Neutrosophic N-Soft Sets with TOPSIS method for Multiple Attribute Decision Making

Muhammad Riaz [1], Khalid Naeem[2], Iqra Zareef [3], Deeba Afzal[4]

[1] Department of Mathematics, University of the Punjab Lahore, Pakistan. E-mail: mriaz.math@pu.edu.pk
[2] Department of Mathematics & Statistics, The University of Lahore, Pakistan. E-mail: khalidnaeem333@gmail.com
[3] Department of Mathematics, University of the Punjab Lahore, Pakistan. E-mail: iqramaan90@yahoo.com
[4] Department of Mathematics & Statistics, The University of Lahore, Pakistan. Pakistan. E-mail: deebafzal@gmail.com

Abstract: The objective of this article is to introduce a new hybrid model of neutrosophic N-soft set which is combination of neutrosophic set and N-soft set. We introduce some basic operations on neutrosophic N-soft sets along with their fundamental properties. For multi-attribute decision-making (MADM) problems with neutrosophic N-soft sets, we propose an extended TOPSIS (technique based on order preference by similarity to ideal solution) method. In this method, we first propose a weighted decision matrix based comparison method to identify the positive and the negative ideal solutions. Afterwards, we define a separation measurement of these solutions. Finally, we calculate relative closeness to identify the optimal alternative. At length, a numerical example is rendered to illustrate the developed scheme in medical diagnosis via hypothetical case study.

Keywords: Neutosophic N-soft set, operations on neutosophic N-soft sets, MADM, TOPSIS, medical diagnosis.

1. Introduction

In contemporary decision-making science, multi-attribute decision-making (MADM) phenomenon plays a significant role in solving many real world problems. To deal with uncertainties, researchers have introduced different theories including, Fuzzy set (FS) [54] that comprises a mapping communicating the degree of association and intuitionistic fuzzy set (IFS) [10, 11] that comprises a pair of mappings communicating the degree of association and the degree of non-association of members of the universe to the unit closed interval with the restriction that sum of degree of association and degree of non-association should not exceed one. Smarandache [46, 47] introduced neutrosophic sets as an extension of IFSs. A neutrosophic object comprises three degrees, namely, degree of association, indeterminacy, and the degree of non-association to each alternative.

Smarandache's Neutrosophic Set [50] is a generalization of Intuitionistic Fuzzy Set, Inconsistent Intuitionistic Fuzzy Set (Picture Fuzzy Set, Ternary Fuzzy Set), Pythagorean Fuzzy Set (Atanassov's Intuitionistic Fuzzy Set of second type), *q*-Rung Orthopair Fuzzy Set, Spherical Fuzzy Set, and *n*-Hyper-Spherical Fuzzy Set; while Neutrosophication is a generalization of Regret Theory, Grey System Theory, and Three-Ways Decision. In 1999, Molodtsov [32] presented the notion of soft set as an important mathematical tool to deal with uncertainties. In 2007, Aktas and Cagman [6] extended the idea of soft sets to soft groups. In 2010, Feng *et al*. [18, 19] presented several results on soft sets, fuzzy soft sets and rough sets. In 2009 and 2011, Ali *et al*. [7, 8] introduced various properties of soft sets, fuzzy soft sets and rough sets. In 2011, Cagman *et al*. [12], and Shabir and Naz [51] independently presented soft topological spaces. Arockiarani *et al*. [9], in 2013, introduced the notion of fuzzy

neutrosophic soft toplogical spaces. In 2016, Davvaz and Sadrabadi [16] presented an interesting application of IFSs in medicine. Nabeeh *et al*. [33, 34] worked on neutrosophic multi-criteria decision making approach for IoT-based enterprises and for personnel selection used the neutrosophic-TOPSIS approach in 2019. Chang *et al*. [35] worked towards a reuse strategic decision pattern framework-from theories to practices. Garg and Arora [20]-[23] introduced generalized intuitionistic fuzzy soft power aggregation operator, Dual hesitant fuzzy soft aggregation operators, a novel scaled prioritized intuitionistic fuzzy soft interaction averaging aggregation operators and their application to multi criteria decision-making. Peng and Dai [36] presented some approaches to single-valued neutrosophic MADM based on MABAC, TOPSIS and new similarity measure with score function. Hashmi *et al*. [24] introduced *m*-polar neutrosophic topology with applications to multi-criteria decision-making in medical diagnosis and clustering analysis. In 2019, Naeem *et al*. [29] presented pythagorean fuzzy soft MCGDM methods based on TOPSIS, VIKOR and aggregation operators. In 2019, Naeem *et al*. [30] established pythagorean *m*-polar fuzzy sets and TOPSIS method for the selection of advertisement mode. In 2019, Riaz *et al*. [37] introduced N-soft topology and its applications to multi-criteria group decision making (MCGDM). Riaz and Hashmi [38] introduced the concept of cubic *m*-polar fuzzy set and presented multi-attribute group decision making (MAGDM) method for agribusiness in the environment of various cubic *m*-polar fuzzy averaging aggregation operators. Riaz and Hashmi [39] introduced the notion of linear Diophantine fuzzy Set (LDFS) and its applications towards multi-attribute decision making problems. Riaz and Hashmi [40] introduced soft rough Pythagorean *m*-polar fuzzy sets and Pythagorean *m*-polar fuzzy soft rough sets with application to decision-making. Riaz and Tehrim [41, 42, 43] substantiated the idea of bipolar fuzzy soft topology, cubic bipolar fuzzy set and cubic bipolar fuzzy ordered weighted geometric aggregation operators and their application using internal and external cubic bipolar fuzzy data. Riaz and Tahrim [44] introduced the concept of bipolar fuzzy soft mappings with application to bipolar disorders.

Smarandache [48] introduced a unifying field in logics: Neutrosophic Logic. Neutrosophy, Neutrosophic Set, Neutrosophic Probability and Statistics. Smarandache [49] introduced Neutrosophic Overset, Neutrosophic Underset, and Neutrosophic Offset. Similarly for Neutrosophic Over-/Under-/Off- Logic, Probability, and Statistics.

Soft sets provide binary evaluation of the objects and other mathematical models like fuzzy sets, intuitionistic fuzzy sets and neutrosophic sets associate values in the interval [0,1]. These models fail to deal with the situation when modeling on real world problems associate non-binary evaluations. Non-binary evaluations are also expected in rating or ranking positions. The ranking can be expressed in multinary values in the form of number of stars, dots, grades or any generalized notation. Motivated by these concerns, in 2017, Fatimah *et al*. [17] floated the idea of N-soft set as an extended model of soft set, in order to describe the importance of grades in real life. In 2018 and 2019, Akram *et al*. [1]-[3] introduced group decision-making methods based on hesitant N-soft sets and intuitionistic fuzzy N-soft rough set.

The technique for the order of preference by similarity to ideal solution (TOPSIS) was initially developed by Hwang and Yoon [26] in 1981. The core idea in the TOPSIS method is that selected alternative should have least geometric distance from positive ideal solution and maximum geometric distance from negative ideal solution. Positive ideal solution represents the condition for

best solution whereas negative ideal solution represents the condition for the worst. In 2000, Chen [13] extended the TOPSIS method to fuzzy environment and solved a decision making problem based on fuzzy information. Later, in 2008, Chen and Tsao [14] developed interval-valued fuzzy TOPSIS method. TOPSIS method in intuitionistic fuzzy framework was proposed by Li and Nan [31] in 2011. Joshi and Kumar [28] discussed TOPSIS method based on intuitionistic fuzzy entropy and distance measure for multi-criteria decision making. Recently, in 2016 Dey *et al.* [15] employed TOPSIS method for solving decision making problem under bipolar neutrosophic environment. In 2013, Xu and Zhang [53] developed a novel approach based on maximizing deviation and TOPSIS method for the explanation of multi-attribute decision making problems. In 2014, Zhang and Xu [55] presented an extension of TOPSIS in multiple criteria decision making with the help of Pythagorean fuzzy sets. Chen and Tsao [14] proposed interval-valued fuzzy TOPSIS method and its experimental analysis in 2016. In 2018, Akram and Arshad [4] presented a novel trapezoidal bipolar fuzzy TOPSIS method for group decision-making. In 2019, Akram and Adeel [5] presented TOPSIS approach for MAGDM based on interval-valued hesitant fuzzy N-soft environment. In 2019, Tehrim and Riaz [45] presented a novel extension of TOPSIS method with bipolar neutrosophic soft topology and its applications to multi-criteria group decision making (MCGDM). Riaz *et al.* [56]-[57] introduced novel concepts of soft rough topology with applications to MAGDM.

The goal of this paper is to present a new hybrid model "neutrosophic N-soft set" and their applications to the decision making (DM). Neutrosophic N-soft set is the generalization of N-soft set, fuzzy N-soft set and intuitionistic fuzzy N-soft.

The comparison analysis of the proposed model with some existing models is given in Table 1.

Sets	Parametrization	Non Binary Evaluation	Truth Membership	Falsity Membership	Indeterminacy
Fuzzy set [54]	×	×		×	×
Intutionistic fuzzy set [10]	×	×		✓	×
Neutrosophic set [46]	×	×		✓	✓
Soft Set [12]	✓	×	×	×	×
N-soft Set [17]	✓		×	×	×
Fuzzy N-soft Set[1]	✓		✓	×	×
Intutionistic N-soft Set [3]	✓		✓	✓	×
Neutrosophic N-soft Set (Proposed)	✓		✓	✓	✓

Table 1: Comparison with other existing theories

The rest of paper is organized as follows. In Section 2, we recall some fundamental concepts of N-soft set, fuzzy neutrosophic set and fuzzy neutrosophic soft set. In Section 3, we propose our new hybrid model fuzzy neutrosophic N-soft set along with their examples. We also present some basic operations on fuzzy neutrosophic N-soft set with illustrations. We also investigate fundamental

properties of the proposed model by using defined operations. In Section 4, we construct relations by using fuzzy neutrosophic N-soft set and define composition of fuzzy neutrosophic N-soft sets using relations. We also define some new choice functions and score functions in connection with fuzzy neutrosophic N-soft sets. In Section 5, we proposed DM method for medical diagnosis by the model. In Section 6, we give a numerical example of this diagnosis method via conjectural case study. In Section 7, we conclude with some future directions and give suggestions for future work.

2. Preliminaries

In this segment, we review some essential definitions and a few aftereffects of N-soft and neutrosophic sets that would be accommodating in the following segments.

Definition 2.1 [54] A *fuzzy set* ϑ in $\mathbb{X}$ is assessed up by a mapping with $\mathbb{X}$ as domain and membership degree in $[0,1]$. The accumulation of all *fuzzy sets* (FSs) in the universal set $\mathbb{X}$ is signified by $\vartheta(\mathbb{X})$.

Definition 2.2 [46, 47] A *neutrosophic set* (NS) $\mathbb{P}$ over the universe of discourse $\mathbb{X}$ is defined as

$$\mathbb{P} = \{\langle \varphi, (\mathbb{T}_{\mathbb{P}}(\varphi), \mathbb{I}_{\mathbb{P}}(\varphi), \mathbb{F}_{\mathbb{P}}(\varphi)) \rangle : \varphi \in \mathbb{X}\}$$

where $\mathbb{T}_{\mathbb{P}}, \mathbb{I}_{\mathbb{P}}, \mathbb{F}_{\mathbb{P}}: \mathbb{X} \to]^-0, 1^+[$ and $^-0 \leq \mathbb{T}_{\mathbb{P}}(\varphi) + \mathbb{I}_{\mathbb{P}}(\varphi) + \mathbb{F}_{\mathbb{P}}(\varphi) \leq 3^+$.

The mapping $\mathbb{T}_{\mathbb{P}}$ stands for degree of membership, $\mathbb{I}_{\mathbb{P}}$ is the degree of indeterminacy and $\mathbb{F}_{\mathbb{P}}$ is the degree of falsity of points of the given set. From philosophical perspective, the neutrosophic set takes the entries from some subset of $]^-0, 1^+[$. But it many actual applications, it is inconvenient to utilize neutrosophic set with entries from such subsets. Therefore, we consider the neutrosophic set which takes the entries from some subset of $[0,1]$.

Definition 2.3 [9] Let $\mathbb{X}$ be a space of objects (points). A *fuzzy neutrosophic set* (FNS) $\mathbb{P}$ in $\mathbb{X}$ is dispirit by a truth-membership function $\mathbb{T}_P$, an indeterminacy membership-function $\mathbb{I}_P$ and a falsity-membership function $\mathbb{F}_P$. In mathematical form, this collection is expressed as

$$\mathbb{P} = \{\langle \varphi, (\mathbb{T}_{\mathbb{P}}(\varphi), \mathbb{I}_{\mathbb{P}}(\varphi), \mathbb{F}_{\mathbb{P}}(\varphi)) \rangle : \varphi \in \mathbb{X}, \mathbb{T}_{\mathbb{P}}, \mathbb{I}_{\mathbb{P}}, \mathbb{F}_{\mathbb{P}} \in [0,1]\}$$

with the constraint that sum of $\mathbb{T}_{\mathbb{P}}(\varphi)$, $\mathbb{I}_{\mathbb{P}}(\varphi)$ and $\mathbb{F}_{\mathbb{P}}(\varphi)$ should fall in $[0,3]$ i.e.

$$0 \leq \mathbb{T}_{\mathbb{P}}(\varphi) + \mathbb{I}_{\mathbb{P}}(\varphi) + \mathbb{F}_{\mathbb{P}}(\varphi) \leq 3$$

Definition 2.4 [32] Let $\mathbb{X}$ be the set of points and E be the set of attributes with $\mathcal{L}$ in E. Assume that $P(\mathbb{X})$ denotes collection of subsets of $\mathbb{X}$. The pair $(\zeta, \mathcal{L})$ is said to be a *soft set* (SS) over $\mathbb{X}$, where ζ is a function given by $\zeta: \mathcal{L} \to P(\mathbb{X})$

Thus, an SS is expressed in mathematical form as

$$(\zeta, \mathcal{L}) = \{(\xi, \zeta(\xi)): \xi \in \mathcal{L}\}.$$

Definition 2.5 [9] Let $\mathbb{X}$ be the initial universal set and E be the set of parameters. We consider the non-empty set $\mathcal{L} \subseteq E$. Let $\widehat{\mathbb{P}}(\mathbb{X})$ signifies the set of all NSs of $\mathbb{X}$. The accretion $\Omega_{\mathcal{L}}$ is called the *neutrosophic soft set* (NSS) over $\mathbb{X}$, where $\Omega_{\mathcal{L}}$ is a function given by $\Omega_{\mathcal{L}}: \mathcal{L} \to \widehat{\mathbb{P}}(\mathbb{X})$. We can write it as

$$\Omega_{\mathcal{L}} = \{(\xi, \{\langle \varphi, \mathbb{T}_{\mathcal{L}(\xi)}(\varphi), \mathbb{I}_{\mathcal{L}(\xi)}(\varphi), \mathbb{F}_{\mathcal{L}(\xi)}(\varphi) \rangle : \varphi \in \mathbb{X}\}): \xi \in E\}$$

Notice that if $\Omega_{\mathcal{L}}(\xi) = \{\langle \varphi, 0, 1, 1 \rangle : \varphi \in \mathbb{X}\}$, then NS-element $(\xi, \Omega_{\mathcal{L}}(\xi))$ does not seem to appear in the NSS $\Omega_{\mathcal{L}}$. The set of all NSSs over $\mathbb{X}$ is symbolized by $NS(\mathbb{X}_E)$.

