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Introduction to anti-bitopological spaces

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Abstract. The main aim of this study is to introduce the notion of anti-bitopological space and to reintroduce some basic concepts of topology in this novel framework. We point out that there are at least three possible and not necessarily equivalent methods of defining *openness* (and thus, the notion of *open set*) with respect to two anti-topologies simultaneously. We choose one of these approaches and concentrate on it. This allows us to define anti-bitopological interior and closure in some specific manner. Moreover, we prove some initial lemmas on anti-bitopological boundary. Finally, we study the problem of subspaces.

Keywords: Anti-Bitopological space; anti-topological space; anti-interior; anti-closure; anti-boundary.

1. Introduction

From the purely historical point of view, it would be fair to mention that the whole concept of *topological space* arose from the observation that some very natural structures (like open intervals on real line or open balls on real plane) can be analyzed in the light of more general definitions and properties. This allowed mathematicians to introduce and study many basic topological notions, e.g. interior, closure, density, compactness or connectedness.

Modern times reversed this initial approach (at least to certain extent). Many authors started to redefine and, in particular, to generalize the very concept of topological space. It seems that the main idea is to check what happens when some natural assumptions (like closure of the family under finite intersection or openness of the whole universal set) are dropped. This led to the development of generalized and supra-topologies (see [5] and [11]), infra-topologies (known as infi-topologies too; see [3] and [12]), minimal structures ([15]), weak structures ([6]) and generalized weak structures ([1], [7]). The latter are the weakest: they are just arbitrary collections of subsets of X. In fact, this concept was introduced already in 1966 by Kim-Leong Lim in [10].

One should be aware that all these studies are more or less important. They allow us to recognize which assumptions are necessary to achieve some expected results (and which are superfluous). In fact, they confront us with some questions from the area of philosophy of mathematics. For example: what does it really mean that a function is *continuous*? Should the interior of a set always be open? Should we always assume that empty set is open? Finally, should there be any "authentic" connection between the abstract notion of *openness* and "natural" openness of open interval on real line? Or maybe open sets in various generalized structures are just some *arbitrarily chosen* (or *distinguished*) sets?

Weakening of the initial definition of topological space is not the only possible direction. Recently, some authors started to investigate anti-topological structures (see [2], [17], [22] and [23]). These families are characterized by the fact that any finite intersections or any unions of their elements *does not* belong to such a family. Moreover, $\emptyset$ and X are never open in this specific sense. Clearly, each element of anti-topological space (that is, each anti-open set in a given space) is maximal and minimal at the same time. It cannot have proper anti-open subsets or supersets. However, non-empty intersections of anti-open sets are possible.

As for the bitopological spaces, it seems that their study was started by Kelly in 1963 (see [8]). In 1967 Pervin (see [14]) analyzed the notion of connectedness in this setting. Later there were many papers on bitopologies and this concept has been reintroduced in some generalized frameworks. For example, biminimal (see [4]) and biweak structures (see [9]) have been already studied. Moreover, some authors analyze spaces equipped with three, four, five or even six topologies.

In this paper we would like to introduce anti-bitopological spaces. We define some basic notions and we point out several important subtleties. Some of them are not obvious at the first glance (even if they are not necessarily technically complicated). Finally, we obtain some kind of general framework which may be used in further research.

In 2019 Smarandache (see [18]) generalized the classical Algebraic Structures to NeutroAlgebraic Structures (or NeutroAlgebras) whose operations and axioms are partially true, partially indeterminate, and partially false as extensions of Partial Algebra, and to AntiAlgebraic Structures (or AntiAlgebras) whose operations and axioms are totally false and on 2020 he continued to develop them e.g. in [19], [20] and [21].

The NeutroAlgebras and AntiAlgebras are a new field of research, which is inspired from our real world.

In classical algebraic structures, all operations are 100 % well-defined, and all axioms are 100 % true, but in real life and in many cases these restrictions are too harsh, since in our world we have things that only partially verify some operations or some laws.

Using the process of NeutroSophication of a classical algebraic structure we produce a NeutroAlgebra, while the process of AntiSophication of a classical algebraic structure produces an AntiAlgebra. NeutroTopology is a particular case of NeutroAlgebra and AntiTopology is a particular case of the AntiAlgebra.

2. Preliminaries

Let us recall the definition of anti-topological space.

Definition 2.1. [22] Assume that X is a non-empty universe and $\mathcal{T}$ be a collection of subsets of X. We say that $(X, \mathcal{T})$ is an anti-topological space if and only if the following conditions are satisfied:

(1) $\emptyset, X \notin \mathcal{T}$.
(2) For any $n \in \mathbb{N}$, if $A_1, A_2, ..., A_n \in \mathcal{T}$, then $\bigcap_{i=1}^{n} A_i \notin \mathcal{T}$. Here we assume that this intersection is *non-trivial*, i.e. that the sets in question are not all identical.
(3) For any $\{A_i\}_{i \in J \neq \emptyset}$ such that $A_i \in \mathcal{T}$ for each $i \in J$, we have $\bigcup_{i \in J} A_i \notin \mathcal{T}$. We assume that this union is non-trivial, i.e. that is, the sets not all identical.

The elements of $\mathcal{T}$ are called *anti-open sets* and their complements are *anti-closed sets*. The set of all anti-closed sets with respect to a given $\mathcal{T}$ is denoted by $\mathcal{T}_{Cl}$. We say that $\mathcal{T}$ is *anti-closed* under finite intersections and arbitrary unions. In fact, one can prove stronger result:

Lemma 2.2. *(see Lemma 2.3 in [22]).*

If $(X, \mathcal{T})$ is an anti-topological space, then it is anti-closed under arbitrary non-trivial intersections.

Moreover, we have:

Theorem 2.3. *(compare with Lemma 2.5 and Lemma 2.7 in [22]).*

If $(X, \mathcal{T})$ is an anti-topological space, then $\mathcal{T}_{Cl}$ is also an anti-topology on X.

We may also define anti-interior and anti-closure.

Definition 2.4. (see Def. 3.1 in [22] and Def. 3.1. and Def. 3.3 in [2]).

Let $(X, \mathcal{T})$ be an anti-topological space and $A \subseteq X$. Then we define *anti-interior* of A and its *anti-closure* as:

(1) $AntiInt(A) = \bigcup\{U; U \subseteq A \text{ and } U \in \mathcal{T}\}$

(2) $AntiCl(A) = \bigcap\{F; A \subseteq F \text{ and } F \in \mathcal{T}_{Cl}\}$

One can easily prove (again, see [2] and [22, 23]) that $AntiInt(A)$ need not be the biggest anti-open set contained in A and $AntiCl(A)$ may not be the smallest anti-closed set containing A. Clearly, the reason is that anti-interior (resp. anti-closure) need not to be anti-open (resp. anti-closed) at all.

The following two lemmas and remark appearing after them will be important in the next section.

Lemma 2.5. *(see Lemma 8 in [22]).*

The intersection of two anti-topologies (established on the same universe X) is an anti-topological space too.

Lemma 2.6. *(see Lemma 9 in [23]). The union of two anti-topologies (on the same universe) need not to be an anti-topological space.*

Remark 2.7. Clearly, the lemma above does not imply that the union of two different anti-topologies *cannot* be an anti-topology too. Take for example $X = \mathbb{Z}^+ \cup \mathbb{Z}^-$. Suppose $\mathcal{T}_1$ consists of the finite subsets of $\mathbb{Z}^+$ with cardinality 3 (e.g. $\{2, 5, 7\}$, $\{30, 300, 3000\}$, while $\mathcal{T}_2$ consists of the finite subsets of $\mathbb{Z}^-$ with cardinality 3 (e.g. $\{-1, -2, -3\}$). Both these structures are anti-topologies and their union is an anti-topology too. Another example: $X = \{a, b, c\}$, $\mathcal{T}_1 = \{\{a\}, \{b\}\}$ and $\mathcal{T}_2 = \{\{c\}\}$.

