa^1 Int(\{d\}) = \emptyset$. This shows us that in general $\kappa^1 Int(\kappa^2 Int(A))$ may not be identical with $\kappa^2 Int(\kappa^1 Int(A))$.

However, the fact mentioned in the remark above should not be confused with the following theorem:

Theorem 4.9. *Assume that* (X, κ^1, κ^2) *is an anti-biminimal space. Then* $A \subseteq X$ *is* $\kappa^1\kappa^2$*-open if and only if it is* $\kappa^2\kappa^1$*-open.*

Proof. Let A be $\kappa^1\kappa^2$-open. Then $\kappa^1 Int(\kappa^2 Int(A)) = A$. However, it must be that $\kappa^2 Int(A)) = A$. Assume the contrary. Clearly, it means that $\kappa^2 Int(A) \subseteq A$ and $\kappa^2 Int(A) \neq A$. Thus there is some $x \in A$ such that $x \notin \kappa^2 Int(A)$. Suppose that $x \in \kappa^1 Int(\kappa 2 Int(A))$. But $\kappa^1 Int(\kappa^2 Int(A)) \subseteq \kappa^2 Int(A)$, so we obtain contradiction. Thus $x \notin \kappa^1 Int(\kappa^2 Int(A)) = A$. Hence $x \notin A$. Contradiction.

Now, if we already know that $\kappa^2 Int(A) = A$, then $\kappa^2 Int(\kappa^1 Int(\kappa^2 Int(A))) = \kappa^2 Int(A) = A$. But on the left side we have (by the assumption) $\kappa^2 Int(\kappa^1 Int(A))$. Finally, $\kappa^2 Int(\kappa^1 Int(A)) = A$. Hence A is $\kappa^1\kappa^2$-open.

Clearly, the other direction of the proof is similar.

Note that this proof would be true for any generalized weak structure: we did not use the fact that $\emptyset$ and X are not open in κ^1 and κ^2. □

Analogously, we have:

Theorem 4.10. *Assume that* (X, κ^1, κ^2) *is an anti-biminimal space. Then* $A \subseteq X$ *is* $\kappa^1\kappa^2$*-closed if and only if it is* $\kappa^2\kappa^1$*-closed.*

Proof. Assume that $\kappa^1 Cl(\kappa^2 Cl(A)) = A$. Then $\kappa^2 Cl(A)) = A$. Suppose the contrary. It means that $A \subseteq \kappa^2 Cl(A)$ but $\kappa^2 Cl(A) \neq A$. Hence there is some $x \in \kappa^2 Cl(A)$ such that $x \notin A$. Suppose that $x \in \kappa^1 Cl(\kappa^2 Cl(A))$. But $\kappa^2 Cl(A) \subseteq \kappa^1 Cl(\kappa^2 Cl(A)) = A$ which gives us that $x \in A$ and this is contradiction.

Now we see that $\kappa^2 Cl(\kappa^1 Cl(\kappa^2 Cl(A))) = \kappa^2 Cl(A) = A$. But on the left side we have $\kappa^2 Cl(\kappa^1(A))$, so finally we get $\kappa^2 Cl(\kappa^1(A)) = A$ which means that A is $\kappa^2\kappa^1$-closed. □

As for the empty set, we may prove the following theorem.

Theorem 4.11. *Let* (X, κ^1, κ^2) *be an anti-biminimal space. Then* $\emptyset$ *is* $\kappa^1\kappa^2$*-closed if and only if* $\bigcap \kappa^1_{Cl} = \emptyset$ *and* $\bigcap \kappa^2_{Cl} = \emptyset$*.*

Proof. $(\Rightarrow)$. Assume that $\bigcap \kappa^2_{Cl} = L \neq \emptyset$. Now let us calculate: $\kappa^1 Cl(\kappa^2 Cl(\emptyset)) = \kappa^1 Cl(\bigcap\{A \in \kappa^2_{Cl}; \emptyset \subseteq A\}) = \kappa^1 Cl(\bigcap\{A \in \kappa^2_{Cl}\}) = \kappa^1 Cl(\bigcap \kappa^2_{Cl}) = \kappa^1 Cl(L)$. However, if L is non-empty, as we assumed, then its κ^1-closure must be non-empty too. Finally, $\kappa^1 Cl(\kappa^2 Cl(\emptyset)) \neq \emptyset$, so $\emptyset$ is not $\kappa^1\kappa^2$-closed.