Definition 2.6 [17] Let $\mathbb{X}$ be a set of points and E be a set of attributes with $\mathcal{L}$ in E. Let $\mathcal{G} = \{0, 1, 2, \cdots, N-1\}$ be the set of ordered grades where $N \in \{2, 3, \cdots\}$. The *N-soft set* (NSS) on $\mathbb{X}$ is denoted by $(\zeta, \mathcal{L}, N)$ where $\zeta: \mathcal{L} \to 2^{\mathbb{X} \times \mathcal{G}}$ is a map characterized by

$$\zeta(\xi) = (\varphi, r_{\mathcal{L}(\xi)})$$

$\forall \varphi \in \mathbb{X}, \xi \in \mathcal{L}, r_{\mathcal{L}(\xi)} \in \mathcal{G}$.

Definition 2.7 [17] A *weak complement of N-soft set* $(\zeta, \mathcal{L}, N)$ is another N-soft set $(\zeta^C, \mathcal{L}, N)$ gratifying $\zeta(\xi)^C \sqcap \zeta(\xi) = \phi, \ \forall \xi \in \mathbb{X}$.

Definition 2.8 [17] A *top weak complement of N-soft set* $(\zeta, \mathcal{L}, N)$ is an N-soft set $(\zeta^\star, \mathcal{L}, N)$, where

$$(\zeta^\star, \mathcal{L}, N) = \begin{cases} \zeta(\xi) = (\varphi, N-1), & if \ r_{\mathcal{L}(\xi)}(\varphi) < N-1 \\ \zeta(\xi) = (\varphi, 0), & if \ r_{\mathcal{L}(\xi)}(\varphi) = N-1 \end{cases}$$

Definition 2.9 [17] A *bottom weak complement of N-soft set* $(\zeta, \mathcal{L}, N)$ is one more N-soft set $(\zeta_\star, \mathcal{L}, N)$, where

$$(\zeta_\star, \mathcal{L}, N) = \begin{cases} \zeta(\xi) = (\varphi, 0), & if \ r_{\mathcal{L}(\xi)}(\varphi) > 0, \\ \zeta(\xi) = (\varphi, N-1), & if \ r_{\mathcal{L}(\xi)}(\varphi) = 0. \end{cases}$$

3 Neutrosophic N-soft Set

In this section, we propose a novel structure *neutrosophic N-soft set* (NNSS), which is blend of NS and NSS. We present some definitions and operations on NNSS too. Some properties of NNSS associated with these operations also have been set up.

Definition 3.1 Let $\mathbb{X}$ be the initial universe set, E the set of attributes and $\mathcal{G}$ the aggregate of ordered grades. We consider non-empty subset $\mathcal{L}$ of E. Let $\widehat{P}(\mathbb{X} \times \mathcal{G})$ be the collection of all NSSs of $\mathbb{X} \times \mathcal{G}$. A *neutrosophic N-soft set* (NNSS) is signified by (λ, Ω, N), where $\Omega = (\zeta, \mathcal{L}, N)$ is an NSS. If there is no ambiguity, we can abbreviate it as $\lambda_{\mathcal{L}}$ represented by the mapping

$$\lambda_{\mathcal{L}} : \mathcal{L} \to \widehat{P}(\mathbb{X} \times \mathcal{G})$$

Mathematically,

$$\lambda_{\mathcal{L}} = \{(\xi, \Gamma_{\mathcal{L}}(\xi)) : \Gamma_{\mathcal{L}}(\xi) = \{(\langle \varphi, \mathbb{T}_{\mathcal{L}(\xi)}(\varphi), \mathbb{I}_{\mathcal{L}(\xi)}(\varphi), \mathbb{F}_{\mathcal{L}(\xi)}(\varphi)\rangle, r_{\mathcal{L}(\xi)}(\varphi)), r_{\mathcal{L}} \in \mathcal{G},$$
$$\varphi \in \mathbb{X}, \mathbb{T}_{\mathcal{L}}, \mathbb{I}_{\mathcal{L}}, \mathbb{F}_{\mathcal{L}} \in [0,1]\}, \xi \in E\}$$

In short form, we may write

$$\lambda_{\mathcal{L}} = \{(\xi, \Gamma_{\mathcal{L}}(\xi)) : \xi \in E\}$$

where

$$\Gamma_{\mathcal{L}}(\xi) = \{(\langle \varphi, \mathbb{T}_{\mathcal{L}(\xi)}(\varphi), \mathbb{I}_{\mathcal{L}(\xi)}(\varphi), \mathbb{F}_{\mathcal{L}(\xi)}(\varphi)\rangle, r_{\mathcal{L}(\xi)}(\varphi)) : r_{\mathcal{L}} \in \mathcal{G}, \varphi \in \mathbb{X}, \mathbb{T}_{\mathcal{L}}, \mathbb{I}_{\mathcal{L}}, \mathbb{F}_{\mathcal{L}} \in [0,1]\}$$

The accretion of all NNSSs is denoted by $\text{NNS}(\mathbb{X})$.

Our proposed structure is more generalized then other existing models. The existing models are special cases of our proposed model, as shown in Table 2

Neutrosophic N-soft Set (Proposed)	$(\xi, (\langle \varphi, \mathbb{T}_{\mathcal{L}(\xi)}(\varphi), \mathbb{I}_{\mathcal{L}(\xi)}(\varphi), \mathbb{F}_{\mathcal{L}(\xi)}(\varphi)\rangle, r_{\mathcal{L}(\xi)}(\varphi))$
Intutionistic N-soft Set [3]	$(\xi, (\langle \varphi, \mathbb{T}_{\mathcal{L}(\xi)}(\varphi), 0, \mathbb{F}_{\mathcal{L}(\xi)}(\varphi)\rangle, r_{\mathcal{L}(\xi)}(\varphi))$
Fuzzy N-soft Set [1]	$(\xi, (\langle \varphi, \mathbb{T}_{\mathcal{L}(\xi)}(\varphi), 0, 0\rangle, r_{\mathcal{L}(\xi)}(\varphi))$
N-soft Set [17]	$(\xi, r_{\mathcal{L}(\xi)}(\varphi))$

Table 2: Comparison with N-soft set and it's other existing generalization

Example 3.2 Let $\mathbb{X} = \{\varphi_1, \varphi_2\}$ and $E = \{\xi_1, \xi_2, \xi_3\}$. Consider $E \supseteq \mathcal{L} = \{\xi_1, \xi_2\}$. Define N8SS as $\lambda_{\mathcal{L}} = \{(\xi_i, \Gamma_{\mathcal{L}}(\xi_i)) : \xi_i \in \mathcal{L}, i = 1, 2\}$, where 8SS is given in Table 3 below:

$(\zeta, \mathcal{L}, 8)$	ξ_1	ξ_2
φ_1	6	3
φ_2	4	5

<div align="center">Table 3: Tabular representation of 8SS</div>

Now, we define N8SS as

$$\Gamma_{\mathcal{L}}(\xi_1) = \{(\langle\varphi_1, 0.8,0.5,0.1\rangle,6), (\langle\varphi_2, 0.6,0.2,0.9\rangle,4)\}$$
$$\Gamma_{\mathcal{L}}(\xi_2) = \{(\langle\varphi_1, 0.5,0.7,0.3\rangle,3), (\langle\varphi_2, 0.7,0.4,0.8\rangle,5)\}$$

The tabular representation of N8SS is given in Table 4.

$\lambda_{\mathcal{L}}$	ξ_1	ξ_2
φ_1	$(\langle 0.8,0.5,0.1\rangle,6)$	$(\langle 0.5,0.7,0.3\rangle,3)$
φ_2	$(\langle 0.6,0.2,0.9\rangle,4)$	$(\langle 0.7,0.4,0.8\rangle,5)$

<div align="center">Table 4: Tabular representation of N8SS</div>

Remarks:

1. Every N2SS $(\lambda, \Omega, 2)$ is generally equal to NSS.

2. Any arbitrary NNSS over the universe $\mathbb{X}$ can also be thought of as N$(N + 1)$-soft set. For example an N8SS can also be treated as an N9SS for the grade 8 is never used as can be seen in Table 4. This observation may be extended on the parallel track.

Now, we head towards presenting some arithmetical notions related to NNSS.

Definition 3.3 Let $\lambda_{\mathcal{L}}, \lambda_{\mathcal{M}} \in$NNS$(\mathbb{X})$. $\lambda_{\mathcal{L}}$ is said to be NNS-*subset* of $\lambda_{\mathcal{M}}$, if

$$\mathcal{L} \sqsubseteq \mathcal{M},$$
$$\mathbb{T}_{\mathcal{L}(\xi)}(\varphi) \le \mathbb{T}_{\mathcal{M}(\xi)}(\varphi),$$
$$\mathbb{I}_{\mathcal{L}(\xi)}(\varphi) \ge \mathbb{I}_{\mathcal{M}(\xi)}(\varphi),$$
$$\mathbb{F}_{\mathcal{L}(\xi)}(\varphi) \ge \mathbb{F}_{\mathcal{M}(\xi)}(\varphi),$$
$$r_{\mathcal{L}(\xi)}(\varphi) \le r_{\mathcal{M}(\xi)}(\varphi)$$

$\forall \xi \in E, \varphi \in X, r_{\mathcal{L}} \in \mathcal{G}$. We demonstrate it by $\lambda_{\mathcal{L}} \sqsubseteq \lambda_{\mathcal{M}}$. $\lambda_{\mathcal{M}}$ is said to be NNS-*superset* of $\lambda_{\mathcal{L}}$.

Example 3.4 Let $\mathbb{X} = \{\varphi_1, \varphi_2\}$ and $E = \{\xi_1, \xi_2, \xi_3\}$. Consider $E \supseteq \mathcal{L} = \{\xi_1, \xi_2\}$. Consider N8SS $\lambda_{\mathcal{L}}$ as given in Example 3.2. Let $\mathcal{M} = E$. Define N8SS $\lambda_{\mathcal{M}}$ as

$$\lambda_{\mathcal{M}} = \{(\xi_i, \Gamma_{\mathcal{M}}(\xi_i)): \xi_i \in \mathcal{M}, i = 1,2,3\}$$

where 8SS is given in Table 5 below.

$(\zeta, \mathcal{M}, 8)$	ξ_1	ξ_2	ξ_3
φ_1	7	4	6
φ_2	5	7	3

<div align="center">Table 5: Tabular representation of 8SS</div>

Now, we define N8SS

$$\Gamma_{\mathcal{M}}(\xi_1) = \{(\langle\varphi_1, 0.9,0.4,0.0\rangle,7), (\langle\varphi_2, 0.7,0.1,0.8\rangle,5)\}$$
$$\Gamma_{\mathcal{M}}(\xi_2) = \{(\langle\varphi_1, 0.6,0.5,0.2\rangle,4), (\langle\varphi_2, 0.9,0.3,0.8\rangle,7)\}$$
$$\Gamma_{\mathcal{M}}(\xi_3) = \{(\langle\varphi_1, 0.8,0.5,0.1\rangle,6), (\langle\varphi_3, 0.5,0.7,0.3\rangle,3)\}$$

having tabular form

$\lambda_{\mathcal{M}}$	ξ_1	ξ_2	ξ_3
φ_1	$(\langle 0.9, 0.4, 0.0 \rangle, 7)$	$(\langle 0.6, 0.5, 0.2 \rangle, 4)$	$(\langle 0.8, 0.5, 0.1 \rangle, 6)$
φ_2	$(\langle 0.7, 0.1, 0.8 \rangle, 5)$	$(\langle 0.9, 0.3, 0.8 \rangle, 7)$	$(\langle 0.5, 0.7, 0.3 \rangle, 3)$

Table 6: Tabular representation of N8SS

It can be seen from Table 4 and Table 6 that $\lambda_{\mathcal{L}} \sqsubseteq \lambda_{\mathcal{M}}$.

Definition 3.5 Let $\lambda_{\mathcal{L}}, \lambda_{\mathcal{M}} \in$ NNS($\mathbb{X}$). Then $\lambda_{\mathcal{L}}$ and $\lambda_{\mathcal{M}}$ are said to be NNS- *equal*, if

$$\mathcal{L} = \mathcal{M},$$
$$\mathbb{T}_{\mathcal{L}(\xi)}(\varphi) = \mathbb{T}_{\mathcal{M}(\xi)}(\varphi),$$
$$\mathbb{I}_{\mathcal{L}(\xi)}(\varphi) = \mathbb{I}_{\mathcal{M}(\xi)}(\varphi),$$
$$\mathbb{F}_{\mathcal{L}(\xi)}(\varphi) = \mathbb{F}_{\mathcal{M}(\xi)}(\varphi),$$
$$r_{\mathcal{L}(\xi)}(\varphi) = r_{\mathcal{M}(\xi)}(\varphi)$$

$\forall \xi \in E, \varphi \in \mathcal{X}, r_{\mathcal{L}} \in \mathcal{G}$. We demonstrate it by $\lambda_{\mathcal{L}} = \lambda_{\mathcal{M}}$.

Definition 3.6 Let $\lambda_{\mathcal{L}} \in$ NNS($\mathbb{X}$). If $\mathbb{T}_{\mathcal{L}(\xi)}(\varphi) = 0, \mathbb{I}_{\mathcal{L}(\xi)}(\varphi) = 1, \mathbb{F}_{\mathcal{L}(\xi)}(\varphi) = 1$ and $r_{\mathcal{L}(\xi)}(\varphi) = 0, \forall \xi \in E, \varphi \in \mathcal{X}, r_{\mathcal{L}} \in \mathcal{G}$; then $\lambda_{\mathcal{L}}$ is called *null* NNSS and symbolized by $\lambda_{\mathcal{L}_\phi}$.