On the other hand, it is possible that the union of two anti-topologies:

(1) is closed under non-empty intersections. Take $X = \{a, b, c, d\}$, $\mathcal{T}_1 = \{\{a, b\}, \{b, c\}, \{c, d\}$ and $\mathcal{T}_2 = \{\{b\}, \{c, \}\}$.
(2) is closed under unions. Take $X = \{a, b, c, d\}$, $\mathcal{T}_1 = \{\{a, b\}, \{b, c\}\}$ and $\mathcal{T}_2 = \{\{a, b, c\}\}$.
(3) is a neutro-topology. Take neutro-topological space presented in [17] (Example 9). It is $X = \{a, b, c, d\}$ with neutro-topology $\tau = \{\{a\}, \{a, b\}, \{c, d\}, \{b, c\}, \emptyset\}$. Assume now that we remove empty set from our collection. We stay with $\tau_1 = \{\{a\}, \{a, b\}, \{c, d\}, \{b, c\}\}$. This structure can be easily presented as a union of two anti-topologies, namely $\mathcal{T}_1 = \{\{a\}, \{c, d\}\}$ and $\mathcal{T}_2 = \{\{a, b\}, \{b, c\}\}$ (and this decomposition is not necessarily unique).

However, it is clear that X and $\emptyset$ never belong to $\mathcal{T}_1 \cup \mathcal{T}_2$.

Remark 2.8. Anti-topologies can be considered as a special subclass of anti-minimal spaces which could be defined in the following way: let X be a non-empty universe and $\mathcal{M} \subseteq P(X)$. If $\emptyset, X \notin \mathcal{M}$, then we say that $\mathcal{M}$ is anti-minimal structure on X.

3. Anti-bitopological spaces: introductory notes

In this section we introduce anti-bitopological spaces.

Definition 3.1. If X is a non-empty set endowed with two anti-topologies $\mathcal{T}_1$ and $\mathcal{T}_2$, then $(X, \mathcal{T}_1, \mathcal{T}_2)$ is called an anti-bitopological space.

Remark 3.2. It is always an interesting question: if we have two or more structures (e.g. topologies, infra-topologies, minimal structures, weak structures, anti-topologies) on X, then how should we define *open* sets. We mean those sets which will be considered as open with respect to the whole structure, that is: "bi-open" ("tri-open" and in general, n-open). There are several approaches in the literature. For example:

(1) In case of biweak structures some authors (see [9]) assume that A is open if and only if $Int_{w_1}(Int_{w_2}(A)) = A$, where w_1, w_2 are weak structures on X and Int_{w_1}, Int_{w_2} are interior operations relying on these structures. The same approach has been presented in [4] but in the context of biminimal spaces.

(2) In case of tri-topological spaces Palaniammal defined (in [13]) A as tri-open if and only if $A \in \tau_1 \cap \tau_2 \cap \tau_3$. Hence, in his opinion tri-open set should be open in *each* of these three topologies. Alternatively, we could say that A should be open in so-called induced topology $\tau_1 \cap \tau_2 \cap \tau_3$.

(3) However, Priyadharsini and Parvathi assumed in [16] that A is tri-open if and only if $A \in \tau_1 \cup \tau_2 \cup \tau_3$. Hence, A should be open in *at least one* topology.

These approaches are not necessarily equivalent. In particular, the first one need not to be equivalent with the third one. This will be shown in the context of anti-bitopological spaces. The third approach will be fundamental for us. However, we will introduce a convenient notation that will allow us to avoid any confusion.

Definition 3.3. If $(X, \mathcal{T}_1, \mathcal{T}_2)$ is an anti-bitopological space with $A \in \mathcal{T}_1 \cup \mathcal{T}_2$, then we say that A is $(\mathcal{T}_1 \cup \mathcal{T}_2)$-*anti-open* and its complement is $(\mathcal{T}_1 \cup \mathcal{T}_2)$-*anti-closed*.

Now we have the following idea:

Definition 3.4. If $(X, \mathcal{T}_1, \mathcal{T}_2)$ is an anti-bitopological space with $A \in \mathcal{T}_1 \cap \mathcal{T}_2$, then we will say that A is $(\mathcal{T}_1 \cap \mathcal{T}_2)$-*anti-open* and its complement is $(\mathcal{T}_1 \cap \mathcal{T}_2)$-*anti-closed*.

We define two basic but natural and important notions:

Definition 3.5. If $(X, \mathcal{T}_1, \mathcal{T}_2)$ is an anti-bitopological space with $A \subseteq X$, then the anti-interior and anti-closure of A (with respect to $\mathcal{T}_1 \cup \mathcal{T}_2$) are defined as follows:

(1) $(\mathcal{T}_1 \cup \mathcal{T}_2)AntiInt(A) = \bigcup\{B; B \subseteq A, B \in \mathcal{T}_1 \cup \mathcal{T}_2\}$.

(2) $(\mathcal{T}_1 \cup \mathcal{T}_2)AntiCl(A) = \bigcap\{C; A \subseteq C, C \in (\mathcal{T}_1 \cup \mathcal{T}_2)_{Cl}\}$.

Clearly, analogous definition can be derived with respect to $(\mathcal{T}_1 \cap \mathcal{T}_2)$-anti-open (anti-closed) sets.

However, we shall not use it in this paper.

Example 3.6. Let $X = \{1, 2, 3, 4, 5\}$, $\mathcal{T}_1 = \{\{3\}, \{1, 2\}, \{1, 4\}\}$ and $\mathcal{T}_2 = \{\{2\}, \{1, 4\}, \{3, 4\}\}$. It can be observed that both $\mathcal{T}_1$ and $\mathcal{T}_2$ are anti-topologies on X. Now, $\mathcal{T}_1 \cup \mathcal{T}_2 = \{\{2\}, \{3\}, \{1, 2\}, \{1, 4\}, \{3, 4\}\}$. This is not an anti-topology: note that $\{2\} \cap \{2, 4\} = \{2\} \in \mathcal{T}_1 \cup \mathcal{T}_2$. However, it is not closed under intersections (note that $\{1, 2\} \cap \{1, 4\} = \{1\} \notin (\mathcal{T}_1 \cup \mathcal{T}_2)$ nor unions (observe that $\{1, 2\} \cup \{1, 4\} = \{1, 2, 4\} \notin (\mathcal{T}_1 \cup \mathcal{T}_2)$.

Now take $A = \{3, 4\}$. Clearly, according to our definition this set is $(\mathcal{T}_1 \cup \mathcal{T}_2)$-anti-open. However, $AntiInt_{\mathcal{T}_1}(AntiInt_{\mathcal{T}_2}(A)) = AntiInt_{\mathcal{T}_1}(\{3, 4\}) = \{3\} \neq A$. Moreover, $A \notin \mathcal{T}_1 \cap \mathcal{T}_2$.

On the other hand, let $B = \{1, 4, 5\}$. Now $B \notin \mathcal{T}_1 \cup \mathcal{T}_2$ but $AntiInt_{\mathcal{T}_1}(AntiInt_{\mathcal{T}_2}(B)) = AntiInt_{\mathcal{T}_1}(\{1, 4\}) = \{1, 4\}$.

However, $C = \{1, 4\}$ belongs to $\mathcal{T}_1 \cup \mathcal{T}_2$ and $AntiInt_{\mathcal{T}_1}(AntiInt_{\mathcal{T}_2}(C)) = AntiInt_{\mathcal{T}_1}(\{1, 4\}) = \{1, 4\} = C$.

The example above suggests that the following definition can be useful:

Definition 3.7. Let $(X, \mathcal{T}_1, \mathcal{T}_2)$ be an anti-bitopological space. Then, we say that $A \subseteq X$ is $\mathcal{T}_1\mathcal{T}_2$-*anti-open* if and only if $AntiInt_{\mathcal{T}_1}(AntiInt_{\mathcal{T}_2}(A)) = A$.

One can prove the following theorem.

