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Theory on Duplicity of Finite Neutrosophic Rings

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Abstract

This article introduces the notion of duplex elements of the finite rings and corresponding neutrosophic rings. The authors establish duplex ring $Dup(R)$ and neutrosophic duplex ring $Dup(R(I))$ by way of various illustrations. The tables of different duplicities are constructed to reveal the comparison between rings $Dup(Z_n)$, $Dup(Dup(Z_n))$ and $Dup(Dup(Dup(Z_n)))$ for the cyclic ring Z_n. The proposed duplicity structures have several algebraic systems with dissimilar consequences. Author's characterize finite rings with $R+R$ is different from the duplex ring $Dup(R)$. However, this characterization supports that $R+R=Dup(R)$ for some well known rings, namely zero rings and finite fields.

Keywords: Multiplicative function, Duplex form; Duplex ring, neutrosophic duplex element, neutrosophic duplex ring

1. Introduction

In the most general sense, elementary number theory deals with and manages the results and properties of different sets of numbers. In this paper, we will examine and discuss some significant sets of numbers in Z_n, called duplexity. We will briefly present the notion of duplexity of Z_n and enumerate how many number of duplex elements are there in Z_n. For the integer x, the element form $x+x$ is called a *duplex form* of x. The most important problem in the elementary theory of integers is to determine the possible forms of duplexes among the integers. For instance, it is clear to see that any duplex form must be of form $2k$, or $2k+2$ in Z, because every even integer is a multiple of 2. This illustration specifies that the ring of integers Z satisfies the conditions: $x+x=2x$, $x+2x=3x$ and so on, but $Z+Z\neq 2Z$, $Z+2Z\neq 3Z$ and so on, where the operation addition '$+$'defined on Z. In general, a duplex form $x+x$ exists in the ring Z of integers. Now, we shall study the enumeration of duplex elements in the finite commutative ring Z_n, and which are finitely many duplex forms $x+x$ in Z_n, where the operation addition '$+$' defined on Z_n.

First, we can generally describe a ring R is an algebraic structure $(R,+,\cdot)$ as an additive abelian group with a multiplicative binary operation such that the structure $(R,+,\cdot)$ is associative and fulfils distributive axioms $a(b+c)=ab+ac$ and $(b+c)a=ba+ca$. A ring R is finite commutative if $|R|<\infty$ and $ab=ba$ for all a, b in R, see [1]. An element u in a commutative ring with unity 1 is called a unit if there exists an element x in R such that $xu=1=ux$, and specifically x is called a multiplicative inverse of u, and vice versa.

All the elements in R which are not multiplicative inverse elements are said to be zero divisors. Note that set of units and zero divisors of R are usually denoted by $R^{\times}$ and $Z(R)$, respectively, and R can be partitioned by the disjoint sets $Z(R)$ and $R^{\times}$. For any subring R of, the set R/S denotes the quotient ring. Now our attention is shifted to focus on the ring Z_n which is isomorphic to the quotient ring Z/nZ, which are the main tools in this paper for various values of $n \geq 1$. For $a, n \in Z$ with $n > 0$, we represent the congruence class of modulo n by the notion $[a]_n$, and the ring Z_n is the set $\{[a]_n : a \in Z\}$, or equivalently $\{[0],[1],[2],...,[n-1]\}$. But there is a one-to-one correspondence between the complete residue systems $\{[0],[1],[2],...,[n-1]\}$ and $\{0,1,2,...,n-1\}$, and thus Z_n can be written simply as $Z_n = \{0,1,2,...,n-1\}$ for complete residue systems modulo n [2]. It is worth clarifying that the ring $Z_n = \{0,1,2,...,n-1\}$ is a commutative ring with unity 1 under addition and multiplication modulo n. We are happy to say that the ring Z_n has countably many applications in various fields such as algebraic number theory, algebraic coding theory, Cryptography, algebraic circuit theory, Antenna theory and algebraic design theory [3-8]. Further, the problem of enumeration of various types of elements in Z_n up to countably finite has received considerable attention in recent years; see for examples [9-14].

Now starts the basic notions, definitions and results of classical rings.

Let R be a finite commutative ring with nonzero identity and $R^{\times}$ be the set of group units of R. Given a finite commutative ring R, the ring $R+R = \{r+r : r \in R, r \in R\}$ is known as *duplex form* of R. However, the problem of characterizing finite commutative rings up to isomorphism has established considerable attention in recent years [see 15 and 16] initiating from the research works of Eldridge [17]. In this chapter, authors characterize finite commutative rings in terms of their duplexes. First, write the notion $Char(R)$ to denote a positive integer n such that $na = 0$ for every a in R, where $na = a + a + ... + a$ (n copies). Recall that the ring Z_n is a finite commutative ring with nonzero unity 1 under addition and multiplication modulo n. Also, the number of the form $a+ib$, $a, b \in Z_n$, is called Gaussian integer, and the set of Gaussian integers represented by $Z_n[i]$, and defined as $Z_n[i] = \{a+bi : a,b \in Z_n, i^2 = -1\}$. Further, note that $|Z_n| = n$ and $|Z_n[i]| = n^2$.