Example 3.7 Let $\mathbb{X} = \{\varphi_1, \varphi_2\}$ and $E = \{\xi_1, \xi_2, \xi_3\}$. Consider $E \supseteq \mathcal{L} = \{\xi_1, \xi_2\}$. Define null N8SS as $\lambda_{\mathcal{L}_\phi} = \{(\xi_i, \Gamma_{\mathcal{L}_\phi}(\xi_i)) : \xi_i \in \mathcal{L}, i = 1,2\}$ where

$$\Gamma_{\mathcal{L}_\phi}(\xi_1) = \{(\langle \varphi_1, 0,1,1 \rangle, 0), (\langle \varphi_2, 0,1,1 \rangle, 0)\}$$
$$\Gamma_{\mathcal{L}_\phi}(\xi_2) = \{(\langle \varphi_1, 0,1,1 \rangle, 0), (\langle \varphi_2, 0,1,1 \rangle, 0)\}$$

The tabular form given in Table 7

$\lambda_{\mathcal{L}_\phi}$	ξ_1	ξ_2
φ_1	$(\langle 0,1,1 \rangle, 0)$	$(\langle 0,1,1 \rangle, 0)$
φ_2	$(\langle 0,1,1 \rangle, 0)$	$(\langle 0,1,1 \rangle, 0)$

Table 7: Tabular representation of null N8SS

Definition 3.8 Let $\lambda_{\mathcal{L}} \in$ NNS($\mathbb{X}$). If $\mathbb{T}_{\mathcal{L}(\xi)}(\varphi) = 1, \mathbb{I}_{\mathcal{L}(\xi)}(\varphi) = 0, \mathbb{F}_{\mathcal{L}(\xi)}(\varphi) = 0$ and $r_{\mathcal{L}(\xi)}(\varphi) = N - 1$, $\forall \xi \in E, \varphi \in \mathbb{X}, r_{\mathcal{L}} \in \mathcal{G}$, then $\lambda_{\mathcal{L}}$ is called *absolute* NNSS and symbolized by $\lambda_{\hat{\mathcal{L}}}$.

Example 3.9 Let $\mathbb{X} = \{\varphi_1, \varphi_2\}$ and $E = \{\xi_1, \xi_2, \xi_3\}$. Consider $E \supseteq \mathcal{L} = \{\xi_1, \xi_2\}$. Define absolute N8SS as $\lambda_{\hat{\mathcal{L}}} = \{(\xi_i, \Gamma_{\hat{\mathcal{L}}}(\xi_i)) : \xi_i \in \mathcal{L}, i = 1,2\}$ where

$$\Gamma_{\hat{\mathcal{L}}}(\xi_1) = \{(\langle \varphi_1, 1,0,0 \rangle, 7), (\langle \varphi_2, 1,0,0 \rangle, 7)\}$$
$$\Gamma_{\hat{\mathcal{L}}}(\xi_2) = \{(\langle \varphi_1, 1,0,0 \rangle, 7), (\langle \varphi_2, 1,0,0 \rangle, 7)\}$$

having tabular representation that is given in Table 8:

$\lambda_{\hat{\mathcal{L}}}$	ξ_1	ξ_2
φ_1	$(\langle 1,0,0 \rangle, 7)$	$(\langle 1,0,0 \rangle, 7)$
φ_2	$(\langle 1,0,0 \rangle, 7)$	$(\langle 1,0,0 \rangle, 7)$

Table 8: Tabular representation of absolute N8SS

Proposition 3.10 *Let* $\lambda_{\mathcal{K}}, \lambda_{\mathcal{L}}, \lambda_{\mathcal{M}} \in NNS(\mathbb{X})$. *Then,*

M. Riaz, K. Naeem, I. Zareef and D. Afzal, Neutrosophic N-Soft Sets with TOPSIS method

1. $\lambda_{\mathcal{L}} \sqsubseteq \lambda_{\hat{\mathcal{L}}}$.
2. $\lambda_{\mathcal{L}_\phi} \sqsubseteq \lambda_{\mathcal{L}}$.
3. $\lambda_{\mathcal{L}} \sqsubseteq \lambda_{\mathcal{L}}$.
4. $\lambda_{\mathcal{K}} \sqsubseteq \lambda_{\mathcal{L}}$ and $\lambda_{\mathcal{L}} \sqsubseteq \lambda_{\mathcal{M}} \Rightarrow \lambda_{\mathcal{K}} \sqsubseteq \lambda_{\mathcal{M}}$.

Proof. The proof follows directly from definitions of related terms.

Proposition 3.11 *Let* $\lambda_{\mathcal{K}}, \lambda_{\mathcal{L}}, \lambda_{\mathcal{M}} \in NNS(\mathbb{X})$. *Then,*

1. $\lambda_{\mathcal{K}} = \lambda_{\mathcal{L}}$ and $\lambda_{\mathcal{L}} = \lambda_{\mathcal{M}} \Rightarrow \lambda_{\mathcal{K}} = \lambda_{\mathcal{M}}$.
2. $\lambda_{\mathcal{L}} \sqsubseteq \lambda_{\mathcal{M}}$ and $\lambda_{\mathcal{M}} \sqsubseteq \lambda_{\mathcal{L}} \Rightarrow \lambda_{\mathcal{L}} = \lambda_{\mathcal{M}}$.

Proof. Straight forward.

Definition 3.12 *Let* $\lambda_{\mathcal{L}} \in NNS(\mathbb{X})$. *Then weak complement of NNSS* $\lambda_{\mathcal{L}}$ *is symbolized by* $\lambda_{\mathcal{L}}^{\mathsf{C}}$ *and defined as*

$$\lambda_{\mathcal{L}}^{\mathsf{C}} = \{(\xi, \Gamma_{\mathcal{L}}^{\mathsf{C}}) : \xi \in E\}$$

where

$$\Gamma_{\mathcal{L}}^{\mathsf{C}} = \{((\varphi, \mathbb{F}_{L(\xi)}(\varphi), 1 - \mathbb{I}_{L(\xi)}(\varphi), \mathbb{T}_{L(\xi)}(\varphi)), r_{L(\xi)}^{\mathsf{C}}(\varphi)) : \varphi \in X\}$$

Here $r_{L(\xi)}^{\mathsf{C}}(\varphi)$ denotes weak complement defined in Definition 2.7.

Example 3.13 Let $\mathbb{X} = \{\varphi_1, \varphi_2\}$ and $E = \{\xi_1, \xi_2, \xi_3\}$. Consider $E \supseteq \mathcal{L} = \{\xi_1, \xi_2\}$. Define complement of N8SS $\lambda_{\mathcal{L}}$ given in Example 3.2 as $\lambda_{\mathcal{L}}^{\mathsf{C}} = \{(\xi_i, \Gamma_{\mathcal{L}}^{\mathsf{C}}(\xi_i)) : \xi_i \in \mathcal{L}, i = 1,2\}$ i.e.

$$\Gamma_{\mathcal{L}}^{\mathsf{C}}(\xi_1) = \{((\varphi_1, 0.1, 0.5, 0.8), 5), ((\varphi_2, 0.9, 0.8, 0.6), 7)\}$$
$$\Gamma_{\mathcal{L}}^{\mathsf{C}}(\xi_2) = \{((\varphi_1, 0.3, 0.3, 0.5), 4), ((\varphi_2, 0.8, 0.6, 0.7), 2)\}$$

The tabular form is given in Table 9.

$\lambda_{\mathcal{L}}^{\mathsf{C}}$	ξ_1	ξ_2
φ_1	$(\langle 0.8, 0.5, 0.1 \rangle, 5)$	$(\langle 0.5, 0.7, 0.3 \rangle, 4)$
φ_2	$(\langle 0.6, 0.2, 0.9 \rangle, 7)$	$(\langle 0.7, 0.4, 0.8 \rangle, 2)$

Table 9: Tabular representation of weak complement of N8SS

Proposition 3.14 *Let* $\lambda_{\mathcal{L}} \in NNS(\mathbb{X})$, *then*

1. $(\lambda_{\mathcal{L}}^{\mathsf{C}})^{\mathsf{C}} \neq \lambda_{\mathcal{L}}$.
2. $\lambda_{\mathcal{L}_\phi}^{\mathsf{C}} \neq \lambda_{\hat{\mathcal{L}}}$.
3. $\lambda_{\hat{\mathcal{L}}}^{\mathsf{C}} \neq \lambda_{\mathcal{L}_\phi}$.

Proof. Straight forward.

Definition 3.15 *Let* $\lambda_{\mathcal{L}} \in NNS(\mathbb{X})$. *Then top weak complement of NNSS* $\lambda_{\mathcal{L}}$ *is symbolized by* $\lambda_{\mathcal{L}}^{\star}$ *and defined as*

$$\lambda_{\mathcal{L}}^{\star} = \{(\xi, \Gamma_{\mathcal{L}}^{\star}) : \xi \in E\}$$

Where,

$$\Gamma_{\mathcal{L}}^{\star} = \{((\varphi, \mathbb{F}_{L(\xi)}(\varphi), 1 - \mathbb{I}_{L(\xi)}(\varphi), \mathbb{T}_{L(\xi)}(\varphi)), r_{L(\xi)}^{\star}(\varphi)) : \varphi \in X\}$$

where, $r_{L(\xi)}^{\star}(\varphi)$ denotes top weak complement defined in Definition 2.8.

Example 3.16 Let $\mathbb{X} = \{\varphi_1, \varphi_2\}$ and $E = \{\xi_1, \xi_2, \xi_3\}$. Consider $E \supseteq \mathcal{L} = \{\xi_1, \xi_2\}$. Define complement of N8SS $\lambda_{\mathcal{L}}$ given in Example 3.2 as $\lambda_{\mathcal{L}}^{\star} = \{(\xi_i, \Gamma_{\mathcal{L}}^{\star}(\xi_i)) : \xi_i \in \mathcal{L}, i = 1,2\}$ i.e.

$$\Gamma_{\mathcal{L}}^{\star}(\xi_1) = \{((\varphi_1, 0.1, 0.5, 0.8), 7), ((\varphi_2, 0.9, 0.8, 0.6), 7)\}$$

$$\Gamma_{\mathcal{L}}^{\star}(\xi_2) = \{((\langle \varphi_1, 0.3, 0.3, 0.5 \rangle, 7), (\langle \varphi_2, 0.8, 0.6, 0.7 \rangle, 7)\}$$

In tabular form given in Table 10.

$\lambda_{\mathcal{L}}^{\star}$	ξ_1	ξ_2
φ_1	$(\langle 0.8, 0.5, 0.1 \rangle, 7)$	$(\langle 0.5, 0.7, 0.3 \rangle, 7)$
φ_2	$(\langle 0.6, 0.2, 0.9 \rangle, 7)$	$(\langle 0.7, 0.4, 0.8 \rangle, 7)$

Table 10: Tabular representation of top weak complement of N8SS

Proposition 3.17 *Let* $\lambda_{\mathcal{L}} \in NNS(\mathbb{X})$. *Then,*

1. $(\lambda_{\mathcal{L}}^{\star})^{\star} \neq \lambda_{\mathcal{L}}$.
2. $\lambda_{\mathcal{L}_\phi}^{\star} = \lambda_{\hat{\mathcal{L}}}$.
3. $\lambda_{\hat{\mathcal{L}}}^{\star} = \lambda_{\mathcal{L}_\phi}$.

Proof. The proof follows quickly from definitions of relevant terms.

Definition 3.18 *Let* $\lambda_{\mathcal{L}} \in NNS(\mathbb{X})$. *Then bottom weak complement of NNSS* $\lambda_{\mathcal{L}}$ *is symbolized by* $\lambda_{\mathcal{L}_\star}$ *and defined as follows*

$$\lambda_{\mathcal{L}_\star} = \{(\xi, \Gamma_{\mathcal{L}_\star}): \xi \in E\}$$

where

$$\Gamma_{\mathcal{L}_\star} = \{((\langle \varphi, \mathbb{F}_{\mathcal{L}(\xi)}(\varphi), 1 - \mathbb{I}_{\mathcal{L}(\xi)}(\varphi), \mathbb{T}_{\mathcal{L}(\xi)}(\varphi) \rangle, r_{\mathcal{L}(\xi)_\star}(\varphi)): \varphi \in X\}$$

Here $r_{\mathcal{L}(\xi)_\star}(\varphi)$ denotes top weak complement defined in Definition 2.9.

Example 3.19 Let $\mathbb{X} = \{\varphi_1, \varphi_2\}$ and $E = \{\xi_1, \xi_2, \xi_3\}$. Consider $E \supseteq \mathcal{L} = \{\xi_1, \xi_2\}$. Bottom weak complement of N8SS $\lambda_{\mathcal{L}}$ defined in Example 3.2 as $\lambda_{\mathcal{L}_\star} = \{(\xi_i, \Gamma_{\mathcal{L}_\star}(\xi_i)): \xi_i \in \mathcal{L}, i = 1,2\}$ where

$$\Gamma_{\mathcal{L}_\star}(\xi_1) = \{((\langle \varphi_1, 0.1, 0.5, 0.8 \rangle, 7), (\langle \varphi_2, 0.9, 0.8, 0.6 \rangle, 7)\}$$

$$\Gamma_{\mathcal{L}_\star}(\xi_2) = \{((\langle \varphi_1, 0.3, 0.3, 0.5 \rangle, 7), (\langle \varphi_2, 0.8, 0.6, 0.7 \rangle, 7)\}$$

In tabular form the bottom weak complement of N8SS is given in Table 11.

$\lambda_{\mathcal{L}_\star}$	ξ_1	ξ_2
φ_1	$(\langle 0.8, 0.5, 0.1 \rangle, 0)$	$(\langle 0.5, 0.7, 0.3 \rangle, 0)$
φ_2	$(\langle 0.6, 0.2, 0.9 \rangle, 0)$	$(\langle 0.7, 0.4, 0.8 \rangle, 0)$

Table 11: Tabular representation of bottom weak complement of N8SS

Proposition 3.20 *Let* $\lambda_{\mathcal{L}} \in NNS(\mathbb{X})$. *Then,*

1. $(\lambda_{\mathcal{L}_\star})_\star \neq \lambda_{\mathcal{L}}$.
2. $(\lambda_{\mathcal{L}_\phi})_\star = \lambda_{\hat{\mathcal{L}}}$.
3. $\lambda_{\hat{\mathcal{L}}_\star} = \lambda_{\mathcal{L}_\phi}$.

Proof. Straight forward.