Theorem 3.8. *If $(X, \mathcal{T}_1, \mathcal{T}_2)$ is an anti-bitopological space and $\{A_i\}_{i \in J \neq \emptyset}$ is a collection of $\mathcal{T}_1\mathcal{T}_2$-anti-open sets. Then $\bigcup_{i \in J} A_i$ is $\mathcal{T}_1\mathcal{T}_2$-anti-open too.*

Proof. $(\subseteq)$. Let $x \in AntiInt_{\mathcal{T}_1}(AntiInt_{\mathcal{T}_2}(\bigcup_{i \in J} A_i))$. It means that there exists some $B \in \mathcal{T}_1$ such that $x \in B$ and $B \subseteq AntiInt_{\mathcal{T}_2}(\bigcup_{i \in J} A_i))$. Hence, $x \in AntiInt_{\mathcal{T}_2}(\bigcup_{i \in J} A_i)$. But then there is some $C \in \mathcal{T}_2$ such that $x \in C \subseteq \bigcup_{i \in J} A_i$. Then $x \in \bigcup_{i \in J} A_i$.

$(\supseteq)$. Let $x \in \bigcup_{i \in J} A_i$. Assume that $x \notin AntiInt_{\mathcal{T}_1}(AntiInt_{\mathcal{T}_2}(\bigcup_{i \in J} A_i))$. Then there is some $k \in J$ such that $x \in A_k$ but for any $\mathcal{T}_1$-anti-open $B \subseteq AntiInt_{\mathcal{T}_2}(\bigcup_{i \in J} A_i)$, $x \notin B$. But $A_k = AntiInt_{\mathcal{T}_1}(AntiInt_{\mathcal{T}_2}(A_k))$, so there is some $C \in \mathcal{T}_1$ such that $x \in C \subseteq AntiInt_{\mathcal{T}_2}(A_k)$. However, $AntiInt_{\mathcal{T}_2}(A_k) \subseteq AntiInt_{\mathcal{T}_2}(\bigcup_{i \in J} A_i)$. Assume the contrary. Then there is some $y \in AntiInt_{\mathcal{T}_2}(A_k)$ such that $y \notin AntiInt_{\mathcal{T}_2}(\bigcup_{i \in J} A_i)$. Hence there is $D \in \mathcal{T}_2$ such that $y \in D \subseteq A_k$ but for any $G \in \mathcal{T}_2$ such that $G \subseteq \bigcup_{i \in J} A_i$, $y \notin G$. But $D \subseteq A_k \subseteq \bigcup_{i \in J} A_i$. This is contradiction. □

Example 3.6 shows us that $(\mathcal{T}_1 \cup \mathcal{T}_2)$-anti-open set need not to be $\mathcal{T}_1\mathcal{T}_2$-anti-open. At first glance, this result is strange. One could say that the very definition of $\mathcal{T}_1\mathcal{T}_2$-anti-open sets resembles the concept of pseudo-anti-open sets in anti-topological spaces: A is pseudo-anti-open if and only if $AntiInt(A) = A$. And we observed in [22] that each anti-open set is pseudo-anti-open too. However, both anti-open and pseudo-anti-open sets rely on the same interior. Now the situation is different. This is because $(\mathcal{T}_1 \cup \mathcal{T}_2)AntiInt(A)$ need not to be identical with $AntiInt_{\mathcal{T}_1}(AntiInt_{\mathcal{T}_2}(A))$. Of course, we can define the following class:

Definition 3.9. Assume that $(X, \mathcal{T}_1, \mathcal{T}_2)$ is an anti-bitopological space. Let $A \subseteq X$. We say that A is $(\mathcal{T}_1 \cup \mathcal{T}_2)$-*pseudo-anti-open* if and only if $A = (\mathcal{T}_1 \cup \mathcal{T}_2)AntiInt(A)$.

Now the following lemma is clear:

Lemma 3.10. *Let $(X, \mathcal{T}_1, \mathcal{T}_2)$ be an anti-bitopological space. Every $(\mathcal{T}_1 \cup \mathcal{T}_2)$-anti-open set is $(\mathcal{T}_1 \cup \mathcal{T}_2)$-pseudo-anti-open too.*

Remark 3.11. The converse need not to be true. Let us go back to Example 3.6. Now $D = \{1, 2, 3\}$ is $(\mathcal{T}_1 \cup \mathcal{T}_2)$-pseudo-anti-open (its $(\mathcal{T}_1 \cup \mathcal{T}_2)$-interior is $\{1, 2\} \cup \{3\} = D$) but it does not belong to $\mathcal{T}_1 \cup \mathcal{T}_2$. Besides, note that $AntiInt_{\mathcal{T}_1}(AntiInt_{\mathcal{T}_2}(D)) = AntiInt_{\mathcal{T}_1}(\{2\}) = \emptyset$.

The last thing in this section is another one example of anti-bitopological space.

Example 3.12. Let $X = \mathbb{R}$ and assume that $\mathcal{T}_1$ consists of all those open intervals (a, b) of the length 1 such that $a \geq 0$ (e.g. $(0, 1)$, $(2, 3)$ or $(\pi, \pi + 1)$). In fact, b must be $a + 1$. Assume that $\mathcal{T}_2$ consists of all those open intervals (a, b) of the length 1 such that $a < 0$ (e.g. $(-10, -9)$ or $(-0.50, 0.50)$. Again, $b = a + 1$.

Consider $(\mathcal{T}_1 \cup \mathcal{T}_2)$. This is an anti-topology on X and it consists of all open intervals of the length exactly 1. Each $(\mathcal{T}_1 \cup \mathcal{T}_2)$-anti-closed set is a complement of some $(a, a + 1)$ (where $a \in \mathbb{R}$). Hence, it is of the form $(-\infty, a] \cup [a+1, +\infty)$. Now take $A = (0, 1)$. Each $(\mathcal{T}_1 \cup \mathcal{T}_2)$-anti-closed set B such that $A \subseteq B$ is in one of the two forms. First, it can be $(-\infty, k] \cup [k + 1, +\infty)$ for some $k \leq -1$ (e.g. $B = (-\infty, -2] \cup [-1, +\infty)$). Second, it can be $(-\infty, m] \cup [m + 1, +\infty)$ where $m \geq 1$ (e.g. $B = (-\infty, 3] \cup [4, +\infty)$). As for the intersection of the first subfamily, it is $[0, +\infty)$. An intersection of the second subfamily is $(-\infty, 1]$. Hence, we may calculate an intersection of intersections of these families to find $(\mathcal{T}_1 \cup \mathcal{T}_2)AntiCl((0, 1))$. This will be $[0, +\infty) \cap (-\infty, 1] = [0, 1]$. Besides, we see that this last set is not $(\mathcal{T}_1 \cup \mathcal{T}_2)$-anti-closed. This will be later generalized in a separate lemma.

4. Further investigation of $(\mathcal{T}_1 \cup \mathcal{T}_2)$-anti-open sets

In this section we shall investigate some properties of our sets. Moreover, we will introduce the notion of boundary in anti-bitopological context. Some of the results are more general and they are true even for arbitrary generalized weak structures.