Neutrosophic Duplex elements are the solutions of some specific neutrosophic equation, and which are main mathematical tools for studying additive elements and their additive reciprocals of an object and their mutual symmetries, which are logically related to neutrosophic systems and their automorphisms. The characterizations of the duplex elements of any finite commutative ring have not been done in general theory of neutrosophic mathematics. But in recent years, the interplay between additive self inverses and group units of a classical ring and its corresponding neutrosophic ring was studied by Chalapathi and co-authors [18-20].

Now reconsider some notations, preliminaries and results of neutrosophic ring theory.

Let 0, 1 and I be three distinct components of any neutrosophic logical system with $0^2 = 0$, $1^2 = 1$ and $I^2 = I$. Then the component I is called the *indeterminate* of a system with some specific algebraic axioms: $0I = 0$, $1I = I$, $I + I = 2I$, and I^{-1} does not exists under usual neutrosophic addition and neutrosophic Multiplication defined on the required system. The component I is a concrete mathematical tool to deal with inconsistent, incomplete and indeterminate information which exist in the real world systems. A nonempty set

N together with I is denoted by $N(I)$ and defined as $N(I)=\{a+bI:a,b\in N, I^2=I\}$, which is called neutrosophic set. Neutrosophic is an innovative research field of philosophy with the composition of indeterminacy founded by Smarandache to develop and deal indeterminacy of a system in nature and science [21]. In addition, the neutrosophic set and their interactions play an important role in classical and modern algebra, and generate a specific theory in modern mathematics called *neutrosophic algebraic theory*, and it contains many algebraic structures, like neutrosophic groups, neutrosophic rings, neutrosophic Boolean rings, neutrosophic zero rings and neutrosophic field [22-25].First, classical rings and their useful results are standard and follow those from [26]. Next, the other neutrosophic concepts and further terminology with corresponding notions will be explained in detail as follows. For any finite commutative ring R, the nonempty neutrosophic set $R(I)=\{a+bI:a,b\in R, I^2=I\}$ is called a Neutrosophic ring generated by R and I under the following neutrosophic binary operations:

$$(a+bI)+(c+dI)=(a+c)+(b+d)I,\ (a+bI)(c+dI)=(ac)+(ad+bc+bd)I.$$

Particularly, $0=0+0I$, $1=1+0I$, $I=0+1I$ are main components of the neutrosophic ring $R(I)$ with $R(I)=R+RI=\langle R\cup I\rangle$. Note that, if R is finite, and then $|R|$ denotes the number of elements in R, consequently that $|R(I)|=|R|^2$.

The contributions of this manuscript are three folds.

First, we propose the use of modular arithmetic to determine the duplex elements for the finite ring Z_n. The number of duplex elements $D(n)$ over Z_n is distributed in $Z_m\times Z_n$. Thus, the enumeration of this procedure is suitable for enumerating the number of duplex elements in $Z_m\times Z_n$. *Second*, we thoroughly characterize finite rings over their duplicities. We provide necessary constructive conditions on various finite rings and weights to achieve their related consequences. *Third* and finally, we establish systematic procedure to construct neutrosophic duplex rings over given classical rings. We prove that neutrosophic duplex rings generated by our basic neutrosophic rule $R(I)=R+RI=\langle R\cup I\rangle$ exhibit a specific structure, and maintain the basic neutrosophic properties of $R(I)$.

2. **Enumeration of Duplex Elements in Z_n**

As the heading suggests, the present section has as its goal is another simple contribution of Z_n, called duplex of Z_n. For those who consider the theory of integers and basic number theory. The intrinsic beauty of the duplex of Z_n has a strange fascination for modern mathematicians. Generally speaking, the duplex of Z_n deals with the characterization of Z_n with $Z_n+Z_n\neq 2Z_n$, or $Z_n+Z_n=2Z_n$.

This section enumerates all duplex elements which are in Z_n, and also demonstrate a number-theoretic connection between the finite number of positive integers and duplex elements in Z_n. Also, this section generates the function $D(n)$ which is a multiplicative function but not complete. Additionally, prove that $|D(Z_n)|=\dfrac{n}{(2,n)}$ and $|D(Z_m\times Z_n)|=\dfrac{mn}{(2,m)(2,n)}$.

Before moving on to the other important concepts and results of the duplex of Z_n, let us define duplex elements of Z_n with different illustrations.

First, we prove that $D(n)$ is a multiplicative function but not complete with an illustration.

Definition 2.1.
An element a in Z_n is called a *duplex element* in Z_n if and only if the equation $x+x=a$ has a solution in Z_n.

The set of all duplex elements in Z_n is denoted by $D(Z_n)$, and $D(n)$ denotes the number of duplex elements in Z_n with $D(n) \neq 0$, since $x+x=0$ is solvable in Z_n. The function $D(n)$ is called the *duplex function* of n.

