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A study on n-HyperSpherical Neutrosophic matrices

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Abstract. The n-HyperSpherical Neutrosophic matrices (n-HSNMs), an extension of the Spherical Neutrosophic matrices, are proposed in this study. We investigate the basic properties of n-HSNMs and compare the idea n-HSNMs with spherical fuzzy matrices. Then, it looks at the characteristics of specific mathematical operations, like max-min, algebraic product, min-max, algebraic sum, and complement. Additionally, scalar multiplication ($n\mathcal{S}$)and exponentiation ($\mathcal{S}^n$)operations of an n-HSNM $\mathcal{S}$ are created and their advantageous properties are illustrated using algebraic operations. Then, we present a new operation (@) on n-HyperSpherical Neutrosophic matrices and look at the distributional rules that result from combining the operations ($\oplus$, $\otimes$, $\wedge$, and $\vee$).

Keywords: Neutrosophic sets; Spherical fuzzy matrix; n-HyperSpherical Neutrosophic matrix; Algebraic sum and product.

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1. Introduction

Khan et al. [5] and Im et al. [4] both established the notion of an intuitionistic fuzzy matrix (IFM) to broaden the idea of Thomason's [11] fuzzy matrix. Every element in an IFM is represented by $\langle \mu_{a_{ij}}, \nu_{a_{ij}} \rangle$ along with $\mu_{a_{ij}}, \nu_{a_{ij}} \in [0,1]$ and also $0 \leq \mu_{a_{ij}} + \nu_{a_{ij}} \leq 1$. As the introduction of IFM, a limited number of analysts have made major contributions to the advancement of the IFM hypothesis [3, 7, 8]. In such a situation, IFM fails to produce a reasonable solution. In order to deal with this situation, we invented the notion of Pythagorean fuzzy matrices (PyFM) by assigning membership degrees such as $\zeta_{a_{ij}}$ and non-membership degrees such as $\zeta_{a_{ij}}$, with the requirement that $0 \leq \zeta^2_{a_{ij}} + \delta^2_{a_{ij}} \leq 1$.

The design of picture fuzzy matrices (PFM) by Dogra and Pal cite1 is well-known, however decision makers are limited in assigning values because to the conditions on $\eta_{a_{ij}}$, $\zeta_{a_{ij}}$, and $\delta_{a_{ij}}$. In [9], certain algebraic procedures on Picture fuzzy matrices are defined, as well as their desired features. Occasionally, the total of their membership degrees is more than 1. In such a case, PFM fails to produce a plausible result. To illustrate this dilemma, we'll use an example that is both provisional and contradictory to membership degrees. $0.2, 0.6$ and 0.6, respectively, are the choices. This is satisfying in the circumstance where their total is more than 1, and PFM fails to handle such data. In order to deal with such situations, the authors [10] developed a new structure of Spherical fuzzy matrices (SFMs), which increase the degree memberships $\eta_{a_{ij}}$, $\zeta_{a_{ij}}$ and $\delta_{a_{ij}}$ to a size that is somewhat larger than image fuzzy matrices. In SFM, the degree memberships are fulfilling the follows: $0 \leq \zeta^2_{a_{ij}} + \eta^2_{a_{ij}} + \delta^2_{a_{ij}} \leq 1$ $(n \geq 1)$.

Matrixes have a vital role in science and technology, as we all know. However, in some cases, the conventional matrix theory fails to answer problems with uncertainties that arise in an uncertain environment. In [6], fuzzy and neutrosophic relational maps were presented. Square Neutrosophic Fuzzy Matrices with elements of $a + Ib$ type, where a and b are fuzzy numbers from $[0, 1]$, are characterized by Dhar, Broumi, and Smarandache [2].

In this work, we extend the ideas of Spherical Neutrosophic matrices to n-Hyper Spherical Neutrosophic matrix by assigning neutral membership degree say $\eta_{a_{ij}}$ together with positive and negative participation measures say $\zeta_{a_{ij}}$ and $\delta_{a_{ij}}$ with condition that $0 \leq \zeta^n_{a_{ij}} + \eta^n_{a_{ij}} + \delta^n_{a_{ij}} \leq 3$ $(n \geq 1)$.

The following is structure of this work. In Section 2, n-Hyper Spherical Neutrosophic matrices are characterized, as well as their algebraic operations and desired features. In Section 3, we define and study the algebraic characteristics of a new operation(@) on n-Hyper Spherical Neutrosophic matrices. The results are relevant in Section 4, n-Hyper Spherical Neutrosophic matrix and algebraic structure on this matrix. In Section 5, where we compose the paper's conclusion.

Definition 1.1. A Pythagorean fuzzy matrix (PFM) of order $m \times n$ is characterized as $\mathcal{S} = (\langle \zeta_{a_{ij}}, \delta_{a_{ij}} \rangle)$ where $\zeta_{a_{ij}} \in [0, 1]$ and $\delta_{a_{ij}} \in [0, 1]$ whether the membership and non-membership values of the ij^{th} element in $\mathcal{S}$ fulfilling the requirement

$$0 \leq \zeta^n_{a_{ij}} + \delta^n_{a_{ij}} \leq 1, \ \forall \ i, j.$$

Definition 1.2. [9] A Picture fuzzy matrix (PFM) $\mathcal{S}$ with the formula, $\mathcal{S} = \left(\left\langle \zeta_{a_{ij}}, \eta_{a_{ij}}, \delta_{a_{ij}} \right\rangle\right)$ and non-negative real integers $\zeta_{a_{ij}}, \eta_{a_{ij}}, \delta_{a_{ij}} \in [0, 1]$ fulfilling the requirement

$$0 \leq \zeta_{a_{ij}} + \eta_{a_{ij}} + \delta_{a_{ij}} \leq 1, \ \forall \ i, j,$$

where $\zeta_{a_{ij}} \in [0, 1]$, $\eta_{a_{ij}} \in [0, 1]$ and $\delta_{a_{ij}} \in [0, 1]$ represent the degree of membership, degree of neutral membership, and degree of non-membership respectively.

Definition 1.3. [10] A Spherical fuzzy matrix $\mathcal{S}$ with the formula, $\mathcal{S} = \left(\left\langle \zeta_{a_{ij}}, \eta_{a_{ij}}, \delta_{a_{ij}} \right\rangle\right)$ of a non-negative real integers $\zeta_{a_{ij}}, \eta_{a_{ij}}, \delta_{a_{ij}} \in [0, 1]$ fulfilling the requirement

$$0 \leq \zeta^2_{a_{ij}} + \eta^2_{a_{ij}} + \delta^2_{a_{ij}} \leq 1, \ \forall \ i, j,$$

where $\zeta_{a_{ij}} \in [0, 1]$, $\eta_{a_{ij}} \in [0, 1]$ and $\delta_{a_{ij}} \in [0, 1]$ represent the degree of membership, the degree of non-membership.

2. n-Hyper Spherical Neutrosophic matrices and their basic operations

The n-HyperSpherical Neutrosophic matrix and its algebraic operations are characterized in this section and also demonstrated De Morgan's rules over complement, commutativity, Idempotency, absorption law, distributivity, and associativity.

Now we'll describe Algebraic operations of n-HyperSpherical Neutrosophic matrices by limiting the measure of negative membership, neutral membership, and positive membership while retaining their total in the range $[0, \sqrt[n]{3}]$.

In $[5, 6, 9, 10]$, we employ some basic notations to arrive at our main findings.

Definition 2.1. A n-Hyper Spherical Neutrosophic matrix (n-HSNM) M of the form, $M = \left(\left\langle \zeta_{a_{ij}}, \eta_{a_{ij}}, \delta_{a_{ij}} \right\rangle\right)$ of a non negative real numbers $\zeta_{a_{ij}}, \eta_{a_{ij}}, \delta_{a_{ij}} \in [0, 1]$ fulfilling the requirement

$$0 \leq \zeta^n_{a_{ij}} + \eta^n_{a_{ij}} + \delta^n_{a_{ij}} \leq 3, \ \forall \ i, j,$$

where $\zeta_{a_{ij}} \in [0, 1]$, $\eta_{a_{ij}} \in [0, 1]$, and $\delta_{a_{ij}} \in [0, 1]$ represent the degree of membership, the degree of neutral membership, and the degree of non-membership.