4.1. *About closure and interior*

Lemma 4.1. *Let $(X, \mathcal{T}_1, \mathcal{T}_2)$ be an anti-topological space. Then the following observations are true:*

(1) *The union (intersection) of two $(\mathcal{T}_1 \cup \mathcal{T}_2)$-anti-open sets may not be $(\mathcal{T}_1 \cup \mathcal{T}_2)$-anti-open.*
(2) *The union (intersection) of two $(\mathcal{T}_1 \cup \mathcal{T}_2)$-anti-closed sets may not be $(\mathcal{T}_1 \cup \mathcal{T}_2)$-anti-open.*

Proof. Proof is simple albeit we shall present two cases. Take the same universe and anti-topologies as in Example 3.6. Now both $\{2\}$ and $\{3\}$ are $(\mathcal{T}_1 \cup \mathcal{T}_2)$-anti-open but their union, namely $\{2, 3\}$ does not belong to $\mathcal{T}_1 \cup \mathcal{T}_2$. Moreover, their intersection (which is $\emptyset$) is beyond $\mathcal{T}_1 \cup \mathcal{T}_2$. The reader is encouraged to find appropriate $(\mathcal{T}_1 \cup \mathcal{T}_2)$-anti-closed counterexamples. □

Lemma 4.2. *Let $(X, \mathcal{T}_1, \mathcal{T}_2)$ be an anti-bitopological space. Then $(\mathcal{T}_1 \cup \mathcal{T}_2)AntiInt(A) \subseteq A$.*

Proof. Let $x \in (\mathcal{T}_1 \cup \mathcal{T}_2)AntiInt(A)$. Then there is some $(\mathcal{T}_1 \cup \mathcal{T}_2)$-anti-open $B \subseteq X$ such that $x \in B \subseteq A$. Then $x \in A$. □

Remark 4.3. But the converse of the lemma above is not true in general as shown in the example below: let $X = \{1, 2, 3, 4\}$, $\mathcal{T}_1 = \{\{2\}, \{1, 3\}\}$ and $\mathcal{T}_2 = \{\{1\}, \{2, 4\}\}$. Then $(X, \mathcal{T}_1, \mathcal{T}_2)$ is an anti-bitopological space. Let $A = \{2, 3\}$ and then we have $(\mathcal{T}_1 \cup \mathcal{T}_2)AntiInt(A) = \{2\} \neq A$.

On the other hand, it is always true that if A is $(\mathcal{T}_1 \cup \mathcal{T}_2)$-anti-open, then $(\mathcal{T}_1 \cup \mathcal{T}_2)AntiInt(A) = A$.

Lemma 4.4. *Let $(X, \mathcal{T}_1, \mathcal{T}_2)$ be an anti-bitopological space and $A \subseteq X$. Then $(\mathcal{T}_1 \cup \mathcal{T}_2)AntiInt(A)$ need not to be the largest $(\mathcal{T}_1 \cup \mathcal{T}_2)$-anti-open set contained in A and $(\mathcal{T}_1 \cup \mathcal{T}_2)AntiCl(A)$ need not to be the smallest $(\mathcal{T}_1 \cup \mathcal{T}_2)$-anti-closed set contained in A.*

Proof. Let us think about $X = \{a, b, c, d\}$ with $\mathcal{T}_1 = \{\{a\}, \{c\}\}$ and $\mathcal{T}_2 = \{\{b\}, \{c, d\}\}$. Then, clearly, $\mathcal{T}_1$ and $\mathcal{T}_2$ are anti-topologies on X and $(X, \mathcal{T}_1, \mathcal{T}_2)$ is an anti-bitopological space. Let $A = \{b, c, d\}$. Then we have $(\mathcal{T}_1 \cup \mathcal{T}_2)AntiInt(A) = \{c\} \cup \{b\} \cup \{c, d\} = \{b, c, d\}$. But this set is not $(\mathcal{T}_1 \cup \mathcal{T}_2)$-anti-open. Now, we see that $(\mathcal{T}_1 \cup \mathcal{T}_2)_{Cl} = \{\{b, c, d\}, \{a, b, d\}, \{a, c, d\}, \{a, b\}\}$. Take $B = \{b, d\}$. Now $(\mathcal{T}_1 \cup \mathcal{T}_2)AntiCl(B) = \{b, c, d\} \cap \{a, b, d\} = B$. But B is not $(\mathcal{T}_1 \cup \mathcal{T}_2)$-anti-closed at all.

Hence, the proposition above. □

Lemma 4.5. *Let $(X, \mathcal{T}_1, \mathcal{T}_2)$ be an anti-bitopological space and $A \subseteq B$. Then $(\mathcal{T}_1 \cup \mathcal{T}_2)AntiInt(A) \subseteq (\mathcal{T}_1 \cup \mathcal{T}_2)AntiInt(B)$.*

Proof. Let $x \in (\mathcal{T}_1 \cup \mathcal{T}_2)AntiInt(A)$. Then there is some $C \in \mathcal{T}_1 \cup \mathcal{T}_2$ such that $x \in C \subseteq A$. But $A \subseteq B$, hence $C \subseteq B$. Hence, $x \in (\mathcal{T}_1 \cup \mathcal{T}_2)AntiInt(A)$. □

Remark 4.6. The converse of the lemma above need not to be true. Take $X = \mathcal{N}^+$, $\mathcal{T}_1 = \{\{3\}, \{5\}, \{7\}, \{9\}, ...\}$ and $\mathcal{T}_2 = \{\{4\}, \{6\}, \{8\}, \{10\}, ...\}$. Now $(\mathcal{T}_1 \cup \mathcal{T}_2)$ is just a collection of all singletons of X without $\{1\}$ and $\{2\}$. Take $A = \{1, 3, 4\}$ and $B = \{2, 3, 4\}$. Now $(\mathcal{T}_1 \cup \mathcal{T}_2)AntiInt(A) = \{3, 4\} = (\mathcal{T}_1 \cup \mathcal{T}_2)AntiInt(B)$ (so, in particular, $(\mathcal{T}_1 \cup \mathcal{T}_2)AntiInt(A) \subseteq (\mathcal{T}_1 \cup \mathcal{T}_2)AntiInt(B)$). But $A \nsubseteq B$.

Lemma 4.7. *Let* $(X, \mathcal{T}_1, \mathcal{T}_2)$ *be an anti-bitopological space and* $A, B \subseteq X$. *Then* $(\mathcal{T}_1 \cup \mathcal{T}_2)AntiInt(A \cap B) \subseteq (\mathcal{T}_1 \cup \mathcal{T}_2)AntiInt(A) \cap (\mathcal{T}_1 \cup \mathcal{T}_2)AntiInt(B)$.

Proof. We see that $A \cap B \subseteq A$ and $A \cap B \subseteq B$. Then $(\mathcal{T}_1 \cup \mathcal{T}_2)AntiInt(A \cap B) \subseteq (\mathcal{T}_1 \cup \mathcal{T}_2)AntiInt(A)$ and analogously $(\mathcal{T}_1 \cup \mathcal{T}_2)AntiInt(A \cap B) \subseteq (\mathcal{T}_1 \cup \mathcal{T}_2)AntiInt(B)$. Hence our lemma is true. □

Remark 4.8. As for the converse of the lemma above, it does not need to be true. Take $X = \{a, b, c, d\}$, $\mathcal{T}_1 = \{\{a\}, \{b\}\}$ and $\mathcal{T}_2 = \{\{b, c\}, \{c, d\}\}$. Now let $A = \{a, b, c, e\}$ and $B = \{c, d\}$. Then $(\mathcal{T}_1 \cup \mathcal{T}_2)AntiInt(A) = \{a, b, c\}$ and $(\mathcal{T}_1 \cup \mathcal{T}_2)AntiInt(B) = \{c, d\}$. Clearly, the intersection of those $(\mathcal{T}_1 \cup \mathcal{T}_2)$-anti-interiors is $\{c\}$. On the other hand, $A \cap B = \{c\}$ but $(\mathcal{T}_1 \cup \mathcal{T}_2)AntiInt(\{c\}) = \emptyset$ and $\{c\} \nsubseteq \emptyset$.

Lemma 4.9. *Let* $(X, \mathcal{T}_1, \mathcal{T}_2)$ *be an anti-bitopological space and* $A, B \subseteq X$. *Then* $(\mathcal{T}_1 \cup \mathcal{T}_2)AntiInt(A) \cup (\mathcal{T}_1 \cup \mathcal{T}_2)AntiInt(B) \subseteq (\mathcal{T}_1 \cup \mathcal{T}_2)AntiInt(A \cup B)$.