For any $n > 1$, we have $D(Z_{2n}) \neq Z_{2n}$ but $D(Z_{2n-1}) = Z_{2n-1}$. This means that the units of Z_{2n} are not the duplex elements in Z_{2n}. For example $D(8) = 4$ since the equations $x+x=0$, $x+x=2$, $x+x=4$ and $x+x=6$ have an individual solution in Z_8, but $x+x=1$, $x+x=3$, $x+x=5$ and $x+x=7$ do not have a solution in Z_8 .

The following table illustrates the number of duplex elements in Z_1, Z_2, Z_3,..., Z_{10} , respectively.

n	1	2	3	4	5	6	7	8	9	10
$D(n)$	1	1	3	2	5	3	7	4	9	5

For any positive integers m and n, the notation $gcd(m,n)$, or (m,n) denotes the greatest common divisor of m and n . Particularly, $gcd(m,n) = 1$ if and only if m and n are called relatively prime. Suppose $gcd(m,n) = 1$. Then the function $f: N \to \mathbb{R}$ is called a Number-Theoretic function, and it is called *multiplicative* if $f(mn) = f(m)f(n)$. Naturally, many number-theoretic functions exist in the theory of numbers [2], and which are completely characterized by its value of n when $n \geq 1$. Now we show that the duplex function $D(n)$ is a *multiplicative* function.

Theorem 2.2. Let $gcd(m,n) = 1$. Then the number theoretic relation is $D(mn) = D(m)D(n)$.
Proof: First of all we adopt the notation: $D(mn)$ is the number of duplex elements in Z_{mn} and $D(m)D(n)$ is the number of duplex elements in $Z_m \times Z_n$. Because $gcd(m,n) = 1$, the ring Z_{mn} is isomorphic to the ring $Z_m \times Z_n$ by the ring isomorphism ψ: $Z_{mn} \to Z_m \times Z_n$ related by $\psi(t) = (t \ \ mod \ \ m, t \ \ mod \ \ n \ )$ for every element t in Z_{mn} (see [1]).

First, we prove that $D(mn) \leq D(m)D(n)$. For this let a be a duplex element in Z_{mn}, then the equation $x+x=a$ is solvable in Z_{mn}. Consequently, there is an element b in Z_{mn} such that $b+b=a$ is solvable in Z_{mn}. Since ψ is an injective map from Z_{mn} onto $Z_m \times Z_n$, so there exists an element (x,y) in $Z_m \times Z_n$ such that $\psi(b) = (x,y)$. Therefore,

$$\psi(a) = \psi(b+b) = \psi(b) + \psi(b) = (x,y) + (x,y) = (x+x, y+y)$$

is solvable in $Z_m \times Z_n$. This implies that $\psi(a)$ is also a duplex element in $Z_m \times Z_n$. Hence, $D(mn) \leq D(m)D(n)$. On the other hand, we can show that $D(mn) \geq D(m)D(n)$. Suppose c is a duplex element in Z_m and d is a duplex element in Z_n. Then there exists u in Z_m and v in Z_n such that

$$(u+u, v+v) = (c,d) \text{ in } Z_m \times Z_n.$$

So, we have

$\psi^{-1}[(c,d)] = \psi^{-1}[(u+u, v+v)] = \psi^{-1}[(u,v) + (u,v)] = \psi^{-1}[(u,v)] + \psi^{-1}[(u,v)]$ is solvable in Z_{mn}. This implies that the element $\psi^{-1}[(c,d)]$ is also a duplex element in Z_{mn}. This shows that $D(mn) \geq$

$D(m)D(n)$. Combination of inequalities $D(mn) \leq D(m)D(n)$ and $D(mn) \geq D(m)D(n)$ yields that the equality $D(mn) = D(m)D(n)$, and this shows that $D(n)$ is a number-theoretic multiplicative function. ■

Now we continue our study by verifying other generalizations of duplex function. This requires the following:

Example 2.3. Consider $m = 2, n = 4$, we find that $D(2) = 1, D(4) = 2$ and $D(8) = 4$ with $D(8) \neq D(2)D(4)$.

Corollary 2.4. Prove that $D(1) = 1$.

Proof: Because of $0 + 0 = 0$, the element 0 is a duplex element in Z_n. So there exists an n such that $D(n) \neq 0$. But by the Theorem [2.2],

$$D(n) = D(n1) = D(n)D(1).$$

Being $D(n)$ non-zero, $D(n)$ may be cancelled from both sides of the above equation to give $D(1) = 1$. ■

The following theorem plays an important role in studying the duplexity of the ring Z_n.

Theorem 2.5. For every $n \geq 1$, the units of Z_{2n} are not the duplex elements in Z_{2n}.