The n-HyperSpherical Fuzzy matrix (n-HSFM) is a specific instance of the Neutrosophic matrix (NFM). Because, $\zeta_{a_{ij}}, \eta_{a_{ij}}$, and $\delta_{a_{ij}} \in [0, 1]$ imply that one also has $\zeta^n_{a_{ij}}, \eta^n_{a_{ij}}$, and $\delta^n_{a_{ij}} \in [0, 1]$ for $n \geq 1$, they are neutrosophic components and each n-HSFS is a NM. The reciprocal, however, is false because if at least one component has a value of 1 and at least one of the other two components has a value of > 0, as in the case of $\zeta_{a_{ij}} = 1$ and $\eta_{a_{ij}} > 0, \delta_{a_{ij}} \in [0, 1]$, then $\zeta^n_{a_{ij}} + \eta^n_{a_{ij}} + \delta^n_{a_{ij}} > 1$ for $(n \geq 1)$. The number of triplets ζ, η, δ that are NFM components, but not n-HSFM components, is infinite.

When the neutrosophic components $\zeta_{a_{ij}} = 0.9$, $\eta_{a_{ij}} = 0.4$, $\delta_{a_{ij}} = 0.5$, for some given ij^{th} element are used, they are not considered to be spherical fuzzy matrix components since $(0.9)^2 + (0.4)^2 + (0.5)^2 = 1.22 > 1$. For $\zeta_{a_{ij}}, \eta_{a_{ij}}$ and $\delta_{a_{ij}} \in [0,1]$, there exist infinitely many values whose sum of squares is strictly bigger than 1, hence they are neutrosophic components rather than spherical fuzzy matrix components.

Let $N_{m\times n}$ represents the collection of all the n-Hyper Spherical Neutrosophic matrices.

Definition 2.2. The n-Hyper Spherical Neutrosophic matrices $\mathcal{S}$ and $\mathcal{T}$ are of the form, $\mathcal{S} = \left(\left\langle \zeta_{a_{ij}}, \eta_{a_{ij}}, \delta_{a_{ij}} \right\rangle\right)$ and $\mathcal{T} = \left(\left\langle \zeta_{b_{ij}}, \eta_{b_{ij}}, \delta_{b_{ij}} \right\rangle\right)$. Then

- $\mathcal{S} < \mathcal{T}$ iff $\forall i, j, \zeta_{a_{ij}} \leq \zeta_{b_{ij}}, \eta_{a_{ij}} \leq \eta_{b_{ij}}$ or $\eta_{a_{ij}} \geq \eta_{b_{ij}}, \delta_{a_{ij}} \geq \delta_{b_{ij}}$.
- $\mathcal{S}^C = \left(\left\langle \delta_{a_{ij}}, \eta_{a_{ij}}, \zeta_{a_{ij}} \right\rangle\right)$.
- $\mathcal{S} \wedge \mathcal{T} = \left(\left\langle \min\left(\zeta_{a_{ij}}, \zeta_{b_{ij}}\right), \min\left(\eta_{a_{ij}}, \eta_{b_{ij}}\right), \max\left(\delta_{a_{ij}}, \delta_{b_{ij}}\right) \right\rangle\right)$.
- $\mathcal{S} \vee \mathcal{T} = \left(\left\langle \max\left(\zeta_{a_{ij}}, \zeta_{b_{ij}}\right), \min\left(\eta_{a_{ij}}, \eta_{b_{ij}}\right), \min\left(\delta_{a_{ij}}, \delta_{b_{ij}}\right) \right\rangle\right)$.
- $\mathcal{S} \otimes \mathcal{T} = \left(\left\langle \zeta_{a_{ij}}\zeta_{b_{ij}}, \sqrt[n]{\eta^n_{a_{ij}} + \eta^n_{b_{ij}} - \eta^n_{a_{ij}}\eta^n_{b_{ij}}}, \sqrt[n]{\delta^n_{a_{ij}} + \delta^n_{b_{ij}} - \delta^n_{a_{ij}}\delta^n_{b_{ij}}} \right\rangle\right)$.
- $\mathcal{S} \oplus \mathcal{T} = \left(\left\langle \sqrt[n]{\zeta^n_{a_{ij}} + \zeta^n_{b_{ij}} - \zeta^n_{a_{ij}}\zeta^n_{b_{ij}}}, \eta_{a_{ij}}\eta_{b_{ij}}, \delta_{a_{ij}}\delta_{b_{ij}} \right\rangle\right)$.

Definition 2.3. The scalar multiplication operation over n-HSNM $\mathcal{S}$ and is characterized by

$$n\mathcal{S} = \left(\left\langle \sqrt[n]{1 - [1 - \zeta^n_{a_{ij}}]^n}, [\eta_{a_{ij}}]^n, [\delta_{a_{ij}}]^n \right\rangle\right).$$

Definition 2.4. The exponentiation operation over n-HSNM $\mathcal{S}$ and is characterized by

$$\mathcal{S}^n = \left(\left\langle [\zeta_{a_{ij}}]^n, \sqrt[n]{1 - [1 - \eta^n_{a_{ij}}]^n}, \sqrt[n]{1 - [1 - \delta^n_{a_{ij}}]^n} \right\rangle\right).$$

Let $N_{m\times n}$ represents the collection of all the n-Hyper Spherical Neutrosophic matrices.

The algebraic product and algebraic sum of n-HSNMs' are connected by the embracing theorem.

Theorem 2.5. *For $\mathcal{S}, \mathcal{T} \in N_{m\times n}$, then $\mathcal{S} \otimes \mathcal{T} \leq \mathcal{S} \oplus \mathcal{T}$.*

Proof. Let

$$\begin{aligned} \mathcal{S} \oplus \mathcal{T} &= \left(\left\langle \sqrt[n]{\zeta^n_{a_{ij}} + \zeta^n_{b_{ij}} - \zeta^n_{a_{ij}}\zeta^n_{b_{ij}}}, \eta_{a_{ij}}\eta_{b_{ij}}, \delta_{a_{ij}}\delta_{b_{ij}} \right\rangle\right) \\ &\text{and} \\ \mathcal{S} \otimes \mathcal{T} &= \left(\left\langle \zeta_{a_{ij}}\zeta_{b_{ij}}, \sqrt[n]{\eta^n_{a_{ij}} + \eta^n_{b_{ij}} - \eta^n_{a_{ij}}\eta^n_{b_{ij}}}, \sqrt[n]{\delta^n_{a_{ij}} + \delta^n_{b_{ij}} - \delta^n_{a_{ij}}\delta^n_{b_{ij}}} \right\rangle\right). \end{aligned}$$

Assume that,

$$\begin{aligned} \zeta_{a_{ij}}\zeta_{b_{ij}} &\leq \sqrt[n]{\zeta^n_{a_{ij}} + \zeta^n_{b_{ij}} - \zeta^n_{a_{ij}}\zeta^n_{b_{ij}}} \\ (i.e)\zeta_{a_{ij}}\zeta_{b_{ij}} - \sqrt[n]{\zeta^n_{a_{ij}} + \zeta^n_{b_{ij}} - \zeta^n_{a_{ij}}\zeta^n_{b_{ij}}} &\geq 0 \\ (i.e)\zeta^n_{a_{ij}}(1 - \zeta^n_{b_{ij}}) + \zeta^n_{b_{ij}}(1 - \zeta^n_{a_{ij}}) &\geq 0 \end{aligned}$$

which is true as $0 \leq \zeta^n_{a_{ij}} \leq 1$ and $0 \leq \zeta^n_{b_{ij}} \leq 1$ and

$$\begin{aligned} \eta_{a_{ij}}\eta_{b_{ij}} &\leq \sqrt[n]{\eta^n_{a_{ij}} + \eta^n_{b_{ij}} - \eta^n_{a_{ij}}\eta^n_{b_{ij}}} \\ (i.e)\eta_{a_{ij}}\eta_{b_{ij}} - \sqrt[n]{\eta^n_{a_{ij}} + \eta^n_{b_{ij}} - \eta^n_{a_{ij}}\eta^n_{b_{ij}}} &\geq 0 \\ (i.e)\eta^n_{a_{ij}}(1-\eta^n_{b_{ij}}) + \eta^n_{b_{ij}}(1-\eta^n_{a_{ij}}) &\geq 0 \end{aligned}$$

