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Polarity of generalized neutrosophic subalgebras in BCK/BCI-algebras

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Abstract: k-polar generalized neutrosophic set is introduced, and it is applied to BCK/BCI-algebras. The notions of k-polar generalized subalgebra, k-polar generalized $(\in, \in\vee q)$-neutrosophic subalgebra and k-polar generalized $(q, \in\vee q)$-neutrosophic subalgebra are defined, and several properties are investigated. Characterizations of k-polar generalized neutrosophic subalgebra and k-polar generalized $(\in, \in\vee q)$-neutrosophic subalgebra are discussed, and the necessity and possibility operator of k-polar generalized neutrosophic subalgebra are are considered. We show that the generaliged neutrosophic q-sets and the generaliged neutrosophic $\in\vee q$-sets subalgebras by using the k-polar generalized $(\in, \in\vee q)$-neutrosophic subalgebra and the k-polar generalized $(q, \in\vee q)$-neutrosophic subalgebra. A k-polar generalized $(\in, \in\vee q)$-neutrosophic subalgebra is established by using the generaliged neutrosophic $\in\vee q$-sets, conditions for a k-polar generalized neutrosophic set to be a k-polar generalized neutrosophic subalgebra and a k-polar generalized $(q, \in\vee q)$-neutrosophic subalgebra are provided.

Keywords: k-polar generalized neutrosophic subalgebra, k-polar generalized $(\in, \in\vee q)$-neutrosophic subalgebra, k-polar generalized $(q, \in\vee q)$-neutrosophic subalgebra.

1 Introduction

In the fuzzy set which is introduced by Zadeh [35], the membership degree is expressed by only one function so called the truth function. As a generalization of fuzzy set, intuitionistic fuzzy set is introduced by Atanassove by using membership function and nonmembership function. The membership (resp. nonmembership) function represents truth (resp. false) part. Smarandache introduced a new notion so called neutrosophic set by using three functions, i.e., membership function (t), nonmembership function (f) and neutalitic/indeterministic membership function (i) which are independent components. Neutrosophic set is applied to BCK/BCI-algebras which are discussed in the papers [13, 19, 20, 21, 22, 26, 27, 30]. Indeterministic membership function is leaning to one side, membership function or nonmembership function, in the application of neutrosophic set to algebraic structures. In order to divide the role of the indeterministic membership function, Song et al.

[31] introduced the generalized neutralrosophic set, and discussed its application in BCK/BCI-algebras. Borzooei et al. [8] introduced the notion of a commutative generalized neutrosophic ideal in a BCK-algebra, and investigated related properties. They considered characterizations of a commutative generalized neutrosophic ideal. Using a collection of commutative ideals in BCK-algebras, they established a commutative generalized neutrosophic ideal. They also introduced the notion of equivalence relations on the family of all commutative generalized neutrosophic ideals in BCK-algebras, and investigated related properties. Zhang [36] introduced the notion of bipolar fuzzy sets as an extension of fuzzy sets, and it is applied in several (algebraic) structures such as (ordered) semigroups (see [12, 7, 10, 28]), (hyper) BCK/BCI-algebras (see [6, 14, 15, 23, 16, 17]) and finite state machines (see [18, 32, 33, 34]). The bipolar fuzzy set is an extension of fuzzy sets whose membership degree range is $[-1, 1]$. So, it is possible for a bipolar fuzzy set to deal with positive information and negative information at the same time. Chen et al. [9] raised a question: "How to generalize bipolar fuzzy sets to multipolar fuzzy sets and how to generalize results on bipolar fuzzy sets to the case of multipolar fuzzy sets?" To solve their question, they tried to fold the negative part into positive part, that is, they used positive part instead of negative part in bipolar fuzzy set. And then they introduced introduced an m-polar fuzzy set which is an extension of bipolar fuzzy sets. It is applied to BCK/BCI-algebra, graph theory and decision-making problems etc. (see [4, 2, 1, 3, 29, 5, 25]).

In this paper, we introduce k-polar generalized neutrosophic set and apply it to BCK/BCI-algebras to study. We define k-polar generalized neutrosophic subalgebra, k-polar generalized $(\in, \in\vee q)$-neutrosophic subalgebra and k-polar generalized $(q, \in\vee q)$-neutrosophic subalgebra and study various properties. We discuss characterization of k-polar generalized neutrosophic subalgebra and k-polar generalized $(\in, \in\vee q)$-neutrosophic subalgebra. We show that the necessity and possibility operator of k-polar generalized neutrosophic subalgebra are also a k-polar generalized neutrosophic subalgebra. Using the k-polar generalized $(\in, \in\vee q)$-neutrosophic subalgebra, we show that the generaliged neutrosophic q-sets and the generaliged neutrosophic $\in\vee q$-sets subalgebras. Using the k-polar generalized $(q, \in\vee q)$-neutrosophic subalgebra, we show that the generaliged neutrosophic q-sets and the generaliged neutrosophic $\in\vee q$-sets are subalgebras. Using the generaliged neutrosophic $\in\vee q$-sets, we establish a k-polar generalized $(\in, \in\vee q)$-neutrosophic subalgebra. We provide conditions for a k-polar generalized neutrosophic set to be a k-polar generalized neutrosophic subalgebra and a k-polar generalized $(q, \in\vee q)$-neutrosophic subalgebra.

2 Preliminaries

If a set X has a special element 0 and a binary operation $*$ satisfying the conditions:

(I) $(\forall u, v, w \in X)\ (((u * v) * (u * w)) * (w * v) = 0)$,

(II) $(\forall u, v \in X)\ ((u * (u * v)) * v = 0)$,

(III) $(\forall u \in X)\ (u * u = 0)$,

(IV) $(\forall u, v \in X)\ (u * v = 0,\ v * u = 0\ \Rightarrow\ u = v)$,

then we say that X is a *BCI-algebra*. If a BCI-algebra X satisfies the following identity:

(V) $(\forall u \in X)\ (0 * u = 0)$,

then X is called a BCK*-algebra.*

Any BCK/BCI-algebra X satisfies the following conditions:

$$(\forall u \in X)\,(u * 0 = u)\,, \tag{2.1}$$
$$(\forall u, v, w \in X)\,(u \leq v \;\Rightarrow\; u * w \leq v * w,\; w * v \leq w * u)\,, \tag{2.2}$$
$$(\forall u, v, w \in X)\,((u * v) * w = (u * w) * v) \tag{2.3}$$

where $u \leq v$ if and only if $u * v = 0$. A subset S of a BCK/BCI-algebra X is called a *subalgebra* of X if $u * v \in S$ for all $u, v \in S$.

See the books [11] and [24] for more information on BCK/BCI-algeebras.

A fuzzy set μ in a BCK/BCI-algebra X is called a *fuzzy subalgebra* of X if $\mu(u * v) \geq \min\{\mu(u), \mu(v)\}$ for all $u, v \in X$.

For any family $\{a_i \mid i \in \Lambda\}$ of real numbers, we define

$$\bigvee\{a_i \mid i \in \Lambda\} := \begin{cases} \max\{a_i \mid i \in \Lambda\} & \text{if } \Lambda \text{ is finite,} \\ \sup\{a_i \mid i \in \Lambda\} & \text{otherwise.} \end{cases}$$

$$\bigwedge\{a_i \mid i \in \Lambda\} := \begin{cases} \min\{a_i \mid i \in \Lambda\} & \text{if } \Lambda \text{ is finite,} \\ \inf\{a_i \mid i \in \Lambda\} & \text{otherwise.} \end{cases}$$

If $\Lambda = \{1, 2\}$, we will also use $a_1 \vee a_2$ and $a_1 \wedge a_2$ instead of $\bigvee\{a_i \mid i \in \Lambda\}$ and $\bigwedge\{a_i \mid i \in \Lambda\}$, respectively.

3 k-polar generalized neutrosophic subalgebras

A *k-polar generalized neutrosophic set* over a universe X is a structure of the form:

$$\widehat{\mathcal{L}} := \left\{ \frac{z}{\left(\widehat{\ell}_T(z), \widehat{\ell}_{IT}(z), \widehat{\ell}_{IF}(z), \widehat{\ell}_F(z)\right)} \mid z \in X,\; \widehat{\ell}_{IT}(z) + \widehat{\ell}_{IF}(z) \leq \hat{1} \right\} \tag{3.1}$$

where $\widehat{\ell}_T$, $\widehat{\ell}_{IT}$, $\widehat{\ell}_{IF}$ and $\widehat{\ell}_F$ are mappings from X into $[0, 1]^k$. The membership values of every element $z \in X$ in $\widehat{\ell}_T$, $\widehat{\ell}_{IT}$, $\widehat{\ell}_{IF}$ and $\widehat{\ell}_F$ are denoted by

$$\begin{aligned} \widehat{\ell}_T(z) &= \left((\pi_1 \circ \widehat{\ell}_T)(z), (\pi_2 \circ \widehat{\ell}_T)(z), \cdots, (\pi_k \circ \widehat{\ell}_T)(z)\right), \\ \widehat{\ell}_{IT}(z) &= \left((\pi_1 \circ \widehat{\ell}_{IT})(z), (\pi_2 \circ \widehat{\ell}_{IT})(z), \cdots, (\pi_k \circ \widehat{\ell}_{IT})(z)\right), \\ \widehat{\ell}_{IF}(z) &= \left((\pi_1 \circ \widehat{\ell}_{IF})(z), (\pi_2 \circ \widehat{\ell}_{IF})(z), \cdots, (\pi_k \circ \widehat{\ell}_{IF})(z)\right), \\ \widehat{\ell}_F(z) &= \left((\pi_1 \circ \widehat{\ell}_F)(z), (\pi_2 \circ \widehat{\ell}_F)(z), \cdots, (\pi_k \circ \widehat{\ell}_F)(z)\right), \end{aligned} \tag{3.2}$$

respectively, and satisfies the following condition

$$(\pi_i \circ \widehat{\ell}_{IT})(z) + (\pi_i \circ \widehat{\ell}_{IF})(z) \leq 1$$

for all $i = 1, 2, \cdots, k$.

We shall use the ordered quadruple $\widehat{\mathcal{L}} := \left(\widehat{\ell}_T, \widehat{\ell}_{IT}, \widehat{\ell}_{IF}, \widehat{\ell}_F\right)$ for the k-polar generalized neutrosophic set in (3.1).

Note that for every k-polar generalized neutrosophic set $\widehat{\mathcal{L}} := \left(\widehat{\ell}_T, \widehat{\ell}_{IT}, \widehat{\ell}_{IF}, \widehat{\ell}_F\right)$ over X, we have

$$(\forall z \in X)\left(\hat{0} \leq \widehat{\ell}_T(z) + \widehat{\ell}_{IT}(z) + \widehat{\ell}_{IF}(z) + \widehat{\ell}_F(z) \leq \hat{3}\right),$$

that is, $0 \leq (\pi_i \circ \widehat{\ell}_T)(z) + (\pi_i \circ \widehat{\ell}_{IT})(z) + (\pi_i \circ \widehat{\ell}_{IF})(z) + (\pi_i \circ \widehat{\ell}_F)(z) \leq 3$ for all $z \in X$ and $i = 1, 2, \cdots, k$.

Unless otherwise stated in this section, X will represent a BCK/BCI-algebra.

Definition 3.1. A k-polar generalized neutrosophic set $\widehat{\mathcal{L}} := \left(\widehat{\ell}_T, \widehat{\ell}_{IT}, \widehat{\ell}_{IF}, \widehat{\ell}_F\right)$ over X is called a *k-polar generalized neutrosophic subalgebra* of X if it satisfies:

$$(\forall z, y \in X)\begin{pmatrix} \widehat{\ell}_T(z*y) \geq \widehat{\ell}_T(z) \wedge \widehat{\ell}_T(y) \\ \widehat{\ell}_{IT}(z*y) \geq \widehat{\ell}_{IT}(z) \wedge \widehat{\ell}_{IT}(y) \\ \widehat{\ell}_{IF}(z*y) \leq \widehat{\ell}_{IF}(z) \vee \widehat{\ell}_{IF}(y) \\ \widehat{\ell}_F(z*y) \leq \widehat{\ell}_F(z) \vee \widehat{\ell}_F(y) \end{pmatrix}, \tag{3.3}$$

that is,

$$\begin{cases} (\pi_i \circ \widehat{\ell}_T)(z*y) \geq (\pi_i \circ \widehat{\ell}_T)(z) \wedge (\pi_i \circ \widehat{\ell}_T)(y) \\ (\pi_i \circ \widehat{\ell}_{IT})(z*y) \geq (\pi_i \circ \widehat{\ell}_{IT})(z) \wedge (\pi_i \circ \widehat{\ell}_{IT})(y) \\ (\pi_i \circ \widehat{\ell}_{IF})(z*y) \leq (\pi_i \circ \widehat{\ell}_{IF})(z) \vee (\pi_i \circ \widehat{\ell}_{IF})(y) \\ (\pi_i \circ \widehat{\ell}_F)(z*y) \leq (\pi_i \circ \widehat{\ell}_F)(z) \vee (\pi_i \circ \widehat{\ell}_F)(y) \end{cases} \tag{3.4}$$

for $i = 1, 2, \cdots, k$.

Example 3.2. Consider a BCK-algebra $X = \{0, \alpha, \beta, \gamma\}$ with the binary operation "$*$" which is given below.

$*$	0	α	β	γ
0	0	0	0	0
α	α	0	α	α
β	β	β	0	β
γ	γ	γ	γ	0

Let $\widehat{\mathcal{L}} := \left(\widehat{\ell}_T, \widehat{\ell}_{IT}, \widehat{\ell}_{IF}, \widehat{\ell}_F\right)$ be a 4-polar neutrosophic set over X in which $\widehat{\ell}_T$, $\widehat{\ell}_{IT}$, $\widehat{\ell}_{IF}$ and $\widehat{\ell}_F$ are defined as follows:

$$\widehat{\ell}_T : X \to [0,1]^4,\ z \mapsto \begin{cases} (0.6, 0.7, 0.8, 0.9) & \text{if } z = 0, \\ (0.4, 0.4, 0.8, 0.5) & \text{if } z = \alpha, \\ (0.5, 0.6, 0.7, 0.3) & \text{if } z = \beta, \\ (0.3, 0.5, 0.4, 0.7) & \text{if } z = \gamma, \end{cases}$$

$$\widehat{\ell}_{IT} : X \to [0,1]^4,\ z \mapsto \begin{cases} (0.7, 0.6, 0.8, 0.9) & \text{if } z = 0, \\ (0.6, 0.4, 0.7, 0.5) & \text{if } z = \alpha, \\ (0.5, 0.5, 0.4, 0.8) & \text{if } z = \beta, \\ (0.2, 0.6, 0.5, 0.7) & \text{if } z = \gamma, \end{cases}$$

$$\widehat{\ell}_{IF} : X \to [0,1]^4,\ z \mapsto \begin{cases} (0.2, 0.3, 0.4, 0.5) & \text{if } z = 0, \\ (0.4, 0.7, 0.5, 0.8) & \text{if } z = \alpha, \\ (0.5, 0.5, 0.8, 0.6) & \text{if } z = \beta, \\ (0.7, 0.3, 0.6, 0.7) & \text{if } z = \gamma, \end{cases}$$

$$\widehat{\ell}_{F} : X \to [0,1]^4,\ z \mapsto \begin{cases} (0.4, 0.4, 0.3, 0.2) & \text{if } z = 0, \\ (0.8, 0.7, 0.5, 0.3) & \text{if } z = \alpha, \\ (0.6, 0.5, 0.6, 0.6) & \text{if } z = \beta, \\ (0.4, 0.6, 0.8, 0.4) & \text{if } z = \gamma, \end{cases}$$

It is routine to verify that $\widehat{\mathcal{L}} := \left(\widehat{\ell}_T, \widehat{\ell}_{IT}, \widehat{\ell}_{IF}, \widehat{\ell}_F\right)$ is a 4-polar generalized neutrosophic subalgebra of X.