Proof: Suppose $u \in Z_{2n}^{\times}$ be a duplex element in Z_{2n}. Then there exists an element x in Z_{2n} such that $x + x = u$ is solvable in Z_{2n} . By the basic celebrations of Z_{2n}, the number $2n$ divides the element $x + x - u$. So, there exists q in Z such that $x + x - u = 2nq$. But $u \epsilon Z_{2n}^{\times}$ implies that $gcd(u, 2n) = 1$, and it implies that $gcd(x + x - 2nq,\ 2n) = 1$, which is not true because $gcd(x + x - 2nq,\ 2n) > 1$ for every x in Z_{2n}. Hence every unit in Z_{2n} is not a duplex element in Z_{2n}. Particularly, $D(Z_{2n}^{\times}) = \emptyset$. ■

Our next goal is to establish a formula for enumerating the number of duplex elements in Z_n. Once this is established, enumerating formulas in a simple form for the different values of n will complete our enumerating procedure. We start with the trivial observation that the duplex element in Z_1 is 0, so that $D(1) = 1$ because $x + x = 0$ is solvable in Z_1. We are now ready to prove that $D(2^n) = 2^{n-1}$, where $n \geq 2$. Because $x + x = 2x(mod\ \ 2^n)$ for all x in Z_{2^n}, it follows that the duplex element in Z_2n is a multiple of 2 under multiplication modulo 2^n, but the total number of multiples of 2 in Z_{2^n}, is 2^{n-1} since $2^n + 2^n \equiv 0(mod\ \ 2^n)$ and thus $D(2^n) = 2^{n-1}$.

Further, we start with the simple observation that for every x in Z_{p^n}, where $p > 2$ is a prime. This concludes that every element in Z_{p^n} is a duplex element in Z_{p^n}, and thus $D(p^n) = p^n$. Finally, we aim to establish a formula for enumeration number of duplex elements in Z_n whenever $n \geq 1$. For every x in Z_n, we have

$$(2, n)x = (2x, nx) = a(2x) + b(nx) \text{ for some } a \text{ and } b \text{ in } Z_n$$
$$= 2ax \text{ in } Z_n = ax + ax \text{ in } Z_n.$$

This observation shows that x is a duplex element in Z_n if and only if $(2, n)x$ is also a duplex element in Z_n.

As we explored duplex elements in Z_n we were led to specify how many there are. We found the answer in the following way.

Theorem 2.6. The number of duplex elements in Z_n is $D(n) = \frac{n}{(2,n)}$.

Proof: Suppose there is an element x in Z_n such that the duplex form $x + x$ can be written as $x + x = nq + (2, n)r$ in Z. By the Bezout's Theorem (ref.[2]),

$$x + x = nq + (2x + ny)r \quad \text{for some } x, y \text{ in } Z.$$
$$= nq + 2xr + nyr = n(q + yr) + 2xr.$$

Now $x + x < n$, so $x + x = 2xr$ is a duplex in Z_n. Conversely, suppose that there is an element y in Z_n such that $y + y = mn + (2, n)s$ in Z. Then the number $(2, n)$ divides y. Thus there is an element t such that $y = (2, n)t$, and hence an element $(2, n)t$ is a duplex element in Z_n. Therefore the number of duplex elements in Z_n is

$$D(n) = \frac{|Z_n|}{(2,n)} = \frac{n}{(2,n)} \cdot \blacksquare$$

The following example demonstrates the preceding theorem.

Example 2.7. Because $(2, n) = 1 \ or \ 2$, the number of duplex elements in Z_9 is 9 and the number of duplex elements in Z_{10} is 5.

Our next aim is to enumerate the number of duplex elements in the ring $Z_m \times Z_n$ for every positive integer m and n. We recall that $Z_m \times Z_n \cong Z_{mn}$ if and only if $(m, n) = 1$. This relation explores that $D(mn) = D(m)D(n)$. Further, if $(m, n) \neq 1$, then by the Theorem [2.2], the number of duplex elements in $Z_m \times Z_n$ is

$$D(m)D(n) = \frac{m}{(2,m)} \cdot \frac{n}{(2,n)} = \frac{mn}{(2,m)(2,n)} \cdot$$

However, we observe that

$$D(Z_m \times Z_n) = D(Z_m) \times D(Z_n) \neq D(Z_{mn}) \text{ whenever} (m, n) \neq 1.$$

Subsequently, $D(m)D(n) = \frac{mn}{(2,m)(2,n)}$ is not equal to $D(mn) = \frac{mn}{(2,mn)}$. For instance, $D(2) = \frac{2}{(2,2)} = 1$, $D(4) = \frac{4}{(2,4)} = 2$, $D(8) = \frac{8}{(2,8)} = 4$ but $D(2 \cdot 4) \neq D(2)D(4)$.

Theorem 2.8. Let $m, n \in N$. Then $D(Z_m \times Z_n) = D(Z_m) \times D(Z_n)$. Particularly, we have $|D(Z_m \times Z_n)| = \frac{mn}{(2,m)(2,n)}$.

Proof: Because of Definition [2.1], the duplex of $Z_m \times Z_n$ is defined as

$D(Z_m \times Z_n) = \{(a, b) \epsilon \ Z_m \times Z_n : (x, y) + (x, y) = (a, b) \text{ is solvable in } Z_m \times Z_n\}$

$= \{(a, b) \epsilon \ Z_m \times Z_n : (x + x, y + y) = (a, b) \text{ is solvable in } Z_m \times Z_n\}$

$= \{(a, b) \epsilon \ Z_m : x + x = a \text{ is solvable in } Z_m\} \times \{(a, b) \epsilon \ Z_n : y + y = b \text{ is solvable in } Z_n\}$

$= D(Z_m) \times D(Z_n)$.