which is true as $0 \leq \eta^n_{a_{ij}} \leq 1$ and $0 \leq \eta^n_{b_{ij}} \leq 1$, and

$$\begin{aligned} \delta_{a_{ij}}\delta_{b_{ij}} &\leq \sqrt[n]{\delta^n_{a_{ij}} + \delta^n_{b_{ij}} - \delta^n_{a_{ij}}\delta^n_{b_{ij}}} \\ (i.e)\delta_{a_{ij}}\delta_{b_{ij}} - \sqrt[n]{\delta^n_{a_{ij}} + \delta^n_{b_{ij}} - \delta^n_{a_{ij}}\delta^n_{b_{ij}}} &\geq 0 \\ (i.e)\delta^n_{a_{ij}}(1-\delta^n_{b_{ij}}) + \delta^n_{b_{ij}}(1-\delta^n_{a_{ij}}) &\geq 0, \end{aligned}$$

which is true as

$$0 \leq \delta^n_{a_{ij}} \leq 1$$

and

$$0 \leq \delta^n_{b_{ij}} \leq 1.$$

Hence, $\mathcal{S} \otimes \mathcal{T} \leq \mathcal{S} \oplus \mathcal{T}$. □

Theorem 2.6. *For any n-Hyper Spherical Neutrosophic matrix p, then*

(i) $\mathcal{S} \oplus \mathcal{S} \geq \mathcal{S}$.

(ii) $\mathcal{S} \otimes \mathcal{S} \leq \mathcal{S}$.

Proof. (i) Let

$$\begin{aligned} \mathcal{S} \oplus \mathcal{S} &= (\langle \zeta_{a_{ij}}, \eta_{a_{ij}}, \delta_{a_{ij}} \rangle) \oplus (\langle \zeta_{a_{ij}}, \eta_{a_{ij}}, \delta_{a_{ij}} \rangle) \\ \mathcal{S} \oplus \mathcal{S} &= \left(\left\langle \sqrt[n]{2\zeta_{a_{ij}} - (\zeta_{a_{ij}})^n}, (\eta_{a_{ij}})^n, (\delta_{a_{ij}})^n \right\rangle\right) \\ \sqrt[n]{2\zeta_{a_{ij}} - (\zeta_{a_{ij}})^n} &= \sqrt[n]{\zeta_{a_{ij}} + \zeta_{a_{ij}}(1-\zeta_{a_{ij}})} \geq \zeta_{a_{ij}}, \quad \forall\ i,j \end{aligned}$$

and

$$\begin{aligned} (\eta_{a_{ij}})^n &\leq \eta_{a_{ij}}, \quad \forall\ i,j \\ (\delta_{a_{ij}})^n &\leq \delta_{a_{ij}}, \quad \forall\ i,j. \end{aligned}$$

Thus, $\mathcal{S} \oplus \mathcal{S} \geq \mathcal{S}$. Similarly, we can also demonstrate that (ii) $\mathcal{S} \otimes \mathcal{S} \leq \mathcal{S}$. □

Theorem 2.7. *For* $\mathcal{S}, \mathcal{T}, \mathcal{U} \in N_{m\times n}$, *then*

(i) $\mathcal{S} \oplus \mathcal{T} = \mathcal{T} \oplus \mathcal{S}$.

(ii) $\mathcal{S} \otimes \mathcal{T} = \mathcal{T} \otimes \mathcal{S}$.

(iii) $(\mathcal{S} \oplus \mathcal{T}) \oplus \mathcal{U} = \mathcal{S} \oplus (\mathcal{T} \oplus \mathcal{U})$.

(iv) $(\mathcal{S} \otimes \mathcal{T}) \otimes \mathcal{U} = \mathcal{S} \otimes (\mathcal{T} \otimes \mathcal{U})$.

Proof. (i) Let

$$\begin{aligned} \mathcal{S} \oplus \mathcal{T} &= \left(\left\langle \sqrt[n]{\zeta_{a_{ij}}^n + \zeta_{b_{ij}}^n - \zeta_{a_{ij}}^n \zeta_{b_{ij}}^n}, \eta_{a_{ij}}\eta_{b_{ij}}, \delta_{a_{ij}}\delta_{b_{ij}} \right\rangle\right) \\ &= \left(\left\langle \sqrt[n]{\zeta_{b_{ij}}^n + \zeta_{a_{ij}}^n - \zeta_{b_{ij}}^n \zeta_{a_{ij}}^n}, \eta_{b_{ij}}\eta_{a_{ij}}, \delta_{b_{ij}}\delta_{a_{ij}} \right\rangle\right) \\ &= \mathcal{T} \oplus \mathcal{S}. \end{aligned}$$

(ii) Let

$$\begin{aligned} \mathcal{S} \otimes \mathcal{T} &= \left(\left\langle \zeta_{a_{ij}}\zeta_{b_{ij}}, \sqrt[n]{\eta_{a_{ij}}^n + \eta_{b_{ij}}^n - \eta_{a_{ij}}^n \eta_{b_{ij}}^n}, \sqrt[n]{\delta_{a_{ij}}^n + \delta_{b_{ij}}^n - \delta_{a_{ij}}^n \delta_{b_{ij}}^n} \right\rangle\right) \\ &= \left(\left\langle \zeta_{b_{ij}}\zeta_{a_{ij}}, \sqrt[n]{\eta_{b_{ij}}^n + \eta_{a_{ij}}^n - \eta_{b_{ij}}^n \eta_{a_{ij}}^n}, \sqrt[n]{\delta_{b_{ij}}^n + \delta_{a_{ij}}^n - \delta_{b_{ij}}^n \delta_{a_{ij}}^n} \right\rangle\right) \\ &= \mathcal{T} \otimes \mathcal{S}. \end{aligned}$$

(iii) Let

$$\begin{aligned} (\mathcal{S} \oplus \mathcal{T}) \oplus \mathcal{U} &= \left(\left\langle \left(\sqrt[n]{\zeta_{a_{ij}}^n + \zeta_{b_{ij}}^n - \zeta_{a_{ij}}^n \zeta_{b_{ij}}^n}, \eta_{a_{ij}}\eta_{b_{ij}}, \delta_{a_{ij}}\delta_{b_{ij}} \right) \oplus \left(\zeta_{c_{ij}}, \eta_{c_{ij}}, \delta_{c_{ij}}\right) \right\rangle\right) \\ &= \Bigg[\sqrt[n]{\left(\sqrt[n]{\zeta_{a_{ij}}^n + \zeta_{b_{ij}}^n - \zeta_{a_{ij}}^n \zeta_{b_{ij}}^n} \right)^n + \zeta_{c_{ij}}^n - \left(\sqrt[n]{\zeta_{a_{ij}}^n + \zeta_{b_{ij}}^n - \zeta_{a_{ij}}^n \zeta_{b_{ij}}^n} \right)^n \zeta_{c_{ij}}^n}, \\ &\quad \eta_{a_{ij}}\eta_{b_{ij}}\eta_{c_{ij}}, \delta_{a_{ij}}\delta_{b_{ij}}\delta_{c_{ij}} \Bigg] \\ &= \Bigg[\sqrt[n]{\zeta_{a_{ij}}^n + \zeta_{b_{ij}}^n + \zeta_{c_{ij}}^n - \zeta_{a_{ij}}^n \zeta_{b_{ij}}^n \zeta_{c_{ij}}^n - \zeta_{a_{ij}}^n \zeta_{c_{ij}}^n - \zeta_{b_{ij}}^n \zeta_{c_{ij}}^n + \zeta_{a_{ij}}^n \zeta_{b_{ij}}^n \zeta_{c_{ij}}^n}, \\ &\quad \eta_{a_{ij}}\eta_{b_{ij}}\eta_{c_{ij}}, \delta_{a_{ij}}\delta_{b_{ij}}\delta_{c_{ij}} \Bigg] \\ &= \Bigg[\sqrt[n]{\zeta_{a_{ij}}^n + \zeta_{b_{ij}}^n + \zeta_{c_{ij}}^n - \zeta_{a_{ij}}^n \zeta_{b_{ij}}^n - \zeta_{a_{ij}}^n \zeta_{c_{ij}}^n - \zeta_{b_{ij}}^n \zeta_{c_{ij}}^n + \zeta_{a_{ij}}^n \zeta_{b_{ij}}^n \zeta_{c_{ij}}^n}, \\ &\quad \eta_{a_{ij}}\eta_{b_{ij}}\eta_{c_{ij}}, \delta_{a_{ij}}\delta_{b_{ij}}\delta_{c_{ij}} \Bigg]. \end{aligned}$$