If we take $z = y$ in (3.3) and use (III), then we have the following lemma.

Lemma 3.3. *Let $\widehat{\mathcal{L}} := \left(\widehat{\ell}_T, \widehat{\ell}_{IT}, \widehat{\ell}_{IF}, \widehat{\ell}_F\right)$ be a k-polar generalized neutrosophic subalgebra of a BCK/BCI-algebr X. Then*

$$(\forall z, y \in X) \quad \left(\begin{array}{l} \widehat{\ell}_T(0) \geq \widehat{\ell}_T(z),\ \widehat{\ell}_{IT}(0) \geq \widehat{\ell}_{IT}(z) \\ \widehat{\ell}_{IF}(0) \leq \widehat{\ell}_{IF}(z),\ \widehat{\ell}_F(0) \leq \widehat{\ell}_F(z) \end{array} \right). \tag{3.5}$$

Proposition 3.4. *Let $\widehat{\mathcal{L}} := \left(\widehat{\ell}_T, \widehat{\ell}_{IT}, \widehat{\ell}_{IF}, \widehat{\ell}_F\right)$ be a k-polar generalized neutrosophic set over X. If there exists a sequence $\{z_n\}$ in X such that $\lim_{n\to\infty} \widehat{\ell}_T(z_n) = \hat{1} = \lim_{n\to\infty} \widehat{\ell}_{IT}(z_n)$ and $\lim_{n\to\infty} \widehat{\ell}_{IF}(z_n) = \hat{0} = \lim_{n\to\infty} \widehat{\ell}_F(z_n)$, then $\widehat{\ell}_T(0) = \hat{1} = \widehat{\ell}_{IT}(0)$ and $\widehat{\ell}_{IF}(0) = \hat{0} = \widehat{\ell}_F(0)$.*

Proof. Using Lemma 3.3, we have

$$\begin{aligned} \hat{1} &= \lim_{n\to\infty} \widehat{\ell}_T(z_n) \leq \widehat{\ell}_T(0) \leq \hat{1} = \lim_{n\to\infty} \widehat{\ell}_{IT}(z_n) \leq \widehat{\ell}_{IT}(0) \leq \hat{1}, \\ \hat{0} &= \lim_{n\to\infty} \widehat{\ell}_{IF}(z_n) \geq \widehat{\ell}_{IF}(0) \geq \hat{0} = \lim_{n\to\infty} \widehat{\ell}_F(z_n) \geq \widehat{\ell}_F(0) \geq \hat{0}. \end{aligned}$$

This completes the proof. □

Proposition 3.5. *Let $\widehat{\mathcal{L}} := \left(\widehat{\ell}_T, \widehat{\ell}_{IT}, \widehat{\ell}_{IF}, \widehat{\ell}_F\right)$ be a k-polar generalized neutrosophic subalgebra of X such that*

$$(\forall z, y \in X) \quad \left(\begin{array}{l} \widehat{\ell}_T(z * y) \geq \widehat{\ell}_T(y),\ \widehat{\ell}_{IT}(z * y) \geq \widehat{\ell}_{IT}(y) \\ \widehat{\ell}_{IF}(z * y) \leq \widehat{\ell}_{IF}(y),\ \widehat{\ell}_F(z * y) \leq \widehat{\ell}_F(y) \end{array} \right). \tag{3.6}$$

Then $\widehat{\mathcal{L}}$ is constant on X, that is, $\widehat{\ell}_T$, $\widehat{\ell}_{IT}$, $\widehat{\ell}_{IF}$ and $\widehat{\ell}_F$ are constants on X.

Proof. Since $z * 0 = z$ for all $z \in X$, it follows from the condition (3.6) that

$$\widehat{\ell}_T(z) = \widehat{\ell}_T(z * 0) \geq \widehat{\ell}_T(0),\ \widehat{\ell}_{IT}(z) = \widehat{\ell}_{IT}(z * 0) \geq \widehat{\ell}_{IT}(0), \tag{3.7}$$

$$\widehat{\ell}_{IF}(z) = \widehat{\ell}_{IF}(z * 0) \leq \widehat{\ell}_{IF}(0),\ \widehat{\ell}_F(z) = \widehat{\ell}_F(z * 0) \leq \widehat{\ell}_F(0) \tag{3.8}$$

for all $z \in X$. Combining (3.5) and (3.7) induces $\widehat{\ell}_T(z) = \widehat{\ell}_T(0)$, $\widehat{\ell}_{IT}(z) = \widehat{\ell}_{IT}(0)$, $\widehat{\ell}_{IF}(z) = \widehat{\ell}_{IF}(0)$ and $\widehat{\ell}_F(z) = \widehat{\ell}_F(0)$ for all $z \in X$. Therefore $\widehat{\ell}_T$, $\widehat{\ell}_{IT}$, $\widehat{\ell}_{IF}$ and $\widehat{\ell}_F$ are constants on X, that is, $\widehat{\mathcal{L}}$ is constant on X. □

Given a k-polar generalized neutrosophic set $\widehat{\mathcal{L}} := \left(\widehat{\ell}_T, \widehat{\ell}_{IT}, \widehat{\ell}_{IF}, \widehat{\ell}_F\right)$ over a universe X, consider the following cut sets.

$$\begin{aligned}
U(\widehat{\ell}_T, \hat{n}_T) &:= \{z \in X \mid \widehat{\ell}_T(z) \geq \hat{n}_T\},\\
U(\widehat{\ell}_{IT}, \hat{n}_{IT}) &:= \{z \in X \mid \widehat{\ell}_{IT}(z) \geq \hat{n}_{IT}\},\\
L(\widehat{\ell}_{IF}, \hat{n}_{IF}) &:= \{z \in X \mid \widehat{\ell}_{IF}(z) \leq \hat{n}_{IF}\},\\
L(\widehat{\ell}_F, \hat{n}_F) &:= \{z \in X \mid \widehat{\ell}_F(z) \leq \hat{n}_F\}
\end{aligned}$$

for $\hat{n}_T, \hat{n}_{IT}, \hat{n}_{IF}, \hat{n}_F \in [0, 1]^k$, that is,

$$\begin{aligned}
U(\widehat{\ell}_T, \hat{n}_T) &:= \{z \in X \mid (\pi_i \circ \widehat{\ell}_T)(z) \geq \hat{n}_T^i \text{ for all } i = 1, 2, \cdots, k\},\\
U(\widehat{\ell}_{IT}, \hat{n}_{IT}) &:= \{z \in X \mid (\pi_i \circ \widehat{\ell}_{IT})(z) \geq \hat{n}_{IT}^i \text{ for all } i = 1, 2, \cdots, k\},\\
L(\widehat{\ell}_{IF}, \hat{n}_{IF}) &:= \{z \in X \mid (\pi_i \circ \widehat{\ell}_{IF})(z) \leq \hat{n}_{IF}^i \text{ for all } i = 1, 2, \cdots, k\},\\
L(\widehat{\ell}_F, \hat{n}_F) &:= \{z \in X \mid (\pi_i \circ \widehat{\ell}_F)(z) \leq \hat{n}_F^i \text{ for all } i = 1, 2, \cdots, k\}
\end{aligned}$$

where $\hat{n}_T = (n_T^1, n_T^2, \cdots, n_T^k)$, $\hat{n}_{IT} = (n_{IT}^1, n_{IT}^2, \cdots, n_{IT}^k)$, $\hat{n}_{IF} = (n_{IF}^1, n_{IF}^2, \cdots, n_{IF}^k)$ and $\hat{n}_F = (n_F^1, n_F^2, \cdots, n_F^k)$. It is clear that $U(\widehat{\ell}_T, \hat{n}_T) = \bigcap_{i=1}^k U(\widehat{\ell}_T, \hat{n}_T)^i$, $U(\widehat{\ell}_{IT}, \hat{n}_{IT}) = \bigcap_{i=1}^k U(\widehat{\ell}_{IT}, \hat{n}_{IT})^i$, $L(\widehat{\ell}_{IF}, \hat{n}_{IF}) = \bigcap_{i=1}^k L(\widehat{\ell}_{IF}, \hat{n}_{IF})^i$ and $L(\widehat{\ell}_F, \hat{n}_F) = \bigcap_{i=1}^k L(\widehat{\ell}_F, \hat{n}_F)^i$, where

$$\begin{aligned}
U(\widehat{\ell}_T, \hat{n}_T)^i &:= \{z \in X \mid (\pi_i \circ \widehat{\ell}_T)(z) \geq \hat{n}_T^i\},\\
U(\widehat{\ell}_{IT}, \hat{n}_{IT})^i &:= \{z \in X \mid (\pi_i \circ \widehat{\ell}_{IT})(z) \geq \hat{n}_{IT}^i\},\\
L(\widehat{\ell}_{IF}, \hat{n}_{IF})^i &:= \{z \in X \mid (\pi_i \circ \widehat{\ell}_{IF})(z) \leq \hat{n}_{IF}^i\},\\
L(\widehat{\ell}_F, \hat{n}_F)^i &:= \{z \in X \mid (\pi_i \circ \widehat{\ell}_F)(z) \leq \hat{n}_F^i\}
\end{aligned}$$

for $i = 1, 2, \cdots, k$.

We handle the characterization of k-polar generalized neutrosophic subalgebra.

Theorem 3.6. *Let $\widehat{\mathcal{L}} := \left(\widehat{\ell}_T, \widehat{\ell}_{IT}, \widehat{\ell}_{IF}, \widehat{\ell}_F\right)$ be a k-polar generalized neutrosophic set over X. Then $\widehat{\mathcal{L}}$ is a k-polar generalized neutrosophic subalgebra of X if and only if the cut sets $U(\widehat{\ell}_T, \hat{n}_T)$, $U(\widehat{\ell}_{IT}, \hat{n}_{IT})$, $L(\widehat{\ell}_{IF}, \hat{n}_{IF})$ and $L(\widehat{\ell}_F, \hat{n}_F)$ are subalgebras of X for all $\hat{n}_T, \hat{n}_{IT}, \hat{n}_{IF}, \hat{n}_F \in [0, 1]^k$.*

Proof. Assume that $\widehat{\mathcal{L}}$ is a k-polar generalized neutrosophic subalgebra of X. Let $z, y \in X$. If $z, y \in U(\widehat{\ell}_T, \hat{n}_T)$ for all $\hat{n}_T \in [0, 1]^k$, then $(\pi_i \circ \widehat{\ell}_T)(z) \geq n_T^i$ and $(\pi_i \circ \widehat{\ell}_T)(y) \geq n_T^i$ for $i = 1, 2, \cdots, k$. It follows that

$$(\pi_i \circ \widehat{\ell}_T)(z * y) \geq (\pi_i \circ \widehat{\ell}_T)(z) \wedge (\pi_i \circ \widehat{\ell}_T)(y) \geq n_T^i$$

$i = 1, 2, \cdots, k$. Hence $z * y \in U(\widehat{\ell}_T, \hat{n}_T)$, and so $U(\widehat{\ell}_T, \hat{n}_T)$ is a subalgebra of X. If $z, y \in L(\widehat{\ell}_F, \hat{n}_F)$ for all $\hat{n}_F \in [0,1]^k$, then $(\pi_i \circ \widehat{\ell}_F)(z) \leq n_F^i$ and $(\pi_i \circ \widehat{\ell}_F)(y) \leq n_F^i$ for $i = 1, 2, \cdots, k$. Hence

$$(\pi_i \circ \widehat{\ell}_F)(z * y) \leq (\pi_i \circ \widehat{\ell}_F)(z) \vee (\pi_i \circ \widehat{\ell}_F)(y) \leq n_F^i$$

$i = 1, 2, \cdots, k$, and so $z * y \in L(\widehat{\ell}_F, \hat{n}_F)$. Therefore $L(\widehat{\ell}_F, \hat{n}_F)$ is a subalgebra of X. Similarly, we can verify that $U(\widehat{\ell}_{IT}, \hat{n}_{IT})$ and $L(\widehat{\ell}_{IF}, \hat{n}_{IF})$ are subalgebras of X.