This result has summarized the cardinality of $D(Z_m) \times D(Z_n)$. So, we have

$$|D(Z_m \times Z_n)| = |D(Z_m) \times D(Z_n)| = |D(Z_m)||D(Z_n)| = \frac{m}{(2,m)} \cdot \frac{n}{(2,n)} = \frac{mn}{(2,m)(2,n)} \cdot$$

3. Duplicity of finite Rings

A ring R is *cyclic* if the structure $(R, +)$ is a cyclic group, where the additive operation $+$ is defined over the ring R. In [27], the author Buck introduced a special ring structure, called cyclic ring. This algebraic structure establishes various results and it explore different algebraic concepts. Generally, every cyclic ring is commutative but it is a ring with unity or without unity. For instance, Z_9 is a cyclic ring with unity but $R^0 = \{0, 3, 6\}$ is also a cyclic ring without unity under addition and multiplication modulo 9. Further, if R is a cyclic ring then obviously the Cartesian product ring $R \times R$ is not a cyclic ring. For instance, $Z_9 \times Z_9$ is not a cyclic ring, in view of the fact that the structure $(Z_9 \times Z_9, +)$ is not a cyclic group under addition modulo 9. Throughout the paper, authors consider the ring Z_n as a cyclic ring of order n.

Recall that the element $x+x$ is called duplex form an element in Z_n under addition and multiplication defined over Z_n. Under this duplex form, we explore the following connections over Z_n : $x+x=2x$, $x+x+x=3x$ and so on, but $Z_n+Z_n \neq 2Z_n$, $Z_n+Z_n+Z_n \neq 3Z_n$, and so on, where the addition '$+$' defined over the ring Z_n. Now summarize these concepts in the following definitions.

Definition 3.1. An element a in a ring R is called duplex element in R if the equation $x+x=a$ has a solution in R.

For instance, the element 0 is a duplex element in every ring R, since $x+x=0$ is solvable in R.

Definition 3.2. The duplex ring of a ring R is denoted by $Dup(R)$ and defined as

$$Dup(R) = \{a : x+x=a \text{ is solvable in } R\}.$$

For instance, the following short table illustrates the duplex rings of the rings $Z_1, Z_2, \ldots, Z_{10}$.

R	Z_1	Z_2	Z_3	Z_4	Z_5	Z_6	Z_7	Z_8	Z_9	Z_{10}
$Dup(R)$	Z_1	$\{0\}$	Z_3	$\{0,2\}$	Z_5	$\{0,2,4\}$	Z_7	$\{0,2,4,6\}$	Z_9	$\{0,2,4,6,8\}$

With this information available, it is an easy task to prove the following result.

Theorem 3.3. The duplicity of R is a subring of R.

Proof. Let a and b be any two elements in $Dup(R)$. Then there exists x and y in R such that $a=x+x$ and $b=y+y$. It is clear that

$$a+b=(x+x)+(y+y)=(x+y)+(x+y), \quad ab=(x+x)(y+y)=xy+xy+xy+xy=(xy+xy)+(xy+xy),$$

which shows that $a+b$ and ab are both elements in $Dup(R)$, and thus $Dup(R)$ is a subring of R. ■

With the support of the preceding theorem, let us define duplex of duplex.

Definition 3.4. The duplex of duplex of a ring R is denoted by $Dup(Dup(R))$ and defined as

$$Dup(Dup(R)) = \{d \in Dup(R) : x+x=d \text{ is solvable in } Dup(R)\}.$$

Similarly, define $Dup(Dup(Dup(R)))$ as follows.

$$Dup(Dup(Dup(R))) = \{y \in Dup(Dup(R)) : x+x=y \text{ is solvable in } Dup(Dup(R))\}.$$

These notions lead directly to the following tabular information.

R	Z_1	Z_2	Z_3	Z_4	Z_5	Z_6	Z_7	Z_8	Z_9	Z_{10}
$Dup(R)$	Z_1	$\{0\}$	Z_3	$\{0,2\}$	Z_5	$\{0,2,4\}$	Z_7	$\{0,2,4,6\}$	Z_9	$\{0,2,4,6,8\}$
$Dup(Dup(R))$	Z_1	$\{0\}$	Z_3	$\{0\}$	Z_5	$\{0,2,4\}$	Z_7	$\{0,4\}$	Z_9	$\{0,2,4,6,8\}$
$Dup(Dup(Dup(R)))$	Z_1	$\{0\}$	Z_3	$\{0\}$	Z_5	$\{0,2,4\}$	Z_7	$\{0\}$	Z_9	$\{0,2,4,6,8\}$

In vision of the preceding table, authors conclude the following.

1. $Dup(Dup(Dup(Z_n))) = Z_n \Leftrightarrow n$ is odd.

2. $Dup\left(Dup\left(Dup(Z_n)\right)\right) = Dup\left(Z_n\right) \Leftrightarrow n$ is a perfect square.

3. $Dup\left(Dup\left(Dup(Z_n)\right)\right) = \left(0\right) \Leftrightarrow n = 2^k$ for some positive integer k .