Proof. We have $A \subseteq A \cup B$, so $(\mathcal{T}_1 \cup \mathcal{T}_2)AntiInt(A) \subseteq (\mathcal{T}_1 \cup \mathcal{T}_2)AntiInt(A \cup B)$. Also, $B \subseteq A \cup B$, so $(\mathcal{T}_1 \cup \mathcal{T}_2)AntiInt(B) \subseteq (\mathcal{T}_1 \cup \mathcal{T}_2)AntiInt(A \cup B)$. Therefore, the conclusion holds. □

Lemma 4.10. *Let* $(X, \mathcal{T}_1, \mathcal{T}_2)$ *be an anti-bitopological space. Now* $(\mathcal{T}_1 \cup \mathcal{T}_2)AntiInt((\mathcal{T}_1 \cup \mathcal{T}_2)AntiInt(A)) = (\mathcal{T}_1 \cup \mathcal{T}_2)AntiInt(A)$.

Proof. $(\subseteq)$. This is clear in the light of Lemma 4.2.

$(\supseteq)$. Let $x \in (\mathcal{T}_1 \cup \mathcal{T}_2)AntiInt(A)$. Hence there is some $B \in \mathcal{T}_1 \cup \mathcal{T}_2$ such that $x \in B \subseteq A$. But by the very definition of $(\mathcal{T}_1 \cup \mathcal{T}_2)$-anti-interior we can say that $B \subseteq \bigcup\{C; C \subseteq A, C \in \mathcal{T}_1 \cup \mathcal{T}_2\} = (\mathcal{T}_1 \cup \mathcal{T}_2)AntiInt(A)$. Hence $x \in \bigcup\{D; D \subseteq (\mathcal{T}_1 \cup \mathcal{T}_2)AntiInt(A), D \in \mathcal{T}_1 \cup \mathcal{T}_2\}$ (as we could see, B is an example of such D). But this means that $x \in (\mathcal{T}_1 \cup \mathcal{T}_2)AntiInt((\mathcal{T}_1 \cup \mathcal{T}_2)AntiInt(A))$.
□

Now we would like to prove some theorems about $(\mathcal{T}_1 \cup \mathcal{T}_2)$-anti-closure. Some of them seem to be elementary but they are necessary to establish our general framework.

Lemma 4.11. *Let* $(X, \mathcal{T}_1, \mathcal{T}_2)$ *be an anti-bitopological space with* $A \subseteq X$. *Then* $A \subseteq (\mathcal{T}_1 \cup \mathcal{T}_2)AntiCl(A)$.

Proof. This is clear as a result of the definition of $(\mathcal{T}_1 \cup \mathcal{T}_2)$-anti-closure. □

Lemma 4.12. *Assume that* $(X, \mathcal{T}_1, \mathcal{T}_2)$ *is an anti-bitopological space with* $A \subseteq X$. *Suppose that* A *is* $(\mathcal{T}_1 \cup \mathcal{T}_2)$*-anti-closed. Then* $(\mathcal{T}_1 \cup \mathcal{T}_2)AntiCl(A) = A$.

Proof. (⊇). This is obvious.

(⊆). Let $x \in (\mathcal{T}_1 \cup \mathcal{T}_2)AntiCl(A)$. Hence for any $(\mathcal{T}_1 \cup \mathcal{T}_2)$-anti-closed B such that $A \subseteq B$, we have that $x \in B$. In particular, $A \subseteq A$ and A is $(\mathcal{T}_1 \cup \mathcal{T}_2)$-anti-closed. Hence $x \in A$. □

Lemma 4.13. *Let $(X, \mathcal{T}_1, \mathcal{T}_2)$ is an anti-bitopological space with $A, B \subseteq X$. Let $A \subseteq B$. Then $(\mathcal{T}_1 \cup \mathcal{T}_2)AntiCl(A) \subseteq (\mathcal{T}_1 \cup \mathcal{T}_2)AntiCl(B)$.*

Proof. Let $x \in (\mathcal{T}_1 \cup \mathcal{T}_2)AntiCl(A)$. Hence for any $(\mathcal{T}_1 \cup \mathcal{T}_2)$-anti-closed D such that $A \subseteq D$, $x \in D$. Assume that there is some E that is $(\mathcal{T}_1 \cup \mathcal{T}_2)$-anti-closed and $B \subseteq E$ but $x \notin E$. But $A \subseteq B \subseteq E$. This is contradiction. □

Lemma 4.14. *Let $(X, \mathcal{T}_1, \mathcal{T}_2)$ be an anti-bitopological space. Let $A, B \subseteq X$. Then $(\mathcal{T}_1 \cup \mathcal{T}_2)AntiCl(A) \cup (\mathcal{T}_1 \cup \mathcal{T}_2)AntiCl(B) \subseteq (\mathcal{T}_1 \cup \mathcal{T}_2)AntiCl(A \cup B)$.*

Proof. We have $A \subseteq A \cup B$ and $B \subseteq A \cup B$. Therefore $(\mathcal{T}_1 \cup \mathcal{T}_2)AntiCl(A) \subseteq (\mathcal{T}_1 \cup \mathcal{T}_2)AntiCl(A \cup B)$ and $(\mathcal{T}_1 \cup \mathcal{T}_2)AntiCl(B) \subseteq (\mathcal{T}_1 \cup \mathcal{T}_2)AntiCl(A \cup B)$. Hence our conclusion is clear. □

Remark 4.15. Note that the converse is not necessarily true (albeit analogous converse would be true e.g. in topological spaces). Take $X = \{a, b, c\}$, $\mathcal{T}_1 = \{\{b, c\}, \{a, c\}, \{a, b\}\}$ and $\mathcal{T}_2 = \{\{b, c\}\}$. Now $(\mathcal{T}_1 \cup \mathcal{T}_2)_{Cl} = \{\{a\}, \{b\}, \{c\}\}$. Take $A = \{a\}$ and $B = \{b\}$. Both are identical with their $(\mathcal{T}_1 \cup \mathcal{T}_2)$-anti-closures. On the other hand, $A \cup B = \{a, b\}$ and $(\mathcal{T}_1 \cup \mathcal{T}_2)AntiCl(A \cup B) = \bigcap \emptyset = X \nsubseteq (\mathcal{T}_1 \cup \mathcal{T}_2)AntiCl(A) \cup (\mathcal{T}_1 \cup \mathcal{T}_2)AntiCl(B) = \{a, b\}$.

Lemma 4.16. *Let $(X, \mathcal{T}_1, \mathcal{T}_2)$ be an anti-bitopological space. Then $(\mathcal{T}_1 \cup \mathcal{T}_2)AntiCl(A \cap B) \subseteq (\mathcal{T}_1 \cup \mathcal{T}_2)AntiCl(A) \cap (\mathcal{T}_1 \cup \mathcal{T}_2)AntiCl(B)$.*

Proof. We have $A \cap B \subseteq A$ and $A \cap B \subseteq B$. Therefore, $(\mathcal{T}_1 \cup \mathcal{T}_2)AntiCl(A \cap B) \subseteq (\mathcal{T}_1 \cup \mathcal{T}_2)AntiCl(A) \cap (\mathcal{T}_1 \cup \mathcal{T}_2)AntiCl(B)$. □

In the next lemma certain relationships between $(\mathcal{T}_1 \cup \mathcal{T}_2)$-anti-closure and $(\mathcal{T}_1 \cup \mathcal{T}_2)$-anti-interior have been proven.

Lemma 4.17. *If $(X, \mathcal{T}_1, \mathcal{T}_2)$ is an anti-bitopological space with $A \subseteq X$. Then:*

(1) $(\mathcal{T}_1 \cup \mathcal{T}_2)AntiInt(A) = ((\mathcal{T}_1 \cup \mathcal{T}_2)AntiCl(A^c))^c$.
(2) $(\mathcal{T}_1 \cup \mathcal{T}_2)AntiCl(A^c) = ((\mathcal{T}_1 \cup \mathcal{T}_2)AntiInt(A))^c$.
(3) $(\mathcal{T}_1 \cup \mathcal{T}_2)AntiCl(A) = ((\mathcal{T}_1 \cup \mathcal{T}_2)AntiInt(A^c))^c$.