Conversely, suppose that the cut sets $U(\widehat{\ell}_T, \hat{n}_T)$, $U(\widehat{\ell}_{IT}, \hat{n}_{IT})$, $L(\widehat{\ell}_{IF}, \hat{n}_{IF})$ and $L(\widehat{\ell}_F, \hat{n}_F)$ are subalgebras of X for all $\hat{n}_T, \hat{n}_{IT}, \hat{n}_{IF}, \hat{n}_F \in [0,1]^k$. If there exists $\alpha, \beta \in X$ such that $\widehat{\ell}_{IT}(\alpha * \beta) < \widehat{\ell}_{IT}(\alpha) \wedge \widehat{\ell}_{IT}(\beta)$, that is,

$$(\pi_i \circ \widehat{\ell}_{IT})(\alpha * \beta) < (\pi_i \circ \widehat{\ell}_{IT})(\alpha) \wedge (\pi_i \circ \widehat{\ell}_{IT})(\beta)$$

for $i = 1, 2, \cdots, k$, then $\alpha, \beta \in U(\widehat{\ell}_{IT}, \hat{n}_{IT})^i$ and $\alpha * \beta \notin U(\widehat{\ell}_{IT}, \hat{n}_{IT})^i$ where $\hat{n}_{IT}^i = (\pi_i \circ \widehat{\ell}_{IT})(\alpha) \wedge (\pi_i \circ \widehat{\ell}_{IT})(\beta)$ for for $i = 1, 2, \cdots, k$. This is a contradiction, and so

$$\widehat{\ell}_{IT}(z * y) \geq \widehat{\ell}_{IT}(z) \wedge \widehat{\ell}_{IT}(y)$$

for all $z, y \in X$. By the similarly way, we know that $\widehat{\ell}_T(z * y) \geq \widehat{\ell}_T(z) \wedge \widehat{\ell}_T(y)$ for all $z, y \in X$. Now, suppose that $\widehat{\ell}_F(\alpha * \beta) > \widehat{\ell}_F(\alpha) \vee \widehat{\ell}_F(\beta)$ for some $\alpha, \beta \in X$. Then

$$(\pi_i \circ \widehat{\ell}_F)(\alpha * \beta) > (\pi_i \circ \widehat{\ell}_F)(\alpha) \vee (\pi_i \circ \widehat{\ell}_F)(\beta)$$

for $i = 1, 2, \cdots, k$. If we take $n_F^i = (\pi_i \circ \widehat{\ell}_F)(\alpha) \vee (\pi_i \circ \widehat{\ell}_F)(\beta)$ for $i = 1, 2, \cdots, k$, then $\alpha, \beta \in L(\widehat{\ell}_F, \hat{n}_F)^i$ but $\alpha * \beta \notin L(\widehat{\ell}_F, \hat{n}_F)^i$, a contradiction. Hence

$$\widehat{\ell}_F(z * y) \leq \widehat{\ell}_F(z) \vee \widehat{\ell}_F(y)$$

for all $z, y \in X$. Similarly, we can check that $\widehat{\ell}_{IF}(z * y) \leq \widehat{\ell}_{IF}(z) \vee \widehat{\ell}_{IF}(y)$ for all $z, y \in X$. Therefore $\widehat{\mathcal{L}}$ is a k-polar generalized neutrosophic subalgebra of X. □

Theorem 3.7. *Let $\widehat{\mathcal{L}} := \left(\widehat{\ell}_T, \widehat{\ell}_{IT}, \widehat{\ell}_{IF}, \widehat{\ell}_F\right)$ be a k-polar generalized neutrosophic set over X. Then $\widehat{\mathcal{L}}$ is a k-polar generalized neutrosophic subalgebra of X if and only if the fuzzy sets $\pi_i \circ \widehat{\ell}_T$, $\pi_i \circ \widehat{\ell}_{IT}$, $\pi_i \circ \widehat{\ell}_F^c$ and $\pi_i \circ \widehat{\ell}_{IF}^c$ are fuzzy subalgebras of X where $(\pi_i \circ \widehat{\ell}_F^c)(z) = 1 - (\pi_i \circ \widehat{\ell}_F)(z)$ and $(\pi_i \circ \widehat{\ell}_{IF}^c)(z) = 1 - (\pi_i \circ \widehat{\ell}_{IF})(z)$ for all $z \in X$ and $i = 1, 2, \cdots, k$.*

Proof. Suppose that $\widehat{\mathcal{L}}$ is a k-polar generalized neutrosophic subalgebra of X. For any $i = 1, 2, \cdots, k$, it is clear that $\pi_i \circ \widehat{\ell}_T$ and $\pi_i \circ \widehat{\ell}_{IT}$ are fuzzy subalgebras of X. For any $z, y \in X$, we get

$$\begin{aligned}(\pi_i \circ \widehat{\ell}_F^c)(z * y) &= 1 - (\pi_i \circ \widehat{\ell}_F)(z * y) = 1 - (\pi_i \circ \widehat{\ell}_F)(z) \vee (\pi_i \circ \widehat{\ell}_F)(y) \\ &= (1 - (\pi_i \circ \widehat{\ell}_F)(z)) \wedge (1 - (\pi_i \circ \widehat{\ell}_F)(y)) \\ &= (\pi_i \circ \widehat{\ell}_F^c)(z) \wedge (\pi_i \circ \widehat{\ell}_F^c)(y)\end{aligned}$$

and

$$\begin{aligned}(\pi_i \circ \widehat{\ell}^c_{IF})(z*y) &= 1-(\pi_i \circ \widehat{\ell}_{IF})(z*y) = 1-(\pi_i \circ \widehat{\ell}_{IF})(z) \vee (\pi_i \circ \widehat{\ell}_{IF})(y)\\ &= (1-(\pi_i \circ \widehat{\ell}_{IF})(z)) \wedge (1-(\pi_i \circ \widehat{\ell}_{IF})(y))\\ &= (\pi_i \circ \widehat{\ell}^c_{IF})(z) \wedge (\pi_i \circ \widehat{\ell}^c_{IF})(y).\end{aligned}$$

Hence $\pi_i \circ \widehat{\ell}^c_F$ and $\pi_i \circ \widehat{\ell}^c_{IF}$ are fuzzy subalgebras of X .

Conversely, suppose that the fuzzy sets $\pi_i \circ \widehat{\ell}_T$, $\pi_i \circ \widehat{\ell}_{IT}$, $\pi_i \circ \widehat{\ell}^c_F$ and $\pi_i \circ \widehat{\ell}^c_{IF}$ are fuzzy subalgebras of X for $i = 1, 2, \cdots, k$ and let $z, y \in X$. Then

$$\begin{aligned}(\pi_i \circ \widehat{\ell}_T)(z*y) &\geq (\pi_i \circ \widehat{\ell}_T)(z) \wedge (\pi_i \circ \widehat{\ell}_T)(y),\\ (\pi_i \circ \widehat{\ell}_{IT})(z*y) &\geq (\pi_i \circ \widehat{\ell}_{IT})(z) \wedge (\pi_i \circ \widehat{\ell}_{IT})(y)\end{aligned}$$

for all $i = 1, 2, \cdots, k$. Also we have

$$\begin{aligned}1-(\pi_i \circ \widehat{\ell}_F)(z*y) &= (\pi_i \circ \widehat{\ell}^c_F)(z*y) \geq (\pi_i \circ \widehat{\ell}^c_F)(z) \wedge (\pi_i \circ \widehat{\ell}^c_F)(y)\\ &= (1-(\pi_i \circ \widehat{\ell}_F)(z)) \wedge (1-(\pi_i \circ \widehat{\ell}_F)(y))\\ &= 1-((\pi_i \circ \widehat{\ell}_F)(z) \vee (\pi_i \circ \widehat{\ell}_F)(y))\end{aligned}$$

and

$$\begin{aligned}1-(\pi_i \circ \widehat{\ell}_{IF})(z*y) &= (\pi_i \circ \widehat{\ell}^c_{IF})(z*y) \geq (\pi_i \circ \widehat{\ell}^c_{IF})(z) \wedge (\pi_i \circ \widehat{\ell}^c_{IF})(y)\\ &= (1-(\pi_i \circ \widehat{\ell}_{IF})(z)) \wedge (1-(\pi_i \circ \widehat{\ell}_{IF})(y))\\ &= 1-((\pi_i \circ \widehat{\ell}_{IF})(z) \vee (\pi_i \circ \widehat{\ell}_{IF})(y))\end{aligned}$$

which imply that $(\pi_i \circ \widehat{\ell}_F)(z*y) \leq (\pi_i \circ \widehat{\ell}_F)(z) \vee (\pi_i \circ \widehat{\ell}_F)(y)$ and

$$(\pi_i \circ \widehat{\ell}_{IF})(z*y) \leq (\pi_i \circ \widehat{\ell}_{IF})(z) \vee (\pi_i \circ \widehat{\ell}_{IF})(y)$$

for all $i = 1, 2, \cdots, k$. Hence $\widehat{\mathcal{L}}$ is a k-polar generalized neutrosophic subalgebra of X . □

Theorem 3.8. *If $\widehat{\mathcal{L}} := \left(\widehat{\ell}_T, \widehat{\ell}_{IT}, \widehat{\ell}_{IF}, \widehat{\ell}_F\right)$ is a k-polar generalized neutrosophic subalgebra of X, then so are $\Box\widehat{\mathcal{L}} := \left(\widehat{\ell}_T, \widehat{\ell}_{IT}, \widehat{\ell}^c_{IT}, \widehat{\ell}^c_T\right)$ and $\Diamond\widehat{\mathcal{L}} := \left(\widehat{\ell}^c_{IF}, \widehat{\ell}^c_F, \widehat{\ell}_F, \widehat{\ell}_{IF}\right)$.*

Proof. Note that $(\pi_i \circ \widehat{\ell}_{IT})(z) + (\pi_i \circ \widehat{\ell}^c_{IT})(z) = (\pi_i \circ \widehat{\ell}_{IT})(z) + 1 - (\pi_i \circ \widehat{\ell}_{IT})(z) = 1$ and $(\pi_i \circ \widehat{\ell}_F)(z) + (\pi_i \circ \widehat{\ell}^c_F)(z) = (\pi_i \circ \widehat{\ell}_F)(z) + 1 - (\pi_i \circ \widehat{\ell}_F)(z) = 1$, that is, $\widehat{\ell}_{IT}(z) + \widehat{\ell}^c_{IT}(z) = \hat{1}$ and $\widehat{\ell}_F(z) + \widehat{\ell}^c_F(z) = \hat{1}$ for all $z \in X$. Hence $\Box\widehat{\mathcal{L}} := \left(\widehat{\ell}_T, \widehat{\ell}_{IT}, \widehat{\ell}^c_{IT}, \widehat{\ell}^c_T\right)$ and $\Diamond\widehat{\mathcal{L}} := \left(\widehat{\ell}^c_{IF}, \widehat{\ell}^c_F, \widehat{\ell}_F, \widehat{\ell}_{IF}\right)$ are k-polar generalized neutrosophic sets over X. For any $z, y \in X$, we get

$$\begin{aligned}(\pi_i \circ \widehat{\ell}^c_{IT})(z*y) &= 1-(\pi_i \circ \widehat{\ell}_{IT})(z*y) \leq 1-((\pi_i \circ \widehat{\ell}_{IT})(z) \wedge (\pi_i \circ \widehat{\ell}_{IT})(y))\\ &= (1-(\pi_i \circ \widehat{\ell}_{IT})(z)) \vee (1-(\pi_i \circ \widehat{\ell}_{IT})(y))\\ &= (\pi_i \circ \widehat{\ell}^c_{IT})(z) \vee (\pi_i \circ \widehat{\ell}^c_{IT})(y),\end{aligned}$$

$$\begin{aligned}(\pi_i \circ \widehat{\ell}^c_T)(z*y) &= 1-(\pi_i \circ \widehat{\ell}_T)(z*y) \leq 1-((\pi_i \circ \widehat{\ell}_T)(z) \wedge (\pi_i \circ \widehat{\ell}_T)(y)) \\ &= (1-(\pi_i \circ \widehat{\ell}_T)(z)) \vee (1-(\pi_i \circ \widehat{\ell}_T)(y)) \\ &= (\pi_i \circ \widehat{\ell}^c_T)(z) \vee (\pi_i \circ \widehat{\ell}^c_T)(y),\end{aligned}$$

$$\begin{aligned}(\pi_i \circ \widehat{\ell}^c_{IF})(z*y) &= 1-(\pi_i \circ \widehat{\ell}_{IF})(z*y) \geq 1-((\pi_i \circ \widehat{\ell}_{IF})(z) \vee (\pi_i \circ \widehat{\ell}_{IF})(y)) \\ &= (1-(\pi_i \circ \widehat{\ell}_{IF})(z)) \wedge (1-(\pi_i \circ \widehat{\ell}_{IF})(y)) \\ &= (\pi_i \circ \widehat{\ell}^c_{IF})(z) \wedge (\pi_i \circ \widehat{\ell}^c_{IF})(y),\end{aligned}$$

and

$$\begin{aligned}(\pi_i \circ \widehat{\ell}^c_F)(z*y) &= 1-(\pi_i \circ \widehat{\ell}_F)(z*y) \geq 1-((\pi_i \circ \widehat{\ell}_F)(z) \vee (\pi_i \circ \widehat{\ell}_F)(y)) \\ &= (1-(\pi_i \circ \widehat{\ell}_F)(z)) \wedge (1-(\pi_i \circ \widehat{\ell}_F)(y)) \\ &= (\pi_i \circ \widehat{\ell}^c_F)(z) \wedge (\pi_i \circ \widehat{\ell}^c_F)(y).\end{aligned}$$

Therefore $\Box\widehat{\mathcal{L}} := \left(\widehat{\ell}_T, \widehat{\ell}_{IT}, \widehat{\ell}^c_{IT}, \widehat{\ell}^c_T\right)$ and $\Diamond\widehat{\mathcal{L}} := \left(\widehat{\ell}^c_{IF}, \widehat{\ell}^c_F, \widehat{\ell}_F, \widehat{\ell}_{IF}\right)$ are $k$polar generalized neutrosophic subalgebras of X. $\square$

Theorem 3.9. *Let $\Lambda_1 \times \Lambda_2 \times \cdots \times \Lambda_k \subseteq [0,1]^k$, that is, $\Lambda_i \subseteq [0,1]$ for $i = 1, 2, \cdots, k$. Let $\mathcal{S}_i := \{S_{t_i} \mid t_i \in \Lambda_i\}$ be a family of subalgebras of X for $i = 1, 2, \cdots, k$ such that*

$$X = \bigcup_{t_i \in \Lambda_i} S_i, \tag{3.9}$$

$$(\forall s_i, t_i \in \Lambda_i)\,(s_i > t_i \;\Rightarrow\; S_{s_i} \subset S_{t_i}) \tag{3.10}$$

for $i = 1, 2, \cdots, k$. Let $\widehat{\mathcal{L}} := \left(\widehat{\ell}_T, \widehat{\ell}_{IT}, \widehat{\ell}_{IF}, \widehat{\ell}_F\right)$ be a k-polar generalized neutrosophic set over X defined by

$$(\forall z \in X) \quad \left(\begin{array}{l}(\pi_i \circ \widehat{\ell}_T)(z) = \bigvee\{q_i \in \Lambda_i \mid z \in S_{q_i}\} = (\pi_i \circ \widehat{\ell}_{IT})(z), \\ (\pi_i \circ \widehat{\ell}_{IF})(z) = \bigwedge\{r_i \in \Lambda_i \mid z \in S_{r_i}\} = (\pi_i \circ \widehat{\ell}_F)(z)\end{array}\right) \tag{3.11}$$

for $i = 1, 2, \cdots, k$. Then $\widehat{\mathcal{L}} := \left(\widehat{\ell}_T, \widehat{\ell}_{IT}, \widehat{\ell}_{IF}, \widehat{\ell}_F\right)$ is a k-polar generalized neutrosophic subalgebra of X.