Definition 3.21 Let $\lambda_{\mathcal{L}}, \lambda_{\mathcal{M}} \in NNS(\mathbb{X})$. Then *difference* of $\lambda_{\mathcal{L}}$ and $\lambda_{\mathcal{M}}$ is symbolized by $\lambda_{\mathcal{L}} \backslash \lambda_{\mathcal{M}}$ and is defined as

$$\lambda_{\mathcal{L}} \backslash \lambda_{\mathcal{M}} = \{(\xi, \{((\langle \varphi, \mathbb{T}_{\mathcal{L}(\xi) \backslash \mathcal{M}(\xi)}(\varphi), \mathbb{I}_{\mathcal{L}(\xi) \backslash \mathcal{M}(\xi)}(\varphi), \mathbb{F}_{\mathcal{L}(\xi) \backslash \mathcal{M}(\xi)}(\varphi) \rangle, r_{\mathcal{L}(\xi) \backslash \mathcal{M}(\xi)}(\varphi)):$$
$$\varphi \in \mathbb{X}\}): \xi \in E\}$$

where $\mathbb{T}_{\mathcal{L}(\xi) \backslash \mathcal{M}(\xi)}(\varphi), \mathbb{I}_{\mathcal{L}(\xi) \backslash \mathcal{M}(\xi)}(\varphi)$ and $\mathbb{F}_{\mathcal{L}(\xi) \backslash \mathcal{M}(\xi)}(\varphi)$ are defined as

$$\mathbb{T}_{\mathcal{L}(\xi) \backslash \mathcal{M}(\xi)}(\varphi) = \min\{\mathbb{T}_{\mathcal{L}(\xi)}(\varphi), \mathbb{F}_{\mathcal{M}(\xi)}(\varphi)\}$$

$$\mathbb{I}_{\mathcal{L}(\xi) \backslash \mathcal{M}(\xi)}(\varphi) = \max\{\mathbb{I}_{\mathcal{L}(\xi)}(\varphi), 1 - \mathbb{I}_{\mathcal{M}(\xi)}(\varphi)\}$$

$$\mathbb{F}_{\mathcal{L}(\xi) \backslash \mathcal{M}(\xi)}(\varphi) = \max\{\mathbb{F}_{\mathcal{L}(\xi)}(\varphi), \mathbb{T}_{\mathcal{M}(\xi)}(\varphi)\}$$

$$r_{\mathcal{L}(\xi)\backslash\mathcal{M}(\xi)}(\varphi) = \begin{cases} r_{\mathcal{L}(\xi)}(\varphi) - r_{\mathcal{M}(\xi)}(\varphi), & if\ r_{\mathcal{L}(\xi)}(\varphi) > r_{\mathcal{M}(\xi)}(\varphi), \\ 0, & otherwise \end{cases}$$

Definition 3.22 Let $\lambda_{\mathcal{L}}, \lambda_{\mathcal{M}} \in$ NNS$(\mathbb{X})$. Then *addition* of $\lambda_{\mathcal{L}}$ and $\lambda_{\mathcal{M}}$ is symbolized by $\lambda_{\mathcal{L}} \oplus \lambda_{\mathcal{M}}$ and is defined as

$$\lambda_{\mathcal{L}} \oplus \lambda_{\mathcal{M}} = \{(\xi, \{(\langle \varphi, \mathbb{T}_{\mathcal{L}(\xi)\oplus\mathcal{M}(\xi)}(\varphi), \mathbb{I}_{\mathcal{L}(\xi)\oplus\mathcal{M}(\xi)}(\varphi), \mathbb{F}_{\mathcal{L}(\xi)\oplus\mathcal{M}(\xi)}(\varphi)\rangle, r_{\mathcal{L}(\xi)\oplus\mathcal{M}(\xi)}(\varphi)) : \varphi \in \mathbb{X}\}) : \xi \in E\}$$

where $\mathbb{T}_{\mathcal{L}(\xi)\oplus\mathcal{M}(\xi)}(\varphi), \mathbb{I}_{\mathcal{L}(\xi)\oplus\mathcal{M}(\xi)}(\varphi)$ and $\mathbb{F}_{\mathcal{L}(\xi)\oplus\mathcal{M}(\xi)}(\varphi)$ are given as

$$\mathbb{T}_{\mathcal{L}(\xi)\oplus\mathcal{M}(\xi)}(\varphi) = \min\{\mathbb{T}_{\mathcal{L}(\xi)}(\varphi) + \mathbb{T}_{\mathcal{M}(\xi)}(\varphi), 1\}$$

$$\mathbb{I}_{\mathcal{L}(\xi)\oplus\mathcal{M}(\xi)}(\varphi) = \min\{\mathbb{I}_{\mathcal{L}(\xi)}(\varphi) + \mathbb{I}_{\mathcal{M}(\xi)}(\varphi), 1\}$$

$$\mathbb{F}_{\mathcal{L}(\xi)\oplus\mathcal{M}(\xi)}(\varphi) = \min\{\mathbb{F}_{\mathcal{L}(\xi)}(\varphi) + \mathbb{F}_{\mathcal{M}(\xi)}(\varphi), 1\}$$

$$r_{\mathcal{L}(\xi)\oplus\mathcal{M}(\xi)}(\varphi) = \begin{cases} r_{\mathcal{L}(\xi)}(\varphi) + r_{\mathcal{M}(\xi)}(\varphi), & if\ 0 \le r_{\mathcal{L}(\xi)}(\varphi) + r_{\mathcal{M}(\xi)}(\varphi) < N - 1, \\ N - 1, & if\ r_{\mathcal{L}(\xi)}(\varphi) + r_{\mathcal{M}(\xi)}(\varphi) \ge N - 1 \end{cases}$$

Definition 3.23 Let $\lambda_{\mathcal{L}}, \lambda_{\mathcal{M}} \in$ NNS $(\mathbb{X})$ be expressed as $\lambda_{\mathcal{L}} = (\lambda_1, \Omega_1, N)$ and $\lambda_{\mathcal{M}} = (\lambda_2, \Omega_2, N_1)$ where $\Omega_1 = (\zeta_1, \mathcal{L}, N_2)$ and $\Omega_2 = (\zeta_2, \mathcal{M}, N_1)$ are NSSs. Then their *restricted union* is symbolized by $(\lambda_1, \Omega_1, N_2) \sqcup_{\mathfrak{R}} (\lambda_2, \Omega_2, N_1)$ and defined as $(w, \Omega_1 \sqcup_{\mathfrak{R}} \Omega_2, \max(N_1, N_2))$ where $\Omega_1 \sqcup_{\mathfrak{R}} \Omega_2 = (W, \mathcal{L} \sqcap \mathcal{M}, \max(N_1, N_2))$ i.e.

$$(\lambda_1, \Omega_1, N_2) \sqcup_{\mathfrak{R}} (\lambda_2, \Omega_2, N_1)$$
$$= \{(\xi, \{(\langle \varphi, \mathbb{T}_{\mathcal{L}(\xi)}(\varphi) \vee \mathbb{T}_{\mathcal{M}(\xi)}(\varphi), \mathbb{I}_{\mathcal{L}(\xi)}(\varphi) \wedge \mathbb{I}_{\mathcal{M}(\xi)}(\varphi), \mathbb{F}_{\mathcal{L}(\xi)}(\varphi) \wedge \mathbb{F}_{\mathcal{M}(\xi)}(\varphi)\rangle, r_{\mathcal{L}(\xi)}(\varphi)$$
$$\vee r_{\mathcal{M}(\xi)}(\varphi)) : \varphi \in \mathbb{X}\}) : \xi \in \mathcal{L} \sqcap \mathcal{M}\}$$

Example 3.24 Consider again $\lambda_{\mathcal{L}}, \lambda_{\mathcal{M}}$ as given in Examples 3.2 and 3.4 respectively. The restricted union $\lambda_{\mathcal{L}} \sqcup_{\mathfrak{R}} \lambda_{\mathcal{M}}$ is given in Table 12.

$\lambda_{\mathcal{L}} \sqcup_{\mathfrak{R}} \lambda_{\mathcal{M}}$	ξ_1	ξ_2
φ_1	$(\langle 0.9, 0.4, 0.0 \rangle, 7)$	$(\langle 0.6, 0.5, 0.2 \rangle, 4)$
φ_2	$(\langle 0.7, 0.1, 0.8 \rangle, 5)$	$(\langle 0.9, 0.3, 0.8 \rangle, 7)$

Table 12: Tabular representation of restricted union of two N8SSs

Definition 3.25 Let $\lambda_{\mathcal{L}}, \lambda_{\mathcal{M}} \in$ NNS $(\mathbb{X})$ be expressed as $\lambda_{\mathcal{L}} = (\lambda_1, \Omega_1, N)$ and $\lambda_{\mathcal{M}} = (\lambda_2, \Omega_2, N_1)$ where $\Omega_1 = (\zeta_1, \mathcal{L}, N_2)$ and $\Omega_2 = (\zeta_2, \mathcal{M}, N_1)$ are NSSs. Then their *extended union* is symbolized by $(\lambda_1, \Omega_1, N_2) \sqcup_{\mathcal{E}} (\lambda_2, \Omega_2, N_1)$ and defined as $(w, \Omega_1 \sqcup_{\mathcal{E}} \Omega_2, \max(N_1, N_2))$ where $\Omega_1 \sqcup_{\mathcal{E}} \Omega_2 = (W, \mathcal{L} \sqcup \mathcal{M}, \max(N_1, N_2))$ i.e.

$$(\lambda_1, \Omega_1, N_2) \sqcup_{\mathcal{E}} (\lambda_2, \Omega_2, N_1) = \{(\xi, \{(\langle \varphi, \mathbb{T}_{\mathcal{L}(\xi)}(\varphi) \vee \mathbb{T}_{\mathcal{M}(\xi)}(\varphi), \mathbb{I}_{\mathcal{L}(\xi)}(\varphi) \wedge \mathbb{I}_{\mathcal{M}(\xi)}(\varphi), \mathbb{F}_{\mathcal{L}(\xi)}(\varphi) \wedge \mathbb{F}_{\mathcal{M}(\xi)}(\varphi)\rangle,$$
$$r_{\mathcal{L}(\xi)}(\varphi) \vee r_{\mathcal{M}(\xi)}(\varphi)) : \varphi \in \mathbb{X}\}) : \xi \in \mathcal{L} \sqcup \mathcal{M}\}$$

Example 3.26 Consider again $\lambda_{\mathcal{L}}, \lambda_{\mathcal{M}}$ as given in Examples 3.2 and 3.4 respectively. The extended union $\lambda_{\mathcal{L}} \sqcup_{\mathcal{E}} \lambda_{\mathcal{M}}$ is given in Table 13.

$\lambda_{\mathcal{L}} \sqcup_{\mathcal{E}} \lambda_{\mathcal{M}}$	ξ_1	ξ_2	ξ_3
φ_1	$(\langle 0.9, 0.4, 0.0 \rangle, 7)$	$(\langle 0.6, 0.5, 0.2 \rangle, 4)$	$(\langle 0.8, 0.5, 0.1 \rangle, 6)$
φ_2	$(\langle 0.7, 0.1, 0.8 \rangle, 5)$	$(\langle 0.9, 0.3, 0.8 \rangle, 7)$	$(\langle 0.5, 0.7, 0.3 \rangle, 3)$

Table 13: Tabular representation of extended union of two N8SSs

Theorem 3.27 *Let* $\lambda_{\mathcal{L}}, \lambda_{\mathcal{M}} \in NNS(\mathbb{X})$. *Then their extended-union* $\lambda_{\mathcal{L}} \sqcup_{\varepsilon} \lambda_{\mathcal{M}}$ *is the smallest NNSS containing both* $\lambda_{\mathcal{L}}$ *and* $\lambda_{\mathcal{M}}$.

Proof. Straight forward.

Definition 3.28 Let $\lambda_{\mathcal{L}}, \lambda_{\mathcal{M}} \in \text{NNS}(\mathbb{X})$ be expressed as $\lambda_{\mathcal{L}} = (\lambda_1, \Omega_1, N)$ and $\lambda_{\mathcal{M}} = (\lambda_2, \Omega_2, N_1)$ where $\Omega_1 = (\zeta_1, \mathcal{L}, N_2)$ and $\Omega_2 = (\zeta_2, \mathcal{M}, N_1)$ are NSSs. Then their *restricted intersection* is symbolized by $(\lambda_1, \Omega_1, N_2) \sqcap_{\mathfrak{R}} (\lambda_2, \Omega_2, N_1)$ and is defined as $(y, \Omega_1 \sqcup_{\mathfrak{R}} \Omega_2, \min(N_1, N_2))$ where $\Omega_1 \sqcap_{\mathfrak{R}} \Omega_2 = (Y, \mathcal{L} \sqcap \mathcal{M}, \min(N_1, N_2))$ i.e.

$$(\lambda_1, \Omega_1, N_2) \sqcap_{\mathfrak{R}} (\lambda_2, \Omega_2, N_1) = \{(\xi, \{((\varphi, \mathbb{T}_{\mathcal{L}(\xi)}(\varphi) \wedge \mathbb{T}_{\mathcal{M}(\xi)}(\varphi), \mathbb{I}_{\mathcal{L}(\xi)}(\varphi) \vee \mathbb{I}_{\mathcal{M}(\xi)}(\varphi), \mathbb{F}_{\mathcal{L}(\xi)}(\varphi) \vee \mathbb{F}_{\mathcal{M}(\xi)}(\varphi)),$$
$$r_{\mathcal{L}(\xi)}(\varphi) \wedge r_{\mathcal{M}(\xi)}(\varphi)): \varphi \in \mathbb{X}\}): \xi \in \mathcal{L} \sqcap \mathcal{M}\}$$

Example 3.29 Consider again $\lambda_{\mathcal{L}}, \lambda_{\mathcal{M}}$ as given in Examples 3.2, 3.4 respectively. The restricted intersection $\lambda_{\mathcal{L}} \sqcap_{\mathfrak{R}} \lambda_{\mathcal{M}}$ is given in Table 14.

$\lambda_{\mathcal{L}} \sqcap_{\mathfrak{R}} \lambda_{\mathcal{M}}$	ξ_1	ξ_2
φ_1	$(\langle 0.8, 0.5, 0.1 \rangle, 6)$	$(\langle 0.5, 0.7, 0.3 \rangle, 3)$
φ_2	$(\langle 0.6, 0.2, 0.9 \rangle, 4)$	$(\langle 0.7, 0.4, 0.8 \rangle, 5)$

Table 14: Tabular representation of restricted intersection of two N8SSs

Theorem 3.30 *Let* $\lambda_{\mathcal{L}}, \lambda_{\mathcal{M}} \in NNS(\mathbb{X})$. *Then their restricted-intersection* $\lambda_{\mathcal{L}} \sqcap_{\mathfrak{R}} \lambda_{\mathcal{M}}$ *is the largest NNSS contained in both* $\lambda_{\mathcal{L}}$ *and* $\lambda_{\mathcal{M}}$.

Proof. Straight forward.