Proof:

(1) ($\subseteq$). Let $x \in (\mathcal{T}_1 \cup \mathcal{T}_2)AntiInt(A)$. Then $x \in A$ and there is a $B \in (\mathcal{T}_2)$ such that $x \in B \subseteq A$. Then $x \notin B^c$. But $A^c \subseteq B^c$ and $B^c \in (\mathcal{T}_1 \cup \mathcal{T}_2)_{Cl}$. Thus, we can say that $x \notin (\mathcal{T}_1 \cup \mathcal{T}_2)AntiCl(A^c)$. Hence, $x \in ((\mathcal{T}_1 \cup \mathcal{T}_2)AntiCl(A^c))^c$.
($\supseteq$). Let $x \in ((\mathcal{T}_1 \cup \mathcal{T}_2)AntiCl(A^c))^c$. Then, there is some B such that B is $(\mathcal{T}_1 \cup \mathcal{T}_2)$-anti-closed, $A^c \subseteq B$ and $x \notin B$. But B^c is a $(\mathcal{T}_1 \cup \mathcal{T}_2)$-anti-open and $B^c \subseteq (A^c)^c = A$. Moreover, $x \in B^c$, so $x \in (\mathcal{T}_1 \cup \mathcal{T}_2)AntiInt(A)$.

(2) The whole thing to do is to take complements of both sides in (1).
(3) The whole thing to do is to take A^c instead of A in (2).

4.2. *About boundary*

In this section we shall investigate the notion of boundary in our anti-bitopological context.

Definition 4.18. Let $(X, \mathcal{T}_1, \mathcal{T}_2)$ be an anti-bitopological space and $A \subseteq X$. We define $(\mathcal{T}_1 \cup \mathcal{T}_2)$-anti-boundary of A as the set of these points which belong to the intersection of the $(\mathcal{T}_1 \cup \mathcal{T}_2)$-anti-closure of A with the $(\mathcal{T}_1 \cup \mathcal{T}_2)$-anti-closure of the complement of A.

It means that $(\mathcal{T}_1 \cup \mathcal{T}_2)AntiBd(A) = (\mathcal{T}_1 \cup \mathcal{T}_2)AntiCl(A) \cap (\mathcal{T}_1 \cup \mathcal{T}_2)AntiCl(A^c)$.

Example 4.19. Let $X = \{1, 2, 3, 4\}$, $\mathcal{T}_1 = \{\{2\}, \{1, 3\}\}$ and $\mathcal{T}_2 = \{\{1\}, \{2, 4\}\}$. Then $(X, \mathcal{T}_1, \mathcal{T}_2)$ is an anti-bitopological space. Then all the $(\mathcal{T}_1 \cup \mathcal{T}_2)$-anti-closed sets are $\{1, 3, 4\}$, $\{2, 4\}$, $\{2, 3, 4\}$ and $\{1, 3\}$. Let $A = \{2, 3\}$. Then $A^c = \{1, 4\}$. Now, $(\mathcal{T}_1 \cup \mathcal{T}_2)AntiCl(A) = \{2, 3, 4\}$ and $(\mathcal{T}_1 \cup \mathcal{T}_2)AntiCl(A^c) = \{1, 3, 4\}$.

Hence, $(\mathcal{T}_1 \cup \mathcal{T}_2)AntiBd(A) = \{2, 3, 4\} \cap \{1, 3, 4\} = \{3, 4\}$.

Example 4.20. Recall Example 3.12. Consider the same space and the same $A = (0, 1)$. We already know that $(\mathcal{T}_1 \cup \mathcal{T}_2)AntiCl(A) = [0, 1]$. Now think about $A^c = (-\infty, 0] \cup [1, +\infty)$. This set is $(\mathcal{T}_1 \cup \mathcal{T}_2)$-anti-closed hence it is identical with its own $(\mathcal{T}_1 \cup \mathcal{T}_2)$-anti-closure. Now $(\mathcal{T}_1 \cup \mathcal{T}_2)AntiBd(A) = [0, 1] \cap ((-\infty, 0] \cup [1, +\infty)) = \{0, 1\}$.

In case of $B = (0, 2)$ we would obtain $(\mathcal{T}_1 \cup \mathcal{T}_2)AntiCl(A) = [0, 2]$, $(\mathcal{T}_1 \cup \mathcal{T}_2)AntiCl(A^c) = (-\infty, 0] \cup [2, +\infty)$ and finally $(\mathcal{T}_1 \cup \mathcal{T}_2)AntiBd(A) = \{0, 2\}$.

In the next proposition we have some fundamental properties of the operation introduced above.

Lemma 4.21. *Let $(X, \mathcal{T}_1, \mathcal{T}_2)$ be an anti-bitopological space and $A \subseteq X$. Then the following results are true:*

(1) $(\mathcal{T}_1 \cup \mathcal{T}_2)AntiBd(A) = (\mathcal{T}_1 \cup \mathcal{T}_2)AntiCl(A) \setminus (\mathcal{T}_1 \cup \mathcal{T}_2)AntiInt(A)$.
(2) $(\mathcal{T}_1 \cup \mathcal{T}_2)AntiInt(A) = A \setminus (\mathcal{T}_1 \cup \mathcal{T}_2)AntiBd(A)$.
(3) $((\mathcal{T}_1 \cup \mathcal{T}_2)AntiBd(A))^c = (\mathcal{T}_1 \cup \mathcal{T}_2)AntiInt(A) \cup (\mathcal{T}_1 \cup \mathcal{T}_2)AntiInt(A^c)$.

(4) $(\mathcal{T}_1 \cup \mathcal{T}_2)AntiCl(A) = (\mathcal{T}_1 \cup \mathcal{T}_2)AntiInt(A) \cup (\mathcal{T}_1 \cup \mathcal{T}_2)AntiBd(A)$.

Proof:

(1) We may show the following sequence of equivalences: $x \in (\mathcal{T}_1 \cup \mathcal{T}_2)AntiBd(A) \Leftrightarrow x \in (\mathcal{T}_1 \cup \mathcal{T}_2)AntiCl(A) \cap (\mathcal{T}_1 \cup \mathcal{T}_2)AntiCl(A^c) \Leftrightarrow x \in (\mathcal{T}_1 \cup \mathcal{T}_2)AntiCl(A)$ and $x \in (\mathcal{T}_1 \cup \mathcal{T}_2)AntiCl(A^c) \Leftrightarrow x \in (\mathcal{T}_1 \cup \mathcal{T}_2)AntiCl(A)$ and $x \notin (\mathcal{T}_1 \cup \mathcal{T}_2)AntiInt(A) \Leftrightarrow x \in (\mathcal{T}_1 \cup \mathcal{T}_2)AntiCl(A) \setminus (\mathcal{T}_1 \cup \mathcal{T}_2)AntiInt(A)$.

(2) We have $A \setminus (\mathcal{T}_1 \cup \mathcal{T}_2)AntiBd(A) = A \setminus ((\mathcal{T}_1 \cup \mathcal{T}_2)AntiCl(A) \cap (\mathcal{T}_1 \cup \mathcal{T}_2)AntiCl(A^c)) = A \cap ((\mathcal{T}_1 \cup \mathcal{T}_2)AntiCl(A) \cap (\mathcal{T}_1 \cup \mathcal{T}_2)AntiCl(A^c))^c = A \cap (((\mathcal{T}_1 \cup \mathcal{T}_2)AntiCl(A))^c \cup ((\mathcal{T}_1 \cup \mathcal{T}_2)AntiCl(A^c))^c) = (A \cap (\mathcal{T}_1 \cup \mathcal{T}_2)AntiCl(A))^c) \cup (A \cap ((\mathcal{T}_1 \cup \mathcal{T}_2)AntiCl(A^c))^c) = \emptyset \cup (A \cap (\mathcal{T}_1 \cup \mathcal{T}_2)AntiInt(A)) = (\mathcal{T}_1 \cup \mathcal{T}_2)AntiInt(A)$.