Proof. For any $i = 1, 2, \cdots, k$, we consider the following two cases.

$$t_i = \bigvee\{q_i \in \Lambda_i \mid q_i < t_i\} \text{ and } t_i \neq \bigvee\{q_i \in \Lambda_i \mid q_i < t_i\}.$$

The first case implies that

$$z \in U(\widehat{\ell}_T, t_i) \Leftrightarrow (\forall q_i < t_i)(z \in S_{q_i}) \Leftrightarrow z \in \bigcap_{q_i < t_i} S_{q_i},$$

$$z \in U(\widehat{\ell}_{IT}, t_i) \Leftrightarrow (\forall q_i < t_i)(z \in S_{q_i}) \Leftrightarrow z \in \bigcap_{q_i < t_i} S_{q_i}.$$

Hence $U(\widehat{\ell}_T, t_i) = \bigcap_{q_i < t_i} S_{q_i} = U(\widehat{\ell}_{IT}, t_i)$, and so $U(\widehat{\ell}_T, t_i)$ and $U(\widehat{\ell}_{IT}, t_i)$ are subalgebras of X for all $i = 1, 2, \ldots, k$. Hence $U(\widehat{\ell}_T, \hat{t}) = \bigcap_{i=1,2,\ldots,k} U(\widehat{\ell}_T, t_i)$ and $U(\widehat{\ell}_{IT}, \hat{t}) = \bigcap_{i=1,2,\ldots,k} U(\widehat{\ell}_{IT}, t_i)$ are subalgebras of X. For the second case, we will show that $U(\widehat{\ell}_T, t_i) = \bigcup_{q_i \geq t_i} S_{q_i} = U(\widehat{\ell}_{IT}, t_i)$ for all $i = 1, 2, \ldots, k$. If $z \in \bigcup_{q_i \geq t_i} S_{q_i}$, then $z \in S_{q_i}$ for some $q_i \geq t_i$. Hence $(\pi_i \circ \widehat{\ell}_{IT})(z) = (\pi_i \circ \widehat{\ell}_T)(z) \geq q_i \geq t_i$, and so $z \in U(\widehat{\ell}_T, t_i)$ and $z \in U(\widehat{\ell}_{IT}, t_i)$. If $z \notin \bigcup_{q_i \geq t_i} S_{q_i}$, then $z \notin S_{q_i}$ for all $q_i \geq t_i$. The condition $t_i \neq \bigvee\{q_i \in \Lambda_i \mid q_i < t_i\}$ induces $(t_i - \varepsilon_i, t_i) \cap \Lambda_i = \emptyset$ for some $\varepsilon_i > 0$. Hence $z \notin S_{q_i}$ for all $q_i > t_i - \varepsilon_i$, which means that if $z \in S_{q_i}$ then $q_i \leq t_i - \varepsilon_i$. Hence $(\pi_i \circ \widehat{\ell}_{IT})(z) = (\pi_i \circ \widehat{\ell}_T)(z) \leq t_i - \varepsilon_i < t_i$ and so $z \notin U(\widehat{\ell}_{IT}, t_i) = U(\widehat{\ell}_T, t_i)$. Therefore $U(\widehat{\ell}_T, t_i) = U(\widehat{\ell}_{IT}, t_i) \subseteq \bigcup_{q_i \geq t_i} S_{q_i}$. Consequently, $U(\widehat{\ell}_T, t_i) = U(\widehat{\ell}_{IT}, t_i) = \bigcup_{q_i \geq t_i} S_{q_i}$ which is a subalgebra of X, and therefore $U(\widehat{\ell}_T, \hat{t}) = \bigcap_{i=1,2,\ldots,k} U(\widehat{\ell}_T, t_i)$ and $U(\widehat{\ell}_{IT}, \hat{t}) = \bigcap_{i=1,2,\ldots,k} U(\widehat{\ell}_{IT}, t_i)$ are subalgebras of X. Now, we consider the following two cases.

$$s_i = \bigwedge\{r_i \in \Lambda_i \mid r_i > s_i\} \text{ and } s_i \neq \bigwedge\{r_i \in \Lambda_i \mid r_i > s_i\}.$$

For the first case, we get

$$z \in L(\widehat{\ell}_{IF}, s_i) \Leftrightarrow (\forall s_i < r_i)(z \in S_{r_i}) \Leftrightarrow z \in \bigcap_{r_i > s_i} S_{r_i},$$
$$z \in L(\widehat{\ell}_F, s_i) \Leftrightarrow (\forall s_i < r_i)(z \in S_{r_i}) \Leftrightarrow z \in \bigcap_{r_i > s_i} S_{r_i}.$$

It follows that $L(\widehat{\ell}_{IF}, s_i) = L(\widehat{\ell}_F, s_i) = \bigcap_{r_i > s_i} S_{r_i}$, which is a subalgebra of X. The second case induces $(s_i, s_i + \varepsilon_i) \cap \Lambda_i = \emptyset$ for some $\varepsilon_i > 0$. If $z \in \bigcup_{r_i \leq s_i} S_{r_i}$, then $z \in S_{r_i}$ for some $r_i \leq s_i$, and thus $(\pi_i \circ \widehat{\ell}_{IF})(z) = (\pi_i \circ \widehat{\ell}_F)(z) \leq r_i \leq s_i$, i.e., $z \in L(\widehat{\ell}_{IF}, s_i)$ and $z \in L(\widehat{\ell}_F, s_i)$. Hene $\bigcup_{r_i \leq s_i} S_{r_i} \subseteq L(\widehat{\ell}_{IF}, s_i) = L(\widehat{\ell}_F, s_i)$. If $z \notin \bigcup_{r_i \leq s_i} S_{r_i}$, then $z \notin S_{r_i}$ for all $r_i \leq s_i$ which implies that $z \notin S_{r_i}$ for all $r_i \leq s_i + \varepsilon_i$, that is, if $z \in S_{r_i}$ then $r_i \geq s_i + \varepsilon_i$. Thus $(\pi_i \circ \widehat{\ell}_{IF})(z) = (\pi_i \circ \widehat{\ell}_F)(z) \geq s_i + \varepsilon_i \geq s_i$ and so $z \notin L(\widehat{\ell}_{IF}, s_i) = L(\widehat{\ell}_F, s_i)$. This shows that $L(\widehat{\ell}_{IF}, s_i) = L(\widehat{\ell}_F, s_i) = \bigcup_{r_i \leq s_i} S_{r_i}$, which is a subalgebra of X. Therefore $L(\widehat{\ell}_F, \hat{s}) = \bigcap_{i=1,2,\ldots,k} L(\widehat{\ell}_F, s_i)$ and $U(\widehat{\ell}_{IF}, \hat{s}) = \bigcap_{i=1,2,\ldots,k} L(\widehat{\ell}_{IF}, s_i)$ are subalgebras of X. Using Theorem 3.6, we know that $\widehat{\mathcal{L}} := \left(\widehat{\ell}_T, \widehat{\ell}_{IT}, \widehat{\ell}_{IF}, \widehat{\ell}_F\right)$ is a k-polar generalized neutrosophic subalgebra of X. □

4 k-polar generalized $(\in, \in\vee q)$-neutrosophic subalgebras

Let $\hat{n}_T = (n_T^1, n_T^2, \cdots, n_T^k)$, $\hat{n}_{IT} = (n_{IT}^1, n_{IT}^2, \cdots, n_{IT}^k)$, $\hat{n}_{IF} = (n_{IF}^1, n_{IF}^2, \cdots, n_{IF}^k)$ and $\hat{n}_F = (n_F^1, n_F^2, \cdots, n_F^k)$ in $[0, 1]^k$. Given a k-polar generalized neutrosophic set $\widehat{\mathcal{L}} := \left(\widehat{\ell}_T, \widehat{\ell}_{IT}, \widehat{\ell}_{IF}, \widehat{\ell}_F\right)$ over a universe X, we consider the following sets.

$$\begin{aligned}
&T_q(\widehat{\ell}_T, \hat{n}_T) := \{z \in X \mid \widehat{\ell}_T(z) + \hat{n}_T > \hat{1}\},\\
&IT_q(\widehat{\ell}_{IT}, \hat{n}_{IT}) := \{z \in X \mid \widehat{\ell}_{IT}(z) + \hat{n}_{IT} > \hat{1}\},\\
&IF_q(\widehat{\ell}_{IF}, \hat{n}_{IF}) := \{z \in X \mid \widehat{\ell}_{IF}(z) + \hat{n}_{IF} < \hat{1}\},\\
&F_q(\widehat{\ell}_F, \hat{n}_F) := \{z \in X \mid \widehat{\ell}_F(z) + \hat{n}_F < \hat{1}\},
\end{aligned}$$

which are called *generaliged neutrosophic q-sets*, and

$$\begin{aligned}
&T_{\in\vee q}(\widehat{\ell}_T, \hat{n}_T) := \{z \in X \mid \widehat{\ell}_T(z) \geq \hat{n}_T \text{ or } \widehat{\ell}_T(z) + \hat{n}_T > \hat{1}\},\\
&IT_{\in\vee q}(\widehat{\ell}_{IT}, \hat{n}_{IT}) := \{z \in X \mid \widehat{\ell}_{IT}(z) \geq \hat{n}_{IT} \text{ or } \widehat{\ell}_{IT}(z) + \hat{n}_{IT} > \hat{1}\},\\
&IF_{\in\vee q}(\widehat{\ell}_{IF}, \hat{n}_{IF}) := \{z \in X \mid \widehat{\ell}_{IF}(z) \leq \hat{n}_{IF} \text{ or } \widehat{\ell}_{IF}(z) + \hat{n}_{IF} < \hat{1}\},\\
&F_{\in\vee q}(\widehat{\ell}_F, \hat{n}_F) := \{z \in X \mid \widehat{\ell}_F(z) \leq \hat{n}_F \text{ or } \widehat{\ell}_F(z) + \hat{n}_F < \hat{1}\}
\end{aligned}$$

which are called *generaliged neutrosophic $\in\vee q$-sets*. Then

$$T_q(\widehat{\ell}_T, \hat{n}_T) = \bigcap_{i=1}^{k} T_q(\widehat{\ell}_T, \hat{n}_T)^i,\ IT_q(\widehat{\ell}_{IT}, \hat{n}_{IT}) = \bigcap_{i=1}^{k} IT_q(\widehat{\ell}_{IT}, \hat{n}_{IT})^i,$$
$$IF_q(\widehat{\ell}_{IF}, \hat{n}_{IF}) = \bigcap_{i=1}^{k} IF_q(\widehat{\ell}_{IF}, \hat{n}_{IF})^i,\ F_q(\widehat{\ell}_F, \hat{n}_F) = \bigcap_{i=1}^{k} F_q(\widehat{\ell}_F, \hat{n}_F)^i$$

and

$$T_{\in\vee q}(\widehat{\ell}_T, \hat{n}_T) = \bigcap_{i=1}^{k} T_{\in\vee q}(\widehat{\ell}_T, \hat{n}_T)^i,\ IT_{\in\vee q}(\widehat{\ell}_{IT}, \hat{n}_{IT}) = \bigcap_{i=1}^{k} IT_{\in\vee q}(\widehat{\ell}_{IT}, \hat{n}_{IT})^i,$$
$$IF_{\in\vee q}(\widehat{\ell}_{IF}, \hat{n}_{IF}) = \bigcap_{i=1}^{k} IF_{\in\vee q}(\widehat{\ell}_{IF}, \hat{n}_{IF})^i,\ F_{\in\vee q}(\widehat{\ell}_F, \hat{n}_F) = \bigcap_{i=1}^{k} F_{\in\vee q}(\widehat{\ell}_F, \hat{n}_F)^i$$

where

$$\begin{aligned}
&T_q(\widehat{\ell}_T, \hat{n}_T)^i = \{z \in X \mid (\pi_i \circ \widehat{\ell}_T)(z) + n_T^i > 1\},\\
&IT_q(\widehat{\ell}_{IT}, \hat{n}_{IT})^i = \{z \in X \mid (\pi_i \circ \widehat{\ell}_{IT})(z) + n_{IT}^i > 1\},\\
&IF_q(\widehat{\ell}_{IF}, \hat{n}_{IF})^i = \{z \in X \mid (\pi_i \circ \widehat{\ell}_{IF})(z) + n_{IF}^i < 1\},\\
&F_q(\widehat{\ell}_F, \hat{n}_F)^i = \{z \in X \mid (\pi_i \circ \widehat{\ell}_F)(z) + n_F^i < 1\}
\end{aligned}$$

and

$$\begin{aligned}
&T_{\in\vee q}(\widehat{\ell}_T, \hat{n}_T)^i = \{z \in X \mid (\pi_i \circ \widehat{\ell}_T)(z) \geq n_T^i \text{ or } (\pi_i \circ \widehat{\ell}_T)(z) + n_T^i > 1\},\\
&IT_{\in\vee q}(\widehat{\ell}_{IT}, \hat{n}_{IT})^i = \{z \in X \mid (\pi_i \circ \widehat{\ell}_{IT})(z) \geq n_{IT}^i \text{ or } (\pi_i \circ \widehat{\ell}_{IT})(z) + n_{IT}^i > 1\},\\
&IF_{\in\vee q}(\widehat{\ell}_{IF}, \hat{n}_{IF})^i = \{z \in X \mid (\pi_i \circ \widehat{\ell}_{IF})(z) \leq n_{IF}^i \text{ or } (\pi_i \circ \widehat{\ell}_{IF})(z) + n_{IF}^i < 1\},\\
&F_{\in\vee q}(\widehat{\ell}_F, \hat{n}_F)^i = \{z \in X \mid (\pi_i \circ \widehat{\ell}_F)(z) \leq n_F^i \text{ or } (\pi_i \circ \widehat{\ell}_F)(z) + n_F^i < 1\}.
\end{aligned}$$

It is clear that $T_{\in\vee q}(\widehat{\ell}_T, \hat{n}_T) = U(\widehat{\ell}_T, \hat{n}_T) \cup T_q(\widehat{\ell}_T, \hat{n}_T)$, $IT_{\in\vee q}(\widehat{\ell}_{IT}, \hat{n}_{IT}) = U(\widehat{\ell}_{IT}, \hat{n}_{IT}) \cup IT_q(\widehat{\ell}_{IT}, \hat{n}_{IT})$, $IF_{\in\vee q}(\widehat{\ell}_{IF}, \hat{n}_{IF}) = L(\widehat{\ell}_{IF}, \hat{n}_{IF}) \cup IF_q(\widehat{\ell}_{IF}, \hat{n}_{IF})$, and $F_{\in\vee q}(\widehat{\ell}_F, \hat{n}_F) = L(\widehat{\ell}_F, \hat{n}_F) \cup F_q(\widehat{\ell}_F, \hat{n}_F)$.

By routine calculations, we have the following properties.

Proposition 4.1. *Given a k-polar generalized neutrosophic set $\widehat{\mathcal{L}} := \left(\widehat{\ell}_T, \widehat{\ell}_{IT}, \widehat{\ell}_{IF}, \widehat{\ell}_F\right)$ over a universe X, we have*

1. *If $\hat{n}_T, \hat{n}_{IT} \in [0, 0.5]^k$, then $T_{\in\vee q}(\widehat{\ell}_T, \hat{n}_T) = U(\widehat{\ell}_T, \hat{n}_T)$ and $IT_{\in\vee q}(\widehat{\ell}_{IT}, \hat{n}_{IT}) = U(\widehat{\ell}_{IT}, \hat{n}_{IT})$.*
2. *If $\hat{n}_F, \hat{n}_{IF} \in [0.5, 1]^k$, then $IF_{\in\vee q}(\widehat{\ell}_{IF}, \hat{n}_{IF}) = L(\widehat{\ell}_{IF}, \hat{n}_{IF})$ and $F_{\in\vee q}(\widehat{\ell}_F, \hat{n}_F) = L(\widehat{\ell}_F, \hat{n}_F)$.*
3. *If $\hat{n}_T, \hat{n}_{IT} \in (0.5, 1]^k$, then $T_{\in\vee q}(\widehat{\ell}_T, \hat{n}_T) = T_q(\widehat{\ell}_T, \hat{n}_T)$ and $IT_{\in\vee q}(\widehat{\ell}_{IT}, \hat{n}_{IT}) = IT_q(\widehat{\ell}_{IT}, \hat{n}_{IT})$.*
4. *If $\hat{n}_F, \hat{n}_{IF} \in [0, 0.5)^k$, then $IF_{\in\vee q}(\widehat{\ell}_{IF}, \hat{n}_{IF}) = IF_q(\widehat{\ell}_{IF}, \hat{n}_{IF})$ and $F_{\in\vee q}(\widehat{\ell}_F, \hat{n}_F) = F_q(\widehat{\ell}_F, \hat{n}_F)$.*

Unless otherwise stated in this section, X will represent a BCK/BCI-algebra.