Let us assume that

$$\begin{aligned} \mathcal{S} \oplus (\mathcal{T} \oplus \mathcal{U}) &= \Bigg[\sqrt[n]{\zeta_{a_{ij}}^n + \left(\sqrt[n]{\zeta_{b_{ij}}^n + \zeta_{c_{ij}}^n - \zeta_{b_{ij}}^n \zeta_{c_{ij}}^n} \right)^n - \zeta_{a_{ij}}^n \left(\sqrt[n]{\zeta_{b_{ij}}^n + \zeta_{c_{ij}}^n - \zeta_{b_{ij}}^n \zeta_{c_{ij}}^n} \right)^n}, \\ &\quad \eta_{a_{ij}}\eta_{b_{ij}}\eta_{c_{ij}}, \delta_{a_{ij}}\delta_{b_{ij}}\delta_{c_{ij}} \Bigg] \\ &= \Bigg[\sqrt[n]{\zeta_{a_{ij}}^n + \zeta_{b_{ij}}^n + \zeta_{c_{ij}}^n - \zeta_{a_{ij}}^n \zeta_{b_{ij}}^n - \zeta_{a_{ij}}^n \zeta_{c_{ij}}^n - \zeta_{b_{ij}}^n \zeta_{c_{ij}}^n + \zeta_{a_{ij}}^n \zeta_{b_{ij}}^n \zeta_{c_{ij}}^n}, \\ &\quad \eta_{a_{ij}}\eta_{b_{ij}}\eta_{c_{ij}}, \delta_{a_{ij}}\delta_{b_{ij}}\delta_{c_{ij}} \Bigg]. \end{aligned}$$

Thus, $(\mathcal{S} \oplus \mathcal{T}) \oplus \mathcal{U} = \mathcal{S} \oplus (\mathcal{T} \oplus \mathcal{U})$. Similarly, we can also demonstrate that (iv) $(\mathcal{S} \otimes \mathcal{T}) \otimes \mathcal{U} = \mathcal{S} \otimes (\mathcal{T} \otimes \mathcal{U})$. □

Theorem 2.8. *For $\mathcal{S}, \mathcal{T} \in N_{m \times n}$, then*

(i) $\mathcal{S} \oplus (\mathcal{S} \otimes \mathcal{T}) \geq \mathcal{S}$.

(ii) $\mathcal{S} \otimes (\mathcal{S} \oplus \mathcal{T}) \leq \mathcal{S}$.

Proof. (i) Let

$$\begin{aligned}
\mathcal{S} \oplus (\mathcal{S} \otimes \mathcal{T}) &= (\langle \zeta_{a_{ij}}, \eta_{a_{ij}}, \delta_{a_{ij}} \rangle) \oplus \left(\left\langle \zeta_{a_{ij}} \zeta_{b_{ij}},\, \sqrt[n]{\eta^n_{a_{ij}} + \eta^n_{b_{ij}} - \eta^n_{a_{ij}} \eta^n_{b_{ij}}},\, \sqrt[n]{\delta^n_{a_{ij}} + \delta^n_{b_{ij}} - \delta^n_{a_{ij}} \delta^n_{b_{ij}}} \right\rangle \right) \\
&= \Big[\sqrt[n]{\zeta^n_{a_{ij}} + \zeta^n_{a_{ij}} \zeta^n_{b_{ij}} - \zeta^n_{a_{ij}} [\zeta^n_{a_{ij}} \zeta^n_{b_{ij}}]}, \eta_{a_{ij}} [\sqrt[n]{\eta^n_{a_{ij}} + \eta^n_{b_{ij}} - \eta^n_{a_{ij}} \eta^n_{b_{ij}}}], \\
&\quad \delta_{a_{ij}} [\sqrt[n]{\delta^n_{a_{ij}} + \delta^n_{b_{ij}} - \delta^n_{a_{ij}} \delta^n_{b_{ij}}}] \Big] \\
&= \Big[\sqrt[n]{\zeta^n_{a_{ij}} + \zeta^n_{a_{ij}} \zeta^n_{b_{ij}} [1 - \zeta^n_{a_{ij}}]}, \eta_{a_{ij}} \left(\sqrt[n]{1 - [1 - \eta^n_{a_{ij}}][1 - \eta^n_{b_{ij}}]} \right), \\
&\quad \delta_{a_{ij}} \left(\sqrt[n]{1 - [1 - \delta^n_{a_{ij}}][1 - \delta^n_{b_{ij}}]} \right) \Big] \\
&\geq \mathcal{S}.
\end{aligned}$$

Hence $\mathcal{S} \oplus (\mathcal{S} \otimes \mathcal{T}) \geq \mathcal{S}$. Similarly, we can also demonstrate that $(ii)\mathcal{S} \otimes (\mathcal{S} \oplus \mathcal{T}) \leq \mathcal{S}$. □

The theorem that follows is self-evident.

Theorem 2.9. *For $\mathcal{S}, \mathcal{T} \in N_{m \times n}$, then*

(i) $\mathcal{S} \vee \mathcal{T} = \mathcal{T} \vee \mathcal{S}$.

(ii) $\mathcal{S} \wedge \mathcal{T} = \mathcal{T} \wedge \mathcal{S}$.

Theorem 2.10. *For $\mathcal{S}, \mathcal{T}, \mathcal{U} \in N_{m \times n}$, then*

(i) $\mathcal{S} \oplus (\mathcal{T} \vee \mathcal{U}) = (\mathcal{S} \oplus \mathcal{T}) \vee (\mathcal{S} \oplus \mathcal{U})$.

(ii) $\mathcal{S} \otimes (\mathcal{T} \vee \mathcal{U}) = (\mathcal{S} \otimes \mathcal{T}) \vee (\mathcal{S} \otimes \mathcal{U})$.

(iii) $\mathcal{S} \oplus (\mathcal{T} \wedge \mathcal{U}) = (\mathcal{S} \oplus \mathcal{T}) \wedge (\mathcal{S} \oplus \mathcal{U})$.

(iv) $\mathcal{S} \otimes (\mathcal{T} \wedge \mathcal{U}) = (\mathcal{S} \otimes \mathcal{T}) \wedge (\mathcal{S} \otimes \mathcal{U})$.