Under this information, the following theorem provides the structure of duplex ring of the ring Z_n .

Theorem 3.5. For any positive integer n , there exists duplex ring $Dup(Z_n)$ of the ring Z_n such that $Dup(Z_n) = (2,n)Z_n$, where $(2,n)$ is the greatest common divisor of the numbers 2 and n .

Proof. It is well known that $Z_n + Z_n \neq 2Z_n$ but the duplex equation $x + x = a$ is solvable in the ring Z_n for every positive integer n . So, the calculations

$$Dup(Z_n) = (2,n)\left(\frac{2}{(2,n)}\right)Z_n \subseteq (2,n)Z_n$$

conform that the first set inclusion $Dup(Z_n) \subseteq (2,n)Z_n$ is true. Before proving another way of this result, consider Bezout's Theorem [2], the number $(2,n)$ can be written as $(2,n) = 2x + ny$ for some integers x and y . Applying this Bezout's result,

$$\begin{aligned}(2,n)Z_n &= (2x+ny)Z_n = 2xZ_n, \text{ since } ny \equiv 0(\bmod n)\\ &= (x+x)Z_n \subseteq Dup(Z_n).\end{aligned}$$

Two set inclusions $Dup(Z_n) \subseteq (2,n)Z_n$ and $(2,n)Z_n \subseteq Dup(Z_n)$ finalize that the duplicity of the ring Z_n as $Dup(Z_n) = (2,n)Z_n$.■

As an immediate application of preceding theorem, authors deduce the following results.

Corollary 3.6. $Dup(Z_n) = Z_n$ if and only if n is odd.

Proof. Noting that $(2,n) = 1$ if and only if n is odd, so may write $Dup(Z_n) \subseteq (2,n)Z_n = 1Z_n = Z_n$.■

In the same way, the relation $Dup(Z_n) \subseteq (2,n)Z_n$ yields the following corollary, and it is another basic fact regarding the order of the duplex ring $Dup(Z_n)$.

Corollary 3.7. Let $n \in N$. Then the cardinality of the duplex ring $Dup(Z_n)$ is $\left|Dup(Z_n)\right| = \frac{n}{(2,n)}$.

Proof. It is clear from the Theorem [section2], and additionally there is a one to one correspondence $(a+a) \mapsto \frac{n}{(2,n)}a$ for every element a in Z_n .■

Theorem 3.8. Let $m,n \in N$. Then $Dup(Z_m \times Z_n) = (2,m)(2,n)(Z_m \times Z_n)$.

Proof. By the Theorem [3.5], we have $Dup(Z_m) = (2,m)Z_m$ and $Dup(Z_n) = (2,n)Z_n$. So, it is clear from the calculations $Dup(Z_m \times Z_n) = Dup(Z_m) \times Dup(Z_n) = (2,m)Z_m \times (2,n)Z_n = (2,m)(2,n)(Z_m \times Z_n)$.■

There is an attractive illustration of the finite fields. First, notice that $Dup(Z_2) \neq Z_2$. For any odd prime, it is well known that $Dup(Z_p) = Z_p$. Particularly, if $p \equiv 3(\bmod 4)$ then $Z_p[i]$ is a field of Gaussian integers and $Dup\left(Z_p[i]\right) = Z_p[i]$. Even if R is not a field then there exists R such that $Dup(R) = R$. For instance, $Z_p \times Z_p$

is not a field but $Dup\left(Z_p \times Z_p\right) = Z_p \times Z_p$. Further, there is another attractive ring R with $Dup(R) = R$, those types of rings are called zero rings and it is denoted by $R = R^0$. Now, we show that $Dup\left(R^0\right) = R^0$.

Theorem 3.9. The duplex ring of any zero rings is itself a zero ring.

Proof. The theorem is certainly true for $\left|R^0\right| = 1$, because $\left|R^0\right| = 1$ if and only if $R^0 = (0)$. Thus we may hereafter restrict our attention to nontrivial zero ring $R^0 \neq (0)$. Let $\left|R^0\right| > 1$. Then we have to prove that $Dup\left(R^0\right) = R^0$. By virtue of the first theorem of this section, $Dup\left(R^0\right) \subseteq R^0$.we makes a start by showing that $R^0 \subseteq Dup\left(R^0\right)$. For a proof by contradiction, assume that R^0 Ú $Dup\left(R^0\right)$. Then the element a is in R^0 and the equation $a = x + x$ is not solvable in R^0. Accordingly, $x + x \neq a$ for some x is in R^0. Squaring on both sides of $x + x \neq a$, it gives $(x+x)^2 \neq a^2 \Rightarrow x^2 + x^2 + x^2 + x^2 \neq a^2 \Rightarrow 0 \neq 0$, it is not true in R^0, and thus our assumption is not true. Hence, $R^0 \subseteq Dup\left(R^0\right)$. So, we finish that $Dup\left(R^0\right) = R^0$.

4.Duplicity of Neutrosophic Rings

In this section, we establish duplex rings and their corresponding neutrosophic duplex rings. On the other hand, first we prove some results of this duplicity and which are useful for subsequent results as well as for the next concepts.