Definition 3.31 Let $\lambda_{\mathcal{L}}, \lambda_{\mathcal{M}} \in \text{NNS}(\mathbb{X})$ be expressed as $\lambda_{\mathcal{L}} = (\lambda_1, \Omega_1, N)$ and $\lambda_{\mathcal{M}} = (\lambda_2, \Omega_2, N_1)$ where $\Omega_1 = (\zeta_1, \mathcal{L}, N_2)$ and $\Omega_2 = (\zeta_2, \mathcal{M}, N_1)$ are NSSs. Then their *restricted intersection* is symbolized by $(\lambda_1, \Omega_1, N_2) \sqcap_{\varepsilon} (\lambda_2, \Omega_2, N_1)$ and defined as $(y, \Omega_1 \sqcap_{\varepsilon} \Omega_2, \min(N_1, N_2))$, where $\Omega_1 \sqcap_{\varepsilon} \Omega_2 = (Y, \mathcal{L} \sqcup \mathcal{M}, \min(N_1, N_2))$ i.e.

$$(\lambda_1, \Omega_1, N_2) \sqcap_{\varepsilon} (\lambda_2, \Omega_2, N_1) = \{(\xi, \{((\varphi, \mathbb{T}_{\mathcal{L}(\xi)}(\varphi) \wedge \mathbb{T}_{\mathcal{M}(\xi)}(\varphi), \mathbb{I}_{\mathcal{L}(\xi)}(\varphi) \vee \mathbb{I}_{\mathcal{M}(\xi)}(\varphi), \mathbb{F}_{\mathcal{L}(\xi)}(\varphi) \vee \mathbb{F}_{\mathcal{M}(\xi)}(\varphi)),$$
$$r_{\mathcal{L}(\xi)}(\varphi) \wedge r_{\mathcal{M}(\xi)}(\varphi)): \varphi \in \mathbb{X}\}): \xi \in \mathcal{L} \sqcup \mathcal{M}\}$$

Example 3.32 Consider again $\lambda_{\mathcal{L}}, \lambda_{\mathcal{M}}$ as given in Examples 3.2, 3.4 respectively. The extended intersection $\lambda_{\mathcal{L}} \sqcap_{\varepsilon} \lambda_{\mathcal{M}}$ is given in Table 15.

$\lambda_{\mathcal{L}} \sqcap_{\varepsilon} \lambda_{\mathcal{M}}$	ξ_1	ξ_2	ξ_3
φ_1	$(\langle 0.8, 0.5, 0.1 \rangle, 6)$	$(\langle 0.5, 0.7, 0.3 \rangle, 3)$	$(\langle 0.8, 0.5, 0.1 \rangle, 6)$
φ_2	$(\langle 0.6, 0.2, 0.9 \rangle, 4)$	$(\langle 0.7, 0.4, 0.8 \rangle, 5)$	$(\langle 0.5, 0.7, 0.3 \rangle, 3)$

Table 15: Tabular representation of extended intersection of two N8SSs

For any two NNSS $\lambda_{\mathcal{L}}$ and $\lambda_{\mathcal{M}}$ over same set of points $\mathbb{X}$ and using the operations defined above, we conclude the following proposition:

Proposition 3.33 Let $\lambda_{\mathcal{L}}$ and $\lambda_{\mathcal{M}}$ be two NNSS

    (1) $\lambda_{\mathcal{L}} \sqcup_{\varepsilon} \lambda_{\mathcal{L}} = \lambda_{\mathcal{L}}$

    (2) $\lambda_{\mathcal{L}} \sqcup_{\varepsilon} \lambda_{\mathcal{M}} = \lambda_{\mathcal{M}} \sqcup_{\varepsilon} \lambda_{\mathcal{L}}$

    (3) $\lambda_{\mathcal{L}} \sqcap_{\mathcal{R}} \lambda_{\mathcal{L}} = \lambda_{\mathcal{L}}$

(4) $\lambda_{\mathcal{L}} \sqcap_{\mathcal{R}} \lambda_{\mathcal{M}} = \lambda_{\mathcal{M}} \sqcap_{\mathcal{R}} \lambda_{\mathcal{L}}$

(5) $\lambda_{\mathcal{L}} \sqcup_{\mathcal{E}} \lambda_{\mathcal{L}_\phi} = \lambda_{\mathcal{L}}$

(6) $\lambda_{\mathcal{L}} \sqcap_{\mathcal{R}} \lambda_{\mathcal{L}_\phi} = \lambda_{\mathcal{L}_\phi}$

For any three NNSS $\lambda_{\mathcal{L}}$, $\lambda_{\mathcal{M}}$ and $\lambda_{\mathcal{N}}$ over same set of points $\mathbb{X}$ and using the operations defined above, we conclude the following proposition:

Proposition 3.34 Let $\lambda_{\mathcal{L}}$, $\lambda_{\mathcal{M}}$ and $\lambda_{\mathcal{N}}$ be three NNSS

(1) $\lambda_{\mathcal{L}} \sqcup_{\mathcal{E}} (\lambda_{\mathcal{M}} \sqcup_{\mathcal{E}} \lambda_{\mathcal{N}}) = (\lambda_{\mathcal{L}} \sqcup_{\mathcal{E}} \lambda_{\mathcal{M}}) \sqcup_{\mathcal{E}} \lambda_{\mathcal{N}}$

(2) $\lambda_{\mathcal{L}} \sqcap_{\mathcal{R}} (\lambda_{\mathcal{M}} \sqcap_{\mathcal{R}} \lambda_{\mathcal{N}}) = (\lambda_{\mathcal{L}} \sqcap_{\mathcal{R}} \lambda_{\mathcal{M}}) \sqcap_{\mathcal{R}} \lambda_{\mathcal{N}}$

(3) $\lambda_{\mathcal{L}} \sqcup_{\mathcal{E}} (\lambda_{\mathcal{M}} \sqcap_{\mathcal{R}} \lambda_{\mathcal{N}}) = (\lambda_{\mathcal{L}} \sqcup_{\mathcal{E}} \lambda_{\mathcal{M}}) \sqcap_{\mathcal{R}} (\lambda_{\mathcal{L}} \sqcup_{\mathcal{E}} \lambda_{\mathcal{N}})$

(4) $\lambda_{\mathcal{L}} \sqcap_{\mathcal{R}} (\lambda_{\mathcal{M}} \sqcup_{\mathcal{E}} \lambda_{\mathcal{N}}) = (\lambda_{\mathcal{L}} \sqcap_{\mathcal{R}} \lambda_{\mathcal{M}}) \sqcup_{\mathcal{E}} (\lambda_{\mathcal{L}} \sqcap_{\mathcal{R}} \lambda_{\mathcal{N}})$

Definition 3.35 Let $\lambda_{\mathcal{L}}, \lambda_{\mathcal{M}} \in \text{NNS}(\mathbb{X})$ be expressed as $\lambda_{\mathcal{L}} = (\lambda_1, \Omega_1, N)$ and $\lambda_{\mathcal{M}} = (\lambda_2, \Omega_2, N_1)$ where $\Omega_1 = (\zeta_1, \mathcal{L}, N_2)$ and $\Omega_2 = (\zeta_2, \mathcal{M}, N_1)$ are NSSs. Then *AND Operation* symbolized by $(\lambda_1, \Omega_1, N_2) \wedge (\lambda_2, \Omega_2, N_1)$ or shortly $\lambda_{\mathcal{L}} \wedge \lambda_{\mathcal{M}}$ and is defined as $(\lambda_1, \Omega_1, N_2) \wedge (\lambda_2, \Omega_2, N_1) = (\lambda_{\mathcal{K}}, \mathcal{L} \times \mathcal{M}, \min(N_1, N_2))$, where degree of membership, indeterminacy and non-membership are given as follows:

$$\mathbb{T}_{\mathcal{K}(\xi_i, \xi_j)}(\varphi) = \min\{\mathbb{T}_{\mathcal{L}(\xi_i)}(\varphi), \mathbb{T}_{\mathcal{M}(\xi_j)}(\varphi)\},$$

$$\mathbb{I}_{\mathcal{K}(\xi_i, \xi_j)}(\varphi) = \frac{\{\mathbb{I}_{\mathcal{L}(\xi_i)}(\varphi) + \mathbb{I}_{\mathcal{M}(\xi_j)}(\varphi)\}}{2},$$

$$\mathbb{F}_{\mathcal{K}(\xi_i, \xi_j)}(\varphi) = \max\{\mathbb{F}_{\mathcal{L}(\xi)}(\varphi), \mathbb{F}_{\mathcal{M}(\xi)}(\varphi)\},$$

$$r_{\mathcal{K}(\xi_i, \xi_j)}(\varphi) = \max\{r_{\mathcal{L}(\xi_i)}(\varphi), r_{\mathcal{M}(\xi_j)}(\varphi)\}, \forall \xi_i \in \mathcal{L}, \xi_j \in \mathcal{M}$$

for all $\varphi \in \mathbb{X}$.

Definition 3.36 Let $\lambda_{\mathcal{L}}, \lambda_{\mathcal{M}} \in \text{NNS}(\mathbb{X})$ be two NNS be expressed as $\lambda_{\mathcal{L}} = (\lambda_1, \Omega_1, N)$ and $\lambda_{\mathcal{M}} = (\lambda_2, \Omega_2, N_1)$ where $\Omega_1 = (\zeta_1, \mathcal{L}, N_2)$ and $\Omega_2 = (\zeta_2, \mathcal{M}, N_1)$ are NSSs. Then *OR operation* is symbolized by $(\lambda_1, \Omega_1, N_2) \vee (\lambda_2, \Omega_2, N_1)$ or shortly $\lambda_{\mathcal{L}} \vee \lambda_{\mathcal{M}}$ and is defined as $(\lambda_1, \Omega_1, N_2) \vee (\lambda_2, \Omega_2, N_1) = (\lambda_{\mathcal{K}}, \mathcal{L} \times \mathcal{M}, \min(N_1, N_2))$, where degree of membership, indeterminacy and non-membership are given as follows:

$$\mathbb{T}_{\mathcal{H}(\xi_i, \xi_j)}(\varphi) = \max\{\mathbb{T}_{\mathcal{L}(\xi_i)}(\varphi), \mathbb{T}_{\mathcal{M}(\xi_j)}(\varphi)\},$$

$$\mathbb{I}_{\mathcal{H}(\xi_i, \xi_j)}(\varphi) = \frac{\{\mathbb{I}_{\mathcal{L}(\xi_i)}(\varphi) + \mathbb{I}_{\mathcal{M}(\xi_j)}(\varphi)\}}{2},$$

$$\mathbb{F}_{\mathcal{H}(\xi_i, \xi_j)}(\varphi) = \min\{\mathbb{F}_{\mathcal{L}(\xi)}(\varphi), \mathbb{F}_{\mathcal{M}(\xi)}(\varphi)\},$$

$$r_{\mathcal{H}(\xi_i, \xi_j)}(\varphi) = \min\{r_{\mathcal{L}(\xi_i)}(\varphi), r_{\mathcal{M}(\xi_j)}(\varphi)\}, \forall \xi_i \in \mathcal{L}, \xi_j \in \mathcal{M}$$

for all $\varphi \in \mathbb{X}$.

Definition 3.37 The *Truth-favorite* of an NNSS $\lambda_{\mathcal{L}}$ is denoted by $\lambda_{\mathcal{M}} = \widehat{\triangle} \lambda_{\mathcal{L}}$ and is defined by

$$\mathbb{T}_{\mathcal{L}(\xi)}(\varphi) = \min\{\mathbb{T}_{\mathcal{L}(\xi)}(\varphi) + \mathbb{I}_{\mathcal{L}(\xi)}(\varphi), 1\}$$

$$\mathbb{I}_{\mathcal{L}(\xi)}(\varphi) = 0$$

$$\mathbb{F}_{\mathcal{L}(\xi)}(\varphi) = \mathbb{F}_{\mathcal{M}(\xi)}(\varphi)$$

$$r_{\mathcal{L}(\xi)}(\varphi) = r_{\mathcal{M}(\xi)}(\varphi)$$

for all $\varphi \in \mathbb{X}, \xi \in \mathcal{L}$.

Definition 3.38 *The Falsity-favorite of an NNSS $\lambda_{\mathcal{L}}$ is denoted by $\lambda_{\mathcal{M}} = \widehat{\nabla} \lambda_{\mathcal{L}}$ and is defined by*

$$\mathbb{T}_{\mathcal{L}(\xi)}(\varphi) = \mathbb{T}_{\mathcal{M}(\xi)}(\varphi)$$
$$\mathbb{I}_{\mathcal{L}(\xi)}(\varphi) = 0$$
$$\mathbb{F}_{\mathcal{L}(\xi)}(\varphi) = \min\{\mathbb{F}_{\mathcal{L}(\xi)}(\varphi) + \mathbb{I}_{\mathcal{L}(\xi)}(\varphi), 1\}$$
$$r_{\mathcal{L}(\xi)}(\varphi) = r_{\mathcal{M}(\xi)}(\varphi)$$

for all $\varphi \in \mathbb{X}, \xi \in \mathcal{L}$.

Proposition 3.39 *Let $\lambda_{\mathcal{L}}$ be an NNSS, then*

1. $\widehat{\triangle}\widehat{\triangle} \lambda_{\mathcal{L}} = \widehat{\triangle} \lambda_{\mathcal{L}}$.
2. $\widehat{\nabla}\widehat{\nabla} \lambda_{\mathcal{L}} = \widehat{\nabla} \lambda_{\mathcal{L}}$.

Proof. Follows immediately from definitions.

Definition 3.40 Let $\lambda_{\mathcal{L}} \in$ NNS$(\mathbb{X})$. Then *scalar multiplication* of $\lambda_{\mathcal{L}}$ with α is symbolized by $\lambda_{\mathcal{L}} \otimes \alpha$ and is defined as

$$\lambda_{\mathcal{L}} \otimes \alpha = \{(\xi, \{(\langle \varphi, \mathbb{T}_{\mathcal{L}(\xi)}(\varphi) \otimes \alpha, \mathbb{I}_{\mathcal{L}(\xi)}(\varphi) \otimes \alpha, \mathbb{F}_{\mathcal{L}(\xi)}(\varphi) \otimes \alpha), r_{\mathcal{L}(\xi)}(\varphi) \otimes \alpha): \varphi \in \mathbb{X}\}):$$
$$\xi \in E\}$$

where $\mathbb{T}_{\mathcal{L}(\xi) \otimes \alpha}(\varphi), \mathbb{I}_{\mathcal{L}(\xi) \otimes \alpha}(\varphi) \mathbb{F}_{\mathcal{L}(\xi) \otimes \alpha}(\varphi)$ and $r_{\mathcal{L}(\xi)}(\varphi) \otimes \alpha$ are defined by

$$\mathbb{T}_{\mathcal{L}(\xi)}(\varphi) \otimes \alpha = \min\{\mathbb{T}_{\mathcal{L}(\xi)}(\varphi) \times \alpha, 1\}$$
$$\mathbb{I}_{\mathcal{L}(\xi)}(\varphi) \otimes \alpha = \min\{\mathbb{I}_{\mathcal{L}(\xi)}(\varphi) \times \alpha, 1\}$$
$$\mathbb{F}_{\mathcal{L}(\xi)}(\varphi) \otimes \alpha = \min\{\mathbb{F}_{\mathcal{L}(\xi)}(\varphi) \times \alpha, 1\}$$
$$r_{\mathcal{L}(\xi)}(\varphi) \otimes \alpha = \begin{pmatrix} r_{\mathcal{L}(\xi)}(\varphi) \times \alpha, & if\ 0 \leq r_{\mathcal{L}(\xi)}(\varphi) \times \alpha < N - 1, \\ N - 1, & otherwise \end{pmatrix}$$