Definition 4.2. Let $\widehat{\mathcal{L}} := \left(\widehat{\ell}_T, \widehat{\ell}_{IT}, \widehat{\ell}_{IF}, \widehat{\ell}_F\right)$ be a k-polar generalized neutrosophic set over X. Then $\widehat{\mathcal{L}}$ is called a *k-polar generalized $(\in, \in\vee q)$-neutrosophic subalgebra* of X if it satisfies:

$$\begin{aligned}
&z \in U(\widehat{\ell}_T, \hat{n}_T),\ y \in U(\widehat{\ell}_T, \hat{n}_T) \ \Rightarrow\ z * y \in T_{\in\vee q}(\widehat{\ell}_T, \hat{n}_T),\\
&z \in U(\widehat{\ell}_{IT}, \hat{n}_{IT}),\ y \in U(\widehat{\ell}_{IT}, \hat{n}_{IT}) \ \Rightarrow\ z * y \in IT_{\in\vee q}(\widehat{\ell}_{IT}, \hat{n}_{IT}),\\
&z \in L(\widehat{\ell}_{IF}, \hat{n}_{IF}),\ y \in L(\widehat{\ell}_{IF}, \hat{n}_{IF}) \ \Rightarrow\ z * y \in IF_{\in\vee q}(\widehat{\ell}_{IF}, \hat{n}_{IF}),\\
&z \in L(\widehat{\ell}_F, \hat{n}_F),\ y \in L(\widehat{\ell}_F, \hat{n}_F) \ \Rightarrow\ z * y \in F_{\in\vee q}(\widehat{\ell}_F, \hat{n}_F)
\end{aligned} \tag{4.1}$$

for all $z, y \in X$, $\hat{n}_T, \hat{n}_{IT} \in (0, 1]^k$ and $\hat{n}_F, \hat{n}_{IF} \in [0, 1)^k$.

Example 4.3. Consider a BCI-algebra $X = \{0, 1, 2, \alpha, \beta\}$ with the binary operation "$*$" which is given below.

$*$	0	1	2	α	β
0	0	0	0	α	α
1	1	0	1	β	α
2	2	2	0	α	α
α	α	α	α	0	0
β	β	α	β	1	0

Let $\widehat{\mathcal{L}} := \left(\widehat{\ell}_T, \widehat{\ell}_{IT}, \widehat{\ell}_{IF}, \widehat{\ell}_F\right)$ be a 3-polar neutrosophic set over X in which $\widehat{\ell}_T$, $\widehat{\ell}_{IT}$, $\widehat{\ell}_{IF}$ and $\widehat{\ell}_F$ are defined as follows:

$$\widehat{\ell}_T : X \to [0,1]^3,\ z \mapsto \begin{cases} (0.6, 0.5, 0.5) & \text{if } z = 0, \\ (0.7, 0.7, 0.2) & \text{if } z = 1, \\ (0.7, 0.8, 0.5) & \text{if } z = 2, \\ (0.3, 0.4, 0.5) & \text{if } z = \alpha, \\ (0.3, 0.4, 0.2) & \text{if } z = \beta, \end{cases}$$

$$\widehat{\ell}_{IT} : X \to [0,1]^3,\ z \mapsto \begin{cases} (0.6, 0.5, 0.6) & \text{if } z = 0, \\ (0.4, 0.3, 0.7) & \text{if } z = 1, \\ (0.6, 0.8, 0.4) & \text{if } z = 2, \\ (0.7, 0.4, 0.1) & \text{if } z = \alpha, \\ (0.4, 0.3, 0.1) & \text{if } z = \beta, \end{cases}$$

$$\widehat{\ell}_{IF} : X \to [0,1]^3,\ z \mapsto \begin{cases} (0.3, 0.1, 0.5) & \text{if } z = 0, \\ (0.8, 0.3, 0.7) & \text{if } z = 1, \\ (0.3, 0.8, 0.5) & \text{if } z = 2, \\ (0.7, 0.9, 0.6) & \text{if } z = \alpha, \\ (0.8, 0.9, 0.7) & \text{if } z = \beta, \end{cases}$$

$$\widehat{\ell}_F : X \to [0,1]^3,\ z \mapsto \begin{cases} (0.2, 0.2, 0.5) & \text{if } z = 0, \\ (0.3, 0.9, 0.8) & \text{if } z = 1, \\ (0.5, 0.2, 0.4) & \text{if } z = 2, \\ (0.6, 0.4, 0.6) & \text{if } z = \alpha, \\ (0.6, 0.9, 0.8) & \text{if } z = \beta, \end{cases}$$

It is routine to verify that $\widehat{\mathcal{L}} := \left(\widehat{\ell}_T, \widehat{\ell}_{IT}, \widehat{\ell}_{IF}, \widehat{\ell}_F\right)$ is 3-polar generalized $(\in,\ \in\vee q)$-neutrosophic subalgebra.

Theorem 4.4. *If $\widehat{\mathcal{L}} := \left(\widehat{\ell}_T, \widehat{\ell}_{IT}, \widehat{\ell}_{IF}, \widehat{\ell}_F\right)$ is a k-polar generalized neutrosophic subalgebra of X, then the generaliged neutrosophic q-sets $T_q(\widehat{\ell}_T, \hat{n}_T)$, $IT_q(\widehat{\ell}_{IT}, \hat{n}_{IT})$, $IF_q(\widehat{\ell}_{IF}, \hat{n}_{IF})$ and $F_q(\widehat{\ell}_F, \hat{n}_F)$ are subalgebras of X for all $\hat{n}_T, \hat{n}_{IT} \in (0,1]^k$ and $\hat{n}_F, \hat{n}_{IF} \in [0,1)^k$.*

Proof. Let $z, y \in T_q(\widehat{\ell}_T, \hat{n}_T)$. Then $\widehat{\ell}_T(z) + \hat{n}_T > \hat{1}$ and $\widehat{\ell}_T(y) + \hat{n}_T > \hat{1}$, that is, $(\pi_i \circ \widehat{\ell}_T)(z) + n_T^i > 1$ and $(\pi_i \circ \widehat{\ell}_T)(y) + n_T^i > 1$ for $i = 1, 2, \cdots, k$. It follows that

$$\begin{aligned}(\pi_i \circ \widehat{\ell}_T)(z * y) + n_T^i &\geq ((\pi_i \circ \widehat{\ell}_T)(z) \wedge (\pi_i \circ \widehat{\ell}_T)(y)) + n_T^i \\ &= ((\pi_i \circ \widehat{\ell}_T)(z) + n_T)^i \wedge ((\pi_i \circ \widehat{\ell}_T)(y) + n_T)^i > 1\end{aligned}$$

for $i = 1, 2, \cdots, k$. Hence $\widehat{\ell}_T(z * y) + \hat{n}_T > \hat{1}$, that is, $z * y \in T_q(\widehat{\ell}_T, \hat{n}_T)$. Therefore $T_q(\widehat{\ell}_T, \hat{n}_T)$ is a subalgebra of X. Let $z, y \in IF_q(\widehat{\ell}_{IF}, \hat{n}_{IF})$. Then $(\pi_i \circ \widehat{\ell}_{IF})(z) + n_{IF}^i < 1$ and $(\pi_i \circ \widehat{\ell}_{IF})(y) + n_{IF}^i < 1$ for $i = 1, 2, \cdots, k$.

Hence

$$(\pi_i \circ \widehat{\ell}_{IF})(z*y) + n^i_{IF} \leq ((\pi_i \circ \widehat{\ell}_{IF})(z) \vee (\pi_i \circ \widehat{\ell}_{IF})(y)) + n^i_{IF}$$
$$= ((\pi_i \circ \widehat{\ell}_{IF})(z) + n_{IF})^i \vee ((\pi_i \circ \widehat{\ell}_{IF})(y) + n_{IF})^i < 1$$

for $i = 1, 2, \cdots, k$ and so $\widehat{\ell}_{IF}(z*y) + \hat{n}_{IF} < \hat{1}$. Thus $z*y \in IF_q(\widehat{\ell}_{IF}, \hat{n}_{IF})$ and $IF_q(\widehat{\ell}_{IF}, \hat{n}_{IF})$ is a subalgebra of X. By the similar way, we can verify that $IT_q(\widehat{\ell}_{IT}, \hat{n}_{IT})$ and $F_q(\widehat{\ell}_F, \hat{n}_F)$ are subalgebras of X. □

We handle characterizations of a k-polar generalized $(\in, \in\vee q)$-neutrosophic subalgebra.

Theorem 4.5. *Let* $\widehat{\mathcal{L}} := \left(\widehat{\ell}_T, \widehat{\ell}_{IT}, \widehat{\ell}_{IF}, \widehat{\ell}_F\right)$ *be a k-polar generalized neutrosophic set over X. Then $\widehat{\mathcal{L}}$ is a k-polar generalized $(\in, \in\vee q)$-neutrosophic subalgebra of X if and only if it satisfies:*

$$(\forall z, y \in X) \begin{pmatrix} \widehat{\ell}_T(z*y) \geq \bigwedge\{\widehat{\ell}_T(z), \widehat{\ell}_T(y), \widehat{0.5}\} \\ \widehat{\ell}_{IT}(z*y) \geq \bigwedge\{\widehat{\ell}_{IT}(z), \widehat{\ell}_{IT}(y), \widehat{0.5}\} \\ \widehat{\ell}_{IF}(z*y) \leq \bigvee\{\widehat{\ell}_{IF}(z), \widehat{\ell}_{IF}(y), \widehat{0.5}\} \\ \widehat{\ell}_F(z*y) \leq \bigvee\{\widehat{\ell}_F(z), \widehat{\ell}_F(y), \widehat{0.5}\} \end{pmatrix}, \tag{4.2}$$

that is,

$$\begin{cases} (\pi_i \circ \widehat{\ell}_T)(z*y) \geq \bigwedge\{(\pi_i \circ \widehat{\ell}_T)(z), (\pi_i \circ \widehat{\ell}_T)(y), 0.5\}, \\ (\pi_i \circ \widehat{\ell}_{IT})(z*y) \geq \bigwedge\{(\pi_i \circ \widehat{\ell}_{IT})(z), (\pi_i \circ \widehat{\ell}_{IT})(y), 0.5\}, \\ (\pi_i \circ \widehat{\ell}_{IF})(z*y) \leq \bigvee\{(\pi_i \circ \widehat{\ell}_{IF})(z), (\pi_i \circ \widehat{\ell}_{IF})(y), 0.5\}, \\ (\pi_i \circ \widehat{\ell}_F)(z*y) \leq \bigvee\{(\pi_i \circ \widehat{\ell}_F)(z), (\pi_i \circ \widehat{\ell}_F)(y), 0.5\} \end{cases} \tag{4.3}$$

for all $z, y \in X$ and $i = 1, 2, \cdots, k$.

Proof. Suppose that $\widehat{\mathcal{L}} := \left(\widehat{\ell}_T, \widehat{\ell}_{IT}, \widehat{\ell}_{IF}, \widehat{\ell}_F\right)$ is a k-polar generalized $(\in, \in\vee q)$-neutrosophic subalgebra of X and let $z, y \in X$. For any $i = 1, 2, \ldots, k$, assume that $(\pi_i \circ \widehat{\ell}_{IT})(z) \wedge (\pi_i \circ \widehat{\ell}_{IT})(y) < 0.5$. Then

$$(\pi_i \circ \widehat{\ell}_{IT})(z*y) \geq (\pi_i \circ \widehat{\ell}_{IT})(z) \wedge (\pi_i \circ \widehat{\ell}_{IT})(y)$$

because if $(\pi_i \circ \widehat{\ell}_{IT})(z*y) < (\pi_i \circ \widehat{\ell}_{IT})(z) \wedge (\pi_i \circ \widehat{\ell}_{IT})(y)$, then there exists $n^i_{IT} \in (0, 0.5)$ such that

$$(\pi_i \circ \widehat{\ell}_{IT})(z*y) < n^i_{IT} \leq (\pi_i \circ \widehat{\ell}_{IT})(z) \wedge (\pi_i \circ \widehat{\ell}_{IT})(y).$$

It follows that $z \in U(\widehat{\ell}_{IT}, n_{IT})^i$ and $y \in U(\widehat{\ell}_{IT}, n_{IT})^i$ but $z*y \notin U(\widehat{\ell}_{IT}, n_{IT})^i$. Also $(\pi_i \circ \widehat{\ell}_{IT})(z*y) + n^i_{IT} < 1$, i.e., $z*y \notin IT_q(\widehat{\ell}_{IT}, \hat{n}_{IT})$. Hence $z*y \notin IT_{\in\vee q}(\widehat{\ell}_{IT}, \hat{n}_{IT})$ which is a contradiction. Therefore

$$(\pi_i \circ \widehat{\ell}_{IT})(z*y) \geq \bigwedge\{(\pi_i \circ \widehat{\ell}_{IT})(z), (\pi_i \circ \widehat{\ell}_{IT})(y), 0.5\}$$

for all $z, y \in X$ with $(\pi_i \circ \widehat{\ell}_{IT})(z) \wedge (\pi_i \circ \widehat{\ell}_{IT})(y) < 0.5$. Now suppose that $(\pi_i \circ \widehat{\ell}_{IT})(z) \wedge (\pi_i \circ \widehat{\ell}_{IT})(y) \geq 0.5$. Then $z \in U(\widehat{\ell}_{IT}, 0.5)^i$ and $y \in U(\widehat{\ell}_{IT}, 0.5)^i$, and so $z*y \in IT_{\in\vee q}(\widehat{\ell}_{IT}, 0.5)^i = U(\widehat{\ell}_{IT}, 0.5)^i \cup IT_q(\widehat{\ell}_{IT}, 0.5)^i$.