Now, this study is going to define duplicity of R and $R(I)$, and study their properties with different illustrations. We notice that $R + R \neq 2R$, $R + R + R \neq 3R$, and so on.

Definition 4.1. Let R be a finite commutative ring. Then the structure $Dup(R)$ is called duplex ring, and it is defined as $Dup(R) = \{a : x + x = a \text{ is solvable in } R\}$.

For any ring R, there is a neutrosophic duplex ring $Dup(R(I))$ of the neutrosophic ring $R(I)$, and it is defined as $Dup(R(I)) = \{\alpha : \beta + \beta = \alpha \text{ is solvable in } R(I)\}$, where $\alpha = a + Ib$ and $\beta = c + Id$ are neutrosophic elements in $R(I)$.

For example, $Dup(\mathrm{Z}_2(I)) = \{0 + 0I\}$, $Dup(\mathrm{Z}_5(I)) = \mathrm{Z}_5(I)$ but $Dup(\mathrm{Z}_5(i, I)) \neq \mathrm{Z}_5(i, I)$ where $\mathrm{Z}_5(i)$ is the ring of Gaussian integers and $\mathrm{Z}_5(i, I)$ is the neutrosophic ring of Gaussian integers.

The following is a basic result to the preceding analysis of duplicity.

Theorem 4.2. The duplicity of $R(I)$ is a neutrosophic subring of $R(I)$.

Proof. By the Theorem [3.3], $Dup(R)$ is a subring of R. Further, we have $R(I) = R + RI$, and therefore, $Dup(R(\mathrm{I})) = Dup(R) + Dup(R)I$. This relation explore that $Dup(R(I))$ is generated by $Dup(R)$ and I, and hence $Dup(R(I))$ is a neutrosophic subring of $R(I)$.■

Corollary 4.3. The duplicity of $R(I)$ is a neutrosophic ideal of $R(I)$.

Proof. It is clear from the observation that $Dup(R)$ is an ideal R, and thus $Dup(R(I))$ is a neutrosophic ideal of $R(I)$.■

The following examples are interesting illustrations of the preceding results. Here note that $Dup(Z_2(I)) = \{0+0I\}$.

Example 4.4. For any odd prime p, the neutrosophic duplex ring of $Z_p(I)$ is again $Z_p(I)$, that is $Dup(Z_p(I)) = Z_p(I)$.

Example 4.5. $Dup(Z_4(I)) = \{0, 2, 2I, 2+2I\}$, $Dup(Z_9(I)) = Z_9(I)$.

In this illustration we observed a connection of duplicity of rings and some fundamental concepts of rings. Under this observation, the following theorem provides a necessary and sufficient condition for the characteristic and duplicity of rings.

Theorem 4.6. Let $Char(R(I))$ be the characteristic of $R(I)$ with $R(I) \neq (0)$. Then, $Dup(R(\mathrm{I})) = (0)$ if and only if $Char(R(I)) = 2$.

Proof. It is well known that $|R| > 1$ if and only if $|R(I)| \geq 4$. So, we have $R \neq (0)$ if and only if $R(I) \neq (0)$, and additionally $Char(R) = Char(R(I))$. Thus we finish that

$$Dup(R(\mathrm{I})) = (0) \Leftrightarrow Dup(R) + Dup(R)I = (0) \Leftrightarrow Dup(R) = (0) \Leftrightarrow r + r = 0 \text{ is solvable in the ring } R$$
$$\Leftrightarrow 2r = 0 \text{ for every } r \text{ in } R \Leftrightarrow Char(R) = 2.■$$

The following example explores this theorem.

Example 4.7. $Dup(Z_2) = (0)$; $Dup(Z_2(I)) = (0)$, $Dup(Z_2[i]) = (0)$; $Dup(Z_2(i, I)) = (0)$,

$Dup(Z_2[x]) = (0)$; $Dup(Z_2(x, I)) = (0)$, where $Z_2[i]$ and $Z_2[x]$ are both rings of Gaussian integers and polynomials under addition and multiplication modulo 2, respectively.

The following theorem plays a significant role in characterizing finite neutrosophic rings and neutrosophic fields in terms of their corresponding duplicity of systems. Given a finite field F, there exists a neutrosophic field $F(I)$ with $F(I) = F + FI$. For instance, $Z_2(I)$, $Z_3(I)$, $Z_5(I)$ are all finite neutrosophic fields. Make a note of that $F(I) + F(I) \neq 2F(I)$.

Theorem 4.8. For any finite field F, the system $F(I) + F(I)$ is also equal to itself the neutrosophic field $F(I)$, where $F(I) + F(I)$ is defined as $F(I) + F(I) = \{\alpha + \alpha : \alpha \in F(I), \alpha \in F(I)\}$.