Proof. We'll start by proving (i) and$(ii) - (iv)$ may be demonstrated similarly.
(i) Let

$$\begin{aligned}
\mathcal{S} \oplus (\mathcal{T} \vee \mathcal{U}) &= \Big[\sqrt[n]{\zeta^n_{a_{ij}} + \max\left(\zeta^n_{b_{ij}}, \zeta^n_{c_{ij}}\right) - \zeta^n_{a_{ij}}.\max\left(\zeta^n_{b_{ij}}, \zeta^n_{c_{ij}}\right)}, \\
&\quad \eta_{a_{ij}}.\max\left(\eta_{b_{ij}}, \eta_{c_{ij}}\right), \delta_{a_{ij}}.\max\left(\delta_{b_{ij}}, \delta_{c_{ij}}\right) \Big] \\
&= \Big[\sqrt[n]{\max\left(\zeta^n_{a_{ij}} + \zeta^n_{b_{ij}}, \zeta_{a^n_{ij}} + \zeta^n_{c_{ij}}\right) - \max\left(\zeta^n_{a_{ij}} \zeta^n_{b_{ij}}, \zeta^n_{a_{ij}} \zeta^n_{c_{ij}}\right)}, \\
&\quad \min\left(\eta_{a_{ij}} \eta_{b_{ij}}, \eta_{a_{ij}} \eta_{c_{ij}}\right), \min\left(\delta_{a_{ij}} \delta_{b_{ij}}, \delta_{a_{ij}} \delta_{c_{ij}}\right) \Big] \\
&= \Big[\sqrt[n]{\max\left(\zeta^n_{a_{ij}} + \zeta^n_{b_{ij}} - \zeta^n_{a_{ij}} \zeta^n_{b_{ij}}, \zeta^n_{a_{ij}} + \zeta^n_{c_{ij}} - \zeta^n_{a_{ij}} \zeta^n_{c_{ij}}\right)}, \\
&\quad \min\left(\eta_{a_{ij}} \eta_{b_{ij}}, \eta_{a_{ij}} \eta_{c_{ij}}\right), \min\left(\delta_{a_{ij}} \delta_{b_{ij}}, \delta_{a_{ij}} \delta_{c_{ij}}\right) \Big] \\
&= (\mathcal{S} \oplus \mathcal{T}) \vee (\mathcal{S} \oplus \mathcal{U}).
\end{aligned}$$

□

Theorem 2.11. *For $\mathcal{S}$, $\mathcal{T} \in N_{m\times n}$, then*

(i) $(\mathcal{S} \wedge \mathcal{T}) \oplus (\mathcal{S} \vee \mathcal{T}) = \mathcal{S} \oplus \mathcal{T}$.

(ii) $(\mathcal{S} \wedge \mathcal{T}) \otimes (\mathcal{S} \vee \mathcal{T}) = \mathcal{S} \otimes \mathcal{T}$.

(iii) $(\mathcal{S} \oplus \mathcal{T}) \wedge (\mathcal{S} \otimes \mathcal{T}) = \mathcal{S} \otimes \mathcal{T}$.

(iv) $(\mathcal{S} \oplus \mathcal{T}) \vee (\mathcal{S} \otimes \mathcal{T}) = \mathcal{S} \oplus \mathcal{T}$.

Proof. We'll start by demonstrating (i), and $(ii) - (iv)$ may be demonstrated similarly.
(i) Let

$$\begin{aligned}
(\mathcal{S} \wedge \mathcal{T}) \oplus (\mathcal{S} \vee \mathcal{T}) &= \Bigg[\sqrt[n]{\min\left(\zeta^n_{a_{ij}}, \zeta^n_{b_{ij}}\right) + \max\left(\zeta^n_{a_{ij}}, \zeta^n_{b_{ij}}\right) - \min\left(\zeta^n_{a_{ij}}, \zeta^n_{b_{ij}}\right) . \max\left(\zeta^n_{a_{ij}}, \zeta^n_{b_{ij}}\right)}, \\
&\quad \max\left(\eta_{a_{ij}}, \eta_{b_{ij}}\right) . \min\left(\eta_{a_{ij}}, \eta_{b_{ij}}\right), \quad \max\left(\delta_{a_{ij}}, \delta_{b_{ij}}\right) . \min\left(\delta_{a_{ij}}, \delta_{b_{ij}}\right)\Bigg] \\
&= \left(\left\langle \sqrt[n]{\zeta^n_{a_{ij}} + \zeta^n_{b_{ij}} - \zeta^n_{a_{ij}}\zeta^n_{b_{ij}}}, \eta_{a_{ij}}\eta_{b_{ij}}, \delta_{a_{ij}}\delta_{b_{ij}} \right\rangle\right) \\
&= \mathcal{S} \oplus \mathcal{T}.
\end{aligned}$$

□

The operator complement obeys De Morgan's principles in the following theorems for the operation $\oplus, \otimes, \vee, \wedge$.

Theorem 2.12. *For $\mathcal{S}, \mathcal{T} \in N_{m\times n}$, then*

(i) $(\mathcal{S} \oplus \mathcal{T})^C = \mathcal{S}^C \otimes \mathcal{T}^C$.

(ii) $(\mathcal{S} \otimes \mathcal{T})^C = \mathcal{S}^C \oplus \mathcal{T}^C$.

(iii) $(\mathcal{S} \oplus \mathcal{T})^C \leq \mathcal{S}^C \oplus \mathcal{T}^C$.

(iv) $(\mathcal{S} \otimes \mathcal{T})^C \geq \mathcal{S}^C \otimes \mathcal{T}^C$.

Proof. We'll show that $(iii), (iv)$, and $(i), (ii)$ are simple.
(iii) Let

$$\begin{aligned}
(\mathcal{S} \oplus \mathcal{T})^C &= \left(\left\langle \delta_{a_{ij}}\delta_{b_{ij}}, \sqrt[n]{\eta^n_{a_{ij}} + \eta^n_{b_{ij}} - \eta^n_{a_{ij}}\eta^n_{b_{ij}}}, \sqrt[n]{\zeta^n_{a_{ij}} + \zeta^n_{b_{ij}} - \zeta^n_{a_{ij}}\zeta^n_{b_{ij}}} \right\rangle\right) \\
\mathcal{S}^C \oplus \mathcal{T}^C &= \left(\left\langle \sqrt[n]{\delta^n_{a_{ij}} + \delta^n_{b_{ij}} - \delta^n_{a_{ij}}\delta^n_{b_{ij}}}, \eta_{a_{ij}}\eta_{b_{ij}}, \zeta_{a_{ij}}\zeta_{b_{ij}} \right\rangle\right).
\end{aligned}$$

Since

$$\begin{aligned}
\delta_{a_{ij}}\delta_{b_{ij}} &\leq \sqrt[n]{\delta^n_{a_{ij}} + \delta^n_{b_{ij}} - \delta^n_{a_{ij}}\delta^n_{b_{ij}}} \\
\sqrt[n]{\eta^n_{a_{ij}} + \eta^n_{b_{ij}} - \eta^n_{a_{ij}}\eta^n_{b_{ij}}} &\geq \eta_{a_{ij}}\eta_{b_{ij}} \\
\sqrt[n]{\zeta^n_{a_{ij}} + \zeta^n_{b_{ij}} - \zeta^n_{a_{ij}}\zeta^n_{b_{ij}}} &\geq \zeta_{a_{ij}}\zeta_{b_{ij}}.
\end{aligned}$$

Hence $(\mathcal{S} \oplus \mathcal{T})^C \leq \mathcal{S}^C \oplus \mathcal{T}^C$.

(iv) Let

$$\begin{aligned} (\mathcal{S} \otimes \mathcal{T})^C &= \left(\left\langle \sqrt[n]{\delta_{a_{ij}}^n + \delta_{b_{ij}}^n - \delta_{a_{ij}}^n \delta_{b_{ij}}^n}, \eta_{a_{ij}} \eta_{b_{ij}}, \zeta_{a_{ij}} \zeta_{b_{ij}} \right\rangle\right) \\ \mathcal{S}^C \otimes \mathcal{T}^C &= \left(\left\langle \delta_{a_{ij}} \delta_{b_{ij}}, \sqrt[n]{\eta_{a_{ij}}^n + \eta_{b_{ij}}^n - \eta_{a_{ij}}^n \eta_{b_{ij}}^n}, \sqrt[n]{\zeta_{a_{ij}}^n + \zeta_{b_{ij}}^n - \zeta_{a_{ij}}^n \zeta_{b_{ij}}^n} \right\rangle\right). \end{aligned}$$

Since

$$\begin{aligned} \sqrt[n]{\delta_{a_{ij}}^n + \delta_{b_{ij}}^n - \delta_{a_{ij}}^n \delta_{b_{ij}}^n} &\geq \delta_{a_{ij}} \delta_{b_{ij}} \\ \eta_{a_{ij}} \eta_{b_{ij}} &\leq \sqrt[n]{\eta_{a_{ij}}^n + \eta_{b_{ij}}^n - \eta_{a_{ij}}^n \eta_{b_{ij}}^n} \\ \zeta_{a_{ij}} \zeta_{b_{ij}} &\leq \sqrt[n]{\zeta_{a_{ij}}^n + \zeta_{b_{ij}}^n - \zeta_{a_{ij}}^n \zeta_{b_{ij}}^n}. \end{aligned}$$

Hence $(\mathcal{S} \otimes \mathcal{T})^C \geq \mathcal{S}^C \otimes \mathcal{T}^C$. □

Theorem 2.13. *For $\mathcal{S}, \mathcal{T} \in N_{m \times n}$, then*

(i) $(\mathcal{S}^C)^C = \mathcal{S}$.