Definition 3.41 Let $\lambda_{\mathcal{L}} \in$ NNS$(\mathbb{X})$. Then *scalar division* of $\lambda_{\mathcal{L}}$ by α is symbolized by $\lambda_{\mathcal{L}} \widetilde{/} \alpha$ and is defined as

$$\lambda_{\mathcal{L}} \widetilde{/} \alpha = \{(\xi, \{(\langle \varphi, \mathbb{T}_{\mathcal{L}(\xi)}(\varphi) \widetilde{/} \alpha, \mathbb{I}_{\mathcal{L}(\xi)}(\varphi) \widetilde{/} \alpha, \mathbb{F}_{\mathcal{L}(\xi)}(\varphi) \widetilde{/} \alpha), r_{\mathcal{L}(\xi)}(\varphi) \widetilde{/} \alpha): \varphi \in \mathbb{X}\}): \xi \in E\}$$

where $\mathbb{T}_{\mathcal{L}(\xi) \widetilde{/} \alpha}(\varphi), \mathbb{I}_{\mathcal{L}(\xi) \widetilde{/} \alpha}(\varphi) \mathbb{F}_{\mathcal{L}(\xi) \widetilde{/} \alpha}(\varphi)$ and $r_{\mathcal{L}(\xi)}(\varphi) \widetilde{/} \alpha$ are defined by

$$\mathbb{T}_{\mathcal{L}(\xi)}(\varphi) \widetilde{/} \alpha = \min\{\mathbb{T}_{\mathcal{L}(\xi)}(\varphi)/\alpha, 1\}$$
$$\mathbb{I}_{\mathcal{L}(\xi)}(\varphi) \widetilde{/} \alpha = \min\{\mathbb{I}_{\mathcal{L}(\xi)}(\varphi)/\alpha, 1\}$$
$$\mathbb{F}_{\mathcal{L}(\xi)}(\varphi) \widetilde{/} \alpha = \min\{\mathbb{F}_{\mathcal{L}(\xi)}(\varphi)/\alpha, 1\}$$
$$r_{\mathcal{L}(\xi)}(\varphi) \widetilde{/} \alpha = \begin{pmatrix} r_{\mathcal{L}(\xi)}(\varphi)/\alpha, & if\ 0 \leq r_{\mathcal{L}(\xi)}(\varphi)/\alpha < N - 1, \\ N - 1, & otherwise \end{pmatrix}$$

$$r_{\mathcal{L}(\xi)}(\varphi) \widetilde{/} \alpha = \begin{cases} r_{\mathcal{L}(\xi)}(\varphi)/\alpha, & if\ 0 \leq r_{\mathcal{L}(\xi)}(\varphi)/\alpha < N - 1, \\ N - 1, & otherwise \end{cases}$$

4 Relations On Neutrosophic N-Soft Sets

Definition 4.1 Let $\lambda_{\mathcal{L}}$ and $\lambda_{\mathcal{M}}$ be two NNSSs defined over the universe $(\mathbb{X}, \mathcal{L})$ and $(\mathbb{X}, \mathcal{M})$ respectively. *Neutrosophic N-soft relation* $\breve{\Re}$ is defined as $\breve{\Re}(\xi_i, \xi_j) = \lambda_{\mathcal{L}}(\xi_i) \sqcap_{\mathcal{R}} \lambda_{\mathcal{M}}(\xi_j)$, $\forall \xi_i \in \mathcal{L}$ and $\forall \xi_j \in \mathcal{M}$, where

$$\breve{\mathfrak{R}} : \mathcal{N} \to P(\mathbb{X})$$

is an NNSS over $(\mathbb{X}, \mathcal{N})$, where $\mathcal{N} \sqsubseteq \mathcal{L} \times \mathcal{M}$.

Definition 4.2 The *composition* $\diamond$ of two neutrosophic N-soft relations $\breve{\mathfrak{R}}_1$ and $\breve{\mathfrak{R}}_2$ is defined by

$$(\breve{\mathfrak{R}}_1 \diamond \breve{\mathfrak{R}}_2)(l, n) = \breve{\mathfrak{R}}_1(l, m) \sqcap \breve{\mathfrak{R}}_2(m, n)$$

where $\breve{\mathfrak{R}}_1$ is neutrosophic N-soft relations from $\lambda_{\mathcal{L}}$ to $\lambda_{\mathcal{M}}$ over the universe $(\mathbb{X}, \mathcal{L})$ and $(\mathbb{X}, \mathcal{M})$ respectively and $\breve{\mathfrak{R}}_2$ is neutrosophic N-soft relations from $\lambda_{\mathcal{M}}$ to $\lambda_{\mathcal{N}}$ over the universe $(\mathbb{X}, \mathcal{M})$ and $(\mathbb{X}, \mathcal{N})$ respectively.

Definition 4.3 Let $\breve{\mathfrak{R}}_1$ is neutrosophic N-soft relation over the universe $(\mathbb{X}, \mathcal{L})$ and $\breve{\mathfrak{R}}_2$ is neutrosophic N-soft relation over the universe $(\mathbb{X}, \mathcal{M})$. The *union* and *intersection* of $\breve{\mathfrak{R}}_1$ and $\breve{\mathfrak{R}}_2$ defined as below

$$(\breve{\mathfrak{R}}_1 \sqcup \breve{\mathfrak{R}}_2)(l, m) = \max\{\breve{\mathfrak{R}}_1(l, m), \breve{\mathfrak{R}}_2(l, m)\}$$

$$(\breve{\mathfrak{R}}_1 \sqcap \breve{\mathfrak{R}}_2)(l, m) = \min\{\breve{\mathfrak{R}}_1(l, m), \breve{\mathfrak{R}}_2(l, m)\}$$

where $\breve{\mathfrak{R}}_1 : \mathcal{L} \times \mathcal{M} \to \mathbb{P}(\mathbb{X})$ and $\breve{\mathfrak{R}}_2 : \mathcal{L} \times \mathcal{M} \to \mathbb{P}(\mathbb{X})$.

Definition 4.4 Let $\lambda_{\mathcal{L}}$ in $(\mathbb{X}, \mathcal{L})$ be a neutrosophic N-soft set. Let $\breve{\mathfrak{R}}$ for $\lambda_{\mathcal{L}}$ to $\lambda_{\mathcal{M}}$. Then *max-min-max composition* of neutrosophic N-soft set with $\lambda_{\mathcal{L}}$ is another neutrosophic N-soft set $\lambda_{\mathcal{M}}$ of $(\mathbb{X}, \mathcal{M})$ which is denoted by $\breve{\mathfrak{R}} \diamond \lambda_{\mathcal{L}}$. The membership function, indeterminate function, non-membership function and grading function of $\lambda_{\mathcal{M}}$ are defined, respectively, as

$$\mathbb{T}_{\breve{\mathfrak{R}} \diamond \lambda_{\mathcal{L}}}(m) = \max_l \{\min(\mathbb{T}_{\mathcal{L}}(l), \mathbb{T}_{\mathcal{L}}(l, m))\},$$

$$\mathbb{I}_{\breve{\mathfrak{R}} \diamond \lambda_{\mathcal{L}}}(m) = \min_l \{\max(\mathbb{I}_{\mathcal{L}}(l), \mathbb{I}_{\mathcal{L}}(l, m))\},$$

$$\mathbb{F}_{\breve{\mathfrak{R}} \diamond \lambda_{\mathcal{L}}}(m) = \min_l \{\max(\mathbb{F}_{\mathcal{L}}(l), \mathbb{F}_{\mathcal{L}}(l, m))\},$$

$$r_{\breve{\mathfrak{R}} \diamond \lambda_{\mathcal{L}}}(m) = \max_l \{\min(r_{\mathcal{L}}(l), r_{\mathcal{L}}(l, m))\},$$

$$\forall l \in \mathcal{L}, m \in \mathcal{M}, r_{\mathcal{L}} \in \mathcal{G}.$$

Definition 4.5 Let $\lambda_{\mathcal{L}}$ be a neutrosophic N-soft set. Then the *choice function* of $\lambda_{\mathcal{L}}$ is defined as

$$C(\lambda_{\mathcal{L}}) = r_{\mathcal{L}} + \mathbb{T}_{\mathcal{L}} - \mathbb{I}_{\mathcal{L}} - \mathbb{F}_{\mathcal{L}}$$

Definition 4.6 Let $\lambda_{\mathcal{L}}$ and $\lambda_{\mathcal{M}}$ be two neutrosophic N-soft sets. Then the *score function* of $\lambda_{\mathcal{L}}$ and $\lambda_{\mathcal{M}}$ is defined as

$$\mathcal{S}_{LM} = C(\lambda_{\mathcal{L}}) - C(\lambda_{\mathcal{M}})$$

Definition 4.7 Let $\lambda_{\mathcal{L}}$ be a neutrosophic N-soft set. We define score function for $\lambda_{\mathcal{L}}$ as

$$\mathcal{S}_L = r_i + \mathbb{T}_i - \mathbb{I}_i \mathbb{F}_i$$

5 Application of Neutrosophic N-Soft Set to Medical Diagnosis

In this Section, we discuss the execution of N-soft set and neutrosophic set in medical diagnosis . In some previous studies of the neutrosophic set and neutrosophic soft set, there are many examples of medical diagnosis but all of them have lack of parameterized evaluation characterization. First we propose Algorithm 1 as given below.

Algorithm 1

Step 1: Input a set $\mathfrak{P}$ of patients, a set $\mathcal{S}$ of symptoms as parameter set and a set $\mathfrak{D}$ of diseases .

Step 2: Construct a relation $\mathfrak{L}(\mathfrak{P} \hookrightarrow \mathcal{S})$ between the patients and symptoms.

Step 3: Construct a relation a relation $\mathfrak{M}(\mathcal{S} \hookrightarrow \mathfrak{D})$ between the symptoms and the diseases.

Step 4: Compute the composition relation $\mathfrak{N}(\mathfrak{P} \hookrightarrow \mathfrak{D})$ the relation of patients and diseases by using Definition 4.4.

Step 5: Obtain the choice function of $\mathfrak{N}$ by using Definition 4.5.

Step 6: Choose the highest choice value of patient corresponding to disease gives the higher possibility of the patient affected with the respective disease.

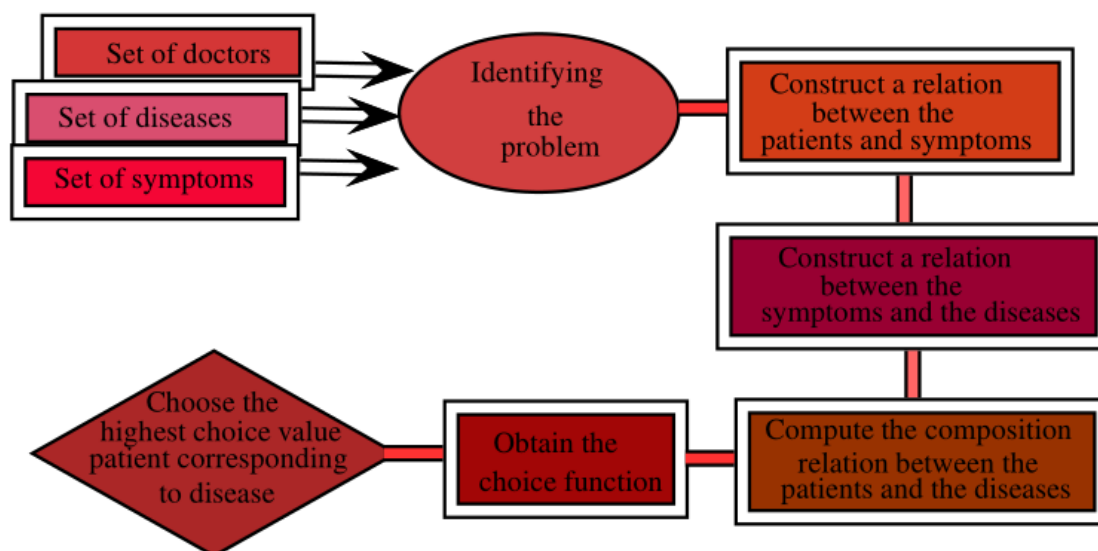Flow chart portrayal of Algorithm 1 is given in Figure 1:

Figure   1: Flow chart representation of Algorithm 1

Now we demonstrate how neutrosophic N-soft set (NNSS) can be efficiently employed in multi-criteria group decision making (MCGDM). First of all, we propose an extension of TOPSIS to NNSS. In this study, we choose TOPSIS because our goal is to solve a medical diagnosis decision making problem. Since medical diagnosis involves similarities (in symptoms) and TOPSIS method is most appropriate method for handling such problems. A detailed study of TOPSIS may be found in [26]. The procedural steps of Neutrosophic N-soft set TOPSIS Method to examine critical situation of each patient is given in Algorithm 2.

Algorithm 2 (Neutrosophic N-soft set TOPSIS Method)

Step 1: Constructing weighed parameter matrix $\mathcal{H}$ by using ranking values obtained in Step 4 of Algorithm 1 composition relation $\mathfrak{N}(\mathfrak{P} \hookrightarrow \mathfrak{D})$ and relates it with linguistic ratings from Table 26.

$$\mathcal{H} = \begin{bmatrix} r_{11} & r_{12} & \cdots & r_{1n} \\ r_{21} & r_{12} & \cdots & r_{2n} \\ \vdots & \vdots & & \vdots \\ r_{i1} & r_{i2} & \cdots & r_{in} \\ \vdots & \vdots & & \vdots \\ r_{m1} & r_{m2} & \cdots & r_{mn} \end{bmatrix} = [r_{ij}]_{m \times n}$$

Step 2: Creating normalized decision matrix $\mathcal{B}$. Throughout from now, we shall use

$L_n = \{1,2,3,\cdots,n\} \ \forall n \in N$

$$b_{ij} = \frac{r_{ij}}{\sqrt{\sum_{k=1}^{m} r_{kj}^2}} \tag{1}$$

$$\mathcal{B} = \begin{bmatrix} b_{11} & b_{12} & \cdots & b_{1n} \\ b_{21} & b_{12} & \cdots & b_{2n} \\ \vdots & \vdots & & \vdots \\ b_{i1} & b_{i2} & \cdots & b_{in} \\ \vdots & \vdots & & \vdots \\ b_{m1} & b_{m2} & \cdots & b_{mn} \end{bmatrix} = [b_{ij}]_{m \times n}$$

Step 3: Creating weighted vector $\mathbf{W} = \{\mathbf{W}_1, \mathbf{W}_2, \mathbf{W}_3, \cdots, \mathbf{W}_n\}$ by using the expression

$$\mathbf{W}_j = \frac{\mathbf{w}_j}{\sum_{k=1}^{m} \mathbf{w}_k}, \mathbf{w}_k = \frac{1}{m} \sum_{i=1}^{m} b_{ij} \qquad (2)$$

Step 4: Constructing weighted decision matrix μ.

$$\mu = \begin{bmatrix} \mu_{11} & \mu_{12} & \cdots & \mu_{1n} \\ \mu_{21} & \mu_{12} & \cdots & \mu_{2n} \\ \vdots & \vdots & & \vdots \\ \mu_{i1} & \mu_{i2} & \cdots & \mu_{in} \\ \vdots & \vdots & & \vdots \\ \mu_{m1} & \mu_{m2} & \cdots & \mu_{mn} \end{bmatrix} = [\mu_{ij}]_{m \times n}$$

where $\mu_{ij} = \mathbf{W}_j b_{ij}$ $\qquad (3)$

Step 5: Finding positive ideal solution (PIS) and negative ideal solution (NIS) by using the Equations

$$PIS = \{\mu_1^+, \mu_2^+, \mu_3^+, \cdots, \mu_j^+ \cdots, \mu_n^+\} = \{\max(\mu_{ij}) : i \in L_n\} \qquad (4)$$

$$NIS = \{\mu_1^-, \mu_2^-, \mu_3^-, \cdots, \mu_j^- \cdots, \mu_n^-\} = \{\min(\mu_{ij}) : i \in L_n\} \qquad (5)$$

Step 6: Calculate separation measurements of PIS ($\mathcal{S}_i^+$) and NIS ($\mathcal{S}_i^-$) for each parameter by using the equations

$$\mathcal{S}_i^+ = \sqrt{\sum_{j=1}^{n} (\mu_{ij} - \mu_j^+)^2}, \quad \forall i \in L_m \qquad (6)$$

and

$$\mathcal{S}_i^- = \sqrt{\sum_{j=1}^{n} (\mu_{ij} - \mu_j^-)^2}, \quad \forall i \in L_m \qquad (7)$$

Step 7: Calculating of relative closeness of alternative to the ideal solution by using the equation

$$\mathcal{C}_i^+ = \frac{\mathcal{S}_i^-}{\mathcal{S}_i^- + \mathcal{S}_i^+}, \quad 0 \le \mathcal{C}_i^+ \le 1, \quad \forall i \in L_m \qquad (8)$$

Step 8: Ranking the preference order.