Analogously, assume that $\bigcap \kappa^1_{Cl} = K \neq \emptyset$. If $\kappa^2 Cl(\emptyset) \neq \emptyset$, then $\kappa^1 Cl(\kappa^2 Cl(\emptyset))$ is non-empty (as a κ^1-closure of non-empty set). However, even if $\kappa^2 Cl(\emptyset) = \emptyset$, then $\kappa^1 Cl(\emptyset) = \bigcap\{A \in \kappa^1; \emptyset \subseteq A\} = \bigcap\{A \in \kappa^1_{Cl}\} = \bigcap \kappa^1_{Cl} = K \neq \emptyset$. Again, in both cases $\emptyset$ is not $\kappa^1\kappa^2$-closed.

($\Leftarrow$).

Assume that $\bigcap \kappa^1_{Cl} = \emptyset$ and $\bigcap \kappa^2_{Cl} = \emptyset$. Let us calculate $\kappa^1 Cl(\kappa^2 Cl(\emptyset)) = \kappa^1 Cl(\bigcap\{A \in \kappa^2_{Cl}; \emptyset \subseteq A\}) = \kappa^1 Cl(\bigcap\{A \in \kappa^2_{Cl}\}) = \kappa^1 Cl(\bigcap \kappa^2_{Cl}) = \kappa^1 Cl(\emptyset) = \bigcap\{A \in \kappa^1_{Cl}; \emptyset \subseteq A\} = \bigcap\{A \in \kappa^1_{Cl}\} = \bigcap \kappa^1_{Cl} = \emptyset$. □

Remark 4.12. Clearly the left side of this theorem can be reformulated as: "X is $\kappa^1\kappa^2$-open".

The theorem above can be illustrated.

Example 4.13. Let (X, κ^1, κ^2) be an anti-biminimal space where $X = \{a, b, c, d, e\}$, $\kappa^1 = \{\{c, d, e\}, \{a, b, e\}\}$, $\kappa^2 = \{\{a, d, e\}, \{a, b, e\}$. Hence $\kappa^1_{Cl} = \{\{a, b\}, \{c, d\}\}$ and $\kappa^2_{Cl} = \{\{b, c\}, \{c, d\}\}$. Clearly, $\bigcap \kappa^2_{Cl} = \{c\} \neq \emptyset$. Now $\kappa^1 Cl(\kappa^2 Cl(\emptyset)) = \kappa^1 Cl(\bigcap \kappa^2_{Cl}) = \kappa^1 Cl(\{c\}) = \bigcap\{A \in \kappa^1_{Cl}; \{c\} \subseteq A\} = \{c, d\} \neq \emptyset$.

This was an example of the situation where $\emptyset$ was not $\kappa^1\kappa^2$-closed. Now take the same universe and $\kappa^1 = \{\{a\}, \{b, c, d, e\}\}$ and $\kappa^2 = \{\{a, d, e\}, \{a, b, c\}\}$. Now $\kappa^1_{Cl} = \kappa^1$ and $\kappa^2_{Cl} = \{\{b, c\}, \{d, e\}\}$. In both these minimal structures the intersection of all closed sets is empty. Now one can check that $\kappa^1 Cl(\kappa^2 Cl(\emptyset)) = \emptyset$ just repeating the reasoning presented in ($\Leftarrow$) part of the last theorem.

Let us go back to the notion of interior. We prove the following lemma.