Proof. Because the element $x + 0$ in $F + F$, we have $x + 0 = x$, which is in F. This implies that $F + F \subseteq F$. To go the other way, let us suppose that $F \not\subset F + F$. Then, $x \in F$ implies that $x \notin F + F$. So, there is an element a in F such that $x \neq a + a$. It is not true for any finite field F, because the structure $(F, +)$ is an abelian group and the equation $x = a + a$ is solvable in $(F, +)$. Thus our point of view $F \subseteq F + F$ is also true. Hence, $F + F = F$. Suppose that F has the duplex form. We end up with the computations

$F(I) + F(I) = (F + FI) + (F + FI) = (F + F) + (IF + FI) = (F + F) + (F + F)\mathrm{I} = F + FI = F(I)$,where $IF = FI$.■

Corollary 4.9. $Dup(F) = F$ and $Dup(F(I)) = F(I)$ whenever $Char(F) \neq 2$.

Proof. It is simply proved from Theorem [above] and Theorem [above]. ■

The following is an example to the preceding analysis of the duplex of F and $F(I)$.

Example 4.10. $Z_2 + Z_2 = Z_2$ but $Dup(Z_2) \neq Z_2$. However, $Dup(Z_2) = (0)$.

$$Z_2(I) + Z_2(I) = Z_2(I) \text{ but } Dup(Z_2(I)) \neq Z_2(I). \text{ However, } Dup(Z_2(I)) = (0+0I).$$

With these results among our tools, we know that the necessary information to now carry out a proof of the fact that duplicity of a zero ring is again itself zero ring. For more information about zero rings, redder refer [24, 27]. A ring $R^0 = (R^0, +, \cdot)$ is called a zero ring if $ab = 0$ for every a and b in R^0. Every finite zero rings is commutative, and also zero ring is a ring without unity. For instance, the ring $R^0 = \{0, 5, 10, 15, 20\}$ is a finite commutative ring without unity under addition and multiplication modulo 25. Additionally, the authors Chalapathi and Madhavi introduced and studied the extended structure of zero rings, called, neutrosophic zero rings [24]. For any zero ring R^0, there exists corresponding neutrosophic zero ring $R^0(I)$, which is also commutative and without unity.

Theorem 4.11. Let R^0 be a finite zero ring. Then, $R^0(I) + R^0(I) = R^0(I)$.

Proof. The theorem is certainly true for $R^0 = (0)$, because $R^0 = (0)$ if and only if $R^0(I) = (0)$. Thus we may here after restrict our attention to nontrivial zero ring $R^0 \neq (0)$. Suppose $|R^0| > 1$ be the positive integer such that $|R^0(I)| \geq 4$. Then, first of all we prove that $R^0 + R^0 = R^0$ for any finite zero ring R^0. By virtue of addition of two rings, $R^0 + R^0 = \{a + a : a \in R^0\}$. The crux of our argument is that $R^0 + R^0$ is a subring of R^0, and this fact fallows that $R^0 + R^0 \subseteq R^0$. For a proof by a contradiction, assume that $R^0 \not\subset R^0 + R^0$ .For some a in R^0, there exists $x \in R^0$ such that $x \neq a + a$. Now squaring on both sides of $x \neq a + a$, we calculate

$$x^2 \neq (a+a)^2 \Rightarrow 0 \neq (a+a)^2, \text{ since } x^2 = 0 \Rightarrow a^2 + a^2 + (a+a) \neq 0$$

$$\Rightarrow a + a \neq 0, \text{ since } a^2 = 0$$

$$\Rightarrow a \neq -a.$$

This means that every element in a finite zero ring R^0 has not mutually additive inverse. This violates the basic condition of the zero rings [24], that means that every nonzero element in nontrivial zero ring has mutually additive inverse, giving us our contradiction. Thus, we have $R^0 \subseteq R^0 + R^0$, and hence $R^0 + R^0 = R^0$. Finally, the theorem follows the following calculations.

$$R^0(I) + R^0(I) = (R^0 + R^0 I) + (R^0 + R^0 I) = (R^0 + R^0) + (R^0 I + R^0 I) = (R^0 + R^0) + (R^0 + R^0) I$$

$$= R^0 + R^0 I = R^0(I). \blacksquare$$

Remark 4.12. From Theorem [4.11], $R^0(I)+R^0(I)=R^0(I)$ but $R^0(I)+R^0(I)=Dup\left(R^0(I)\right)$. This explores that the neutrosophic equation $\alpha+\alpha=\beta$ is solvable in $R^0(I)$ for every neutrosophic elements α and β in $R^0(I)$.

Corollary 4.13. For any zero rings R^0, we have $Dup\left(R^0(I)\right)=R^0(I)$.

Proof. It is full fill from the following calculations.

$$Dup\left(R^0(I)\right)=Dup\left(R^0\right)+Dup\left(R^0\right)I=R^0+R^0I=R^0(I).$$

5. Conclusions

In this paper, we have determined and counted all duplex elements in the finite cyclic ring Z_n. We have established that there is a number theoretic connection between the duplex function $D(n)$ and elements in Z_n, and also prove that $D(n)=\frac{n}{(2,n)}$. More importantly, we have shown that $D(Z_m\times Z_n)=\frac{mn}{(2,m)(2,n)}$. We have also discussed duplicity of finite rings and neutrosophic rings. A short discussion about how this duplex ring could be applied to the neutrosophic rings.

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