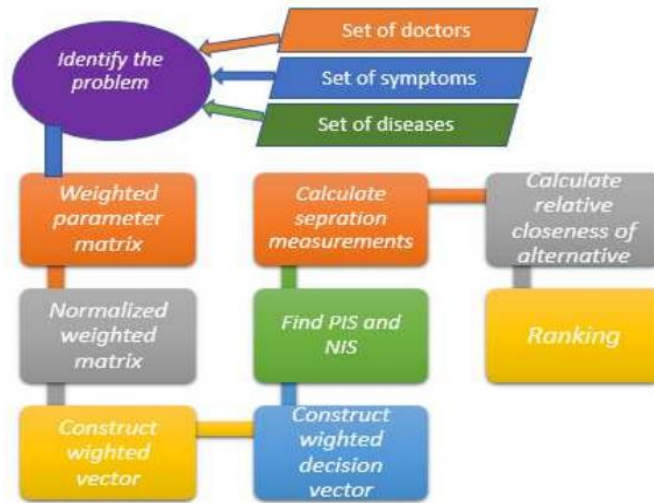Flow chart portrayal of neutrosophic N-soft set TOPSIS method is shown in Figure 2.

Figure 2: Flow chart of neutrosophic N-soft set TOPSIS method

5.1 Numerical Example

Now we employ the above **Algorithm 1** to find the decision factor about the following top four deadliest diseases in the world. Due to the following risk factors, these diseases progress slowly. Here is some detail about these diseases:

D_1: **Coronary artery disease (CAD)**

CAD occurs when the vessels that transfer blood towards heart become narrowed. CAD leads to heart failure, arrhythmias and chest pain. Risk factors for CAD are

High blood pressure	High cholesterol	Smoking
Family history of CAD	Diabetes	Obesity

Table 16: Risk factors for CAD

D_2: **Stroke**

This fatal disease occurs when some artery is in brain blocked or leaks. The risk factors for Stroke are:

High blood pressure	Being female	Smoking
Family history of stroke	Being American	Being African

Table 17: Risk factors for Stroke

D_3: **Lower respiratory infections (LRI)**

This disease occurs due to tuberculosis, pneumonia, influenza, flu, or bronchitis. Risk factors for LRI contain

Poor air quality	Asthma	Smoking
Weak immune system	HIV	Crowded child-care settings

Table 18: Risk factors for LRI

D_4: **Chronic obstructive pulmonary disease (COPD)**

This disease is a long-term, progressive lung disease that makes breathing difficult. Risk factors for COPD are

Family history	Lungs irritation
History of respiratory infections	Smoking

Table 19: Risk factors for COPD

D_5: **Trachea, bronchus and lungs cancers**

Respiratory cancers incorporate diseases of the bronchus, larynx, lungs and trachea. The risk factors for Trachea, bronchus and lungs cancers involve

Use of coal for cooking	Tobacco usage	Poor air quality
Family history of disease	Smoking	Diesel fumes

Table 20: Risk factors for Trachea, bronchus and lungs cancers

Core in certain sense is the most basic part occurring in the considered knowledge. Core can be translated as the arrangement of most trademark some portion of knowledge, which cannot be abstained from when decreasing the data. The core risk factor of all diseases discussed above is "smoking". For computational purpose, let's decide the grading values depending upon the degree of membership function as in Table 21:

Degree of membership function	Grading values
$\mathbb{T} = 0$	0
$0 < \mathbb{T} \leq 0.2$	1
$0.2 < \mathbb{T} \leq 0.4$	2
$0.4 < \mathbb{T} \leq 0.6$	3
$0.6 < \mathbb{T} \leq 0.8$	4
$0.8 < \mathbb{T} \leq 1.0$	5

Table 21: Ranking scale

Table 22 yields relation between symptoms and patients:

$\mathfrak{L}$	Headache($\mathfrak{s}_1$)	Shortness of breath($\mathfrak{s}_2$)	Angina($\mathfrak{s}_3$)
$\mathfrak{p}_1$	$(\langle 0.7,0.2,0.5\rangle,4)$	$(\langle 0.6,0.3,0.4\rangle,3)$	$(\langle 0.4,0.6,0.5\rangle,2)$
$\mathfrak{p}_2$	$(\langle 0.9,0.3,0.1\rangle,5)$	$(\langle 0.7,0.4,0.3\rangle,4)$	$(\langle 0.8,0.5,0.2\rangle,4)$
$\mathfrak{p}_3$	$(\langle 0.6,0.6,0.4\rangle,3)$	$(\langle 0.5,0.5,0.8\rangle,3)$	$(\langle 0.2,0.4,0.8\rangle,1)$
$\mathfrak{p}_4$	$(\langle 0.2,0.5,0.8\rangle,1)$	$(\langle 0.3,0.1,0.7\rangle,2)$	$(\langle 0.7,0.1,0.3\rangle,4)$

Table 22: Relation between symptoms and patients

The relation between the symptoms and the diseases is given in Table 23:

$\mathfrak{M}$	$\mathfrak{D}_1$	$\mathfrak{D}_2$	$\mathfrak{D}_3$	$\mathfrak{D}_4$
Headache($\mathfrak{s}_1$)	$(\langle 0.8,0.4,0.2\rangle,4)$	$(\langle 0.9,0.2,0.1\rangle,5)$	$(\langle 0.6,0.3,0.4\rangle,3)$	$(\langle 0.7,0.5,0.3\rangle,4)$
Shortness of breath($\mathfrak{s}_2$)	$(\langle 0.1,0.8,0.9\rangle,1)$	$(\langle 0.2,0.9,0.8\rangle,1)$	$(\langle 0.5,0.7,0.5\rangle,3)$	$(\langle 0.3,0.7,0.6\rangle,2)$
Angina($\mathfrak{s}_3$)	$(\langle 0.5,0.7,0.5\rangle,3)$	$(\langle 0.4,0.6,0.6\rangle,2)$	$(\langle 0.3,0.5,0.7\rangle,2)$	$(\langle 0.9,0.1,0.1\rangle,5)$

Table 23: Relation between the symptoms and the diseases

The composition relation of patients and diseases in Table 24:

$\mathfrak{N}$	$\mathfrak{D}_1$	$\mathfrak{D}_2$	$\mathfrak{D}_3$	$\mathfrak{D}_4$
$\mathfrak{p}_1$	$(\langle 0.7,0.4,0.5\rangle,4)$	$(\langle 0.7,0.2,0.5\rangle,4)$	$(\langle 0.6,0.3,0.5\rangle,3)$	$(\langle 0.7,0.5,0.5\rangle,4)$
$\mathfrak{p}_2$	$(\langle 0.8,0.4,0.2\rangle,4)$	$(\langle 0.9,0.3,0.1\rangle,5)$	$(\langle 0.6,0.3,0.4\rangle,3)$	$(\langle 0.7,0.5,0.2\rangle,4)$
$\mathfrak{p}_3$	$(\langle 0.6,0.6,0.4\rangle,3)$	$(\langle 0.6,0.6,0.4\rangle,3)$	$(\langle 0.6,0.6,0.4\rangle,3)$	$(\langle 0.60.4,0.4\rangle,3)$
$\mathfrak{p}_4$	$(\langle 0.5,0.5,0.5\rangle,3)$	$(\langle 0.4,0.5,0.6\rangle,2)$	$(\langle 0.3,0.5,0.7\rangle,2)$	$(\langle 0.7,0.5,0.3\rangle,4)$

Table 24: Composition relation of patients and diseases

Table 25 gives choice values of the relation $\mathfrak{N}$:

$\mathfrak{N}$	$\mathfrak{D}_1$	$\mathfrak{D}_2$	$\mathfrak{D}_3$	$\mathfrak{D}_4$
$\mathfrak{p}_1$	3.8	4	2.8	3.7
$\mathfrak{p}_2$	4.2	5.5	2.9	4
$\mathfrak{p}_3$	2.6	2.6	2.6	2.8
$\mathfrak{p}_4$	2.5	1.3	1.1	3.9

Table 25: Choice values of relation $\mathfrak{N}$

From Table 25, we conclude that the patients $\mathfrak{p}_1$ and $\mathfrak{p}_2$ are likely to be suffering from $\mathfrak{D}_2$ whereas $\mathfrak{p}_3$ and $\mathfrak{p}_4$ are suffering from $\mathfrak{D}_4$.

In order to examine the intensity level of the disease of the patients, we use **neutrosophic N-soft TOPSIS** method which is demonstrated in **Algorithm 2**. First, we decide the grading values as a function of linguistic terms as Table 26:

Linguistic Terms	Grading Values
Undetermined (U)	0
Very Stable (VS)	1
Stable (S)	2
Grave (G)	3
Critical (C)	4
Very Critical (VC)	5

Table 26: Linguistic terms for evaluation of parameters

Now we construct weighted parameter matrix by using Step 9 and Table 26 as

$$\mathcal{H} = \begin{bmatrix} 4 & 4 & 3 \\ 4 & 5 & 3 \\ 3 & 3 & 3 \\ 3 & 2 & 2 \end{bmatrix} = \begin{bmatrix} C & C & G \\ C & VC & G \\ G & G & G \\ G & S & S \end{bmatrix}$$

Creating normalized decision matrix $\mathcal{B}$ by using Equation 1

$$\mathcal{B} = \begin{bmatrix} 0.57 & 0.54 & 0.54 & 0.53 \\ 0.57 & 0.68 & 0.54 & 0.53 \\ 0.43 & 0.41 & 0.54 & 0.40 \\ 0.43 & 0.27 & 0.36 & 0.53 \end{bmatrix}$$

Now by using Equation 2 construct weight vector

$$\mathbf{W} = \{\mathbf{W}_1, \mathbf{W}_2, \mathbf{W}_3, \mathbf{W}_4\} = \{0.58, 0.14, 0.14, 0.14\}$$

By using Equation 3 the weighted decision matrix μ is

$$\mu = \begin{bmatrix} 0.33 & 0.07 & 0.07 & 0.07 \\ 0.33 & 0.09 & 0.07 & 0.07 \\ 0.25 & 0.06 & 0.07 & 0.06 \\ 0.25 & 0.04 & 0.05 & 0.07 \end{bmatrix}$$

The positive ideal solution (PIS) and negative ideal solution (NIS) by using the Equations 4 and 5 as

$$PIS = \{0.33, 0.09, 0.07, 0.07\}$$
$$NIS = \{0.25, 0.04, 0.05, 0.06\}$$

The separation measurements of PIS and NIS for each parameter by using the Equations 6 and 7 are

$$\mathcal{S}_1^+ = 0.11$$
$$\mathcal{S}_2^+ = 0.06$$
$$\mathcal{S}_3^+ = 0.02$$
$$\mathcal{S}_4^+ = 0.01$$
$$\mathcal{S}_1^- = 0.11$$
$$\mathcal{S}_2^- = 0.06$$
$$\mathcal{S}_3^- = 0.03$$
$$\mathcal{S}_4^- = 0.02$$

The relative closeness of alternatives to the ideal solution by using Equation 8 are

$$\mathcal{C}_1^+ = 0.5$$
$$\mathcal{C}_2^+ = 0.5$$

$$\mathcal{C}_3^+ = 0.6$$
$$\mathcal{C}_4^+ = 0.7$$

Ranking the preference order is

$$\mathcal{C}_4^+ \geq \mathcal{C}_3^+ \geq \mathcal{C}_2^+ \geq \mathcal{C}_1^+$$

which indicates that condition of patient $\mathfrak{p}_4$ is most critical. The pictorial representation of the rankings of the patients is demonstrated with the assistance of a chart as given in Figure 3.
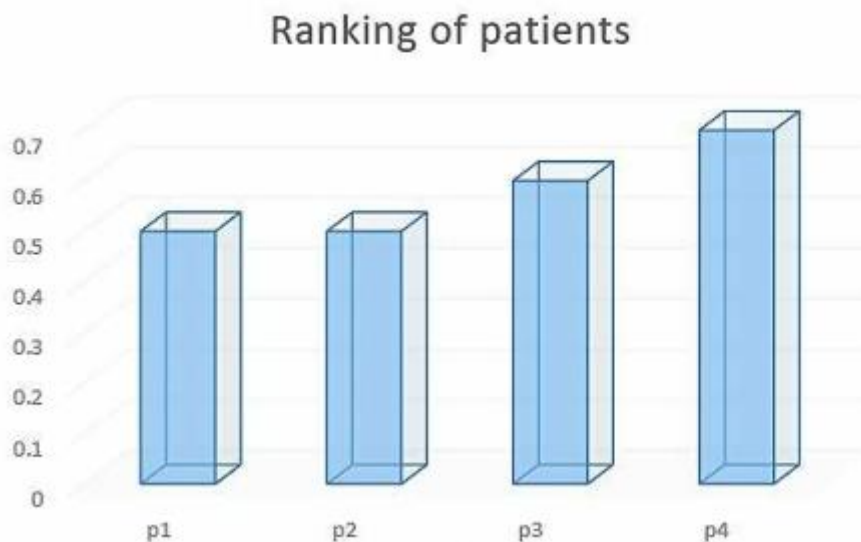

Figure 3: Ranking of patients w.r.t. intensity level of disease

5. Conclusion

The purpose of this work is to lay the foundation of theory of neutrosophic N-soft set as a hybrid model of neutrosophic sets and N-soft sets. We established some basic operations on neutrosophic N-soft sets along with their fundamental properties. We introduced the notions of NNS-subset, null-NNS, absolute-NNS, complements of NNS, truth-favorite, falsity-favorite, relations on NNS, composition of NNSS and score function of NNS. We explained these concepts with the help of illustrations. We presented a novel application of multi-attribute decision-making (MADM) based on neutrosophic N-soft set by using Algorithm 1. We proposed neutrosophic N-soft sets TOPSIS method as demonstrated in Algorithm 2 for MADM in medical diagnosis. We defined separation measurements of positive ideal solution and negative ideal solution to compute a relative closeness to identify the optimal alternative. Lastly, a numerical example is given to illustrate the developed method for medical diagnosis.