Hence $z * y \in U(\widehat{\ell}_{IT}, 0.5)^i$. Otherwise, $(\pi_i \circ \widehat{\ell}_{IT})(z * y) + 0.5 < 0.5 + 0.5 = 1$, a contradiction. Consequently,

$$(\pi_i \circ \widehat{\ell}_{IT})(z * y) \geq \bigwedge\{(\pi_i \circ \widehat{\ell}_{IT})(z), (\pi_i \circ \widehat{\ell}_{IT})(y), 0.5\}$$

for all $z, y \in X$. Similarly, we know that

$$(\pi_i \circ \widehat{\ell}_T)(z * y) \geq \bigwedge\{(\pi_i \circ \widehat{\ell}_T)(z), (\pi_i \circ \widehat{\ell}_T)(y), 0.5\}$$

for all $z, y \in X$. Suppose that $\widehat{\ell}_F(z) \vee \widehat{\ell}_F(y) > \widehat{0.5}$. If $\widehat{\ell}_F(z*y) > \widehat{\ell}_F(z) \vee \widehat{\ell}_F(y) := \hat{n}_F$, then $z, y \in L(\widehat{\ell}_F, \hat{n}_F)$, $z * y \notin L(\widehat{\ell}_F, \hat{n}_F)$ and $\widehat{\ell}_F(z * y) + \hat{n}_F > 2\hat{n}_F > 1$, i.e., $z * y \notin F_q(\widehat{\ell}_F, \hat{n}_F)$. This is a contradiction, and so $\widehat{\ell}_F(z*y) \leq \bigvee\{\widehat{\ell}_F(z), \widehat{\ell}_F(y), \widehat{0.5}\}$ whenever $\widehat{\ell}_F(z) \vee \widehat{\ell}_F(y) > \widehat{0.5}$. Now assume that $\widehat{\ell}_F(z) \vee \widehat{\ell}_F(y) \leq \widehat{0.5}$. Then $z, y \in L(\widehat{\ell}_F, \widehat{0.5})$ and thus $z * y \in F_{\in\vee q}(\widehat{\ell}_F, \widehat{0.5}) = L(\widehat{\ell}_F, \widehat{0.5}) \cup F_q(\widehat{\ell}_F, \widehat{0.5})$. If $z * y \notin L(\widehat{\ell}_F, \widehat{0.5})$, that is, $\widehat{\ell}_F(z * y) > \widehat{0.5}$, then $\widehat{\ell}_F(z * y) + \widehat{0.5} > \widehat{0.5} + \widehat{0.5} = \hat{1}$, i.e., $z * y \notin F_q(\widehat{\ell}_F, \widehat{0.5})$. This is a contradiction. Hence $\widehat{\ell}_F(z*y) \leq \widehat{0.5}$ and so $\widehat{\ell}_F(z*y) \leq \bigvee\{\widehat{\ell}_F(z), \widehat{\ell}_F(y), \widehat{0.5}\}$ whenever $\widehat{\ell}_F(z) \vee \widehat{\ell}_F(y) \leq \widehat{0.5}$. Therefore $\widehat{\ell}_F(z*y) \leq \bigvee\{\widehat{\ell}_F(z), \widehat{\ell}_F(y), \widehat{0.5}\}$ for all $z, y \in X$. By the similar way, we have $\widehat{\ell}_{IF}(z * y) \leq \bigvee\{\widehat{\ell}_{IF}(z), \widehat{\ell}_{IF}(y), \widehat{0.5}\}$ for all $z, y \in X$.

Conversely, let $\widehat{\mathcal{L}} := \left(\widehat{\ell}_T, \widehat{\ell}_{IT}, \widehat{\ell}_{IF}, \widehat{\ell}_F\right)$ be a k-polar generalized neutrosophic set over X which satisfies the condition (4.2). Let $z, y \in X$ and $\hat{n}_T = (n_T^1, n_T^2, \cdots, n_T^k) \in [0, 1]^k$. If $z, y \in U(\widehat{\ell}_T, \hat{n}_T)$, then $\widehat{\ell}_T(z) \geq \hat{n}_T$ and $\widehat{\ell}_T(y) \geq \hat{n}_T$. If $\widehat{\ell}_T(z * y) < \hat{n}_T$, then $\widehat{\ell}_T(z) \wedge \widehat{\ell}_T(y) \geq \widehat{0.5}$. Otherwise, we get

$$\widehat{\ell}_T(z * y) \geq \bigwedge\{\widehat{\ell}_T(z), \widehat{\ell}_T(y), \widehat{0.5}\} = \widehat{\ell}_T(z) \wedge \widehat{\ell}_T(y) \geq \hat{n}_T,$$

which is a contradiction. Hence

$$\widehat{\ell}_T(z * y) + \hat{n}_T > 2\widehat{\ell}_T(z * y) \geq 2\bigwedge\{\widehat{\ell}_T(z), \widehat{\ell}_T(y), \widehat{0.5}\} = \hat{1}$$

and so $z * y \in T_q(\widehat{\ell}_T, \hat{n}_T) \subseteq T_{\in\vee q}(\widehat{\ell}_T, \hat{n}_T)$. Similarly, if $z, y \in U(\widehat{\ell}_{IT}, \hat{n}_{IT})$, then $z * y \in IT_{\in\vee q}(\widehat{\ell}_{IT}, \hat{n}_{IT})$ for $\hat{n}_{IT} = (n_{IT}^1, n_{IT}^2, \cdots, n_{IT}^k) \in [0, 1]^k$. Now, let $z, y \in L(\widehat{\ell}_{IF}, \hat{n}_{IF})$ for $\hat{n}_{IF} = (n_{IF}^1, n_{IF}^2, \cdots, n_{IF}^k) \in [0, 1]^k$. Then $\widehat{\ell}_{IF}(z) \leq \hat{n}_{IF}$ and $\widehat{\ell}_{IF}(y) \leq \hat{n}_{IF}$. If $\widehat{\ell}_{IF}(z * y) > \hat{n}_{IF}$, then $\widehat{\ell}_{IF}(z) \vee \widehat{\ell}_{IF}(z) \leq \widehat{0.5}$ because if not, then $\widehat{\ell}_{IF}(z * y) \leq \bigvee\{\widehat{\ell}_{IF}(z), \widehat{\ell}_{IF}(y), \widehat{0.5}\} \leq \widehat{\ell}_{IF}(z) \vee \widehat{\ell}_{IF}(y) \leq \hat{n}_{IF}$, which is a contradiction. Thus

$$\widehat{\ell}_{IF}(z * y) + \hat{n}_{IF} < 2\widehat{\ell}_{IF}(z * y) \leq 2\bigvee\{\widehat{\ell}_{IF}(z), \widehat{\ell}_{IF}(y), \widehat{0.5}\} = \hat{1}$$

and so $z * y \in IF_q(\widehat{\ell}_{IF}, \hat{n}_{IF}) \subseteq IF_{\in\vee q}(\widehat{\ell}_{IF}, \hat{n}_{IF})$. Similarly, we know that if $z, y \in L(\widehat{\ell}_F, \hat{n}_F)$, then $z * y \in F_q(\widehat{\ell}_F, \hat{n}_F) \subseteq F_{\in\vee q}(\widehat{\ell}_F, \hat{n}_F)$ for $\hat{n}_F = (n_F^1, n_F^2, \cdots, n_F^k) \in [0, 1]^k$. Therefore $\widehat{\mathcal{L}}$ is a k-polar generalized $(\in, \in\vee q)$-neutrosophic subalgebra of X. □

Using the k-polar generalized $(\in, \in\vee q)$-neutrosophic subalgebra, we show that the generaliged neutrosophic q-sets subalgebras.

Theorem 4.6. *If $\widehat{\mathcal{L}} := \left(\widehat{\ell}_T, \widehat{\ell}_{IT}, \widehat{\ell}_{IF}, \widehat{\ell}_F\right)$ is a k-polar generalized $(\in, \in\vee q)$-neutrosophic subalgebra of X, then the generaliged neutrosophic q-sets $T_q(\widehat{\ell}_T, \hat{n}_T)$, $IT_q(\widehat{\ell}_{IT}, \hat{n}_{IT})$, $IF_q(\widehat{\ell}_{IF}, \hat{n}_{IF})$ and $F_q(\widehat{\ell}_F, \hat{n}_F)$ are subalgebras of X for all $\hat{n}_T, \hat{n}_{IT} \in (0.5, 1]^k$ and $\hat{n}_F, \hat{n}_{IF} \in [0, 0.5)^k$.*

Proof. Suppose that $\widehat{\mathcal{L}} := \left(\widehat{\ell}_T, \widehat{\ell}_{IT}, \widehat{\ell}_{IF}, \widehat{\ell}_F\right)$ is a k-polar generalized $(\in, \in\vee q)$-neutrosophic subalgebra of X. Let $z, y \in X$. If $z, y \in IT_q(\widehat{\ell}_{IT}, \hat{n}_{IT})$ for $\hat{n}_{IT} \in (0.5, 1]^k$, then $\widehat{\ell}_{IT}(z) + \hat{n}_{IT} > \hat{1}$ and $\widehat{\ell}_{IT}(y) + \hat{n}_{IT} > \hat{1}$. It follows from Theorem 4.5 that

$$\begin{aligned}\widehat{\ell}_{IT}(z * y) + \hat{n}_{IT} &\geq \bigwedge\{\widehat{\ell}_{IT}(z), \widehat{\ell}_{IT}(y), \widehat{0.5}\} + \hat{n}_{IT} \\ &= \bigwedge\{\widehat{\ell}_{IT}(z) + \hat{n}_{IT}, \widehat{\ell}_{IT}(y) + \hat{n}_{IT}, \widehat{0.5} + \hat{n}_{IT}\} \\ &> \hat{1},\end{aligned}$$

i.e., $z * y \in IT_q(\widehat{\ell}_{IT}, \hat{n}_{IT})$. Thus $IT_q(\widehat{\ell}_{IT}, \hat{n}_{IT})$ is a subalgebra of X. Suppose that $z, y \in F_q(\widehat{\ell}_F, \hat{n}_F)$ for $\hat{n}_F \in [0, 0.5)^k$. Then $(\pi_i \circ \widehat{\ell}_F)(z) + n_F^i < 1$ and $(\pi_i \circ \widehat{\ell}_F)(z) + n_F^i < 1$. Using Theorem 4.5, we have

$$\begin{aligned}(\pi_i \circ \widehat{\ell}_F)(z * y) + n_F^i &\leq \bigvee\{(\pi_i \circ \widehat{\ell}_F)(z), (\pi_i \circ \widehat{\ell}_F)(y), 0.5\} + n_F^i \\ &= \bigvee\{(\pi_i \circ \widehat{\ell}_F)(z) + n_F^i, (\pi_i \circ \widehat{\ell}_F)(y) + n_F^i, 0.5 + n_F^i\} \\ &< 1\end{aligned}$$

and thus $z * y \in F_q(\widehat{\ell}_F, \hat{n}_F)^i$ for all $i = 1, 2, \cdots, k$. Hence $z * y \in \bigcap_{i=1}^k F_q(\widehat{\ell}_F, \hat{n}_F)^i = F_q(\widehat{\ell}_F, \hat{n}_F)$, and therefore $F_q(\widehat{\ell}_F, \hat{n}_F)$ is a subalgebra of X. Similarly, we can induce that $T_q(\widehat{\ell}_T, \hat{n}_T)$ and $IF_q(\widehat{\ell}_{IF}, \hat{n}_{IF})$ are subalgebras of X for $\hat{n}_{IT} \in (0.5, 1]^k$ and $\hat{n}_F \in [0, 0.5)^k$. □

Using the generaliged neutrosophic $\in\vee q$-sets, we establish a k-polar generalized $(\in, \in\vee q)$-neutrosophic subalgebra.

Theorem 4.7. *Given a k-polar generalized neutrosophic set $\widehat{\mathcal{L}} := \left(\widehat{\ell}_T, \widehat{\ell}_{IT}, \widehat{\ell}_{IF}, \widehat{\ell}_F\right)$ over X, if the generaliged neutrosophic $\in\vee q$-sets $T_{\in\vee q}(\widehat{\ell}_T, \hat{n}_T)$, $IT_{\in\vee q}(\widehat{\ell}_{IT}, \hat{n}_{IT})$, $IF_{\in\vee q}(\widehat{\ell}_{IF}, \hat{n}_{IF})$ and $F_{\in\vee q}(\widehat{\ell}_F, \hat{n}_F)$ are subalgebras of X for all $\hat{n}_T, \hat{n}_{IT} \in (0, 1]^k$ and $\hat{n}_F, \hat{n}_{IF} \in [0, 1)^k$, then $\widehat{\mathcal{L}}$ is a k-polar generalized $(\in, \in\vee q)$-neutrosophic subalgebra of X.*

Proof. Assume that there exist $\alpha, \beta \in X$ such that

$$(\pi_i \circ \widehat{\ell}_T)(\alpha * \beta) < \bigwedge\{(\pi_i \circ \widehat{\ell}_T)(\alpha), (\pi_i \circ \widehat{\ell}_T)(\beta), 0.5\}$$

for $i = 1, 2, \cdots, k$. Then there exists $n_T^i \in (0, 0.5]$ such that

$$(\pi_i \circ \widehat{\ell}_T)(\alpha * \beta) < n_T^i \leq \bigwedge\{(\pi_i \circ \widehat{\ell}_T)(\alpha), (\pi_i \circ \widehat{\ell}_T)(\beta), 0.5\}.$$

Hence $\alpha, \beta \in U(\widehat{\ell}_T, \hat{n}_T)^i$, and so $\alpha, \beta \in \bigcap_{i=1}^k U(\widehat{\ell}_T, \hat{n}_T)^i = U(\widehat{\ell}_T, \hat{n}_T) \subseteq T_{\in\vee q}(\widehat{\ell}_T, \hat{n}_T)$. Since $T_{\in\vee q}(\widehat{\ell}_T, \hat{n}_T)$ is a subalgebra of X, it follows that $\alpha * \beta \in T_{\in\vee q}(\widehat{\ell}_T, \hat{n}_T) = \bigcap_{i=1}^k T_{\in\vee q}(\widehat{\ell}_T, \hat{n}_T)^i$. Thus $(\pi_i \circ \widehat{\ell}_T)(\alpha * \beta) \geq n_T^i$ or $(\pi_i \circ \widehat{\ell}_T)(\alpha * \beta) + n_T^i > 1$ for $i = 1, 2, \cdots, k$. This is a contradiction, and thus $(\pi_i \circ \widehat{\ell}_T)(z * y) \geq \bigwedge\{(\pi_i \circ \widehat{\ell}_T)(z), (\pi_i \circ \widehat{\ell}_T)(y), 0.5\}$ for all $z, y \in X$ and $i = 1, 2, \cdots, k$. Now, if there exist $\alpha, \beta \in X$ such that

$$(\pi_i \circ \widehat{\ell}_{IF})(\alpha * \beta) > \bigvee\{(\pi_i \circ \widehat{\ell}_{IF})(\alpha), (\pi_i \circ \widehat{\ell}_{IF})(\beta), 0.5\}$$

for $i = 1, 2, \cdots, k$, then

$$(\pi_i \circ \widehat{\ell}_{IF})(\alpha * \beta) > n^i_{IF} \geq \bigvee\{(\pi_i \circ \widehat{\ell}_{IF})(\alpha), (\pi_i \circ \widehat{\ell}_{IF})(\beta), 0.5\} \tag{4.4}$$