(3) Using Lemma 4.17 we may write:
$((\mathcal{T}_1 \cup \mathcal{T}_2)AntiBd(A))^c = ((\mathcal{T}_1 \cup \mathcal{T}_2)AntiCl(A) \cap (\mathcal{T}_1 \cap \mathcal{T}_2)AntiCl(A^c))^c = ((\mathcal{T}_1 \cup \mathcal{T}_2)AntiCl(A))^c \cup ((\mathcal{T}_1 \cup \mathcal{T}_2)AntiCl(A^c))^c = (\mathcal{T}_1 \cup \mathcal{T}_2)AntiInt(A^c) \cup (\mathcal{T}_1 \cup \mathcal{T}_2)AntiInt(A)$, as expected.

(4) This can be proved in a similar manner.

Remark 4.22. Note that $(\mathcal{T}_1 \cup \mathcal{T}_2)AntiBd(A)$ need not to be equal with $(\mathcal{T}_1 \cup \mathcal{T}_2)AntiCl(A) \cap A^c$. This condition is too weak. Let $X = \{1, 2, 3, 4\}$, $\mathcal{T}_1 = \{\{1, 3\}, \{2, 4\}, \{5\}\}$ and $\mathcal{T}_2 = \{\{1\}, \{2, 4\}, \{3, 5\}\}$. Consider $A = \{1, 2, 3\}$. Then $A^c = \{4, 5\}$. As for the $(\mathcal{T}_1 \cup \mathcal{T}_2)$-anti-closed sets, these are $\{2, 4, 5\}$, $\{1, 3, 5\}$, $\{1, 2, 3, 4\}$, $\{2, 3, 4, 5\}$ and $\{1, 2, 4\}$. Now, $(\mathcal{T}_1 \cup \mathcal{T}_2)AntiInt(A) = \{1, 3\}$, $(\mathcal{T}_1 \cup \mathcal{T}_2)AntiCl(A) = \{1, 2, 3, 4\}$ and $(\mathcal{T}_1 \cup \mathcal{T}_2)AntiCl(A^c) = \{2, 4, 5\} \cap \{2, 3, 4, 5\} = \{2, 4, 5\}$. So $(\mathcal{T}_1 \cup \mathcal{T}_2)AntiBd(A) = \{1, 2, 3, 4\} \cap \{2, 4, 5\} = \{2, 4\}$. But this set is different than $(\mathcal{T}_1 \cup \mathcal{T}_2)AntiCl(A) \cap A^c = \{1, 2, 3, 4\} \cap \{4, 5\} = \{4\}$.

Moreover, it is clear (in the light of the example above) that $(\mathcal{T}_1 \cup \mathcal{T}_2)AntiBd(A)$ need not to be equal with $((\mathcal{T}_1 \cup \mathcal{T}_2)AntiCl(A) \cap A^c) \setminus (\mathcal{T}_1 \cup \mathcal{T}_2)AntiInt(A)$. In our case this last set would be equal to $(\{1, 2, 3, 4\} \cap \{4, 5\}) \setminus \{1, 3\} = \{4\}$.

The next theorem gives us some additional information about boundary.

Lemma 4.23. *Let $(X, \mathcal{T}_1, \mathcal{T}_2)$ be an anti-bitopological space and $A \subseteq X$. Then the following results hold:*

(1) $(\mathcal{T}_1 \cup \mathcal{T}_2)AntiBd((\mathcal{T}_1 \cup \mathcal{T}_2)AntiInt(A)) \subseteq (\mathcal{T}_1 \cup \mathcal{T}_2)AntiBd(A)$.

(2) $(\mathcal{T}_1 \cup \mathcal{T}_2)AntiBd((\mathcal{T}_1 \cup \mathcal{T}_2)AntiCl(A)) \subseteq (\mathcal{T}_1 \cup \mathcal{T}_2)AntiBd(A)$.

Proof:

(1) We have: $(\mathcal{T}_1 \cup \mathcal{T}_2)AntiBd((\mathcal{T}_1 \cup \mathcal{T}_2)AntiInt(A)) = (\mathcal{T}_1 \cup \mathcal{T}_2)AntiCl((\mathcal{T}_1 \cup \mathcal{T}_2)AntiInt(A)) \cap (\mathcal{T}_1 \cup \mathcal{T}_2)AntiCl(((\mathcal{T}_1 \cup \mathcal{T}_2)AntiInt(A))^c) = (\mathcal{T}_1 \cup \mathcal{T}_2)AntiCl((\mathcal{T}_1 \cup \mathcal{T}_2)AntiInt(A)) \cap (\mathcal{T}_1 \cup \mathcal{T}_2)AntiCl((((\mathcal{T}_1 \cup \mathcal{T}_2)AntiCl(A^c)^c)^c) = (\mathcal{T}_1 \cup \mathcal{T}_2)AntiCl((\mathcal{T}_1 \cup$

$\mathcal{T}_2)AntiInt(A)) \cap (\mathcal{T}_1 \cup \mathcal{T}_2)AntiCl(A^c) \subseteq (\mathcal{T}_1 \cup \mathcal{T}_2)AntiCl(A) \cap (\mathcal{T}_1 \cup \mathcal{T}_2)AntiCl(A^c) = (\mathcal{T}_1 \cup \mathcal{T}_2)AntiBd(A)$.

(2) We have: $(\mathcal{T}_1 \cup \mathcal{T}_2)AntiBd((\mathcal{T}_1 \cup \mathcal{T}_2)AntiCl(A)) = (\mathcal{T}_1 \cup \mathcal{T}_2)AntiCl((\mathcal{T}_1 \cup \mathcal{T}_2)AntiCl(A)) \cap (\mathcal{T}_1 \cup \mathcal{T}_2)AntiCl(((\mathcal{T}_1 \cup \mathcal{T}_2)AntiCl(A))^c) = (\mathcal{T}_1 \cup \mathcal{T}_2)AntiCl(A) \cap (\mathcal{T}_1 \cup \mathcal{T}_2)AntiCl(((\mathcal{T}_1 \cup \mathcal{T}_2)AntiCl(A))^c)$.

However, we know that $A \subseteq (\mathcal{T}_1 \cup \mathcal{T}_2)AntiCl(A)$. Hence $((\mathcal{T}_1 \cup \mathcal{T}_2)AntiCl(A))^c \subseteq A^c$. Thus $(\mathcal{T}_1 \cup \mathcal{T}_2)AntiCl(A) \cap (\mathcal{T}_1 \cup \mathcal{T}_2)AntiCl(((\mathcal{T}_1 \cup \mathcal{T}_2)AntiCl(A))^c) \subseteq (\mathcal{T}_1 \cup \mathcal{T}_2)AntiCl(A) \cap (\mathcal{T}_1 \cup \mathcal{T}_2)AntiCl(A^c) = (\mathcal{T}_1 \cup \mathcal{T}_2)AntiBd(A)$.

Remark 4.24. Note that it is not necessarily true that $(\mathcal{T}_1 \cup \mathcal{T}_2)AntiBd(A \cup B) \subseteq (\mathcal{T}_1 \cup \mathcal{T}_2)AntiBd(A) \cup (\mathcal{T}_1 \cup \mathcal{T}_2)AntiBd(B)$.

Counterexample: take $X = \{a, b, c\}$, $\mathcal{T}_1 = \{\{b, c\}, \{c, a\}, \{a, b\}\}$ and $\mathcal{T}_2 = \{\{b, c\}\}$. Now $(\mathcal{T}_1 \cup \mathcal{T}_2)_{Cl} = \{\{a\}, \{b\}, \{c\}\}$. Take $A = \{a\}$ and $B = \{b\}$. Clearly, $(\mathcal{T}_1 \cup \mathcal{T}_2)AntiCl(A) = A$ and $(\mathcal{T}_1 \cup \mathcal{T}_2)AntiCl(B) = B$. Moreover, $A^c = \{b, c\}$ and $B^c = \{a, c\}$. Thus, $(\mathcal{T}_1 \cup \mathcal{T}_2)AntiCl(A^c) = \bigcap \emptyset = X$ and $(\mathcal{T}_1 \cup \mathcal{T}_2)AntiCl(B^c) = \bigcap \emptyset = X$.