(ii) $(\mathcal{S} \vee \mathcal{T})^C = \mathcal{S}^C \wedge \mathcal{T}^C$.

(iii) $(\mathcal{S} \wedge \mathcal{T})^C = \mathcal{S}^C \vee \mathcal{T}^C$.

Proof. We'll prove only (ii), (i) is self-evident.

$$\begin{aligned} \mathcal{S} \vee \mathcal{T} &= \left(\left\langle \max\left(\zeta_{a_{ij}}, \zeta_{b_{ij}}\right), \min\left(\eta_{a_{ij}}, \eta_{b_{ij}}\right), \min\left(\delta_{a_{ij}}, \delta_{b_{ij}}\right) \right\rangle\right) \\ (\mathcal{S} \vee \mathcal{T})^C &= \left(\left\langle \min\left(\delta_{a_{ij}}, \delta_{b_{ij}}\right), \min\left(\eta_{a_{ij}}, \eta_{b_{ij}}\right), \max\left(\zeta_{a_{ij}}, \zeta_{b_{ij}}\right) \right\rangle\right) \\ \Rightarrow \mathcal{S}^C &= \left(\left\langle \delta_{a_{ij}}, \eta_{a_{ij}}, \zeta_{a_{ij}} \right\rangle\right) \\ \mathcal{T}^C &= \left(\left\langle \delta_{b_{ij}}, \eta_{b_{ij}}, \zeta_{b_{ij}} \right\rangle\right) \\ \Rightarrow \mathcal{S}^C \wedge \mathcal{T}^C &= \left(\left\langle \min\left(\delta_{a_{ij}}, \delta_{b_{ij}}\right), \min\left(\eta_{a_{ij}}, \eta_{b_{ij}}\right), \max\left(\zeta_{a_{ij}}, \zeta_{b_{ij}}\right) \right\rangle\right). \end{aligned}$$

Hence $(\mathcal{S} \vee \mathcal{T})^C = \mathcal{S}^C \wedge \mathcal{T}^C$. Similarly, we can also demonstrate that $(iii)(\mathcal{S} \wedge \mathcal{T})^C = \mathcal{S}^C \vee \mathcal{T}^C$.

□

We'll show the algebraic characteristics of n-Hyper Spherical Neutrosophic matrices under scalar multiplication and exponentiation using the definitions 1.1, 1.2 and 1.3.

Theorem 2.14. *If $\mathcal{S}$, $\mathcal{T} \in N_{m \times n}$, then $n > 0$,*

(i) $n(\mathcal{S} \oplus \mathcal{T}) = n\mathcal{S} \oplus n\mathcal{T},\ n > 0$.

(ii) $n_1\mathcal{S} \oplus n_2\mathcal{S} = (n_1 + n_2)\mathcal{S},\ n_1, n_2 > 0$.

(iii) $(\mathcal{S} \otimes \mathcal{T})^n = \mathcal{S}^n \otimes \mathcal{T}^n,\ n > 0$.

(iv) $\mathcal{S}_1^n \otimes \mathcal{S}_2^n = \mathcal{S}^{(n_1+n_2)},\ n_1, n_2 > 0$.

Proof. According to the concept, for the two n-HSNMs $\mathcal{S}$ and $\mathcal{T}$, and $n, n_1, n_2 > 0$, we have

(i) Let

$$\begin{aligned}
n(\mathcal{S} \oplus \mathcal{T}) &= n\left(\left\langle \sqrt[n]{\zeta_{a_{ij}}^n + \zeta_{b_{ij}}^n - \zeta_{a_{ij}}^n \zeta_{b_{ij}}^n}, \eta_{a_{ij}}\eta_{b_{ij}}, \delta_{a_{ij}}\delta_{b_{ij}} \right\rangle\right) \\
&= \left(\left\langle \sqrt[n]{1 - [1-\zeta_{a_{ij}}^n]^n [1-\zeta_{a_{ij}}^n]^n}, [\eta_{a_{ij}}\eta_{b_{ij}}]^n, [\delta_{a_{ij}}\delta_{b_{ij}}]^n \right\rangle\right) \\
&= \left(\left\langle \sqrt[n]{1 - [1-\zeta_{a_{ij}}^n + \zeta_{b_{ij}}^n - \zeta_{a_{ij}}^n \zeta_{b_{ij}}^n]^n}, [\eta_{a_{ij}}\eta_{b_{ij}}]^n, [\delta_{a_{ij}}\delta_{b_{ij}}]^n \right\rangle\right) \\
n\mathcal{S} \oplus n\mathcal{T} &= \left(\left\langle \left(\sqrt[n]{1 - [1-\zeta_{a_{ij}}^n]^n}, [\eta_{a_{ij}}]^n, [\delta_{a_{ij}}]^n \right) \oplus \left(\sqrt[n]{1 - [1-\zeta_{b_{ij}}^n]^n}, [\eta_{b_{ij}}]^n, [\delta_{b_{ij}}]^n \right) \right\rangle\right) \\
&= \left(\left\langle \sqrt[n]{1 - [1-\zeta_{a_{ij}}^n]^n [1-\zeta_{b_{ij}}^n]^n}, [\eta_{a_{ij}}\eta_{b_{ij}}]^n, [\delta_{a_{ij}}\delta_{b_{ij}}]^n \right\rangle\right) \\
&= \left(\left\langle \sqrt[n]{1 - [1-\zeta_{a_{ij}}^n + \zeta_{b_{ij}}^n - \zeta_{a_{ij}}^n \zeta_{b_{ij}}^n]^n}, [\eta_{a_{ij}}\eta_{b_{ij}}]^n, [\delta_{a_{ij}}\delta_{b_{ij}}]^n \right\rangle\right) \\
&= n(\mathcal{S} \oplus \mathcal{T}).
\end{aligned}$$

(ii)Let

$$\begin{aligned}
n_1\mathcal{S} \oplus n_2\mathcal{T} &= \left(\left\langle \left(\sqrt[n]{1 - [1-\zeta_{a_{ij}}^n]^{n_1}}, [\eta_{a_{ij}}]^{n_1}, [\delta_{a_{ij}}]^{n_1} \right) \oplus \left(\sqrt[n]{1 - [1-\zeta_{a_{ij}}^n]^{n_2}}, [\eta_{a_{ij}}]^{n_2}, [\delta_{a_{ij}}]^{n_2} \right) \right\rangle\right) \\
&= \left(\left\langle \sqrt[n]{1 - [1-\zeta_{a_{ij}}^n]^{n_1+n_2}}, [\eta_{a_{ij}}]^{n_1+n_2}, [\delta_{a_{ij}}]^{n_1+n_2} \right\rangle\right) \\
&= (n_1 + n_2)\mathcal{S}.
\end{aligned}$$