Lemma 4.14. *Let (X, κ^1, κ^2) be an anti-biminimal structure space. Assume that $A, B \subseteq X$ are $\kappa^1\kappa^2$-open subsets of X. Then $A \cup B$ is $\kappa^1\kappa^2$-open too.*

Proof. Assume that both A and B are $\kappa^1\kappa^2$-open. Hence, $\kappa^1 Int(\kappa^2 Int(A)) = A$ and analogously $\kappa^1 Int(\kappa^2 Int(B)) = B$. Clearly, $A \subseteq A \cup B$ and $B \subseteq A \cup B$. Hence, $\kappa^1 Int(\kappa^2 Int(A)) \subseteq \kappa^1 Int(\kappa^2 Int(A \cup B))$ and $\kappa^1 Int(\kappa^2 Int(B)) \subseteq \kappa^1 Int(\kappa^2 Int(A \cup B))$. Thus, $A \cup B = \kappa^1 Int(\kappa^2 Int(A)) \cup \kappa^1 Int(\kappa^2 Int(B)) \subseteq \kappa^1 Int(\kappa^2(A \cup B))$. However, on the other hand, $\kappa^1 Int(\kappa^2 Int(A \cup B)) \subseteq A \cup B$. Finally, $\kappa^1 Int(\kappa^2 Int(A \cup B)) = A \cup B$. Thus, $A \cup B$ is $\kappa^1\kappa^2$-open. □

As for the whole universe, we have a theorem analogous to Theorem 4.11.

Theorem 4.15. *Let (X, κ^1, κ^2) be an anti-biminimal space. Then X is $\kappa^1\kappa^2$-open if and only if $\bigcup \kappa^1 = X$ and $\bigcup \kappa^2 = X$.*

Proof. ($\Rightarrow$). Assume that $\bigcup \kappa^2 = M \neq X$. Now let us calculate: $\kappa^1 Int(\kappa^2 Int(X)) = \kappa^1 Int(\bigcup\{A \in \kappa^2; A \subseteq X\}) = \kappa^1 Int(\bigcup \kappa^2) = \kappa^1 Int(M)$. However, it M is properly contained in X, then its κ^1-interior must be properly contained in X too. Finally, $\kappa^1 Int(\kappa^2 Int(X)) \neq X$, so X is not $\kappa^1\kappa^2$-open.

Assume now that $\bigcup \kappa^1 = N \neq X$. If $\kappa^2 Int(X) = M \neq X$, then $\kappa^1 Int(\kappa^2 Int(X)) = \kappa^1 Int(M) \neq X$ (being contained in M). However, even if $\kappa^2 Int(X) = X$, then $\kappa^1 Int(X) = \bigcup\{A \in \kappa^1; A \subseteq X\} = \bigcup \kappa^1 = N \neq X$. In both cases X is not $\kappa^1\kappa^2$-open.

($\Leftarrow$). Suppose that $\bigcup \kappa^1 = X$ and $\bigcup \kappa^2 = X$. Let us calculate $\kappa^1 Int(\kappa^2 Int(X)) = \kappa^1 Int(\bigcup\{A \in \kappa^2; A \subseteq X\}) = \kappa^1 Int(\bigcup \kappa^2) = \kappa^1 Int(X) = \bigcup\{A \in \kappa^1; A \subseteq X\} = \bigcup\{A \in \kappa^1\} = \bigcup \kappa^1 = X$. □

This was direct proof but it was enough to use Theorem 4.11, Remark 4.12 and the fact that $\bigcup \kappa = -\bigcap \kappa_{Cl}$.

5. Conclusion

In this paper we presented anti-minimal and anti-biminimal spaces. We proved some initial claims about these structures. Now our idea is to analyze the notion of nowhere density and to introduce the idea of continuous functions (in both frameworks). Moreover, we think that it would be valuable to analyze those somewhat untypical operators that have been mentioned in Remark 2.8.

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