Then we see that $A \cup B = \{a, b\}$ and $(\mathcal{T}_1 \cup \mathcal{T}_2)AntiCl(\{a, b\}) = \bigcap \emptyset = X$. Moreover, $(A \cup B)^c = \{c\}$.

Now $(\mathcal{T}_1 \cup \mathcal{T}_2)AntiBd(A) = \{a\} \cap X = \{a\}$ and $(\mathcal{T}_1 \cup \mathcal{T}_2)AntiBd(B) = \{b\} \cap X = \{b\}$. The union of these two sets is $\{a, b\}$. However, $(\mathcal{T}_1 \cup \mathcal{T}_2)AntiBd(A \cup B) = X \cap (\mathcal{T}_1 \cup \mathcal{T}_2)AntiCl(\{c\}) = X \cap \{c\} = \{c\}$. But $\{c\} \nsubseteq \{a, b\}$.

However, analogous property is true in topological spaces.

Remark 4.25. Note that it is not true in general that $(\mathcal{T}_1 \cup \mathcal{T}_2)AntiBd(A \cap B) \subseteq (\mathcal{T}_1 \cup \mathcal{T}_2)AntiBd(A) \cup (\mathcal{T}_1 \cup \mathcal{T}_2)AntiBd(B)$.

Take the same space as in Remark 4.22. Consider $A = \{1, 2, 3, 4\}$ and $B = \{2, 3, 4, 5\}$. Calculate $A \cap B = \{2, 3, 4\}$. Then $(\mathcal{T}_1 \cup \mathcal{T}_2)AntiCl(A \cap B) = (\mathcal{T}_1 \cup \mathcal{T}_2)AntiCl(\{2, 3, 4\}) = \{1, 2, 3, 4\} \cap \{2, 3, 4, 5\} = \{2, 3, 4\}$. Then $(\mathcal{T}_1 \cup \mathcal{T}_2)AntiCl((A \cap B)^c) = (\mathcal{T}_1 \cup \mathcal{T}_2)AntiCl(\{1, 5\}) = \{1, 3, 5\}$.

Now $(\mathcal{T}_1 \cup \mathcal{T}_2)AntiBd(A \cap B) = \{2, 3, 4\} \cap \{1, 3, 5\} = \{3\}$.

Then we calculate $(\mathcal{T}_1 \cup \mathcal{T}_2)AntiCl(A) = \{1, 2, 3, 4\}$, $(\mathcal{T}_1 \cup \mathcal{T}_2)AntiCl(A^c) = (\mathcal{T}_1 \cup \mathcal{T}_2)AntiCl(\{5\}) = \{2, 4, 5\} \cap \{1, 3, 5\} \cap \{2, 3, 4, 5\} = \{5\}$. Thus $(\mathcal{T}_1 \cup \mathcal{T}_2)AntiBd(A) = \{1, 2, 3, 4\} \cap \{5\} = \emptyset$.

Moreover, $(\mathcal{T}_1 \cup \mathcal{T}_2)AntiCl(B) = \{2, 3, 4, 5\}$, $(\mathcal{T}_1 \cup \mathcal{T}_2)AntiCl(B^c) = (\mathcal{T}_1 \cup \mathcal{T}_2)AntiCl(\{1\}) = \{1, 3, 5\} \cap \{1, 2, 3, 4\} \cap \{1, 2, 4\} = \{1\}$ and thus $(\mathcal{T}_1 \cup \mathcal{T}_2)AntiBd(B) = \{2, 3, 4, 5\} \cap \{1\} = \emptyset$.

If so, then $(\mathcal{T}_1 \cup \mathcal{T}_2)AntiBd(A) \cup (\mathcal{T}_1 \cup \mathcal{T}_2)AntiBd(B) = \emptyset \cup \emptyset = \emptyset$. But clearly, $\{3\} \nsubseteq \emptyset$.

However, analogous property is true in topological spaces.

Theorem 4.26. *Let $(X, \mathcal{T}_1, \mathcal{T}_2)$ be an anti-topological space and $A \subseteq X$. Then the following results have been found:*

(1) *If A is $(\mathcal{T}_1 \cup \mathcal{T}_2)$-anti-open, then $(\mathcal{T}_1 \cup \mathcal{T}_2)AntiCl(A) \setminus A = (\mathcal{T}_1 \cup \mathcal{T}_2)AntiBd(A)$.*
(2) *If A is $(\mathcal{T}_1 \cup \mathcal{T}_2)$-anti-closed then $(\mathcal{T}_1 \cup \mathcal{T}_2)AntiBd(A) \subseteq A$.*

Proof:

Since A is $(\mathcal{T}_1 \cup \mathcal{T}_2)$-anti-open, therefore $(\mathcal{T}_1 \cup \mathcal{T}_2)AntiInt(A) = A$ and $(\mathcal{T}_1 \cup \mathcal{T}_2)AntiInt(A) = ((\mathcal{T}_1 \cup \mathcal{T}_2)AntiCl(A^c))^c$. Then:

(1) $(\mathcal{T}_1 \cup \mathcal{T}_2)AntiCl(A) \setminus A = (\mathcal{T}_1 \cup \mathcal{T}_2)AntiCl(A) \setminus (\mathcal{T}_1 \cup \mathcal{T}_2)AntiInt(A) = (\mathcal{T}_1 \cup \mathcal{T}_2)AntiCl(A) \setminus ((\mathcal{T}_1 \cup \mathcal{T}_2)AntiCl(A^c))^c = (\mathcal{T}_1 \cup \mathcal{T}_2)AntiCl(A) \cap (((\mathcal{T}_1 \cup \mathcal{T}_2)AntiCl(A^c))^c)^c = (\mathcal{T}_1 \cup \mathcal{T}_2)AntiCl(A) \cap (\mathcal{T}_1 \cup \mathcal{T}_2)AntiCl(A^c) = (\mathcal{T}_1 \cup \mathcal{T}_2)AntiBd(A)$.
(2) If A is $(\mathcal{T}_1 \cup \mathcal{T}_2)$-anti-closed then $(\mathcal{T}_1 \cup \mathcal{T}_2)AntiCl(A) = A$. So, $(\mathcal{T}_1 \cup \mathcal{T}_2)AntiBd(A) = (\mathcal{T}_1 \cup \mathcal{T}_2)AntiCl(A) \cap (\mathcal{T}_1 \cup \mathcal{T}_2)AntiCl(A^c) = A \cap (\mathcal{T}_1 \cup \mathcal{T}_2)AntiCl(A^c) \subseteq A$.

5. **Conclusions**

Here, we have introduced the basics of anti-bitopological space. The whole study of anti-topological spaces is still in its seminal form. This applies even more to anti-bitopologies. Thus, it is important to analyze many standard notions in both these novel frameworks. Some typical properties of interior, closure or boundary which are true in topological spaces are not necessarily true in weaker or just different structures. For example, in topological structures we can prove that $Int(A \cap B) = Int(A) \cap Int(B)$ but in anti-topologies (and anti-bitopologies) only left-to-right inclusion (that is, $(\subseteq)$) holds. Analogously, in topology we can say that $Cl(A \cup B) = Cl(A) \cup Cl(B)$, but in our structures only right-to-left inclusion is true.

In Remark 3.2 we have shown that there at least three possible approaches to the notion of "bi-anti-open" set. We ourselves focused on the family $(\mathcal{T}_1 \cup \mathcal{T}_2)$ but we presented some additional remarks on $(\mathcal{T}_1 \cap \mathcal{T}_2)$ and on those sets which satisfy the condition $A = AntiInt_{\mathcal{T}_1}(AntiInt_{\mathcal{T}_2}(A))$. In general, these three classes are not identical.

Now we would like to investigate other topological notions in anti-bitopological framework. For example, it would be reasonable to analyze density, nowhere density and connectedness.

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