(iii) Let

$$\begin{aligned}
(\mathcal{S} \otimes \mathcal{T})^n &= \left[\left(\zeta_{a_{ij}}\zeta_{b_{ij}}\right)^n, \sqrt[n]{1 - [1-\eta_{a_{ij}}^n + \eta_{b_{ij}}^n - \eta_{a_{ij}}^n \eta_{b_{ij}}^n]^n}, \sqrt[n]{1 - [1-\delta_{a_{ij}}^n + \delta_{b_{ij}}^n - \delta_{a_{ij}}^n \delta_{b_{ij}}^n]^n} \right] \\
&= \left[\left(\zeta_{a_{ij}}\zeta_{b_{ij}}\right)^n, \sqrt[n]{1 - [1-\eta_{a_{ij}}^n]^n [1-\eta_{b_{ij}}^n]^n, 1 - [1-\delta_{a_{ij}}^n]^n [1-\delta_{b_{ij}}^n]^n} \right] \\
\mathcal{S}^n \otimes \mathcal{T}^n &= \left[\left(\zeta_{a_{ij}}\zeta_{b_{ij}}\right)^n, \sqrt[n]{1 - [1-\eta_{a_{ij}}^n]^n + 1 - [1-\eta_{b_{ij}}^n]^n - \left(1 - [1-\eta_{a_{ij}}^n]^n\right)\left(1 - [1-\eta_{b_{ij}}^n]^n\right)} \right.
\end{aligned}$$

$$\left. \sqrt[n]{1 - [1-\delta_{a_{ij}}^n]^n + 1 - [1-\delta_{b_{ij}}^n]^n - \left(1 - [1-\delta_{a_{ij}}^n]^n\right)\left(1 - [1-\delta_{b_{ij}}^n]^n\right)} \right]$$

$$\begin{aligned}
&= \left(\left\langle \left(\zeta_{a_{ij}}\zeta_{b_{ij}}\right)^n, \sqrt[n]{1 - [1-\eta_{a_{ij}}^n]^n [1-\eta_{b_{ij}}^n]^n}, \sqrt[n]{1 - [1-\delta_{a_{ij}}^n]^n [1-\delta_{b_{ij}}^n]^n} \right\rangle\right) \\
&= (P \otimes Q)^n.
\end{aligned}$$

(iv) Let

$$\begin{aligned}
\mathcal{S}^{n_1} \otimes \mathcal{S}^{n_2} &= \left[\left(\zeta_{a_{ij}}\right)^{n_1+n_2}, \sqrt[n]{1 - [1-\eta_{a_{ij}}^n]^{n_1} + 1 - [1-\eta_{a_{ij}}^n]^{n_2} - \left(1 - [1-\eta_{a_{ij}}^n]^{n_1}\right)\left(1 - [1-\eta_{a_{ij}}^n]^{n-2}\right)}, \right. \\
&\quad \left. \sqrt[n]{1 - [1-\delta_{a_{ij}}^n]^{n_1} + 1 - [1-\delta_{a_{ij}}^n]^{n-2} - \left(1 - [1-\delta_{a_{ij}}^n]^{n_1}\right)\left(1 - [1-\delta_{a_{ij}}^n]^{n_2}\right)} \right] \\
&= \left(\left\langle \left(\zeta_{a_{ij}}\right)^{n_1+n_2}, \sqrt[n]{1 - [1-\eta_{a_{ij}}^n]^{n_1+n_2}}, \sqrt[n]{1 - [1-\delta_{a_{ij}}^n]^{n_1+n_2}} \right\rangle\right) \\
&= \mathcal{S}^{(n_1+n_2)}.
\end{aligned}$$

Hence proved. □

Theorem 2.15. *Suppose $\mathcal{S}$, $\mathcal{T} \in N_{m\times n}$, then $n > 0$,*

(i) $n\mathcal{S} \leq n\mathcal{T}$.

(ii) $\mathcal{S}^n \leq \mathcal{T}^n$.

Proof. (i) Let $\mathcal{S} \leq \mathcal{T}$. Then, we have

$$\zeta_{a_{ij}} \leq \zeta_{b_{ij}} \quad \eta_{a_{ij}} \geq \eta_{b_{ij}} \quad \text{and} \quad \delta_{a_{ij}} \geq \delta_{b_{ij}} \quad \forall\, i, j.$$

Now,

$$\begin{aligned} \sqrt[n]{1-[1-\zeta_{a_{ij}}^n]^n} &\leq \sqrt[n]{1-[1-\zeta_{b_{ij}}^n]^n} \\ \left[\eta_{a_{ij}}\right]^n &\geq \left[\eta_{b_{ij}}\right]^n \\ \left[\delta_{a_{ij}}\right]^n &\geq \left[\delta_{b_{ij}}\right]^n \quad \forall\, i, j. \end{aligned}$$

(ii) Also,

$$\begin{aligned} [\zeta_{a_{ij}}]^n &\geq [\zeta_{b_{ij}}]^n \\ \sqrt[n]{1-[1-\eta_{a_{ij}}^n]^n} &\leq \sqrt[n]{1-[1-\eta_{b_{ij}}^n]^n} \\ \sqrt[n]{1-[1-\delta_{a_{ij}}^n]^n} &\leq \sqrt[n]{1-[1-\delta_{b_{ij}}^n]^n} \quad \forall\, i, j. \end{aligned}$$

□

Similarly, we can prove the following theorems.

Theorem 2.16. *For $\mathcal{S}$, $\mathcal{T} \in N_{m\times n}$, then $n > 0$,*

(i) $n(\mathcal{S} \wedge \mathcal{T}) = n\mathcal{S} \wedge n\mathcal{T}$.

(ii) $n(\mathcal{S} \vee \mathcal{T}) = n\mathcal{S} \vee n\mathcal{T}$.

Theorem 2.17. *Suppose $\mathcal{S}$, $\mathcal{T} \in N_{m\times n}$, then $n > 0$,*

(i) $(\mathcal{S} \wedge \mathcal{T})^n = \mathcal{S}^n \wedge \mathcal{T}^n$.

(ii) $(\mathcal{S} \vee \mathcal{T})^n = \mathcal{S}^n \vee \mathcal{T}^n$.

Theorem 2.18. *Suppose $\mathcal{S}$, $\mathcal{T} \in N_{m\times n}$, then $n > 0$,*
$(\mathcal{S} \oplus \mathcal{T})^n \neq \mathcal{S}^n \oplus \mathcal{T}^n$.

Proof. Let

$$\begin{aligned} (\mathcal{S} \oplus \mathcal{T})^n &= \left[\left(\sqrt[n]{\zeta_{a_{ij}}^n + \zeta_{b_{ij}}^n - \zeta_{a_{ij}}^n \zeta_{b_{ij}}^n}\right)^n, \sqrt[n]{1-[1-\eta_{a_{ij}}^n \eta_{b_{ij}}^n]^n}, \sqrt[n]{1-[1-\delta_{a_{ij}}^n \delta_{b_{ij}}^n]^n}\right] \\ \mathcal{S}^n \oplus \mathcal{T}^n &= \left[\sqrt[n]{[\zeta_{a_{ij}}^n]^n + [\zeta_{b_{ij}}^n]^n - [\zeta_{a_{ij}}^n]^n [\zeta_{b_{ij}}^n]^n}, \left(\sqrt[n]{1-[1-\eta_{a_{ij}}^n]^n}\right)^n . \left(\sqrt[n]{1-[1-\eta_{b_{ij}}^n]^n}\right)^n, \right. \\ &\left. \left(\sqrt[n]{1-[1-\delta_{a_{ij}}^n]^n}\right)^n . \left(\sqrt[n]{1-[1-\delta_{a_{ij}}^n]^n}\right)^n \right]. \end{aligned}$$

Hence $(\mathcal{S} \oplus \mathcal{T})^n \neq \mathcal{S}^n \oplus \mathcal{T}^n$. □

3. New operation (@) on n-HyperSpherical Neutrosophic matrices

In this part, we describe and show the algebraic properties of a new operation(@) on n-HyperSpherical Neutrosophic matrices. We also go through the Disstributivity rules in the situation when the $\oplus, \otimes, \vee$ and $\wedge$ operations are combined.

Definition 3.1. A n-HyperSpherical Neutrosophic matrices $\mathcal{S}$ and $\mathcal{T}$ are of the form, $\mathcal{S} = \left(\langle \zeta_{a_{ij}}, \eta_{a_{ij}}, \delta_{a_{ij}} \rangle\right)$ and $\mathcal{T} = \left(\langle \zeta_{b_{ij}}, \eta_{b_{ij}}, \delta_{b_{ij}} \rangle\right)$. Then

$$\mathcal{S}@\mathcal{T} = \left(\left\langle \sqrt[n]{\frac{\zeta^n_{a_{ij}} + \zeta^n_{b_{ij}}}{2}}, \sqrt[n]{\frac{\eta^n_{a_{ij}} + \eta^n_{b_{ij}}}{2}}, \sqrt[n]{\frac{\delta^n_{a_{ij}} + \delta^n_{b_{ij}}}{2}} \right\rangle\right).$$

Remark 3.2. Obviously, for every two n-HyperSpherical Neutrosophic matrices $\mathcal{S}$ and $\mathcal{T}$, then $\mathcal{S}@\mathcal{T}$ is a n-HyperSpherical Neutrosophic matrix.