for some $n^i_{IF} \in [0.5, 1)$. Hence $\alpha, \beta \in L(\widehat{\ell}_{IF}, \hat{n}_{IF})^i$, and so $\alpha, \beta \in \bigcap_{i=1}^k L(\widehat{\ell}_{IF}, \hat{n}_{IF})^i = L(\widehat{\ell}_{IF}, \hat{n}_{IF}) \subseteq IF_{\in\vee q}(\widehat{\ell}_{IF}, \hat{n}_{IF})$. This implies that $\alpha * \beta \in IF_{\in\vee q}(\widehat{\ell}_{IF}, \hat{n}_{IF})$, and (4.4) induces $\alpha * \beta \notin L(\widehat{\ell}_{IF}, \hat{n}_{IF})^i$ and $(\pi_i \circ \widehat{\ell}_{IF})(\alpha * \beta) + n^i_{IF} > 2n^i_{IF} > 1$ for $i = 1, 2, \cdots, k$. Thus $\alpha * \beta \notin \bigcap_{i=1}^k L(\widehat{\ell}_{IF}, \hat{n}_{IF})^i = L(\widehat{\ell}_{IF}, \hat{n}_{IF})$ and $\alpha * \beta \notin \bigcap_{i=1}^k IF_q(\widehat{\ell}_{IF}, \hat{n}_{IF})^i = IF_q(\widehat{\ell}_{IF}, \hat{n}_{IF})$. Hence $\alpha * \beta \notin IF_{\in\vee q}(\widehat{\ell}_{IF}, \hat{n}_{IF})$ which is a contradiction. Therefore

$$(\pi_i \circ \widehat{\ell}_{IF})(z * y) \leq \bigvee\{(\pi_i \circ \widehat{\ell}_{IF})(z), (\pi_i \circ \widehat{\ell}_{IF})(y), 0.5\}$$

for for all $z, y \in X$ and $i = 1, 2, \cdots, k$, i.e., $\widehat{\ell}_{IF}(z * y) \leq \bigvee\{\widehat{\ell}_{IF}(z), \widehat{\ell}_{IF}(y), \widehat{0.5}\}$ for all $z, y \in X$. Similarly, we show that $(\pi_i \circ \widehat{\ell}_{IT})(z*y) \geq \bigwedge\{(\pi_i \circ \widehat{\ell}_{IT})(z), (\pi_i \circ \widehat{\ell}_{IT})(y), 0.5\}$ and $(\pi_i \circ \widehat{\ell}_F)(z*y) \leq \bigvee\{(\pi_i \circ \widehat{\ell}_F)(z), (\pi_i \circ \widehat{\ell}_F)(y), 0.5\}$ for all $z, y \in X$ and $i = 1, 2, \cdots, k$. Using Theorem 4.5, we conclude that $\widehat{\mathcal{L}}$ is a k-polar generalized $(\in, \in\vee q)$-neutrosophic subalgebra of X. □

Using the k-polar generalized $(\in, \in\vee q)$-neutrosophic subalgebra, we show that the generaliged neutrosophic $\in\vee q$-sets subalgebras.

Theorem 4.8. *If $\widehat{\mathcal{L}} := \left(\widehat{\ell}_T, \widehat{\ell}_{IT}, \widehat{\ell}_{IF}, \widehat{\ell}_F\right)$ is a k-polar generalized $(\in, \in\vee q)$-neutrosophic subalgebra of X, then the generaliged neutrosophic $\in\vee q$-sets $T_{\in\vee q}(\widehat{\ell}_T, \hat{n}_T)$, $IT_{\in\vee q}(\widehat{\ell}_{IT}, \hat{n}_{IT})$, $IF_{\in\vee q}(\widehat{\ell}_{IF}, \hat{n}_{IF})$ and $F_{\in\vee q}(\widehat{\ell}_F, \hat{n}_F)$ are subalgebras of X for all $\hat{n}_T, \hat{n}_{IT} \in (0, 0.5]^k$ and $\hat{n}_F, \hat{n}_{IF} \in [0.5, 1)^k$.*

Proof. Let $z, y \in IT_{\in\vee q}(\widehat{\ell}_{IT}, \hat{n}_{IT})$. Then

$$z \in U((\widehat{\ell}_{IT}, \hat{n}_{IT})^i \text{ or } z \in IT_q((\widehat{\ell}_{IT}, \hat{n}_{IT})^i$$

and

$$y \in U((\widehat{\ell}_{IT}, \hat{n}_{IT})^i \text{ or } y \in IT_q((\widehat{\ell}_{IT}, \hat{n}_{IT})^i$$

for $i = 1, 2, \cdots, k$. Thus we get the following four cases:

(i) $z \in U((\widehat{\ell}_{IT}, \hat{n}_{IT})^i$ and $y \in U((\widehat{\ell}_{IT}, \hat{n}_{IT})^i$,

(ii) $z \in U((\widehat{\ell}_{IT}, \hat{n}_{IT})^i$ and $y \in IT_q((\widehat{\ell}_{IT}, \hat{n}_{IT})^i$,

(iii) $z \in IT_q((\widehat{\ell}_{IT}, \hat{n}_{IT})^i$ and $y \in U((\widehat{\ell}_{IT}, \hat{n}_{IT})^i$,

(iv) $z \in IT_q((\widehat{\ell}_{IT}, \hat{n}_{IT})^i$ and $y \in IT_q((\widehat{\ell}_{IT}, \hat{n}_{IT})^i$.

For the first case, we have $z * y \in IT_{\in\vee q}((\widehat{\ell}_{IT}, \hat{n}_{IT})^i$ for $i = 1, 2, \cdots, k$ and so

$$z * y \in \bigcap_{i=1}^{k} IT_{\in\vee q}((\widehat{\ell}_{IT}, \hat{n}_{IT})^i = IT_{\in\vee q}(\widehat{\ell}_{IT}, \hat{n}_{IT}).$$

In the the case (ii) (resp., (iii)), $y \in IT_q((\widehat{\ell}_{IT}, \hat{n}_{IT})^i$ (resp., $z \in IT_q((\widehat{\ell}_{IT}, \hat{n}_{IT})^i)$ induce $\widehat{\ell}_{IT}(y) > 1 - n^i_{IT} \geq n^i_{IT}$ (resp., $\widehat{\ell}_{IT}(z) > 1 - n^i_{IT} \geq n^i_{IT}$), that is, $y \in U((\widehat{\ell}_{IT}, \hat{n}_{IT})^i$ (resp., $z \in U((\widehat{\ell}_{IT}, \hat{n}_{IT})^i)$. Thus $z * y \in IT_{\in\vee q}((\widehat{\ell}_{IT}, \hat{n}_{IT})^i$ for $i = 1, 2, \cdots, k$ which implies that

$$z * y \in \bigcap_{i=1}^{k} IT_{\in\vee q}((\widehat{\ell}_{IT}, \hat{n}_{IT})^i = IT_{\in\vee q}(\widehat{\ell}_{IT}, \hat{n}_{IT}).$$

The last case induces $\widehat{\ell}_{IT}(z) > 1 - n^i_{IT} \geq n^i_{IT}$ and $\widehat{\ell}_{IT}(y) > 1 - n^i_{IT} \geq n^i_{IT}$, i.e., $z, y \in U((\widehat{\ell}_{IT}, \hat{n}_{IT})^i$ for $i = 1, 2, \cdots, k$. It follows that

$$z * y \in \bigcap_{i=1}^{k} IT_{\in\vee q}((\widehat{\ell}_{IT}, \hat{n}_{IT})^i = IT_{\in\vee q}(\widehat{\ell}_{IT}, \hat{n}_{IT}).$$

Therefore $IT_{\in\vee q}(\widehat{\ell}_{IT}, \hat{n}_{IT})$ is a subalgebra of X for all $\hat{n}_{IT} \in (0, 0.5]^k$. Similarly, we can show that the set $T_{\in\vee q}(\widehat{\ell}_{T}, \hat{n}_{T})$ is a subalgebra of X for all $\hat{n}_{T} \in (0, 0.5]^k$. Let $z, y \in F_{\in\vee q}(\widehat{\ell}_{F}, \hat{n}_{F})$. Then

$$\widehat{\ell}_F(z) \leq \hat{n}_F \text{ or } \widehat{\ell}_F(z) + \hat{n}_F < \hat{1}$$

and

$$\widehat{\ell}_F(y) \leq \hat{n}_F \text{ or } \widehat{\ell}_F(y) + \hat{n}_F < \hat{1}.$$

If $\widehat{\ell}_F(z) \leq \hat{n}_F$ and $\widehat{\ell}_F(y) \leq \hat{n}_F$, then

$$\widehat{\ell}_F(z * y) \leq \bigvee\{\widehat{\ell}_F(z), \widehat{\ell}_F(y), \widehat{0.5}\} \leq \hat{n}_F \vee \widehat{0.5} = \hat{n}_F$$

by Theorem 4.5, and so $z * y \in L(\widehat{\ell}_F, \hat{n}_F) \subseteq F_{\in\vee q}(\widehat{\ell}_F, \hat{n}_F)$. If $\widehat{\ell}_F(z) \leq \hat{n}_F$ or $\widehat{\ell}_F(y) + \hat{n}_F < \hat{1}$, then

$$\widehat{\ell}_F(z * y) \leq \bigvee\{\widehat{\ell}_F(z), \widehat{\ell}_F(y), \widehat{0.5}\} \leq \bigvee\{\hat{n}_F, \hat{1} - \hat{n}_F, \widehat{0.5}\} = \hat{n}_F$$

by Theorem 4.5. Hence $z * y \in L(\widehat{\ell}_F, \hat{n}_F) \subseteq F_{\in\vee q}(\widehat{\ell}_F, \hat{n}_F)$. Similarly, if $\widehat{\ell}_F(z) + \hat{n}_F < \hat{1}$ and $\widehat{\ell}_F(y) \leq \hat{n}_F$, then $z * y \in F_{\in\vee q}(\widehat{\ell}_F, \hat{n}_F)$. If $\widehat{\ell}_F(z) + \hat{n}_F < \hat{1}$ and $\widehat{\ell}_F(y) + \hat{n}_F < \hat{1}$, then

$$\widehat{\ell}_F(z * y) \leq \bigvee\{\widehat{\ell}_F(z), \widehat{\ell}_F(y), \widehat{0.5}\} \leq (\hat{1} - \hat{n}_F) \vee \widehat{0.5} = \widehat{0.5} < \hat{n}_F$$

by Theorem 4.5. Thus $z * y \in L(\widehat{\ell}_F, \hat{n}_F) \subseteq F_{\in\vee q}(\widehat{\ell}_F, \hat{n}_F)$. Consequently, $F_{\in\vee q}(\widehat{\ell}_F, \hat{n}_F)$ is a subalgebra of X for all $\hat{n}_F \in [0.5, 1)^k$. By the similar way, we can verify that $IF_{\in\vee q}(\widehat{\ell}_{IF}, \hat{n}_{IF})$ is a subalgebra of X for all $\hat{n}_{IF} \in [0.5, 1)^k$. □

5 k-polar generalized $(q, \in\!\vee q)$-neutrosophic subalgebras

Definition 5.1. Let $\widehat{\mathcal{L}} := \left(\widehat{\ell}_T, \widehat{\ell}_{IT}, \widehat{\ell}_{IF}, \widehat{\ell}_F\right)$ be a k-polar generalized neutrosophic set over X. Then $\widehat{\mathcal{L}}$ is called a *k-polar generalized $(q, \in\!\vee q)$-neutrosophic subalgebra* of X if it satisfies:

$$\begin{aligned}
&z \in T_q(\widehat{\ell}_T, \hat{n}_T),\ y \in T_q(\widehat{\ell}_T, \hat{n}_T) \ \Rightarrow\ z * y \in T_{\in\vee q}(\widehat{\ell}_T, \hat{n}_T),\\
&z \in IT_q(\widehat{\ell}_{IT}, \hat{n}_{IT}),\ y \in IT_q(\widehat{\ell}_{IT}, \hat{n}_{IT}) \ \Rightarrow\ z * y \in IT_{\in\vee q}(\widehat{\ell}_{IT}, \hat{n}_{IT}),\\
&z \in IF_q(\widehat{\ell}_{IF}, \hat{n}_{IF}),\ y \in IF_q(\widehat{\ell}_{IF}, \hat{n}_{IF}) \ \Rightarrow\ z * y \in IF_{\in\vee q}(\widehat{\ell}_{IF}, \hat{n}_{IF}),\\
&z \in F_q(\widehat{\ell}_F, \hat{n}_F),\ y \in F_q(\widehat{\ell}_F, \hat{n}_F) \ \Rightarrow\ z * y \in F_{\in\vee q}(\widehat{\ell}_F, \hat{n}_F)
\end{aligned} \tag{5.1}$$

for all $z, y \in X$, $\hat{n}_T, \hat{n}_{IT} \in (0,1]^k$ and $\hat{n}_F, \hat{n}_{IF} \in [0,1)^k$.

Example 5.2. Let $X = \{0, 1, 2, \alpha, \beta\}$ be the BCI-algebra which is given in Example 4.3. Let $\widehat{\mathcal{L}} := (\widehat{\ell}_T, \widehat{\ell}_{IT}, \widehat{\ell}_{IF}, \widehat{\ell}_F)$ be a 3-polar generalized neutrosophic set over X in which $\widehat{\ell}_T$, $\widehat{\ell}_{IT}$, $\widehat{\ell}_{IF}$ and $\widehat{\ell}_F$ are defined as follows:

$$\widehat{\ell}_T : X \to [0,1]^3,\ z \mapsto \begin{cases} (0.6, 0.7, 0.8) & \text{if } z = 0,\\ (0.7, 0.0, 0.0) & \text{if } z = 1,\\ (0.0, 0.0, 0.9) & \text{if } z = 2,\\ (0.0, 0.0, 0.0) & \text{if } z = \alpha,\\ (0.0, 0.0, 0.0) & \text{if } z = \beta, \end{cases}$$

$$\widehat{\ell}_{IT} : X \to [0,1]^3,\ z \mapsto \begin{cases} (0.6, 0.7, 0.8) & \text{if } z = 0,\\ (0.7, 0.0, 0.0) & \text{if } z = 1,\\ (0.5, 0.8, 0.9) & \text{if } z = 2,\\ (0.0, 0.0, 0.7) & \text{if } z = \alpha,\\ (0.0, 0.0, 0.0) & \text{if } z = \beta, \end{cases}$$

$$\widehat{\ell}_{IF} : X \to [0,1]^3,\ z \mapsto \begin{cases} (0.2, 0.3, 0.1) & \text{if } z = 0,\\ (1.0, 1.0, 0.2) & \text{if } z = 1,\\ (0.3, 0.4, 1.0) & \text{if } z = 2,\\ (0.4, 1.0, 1.0) & \text{if } z = \alpha,\\ (1.0, 1.0, 1.0) & \text{if } z = \beta, \end{cases}$$

$$\widehat{\ell}_F : X \to [0,1]^3,\ z \mapsto \begin{cases} (0.2, 0.4, 0.4) & \text{if } z = 0,\\ (0.4, 1.0, 1.0) & \text{if } z = 1,\\ (1.0, 0.2, 0.1) & \text{if } z = 2,\\ (1.0, 0.3, 1.0) & \text{if } z = \alpha,\\ (1.0, 1.0, 1.0) & \text{if } z = \beta, \end{cases}$$

It is routine to verify that $\widehat{\mathcal{L}} := \left(\widehat{\ell}_T, \widehat{\ell}_{IT}, \widehat{\ell}_{IF}, \widehat{\ell}_F\right)$ is a 3-polar generalized $(q, \in\!\vee q)$-neutrosophic subalgebra of X.