This may be the starting point for neutrosophic N-soft set mathematical concepts and information structures that are based on neutrosophic set and N-soft set theoretic operations. We have studied a few concepts only, it will be necessary to carry out more theoretical research to recognize a general framework for the practical applications. The proposed model of neutrosophic N-soft set can be elaborated with new research topics such as image processing, expert systems, soft computing techniques, fusion rules, cognitive maps, graph theory and decision-making of real world problems.

We hope that this study will prove a ground-breaking and will open new doors for the vibrant researchers in this field.

Acknowledgement

The authors are highly thankful to the Editor-in-chief and the referees for their valuable comments and suggestions for improving the quality of our paper.

Conflicts of Interest

The authors declare that they have no conflict of interest.

References

[1] M. Akram, A. Adeel, J. C. R. Alcantud, *Fuzzy N-soft sets: A novel model with applications*, Journal of Intelligent & Fuzzy Systems, **35**(4)(2018), 4757-4771.

[2] M. Akram, A. Adeel, J. C. R. Alcantud, *Group decision-making methods based on hesitant N-soft sets*, Expert System With Applications, **115**(2019), 95-105.

[3] M. Akram, G.Ali, J. C. R. Alcantud, *New decision-making hybrid model: intuitionistic fuzzy N-soft rough sets*, Soft Computing, (2019), doi:/10.1007/s00500-019-03903-w.

[4] M. Akram and M. Arshad, *A Novel Trapezoidal Bipolar Fuzzy TOPSIS Method for Group Decision-Making*, Group Decision and Negotiation, (2018) doi.org/10.1007/s10726-018-9606-6.

[5] M. Akram and A. Adeel, *TOPSIS Approach for MAGDM Based on Interval-Valued Hesitant Fuzzy N-Soft Environment*, International Journal of Fuzzy Systems, (2018) doi.org/10.1007/s40815-018-0585-1.

[6] H. Aktas and N. Çağman, *Soft sets and soft group*, Information Sciences, **1**(77)(2007), 2726-2735.

[7] M. I. Ali, F. Feng, X. Y. Liu, W. K. Min and M. Shabir, *On some new operations in soft set theory*, Computers and Mathematics with Applications, **57**(2009), 1547-1553.

[8] M. I. Ali, *A note on soft sets, rough soft sets and fuzzy soft sets*, Applied Soft Computing, **11**(2011), 3329-3332.

[9] I. Arockiarani, I. R. Sumathi, J. Martina, Jency, *Fuzzy neutrosophic soft toplogical spaces*, Inf. J. Math. Arch., **4**(10) (2013), 225-238.

[10] K. T. Atanassov, *Intuitionistic fuzzy sets*, in: V. Sgurev, Ed., VII ITKRs Session, Sofia, June 1983 (Central Sci. and Techn. Library, Bulg. Academy of Sciences, 1984).

[11] K. T. Atanassov, *Intuitionistic fuzzy sets*, Fuzzy Sets and Systems, **20**(1986), 87-96.

[12] N. Çağman, S. Karataş and S. Enginoglu, *Soft topology*, Computers and Mathematics with Applications, **62**(2011), 351-358.

[13] C. T. Chen, *Extensions of the TOPSIS for group decision-making under fuzzy environment*, Fuzzy Sets Syst, **1**(114)(2000), 1-9.

[14] T. Y. Chen and C. Y. Tsao, The interval-valued fuzzy TOPSIS method and experimental analysis. Fuzzy Sets Syst, **159**(11)(2008), 1410-1428.

[15] P. P. Dey, S. Pramanik and B. C. Giri, TOPSIS for solving multi-attribute decision making problems under bipolar neutrosophic environment. In: Smarandache F, Pramanik S (eds) New trends in neutrosophic theory and applications. Pons Editions, Brussels (2016), 65-77.

[16] B. Davvaz and E. H. Sadrabadi, *An application of intuitionistic fuzzy sets in medicine*, International Journal of Biomathematics, **9**(3)(2016), 1-15.

[17] F. Fatimah, D. Rosadi, R. B. F. Hakim and J. C. R. Alcantud, *N-soft sets and their decision-making algoritms*, Soft Computing, **22**(2018), 3829-3842.

[18] F. Feng, Y. B. Jun, X. Liu and L. Li, *An adjustable approach to fuzzy soft set based decision making*, Journal of Computational and Applied Mathematics, **234**(1)(2010), 10-20.

[19] F. Feng, C. Li, B. Davvaz, M. I. Ali, *Soft sets combined with fuzzy sets and rough sets*, a tentative approach, Soft Computing, **14** (9)(2010) 899-911.

[20] H. Garg and R. Arora, *Generalized intuitionistic fuzzy soft power aggregation operator based on t-norm and their application in multicriteria decision-making*, International Journal of Intelligent Systems, **34**(2)(2019), 215-246.

[21] H. Garg and R. Arora, *Dual hesitant fuzzy soft aggregation operators and their applicatio in decision-making*, Cognitive Computation, **10**(5)(2018), 769-789.

[22] H. Garg and R. Arora, *A nonlinear-programming methodology for multi-attribute decision-making problem with interval-valued intuitionistic fuzzy soft sets information*, Applied Intelligence, **48**(8)(2018), 2031-2046.

[23] H. Garg and R. Arora, *Novel scaled prioritized intuitionistic fuzzy soft interaction averaging aggregation operators and their application to multi criteria decision-making*, Engineering Applications of artificial Intelligence (EAAI), **71**(2018), 100-112.

[24] M. R. Hashmi, M. Riaz and F. Smarandache, *m-polar Neutrosophic Topology with Applications to Multi-Criteria Decision-Making in Medical Diagnosis and Clustering Analysis*, International Journal of Fuzzy Systems, **22**(1)(2020), 273-292. doi.org/10.1007/s40815-019-00763-2.

[25] M. R. Hashmi and M. Riaz, *A Novel Approach to Censuses Process by using Pythagorean m-polar Fuzzy Dombi's Aggregation Operators*, Journal of Intelligent & Fuzzy Systems, **38**(2)(2020), 1977-1995. DOI: 10.3233/JIFS-190613.

[26] C. L. Hwang and K. Yoon, *Methods for multiple attribute decision-making*, In: Fandel G, TrockelW(eds) Multiple attribute decision making. Springer, Berlin, (1981), 58-191.

[27] M. B. Gorzalzany, *A method of inference in approximate reasoning based on interval-valued fuzzy sets*, Fuzzy Sets and Systems, **21**(1987), 1-17.

[28] D. Joshi and S. Kumar, *Intuitionistic fuzzy entropy and distance measure based TOPSIS method for multi-criteria decision making*, Egypt Inf J, **15**(2014), 97–104.

[29] K. Naeem, M. Riaz, X. D. Peng and D. Afzal, *Pythagorean Fuzzy Soft MCGDM Methods Based on TOPSIS, VIKOR and Aggregation Operators*, Journal of Intelligent & Fuzzy Systems, **37**(5)(2019), 6937-6957. DOI:10.3233/JIFS- 190905.

[30] K. Naeem, M. Riaz, and D. Afzal, *Pythagorean m-polar Fuzzy Sets and TOPSIS method for the Selection of Advertisement Mode*, Journal of Intelligent & Fuzzy Systems, **37**(6)(2019), 8441-8458. DOI: 10.3233/JIFS-191087.

[31] D. F. Li and J. X. Nan, *Extension of the TOPSIS for muti-attribute group decision making under Atanassov IFS environments*, Int J Fuzzy Syst Appl, **1**(4)(2011), 47-61.

[32] D. Molodtsov, *Soft set theoty-first results*, Computers and Mathematics with Applications, **37**(4-5)(1999), 19-31.

[33] N. A. Nabeeh, M. A. Basset, H. A. El-Ghareeb and A. Aboelfetouh, *Neutrosophic Multi-Criteria Decision Making Approach for IoT-Based Enterprises*, IEEE Access, **7**(2019), 59559-59574, DOI:10.1109/ACCESS.2019.2908919.

[34] N. A. Nabeeh, F. Smarandache, M. A. Basset, H. A. El-Ghareeb and A. Aboelfetouh, *An Integrated Neutrosophic-TOPSIS Approach and Its Application to Personnel Selection: A New Trend in Brain Processing and Analysis*, IEEE Access, **7**(2019), 29734-29744, DOI: 10.1109/ACCESS.2019.2899841.

[35] V. Chang, M. A. Basset and M. Ramachandran, *Towards a Reuse Strategic Decision Pattern Framework-from Theories to Practices*, IEEE Access, **21**(1)(2019), 27-44, DOI:10.1007/s10796-018-9853-8.

[36] X. D. Peng and J. Dai, *Approaches to single-valued neutrosophic MADM based on MABAC, TOPSIS and new similarity measure with score function*, Neural Computing and Applications, **29**(10)(2018), 939-954.

[37] M. Riaz, N. Çağman, I. Zareef and M. Aslam, *N-Soft Topology and its Applications to Multi-Criteria Group Decision Making*, Journal of Intelligent & Fuzzy Systems, **36**(6)(2019), 6521-6536. DOI:10.3233/JIFS-182919. DOI:10.3233/JIFS-182919.

[38] M. Riaz and M. R. Hashmi, *MAGDM for agribusiness in the environment of various cubic m-polar fuzzy averaging aggregation operators*, Journal of Intelligent and Fuzzy Systems, (2019). DOI:10.3233/JIFS-182809.

[39] M. Riaz and M. R. Hashmi, *Linear Diophantine Fuzzy Set and its Applications towards Multi-Attribute Decision Making Problems*, Journal of Intelligent and Fuzzy Systems, (2019). DOI:10.3233/JIFS-190550.

[40] M. Riaz and M. R. Hashmi, *Soft Rough Pythagorean m-Polar Fuzzy Sets and Pythagorean m-Polar Fuzzy Soft Rough Sets with Application to Decision-Making*, Computational and Applied Mathematics, (2019), DOI: 10.1007/s40314-019-0989-z.

[41] M. Riaz and S. T. Tehrim, *Certain properties of bipolar fuzzy soft topology via Q-neighborhood*, Punjab University Journal of Mathematics, **51**(3)(2019), 113-131.

[42] M. Riaz and S. T. Tehrim, *Cubic bipolar fuzzy ordered weighted geometric aggregation operators and their application using internal and external cubic bipolar fuzzy data*, Computational & Applied Mathematics, **38**(2)(2019), 1-25.

[43] M. Riaz and S. T. Tehrim, *Multi-attribute group decision making based cubic bipolar fuzzy information using averaging aggregation operators*, Journal of Intelligent & Fuzzy Systems, **37**(2)(2019), 2473-2494.

[44] M. Riaz and S. T. Tehrim, *Bipolar Fuzzy Soft Mappings with Application to Bipolar Disorders*, International Journal of Biomathematics, (2019). Doi.org/10.1142/S1793524519500803.

[45] S. T. Tehrim and M. Riaz, *A novel extension of TOPSIS to MCGDM with Bipolar Neutrosophic soft topology*, Journal of Intelligent and Fuzzy Systems, (2019). DOI:10.3233/JIFS-190668.

[46] F. Smarandache, *Neutrosophy Neutrosophic Probability, Set and Logic*, American Research Press, (1998), Rehoboth, USA.

[47] F. Smarandache, *Neutrosophic set-A generalisation of the intuitionistic fuzzy sets*, International Journal of Pure and Applied Mathematics, **24**(3)(2005), 287-297.

[48] F. Smarandache, *A Unifying Field in Logics: Neutrosophic Logic. Neutrosophy, Neutrosophic Set, Neutrosophic Probability and Statistics* (second, third, fourth respectively fifth edition), American Research Press, 155 p., 1999, 2000, 2005, 2006.

[49] F. Smarandache, *Neutrosophic Overset, Neutrosophic Underset, and Neutrosophic Offset. Similarly for Neutrosophic Over-/Under-/Off- Logic, Probability, and Statistics*, 168 p., Pons Editions, Bruxelles, Belgique (2016).

[50] F. Smarandache, *Neutrosophic Set is a Generalization of Intuitionistic Fuzzy Set, Inconsistent Intuitionistic Fuzzy Set (Picture Fuzzy Set, Ternary Fuzzy Set), Pythagorean Fuzzy Set (Atanassov's Intuitionistic Fuzzy Set of second type), q-Rung Orthopair Fuzzy Set, Spherical Fuzzy Set, and n-HyperSpherical Fuzzy Set, while Neutrosophication is a Generalization of Regret Theory, Grey System Theory, and Three-Ways Decision*, Journal of New Theory, **29**(2019), 01-35. https://arxiv.org/ftp/arxiv/papers/1911/1911.07333.pdf.

[51] M. Shabir and M. Naz, *On soft topological spaces*, Computers and Mathematics with Applications, **61**(2011), 1786-1799.

[52] H. Wang, F. Smarandache, Y. Q. Zhang, R. Sundrraman, *Single valued neutrosophic sets,* Multispace and Multistructure, **4** (2010), 410-413.

[53] Z. Xu and X. Zhang, *Hesitant fuzzy multi-attribute decision-making based on TOPSIS with incomplete weight information*. Knowl. Based Syst., **52**(2013), 53-64.

[54] L. A. Zadeh, *Fuzzy sets*, Information and Control, **8**(1965), 338-353.

[55] X. Zhang, Z. S. Xu, *Extension of TOPSIS to multiple criteria decision making with pythagorean fuzzy sets*, International Journal of Intelligent Systems, **29**(2014), 1061-1078.

[56] M. Riaz, F. Smarandache, A. Firdous and A. Fakhar, On Soft Rough Topology with Multi-Attribute Group Decision Making, Mathematics 7(1)(2019), 1-18.

[57] M. Riaz, B. Davvaz, A. Firdous and A. Fakhar, Novel Concepts of Soft Rough Set Topology with Applications, Journal of Intelligent & Fuzzy Systems 36(4) (2019) 3579-3590. DOI:10.3233/JIFS-181648.