Simple illustration given: For $\mathcal{S}@\mathcal{T}$,

$$\begin{aligned} 0 &\leq \frac{\zeta_{a_{ij}} + \zeta_{b_{ij}}}{2} + \frac{\eta_{a_{ij}} + \eta_{b_{ij}}}{2} + \frac{\delta_{a_{ij}} + \delta_{b_{ij}}}{2} \\ &\leq \frac{\zeta_{a_{ij}} + \eta_{a_{ij}} + \delta_{a_{ij}}}{2} + \frac{\zeta_{b_{ij}} + \eta_{b_{ij}} + \delta_{b_{ij}}}{2} \\ &\leq \frac{1}{2} + \frac{1}{2} = 1. \end{aligned}$$

Theorem 3.3. *For any n-HyperSpherical Neutrosophic matrix $\mathcal{S}$, then $\mathcal{S}@\mathcal{S} = \mathcal{S}$.*

Proof. Let

$$\begin{aligned} \mathcal{S}@\mathcal{S} &= \left(\left\langle \sqrt[n]{\frac{\zeta^n_{a_{ij}} + \zeta^n_{a_{ij}}}{2}}, \sqrt[n]{\frac{\eta^n_{a_{ij}} + \eta^n_{a_{ij}}}{2}}, \sqrt[n]{\frac{\delta^n_{a_{ij}} + \delta^n_{a_{ij}}}{2}} \right\rangle\right) \\ &= \left(\left\langle \left(\sqrt[n]{\frac{\zeta^n_{a_{ij}} + \zeta^n_{a_{ij}}}{2}}\right)^n, \left(\sqrt[n]{\frac{\eta^n_{a_{ij}} + \eta^n_{a_{ij}}}{2}}\right)^n, \left(\sqrt[n]{\frac{\delta^n_{a_{ij}} + \delta^n_{a_{ij}}}{2}}\right)^n \right\rangle\right) \\ &= \left(\left\langle \frac{2\zeta^n_{a_{ij}}}{2}, \frac{2\eta^n_{a_{ij}}}{2}, \frac{2\delta^n_{a_{ij}}}{2} \right\rangle\right) \\ &= \left(\langle \zeta_{a_{ij}}, \eta_{a_{ij}}, \delta_{a_{ij}} \rangle\right). \end{aligned}$$

Since $\zeta^n_{a_{ij}} \leq \zeta_{a_{ij}}, \eta^n_{a_{ij}} \leq \eta_{a_{ij}}, \delta^n_{a_{ij}} \leq \delta_{a_{ij}}$

$$\mathcal{S}@\mathcal{S} = \mathcal{S}.$$ □

Remark 3.4. If $v, w \in [0,1]$, then $vw \leq \frac{v+w}{2}, \frac{v+w}{2} \leq v + w - vw$.

Theorem 3.5. *Suppose $\mathcal{S}$, $\mathcal{T} \in N_{m \times n}$, then*

(i) $(\mathcal{S} \oplus \mathcal{T}) \vee (\mathcal{S}@\mathcal{T}) = \mathcal{S} \oplus \mathcal{T}$.
(ii) $(\mathcal{S} \otimes \mathcal{T}) \wedge (\mathcal{S}@\mathcal{T}) = \mathcal{S} \otimes \mathcal{T}$.
(iii) $(\mathcal{S} \oplus \mathcal{T}) \wedge (\mathcal{S}@\mathcal{T}) = \mathcal{S}@\mathcal{T}$.
(iv) $(\mathcal{S} \otimes \mathcal{T}) \vee (\mathcal{S}@\mathcal{T}) = \mathcal{S}@\mathcal{T}$.

Proof. We'll show that (i) and (iii), as well as (ii) and (iv), may be demonstrated in the similar manner. (i) Let

$$\begin{aligned}(\mathcal{S} \oplus \mathcal{T}) \vee (\mathcal{S}@\mathcal{T}) &= \Bigg[\max\left(\sqrt[n]{\zeta_{a_{ij}}^n + \zeta_{b_{ij}}^n - \zeta_{a_{ij}}^n \zeta_{b_{ij}}^n}, \sqrt[n]{\frac{\zeta_{a_{ij}}^n + \zeta_{b_{ij}}^n}{2}}\right), \min\left(\eta_{a_{ij}}\eta_{b_{ij}}, \sqrt[n]{\frac{\eta_{a_{ij}}^n + \eta_{b_{ij}}^n}{2}}\right), \\ &\quad \min\left(\delta_{a_{ij}}\delta_{b_{ij}}, \sqrt[n]{\frac{\delta_{a_{ij}}^n + \delta_{b_{ij}}^n}{2}}\right)\Bigg] \\ &= \left(\left\langle \sqrt[n]{\zeta_{a_{ij}}^n + \zeta_{b_{ij}}^n - \zeta_{a_{ij}}^n \zeta_{b_{ij}}^n}, \eta_{a_{ij}}\eta_{b_{ij}}, \delta_{a_{ij}}\delta_{b_{ij}} \right\rangle\right) \\ &= \mathcal{S} \oplus \mathcal{T}.\end{aligned}$$

(iii) Let

$$\begin{aligned}(\mathcal{S} \oplus \mathcal{T}) \wedge (\mathcal{S}@\mathcal{T}) &= \Bigg[\min\left(\sqrt[n]{\zeta_{a_{ij}}^n + \zeta_{b_{ij}}^n - \zeta_{a_{ij}}^n \zeta_{b_{ij}}^n}, \sqrt[n]{\frac{\zeta_{a_{ij}}^n + \zeta_{b_{ij}}^n}{2}}\right), \max\left(\eta_{a_{ij}}\eta_{b_{ij}}, \sqrt[n]{\frac{\eta_{a_{ij}}^n + \eta_{b_{ij}}^n}{2}}\right), \\ &\quad \max\left(\delta_{a_{ij}}\delta_{b_{ij}}, \sqrt[n]{\frac{\delta_{a_{ij}}^n + \delta_{b_{ij}}^n}{2}}\right)\Bigg] \\ &= \left(\left\langle \sqrt[n]{\frac{\zeta_{a_{ij}}^n + \zeta_{b_{ij}}^n}{2}}, \sqrt[n]{\frac{\eta_{a_{ij}}^n + \eta_{b_{ij}}^n}{2}}, \sqrt[n]{\frac{\delta_{a_{ij}}^n + \delta_{b_{ij}}^n}{2}} \right\rangle\right) \\ &= \mathcal{S}@\mathcal{T}.\end{aligned}$$

Hence proved. □

Remark 3.6. Under the n-Hyper Spherical Neutrosophic matrix operations of algebraic sum and algebraic product, the n-Hyper Spherical Neutrosophic matrix forms a semi-lattice, associativity, commutativity, and idempotency. When $\oplus, \otimes$ and $\wedge, \vee, @$ are combined, the distributive law also holds.

4. **Applications**

The results are relevant to the development of n-Hyper Spherical Neutrosophic semi-lattice structure, n-Hyper Spherical Neutrosophic matrix, and algebraic structure on this matrix.

5. Conclusion

n-Hyper Spherical Neutrosophic matrices and their algebraic operations are characterized in this study. Then various qualities are demonstrated, including associativity, idempotency, distributivity, commutativity, absorption law, and De Morgan's laws over complement. Lastly, we established a new operation(@) on n-Hyper Spherical Neutrosophic matrices and studied distributive laws in the situation of combining the operations of $\oplus, \otimes, \wedge$, and $\vee$. This finding can be used to the n-Hyper Spherical Neutrosophic matrix theory in the future. The conclusions of this work will be useful in the creation of the n-Hyper Spherical Neutrosophic semilattice and its algebraic property. The applicability of the suggested aggregating operators of n-HSNMs in risk analysis, decision making and many other fuzzy environments will need to be studied in the future.

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