Using the k-polar generalized $(q, \in\vee q)$-neutrosophic subalgebra, we show that the generaliged neutrosophic q-sets and the generaliged neutrosophic $\in\vee q$-sets are subalgebras.

Theorem 5.3. *If $\widehat{\mathcal{L}} := \left(\widehat{\ell}_T, \widehat{\ell}_{IT}, \widehat{\ell}_{IF}, \widehat{\ell}_F\right)$ is a k-polar generalized $(q, \in\vee q)$-neutrosophic subalgebra of X, then the generaliged neutrosophic q-sets $T_q(\widehat{\ell}_T, \hat{n}_T)$, $IT_q(\widehat{\ell}_{IT}, \hat{n}_{IT})$, $IF_q(\widehat{\ell}_{IF}, \hat{n}_{IF})$ and $F_q(\widehat{\ell}_F, \hat{n}_F)$ are subalgebras of X for all $\hat{n}_T, \hat{n}_{IT} \in (0.5, 1]^k$ and $\hat{n}_F, \hat{n}_{IF} \in [0, 0.5)^k$.*

Proof. Let $z, y \in T_q(\widehat{\ell}_T, \hat{n}_T)$. Then $z * y \in T_{\in\vee q}(\widehat{\ell}_T, \hat{n}_T)$, and so $z * y \in U(\widehat{\ell}_T, \hat{n}_T)$ or $z * y \in T_q(\widehat{\ell}_T, \hat{n}_T)$. If $z * y \in U(\widehat{\ell}_T, \hat{n}_T)$, then $(\pi_i \circ \widehat{\ell}_T)(z * y) \geq n_T^i > 1 - n_T^i$ since $n_T^i > 0.5$ for all $i = 1, 2, \cdots, k$. Hence $z*y \in T_q(\widehat{\ell}_T, \hat{n}_T)$, and so $T_q(\widehat{\ell}_T, \hat{n}_T)$ is a subalgebra of X. By the similar way, we can verify that $IT_q(\widehat{\ell}_{IT}, \hat{n}_{IT})$ is a subalgebra of X. Let $z, y \in F_q(\widehat{\ell}_F, \hat{n}_F)$. Then $z * y \in F_{\in\vee q}(\widehat{\ell}_F, \hat{n}_F)$, and so $z * y \in L(\widehat{\ell}_F, \hat{n}_F)$ of $z * y \in F_q(\widehat{\ell}_F, \hat{n}_F)$. If $z * y \in L(\widehat{\ell}_F, \hat{n}_F)$, then $(\pi_i \circ \widehat{\ell}_F)(z * y) \leq n_F^i < 1 - n_F^i$ since $n_F^i < 0.5$ for all $i = 1, 2, \cdots, k$. Thus $z * y \in F_q(\widehat{\ell}_F, \hat{n}_F)$, and hence $F_q(\widehat{\ell}_F, \hat{n}_F)$ is a subalgebra of X. Similarly, the set $IF_q(\widehat{\ell}_{IF}, \hat{n}_{IF})$ is a subalgebra of X. □

Theorem 5.4. *If $\widehat{\mathcal{L}} := \left(\widehat{\ell}_T, \widehat{\ell}_{IT}, \widehat{\ell}_{IF}, \widehat{\ell}_F\right)$ is a k-polar generalized $(q, \in\vee q)$-neutrosophic subalgebra of X, then the generaliged neutrosophic $\in\vee q$-sets $T_{\in\vee q}(\widehat{\ell}_T, \hat{n}_T)$, $IT_{\in\vee q}(\widehat{\ell}_{IT}, \hat{n}_{IT})$, $IF_{\in\vee q}(\widehat{\ell}_{IF}, \hat{n}_{IF})$ and $F_{\in\vee q}(\widehat{\ell}_F, \hat{n}_F)$ are subalgebras of X for all $\hat{n}_T, \hat{n}_{IT} \in (0.5, 1]^k$ and $\hat{n}_F, \hat{n}_{IF} \in [0, 0.5)^k$.*

Proof. Let $z, y \in T_{\in\vee q}(\widehat{\ell}_T, \hat{n}_T)$ for $\hat{n}_T \in (0.5, 1]^k$. If $z, y \in T_q(\widehat{\ell}_T, \hat{n}_T)$, then obviously $z * y \in T_{\in\vee q}(\widehat{\ell}_T, \hat{n}_T)$. If $z \in U(\widehat{\ell}_T, \hat{n}_T)$ and $y \in T_q(\widehat{\ell}_T, \hat{n}_T)$, then $\widehat{\ell}_T(z) + \hat{n}_T \geq 2\hat{n}_T > \hat{1}$, i.e., $z \in T_q(\widehat{\ell}_T, \hat{n}_T)$. It follows that $z * y \in T_{\in\vee q}(\widehat{\ell}_T, \hat{n}_T)$. We can prove $z * y \in T_{\in\vee q}(\widehat{\ell}_T, \hat{n}_T)$ whenever $y \in U(\widehat{\ell}_T, \hat{n}_T)$ and $z \in T_q(\widehat{\ell}_T, \hat{n}_T)$ in the same way. If $z, y \in U(\widehat{\ell}_T, \hat{n}_T)$, then $\widehat{\ell}_T(z) + \hat{n}_T \geq 2\hat{n}_T > \hat{1}$ and $\widehat{\ell}_T(y) + \hat{n}_T \geq 2\hat{n}_T > \hat{1}$ and so $z, y \in T_q(\widehat{\ell}_T, \hat{n}_T)$. Thus $z * y \in T_{\in\vee q}(\widehat{\ell}_T, \hat{n}_T)$. Therefore $T_{\in\vee q}(\widehat{\ell}_T, \hat{n}_T)$ is a subalgebra of X for $\hat{n}_T \in (0.5, 1]^k$. Now, let $z, y \in F_{\in\vee q}(\widehat{\ell}_F, \hat{n}_F)$ for $\hat{n}_F \in [0, 0.5)^k$. If $z, y \in F_q(\widehat{\ell}_F, \hat{n}_F)$, then obviously $z * y \in F_{\in\vee q}(\widehat{\ell}_F, \hat{n}_F)$. If $z \in L(\widehat{\ell}_F, \hat{n}_F)$ and $y \in F_q(\widehat{\ell}_F, \hat{n}_F)$, then $\widehat{\ell}_F(z) + \hat{n}_F \leq 2\hat{n}_F < \hat{1}$, i.e., $z \in F_q(\widehat{\ell}_F, \hat{n}_F)$. Hence $z * y \in F_{\in\vee q}(\widehat{\ell}_F, \hat{n}_F)$. Similarly, we can prove that if $y \in L(\widehat{\ell}_F, \hat{n}_F)$ and $z \in F_q(\widehat{\ell}_F, \hat{n}_F)$, then $z*y \in F_{\in\vee q}(\widehat{\ell}_F, \hat{n}_F)$. If $z, y \in L(\widehat{\ell}_F, \hat{n}_F)$, then $\widehat{\ell}_F(z) + \hat{n}_F \leq 2\hat{n}_F < \hat{1}$ and $\widehat{\ell}_F(y) + \hat{n}_F \leq 2\hat{n}_F < \hat{1}$, that is, $z, y \in F_q(\widehat{\ell}_F, \hat{n}_F)$. Hence $z*y \in F_{\in\vee q}(\widehat{\ell}_F, \hat{n}_F)$. Therefore $F_{\in\vee q}(\widehat{\ell}_F, \hat{n}_F)$ is a subalgebra of X for all $\hat{n}_F \in [0, 0.5)^k$. In the same way, we can show that $IT_{\in\vee q}(\widehat{\ell}_{IT}, \hat{n}_{IT})$ is a subalgebra of X for $\hat{n}_{IT} \in (0.5, 1]^k$ and $IF_{\in\vee q}(\widehat{\ell}_{IF}, \hat{n}_{IF})$ is a subalgebra of X for all $\hat{n}_{IF} \in [0, 0.5)^k$. □

We provide conditions for a k-polar generalized neutrosophic set to be a k-polar generalized $(q, \in\vee q)$-neutrosophic subalgebra.

Theorem 5.5. *For a subalgebra S of X, let $\widehat{\mathcal{L}} := \left(\widehat{\ell}_T, \widehat{\ell}_{IT}, \widehat{\ell}_{IF}, \widehat{\ell}_F\right)$ be a k-polar generalized neutrosophic set over X such that*

$$(\forall z \in S)(\widehat{\ell}_T(z) \geq \widehat{0.5},\ \widehat{\ell}_{IT}(z) \geq \widehat{0.5},\ \widehat{\ell}_{IF}(z) \leq \widehat{0.5},\ \widehat{\ell}_F(z) \leq \widehat{0.5}), \tag{5.2}$$

$$(\forall z \in X \setminus S)(\widehat{\ell}_T(z) = \widehat{0} = \widehat{\ell}_{IT}(z),\ \widehat{\ell}_{IF}(z) = \widehat{1} = \widehat{\ell}_F(z)). \tag{5.3}$$

Then $\widehat{\mathcal{L}}$ is a k-polar generalized $(q, \in\vee q)$-neutrosophic subalgebra of X.

Proof. Let $z, y \in T_q(\widehat{\ell}_T, \hat{n}_T) = \bigcap_{i=1}^k T_q(\widehat{\ell}_T, \hat{n}_T)^i$. Then $(\pi_i \circ \widehat{\ell}_T)(z) + n_T^i > 1$ and $(\pi_i \circ \widehat{\ell}_T)(y) + n_T^i > 1$ for all $i = 1, 2, \cdots, k$. If $z * y \notin S$, then $z \in X \setminus S$ or $y \in X \setminus S$ since S is a subalgebra of X. Hence $(\pi_i \circ \widehat{\ell}_T)(z) = 0$ or $(\pi_i \circ \widehat{\ell}_T)(y) = 0$, which imply that $n_T^i > 1$, a contradiction. Thus $z * y \in S$ and so $(\pi_i \circ \widehat{\ell}_T)(z * y) \geq 0.5$ by (5.2). If $n_T^i > 0.5$, then $(\pi_i \circ \widehat{\ell}_T)(z * y) + n_T^i > 1$, ie., $z * y \in T_q(\widehat{\ell}_T, \hat{n}_T)^i$ for all $i = 1, 2, \cdots, k$. Hence $z * y \in \bigcap_{i=1}^k T_q(\widehat{\ell}_T, \hat{n}_T)^i = T_q(\widehat{\ell}_T, \hat{n}_T)$. Similarly, if $z, y \in IT_q(\widehat{\ell}_{IT}, \hat{n}_{IT})$, then $z * y \in IT_q(\widehat{\ell}_{IT}, \hat{n}_{IT})$. Let $z, y \in IF_q(\widehat{\ell}_{IF}, \hat{n}_{IF}) = \bigcap_{i=1}^k IF_q(\widehat{\ell}_{IF}, \hat{n}_{IF})^i$. Then $(\pi_i \circ \widehat{\ell}_{IF})(z) + n_{IF}^i < 1$ and $(\pi_i \circ \widehat{\ell}_{IF})(y) + n_{IF}^i < 1$ for all $i = 1, 2, \cdots, k$, which implies that $z * y \in S$. If $n_{IF}^i \geq 0.5$, then $(\pi_i \circ \widehat{\ell}_{IF})(z * y) \leq 0.5 \leq n_{IF}^i$ for all $i = 1, 2, \cdots, k$ which shows that $z * y \in \bigcap_{i=1}^k L(\widehat{\ell}_{IF}, \hat{n}_{IF})^i = L(\widehat{\ell}_{IF}, \hat{n}_{IF})$. If $n_{IF}^i < 0.5$, then $(\pi_i \circ \widehat{\ell}_{IF})(z * y) + n_{IF}^i < 1$ for all $i = 1, 2, \cdots, k$ and so $z * y \in \bigcap_{i=1}^k IF_q(\widehat{\ell}_{IF}, \hat{n}_{IF})^i = IF_q(\widehat{\ell}_{IF}, \hat{n}_{IF})$. Similarly way is to show that if $z, y \in F_q(\widehat{\ell}_F, \hat{n}_F)$, then $z * y \in F_{\in\vee q}(\widehat{\ell}_F, \hat{n}_F)$. Therefore $\widehat{\mathcal{L}}$ is a k-polar generalized $(q, \in\vee q)$-neutrosophic subalgebra of X. □

Combining Theorems 5.3 and 5.5, we have the following corollary.

Corollary 5.6. *If a k-polar generalized neutrosophic set $\widehat{\mathcal{L}} := \left(\widehat{\ell}_T, \widehat{\ell}_{IT}, \widehat{\ell}_{IF}, \widehat{\ell}_F\right)$ satisfies two conditions (5.2) and (5.3) for a subalgebra S of X, then the generaliged neutrosophic q-sets $T_q(\widehat{\ell}_T, \hat{n}_T)$, $IT_q(\widehat{\ell}_{IT}, \hat{n}_{IT})$, $IF_q(\widehat{\ell}_{IF}, \hat{n}_{IF})$ and $F_q(\widehat{\ell}_F, \hat{n}_F)$ are subalgebras of X for all $\hat{n}_T, \hat{n}_{IT} \in (0.5, 1]^k$ and $\hat{n}_F, \hat{n}_{IF} \in [0, 0.5)^k$.*

6 Conclusions

We have introduced k-polar generalized neutrosophic set and have applied it to BCK/BCI-algebras. We have defined k-polar generalized neutrosophic subalgebra, k-polar generalized $(\in, \in\vee q)$-neutrosophic subalgebra and k-polar generalized $(q, \in\vee q)$-neutrosophic subalgebra and have studid various properties. We have discussed characterization of k-polar generalized neutrosophic subalgebra and k-polar generalized $(\in, \in\vee q)$-neutrosophic subalgebra. We have shown that the necessity and possibility operator of k-polar generalized neutrosophic subalgebra are also a k-polar generalized neutrosophic subalgebra. Using the k-polar generalized $(\in, \in\vee q)$-neutrosophic subalgebra, we have shown that the generaliged neutrosophic q-sets and the generaliged neutrosophic $\in\vee q$-sets subalgebras. Using the k-polar generalized $(q, \in\vee q)$-neutrosophic subalgebra, we have shown that the generaliged neutrosophic q-sets and the generaliged neutrosophic $\in\vee q$-sets are subalgebras. Using the generaliged neutrosophic $\in\vee q$-sets, we have established a k-polar generalized $(\in, \in\vee q)$-neutrosophic subalgebra. We have provided conditions for a k-polar generalized neutrosophic set to be a k-polar generalized neutrosophic subalgebra and a k-polar generalized $(q, \in\vee q)$-neutrosophic subalgebra.

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